WAVE ASYMPTOTICS FOR WAVEGUIDES AND MANIFOLDS WITH INFINITE CYLINDRICAL ENDS

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Abstract. We describe wave decay rates associated to embedded resonances and spectral thresholds for waveguides and manifolds with infinite cylindrical ends. We show that if the cut-off resolvent is polynomially bounded at high energies, as is the case in certain favorable geometries, then there is an associated asymptotic expansion, up to a $O(t^{-k_0})$ remainder, of solutions of the wave equation on compact sets as $t \to \infty$. In the most general such case we have $k_0 = 1$, and under an additional assumption on the infinite ends we have $k_0 = \infty$. If we localize the solutions to the wave equation in frequency as well as in space, then our results hold for quite general waveguides and manifolds with infinite cylindrical ends.

To treat problems with and without boundary in a unified way, we introduce a black box framework analogous to the Euclidean one of Sjöstrand and Zworski. We study the resolvent, generalized eigenfunctions, spectral measure, and spectral thresholds in this framework, providing a new approach to some mostly well-known results in the scattering theory of manifolds with cylindrical ends.

1. Introduction

Wave decay rates on a manifold of infinite volume can be related to the geometry of the manifold via the behavior of the resolvent $(-\Delta - z)^{-1}$ in the vicinity of the spectrum. A particularly important and long-studied class of problems is that of compactly supported perturbations of Euclidean space, an example of which is the classical obstacle scattering problem. In that case the dominant contributions to wave decay rates come from the resolvent behavior near the only threshold in the spectrum, $z = 0$, and as $\operatorname{Re} z \to +\infty$. We think of the former as being related to the geometric infinity–here the spatial dimension is especially important. The latter typically reflects the dynamics of the compact, and possibly empty, set of trapped geodesics. In particular, in this setting we can separate the contributions of the geometric infinity and the trapped set. Similar results hold in many situations where infinity is "large" in a suitable sense, such as on asymptotically Euclidean, conic, and hyperbolic manifolds.

In this paper we consider manifolds which are isometric to a cylinder $(0, \infty) \times Y$ (with Y compact) outside of a compact set. Thus we cannot separate the geometric infinity and the trapped geodesics, since the latter occur outside of arbitrarily large compact sets. Also in contrast with the case of obstacle scattering is the relatively complicated nature of the spectrum of the Laplacian. The continuous spectrum has infinitely many thresholds, given by the eigenvalues of the Laplacian on Y, where the multiplicity increases. In addition, there may be up to infinitely many embedded resonances and eigenvalues.

A motivation for the study of such manifolds comes from waveguides and quantum dots connected to leads. These appear in certain models of electron motion in semiconductors and of propagation of electromagnetic and sound waves. We give just a few pointers to the physics and applied math literature here [LCM99, Rai00, RBBH12, EK15, BGW17].

Our main results concern manifolds with infinite cylindrical ends for which the resolvent is well-behaved at high energies, which is the case in some favorable geometric situations discussed below. In this case we can compute asymptotics of the wave equation in terms of the features of the spectrum discussed above. Roughly speaking, a resonance at zero or any eigenvalues contribute non-decaying terms; any other embedded resonances, which can occur only at thresholds, contribute terms decaying like $t^{-1/2-k}$, with $k \in \mathbb{N}_0$; and non-resonant thresholds contribute terms decaying like $t^{-3/2-k}$.

More specifically, in this paper we study asymptotic expansions as $t \to \infty$ of solutions to the wave equation

$$(\partial_t^2 - \Delta)u(t) = 0, \quad u(0) = f_1, \quad \partial_t u(0) = f_2, \tag{1.1}$$

where $\Delta \leq 0$ is the Laplacian on a suitable Riemannian manifold (X, g) with infinite cylindrical ends, and f_1 and f_2 are suitable initial conditions.

Our main results allow us to replace $-\Delta$ with a more general self-adjoint operator H but require an assumption on the high energy behavior of the cut-off resolvent of $-\Delta$ or H. The companion paper [CDa] gives a technique of constructing manifolds with infinite cylindrical ends so that such estimates hold for the resolvent of the Laplacian, or in fact for the resolvent of many Schrödinger operators; see Sections 1.2.1 and 1.2.2 of this paper for examples. The paper [CDb] studies the Dirichlet Laplacian on domains in $\mathbb{R}^d$ which are "star-shaped" in a certain sense, and shows that such estimates hold for the high-energy resolvent; see Section 1.2.3 for examples of domains satisfying these conditions. However, if we apply a spectral cut-off to our solution u of (1.1), the hypothesis on the high energy resolvent is unnecessary–see Proposition 4.2.

Our starting point is the following elementary result for the wave equation on a Riemannian product. We will see below that many aspects of this result carry over to a range of more complicated geometries.

1.1. **The wave equation on a half cylinder.** Let $(X, g) = ((0, \infty)_r \times Y_y, dr^2 + g_Y)$, where (Y, g_Y) is a compact Riemannian manifold without boundary. Let $\Delta_Y \leq 0$ be the Laplacian on (Y, g_Y), and let $\{\phi_j\}_{j=0}^{\infty}$ be a complete orthonormal set of eigenfunctions of Δ_Y, with $-\Delta_Y \phi_j = \sigma_j^2 \phi_j$, $0 = \sigma_0 \leq \sigma_1 \leq \cdots$.

Let $f_1,\ f_2 \in C_c^\infty(X)$. We shall consider solutions u_D and u_N, satisfying Dirichlet and Neumann boundary conditions respectively, to (1.1) on X. Then we can solve (1.1) by separating variables, writing

$$f_\ell = f_\ell(r, y) = \sum_{j=0}^{\infty} f_{\ell,j}(r)\phi_j(y), \qquad u_B(t) = u_B(t, r, y) = \sum_{j=0}^{\infty} u_{j,B}(t, r)\phi_j(y), \tag{1.2}$$

where $\ell \in \{1,\ 2\}$ and B denotes the boundary condition "D" or "N." We can perhaps most easily solve these initial value problems by extending $f_1,\ f_2$ to be odd (Dirichlet) or even (Neumann) functions on $\mathbb{R} \times Y$, and solving the wave equation on the full cylinder. In doing so, we see that we get different expressions for the terms with $\sigma_j = 0$ and those with $\sigma_j \neq 0$. Suppose $f_l(r, y) = 0$ if $r > R_1$, $l = 1,\ 2$. Then if $\sigma_j = 0$

$$u_{j,N}(t, r) = \int_0^\infty f_{2,j}(r')dr', \qquad u_{j,D}(t) = 0 \text{ if } \sigma_j = 0, \text{ for } 0 < r < R \text{ and } t > R + R_1 \tag{1.3}$$

by d'Alembert's formula. On the other hand, if $\sigma_j > 0$ for any nonnegative integer k_0 we have for t sufficiently large

$$u_{j,B}(t, r) = \sum_{k=0}^{k_0} \left[p_{j,k,B}(r)\cos(\sigma_j t + \tfrac{\pi}{4}) + q_{j,k,B}(r)\sin(\sigma_j t + \tfrac{\pi}{4})\right] t^{-k-\frac{1}{2}} + O\left(t^{-k_0-\frac{3}{2}}\right), \quad \text{if } \sigma_j > 0, \tag{1.4}$$

by the method of stationary phase (see also [Hör87] for asymptotics as $r \to \infty$). Here for each j $p_{j,k,B}$ and $q_{j,k,B}$ are polynomials in r of degree at most $2k$, and the remainders are uniform in j and as r varies in a compact set. Moreover, for the Neumann boundary condition

$$p_{j,0,N} = 2\sqrt{\frac{\sigma_j}{2\pi}} \int_0^\infty f_{1,j}(r')dr', \qquad q_{j,0,N} = \frac{2}{\sqrt{2\pi\sigma_j}} \int_0^\infty f_{2,j}(r')dr'. \tag{1.5}$$

With the Dirichlet boundary condition, $p_{j,0,D} = 0 = q_{j,0,D}$, and

$$p_{j,1,D}(r) = 2r\sqrt{\frac{\sigma_j}{2\pi}} \int_0^\infty r' f_{2,j}(r')dr', \qquad q_{j,1,D}(r) = 2\sigma_j r\sqrt{\frac{\sigma_j}{2\pi}} \int_0^\infty r' f_{1,j}(r')dr'. \tag{1.6}$$

To interpret the above result in terms of the spectrum, we can similarly write the resolvent of $-\Delta_B$ in terms of shifted resolvents of $-(\partial_r^2)_B$. If $f(r, y) = \sum_{j=0}^\infty f_j(r)\phi_j(y)$, then

$$(-\Delta_B - z)^{-1} f = \sum_{j=0}^\infty \left(\left(-(\partial_r^2)_B + \sigma_j^2 - z\right)^{-1} f_j\right)\phi_j, \qquad z \in \mathbb{C} \setminus [0, \infty).$$

From this we see that the spectrum of $-\Delta_B$ is $[0, \infty)$, is purely absolutely continuous, and has thresholds (points at which multiplicity jumps) at the eigenvalues of $-\Delta_Y$. For the Neumann Laplacian on the half cylinder, at each threshold the spectrum contains an embedded resonance of multiplicity equal to the multiplicity of the corresponding eigenvalue of $-\Delta_Y$ (and there are no other resonances embedded in the spectrum). The Dirichlet Laplacian on the half-cylinder has no embedded resonances.

We now see that the coefficient of the constant term in the expansion (1.3), given by $u_{j,B}(r)$ with $\sigma_j = 0$ as in (1.5), is determined by the resonant states at zero and a (distributional) pairing with the initial data. The number of states is equal to the number of connected components of Y. The terms of order $t^{-1/2}$ in (1.4) have coefficients given in (1.5) in terms of resonant states at nonzero eigenvalues of $-\Delta_Y$ and (distributional) pairings of these with the initial data. These terms are *zero* in the absence of threshold resonances, as is the case for the Dirichlet half-cylinder. From the example of the Dirichlet half-cylinder, we see that in the absence of eigenvalues or resonances embedded in the continuous spectrum we may have a wave decay like $O(t^{-3/2})$, but we cannot in general expect faster decay.

In the remainder of the paper we adapt the asymptotics above, in a somewhat weaker form, to Schrödinger operators on more general manifolds with cylindrical ends. One difficulty is that such operators can have much nastier behavior of the resolvent near the continuous spectrum, including the possible presence of infinitely many embedded eigenvalues, accumulating at infinity, [CZ95, Par95]. In additional to the possible existence of resonances embedded in the continuous spectrum, there may be additional "complex" resonances, resonances which lie on the Riemann surface $\hat{Z}$ to which the resolvent continues; see Section 2.2. In general settings we cannot rule out the possibility that at high energy these rapidly approach the continuous spectrum. Below we mostly restrict our attention to some particular cases in which the resolvent is better behaved. In these cases, the non-real resonances do not approach the continuous spectrum too rapidly, and they do not appear in the expansions.

1.2. **Two term asymptotics for mildly trapping manifolds or waveguides with cylindrical ends.** Our first extension of the results of Section 1.1 is to manifolds with infinite cylindrical ends for which we have polynomial bounds on the cut-off resolvent. Rather than state the theorem in full generality here, for now we let (X, g) be one of the examples in Sections 1.2.1 and 1.2.2. Alternatively, we allow the waveguides $X \subset \mathbb{R}^d$ to be one of the domains in Section 1.2.3, in which case we consider the case of Dirichlet boundary conditions. In each case, Y is the cross section of the infinite cylindrical ends, $\Delta_Y \leq 0$ is the Laplacian on Y (with Dirichlet boundary conditions for the domains of Section 1.2.3), and $0 \leq \sigma_0^2 \leq \sigma_1^2 \leq \cdots$ are the eigenvalues of $-\Delta_Y$, repeated with multiplicity.

By the results of [CDa, Theorem 1.1] for the examples of Sections 1.2.1 and 1.2.2 and [CDb] for the examples of Section 1.2.3, $-\Delta$ has a finite, possibly empty, set of eigenvalues which we denote $\{E_\ell\}$ (repeated with multiplicity), with corresponding orthonormal L^2 eigenfunctions $\{\eta_\ell\}$: $-\Delta\eta_\ell = E_\ell\eta_\ell$. The generalized eigenfunctions $\{\Phi_j\}$ are defined in (2.17); each Φ_j depends on a parameter λ and a variable $x \in X$, so that $\Phi_j(\lambda) = \Phi_j(\lambda, x)$. The $\Phi_j(\lambda)$ are (certain) elements of the null space of $-\Delta - \lambda^2$ which are not in $L^2(X)$. If $\Phi_j(\sigma_j) \not\equiv 0$ then σ_j corresponds to a threshold resonance and $\Phi_j(\sigma_j)$ is a resonant state; $\Phi_j(\sigma_j) \in C^\infty(X)$. See Section 2.3 for a more detailed discussion of the generalized eigenfunctions.

Our first result is a two term expansion, an example of which is the following theorem. In the statement of the theorem, for $f_j \in C_c^\infty(X)$ and $v \in C^\infty(X)$, $\langle f_j, v\rangle = \int_X f_j\overline{v}$.

Theorem 1.1. *Let (X, g) be as in the examples in Sections 1.2.1 or 1.2.2, or let $X \subset \mathbb{R}^2$ be a domain as in Section 1.2.3. Let $f_1,\ f_2 \in C_c^\infty(X)$ be given and $u(t)$ solve* (1.1)*, and in addition satisfy Dirichlet boundary conditions if X is as in Section 1.2.3. Then we can write*

$$u(t) = u_e(t) + u_{thr}(t) + u_r(t), \tag{1.7}$$

where

$$u_e(t) = u_e(t, x) = \sum_{E_\ell \in \operatorname{spec}_p(-\Delta)} \eta_\ell(x) \left(\cos((E_\ell)^{1/2}t)\langle f_1, \eta_\ell\rangle + \frac{\sin((E_\ell)^{1/2}t)}{(E_\ell)^{1/2}}\langle f_2, \eta_\ell\rangle \right) \tag{1.8}$$

and

$$u_{thr}(t) = u_{thr}(t,x) = \frac{1}{4}\sum_{\sigma_j=0}\Phi_j(0,x)\langle f_2, \Phi_j(0)\rangle + \frac{1}{2\sqrt{t}}\sum_{\sigma_j>0}\sqrt{\frac{\sigma_j}{2\pi}}\cos(\sigma_j t+\pi/4)\Phi_j(\sigma_j,x)\langle f_1,\Phi_j(\sigma_j)\rangle + \frac{1}{2\sqrt{t}}\sum_{\sigma_j>0}\frac{1}{\sqrt{2\pi\sigma_j}}\sin(\sigma_j t+\pi/4)\Phi_j(\sigma_j,x)\langle f_2,\Phi_j(\sigma_j)\rangle \quad (1.9)$$

where $x \in X$*. Moreover, for any* $\chi \in C_c^\infty(X)$*, there is a constant* C *so that the remainder,* $u_r(t)$*, satisfies*

$$\|\chi u_r(t)\|_{L^2(X)} \leq Ct^{-1}, \text{ for } t \text{ sufficiently large.} \quad (1.10)$$

For the manifolds and domains considered here, as a consequence of [CDa, Theorem 1.1] and [CDb, Theorem 3], each of these sums in (1.8) and (1.9) is in fact a finite sum. Since we have assumed the initial data $f_1,\ f_2 \in C_c^\infty(X)$, we could replace the bound (1.10) by $\|\chi u_r(t)\|_{H^m(X)} \leq Ct^{-1}$ for any $m \in \mathbb{N}$, with a new constant C depending on m.

We compare the expansion of u in Theorem 1.1 with that of the solution to the wave equation on the Neumann half cylinder given by (1.2-1.5). From the expression for u_{thr} in (1.9), if $\sigma_j = 0$, then $\langle\Phi_j(0), f_2\rangle$ corresponds to $2\int_0^\infty f_{2,j}(r)dr$ from (1.3) and (1.2). If $\sigma_j > 0$, then $\Phi_j(\sigma_j)\langle f_1, \Phi_j(\sigma_j)\rangle$ corresponds to $2\sqrt{\frac{2\pi}{\sigma_j}}p_{j,0,N}(r)\phi_j$ from (1.4). In contrast, for the Dirichlet half-cylinder there are no embedded resonances. Hence, for the Dirichlet half-cylinder $\Phi_j(\sigma_j) = 0$ for each j.

Theorem 3.2 is a more general version of Theorem 1.1. In Theorem 3.2 we can allow any manifold with infinite cylindrical ends for which we have a polynomial bound on the cut-off resolvent of the Laplacian at high energies. That this condition holds for the manifolds in Sections 1.2.1 and 1.2.2 is shown in [CDa], where such estimates are shown for the resolvent of $-\Delta + V$, for a large class of potentials V. Theorems 1 and 2 of [CDb] give the needed resolvent estimates for the Dirichlet Laplacian for the domains of Section 1.2.3.

In fact, our wave expansion, Theorem 3.2, holds for more general compactly supported perturbations of the Laplacian on a manifold or domain with infinite cylindrical ends, as long as appropriate high-energy resolvent bounds hold. In Section 2 we adapt the Euclidean black box formalism of [SZ91] to the cylindrical end setting. This allows us to treat in a unified way problems with and without a boundary extending to infinity. This formalism also allows us to handle the Klein-Gordon equation in addition to the wave equation. In Section 2 we give a self-contained presentation of some properties of the resolvent, generalized eigenfunctions, and spectral measure in this very general setting. This extends some results known for manifolds with infinite cylindrical ends.

1.2.1. *Examples with minimal trapping.* Let r be the radial coordinate in $\mathbb{R}^d$ for some $d \geq 2$, and let

$$X = \mathbb{R}^d, \qquad g_0 = dr^2 + F(r)dS,$$

where dS is the usual metric on the $(d-1)$-dimensional unit sphere, $F(r) = r^2$ near $r = 0$, and F' is compactly supported on some interval $[0, R]$ and positive on $(0, R)$; see Figure 1.

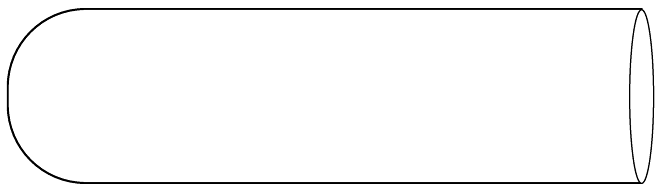

FIGURE 1. A cigar-shaped warped product.

Then all g_0-geodesics obey, for $r(t) \neq 0$,

$$\ddot{r}(t) := \frac{d^2}{dt^2} r(t) = 2|\eta|^2 F'(r(t)) F(r(t))^{-2} \geq 0,$$

where $r(t)$ is the r coordinate of the geodesic at time t and η is the angular momentum, constant on each geodesic. Consequently, the only trapped geodesics (that is, the only maximally extended geodesics with $\sup_{t\in\mathbb{R}} r(t) < +\infty$) are the ones with $\dot{r}(t) \equiv F'(r(t)) \equiv 0$, that is the ones in the cylindrical end that have no radial momentum. This is the smallest amount of trapping a manifold with a cylindrical end can have.

Let g be any metric on X such that $g - g_0$ is supported in $\{(r, y) \mid r < R\}$, and such that g and g_0 have the same trapped geodesics. For example we may take $g = g_0 + cg_1$, where g_1 is any symmetric two-tensor with support in $\{(r, y) \mid r < R\}$, and $c \in \mathbb{R}$ is chosen sufficiently small depending on g_1. Alternatively, we may take $g = dr^2 + g_S(r)$, where $g_S(r)$ is a smooth family of metrics on the sphere such that $g_S(r) = r^2 dS$ near $r = 0$ and $g_S(r) = F(r)dS$ near $r \geq R$, and such that $\partial_r g_S(r) > 0$ on $(0, R)$. This way we can construct examples where $g - g_0$ is not small.

For this set of examples, it is a consequence of the results of [CDa] that there are at most finitely many eigenvalues of the Laplacian on X, and at most finitely many threshold resonances.

1.2.2. *Examples based on convex cocompact manifolds.* Let (X, g_H) be a convex cocompact hyperbolic surface, such as the symmetric hyperbolic 'pair of pants' surface with three funnels depicted in Figure 2.

In particular, there is a compact set $N \subset X$ (the convex core of X) such that

$$X \setminus N = (0, \infty)_r \times Y_y, \qquad g_H|_{X\setminus N} = dr^2 + \cosh^2 r\, dy^2,$$

where Y is a disjoint union of $k \geq 1$ geodesic circles, not necessarily all of the same length.

We construct a metric on X which gives it the structure of a manifold with infinite cylindrical ends by modifying the metric on the funnel ends. Take g such that

$$g|_N = g_H|_N, \qquad g|_{X\setminus N} = dr^2 + F(r)dy^2, \tag{1.11}$$

where $F(r) = \cosh^2 r$ near $r = 0$, and F' is compactly supported and positive on the interior of the convex hull of its support.

It is also possible to construct examples with dimension $d \geq 3$ – see [CDa, Section 2.2].

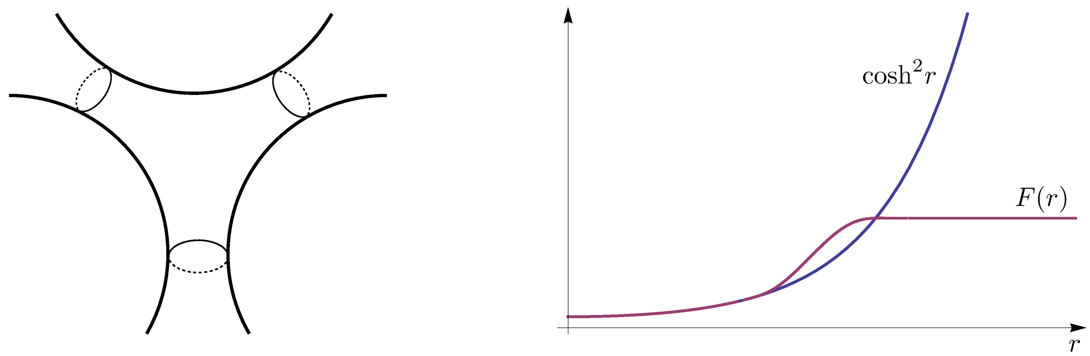

FIGURE 2. A hyperbolic surface (X, g_H) with three funnels, and a modification of the metric which changes the funnel ends to cylindrical ends.

Again, for this set of examples, it is a consequence of the results of [CDa] that there are at most finitely many eigenvalues of the Laplacian on X, and at most finitely many threshold resonances.

1.2.3. *Star-shaped planar waveguides, with the Dirichlet Laplacian.* We next turn to examples of domains $X \subset \mathbb{R}^2$ for which Theorem 1.1 holds for solutions u of the wave equation which in addition satisfy Dirichlet boundary conditions on ∂X. Though here we restrict ourselves to $d = 2$ for simplicity, the paper [CDb] has conditions on domains in $\mathbb{R}^d$ with $d \geq 3$ which ensure the theorem holds, as well as alternative sufficient conditions for planar domains.

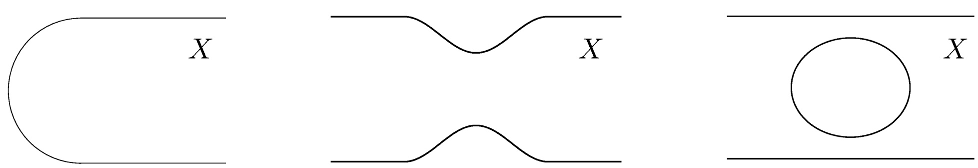

FIGURE 3. A domain with one end, and two domains with two isometric ends.

Let (s, y) be Cartesian coordinates on $\mathbb{R}^2$ and $X \subset \mathbb{R}^2$ be a domain. Three examples of domains we allow are shown in Figure 3. They are defined as follows:

(1) Let X be the union of the open disk of radius 1 centered at $(1, 0)$ with the strip $(1, \infty) \times (-1, 1)$.
(2) Let X be the set of (s, y) such that $f_1(s) - 1 \leq y \leq 1 - f_2(s)$, where f_1 and f_2 are nontrivial compactly supported functions in $C^{1,1}(\mathbb{R})$, which take values in $[0, 1)$, and which are monotonic away from $s = 0$.
(3) Let $X = (\mathbb{R} \times (-1, 1)) \setminus K$, where K is a compact, convex subset of $\mathbb{R} \times (-1, 1)$ which has $C^{1,1}$ boundary and is symmetric about the y axis.

To state more general assumptions, let $\nu = (\nu_s, \nu_y)$ be the outward pointing unit normal to ∂X. We assume $s\nu_s \leq 0$ on ∂X, and call such domains "star-shaped" with respect to $s = \pm\infty$. Given an open interval I and positive constant C_I, define $\Gamma_F = \Gamma_F(I, C_I) \subset \partial X \cap (I \times \mathbb{R})$ via

$$\Gamma_F = \{(s, y) \in \partial X \cap (I \times \mathbb{R}) \mid \; s\nu_s \leq -C_I\}. \tag{1.12}$$

Now we assume that there is an open interval I and a positive constant C_I so that the intersection of X with $I \times \mathbb{R}$ consists of bounded open sets $X_1, \dots, X_K$ with mutually disjoint closures such that for each $k = 1, \dots, K$,

$$(\partial X \cap \partial X_k) \setminus \Gamma_F \subset I \times \{a_k\},$$

for some real a_k.

Moreover, we assume X is a domain with infinite cylindrical ends. Specializing to the case here, by this we mean that there is $R_0 > 0$ such that $X \cap ([-R_0, R_0] \times \mathbb{R})$ is bounded and

$$X \cap ((-\infty, -R_0] \times \mathbb{R}) = (-\infty, -R_0] \times Y_-, \qquad X \cap ([R_0, \infty) \times \mathbb{R}) = [R_0, \infty) \times Y_+, \tag{1.13}$$

where Y_- and Y_+ are (not necessarily connected) bounded open sets in $\mathbb{R}$. We allow the possibility that one, but not both, of $Y_\pm$ is the empty set. Then $Y = Y_+ \sqcup Y_-$.

Finally we assume that each point $p \in \partial X$ has a neighborhood U_p so that either $U_p \cap X$ is convex or $U_p \cap \partial X$ is $C^{1,1}$. Figures 3 and 4 include examples of domains satisfying all of these conditions.

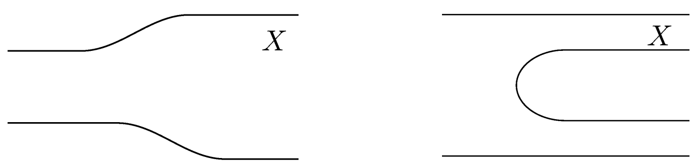

FIGURE 4. Some domains with multiple nonisometric ends.

Resolvent estimates for domains of this type are studied in [CDb]. Results of that paper ensure that for the class of domains in $\mathbb{R}^2$ described here, there are no embedded eigenvalues or resonances. Hence the sums in (1.8) and (1.9) are actually 0, and $\|\chi u(t)\| = O(t^{-1})$.

1.3. **Complete expansions under an additional spacing condition on the thresholds.** In this subsection we suppose that (X, g) is as in Section 1.2 but with an additional assumption on the eigenvalues of $-\Delta_Y$. This assumption holds for all the examples in Section 1.2.1, but only for some of the other examples.

To state it, let $\{\nu_l\}_{l=1}^\infty$ be the sequence of square roots of *distinct positive* eigenvalues of $-\Delta_Y$ in increasing order, so that $0 < \nu_1 < \nu_2 < \cdots$. The assumption is that there are positive constants c_Y and N_Y, such that

$$\nu_{l+1} - \nu_l \geq c_Y \nu_l^{-N_Y}, \tag{1.14}$$

for all $l \in \mathbb{N}$ with $\nu_l \geq 1$. Note that this assumption allows the eigenvalues of $-\Delta_Y$ to have high multiplicities, but forbids *distinct* eigenvalues from clustering too closely together. Since for the examples in Section 1.2.1 $\nu_l = \sqrt{l(l+d-2)}$, it is easy to see (1.14) holds. On the other hand, consider the examples illustrated using Figure 2 in Section 1.2.2–these have $d = 2$ and three connected ends. In this case, Y consists of the disjoint union of three circles. Noting that the function $F(r)$ of (1.11) is the same for each end, if the radii of these three circles are rational multiples of each other, then (1.14) holds. In fact, if the radii are algebraic multiples of each other, this holds by Liouville's Theorem on approximation of algebraic numbers, e.g.

[DKS10, Proposition 7.5.15]. If, however, for some pair of the circles the radius of one circle is a transcendental multiple of the other, then this condition may not hold. In general, this condition holds for some of the examples of Section 1.2.2, but not for all. Turning to the domains in $\mathbb{R}^2$ pictured in Section 1.2.3, we can see that (1.14) holds for all the examples in Figure 3, where each domain either has a single connected end, or two isometric connected ends. For the examples in Figure 4, whether or not (1.14) holds is again subtle, depending on the ratios of the lengths of the cross-sections of the ends.

With the assumption (1.14) and a bound on the cut-off resolvent at high energy we can bound derivatives of the cut-off resolvent at high energy, see Lemmas 4.3 and 4.4. This allows us to refine our expansions.

Theorem 1.2. *Let (X, g) be as in the examples in Sections 1.2.1 or 1.2.2. Alternatively, let $X \subset \mathbb{R}^d$ be a domain as in section 1.2.3, and consider the Dirichlet boundary conditions. In either case, assume (1.14) holds. Let $f_1,\ f_2 \in C_c^\infty(X)$ be given, and let $u(t)$ solve* (1.1) *and satisfy Dirichlet boundary conditions in the case of a domain $X \subset \mathbb{R}^2$. Then for each $k_0 \in \mathbb{N}$ we can write*

$$u(t) = u_e(t) + u_{thr,k_0}(t) + u_{r,k_0}(t),$$

where u_e is still given by (1.8) *and*

$$u_{thr,k_0}(t) = \frac{1}{4}\sum_{\sigma_j=0}\Phi_j(0)\langle f_2, \Phi_j(0)\rangle + \sum_{k=0}^{k_0-1} t^{-1/2-k}\sum_{l=1}^{\infty}(e^{it\nu_l}b_{l,k,+} + e^{-it\nu_l}b_{l,k,-})$$

for some $b_{l,k,\pm} \in \langle r\rangle^{1/2+2k+\epsilon}L^2(X)$. For any $\chi \in C_c^\infty(X)$ there is a constant C so that

$$\sum_{l=1}^{\infty}\|\chi b_{l,k,\pm}\|_{L^2(X)} < C,\ k = 0, 1, 2, ..., k_0 - 1$$

and

$$\|\chi u_{r,k_0}(t)\|_{L^2(X)} \leq Ct^{-k_0} \text{ for } t \text{ sufficiently large.}$$

Moreover, the $b_{l,k,\pm}$ are determined by the value ν_l, the initial data f_1, f_2, and suitable derivatives of elements of the set $\{\Phi_{j'}(\lambda)\}_{0\leq\sigma_{j'}\leq\nu_l}$ evaluated at $\pm\nu_l$.

For further details about how the $b_{l,k,\pm}$ are determined, see Theorem 4.1 and Lemma 4.7 and its proof. We note, however, that for general initial data and $k > 0$ we do not expect the sums $\sum_{l=1}^{\infty} b_{l,k,\pm}$ to be finite sums; compare, for example the Dirichlet half-cylinder case (1.4) and (1.6). Comparing Theorem 1.1 gives, for $l \in \mathbb{N}$

$$b_{l,0,\pm} = \frac{\sqrt{\nu_l}}{4\sqrt{2\pi}}\sum_{\sigma_j=\nu_l}\Phi_j(\sigma_j)\left(e^{\pm i\pi/4}\langle f_1, \Phi_j(\sigma_j)\rangle + \frac{1}{i\nu_l}e^{\pm i\pi/4}\langle f_2, \Phi_j(\sigma_j)\rangle\right).$$

As in Theorem 1.1, because of the smoothness of the initial data we can instead bound $\|\chi u_{r,k_0}(t)\|_{H^m(X)} \leq Ct^{-k_0}$ with the constant depending on m as well as χ and the initial data, and the series $\sum_{l=1}^{\infty}\|\chi b_{l,k,\pm}\|_{H^m(X)}$ converges for each value of m.

Theorems 1.1 and 1.2, and their more general versions, Theorems 3.2 and 4.1, require high energy bounds on the norm of the cut-off resolvent. The bounds are generally proved under some conditions on the trapping, as we have discussed earlier in the introduction. However, for a general manifold with infinite cylindrical ends, without a bound on the high energy behavior of the resolvent or any restrictions on the trapping, we can find an asymptotic expansion of $\chi\psi_{sp}(-\Delta)u(t)$, provided that $\psi_{sp} \in C_c^\infty(\mathbb{R})$, see Proposition 4.2. Here, as before, $u(t)$ is the solution of (1.1), and we must assume the initial data have support in a fixed compact set.

The literature of the study of local energy decay under the assumptions of no trapping or mild trapping is quite large, and we mention only a few papers. The study of local energy decay for nontrapping perturbations of the Laplacian on Euclidean space was initiated by Morawetz in [Mor61] and continued in, for example, [LMP62, MRS77, Vai89]. The question of wave expansions or wave decay on noncompact manifolds with various kinds of ends and different trapping assumptions is a very active area of research; see [Zwo17] and references therein for some more recent results. The most closely related results of which we are aware are for solutions to the wave equation on a planar waveguide without forcing [Lyf76] and with forcing [HW06], where the expansion is found to order $o(1)$. For energy decay of solutions to a dissipative wave equation on a waveguide, see [MR18].

Our results build on studies of the spectral theory of manifolds with cylindrical ends, in particular we mention [Gol73, Lyf76, Gui89, Mel93, Chr95, Par95]. More recent papers include [Chr02, IKL10, MS10, RTdA13] and references therein. We give more precise references as they are used.

1.4. **Notation.** In this section we collect, for reference, some notation introduced in Section 2.1:

- H is an operator (generalizing the Laplacian on a manifold with cylindrical ends) on a Hilbert space $\mathcal{H}$.
- H_Y is a nonnegative operator (generalizing the Laplacian on the cross sections of the cylindrical ends) on a Hilbert space $\mathcal{H}_Y$.
- $\{\sigma_j^2\}_{j\geq 0}$ are the eigenvalues of H_Y, repeated with multiplicity, with $0 \leq \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots$. This set may be finite or infinite.
- $\{\phi_j\}$ are a complete orthonormal set of eigenfunctions of H_Y with $H_Y\phi_j = \sigma_j^2\phi_j$.
- $\{\nu_l^2\}_{l\geq 1}$ are the *distinct*, *positive* eigenvalues of H_Y with $0 < \nu_1 < \nu_2 < \cdots$.

And here is some notation introduced elsewhere in the paper:

- f_1, f_2 are initial data; see (1.1) and (3.1).
- $\tau_j(\lambda) = (\lambda^2 - \sigma_j^2)^{1/2}$, with $\operatorname{Im}\tau_j > 0$ when $\operatorname{Im}\lambda > 0$; see (2.4).
- The resolvent $R(\lambda) = (H - \lambda^2)^{-1} \colon \mathcal{H} \to \mathcal{H}$ is originally defined when $\operatorname{Im}\lambda > 0$, but the cutoff resolvent $\chi R(\lambda)\chi$ continues meromorphically to the Riemann surface $\hat{Z}$, which is a cover of $\mathbb{C}$ with covering map p; see Section 2.2.
- $\mathcal{P}$ is the orthogonal projection onto the span of the eigenfunctions of H, if any; see Section 2.2.
- $\Phi_j(\lambda)$ are generalized eigenfunctions (also sometimes called scattering solutions, or distorted plane waves) of H: they obey $H\Phi_j(\lambda) = \lambda^2\Phi_j(\lambda)$ and also satisfy further conditions; see (2.17).
- $\{\eta_\ell\}$ are orthonormal eigenfunctions of H with eigenvalue E_ℓ: $H\eta_\ell = E_\ell\eta_\ell$; see Section 3.1.

Throughout the paper, C denotes a positive constant whose value may change from line to line.

2. Black box scattering on cylinders

The results we shall prove about local wave expansions are valid for a large class of "black box" perturbations of the Laplacian on a manifold with infinite cylindrical ends, with or without boundary, provided that there are appropriate high-energy bounds on the cut-off resolvent. In fact, the natural setting for our work is more general still, requiring only that 'infinity is one-dimensional' in a sense we make precise below. This more abstract setting also allows low regularity (as in some of the examples from [CDb]) and other settings, like quantum graphs.

We adapt the idea of [SZ91] (see also [DZ19, Chapter 4]) of a compactly supported black box perturbation of the Laplacian on Euclidean space to give a definition of a compactly supported black box perturbation of the Laplacian on a manifold with infinite cylindrical ends, or more generally of an operator on $L^2(\mathbb{R}_+;\mathcal{H}_Y)$, where $\mathcal{H}_Y$ is any Hilbert space. We collect a number of results related to the resolvent, spectrum, generalized eigenfunctions and spectral measure of such operators. We give a mostly self-contained presentation of these results, though many can be found in [Gui89, Mel93, Par95, Chr95] in less general settings.

2.1. **Black box operators.**

In this section we give the main definitions and assumptions we will use throughout the paper, and illustrate them with examples, focusing first on the simpler case of complete manifolds without boundary, and then on the more complicated one where there is a boundary. We conclude with some further examples.

2.1.1. *Manifolds without boundary.* Our basic example of a black box operator is the Laplacian on a manifold with infinite cylindrical ends and no boundary, as in Sections 1.2.1 and 1.2.2. Namely, let (X, g) be a complete Riemannian manifold without boundary, having an open subset X_∞ (the infinite cylindrical ends) with the following properties: $X \setminus X_\infty$ is bounded, and

$$(X_\infty, g{\restriction}_{X_\infty}) = ((0,\infty)_r \times Y, dr^2 + g_Y),$$

where (Y, g_Y) is a compact manifold without boundary. Then let

$$\begin{aligned} &\mathcal{H} = L^2(X), \qquad \mathcal{H}_0 = L^2(X\setminus X_\infty), \qquad \mathcal{H}_Y = L^2(Y),\\ &H = -\Delta \text{ be the Laplacian on } X, \qquad \mathcal{D} = H^2(X),\\ &H_Y = -\Delta_Y \text{ be the Laplacian on } Y, \qquad \mathcal{D}_Y = H^2(Y). \end{aligned} \tag{2.1}$$

In what follows we will use the notation on the left hand sides of (2.1) in a more general way, but the reader should keep in mind this more concrete setting.

2.1.2. *Definitions, notation, and assumptions.* Let $\mathcal{H}$ be a complex Hilbert space with orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus L^2(\mathbb{R}_+;\mathcal{H}_Y),$$

where $\mathcal{H}_0$ and $\mathcal{H}_Y$ are separable Hilbert spaces. We use r as the coordinate for $\mathbb{R}_+ = (0,\infty)$. Let H be a self-adjoint operator on $\mathcal{H}$ with dense domain $\mathcal{D}$. Let H_Y be a self-adjoint operator on $\mathcal{H}_Y$ with dense domain $\mathcal{D}_Y$.

Let $\mathbb{1}_0$ and $\mathbb{1}_\infty$ denote the orthogonal projections $\mathbb{1}_0 : \mathcal{H} \to \mathcal{H}_0$ and $\mathbb{1}_\infty : \mathcal{H} \to L^2(\mathbb{R}_+;\mathcal{H}_Y)$.

If $g \in C([0,\infty))$ and $f \in \mathcal{H}$, then by gf we mean

$$\chi f = g(0)\mathbb{1}_0 f + g\mathbb{1}_\infty f. \tag{2.2}$$

Hence $f \in g\mathcal{H}$ means $f = gh$ for some $h \in \mathcal{H}$. We will most often use this for $g = \chi \in C_c^\infty([0,\infty))$ and $g(r) = \langle r\rangle^p = (1+r^2)^{p/2}$.

Assume that H is lower semi-bounded and that $\mathbb{1}_0(H+i)^{-1}$ is compact. Assume that $H_Y \geq 0$ and that H_Y has no essential spectrum.

Let $\{\phi_0, \phi_1, \dots\}$ be an orthonormal basis of eigenfunctions of H_Y, such that $H_Y\phi_j = \sigma_j^2\phi_j$, with $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots$. Beginning in §3, we will also use $0 < \nu_1 < \nu_2 < \cdots$ to denote the positive, distinct elements in the sequence $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots$. Thus, the σ's are the square roots of *all* the eigenvalues of H_Y *with* multiplicity, and the ν's are the square roots of the *positive* eigenvalues of H_Y *without* multiplicity.

If $\dim \mathcal{H}_Y < \infty$, then these sequences terminate. This occurs if and only if the operator H_Y is bounded.

We identify elements $u \in L^2(\mathbb{R};\mathcal{H}_Y)$ with sequences $(u_j) \in \ell^2(\mathbb{N}_0; L^2(\mathbb{R}_+))$ using the above basis:

$$u(r) = \sum_{j=0}^\infty u_j(r)\phi_j,$$

where, if $\dim \mathcal{H}_Y < \infty$, then once again the sequences must terminate. Let

$$\mathcal{D}_\infty = \left\{(u_j) \in \ell^2(\mathbb{N}_0; L^2(\mathbb{R}_+)) \mid (-u_j'' + \sigma_j^2 u_j) \in \ell^2(\mathbb{N}_0; L^2(\mathbb{R}_+))\right\}. \tag{2.3}$$

Assume that $\mathbb{1}_\infty\mathcal{D} \subset \mathcal{D}_\infty$, and $\mathbb{1}_\infty H = -\partial_r^2 + H_Y$ in the sense that if $f \in \mathcal{D}$, then $\mathbb{1}_\infty Hf = (-\partial_r^2 + H_Y)\mathbb{1}_\infty f$. Assume that if $f \in \mathcal{D}_\infty$ and $f\restriction_{r\leq\epsilon} = 0$ for some $\epsilon > 0$, then $f \in \mathcal{D}$.

2.1.3. *Examples with boundary.* We can modify the basic examples of Section 2.1.1 to allow X and Y to be open manifolds, if we impose appropriate boundary conditions.

The simplest such example is the half cylinder of Section 1.1. In addition to the Dirichlet and Neumann boundary conditions considered there, we could just as well take more general self-adjoint boundary conditions.

We now turn to our primary example with $\partial Y \neq \emptyset$, where X is isometric to a waveguide. Here by a waveguide we mean a domain $\Omega \subset \mathbb{R}^d$ which has a subset $\Omega_\infty \subset \Omega$ so that the closure of $\Omega \setminus \Omega_\infty$ is compact and Ω_∞ is a "cylindrical end." That is, $\Omega_\infty = \bigcup_{l=1}^L \Omega_l$, and, for $l = 1, \dots, L$, Ω_l can, under a rigid motion of $\mathbb{R}^d$, be identified with $(0,\infty)\times U_l$, with $U_l \subset \mathbb{R}^{d-1}$ a bounded domain. In our earlier notation, $Y = \sqcup_{l=1}^L U_l$. The metrics on Ω and Y come from restriction of the usual Euclidean metric. Then put $\mathcal{H} = L^2(\Omega)$, $\mathcal{H}_0 = L^2(\Omega\setminus\Omega_\infty)$, and $\mathcal{H}_Y = L^2(Y)$.

To consider the Dirichlet Laplacian on Ω, we set $H = -\Delta$, where Δ is the Euclidean Laplacian here, and $\mathcal{D} = \{f \in H_0^1(\Omega) : \Delta f \in L^2(\Omega)\}$; see e.g. [Tay11, Section 8.2]. Similarly $H_Y = -\Delta_Y$ and $\mathcal{D}_Y = \{g \in H_0^1(Y) : \Delta_Y g \in L^2(Y)\}$. Then

$$\mathcal{D}_\infty = \left\{f \in H^1(\Omega_\infty) \mid \Delta f \in L^2(\Omega_\infty) \text{ and } f\restriction_{(0,\infty)\times\partial U_l} = 0\right\},$$

and the desired inclusions involving $\mathcal{D}$ and $\mathcal{D}_\infty$ follow. That $\mathbb{1}_{\mathcal{H}_0}(H+i)^{-1}$ is compact follows from e.g. [Bor20, Theorem 6.9], and the desired properties of H_Y can be obtained by e.g. taking an open cube Q with $Y \subset Q$ and using domain monotonicity of eigenvalues as in e.g. [Bor20, Theorem 6.20].

Notice that for the treatment of Dirichlet boundary conditions on Ω we did not require any regularity on the boundary $\partial\Omega$. With appropriate regularity assumptions, the Laplacian on a domain or manifold with Neumann (or certain more general) boundary conditions can also be put into the above black box framework; see e.g. [Tay11, Section 8.2] and [Bor20, Section 6.4].

2.1.4. *Other examples.* We have already mentioned our primary examples of black box operators motivated by the Laplacian on a manifold with infinite cylindrical ends, but we briefly outline some further examples.

We begin with an example in which the space $\mathcal{H}_Y$ is finite-dimensional. Consider the operator $-\partial_r^2 + V$ on $L^2(\mathbb{R})$, where V is a "steplike" potential, as studied in, for example [CK85, Chr06, DS12], and others. For us, this means $V \in L^\infty(\mathbb{R};\mathbb{R})$, and there is an $R > 0$ and constants $V_\pm$ so that $V(r) = V_\pm$ when $\pm r > R$. Here $\mathcal{H}_Y = \mathbb{C}^2 = \mathcal{D}_Y$ with the usual inner product, and if $f = (f_+, f_-) \in \mathcal{H}_Y$, then $H_Y f = (V_+ f_+, V_- f_-)$. Thus we must make the assumption that $V_\pm \geq 0$ in order that the condition that $H_Y \geq 0$ be satisfied. The hypotheses we make on V are more restrictive than those of [CK85]. In a similar fashion, matrix-valued Schrödinger operators on $\mathbb{R}$ with steplike potentials can be put in this framework, provided the matrix-valued potential satisfies analogous conditions.

Another example with $\mathcal{H}_Y$ finite-dimensional is provided by a Schrödinger operator on a quantum graph X which is the union of a finite part and a finite number of half-lines. Let $\mathcal{E}$ be the set of edges, and let $\mathcal{E} = \mathcal{E}_K \cup \mathcal{E}_\infty$, where each edge $e \in \mathcal{E}_K$ has finite length $l(e)$, and each edge in $\mathcal{E}_\infty$ can be identified with $(0,\infty)$. We assume each set $\mathcal{E}_K$ and $\mathcal{E}_\infty$ is finite. Then $\mathcal{H} = \bigoplus_{e\in\mathcal{E}_K} L^2(0,\ell(e)) \bigoplus_{e\in\mathcal{E}_\infty} L^2(0,\infty)$.

If we impose Neumann–Kirchoff boundary conditions at the vertices, then $\mathcal{D}$ is the set of functions $f \in \mathcal{H}$ whose restrictions to all edges are in H^2, which are continuous across vertices, and which satisfy, at each vertex v, $\sum_{e\in\mathcal{E}(v)} \frac{\partial f}{\partial r_e}(v) = 0$. Here $\mathcal{E}(v)$ is the set of edges meeting at vertex v, r_e is the coordinate on edge e, and the derivatives are taken in directions away from the vertex. Now let $V \in L^\infty(X;\mathbb{R})$ have compactly supported (distributional) derivative, and suppose $V \geq 0$ outside of a compact set. Then H is given by $-\partial_{r_e}^2 + V(r_e)$ on each edge. See e.g. [Bor20, Chapter 8] and [EK15, Chapter 8] for more on quantum graphs. In this example, $\mathcal{H}_Y = \mathbb{C}^N$, where N is the number of edges in $\mathcal{E}_\infty$ and the operator H_Y is determined in a manner similar to the step-like Schrödinger case described above, by the values of V "near infinity."

A possibility with $\mathcal{H}_Y$ infinite-dimensional is to let Y be a finite quantum graph. Then using notation as in the previous example, $\mathcal{E} = \mathcal{E}^K$ and $\mathcal{H}_Y = \bigoplus_{e\in\mathcal{E}} L^2(0,\ell(e))$. We can let H_Y be the operator $-\partial_y^2$ on each edge, with domain $\mathcal{D}_Y$ obtained by imposing Neumann-Kirchoff boundary conditions as described above. Then let $X = \mathbb{R}_+ \times Y$, $\mathcal{H} = L^2(X)$, $V \in L_c^\infty(X;\overline{\mathbb{R}_+})$, and $H = -\partial_r^2 + H_Y + V$, with $\mathcal{D} = \mathcal{D}_\infty \cap \{(u_j) \in \ell^2(\mathbb{N}_0; H_0^1(\mathbb{R}_+)\}$, giving the Dirichlet boundary condition at $r = 0$.

Note that if the operator H with domain $\mathcal{D} \subset \mathcal{H}$ satisfies the general conditions listed in Section 2.1.2, so does the operator $H + m^2$ for any $m \in \mathbb{R}$, by replacing H_Y by $H_Y + m^2$ (and hence replacing σ_j^2 by $\sigma_j^2 + m^2$). This implies that Theorems 3.2 and 4.1 apply to the Klein-Gordon equation as well as the wave equation.

2.2. **The resolvent.** Now let H be an operator as described above, and for $\operatorname{Im}\lambda > 0$, $\operatorname{Re}\lambda \neq 0$, define the resolvent

$$R(\lambda) = (H - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{H}.$$

We recall and extend some results (proved in [Gui89], [Mel93, sections 6.7-6.10], [Chr95, Section 2], and [Par95] for manifolds with cylindrical ends) on the behavior of $R(\lambda)$ and of its meromorphic continuation, focusing on the region near the real axis and on the consequences for the spectral measure.

We use the following model resolvent. For $\operatorname{Im}\lambda > 0$, let $R_0(\lambda)$ be the resolvent for $-\partial_r^2 + H_Y$ with Dirichlet boundary condition at $r = 0$, that is to say

$$(-\partial_r^2 + H_Y - \lambda^2)u = f \Longleftrightarrow u = R_0(\lambda)f,$$

for $f \in L^2(\mathbb{R}_+; \mathcal{H}_Y)$ and $u \in \mathcal{D}_\infty$ such that $u\restriction_{r=0}= 0$, where $\mathcal{D}_\infty$ is defined in (2.3). Explicitly, for $\operatorname{Im}\lambda > 0$, $\sigma_j^2 \in \operatorname{spec}(H_Y)$, let

$$\tau_j(\lambda) = (\lambda^2 - \sigma_j^2)^{1/2}, \tag{2.4}$$

where we take the square root to have positive imaginary part. Then for any $(f_j) \in \ell^2(\mathbb{N}_0; L^2(\mathbb{R}_+))$ (recalling sequences terminate if $\dim \mathcal{H} < \infty$),

$$R_0(\lambda) \sum_{j\geq 0} f_j(r)\phi_j = \sum_{j\geq 0} \frac{i}{2\tau_j(\lambda)} \int_0^\infty \left(e^{i\tau_j(\lambda)|r-r'|} - e^{i\tau_j(\lambda)(r+r')}\right) f_j(r')dr'\phi_j. \tag{2.5}$$

From this formula we see that the spectrum of $-\partial_r^2 + H_Y$, with Dirichlet boundary condition at $r = 0$, is absoultely continuous and given by $[\sigma_0^2, \infty)$. However, the multiplicity of the spectrum changes at the points $\sigma_j^2 \in \operatorname{spec}(H_Y)$, because, for any $\lambda^2 \geq 0$, the number of summands in (2.5) which are not bounded $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ equals the number values of j such that $\sigma_j^2 \leq \lambda^2$. We call the points $\{\sigma_j^2\}$ thresholds. When using λ^2 as a spectral parameter, we shall abuse terminology a bit and refer to the points $\{\pm\sigma_j\}$ as thresholds as well.

In the remainder of Section 2.2 we will derive, among other things, the following description of the spectrum of H. The essential spectrum of H is given by the same continuous spectrum $[\sigma_0^2, \infty)$, with the same thresholds and multiplicities. The discrete spectrum of H is countable (possibly empty) with infinity being the only possible accumulation point.

We begin with the upper half plane, where H may have only discrete spectrum.

Lemma 2.1. *The resolvent $R(\lambda)$ is meromorphic for* $\operatorname{Im}\lambda > 0$.

This is fairly clear from our assumptions on H. However, we give a detailed proof based on (2.9), an identity due to Vodev [Vod14], which will also be useful in our analysis later. In [Vod14] the identity is stated only for Schrödinger operators on $\mathbb{R}^d$. However, it in fact holds in far greater generality for operators which are, in an appropriate sense, compactly supported perturbations of each other. The identity (2.9) is a version adapted to our setting.

Proof. Let $\chi_1 = \chi_1(r) \in C_c^\infty([0,\infty))$ be 1 near $r=0$. Let λ and λ_0 have positive imaginary parts and nonzero real parts. To relate $R(\lambda)$ and $R_0(\lambda)$ we start with

$$R(\lambda)(1-\chi_1)\colon L^2(\mathbb{R}_+;\mathcal{H}_Y) \to \mathcal{H},$$

which we write as

$$R(\lambda)(1-\chi_1) = R(\lambda)(1-\chi_1)(-\partial_r^2 + H_Y - \lambda^2)R_0(\lambda).$$

But since $(-\partial_r^2 + H_Y)(1-\chi_1) = H(1-\chi_1)$, we may write

$$\begin{aligned} R(\lambda)(1-\chi_1) &= R(\lambda)\{(H-\lambda^2)(1-\chi_1) + [\chi_1, \partial_r^2]\}R_0(\lambda) \\ &= \{(1-\chi_1) - R(\lambda)[\partial_r^2, \chi_1]\}R_0(\lambda). \end{aligned} \tag{2.6}$$

Likewise

$$(1-\chi_1)R(\lambda_0) = R_0(\lambda_0)\{(1-\chi_1) + [\partial_r^2, \chi_1]R(\lambda_0)\}. \tag{2.7}$$

On the other hand, by the resolvent identity we have

$$\begin{aligned} R(\lambda) - R(\lambda_0) &= (\lambda^2 - \lambda_0^2)R(\lambda)R(\lambda_0) \\ &= (\lambda^2 - \lambda_0^2)\left(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) + R(\lambda)(1-\chi_1)^2R(\lambda_0)\right). \end{aligned} \tag{2.8}$$

Inserting (2.6) and (2.7) into (2.8) gives

$$\begin{aligned} R(\lambda) - R(\lambda_0) =&(\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) \\ &+ \{(1-\chi_1) - R(\lambda)[\partial_r^2, \chi_1]\}R_0(\lambda)R_0(\lambda_0)\{(1-\chi_1) + [\partial_r^2, \chi_1]R(\lambda_0)\}\Big). \end{aligned} \tag{2.9}$$

Bringing the $R(\lambda)$ terms to the left, the remaining terms to the right, and factoring, gives

$$R(\lambda)(I + K(\lambda,\lambda_0)) = F(\lambda,\lambda_0),$$

for any λ and λ_0 with positive imaginary parts and nonzero real parts, where

$$K(\lambda,\lambda_0) = (\lambda_0^2 - \lambda^2)\Big(\chi_1(2-\chi_1)R(\lambda_0) - [\partial_r^2, \chi_1]R_0(\lambda)R_0(\lambda_0)\{(1-\chi_1) + [\partial_r^2, \chi_1]R(\lambda_0)\}\Big),$$

and

$$F(\lambda,\lambda_0) = R(\lambda_0) + (\lambda^2 - \lambda_0^2)(1-\chi_1)R_0(\lambda)R_0(\lambda_0)\{(1-\chi_1) + [\partial_r^2, \chi_1]R(\lambda_0)\}.$$

For λ_0 fixed and λ sufficiently close to λ_0, we can invert $I + K(\lambda,\lambda_0)$ using a Neumann series. Observe that $K(\lambda,\lambda_0)$ is compact, because of our assumption that $\mathbb{1}_0(H+i)^{-1}$ is compact; see e.g. [DZ19, Lemma 4.2]. Moreover $\lambda \mapsto K(\lambda,\lambda_0)$ continues analytically to the upper half plane. Consequently, by the analytic Fredholm theorem (see e.g. [DZ19, Theorem C.8]),

$$\lambda \mapsto R(\lambda) = F(\lambda,\lambda_0)(I + K(\lambda,\lambda_0))^{-1}, \tag{2.10}$$

continues meromorphically to the upper half plane, for any fixed λ_0 with positive imaginary part and nonzero real part. ☐

The statement of Lemma 2.1 is equivalent to the statement that the essential spectrum of H is contained in $[0,\infty)$. Its proof can be easily adapted to show that the essential spectrum of H equals $[\sigma_0^2,\infty)$. More specifically, if $\sigma_0 > 0$, then use the fact that $R_0(\lambda)$ continues analytically to $\mathbb{C} \setminus \{\lambda \in \mathbb{R} \mid |\lambda| \geq \sigma_0\}$ to show that the essential spectrum of H is contained in $[\sigma_0^2,\infty)$. To show the reverse containment, reverse the roles of $R(\lambda)$ and $R_0(\lambda)$ in the proof.

To study the essential spectrum in more detail, we use the minimal complete Riemann surface to which the functions τ_j (defined by (2.4)) continue analytically for each $\sigma_j^2 \in \operatorname{spec}(H_Y)$. We denote this Riemann surface by $\hat{Z}$. We use the term *physical space* to refer to a certain copy of the upper half plane $\{\operatorname{Im}\lambda > 0\}$ in $\hat{Z}$, defined as follows: if $\sigma_0 = 0$, then the physical space is the unique copy of the upper half plane on which $\operatorname{Im}\tau_j(\lambda) > 0$ for each $\sigma_j^2 \in \operatorname{spec}(H_Y)$, and if $\sigma_0 > 0$, then there are two such copies, and we let the physical space be one of them (it doesn't matter which one). Throughout this paper we will identify the upper half plane in $\mathbb{C}$ with the physical space in $\hat{Z}$, and $\mathbb{R}$ in $\mathbb{C}$ with the boundary of the physical space in $\hat{Z}$.

We define a projection $p : \hat{Z} \to \mathbb{C}$ as follows: When λ lies in the physical space, we put $p(\lambda) = \lambda$, and for general $\lambda \in \hat{Z}$, extend $p(\lambda)$ by analytic continuation. Then p is an countably-sheeted covering map which is regular over $\mathbb{C} \setminus \{\pm\sigma_j : j \in \mathbb{N}_0,\ \sigma_j > 0\}$, and which has points of ramification of order two over each distinct $\pm\sigma_j$ with $\sigma_j > 0$. If $\sigma_0 = 0$, then $p = \tau_0$. Regardless of the value of σ_0, the physical space is the unique preimage of the upper half plane in $\mathbb{C}$ under p on which $\operatorname{Im}\tau_j(\lambda) > 0$ for each $\sigma_j^2 \in \operatorname{spec}(H_Y)$. The global structure of $\hat{Z}$ is complicated, but we shall only need to work with $\lambda \in \hat{Z}$ in a neighborhood of $\mathbb{R}$. For more about $\hat{Z}$, see e.g. [Mel93, Section 6.7]

From the explicit formula (2.5), we see that, for any $\chi = \chi(r) \in C_c^\infty([0,\infty))$, the cut-off resolvent $\chi R_0(\lambda)\chi$ continues analytically to $\hat{Z}$. The physical space is that preimage of the upper half plane in $\mathbb{C}$ under p on which $R_0(\lambda)$ is bounded $\mathcal{H} \to \mathcal{H}$. We now prove meromorphic continuation of $\chi R(\lambda)\chi$ to the same Riemann surface.

Lemma 2.2. *For any $\chi = \chi(r) \in C_c^\infty([0,\infty))$ the cut-off resolvent $\chi R(\lambda)\chi$ continues meromorphically to $\hat{Z}$.*

Proof. The proof is straightforward because the preliminary work has been done in the proof of Lemma 2.1.

Without loss of generality we may assume that χ is 1 near $r = 0$. Fix $\chi_1 = \chi_1(r) \in C_c^\infty([0,1))$, also 1 near $r = 0$, so that $\chi\chi_1 = \chi_1$. Then, multiplying (2.10) on the left and right by χ, we obtain

$$\chi R(\lambda)\chi = \chi F(\lambda,\lambda_0)(I + K(\lambda,\lambda_0))^{-1}\chi = \chi F(\lambda,\lambda_0)\chi(I + K(\lambda,\lambda_0)\chi)^{-1}, \tag{2.11}$$

for λ_0 fixed with positive imaginary part and nonzero real part, and for λ sufficiently close to λ_0, where for the second equality we used the Neumann series for $(I+K)^{-1}$ and $\chi K = K$. From the fact that $\chi R_0(\lambda)\chi$ continues analytically to $\hat{Z}$, and the resolvent identity $(\lambda^2 - \lambda_0^2)R_0(\lambda)R_0(\lambda_0) = R_0(\lambda) - R_0(\lambda_0)$, we see that $\lambda \mapsto \chi F(\lambda,\lambda_0)\chi$ and $\lambda \mapsto K(\lambda,\lambda_0)\chi$ both continue analytically to $\hat{Z}$. Hence, by the analytic Fredholm theorem, $\chi R(\lambda)\chi$ continues meromorphically to $\hat{Z}$. $\square$

We now refine Lemma 2.2 when $\operatorname{Im}\lambda = 0$, that is to say on the boundary of the physical space. Our arguments here mostly follow [Mel93, Propositions 6.27 and 6.28]. We denote by $\mathcal{P}_{\lambda_0} : \mathcal{H} \to \mathcal{H}$ the orthogonal projection onto the eigenspace of H at λ_0^2, with the convention that if λ_0^2 is not an eigenvalue of H, then $\mathcal{P}_{\lambda_0} = 0$.

Lemma 2.3. *Let $\lambda_0 \in \mathbb{R}$ and $\epsilon > 0$.*

- *If λ_0 is not a threshold, then there exists $B(\lambda)\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{1/2+\epsilon}\mathcal{H}$, continuous with respect to λ, such that*
$$R(\lambda) = -\frac{\mathcal{P}_{\lambda_0}}{\lambda^2-\lambda_0^2} + B(\lambda), \tag{2.12}$$
when $\operatorname{Im}\lambda \geq 0$ and λ is sufficiently close to λ_0.
- *If λ_0 is a threshold, that is to say $\lambda_0 = \sigma_j$ for some $\sigma_j^2 \in \operatorname{spec}(H_Y)$, then there exist $A_0,\ B(\lambda)\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{1/2+\epsilon}\mathcal{H}$, with A_0 independent of λ and B continuous with respect to λ, such that*
$$R(\lambda) = -\frac{\mathcal{P}_{\lambda_0}}{\tau_j(\lambda)^2} + \frac{A_0}{\tau_j(\lambda)} + B(\lambda), \tag{2.13}$$
when $\operatorname{Im}\lambda \geq 0$ and λ is sufficiently close to λ_0.

Before giving the proof, we clarify what we mean by Hf if f lies in a weighted space, rather than $\mathcal{H}$. Let $g(r) = \langle r\rangle^p$ or $g(r) = e^{pr}$ for some $p \in \mathbb{R}$. Suppose $f \in g\mathcal{D}$, with $\mathbb{1}_\infty f = \sum f_j\phi_j$. Then $(f_j),\ (-f_j'' + \sigma_j^2 f_j) \in \ell^2(\mathbb{N}_0; gL^2(\mathbb{R}_+))$. If $\chi = \chi(r) \in C_c^\infty(\mathbb{R}_+)$ is 1 in a neighborhood of $r=0$, then $H(1-\chi)f \in g\mathcal{H}$ is given by $H(1-\chi)f = \sum(-\partial_r^2 + \sigma_j^2)((1-\chi)f_j)\phi_j$. Moreover, $Hf = H\chi f + H(1-\chi)f$.

Proof. We use a more elaborate version of (2.11). Fix $\chi_0,\ \chi_1 \in C_c^\infty([0,1))$, both 1 near $r=0$, such that $\chi_0\chi_1 = \chi_1$. Then, abbreviating $K(\lambda,\lambda_0)$ to K and using $\chi_0 K = K$, we have
$$(I - K(1-\chi_0))(I+K) = I + K\chi_0 \quad\Longrightarrow\quad (I+K)^{-1} = (I+K\chi_0)^{-1}(I-K(1-\chi_0)),$$
so that
$$R(\lambda) = F(\lambda,\lambda_0)(I + K(\lambda,\lambda_0)\chi_0)^{-1}(I - K(\lambda,\lambda_0)(1-\chi_0)), \tag{2.14}$$
for λ and λ_0 in the upper half plane, away from any poles of R. By the explicit formula (2.5), we see that $\lambda \mapsto R_0(\lambda)\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{1/2+\epsilon}\mathcal{H}$ extends continuously from the physical space to its boundary. Hence $\lambda \mapsto (I - K(\lambda,\lambda_0)(1-\chi_0))\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{-1/2-\epsilon}\mathcal{H}$ and $\lambda \mapsto F(\lambda,\lambda_0)\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{1/2+\epsilon}\mathcal{H}$ also extend continuously. By the analytic Fredholm theorem, $\lambda \mapsto (I + K(\lambda,\lambda_0)\chi_0)^{-1}\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{-1/2-\epsilon}\mathcal{H}$ also extends continuously, except possibly on a discrete set where is has poles.

To describe these poles, we use the fact that if $\operatorname{Im}\lambda > 0$, then $\|R(\lambda)\|^{-1}$ equals the distance from λ^2 to the spectrum of H, where $\|\cdot\|$ denotes the operator norm $\mathcal{H} \to \mathcal{H}$. In particular,
$$0 < \operatorname{Im}\lambda \leq |\operatorname{Re}\lambda| \Longrightarrow \|R(\lambda)\| \leq 1/(2\operatorname{Im}\lambda|\operatorname{Re}\lambda|). \tag{2.15}$$
Hence, if $\lambda_0 \in \mathbb{R}$ is not a threshold, then there exist $A\colon \mathcal{H} \to \mathcal{H}$ and $B(\lambda)\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{1/2+\epsilon}\mathcal{H}$ such that
$$R(\lambda) = \frac{A}{\lambda^2-\lambda_0^2} + B(\lambda),$$
when $\operatorname{Im}\lambda \geq 0$ and λ is sufficiently close to λ_0. Composing on the left with $H-\lambda^2$, and matching powers of $\lambda^2-\lambda_0^2$ as $\lambda \to \lambda_0$, gives
$$(H-\lambda_0^2)A = 0, \qquad I = -A + (H-\lambda_0^2)B(\lambda_0), \tag{2.16}$$

which implies $A = -\mathcal{P}_{\lambda_0}$. Indeed, the first of (2.16) implies that the image of A is contained in the image of $\mathcal{P}_{\lambda_0}$, so that $\mathcal{P}_{\lambda_0} A = A$. Then apply $\mathcal{P}_{\lambda_0}$ to the second of (2.16) and use $\mathcal{P}_{\lambda_0}(H - \lambda_0^2) = 0$ to see that $\mathcal{P}_{\lambda_0} = -\mathcal{P}_{\lambda_0} A$.

If $\lambda_0 = \sigma_j$, then near λ_0 we must use the coordinate $\tau_j(\lambda)$ to describe the possible pole there, and it follows from (2.15) that there are operators $A\colon \mathcal{H} \to \mathcal{H}$ and $A_0,\ B(\lambda)\colon \langle r\rangle^{-1/2-\epsilon}\mathcal{H} \to \langle r\rangle^{1/2+\epsilon}\mathcal{H}$ such that

$$R(\lambda) = \frac{A}{\tau_j(\lambda)^2} + \frac{A_0}{\tau_j(\lambda)} + B(\lambda),$$

when $\operatorname{Im}\lambda \geq 0$ and λ is sufficiently close to λ_0. Composing on the left with $H - \lambda^2$, and matching powers of $\tau_j(\lambda)$ as $\lambda \to \sigma_j$, gives (2.16) and hence $A = -\mathcal{P}_{\lambda_0}$. $\square$

Let $\mathcal{P}$ denote the orthogonal projection onto the span of the eigenfunctions of H, if any. Then $R(\lambda)\mathcal{P}$ continues meromorphically from the physical space to $\hat{Z}$ (or even just to $\mathbb{C}$), and for any $\chi = \chi(r) \in C_c^\infty([0,\infty))$, $\chi R(\lambda)(I - \mathcal{P})\chi$ is continuous for λ on the boundary of the physical space, except, perhaps, at the thresholds $\{\pm\sigma_j \mid \sigma_j^2 \in \operatorname{spec}(H_Y)\}$. We will further describe the possible singularities at the thresholds in Section 2.5.

2.3. **The generalized eigenfunctions.** To describe $R(\lambda)$ in more detail, we define a family of generalized eigenfunctions Φ_j of H depending on the parameter $\lambda \in \hat{Z}$, though we shall be most interested in them for $\lambda \in \mathbb{R}$, that is, λ on the boundary of the physical space. Recall that $\{\phi_0,\ \phi_1, \dots\} \subset \mathcal{D}_Y$ is a complete orthonormal set of eigenfunctions of H_Y, with $H_Y\phi_j = \sigma_j^2\phi_j$. Let $\chi = \chi(r) \in C_c^\infty([0,\infty))$ be 1 for $r \leq 1$, and set

$$\Phi_j^\infty(\lambda) = e^{-i\tau_j(\lambda)r}\phi_j \in C^\infty(\mathbb{R}_+; \mathcal{H}_Y).$$

Now, for λ in the physical space and away from poles of the resolvent we define a function $\Phi_j = \Phi_j(\lambda) \in \langle r\rangle^{1/2+\epsilon} e^{\operatorname{Im}\tau_j(\lambda)r}\mathcal{D}$:

$$\Phi_j(\lambda) = (1-\chi)\Phi_j^\infty(\lambda) - R(\lambda)[\partial_r^2, \chi]\Phi_j^\infty(\lambda), \text{ for } \operatorname{Im}\lambda > 0. \tag{2.17}$$

Then (in the sense of the comment after the statement of Lemma 2.3)

$$(H - \lambda^2)\Phi_j(\lambda) = 0. \tag{2.18}$$

It is easy to see that when $\operatorname{Im}\lambda > 0$ and λ^2 is not an eigenvalue of H then $\Phi_j(\lambda)$ is independent of the choice of $\chi \in C_c^\infty([0,\infty))$ which is 1 for $r \leq 1$: If Φ_j, $\tilde{\Phi}_j$ are defined as in (2.17) with two different such functions $\chi,\ \tilde{\chi}$, then $\Phi_j - \tilde{\Phi}_j \in \mathcal{H}$ is in the null space of $H - \lambda^2$, and hence is 0 by necessity. The functions Φ_j have a meromorphic continuation to $\hat{Z}$ (with the weighted space to which they belong depending not only on the index j but also on the point in the Riemann surface $\hat{Z}$), which we continue to denote in the same way. Moreover, we shall show below that $\Phi_j(\lambda)$ is a continuous function of λ when λ is on the boundary of the physical space and $|\lambda| \geq \sigma_j$. When λ is in the boundary of the physical space and $\sigma_j \geq |\lambda|$, $\Phi_j(\lambda) \in \langle r\rangle^{1/2+\epsilon}\mathcal{H}$ for any $\epsilon > 0$.

From (2.17), away from poles of the resolvent we have

$$[\mathbb{1}_\infty \Phi_j(\lambda)](r) = e^{-i\tau_j(\lambda)r}\phi_j + \sum_{\sigma_m^2 \in \operatorname{spec}(H_Y)} S_{mj}(\lambda) e^{i\tau_m(\lambda)r}\phi_m, \tag{2.19}$$

for some functions $S_{mj}(\lambda)$ which determine the scattering matrix. For each λ away from poles of the resolvent the series in (2.19) converges absolutely, uniformly for r varying in a compact set, as do its derivatives with respect to r and its 'derivatives' with respect to H_Y. For $\lambda \in \mathbb{R}$ (ie., on the boundary of the physical space) with $|\lambda| > \sigma_j$ and away from the thresholds and the poles of the resolvent, one can equivalently define $\Phi_j(\lambda)$ to be the element of $\langle r\rangle^{1/2+\epsilon}\mathcal{H}$ which satisfies $(H-\lambda^2)\Phi_j = 0$ and which has an expansion of the form (2.19) for some S_{mj}. Note that in the expansion (2.19), the terms with $\sigma_m > |\lambda|$ are exponentially decaying; these correspond to evanescent modes. Those with $\sigma_m < |\lambda|$ correspond to outgoing propagating modes.

Both the generalized eigenfunctions Φ_j and the functions S_{mj} are meromorphic functions on $\hat{Z}$, as can be seen from (2.17). In particular, near a threshold corresponding to σ_k, both Φ_j and S_{mj} are locally meromorphic functions of $\tau_k(\lambda)$.

We give a self-contained proof of the following lemma, which is known (at least in specific cases) but perhaps not explicitly in the literature.

Lemma 2.4. *If $\lambda_0 \in \mathbb{R}$ and $|\lambda_0| \geq \sigma_j \geq 0$, then $\Phi_j(\lambda)$ and $S_{mj}(\lambda)$ are continuous at λ_0.*

We comment that there is no restriction on the size of σ_m compared to $|\lambda_0|$.

Proof. We give the proof for $\lambda_0 \geq \sigma_j$, as the proof for $\lambda_0 \leq -\sigma_j$ is similar.

From [Par95, (3.4)], or [Chr95, Lemma 1.2],

$$\sum_{0\leq\sigma_m\leq\lambda} \tau_m(\lambda)S_{mj}(\lambda)\overline{S}_{mk}(\lambda) = \tau_j(\lambda)\delta_{jk}, \text{ if } 0 \leq \sigma_j, \sigma_k \leq \lambda. \tag{2.20}$$

In particular, this implies

$$\sum_{0\leq\sigma_m\leq\lambda} \tau_m(\lambda)|S_{mj}(\lambda)|^2 = \tau_j(\lambda), \text{ if } 0 \leq \sigma_j \leq \lambda.$$

Thus $S_{mj}(\lambda)(\tau_m(\lambda)/\tau_j(\lambda))^{1/2}$ is bounded for $\lambda \geq \sigma_m, \sigma_j$. Since S_{mj} is meromorphic on $\hat{Z}$, $S_{mj}(\lambda)$ is actually continuous in this region. Thus far we have proved $S_{mj}(\lambda)$ has no poles if $0 \leq \max(\sigma_j, \sigma_m) \leq \lambda$.

Now suppose there is a $\lambda_0 \geq \sigma_j \geq 0$ so that Φ_j has a pole at λ_0. Let σ_{k_0} be any minimizer of $\{|\sigma_k - \lambda_0| : | \sigma_k^2 \in \operatorname{spec}(H_Y)\}$, and use the coordinate $\tau_{k_0}(\lambda)$ near λ_0. Suppose that, with respect to this coordinate $\Phi_j(\lambda)$, has a pole of order $p_0 > 0$ at λ_0. In particular this implies

$$f(\lambda) := \Phi_j(\lambda)(\tau_{k_0}(\lambda))^{p_0} \tag{2.21}$$

is analytic with respect to the coordinate $\tau_{k_0}(\lambda)$ near λ_0, and $f(\lambda_0)$ is nontrivial. Since we have already shown $S_{mj}(\lambda)$ is regular at λ_0 if $0 \leq \max(\sigma_j, \sigma_m) \leq \lambda_0$, this ensures by (2.19) that $f(\lambda_0) \in \mathcal{H}$. Using (2.18), we have that $(H-\lambda_0^2)f(\lambda_0) = 0$, so that $f(\lambda_0)$ is an eigenfunction of H with eigenvalue λ_0^2.

Let $\mathcal{P}_{\lambda_0} : \mathcal{H} \to \mathcal{H}$ denote orthogonal projection onto the span of the eigenfunctions of H with eigenvalue λ_0^2. Note that if $\eta \in \mathcal{H}$ is an eigenfunction of H with eigenvalue E and $\sigma_l^2 \leq E$, then

$$\langle \mathbb{1}_\infty \eta, \phi_l\rangle_{\mathcal{H}_Y}(r) = 0. \tag{2.22}$$

Hence $\mathcal{P}_{\lambda_0}[\partial_r^2,\chi]\Phi_j^\infty(\lambda)=0$ for any λ. This in turn means

$$f(\lambda_0)=\lim_{\lambda\to\lambda_0\colon \operatorname{Im}\lambda>0} f(\lambda)=\lim_{\lambda\to\lambda_0,\, \operatorname{Im}\lambda>0}(\tau_{k_0}(\lambda))^{p_0}\mathcal{P}_{\lambda_0}R(\lambda)[\partial_r^2,\chi]\Phi_j^\infty(\lambda)=0$$

since $R(\lambda)$ commutes with $\mathcal{P}_{\lambda_0}$ when λ is in the physical space. But this contradicts that $f(\lambda_0)$ is nontrivial. Hence $\Phi_j(\lambda)$ does not have a pole at λ_0. The expansion (2.19) then shows that all the $S_{mj}(\lambda)$ are regular at λ_0. Hence Φ_j and S_{mj} are continuous at λ_0. $\square$

This lemma implies that if $\sigma_k\geq\sigma_j$ and $\delta_0>0$ is sufficiently small, then $\Phi_j(\lambda)$ is a smooth function of $\tau_k(\lambda)$ on $(\pm\sigma_k-\delta_0,\pm\sigma_k+\delta_0)$.

2.4. **The spectral measure.** The spectral measure for $(I-\mathcal{P})H$ can be written in terms of the generalized eigenfunctions $\{\Phi_j\}$. When H is the Laplacian on a manifold with infinite cylindrical ends (or in fact on a more general b-manifold), Lemma 2.5 below follows from the results of [Mel93, Section 6.9] and [Chr95, Section 2]. (See particularly [Chr95, (2.2)] and the end of the proof of Lemma 2.5. See also [Lyf75, Section 5].)

Below we use the notation

$$(g\otimes h)f=g\langle f,h\rangle_{\mathcal{H}}.$$

We remark that if $f\in\langle r\rangle^{-p}\mathcal{H}$ and $h\in\langle r\rangle^{p}\mathcal{H}$ then we can make sense of $\langle f,h\rangle_{\mathcal{H}}$ by writing $\langle f,h\rangle_{\mathcal{H}}=\langle\langle r\rangle^p f,\langle r\rangle^{-p}h\rangle_{\mathcal{H}}$. Recall that when $\lambda\in\mathbb{R}$, $R(\lambda)=R(\lambda+i0)$, where $\lambda+i0$ is a (particular) point on the boundary of the physical space.

The next lemma will be used to give an expression for the part of the spectral measure corresponding to the continuous spectrum of H. In order to prove it, we use another version of Vodev's identity. Let $\chi_1=\chi_1(r)\in C_c^\infty(\mathbb{R}_+)$ be 1 in a neighborhood of the origin, and let $\chi=\chi(r)\in C_c^\infty(\mathbb{R}_+)$, with $\chi\chi_1=\chi_1$. Then for λ in the physical space, starting with (2.9), using $\lambda^2-\lambda_0^2=p^2(\lambda)-p^2(\lambda_0)$ and $(\lambda^2-\lambda_0^2)R_0(\lambda)R_0(\lambda_0)=R_0(\lambda)-R_0(\lambda_0)$ and multiplying on both the left and right by χ gives

$$\begin{aligned}\chi R(\lambda)\chi-\chi R(\lambda_0)\chi&=(p^2(\lambda)-p^2(\lambda_0))\chi R(\lambda)\chi\chi_1(2-\chi_1)\chi R(\lambda_0)\chi\\&\quad+(1-\chi_1-\chi R(\lambda)\chi[\partial_r^2,\chi_1])\left(\chi R_0(\lambda)\chi-\chi R_0(\lambda_0)\chi\right)(1-\chi_1+[\partial_r^2,\chi_1]\chi R(\lambda_0)\chi).\end{aligned}\tag{2.23}$$

The identity holds on all of $\hat{Z}$ by meromorphic continuation.

Lemma 2.5. *Let $\chi=\chi(r)\in C_c^\infty([0,\infty))$. Then for $\lambda\in\mathbb{R}$, $\lambda\neq\pm\sigma_k$, $k\in\mathbb{N}_0$, we have*

$$\frac{1}{i}\chi[R(\lambda)-R(-\lambda)](I-\mathcal{P})\chi=\frac{1}{2}\sum_{0\leq\sigma_j^2\leq\lambda^2}\frac{1}{\tau_j(\lambda)}\chi\Phi_j(\lambda)\otimes\Phi_j(\lambda)\chi.\tag{2.24}$$

Note that by Lemma 2.3,

$$\chi[R(\lambda)-R(-\lambda)](I-\mathcal{P})\chi=\chi[R(\lambda)-R(-\lambda)]\chi.$$

We include the $(I-\mathcal{P})$ here to emphasize that this combination does not have poles arising from the embedded eigenvalues of H.

Proof. Without loss of generality we may assume that χ is real valued, and that $\chi(r) = 1$ for $r \leq 2$. Choose $\chi_1 \in C_c^\infty([0,\infty))$, also real valued, so that $\chi_1\chi = \chi_1$ and $\chi_1 = 1$ for $r \leq 1$. We shall use (2.23). We identify points on the open upper half plane (with λ as the parameter) with the physical space of $\hat{Z}$. Thus $\lambda > 0$ corresponds to approaching the spectral parameter λ^2 from the upper half plane, and $\lambda < 0$ corresponds to approaching the spectral parameter λ^2 from the lower half plane.

Recall that in (2.5) we defined $R_0(\lambda)$ to be the resolvent for $-\partial_r^2 + H_Y$ with Dirichlet boundary conditions at $r = 0$. By explicit computation,

$$R_0(\lambda) - R_0(-\lambda) = \frac{i}{2} \sum_{0 \leq \sigma_j \leq \lambda} \frac{1}{\tau_j(\lambda)} \Phi_j^0(\lambda) \otimes \Phi_j^0(\lambda) \tag{2.25}$$

where

$$\Phi_j^0(\lambda) = \Phi_j^0(\lambda, r) = (e^{-i\tau_j(\lambda)r} - e^{i\tau_j(\lambda)r})\phi_j. \tag{2.26}$$

We remark here that if $|\lambda| > \sigma_j$, then $\Phi_j^0(-\lambda) = -\Phi_j^0(\lambda)$, while for $|\lambda| < \sigma_j$, $\Phi_j^0(-\lambda) = \Phi_j^0(\lambda)$, and $\Phi_j^0(\sigma_j) = 0$. From (2.23) and (2.25), if λ^2 is not an eigenvalue of H,

$$\begin{aligned}
&\chi R(\lambda)\chi - \chi R(-\lambda)\chi \\
&= (1 - \chi_1 - \chi R(\lambda)\chi[\partial_r^2, \chi_1])(\chi R_0(\lambda)\chi - \chi R_0(-\lambda)\chi)(1 - \chi_1 + [\partial_r^2, \chi_1]\chi R(-\lambda)\chi) \\
&= \frac{i}{2}(1 - \chi_1 - \chi R(\lambda)\chi[\partial_r^2, \chi_1]) \left(\sum_{0 \leq \sigma_j \leq \lambda} \frac{1}{\tau_j(\lambda)} \chi \Phi_j^0(\lambda) \otimes \Phi_j^0(\lambda)\chi \right) (1 - \chi_1 + [\partial_r^2, \chi_1]\chi R(-\lambda)\chi).
\end{aligned}$$

Now we claim that

$$\Phi_j(\lambda) = (1 - \chi_1)\Phi_j^0(\lambda) - R(\lambda)[\partial_r^2, \chi_1]\Phi_j^0(\lambda). \tag{2.27}$$

That (2.27) holds can be checked by using the comment following (2.19). Alternately, one may use that for λ in the physical space, the functions defined in (2.17) and by the right hand side of (2.27) both are in the null space of $H - \lambda^2$, and differ by an element of $\mathcal{H}$, and hence must agree if λ^2 not an eigenvalue of H. The identity then holds for all λ by analytic continuation.

This finishes the proof if λ^2 is not an eigenvalue of H. If λ^2 is an eigenvalue, the result follows from the fact that both sides of (2.24) are continuous functions of λ away from the thresholds. $\square$

We note that a related proof of an analogous result for the Schrödinger operator on $\mathbb{R}$ can be found in, for example, [RS79, Appendix to XI.6].

2.5. **Threshold behavior.** We now discuss in more detail the behavior of the resolvent at thresholds. Even after projecting away from the eigenfunctions, the resolvent near the threshold corresponding to σ_j may have a singularity which is like $1/\tau_j$, and it is this singularity we wish to better understand. Lemma 2.5 can be combined with a result of [Mel93] to obtain the following corollary. We note that we are giving a somewhat different formulation, and a rather different proof, of part of [Mel93, Proposition 6.28] in this more general setting.

Lemma 2.6. *For λ sufficiently near $\pm\sigma_j$,*

$$\chi\left(R(\lambda)(I-\mathcal{P}) - \frac{i}{4\tau_j(\lambda)}\sum_{l:\sigma_l=\sigma_j}\Phi_l(\sigma_j)\otimes\Phi_l(\sigma_j)\right)\chi$$

is bounded if $\chi=\chi(r)\in C_c^\infty([0,\infty))$. Moreover,

$$\Phi_j(\sigma_j)=\Phi_j(-\sigma_j). \tag{2.28}$$

Proof. By the Laurent expansion (2.13), there is an operator A_0 so that

$$\chi\left(R(\lambda)(I-\mathcal{P})-\frac{A_0}{\tau_j(\lambda)}\right)\chi$$

is bounded near $\lambda=\sigma_j$. Likewise, there is a B_0 so that

$$\chi\left(R(\lambda)(I-\mathcal{P})-\frac{B_0}{\tau_j(\lambda)}\right)\chi$$

is bounded near $\lambda=-\sigma_j$.

If $\sigma_j=0$, then trivially $A_0=B_0$. So suppose temporarily $\sigma_j>0$. If $\lambda\in\mathbb{R}$ with $0<\lambda<\sigma_j$, then $\tau_j(-\lambda)=\tau_j(\lambda)$. Thus we find

$$\lim_{\lambda\uparrow\sigma_j}\tau_j(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})\chi=\chi(A_0-B_0)\chi.$$

However, since by Lemma 2.4 $\Phi_k(\lambda)$ is continuous at $\lambda=\sigma_j$ if $\sigma_k\le\sigma_j$, we find from (2.24) that

$$\lim_{\lambda\uparrow\sigma_j}\tau_j(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})\chi=0.$$

Thus $A_0=B_0$.

Now let $\sigma_j\ge 0$. If $\lambda\in\mathbb{R}$, $\lambda>\sigma_j$, then $\tau_j(-\lambda)=-\tau_j(\lambda)$, so that

$$\lim_{\lambda\downarrow\sigma_j}\tau_j(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})\chi=\chi(A_0+B_0)\chi=2\chi A_0\chi.$$

Comparing (2.24) this means

$$A_0=\frac{i}{4}\sum_{l:\sigma_l=\sigma_j}\Phi_l(\sigma_j)\otimes\Phi_l(\sigma_j).$$

Now we turn to proving (2.28). Let $\chi,\chi_1\in C_c^\infty([0,\infty))$ be 1 for $r\le 1$, and let $\mathcal{P}_{\sigma_j}$ denote projection onto the eigenfunctions of H with eigenvalue σ_j^2, if any. By (2.13),

$$\chi\left(R(\lambda)-\frac{A_0}{\tau_j(\lambda)}+\frac{\mathcal{P}_{\sigma_j}}{(\tau_j(\lambda))^2}\right)\chi$$

is bounded for λ near $\pm\sigma_j$. By (2.22) we have $\mathcal{P}_{\sigma_j}[\partial_r^2,\chi_1]\Phi_j^0(\lambda)=0$, where $\Phi_j^0(\lambda)$ is given by (2.26). Now use this and (2.27) to find

$$\chi\Phi_j(\pm\sigma_j)=-\lim_{\lambda\to\pm\sigma_j}\chi R(\lambda)[\partial_r^2,\chi_1]\Phi_j^0(\lambda)=-\chi A_0[\partial_r^2,\chi_1]\lim_{\lambda\to\pm\sigma_j}\frac{\Phi_j^0(\lambda)}{\tau_j(\lambda)}=2i\chi A_0[\partial_r^2,\chi_1]r\phi_j.$$

$\square$

Given $\sigma_{j_0}^2 \in \operatorname{spec}(H_Y)$, consider the set

$$\mathcal{G}_{j_0} := \{\Phi_j(\sigma_j) \mid \sigma_j = \sigma_{j_0}\}. \tag{2.29}$$

If $\mathcal{G}_{j_0}$ contains at least one nonzero element, we say $\pm\sigma_{j_0}$ is a threshold resonance, and the nonzero elements of this set are *resonant states* associated with $\pm\sigma_j$. It follows from (2.20) that they have an expansion on the ends as in (2.19). The set $\mathcal{G}_{j_0}$ contains a nonzero element if and only if $R(\lambda)(I-\mathcal{P})$ has a pole at σ_{j_0} on the boundary of the physical space. Indeed, we can see from Lemma 2.6 that if the set $\mathcal{G}_{j_0}$ contains only 0, then $R(\lambda)(I-\mathcal{P})$ is continuous at $\lambda = \sigma_{j_0}$. If $\sigma_{j_0}^2$ is a simple eigenvalue of H_Y, then the other direction is immediate. Otherwise, if $\sigma_{j_0}^2$ is an eigenvalue of H_Y of multiplicity at least two, see [Mel93, Proposition 6.28] to see that the singularity at σ_{j_0} of $R(\lambda)(I-\mathcal{P})$ is nontrivial.

The threshold resonant states are analogous to the familiar half-bound states of Euclidean scattering theory, see e.g. [New02].

3. Two term wave expansions

Let H be an operator as in Section 2 and let $u(t)$ be the solution to the wave equation

$$(\partial_t^2 + H)u = 0,\ u(0) = f_1,\ u_t(0) = f_2; \tag{3.1}$$

that is, $u(t) = \cos(t\sqrt{H})f_1 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_2$. Here $f_1, f_2 \in \mathcal{H}$, though later we shall impose more stringent conditions on $f_1,\ f_2$.

We begin by recalling the contribution of the eigenvalues to u. In Section 3.2 we state the main theorem, the two term asymptotics result. In the remainder of Section 3 we give the proof of Theorem 3.2.

3.1. **Projection onto the eigenfunctions.**

We begin by recalling the contribution of the eigenvalues to the behavior of u. This requires only that H is self-adjoint.

Let $\{E_\ell\}$ denote the eigenvalues of H, repeated with multiplicity, with corresponding orthonormal eigenfunctions $\{\eta_\ell\}$: $H\eta_\ell = E_\ell \eta_\ell$. For a general black box operator H the set $\{E_\ell\}$ could be empty, nonempty but finite, or infinite, and examples are known of each. However, the assumptions on H which we make in Theorem 3.2 imply that H does not have infinitely many eigenvalues. On the other hand, [CDa] contains examples of Schrödinger operators on a manifold with infinite cylindrical ends which have finitely many embedded eigenvalues but which still have the type of high-energy resolvent estimate which we need for Theorem 3.2.

The following lemma, however, does not require high energy estimates on the cut-off resolvent, and holds whether the set of eigenvalues is finite or infinite.

Lemma 3.1. *Let $u(t)$ be the solution of (3.1). Then, with $u_e(t) = \mathcal{P}u(t)$,*

$$u_e(t) = \sum_{\substack{E_\ell \in \mathrm{spec}_p(H) \\ E_\ell \neq 0}} \eta_\ell \left(\cos((E_\ell)^{1/2} t) \langle f_1, \eta_\ell \rangle_{\mathcal{H}} + \frac{\sin((E_\ell)^{1/2} t)}{(E_\ell)^{1/2}} \langle f_2, \eta_\ell \rangle_{\mathcal{H}} \right) + \sum_{\substack{E_\ell \in \mathrm{spec}_p(H) \\ E_\ell = 0}} \eta_\ell \left(\langle f_1, \eta_\ell \rangle_{\mathcal{H}} + t \langle f_2, \eta_\ell \rangle_{\mathcal{H}} \right). \quad (3.2)$$

Proof. For the initial data f_1, f_2 we can write $f_j = \mathcal{P}f_j + (I - \mathcal{P})f_j$. Then, since $\mathcal{P}$ commutes with H, $\mathcal{P}u(t) = u_e(t)$, where u_e satisfies

$$\begin{aligned} (\partial_t^2 + H)u_e(t) &= 0 \\ u_e(0) &= \mathcal{P}f_1, \\ (\partial_t u_e)(0) &= \mathcal{P}f_2. \end{aligned}$$

Then a straightforward computation shows that the explicit expression in (3.2) solves this initial value problem. ☐

3.2. **Statement of Theorem 3.2.** Let

$$\mathbb{C}_{\mathrm{slit}} := \mathbb{C} \setminus \left(\bigcup_{l>0} \bigcup_{\pm} \{\pm\nu_l - is,\ s \geq 0\} \right) = \mathbb{C} \setminus \left(\bigcup_{j:\sigma_j>0} \bigcup_{\pm} \{\pm\sigma_j - is,\ s \geq 0\} \right). \quad (3.3)$$

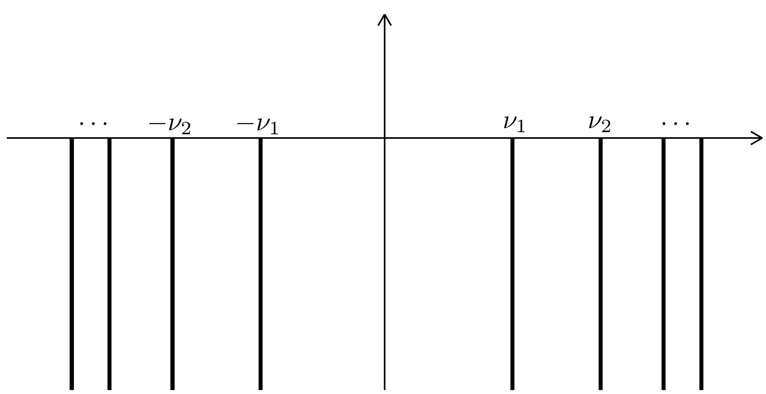

FIGURE 5. The set $\mathbb{C}_{\mathrm{slit}}$ is the complex plane with downward half-lines removed at the square roots of the nonzero eigenvalues of H_Y.

The operator $\chi R(\lambda)\chi$ continues meromorphically from $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$ to $\mathbb{C}_{\mathrm{slit}}$. In fact, we can identify $\mathbb{C}_{\mathrm{slit}}$ with a subset of the Riemann surface $\hat{Z}$. We use the same notation for the continuation of $R(\lambda)$ to $\mathbb{C}_{\mathrm{slit}}$, so that when $\lambda \in \mathbb{R}$, $R(\lambda) = R(\lambda + i0)$. We note that $\mathbb{C}_{\mathrm{slit}}$ does not require a cut at 0, because if $\sigma_j = 0$, then $\tau_j(\lambda) = \tau_0(\lambda) = \lambda$ is already analytic on $\mathbb{C}$. This does not mean, however, that $R(\lambda)$ must be analytic at the origin, as it may still have a pole there.

In the following theorem, u_e encodes the contribution of the eigenvalues of H as in Lemma 3.1 and u_{thr} is the leading order contribution from the threshold resonances (if any). The expansion for $u_e(t)$ is given in Lemma 3.1. Recall that we have assumed that H is lower semibounded. Here we choose $M_0 \in (0,\infty)$ so that $H + M_0 > 0$.

Theorem 3.2. *Let H be a black box operator as defined in Section 2.1, and suppose that for some $N_1,\ N_2 \in [0,\infty)$, $\lambda_0 > 1$, and any $\tilde{\chi} \in C_c^\infty([0,\infty))$ there are C_0, C_1 so that $\tilde{\chi}R(\lambda)\tilde{\chi}$ is analytic on the set*

$$\{\lambda \in \mathbb{C}_{\text{slit}} \mid\ \operatorname{Re}\lambda > \lambda_0 - 1 \textit{ and } \operatorname{Im}\lambda > -C_0(\operatorname{Re}\lambda)^{-N_1}\} \tag{3.4}$$

in $\hat{Z}$, and that in this region

$$\|\tilde{\chi}R(\lambda)\tilde{\chi}\| \le C_1(1+|\lambda|)^{N_2}. \tag{3.5}$$

Fix $r_1 > 0$. For $k = 1,\ 2$, suppose $f_k \in (H+M_0)^{-m_k}\mathcal{H}$ with $m_k \in \mathbb{N}$, $m_k > (N_2+4-k)/2$ and $m_k \ge (N_1+3-k)/2$. If the set $\{\nu_l\}$ is infinite, assume in addition there are positive constants N_3, C_2 and L_0 so that

$$l^{1/N_3}/C_2 \le \nu_l,\ \textit{for } l \ge L_0 \tag{3.6}$$

and $m_k > (N_2+N_3+3-k)/2$ for $k = 1,2$. Suppose $\mathbb{1}_\infty f_k$ vanishes for $r > r_1 > 0$. Let $u(t)$ be the solution of (3.1). Then

$$u(t) = u_e(t) + u_{thr}(t) + u_r(t),$$

where $u_e(t) = \mathcal{P}u(t)$ has an expansion as given in Lemma 3.1, and

$$\begin{aligned} u_{thr}(t) = \frac{1}{4}\sum_{\sigma_j=0}\Phi_j(0)\langle f_2,\Phi_j(0)\rangle_{\mathcal{H}} + \frac{1}{2\sqrt{t}}\sum_{\sigma_j>0}\sqrt{\frac{\sigma_j}{2\pi}}\cos(\sigma_j t+\pi/4)\Phi_j(\sigma_j)\langle f_1,\Phi_j(\sigma_j)\rangle_{\mathcal{H}} \\ + \frac{1}{2\sqrt{t}}\sum_{\sigma_j>0}\frac{1}{\sqrt{2\pi\sigma_j}}\sin(\sigma_j t+\pi/4)\Phi_j(\sigma_j)\langle f_2,\Phi_j(\sigma_j)\rangle_{\mathcal{H}}. \end{aligned} \tag{3.7}$$

Moreover, if $\chi \in C_c^\infty([0,\infty))$, then

$$\|\chi(H+M_0)^{1/2}u_r(t)\|_{\mathcal{H}} + \|\chi(\partial_t u_r)(t)\|_{\mathcal{H}} \le Ct^{-1}\sum_{k=1}^{2}\|(H+M_0)^{m_k}(I-\mathcal{P})f_k\|_{\mathcal{H}},$$

if t is sufficiently large.

Remark 3.1. Note that it is possible that there are no σ_j which are zero, in which case the first sum in (3.7) is 0.

Remark 3.2. We note that our assumptions on H in this theorem ensure that there are no threshold resonances larger than $\lambda_0 - 1$, or eigenvalues larger than $(\lambda_0-1)^2$. Hence the sums in u_e and u_{thr}, see (3.2) and (3.7), are finite.

Remark 3.3. The identity $R(-\overline{\lambda}) = R(\lambda)^*$, which is a consequence of the self-adjointness of H, and the consequent symmetry of the resonances mean that $\tilde{\chi}R(\lambda)\tilde{\chi}$ is analytic in the region $\{\lambda \in \mathbb{C}_{\text{slit}} \mid\ \operatorname{Re}\lambda < -(\lambda_0-1) \text{ and } \operatorname{Im}\lambda > -C_0(-\operatorname{Re}\lambda)^{-N_1}\}$, and satisfies $\|\tilde{\chi}R(\lambda)\tilde{\chi}\| \le C_1(1+|\lambda|)^{N_2}$ there.

Remark 3.4. The assumption that there is a resonance-free region of the form (3.4) follows from the seemingly weaker assumption that the bound (3.5) on the cut-off resolvent holds for $\lambda \in \mathbb{R}$, $|\lambda| > \lambda_0 - 1$. By [CDa, Theorem 5.6] this implies the existence of a resonance-free region of the form (3.4) with $N_1 = N_2 + 1$, with a corresponding estimate on the cut-off resolvent there. Note that by [CDa, Theorem 3.1] and [CDa, Sections 3.2, 3.3] this bound on the resolvent for $-\Delta_X$ (or $-\Delta_X + V$, for a large class of $V \in C_c^\infty(X;\mathbb{R})$) holds for the examples of manifolds X in Sections 1.2.1 and 1.2.2. Moreover, [CDa, Theorem 3.1] gives a more general method of constructing manifolds and Schrödinger operators for which such an estimate holds. The paper [CDb] includes classes of domains $\Omega \subset \mathbb{R}^d$ for which the Dirichlet Laplacian satisfies the conditions of the theorem. These include the planar waveguides described in Section 1.2.3. Thus Theorem 1.1 follows from Theorem 3.2.

Remark 3.5. The assumption (3.6) is a weak substitute for a Weyl law for the eigenvalues of H_Y. For example, if (Y, g_Y) is a smooth compact manifold of dimension $d-1$ and $H_Y = -\Delta_Y$, then the Weyl law implies (3.6) holds with $N_3 = d-1$. But if the eigenvalues of H_Y have high multiplicity, then (3.6) may also hold for a smaller value of N_3. For example, let $\beta > 0$ be a fixed real number, $\mu_0 \in \mathbb{N}$, and let $(Y, g_Y) = \sqcup_{\mu=1}^{\mu_0} (\mathbb{S}^{d-1}, \beta g_{\mathbb{S}^{d-1}})$ where $\mathbb{S}^{d-1}$ is the $d-1$-dimensional unit sphere, and $g_{\mathbb{S}^{d-1}}$ is the usual metric on it. Then $\nu_l = \sqrt{l(d+l-2)/\beta}$ and (3.6) holds with $N_3 = 1$, regardless of the value of the dimension d.

Note that without loss of generality we may assume that $\chi(r) = 1$ for $r < r_1$, where χ is as in the statement of the theorem. We do so in the remainder of this section. In particular, this implies that $\chi f_k = f_k$, $k = 1, 2$.

3.3. **Reduction to Propositions 3.5 and 3.6.** In this section we prove Theorem 3.2 modulo the proofs of two propositions. We have already found the contribution of the discrete spectrum to $u(t)$ in Lemma 3.1. We use the spectral theorem to write $(I - \mathcal{P})u(t)$ as an integral. This, in turn, we write as the sum of three integrals depending on the size of the spectral parameter. Each of these three will be evaluated or bounded using a different technique.

Lemma 3.3. *Let $u(t)$ be the solution of (3.1). Then*

$$\begin{pmatrix} (I-\mathcal{P})u(t) \\ \partial_t (I-\mathcal{P})u(t) \end{pmatrix} = \mathrm{PV}\, \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{it\lambda} A(\lambda)(R(\lambda) - R(-\lambda))(I-\mathcal{P}) d\lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where

$$A(\lambda) = \begin{pmatrix} \lambda & -i \\ i\lambda^2 & \lambda \end{pmatrix} \tag{3.8}$$

and PV *is the principal value.*

Proof. We have

$$u(t) = \cos(t\sqrt{H}) f_1 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_2.$$

By the functional calculus and Stone's formula,

$$\begin{aligned}\cos(t\sqrt{H})(I-\mathcal{P}) &= \frac{1}{2\pi i}\int_0^\infty \cos(t\sqrt{\tau})[(H-\tau-i0)^{-1}-(H-\tau+i0)^{-1}](I-\mathcal{P})d\tau \\ &= \frac{1}{2\pi i}\int_0^\infty [e^{it\lambda}+e^{-it\lambda}][R(\lambda)-R(-\lambda)](I-\mathcal{P})\lambda d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^\infty e^{it\lambda}[R(\lambda)-R(-\lambda)](I-\mathcal{P})\lambda d\lambda. \end{aligned} \tag{3.9}$$

As in the statement of Lemma 2.5, the factor $I-\mathcal{P}$ can be omitted in the last two instances, but it is needed in the first two. We continue to often include it even when it is unnecessary, both to emphasize that there are no poles due to eigenvalues, and because some later manipulations will require $I-\mathcal{P}$.

Similarly

$$\begin{aligned}\frac{\sin(t\sqrt{H})}{\sqrt{H}}(I-\mathcal{P}) &= \frac{-1}{2\pi}\int_0^\infty [e^{it\lambda}-e^{-it\lambda}][R(\lambda)-R(-\lambda)](I-\mathcal{P})d\lambda \\ &= -\,\mathrm{PV}\,\frac{1}{2\pi}\int_{-\infty}^\infty e^{it\lambda}[R(\lambda)-R(-\lambda)](I-\mathcal{P})d\lambda.\end{aligned}$$

Here we do need the principal value if H has 0 as a threshold resonance, since in that case $(R(\lambda)-R(-\lambda))(I-\mathcal{P})$ has a pole of order 1 at 0. $\square$

We use the integral representation from Lemma 3.3 to write $(I-\mathcal{P})u(t)$ as the sum of three terms:

$$(I-\mathcal{P})\begin{pmatrix} u(t) \\ \partial_t u(t)\end{pmatrix} = (I_s(t)+I_m(t)+I_l(t))\begin{pmatrix} f_1 \\ f_2\end{pmatrix} \tag{3.10}$$

where

$$\begin{aligned} I_s(t) &= \mathrm{PV}\,\frac{1}{2\pi i}\int_{|\lambda|<\lambda_0} e^{it\lambda}A(\lambda)(R(\lambda)-R(-\lambda))(I-\mathcal{P})d\lambda \\ I_m(t) &= \frac{1}{2\pi i}\int_{\lambda_0<|\lambda|<t^\epsilon} e^{it\lambda}A(\lambda)(R(\lambda)-R(-\lambda))(I-\mathcal{P})d\lambda \\ I_l(t) &= \frac{1}{2\pi i}\int_{t^\epsilon<|\lambda|} e^{it\lambda}A(\lambda)(R(\lambda)-R(-\lambda))(I-\mathcal{P})d\lambda. \end{aligned}$$

Here $\epsilon>0$ is a constant to be determined later and $\lambda_0>1$ is as in the statement of Theorem 3.2. We shall later add an additional assumption on λ_0 for convenience. The subscripts s, m, l stand for small, medium, and large, and refer to the size of $|\lambda|$. We shall bound each of these in turn, beginning with the easiest. Note that for I_m, I_l we may omit the $I-\mathcal{P}$ as H has no eigenvalues which are greater than or equal to λ_0^2.

Recall $M_0\in(0,\infty)$ was chosen so that $H+M_0>0$.

Lemma 3.4. *If* $\min(2m_1-1, 2m_2) \geq 0$, *then*

$$\left\| \begin{pmatrix} (H+M_0)^{1/2} & 0 \\ 0 & I \end{pmatrix} I_l(t) \begin{pmatrix} (H+M_0)^{-m_1} & 0 \\ 0 & (H+M_0)^{-m_2} \end{pmatrix} \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} = O(t^{-\epsilon(\min(2m_1-1,2m_2))}) \quad (3.11)$$

Proof. We note that for $m \in \mathbb{N}$,

$$\begin{aligned} \chi(R(\lambda)-R(-\lambda))\chi &= \chi(R(\lambda)-R(-\lambda))(H+M_0)^{-m}(H+M_0)^m\chi \\ &= \chi(R(\lambda)-R(-\lambda))(\lambda^2+M_0)^{-m}(H+M_0)^m\chi. \quad (3.12) \end{aligned}$$

By the spectral theorem, using that for $\lambda \gg 0$, $\lambda(R(\lambda)-R(-\lambda))d\lambda$ is, up to a constant multiple, the spectral measure for H, we find for $m,\ j,\ k \in \mathbb{Z}$

$$\begin{aligned} &\left\|(H+M_0)^{j/2}\int_{|\lambda|>t^\epsilon} e^{it\lambda}\lambda^{k+1}(R(\lambda)-R(-\lambda))d\lambda(H+M_0)^{-m}\right\|_{\mathcal{H}\to\mathcal{H}} \\ &= \left\|\int_{|\lambda|>t^\epsilon} e^{it\lambda}\lambda^k(\lambda^2+M_0)^{-m+j/2}\lambda(R(\lambda)-R(-\lambda))d\lambda\right\|_{\mathcal{H}\to\mathcal{H}} \\ &= 4\pi \sup_{|\lambda|>t^\epsilon}|\lambda|^k(M_0+\lambda^2)^{-m+j/2} = O(t^{(-2m+k+j)\epsilon}) \end{aligned}$$

if $k+j-2m \leq 0$. Here we use that a nontrivial spectral projector has norm 1.

The lemma follows from this and the expression for $I_l(t)$. □

Evaluating and bounding the contributions of I_s and I_m requires more effort and each will be studied separately. We state the results here.

Proposition 3.5. *Let* $f_1,\ f_2, \chi,\ \lambda_0$ *and* u_{thr} *be as in Theorem 3.2. In addition, suppose* $\lambda_0 \neq \sigma_j$ *for any* j. *Then there is a constant* C *(depending on* χ*) so that*

$$\left\|\chi \begin{pmatrix} (H+M_0)^{1/2} & 0 \\ 0 & I \end{pmatrix} \left(I_s(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \begin{pmatrix} u_{thr}(t) \\ \partial_t u_{thr}(t) \end{pmatrix} \right) \right\|_{\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-1} \left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_{\mathcal{H}\oplus\mathcal{H}} \quad (3.13)$$

when t *is sufficiently large.*

The term u_{thr} corresponds to possible resonances in $[0, \lambda_0)$ and if it is nontrivial decays at fastest at a rate proportional to $t^{-1/2}$. We remark that the error $O(t^{-1})$ in the estimate of Proposition 3.5 is sharp and is due to the discontinuous nature of the cut-off at $\lambda = \pm\lambda_0$ in the definition of $I_s(t)$. The error could be improved by instead using a smooth cut-off function in the λ variable to define $I_s(t)$ and $I_m(t)$; we do something similar in Section 4. However, since our methods for estimating the contribution of $I_m(t)$ result in an error of size $O(t^{-1})$ even with this change, we would not gain by taking this alternate approach here.

For $I_m(t)$, the corresponding result is

Proposition 3.6. *Assume (3.4), (3.5), and (3.6). Let* $0 < \epsilon < 1/N_1$ *and let* λ_0 *be as in the statement of Theorem 3.2. If* H_Y *is bounded so that the set* $\{\nu_l\}$ *is finite, by increasing* λ_0 *if necessary, choose* $\lambda_0^2 > \|H_Y\|$. *Choose* $m_k \in \mathbb{N}$ *so that* $m_k > (N_2+4-k)/2$ *for* $k = 1, 2$, *and,* *if the set $\{\nu_l\}$ is infinite, assume in addition $m_k > (N_2+N_3+3-k)/2$. Then for t sufficiently large*

$$\left\| \chi \begin{pmatrix} (H+M_0)^{1/2} & 0 \\ 0 & I \end{pmatrix} I_m(t) \begin{pmatrix} (H+M_0)^{-m_1} & 0 \\ 0 & (H+M_0)^{-m_2} \end{pmatrix} \chi \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-1}. \tag{3.14}$$

It is only in the proof of Proposition 3.6 that we use the assumptions (3.4), (3.5) and (3.6). We prove Proposition 3.5 in Section 3.4 and Proposition 3.6 in Section 3.5.

Assuming these two propositions, we may now prove Theorem 3.2.

Proof of Theorem 3.2. Writing $u(t) = \mathcal{P}u(t) + (I-\mathcal{P})u(t)$, Lemma 3.1 gives an explicit expression for $u_e(t) = \mathcal{P}u(t)$.

Recall that $\mathbb{1}_\infty f_1,\ \mathbb{1}_\infty f_2$ vanish for r sufficiently large, and without loss of generality we have chosen χ so that $\chi f_k = f_k$; in particular, $(H+M_0)^{m_k}\chi f_k = (H+M_0)^{m_k} f_k = \chi(H+M_0)^{m_k} f_k$. From (3.10) we see that to understand $(I-\mathcal{P})u(t)$ it suffices to understand the contributions of $I_s(t)$, $I_m(t)$, and $I_l(t)$. Here we choose $\epsilon = 1/(N_1+1)$. If H_Y is bounded, we choose λ_0 both satisfying the conditions of the theorem and so that $\lambda_0^2 > \|H_Y\|$. If H_Y is unbounded, choose λ_0 satisfying the conditions of the theorem, and so that $\lambda_0 \neq \sigma_j$ for any j.

Then, with $m_k \geq (N_1+3-k)/2$, $k=1,2$, the bound of $Ct^{-\epsilon\min(2m_1-1,2m_2)}$ on $I_l(t)$ from (3.11) is less than or equal to Ct^{-1}. The results of Propositions 3.5 and 3.6 complete the proof. □

3.4. **The contribution of $I_s(t)$.** The goal of this section is to prove Proposition 3.5. We remark that to prove this proposition, we do not need to assume bounds on the resolvent of H at high energy. Moreover, note that the bound is given in terms of $\|f_1\|_{\mathcal{H}}$, $\|f_2\|_{\mathcal{H}}$, so that we do not need the Sobolev-type norms $\|(H+M_0)^{m_k} f_k\|_{\mathcal{H}}$ in this section.

In order to prove the proposition, we shall write the integral defining I_s as the sum of three types of terms: an integral over a small neighborhood of 0, an integral over a small neighborhood of ν_l or $-\nu_l$ for each $\nu_l \in (0,\lambda_0)$, and an integral of a function with support disjoint from all $\pm\sigma_j$ with $0 \leq \sigma_j < \lambda_0$. Recall that the integrand is continuous near λ_0, since we have assumed that no embedded resonances exceed $\lambda_0 - 1$.

Let $\psi \in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of 0, and set

$$I_0(t) = \mathrm{PV}\,\frac{1}{2\pi i}\int \psi(\lambda)e^{it\lambda}A(\lambda)(R(\lambda)-R(-\lambda))(I-\mathcal{P})d\lambda.$$

Note that if $\sigma_0 > 0$, then $I_0(t) = 0$ if the support of ψ is chosen sufficiently small.

We emphasize that the following lemma does not require high energy resolvent estimates for H, and is valid for any $f_1,\ f_2 \in \mathcal{H}$ which have $\mathbb{1}_\infty f_1,\ \mathbb{1}_\infty f_2$ both supported for r in a fixed compact subset of $[0,\infty)$.

Lemma 3.7. *Let H be any operator satisfying the hypotheses of Section 2, and let $f_1,\ f_2 \in \mathcal{H}$ have $\mathbb{1}_\infty f_1,\ \mathbb{1}_\infty f_2$ both supported in $r \leq r_1$. Then with the support of ψ chosen sufficiently small,* *for any* $q \in \mathbb{N}_0$, $k \in \mathbb{N}$ *there is a constant* C *depending on* q, k *and the support of* f_1, f_2 *so that*

$$\left\| \chi(H+M_0)^{q/2} I_0(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \chi \sum_{\sigma_j=0} M_0^{q/2} \Phi_j(0) \langle f_2, \Phi_j(0) \rangle \\ 0 \end{pmatrix} \right\|_{\mathcal{H}\oplus\mathcal{H}} \leq C t^{-k} (\|f_1\|_{\mathcal{H}} + \|f_2\|_{\mathcal{H}})$$

when t *is sufficiently large.*

Proof. Recall that $\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi$ has a singularity at worst like $1/\lambda$ at $\lambda = 0$. Thus, if $p \in \mathbb{N}$, then $\lambda^p(\lambda^2 + M_0)^{q/2}\psi(\lambda)\chi(R(\lambda) - R(-\lambda))(1 - \mathcal{P})\chi$ is a smooth function of $\lambda \in \mathbb{R}$ if the support of ψ is chosen sufficiently small that it contains no $\pm\sigma_j$ with $\sigma_j \neq 0$. Hence for $p \in \mathbb{N}$, by integrating by parts k times, we find for $t > 0$

$$\begin{aligned} &\left\| \int_{-\infty}^{\infty} e^{it\lambda} \lambda^p (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi d\lambda \right\|_{\mathcal{H}\to\mathcal{H}} \\ &\leq t^{-k} \int_{-\infty}^{\infty} \left\| \frac{d^k}{d\lambda^k} \Big(\lambda^p (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi \Big) \right\|_{\mathcal{H}\to\mathcal{H}} d\lambda = O(t^{-k}),\ p \in \mathbb{N} \end{aligned} \tag{3.15}$$

for any $k \in \mathbb{N}$.

Using (3.15) and considering the expression (3.8) for A, this means we need only consider more carefully the entry corresponding to the upper right-hand corner of A. We also will use, see Lemma 2.5,

$$(R(\lambda) - R(-\lambda))(I - \mathcal{P}) = \frac{i}{2\lambda} \sum_{\sigma_j=0} \Phi_j(0) \otimes \Phi_j(0) + B(\lambda)$$

where $\chi B(\lambda)\chi$ is analytic in a neighborhood of $\lambda = 0$. Hence, using another integration by parts argument,

$$\begin{aligned} &\mathrm{PV} \int_{-\infty}^{\infty} e^{it\lambda} (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi d\lambda = \\ &\qquad \frac{i M_0^{q/2}}{2} \sum_{\sigma_j=0} \chi\Phi_j(0) \otimes \chi\Phi_j(0)\, \mathrm{PV} \int_{-\infty}^{\infty} e^{it\lambda} \frac{\psi(\lambda)}{\lambda} d\lambda + O(t^{-k}) \end{aligned} \tag{3.16}$$

when the support of ψ is sufficiently small.

Now we use

$$\mathrm{PV} \int_{-\infty}^{\infty} e^{it\lambda} \frac{1}{\lambda} d\lambda = i\pi,\ t > 0$$

and the fact that ψ is 1 in a small neighborhood of the origin to find that

$$\begin{aligned} &\mathrm{PV} \int_{-\infty}^{\infty} e^{it\lambda} (\lambda^2 + M_0)^{q/2} \psi(\lambda) \chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi d\lambda \\ &\qquad = \frac{-\pi M_0^{q/2}}{2} \sum_{\sigma_j=0} \chi\Phi_j(0) \otimes \chi\Phi_j(0) + O(t^{-k}) \end{aligned}$$

for t sufficiently large. $\square$

The next lemma follows directly from the more general Lemma A.1.

Lemma 3.8. *Let $\mathcal{X}$ be a Banach space, $\sigma_j > 0$, and set $B_0 = \{z \in \mathbb{C} \mid |z| < \min(\sigma_j, 1)/2\}$. If $F \in C_c^\infty(B_0; \mathcal{X})$ then there is a $C > 0$ so that*

$$\left\| \int_0^\infty e^{-i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - (\sigma_j t)^{-1/2} e^{-i\pi/4} \sqrt{2\pi} F(0) e^{-i\sigma_j t} \right\|_{\mathcal{X}} \leq Ct^{-1},\ t > 0.$$

Moreover,

$$\left\| \int_0^\infty e^{i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \right\|_{\mathcal{X}} \leq Ct^{-1},\ t > 0.$$

Proof of Proposition 3.5. Let $\psi \in C_c^\infty(\mathbb{R}; [0,1])$ be equal to 1 in a small neighborhood of the origin. Set $L = \max\{l \mid \nu_l < \lambda_0\}$, and, for $l = 1, ..., L$, set $\psi_l(\lambda) = \psi(|\lambda^2 - \nu_l^2|)$. Note that ψ_l is smooth since ψ is 1 in a neighborhood of the origin. Choose the support of ψ sufficiently small that

$$0 < l,\ l' \leq L,\ l \neq l' \Rightarrow \operatorname{supp} \psi_l \cap \operatorname{supp} \psi_{l'} = \emptyset \text{ and} \operatorname{supp} \psi \cap \operatorname{supp} \psi_l = \emptyset.$$

Then set $\psi_s(\lambda) = \psi(\lambda) + \sum_{l=1}^L \psi_l(\lambda)$. By shrinking the support of ψ if necessary, we can assume that ψ_s is 0 in a neighborhood of $\pm\lambda_0$. Note that $(R(\lambda) - R(-\lambda))(I - \mathcal{P})(1 - \psi_s(\lambda))$ is smooth on $[-\lambda_0, \lambda_0]$ by our choice of ψ_s and since λ_0^2 is not an eigenvalue of H_Y.

To simplify notation, for $q \in \mathbb{N}_0$, set

$$A_q(\lambda) = (\lambda^2 + M_0)^{q/2} A(\lambda).$$

Hence, integrating by parts,

$$\begin{aligned}
&\left\| \int_{-\lambda_0}^{\lambda_0} e^{it\lambda} A_q(\lambda)(1 - \psi_s(\lambda))\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi d\lambda \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \\
&= \Big\| \frac{-i}{t} [e^{it\lambda_0} A_q(\lambda_0) + e^{-it\lambda_0} A_q(-\lambda_0)]\chi(R(\lambda_0) - R(-\lambda_0))(I - \mathcal{P})\chi \\
&\quad + \frac{i}{t} \int_{-\lambda_0}^{\lambda_0} e^{it\lambda} \frac{d}{d\lambda} \left(A_q(\lambda)(1 - \psi_s(\lambda))\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi\right) d\lambda \Big\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \\
&= O(t^{-1})
\end{aligned} \tag{3.17}$$

since

$$\left\| \frac{d}{d\lambda} \left(A_q(\lambda)(1 - \psi_s(\lambda))\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi\right) \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \in L^1([-\lambda_0, \lambda_0]).$$

We will now focus on neighborhoods of $\pm\nu_l$. Consider $\int_{-\lambda_0}^{\lambda_0} e^{it\lambda} A_q(\lambda)\psi_l(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P})d\lambda$ with $0 < l \leq L$. Let $j = j(l) \in \mathbb{N}$ be such that $\nu_l = \sigma_j$. By our choice of the support properties of ψ, $\tau_j(\lambda)\psi_l(\lambda)R(\lambda)(I - \mathcal{P})$ is a smooth function of $\tau_j(\lambda)$ for $\lambda \in \mathbb{R}$, but $\tau_j(\lambda)\psi_l(\lambda)R(-\lambda)(I - \mathcal{P})$ is not. The reason for this second is that for $\lambda \in \mathbb{R}$, $\tau_j(-\lambda)/\tau_j(\lambda) = 1$ if

$0<|\lambda|<\sigma_j$, and $\tau_j(-\lambda)/\tau_j(\lambda)=-1$ for $|\lambda|>\sigma_j$. Hence we do a change of variable:

$$\begin{aligned}\int_{-\lambda_0}^{\lambda_0} & e^{it\lambda}A_q(\lambda)\psi_l(\lambda)(R(\lambda)-R(-\lambda))(I-\mathcal{P})d\lambda \\ &= \int_{-\lambda_0}^{0}(e^{it\lambda}A_q(\lambda)-e^{-it\lambda}A_q(-\lambda))\psi_l(\lambda)R(\lambda)(I-\mathcal{P})d\lambda \\ &\quad + \int_0^{\lambda_0}(e^{it\lambda}A_q(\lambda)-e^{-it\lambda}A_q(-\lambda))\psi_l(\lambda)R(\lambda)(I-\mathcal{P})d\lambda. \end{aligned} \tag{3.18}$$

By shrinking the support of ψ if necessary, we may apply Lemma 3.8 to the second integral on the right-hand side, with $F(\tau)=\chi\tau A_q((\tau^2+\sigma_j^2)^{1/2})\psi(|\tau|^2)R(\lambda(\tau))(I-\mathcal{P})\chi$, where $\lambda(\tau)$ is the locally well-defined inverse of $\hat{Z}\ni\lambda\mapsto\tau_j(\lambda)\in\mathbb{C}$. Thus, using the meromorphic continuation of $\chi R\chi$ to $\hat{Z}$ and Lemma 2.6, F is a smooth function supported in a complex neighborhood of the origin.

From Lemma 3.8, for $t>0$

$$\begin{aligned}\int_0^{\lambda_0} e^{-it\lambda}A_q(-\lambda)\psi_l(\lambda)\chi R(\lambda)(I-\mathcal{P})\chi d\lambda &= \int_0^{\lambda_0} e^{-it\lambda}A_q(-\lambda)\psi(|\tau_j(\lambda)|^2)\chi R(\lambda)(I-\mathcal{P})\chi d\lambda \\ &= \sqrt{2\pi}e^{-i(\sigma_j t+\pi/4)}A_q(-\sigma_j)[\chi R(\lambda)(I-\mathcal{P})\chi\tau_j(\lambda)]\restriction_{\lambda=\sigma_j}(\sigma_j t)^{-1/2}+B_{l,1}(t)\end{aligned} \tag{3.19}$$

where $\|B_{l,1}(t)\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}}=O(t^{-1})$. By a result parallel to Lemma 3.8, for $t>0$,

$$\begin{aligned}\int_{-\lambda_0}^{0} e^{-it\lambda}A_q(-\lambda)\psi_l(\lambda)\chi R(\lambda)(I-\mathcal{P})\chi d\lambda &= \int_{-\lambda_0}^{0} e^{-it\lambda}A_q(-\lambda)\psi(|\tau_j(\lambda)|^2)\chi R(\lambda)(I-\mathcal{P})\chi d\lambda \\ &= -\sqrt{2\pi}e^{i(\sigma_j t+\pi/4)}A_q(\sigma_j)[\chi R(\lambda)(I-\mathcal{P})\chi\tau_j(\lambda)]\restriction_{\lambda=-\sigma_j}(\sigma_j t)^{-1/2}+B_{l,2}(t)\end{aligned} \tag{3.20}$$

where $\|B_{l,2}(t)\|=O(t^{-1})$. From Lemma 3.8 (and the analogous result for the integral over $\lambda<0$)

$$\left\|\chi\int_{-\lambda_0}^{\lambda_0} e^{it\lambda}A_q(\lambda)\psi_l(\lambda)R(\lambda)(I-\mathcal{P})\chi d\lambda\right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}}=O(t^{-1}),\ t\to\infty. \tag{3.21}$$

It follows from Lemma 2.6 that

$$(\tau_j(\lambda)\chi R(\lambda)(I-\mathcal{P})\chi)\restriction_{\lambda=\pm\sigma_j}=\frac{i}{4}\sum_{j':\sigma_{j'}=\sigma_j=\nu_l}\chi\Phi_{j'}(\sigma_j)\otimes\chi\Phi_{j'}(\sigma_j). \tag{3.22}$$

Hence from (3.19-3.22), we have

$$\begin{aligned}&\int_{-\lambda_0}^{\lambda_0} e^{it\lambda}A_q(\lambda)\psi_l(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})\chi d\lambda \\ &=(\sigma_j t)^{-1/2}\frac{i}{2}\sqrt{\pi/2}\sum_{j:\sigma_j=\nu_l}\left[e^{i(\sigma_j t+\pi/4)}A_q(\sigma_j)-e^{-i(\sigma_j t+\pi/4)}A_q(-\sigma_j)\right]\chi\Phi_j(\sigma_j)\otimes\chi\Phi_j(\sigma_j)+B_{l,\chi}\end{aligned} \tag{3.23}$$

where $\|B_{l,\chi}\|=O(t^{-1})$.

Using (3.17), (3.23) and Lemma 3.7 proves Proposition 3.5. ∎

3.5. **The contribution of $I_m(t)$.** The main result of this section is Proposition 3.6, which provides the needed bound on $I_m(t)$. This is the portion of the proof of Theorem 3.2 for which we use the resonance-free region and the high energy resolvent estimate, and we assume these hold for all results in this section. We have some freedom in our choice of λ_0 – we can always choose a larger value. The choice of λ_0 in Proposition 3.6 is made to simplify the proof a bit.

In order to prove the bound, we shall perform a contour deformation into $\mathbb{C}_{\text{slit}}$. We recall that

$$I_m(t) = \frac{1}{2\pi i}\int_{\lambda_0<|\lambda|<t^\epsilon} e^{it\lambda}A(\lambda)[R(\lambda)-R(-\lambda)]d\lambda. \tag{3.24}$$

There is no need to compose on the right with $(I-\mathcal{P})$ here, since λ_0^2 exceeds the largest eigenvalue of H.

In order to simplify notation, set

$$G(\lambda) = \begin{pmatrix} (\lambda^2+M_0)^{1/2} & 0 \\ 0 & 1 \end{pmatrix} A(\lambda) \begin{pmatrix} (\lambda^2+M_0)^{-m_1} & 0 \\ 0 & (\lambda^2+M_0)^{-m_2} \end{pmatrix} \tag{3.25}$$

and

$$R_\chi(\lambda) = \chi R(\lambda)\chi.$$

Then

$$\begin{aligned}\left\| \chi \begin{pmatrix} (H+M_0)^{1/2} & 0 \\ 0 & I \end{pmatrix} I_m(t) \begin{pmatrix} (H+M_0)^{-m_1} & 0 \\ 0 & (H+M_0)^{-m_2} \end{pmatrix}\chi \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \\ = \left\| \frac{1}{2\pi i}\int_{\lambda_0<|\lambda|<t^\epsilon} e^{it\lambda}G(\lambda)[R_\chi(\lambda)-R_\chi(-\lambda)]d\lambda \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}}.\end{aligned} \tag{3.26}$$

We treat the contributions from the terms $R_\chi(\lambda)$ and $R_\chi(-\lambda)$ separately; the second is substantially more difficult than the first.

To bound the term in (3.24) with $e^{it\lambda}R_\chi(\lambda)$ we shall use the following lemma.

Lemma 3.9. *Let $\lambda_0>0$ be as in the statement of Theorem 3.2. Then, if ϵ, $t>0$ then there is a constant C so that*

$$\left\| \int_{\lambda_0<|\lambda|<t^\epsilon} e^{it\lambda}G(\lambda)R_\chi(\lambda)d\lambda \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-1} \tag{3.27}$$

if $N_2+\max(2-2m_1, 1-2m_2)\leq 0$ and t is sufficiently large.

We postpone the proof of this lemma to Section 3.6.

The proof of Lemma 3.9 uses a contour deformation argument, deforming the contour into the upper half-plane where $e^{it\lambda}$ decays in t. To bound an integral of the form

$$\int_{-t^\epsilon}^{-\lambda_0} G(\lambda)e^{it\lambda}R_\chi(-\lambda)d\lambda \text{ or } \int_{\lambda_0}^{t^\epsilon} G(\lambda)e^{it\lambda}R_\chi(-\lambda)d\lambda \tag{3.28}$$

in a similar way, we run into the problem that $R_\chi(-\lambda)$ must be evaluated at a point with $\operatorname{Im}(-\lambda)<0$. Since the continuation of R_χ is to $\mathbb{C}_{\text{slit}}$, we see that this is complicated. Each *distinct* value of $\sigma_j>0$ gives ramification points at $\pm\nu_l$ in $\hat{Z}$; this corresponds to the omitted rays in the lower half plane in $\mathbb{C}_{\text{slit}}$, and we must stay away from these omitted rays. Instead, a contour deformation argument gives us the following proposition, which we prove in Section 3.6. The values of a and b are chosen so that we can avoid the omitted rays in $\mathbb{C}_{\mathrm{slit}}$.

Proposition 3.10. *Let $0 < \epsilon < 1/N_1$ and set*

$$p = N_2 + \max(2 - 2m_1, 1 - 2m_2)$$

where N_1, N_2 are as in the statement of Theorem 3.2. Then there are constants T, $C > 0$ so that if $t > T$ then

$$\left\| \int_{a \leq |\lambda| \leq b} e^{i\lambda t} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}} \leq C t^{-1} \left(a^p + b^p + \int_a^b s^p ds \right) \tag{3.29}$$

whenever a and b satisfy $\max(\lambda_0, \nu_l) < a < b < \min(\nu_{l+1}, t^\epsilon)$ for some $l \in \mathbb{N}_0$. Moreover, if H_Y is bounded so that $\{\nu_l\}$ is finite, then (3.29) holds whenever $\max(\lambda_0, \|H_Y\|) < a < b < t^\epsilon$.

Note that, by assumption, $\lambda_0 > 1$ and $p < -1$.

For the proofs of both Propositions 3.10 and 3.11, we focus on the integrals over negative values of λ, as these are notationally slightly easier to handle after a change of variable as in (3.35). The integrals over positive values of λ can be handled in almost the same way. Alternatively, one can use that for $\lambda \in \mathbb{R}$, $R_\chi(\lambda) = R_\chi(-\lambda)^*$ for real-valued χ. We postpone the proof of Proposition 3.10 to Section 3.6, and instead turn to the consequences of the proposition.

Proposition 3.11. *Let λ_0 be as in the statement of of Proposition 3.6, and assume $m_k > (N_2 + 4 - k)/2$ for $k = 1, 2$. If the set $\{\nu_l\}$ is infinite, assume in addition that $m_k > (N_2 + N_3 + 3 - k)/2$ for $k = 1, 2$. Then, if $0 < \epsilon < 1/N_1$ and t is sufficiently large, there is a constant C so that*

$$\left\| \int_{\lambda_0 < |\lambda| < t^\epsilon} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\|_{\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}} \leq C t^{-1}. \tag{3.30}$$

Proof. If H_Y is bounded, so that the set $\{\nu_l\}$ is finite, then this proposition follows almost directly from Proposition 3.10, using the choice of $\lambda_0^2 > \|H_Y\|$. We write

$$\int_{\lambda_0 < |\lambda| < t^\epsilon} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda = \lim_{\delta \downarrow 0} \int_{\lambda_0 + \delta < |\lambda| < t^\epsilon - \delta} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda,$$

apply the estimate of Proposition 3.10, and use the fact that $\int_1^\infty s^p ds$ converges since $p < -1$.

Now suppose the set $\{\nu_l\}$ is infinite. We give the proof for the integral over $(-t^\epsilon, -\lambda_0)$, as the proof for the integral over (λ_0, t^ϵ) is essentially identical. Choose $l_0 \in \mathbb{N}$ so that $\nu_{l_0} \geq \lambda_0$ but $\nu_{l_0 - 1} \leq \lambda_0$, and let $L(t) \in \mathbb{N}$ be such that $\nu_{L(t)} \leq t^\epsilon$, but $\nu_{L(t)+1} \geq t^\epsilon$. Then we write

$$\begin{aligned} \int_{-t^\epsilon}^{-\lambda_0} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda = \lim_{\delta \downarrow 0} \Bigg(& \sum_{l = l_0}^{L(t)-1} \int_{-\nu_{l+1} + \delta}^{-\nu_l - \delta} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \\ & + \int_{-t^\epsilon + \delta}^{-\nu_{L(t)} - \delta} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda + \int_{-\nu_{l_0} + \delta}^{-\lambda_0 - \delta} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \Bigg). \end{aligned} \tag{3.31}$$

Using (3.31) and Proposition 3.10 we find that

$$\left\| \int_{-t^\epsilon}^{-\lambda_0} e^{it\lambda} G(\lambda) R_\chi(-\lambda) d\lambda \right\| \leq C t^{-1} \left(\sum_{l=l_0}^{L(t)-1} \left(\nu_l^p + \nu_{l+1}^p \right) + \int_{\lambda_0}^{t^\epsilon} s^p ds + \lambda_0^p + t^{\epsilon p} \right). \tag{3.32}$$

Using the lower bound (3.6): $\nu_l \geq l^{1/N_3}/C_2$,

$$\sum_{l=l_0}^{L(t)-1} \left(\nu_l^p + \nu_{l+1}^p \right) \leq 2 \sum_{l=l_0}^{L(t)} (l^{1/N_3}/C_2)^p. \tag{3.33}$$

This sum is bounded independently of t because $p/N_3 < -1$, or $m_k > (N_2 + N_3 + 3 - k)/2$ for $k = 1, 2$. The integral in (3.32) is bounded independently of t because $p < -1$. $\square$

Proposition 3.6 follows directly from (3.26), Proposition 3.11, and Lemma 3.9.

3.6. **Proofs of Proposition 3.10 and Lemma 3.9.** It remains to prove Proposition 3.10 and Lemma 3.9. The proofs are similar, involving contour deformations off the real axis to take advantage of the exponential decay of $e^{\pm it\lambda}$ in the appropriate half-plane. That this and the resulting estimates are possible are due to the assumptions of a resonance-free region in which we have an estimate on the cut-off resolvent, our assumptions (3.4) and (3.5). As the proof of Proposition 3.10 is more complicated, we focus on it. In particular, we prove (3.29) for the integral over $[-b, -a]$ carefully, as the proof for the integral over $[a, b]$ is completely analogous. A first step in our proof of Proposition 3.10 is

Lemma 3.12. *Let $\epsilon < 1/N_1$. Suppose $\max(\lambda_0, \nu_l) < a < b < \min(\nu_{l+1}, t^\epsilon)$ for some $l \in \mathbb{N}_0$. Alternatively, if H_Y is bounded, we allow the possibility that $\max(\lambda_0, \|H_Y\|) < a < b < t^\epsilon$. For each $t > 1$, let $\mathfrak{R}_t$ be the closed rectangle in $\mathbb{C}_{\text{slit}}$ with vertices*

$$a,\ b,\ b - i(\log t)/t, \quad \text{and } a - i(\log t)/t.$$

Let $\gamma_\downarrow$, $\gamma_\rightarrow$, and $\gamma_\uparrow$ be the left, bottom, and right sides of $\mathfrak{R}_t$, oriented counterclockwise.

Then there is a $T > 1$, independent of a, b, and l, such that for $t > T$

$$\int_{-b}^{-a} e^{i\lambda t} G(\lambda) R_\chi(-\lambda) d\lambda = I_\downarrow + I_\rightarrow + I_\uparrow,$$

where

$$I_\bullet = \int_{\gamma_\bullet} e^{-i\lambda t} G(-\lambda) R_\chi(\lambda) d\lambda, \tag{3.34}$$

with $\bullet$ denoting one of $\downarrow$, $\rightarrow$, or $\uparrow$.

Proof. By a change of variable,

$$\int_{-b}^{-a} e^{i\lambda t} G(\lambda) R_\chi(-\lambda) d\lambda = \int_a^b e^{-i\lambda t} G(-\lambda) R_\chi(\lambda) d\lambda. \tag{3.35}$$

Note that $G(-\lambda)$ is analytic in λ in a neighborhood of $\mathfrak{R}_t$, for any $t > 1$. Moreover, there is a $T > 1$, independent of a, b and l satisfying conditions of the lemma, such that $R_\chi(\lambda)$ is analytic in a neighborhood of $\mathfrak{R}_t$ when $t > T$. It is here that we use $\epsilon < 1/N_1$ and the fact that $R_\chi(\lambda)$

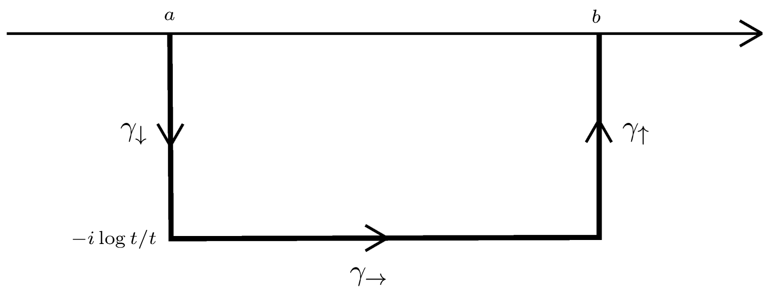

FIGURE 6. The contour used in Lemma 3.12.

has an analytic continuation to $\{\lambda \in \mathbb{C}_{\text{slit}} \mid \operatorname{Re}\lambda > \lambda_0 \text{ and } \operatorname{Im}\lambda > -C_0(\operatorname{Re}\lambda)^{-N_1}\}$. Hence by Cauchy's theorem

$$\int_{-\partial\mathfrak{R}_t} e^{-i\lambda t} G(-\lambda) R_\chi(\lambda) d\lambda = 0.$$

The integral over the top side of the rectangle is the integral in (3.35), hence the lemma follows directly. □

The next lemma bounds the integrals over the vertical sides of the rectangle.

Lemma 3.13. *Set $p = N_2 + \max(2 - 2m_1, 1 - 2m_2)$. With the notation and assumptions as in Lemma 3.12, there is a constant $C > 0$ independent of a, b and l so that for $t > T$*

$$\|I_\downarrow\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-1}a^p,$$

and

$$\|I_\uparrow\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-1}b^p.$$

Here N_2 is as in the statement of Theorem 3.2.

Proof. The proofs of the two inequalities are essentially the same, so we prove only the first one. We use that $|e^{-i\lambda t}| = e^{t\operatorname{Im}\lambda}$. Hence

$$\begin{aligned}\|I_\downarrow\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} &= \left\| \int_{\gamma_\downarrow} e^{-i\lambda t} G(-\lambda) R_\chi(\lambda) d\lambda \right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \\ &\leq \int_{-(\log t)/t}^{0} e^{ts} \|G(-(a+is)) R_\chi(a+is)\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} ds.\end{aligned}$$

Recall from the definition of G in (3.25) that for $\lambda \gg 1$,

$$G(-\lambda) \sim \begin{pmatrix} -\lambda^{2-2m_1} & -i\lambda^{1-2m_2} \\ i\lambda^{2-2m_1} & -\lambda^{1-2m_2} \end{pmatrix}.$$

Moreover, for λ lying on the image of $\gamma_\downarrow$, $|\lambda|$ is quite close to a, so that on $\gamma_\downarrow$, $\|G(-\lambda)\| \leq Ca^{\max(2-2m_1,1-2m_2)}$ and $\|R_\chi(\lambda)\|_{\mathcal{H}\to\mathcal{H}} \leq Ca^{N_2}$ by our assumptions on R_χ in the statement of the theorem. The constants here are independent of a and l. Thus

$$\|I_\downarrow\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \leq C \int_{-(\log t)/t}^{0} e^{ts} a^{N_2+\max(2-2m_1,1-2m_2)} ds \leq Ca^p\, t^{-1}.$$

□

Next we bound the integral over the bottom side of the rectangle.

Lemma 3.14. *With the notation and assumptions of Lemma 3.12, there is a constant $C > 0$ independent of a, b, and l so that for $t > T$*

$$\|I_{\rightarrow}\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-1}\int_a^b s^{N_2+\max(2-2m_1,1-2m_2)}ds$$

with N_2 as in the statement of Theorem 3.2.

Proof. Arguing as in the proof of the previous lemma,

$$\begin{aligned}\|I_{\rightarrow}\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} &= \left\|\int_a^b e^{-i(s-i(\log t)/t)t}G(-s+i(\log t)/t)R_\chi(s-i(\log t)/t)ds\right\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}}\\ &\leq C\int_a^b e^{-\log t}s^{N_2+\max(2-2m_1,1-2m_2)}ds\end{aligned}$$

which proves the lemma. □

Proof of Proposition 3.10. The proof of the estimate on the the integral over $[-b,-a]$ in (3.29) follows by combining the results of Lemmas 3.12, 3.13, and 3.14. The proof of the estimate for the integral over $[a,b]$ follows in a completely analogous way, using the consequences of the self-adjointness of H for the (continued) resolvent $R(\lambda)$. □

Proof of Lemma 3.9. We can use Cauchy's theorem to write the integral

$$\int_{\lambda_0<\lambda<t^\epsilon} e^{it\lambda}G(\lambda)\chi R(\lambda)\chi d\lambda \tag{3.36}$$

as the sum of integrals over the three line segments $\lambda_0+i[0,3t^{-1}\log t]$, $[\lambda_0,t^\epsilon]+i3t^{-1}\log t$, and $t^\epsilon+i[0,3t^{-1}\log t]$, where we reverse the orientation on the last interval. We are deforming into the upper half plane, the physical region, where $R(\lambda)$ is a bounded operator on $\mathcal{H}$ when λ^2 is not an eigenvalue of H–hence the assumption $\lambda_0>1$. Note that $|e^{it\lambda}| = e^{-t\operatorname{Im}\lambda}$. Now we can bound the integrals over the vertical segments as in Lemma 3.13 and the integral over the top as in Lemma 3.14. To bound the integral over the sides, it suffices to have $N_2+\max(2-2m_1,1-2m_2)\leq 0$. To bound the integral over the top, where we can use $\|R(\lambda)\|_{\mathcal{H}\to\mathcal{H}}\leq 1/|\operatorname{Im}\lambda^2|$, it would suffice to take $m_1=m_2=1$.

The bound for the portion of the integral over $-t^\epsilon<\lambda<-\lambda_0$ is proved in a similar way. □

4. A wave expansion under a hypothesis on the distinct eigenvalues of H_Y

Under an assumption on the distinct eigenvalues of H_Y, we can find an asymptotic expansion of $u(t)$ to order t^{-k_0} for any $k_0\in\mathbb{N}$. This expansion involves an infinite sum, see (4.4). If multiplied by the cut-off function χ, the sum over l converges absolutely, see (4.18). The main result of this section is Theorem 4.1.

In order to state the theorem, we introduce the notion of a distance on $\hat{Z}$. For two points $\lambda,\ \lambda'\in\hat{Z}$ we define $d_{\hat{Z}}(\lambda,\lambda') = \sup_j|\tau_j(\lambda)-\tau_j(\lambda')|$. That this is well-defined is shown in [CDa, Lemma 5.1]. In the statement of Theorem 4.1 below, by $\lambda'\in\mathbb{R}$ we mean that λ' lies on the boundary of the physical space. We also recall that ν_l^2 denote the *distinct* positive eigenvalues of H_Y, with $0 < \nu_1 < \nu_2 ...$

Theorem 4.1. *Let H be a black box operator as in Section 2.1, and suppose that for some $N_1,\ N_2 \in [0,\infty)$, $\lambda_0 > 1$, and any $\tilde{\chi} \in C_c^\infty([0,\infty))$ with $\tilde{\chi}(r) = 1$ for $r \leq 1$ there are $C_0,\ C_1$ so that $\tilde{\chi} R(\lambda)\tilde{\chi}$ is analytic on the set*

$$\{\lambda \in \hat{Z} :\ d_{\hat{Z}}(\lambda, \lambda') < C_0(1+\lambda')^{-N_1} \text{ for some } \lambda' \in \mathbb{R},\ \lambda' > \lambda_0 - 1\} \tag{4.1}$$

and that in this region

$$\|\tilde{\chi} R(\lambda)\tilde{\chi}\| \leq C_1(1+|\lambda|)^{N_2}. \tag{4.2}$$

In addition, suppose that there are $c_Y > 0$, $N_Y \geq 0$ so that

$$\nu_{l+1} - \nu_l > c_Y \nu_l^{-N_Y} \quad \text{when } \nu_l > 1. \tag{4.3}$$

Let $k_0 \in \mathbb{N}$ be given, and $\chi \in C_c^\infty([0,\infty))$ be one for $r \leq 1$. Let $u(t)$ be the solution of (3.1), with $f_1,\ f_2 \in (H+M_0)^{-m}\mathcal{H}$ for any $m \in \mathbb{N}_0$ and $\mathbb{1}_\infty f_1,\ \mathbb{1}_\infty f_2$ supported in $r \leq r_1 < \infty$. Then there are $b_{l,k,\pm} \in \langle r\rangle^{1/2+2k+\epsilon}\mathcal{H}$, depending on $f_1,\ f_2$ so that if we set

$$u_{thr,k_0}(t) = \frac{1}{4}\sum_{\sigma_j=0} \Phi_j(0)\langle f_2, \Phi_j(0)\rangle_{\mathcal{H}} + \sum_{k=0}^{k_0-1} t^{-1/2-k} \sum_{l>0} (e^{it\nu_l} b_{l,k,+} + e^{-it\nu_l} b_{l,k,-}) \tag{4.4}$$

then there are $m \in \mathbb{N}$, $C > 0$ so that

$$\|\chi(u(t) - u_e(t) - u_{thr,k_0}(t))\|_{\mathcal{H}} \leq C t^{-k_0} \left(\|(H+M_0)^m f_1\|_{\mathcal{H}} + \|(H+M_0)^m f_2\|_{\mathcal{H}} \right)$$

if t is sufficiently large. Moreover, for $k = 0,\ 1, ..., k_0 - 1$

$$\sum_{l>0} \left(\|\chi b_{l,k,+}\|_{\mathcal{H}} + \|\chi b_{l,k,-}\|_{\mathcal{H}} \right) \leq C \left(\|(H+M_0)^m f_1\|_{\mathcal{H}} + \|(H+M_0)^m f_2\|_{\mathcal{H}} \right). \tag{4.5}$$

The value of m needed depends polynomially on k_0, and also depends on $N_1,\ N_2$, and N_Y. The $b_{l,k,\pm}$ are determined by the value of ν_l, the initial data $f_1,\ f_2$, and the derivatives with respect to τ_j of order at most $2k$ of elements of the set $\{\Phi_{j'}\}_{0 \leq \sigma_{j'} \leq \nu_l}$ evaluated at $\pm\nu_l$, where $\sigma_j = \nu_l$. Recall u_e is given in (3.2).

As in Remark 3.4, the assumption that (4.2) holds in a region of the form (4.1) follows from the seemingly weaker assumption that the bound (4.2) on the cut-off resolvent holds for all sufficiently large $\lambda \in \mathbb{R}$. This follows from [CDa, Theorem 5.6]. Thus, the primary difference in the hypotheses of Theorems 3.2 and Theorem 4.1 is the assumption that (4.3) holds. If H_Y is bounded, then (4.3) always holds.

The paper [CDa] includes examples of manifolds X and large classes of potentials $V \in C_c^\infty(X;\mathbb{R})$ so that the hypotheses of this theorem hold for $H = -\Delta + V$ on X; see [CDa, Theorems 3.1 and 5.6, and Section 3.2]. These manifolds include the manifolds in Section 1.2.1 and any of the manifolds in Section 1.2.2 which satisfy (4.3). The question of spacing of distinct eigenvalues of the Laplacian on a compact manifold is complicated, even for manifolds of dimension 1 which are not connected; see the discussion after (1.14). For general manifolds in higher dimensions this is, as far as we know, an open problem. However, examples of compact manifolds satisfying the condition (4.3) include spheres and flat tori.

The paper [CDb] includes classes of open subsets of $\mathbb{R}^d$ with cylindrical ends for which the resolvent of the Dirichlet Laplacian satisfies the needed estimate, and these include the examples of Section 1.2.3. The Dirichlet eigenvalues on the cross section satisfy (4.3) for all of the examples of Figure 3 but only sometimes for the examples of Figure 4.

The sums over $l > 0$ in (4.4) and (4.5) are sums over all values of $l \in \mathbb{N}$ so that $\nu_l^2 \in \operatorname{spec}(H_Y) \setminus \{0\}$. Hence the sums in l are finite sums if H_Y is bounded, and otherwise are sums over $l \in \mathbb{N}$. We use this convention throughout this section.

Our Theorems 3.2 and 4.1 require a polynomial bound on the cut-off resolvent at high energies in order to handle the large energy contribution to solutions of the wave equation. If instead we consider only the solution localized in a finite energy range, neither the bound on the cut-off resolvent nor the assumption made in Theorem 4.1 on the distinct eigenvalues of H_Y is necessary. We prove the following Proposition naturally in the course of the proof of Theorem 4.1.

Proposition 4.2. *Let H be any operator satisfying all the conditions on the black box operator outlined in Section 2.1. Let $\psi_{sp} \in C_c^\infty(\mathbb{R};\mathbb{R})$, $r_1 > 0$, $f_1, f_2 \in \mathcal{H}$ satisfy $\mathbb{1}_\infty f_1$, $\mathbb{1}_\infty f_2$ vanish for $r > r_1$. Let $\psi_{sp,1} \in C_c^\infty(\mathbb{R};\mathbb{R})$ satisfy $\psi_{sp,1}\psi_{sp} = \psi_{sp}$. Then if $k_0 \in \mathbb{N}$, for each ν_l with $\nu_l \in \operatorname{supp}\psi_{sp,1}$ there are $b_{l,k,\pm} = b_{l,k,\pm}(f_1, f_2, \psi_{sp}) \in r^{2k+1/2+\epsilon}\mathcal{H}$, $k = 0, ..., k_0 - 1$, so that*

$$\chi\psi_{sp}(H)u(t) = \chi\psi_{sp}(H)u_e(t) + \frac{1}{4}\psi_{sp}(0)\sum_{\sigma_j=0}\chi\Phi_j(0)\langle f_2, \Phi_j(0)\rangle_{\mathcal{H}}$$
$$+ \sum_{l>0}\psi_{sp,1}(\nu_l^2)\sum_{k=0}^{k_0-1}\chi(b_{l,k,+}e^{it\nu_l} + b_{l,k,-}e^{-it\nu_l})t^{-1/2-k} + \chi u_{r,k_0,\psi_{sp}}(t) \quad (4.6)$$

with $\|\chi u_{r,k_0,\psi_{sp}}(t)\|_{\mathcal{H}} \le Ct^{-k_0}$ for sufficiently large t. Here $u_e(t)$ is as given in (3.2).

Note that the assumption that $\psi_{sp,1}$ has compact support means that the sum in (4.6) is finite. Related results for spectrally cut-off solutions of the wave equation (though with quite different geometry) can be found in [GHS13, VW13]. Again, we note that we need neither the assumption of high-energy bounds on the (cut-off) resolvent nor an assumption on the spacing of the distinct eigenvalues ν_l^2 in the hypotheses of Proposition 4.2.

4.1. **Bounds on the derivatives of the cut-off resolvent.** Our proof of Theorem 4.1 will require bounds of the derivatives of the cut-off resolvent along the real axis. It is here that we will use our assumption on the spacing of the distinct eigenvalues of H_Y. Our first lemma, however, does not need these hypotheses as we bound the derivatives away from the thresholds.

Lemma 4.3. *Suppose the hypotheses of Theorem 4.1 hold. Let $N \ge N_1$ be fixed and let $\beta \ge \max(1, 2/C_0)$. If $\lambda' \in \mathbb{R}$, $|\lambda'| > \lambda_0$ and $\inf_{l,\pm}|\lambda' \pm \nu_l| > |\lambda'|^{-N}/\beta$, then there is a $C > 0$ so that*

$$\left\| \frac{\partial^k}{\partial\lambda^k}\chi R(\lambda)\chi\restriction_{\lambda=\lambda'} \right\|_{\mathcal{H}\to\mathcal{H}} \le Ck!(1 + |\lambda'|)^{N_2+kN}\beta^k,\ k \in \mathbb{N}. \quad (4.7)$$

Proof. There is a ball in $\mathbb{C}_{\text{slit}}$ centered at λ' of radius $|\lambda'|^{-N}/\beta$ on which $\chi R(\lambda)\chi$ is analytic, with norm bounded by $C|\lambda|^{N_2}$. Hence the estimate (4.7) follows immediately from the Cauchy estimates. □

Away from the thresholds we can use λ as a coordinate, as we did in Lemma 4.3. Near a threshold we need to introduce a different local coordinate. In particular, near the threshold σ_j (and $-\sigma_j$) we shall use τ_j as a local coordinate.

For the setting of the next lemma, we think of $\{\nu_l\}$ as denoting not just the square roots of the distinct eigenvalues of H_Y, but also corresponding to a point on the boundary of the physical space in $\hat{Z}$. Given ν_l, choose $\epsilon = \epsilon(l) > 0$ so that $|\nu_l^2 - \nu_{l\pm1}^2| > \epsilon^2$ and let $j = j(l) \in \mathbb{N}$ be such that $\nu_l = \sigma_j$. Then we may, in a natural way, identify $B(0;\epsilon) = \{z \in \mathbb{C} : |z| < \epsilon\}$ with a (particular) neighborhood $U_{\nu_l}(\epsilon)$ of $\nu_l \in \hat{Z}$ by using

$$U_{\nu_l}(\epsilon) \ni \lambda \mapsto \tau_j(\lambda) \in B(0;\epsilon); \tag{4.8}$$

$U_{\nu_l}(\epsilon)$ is defined to be the connected component of $\tau_j^{-1}(B(0;\epsilon))$ containing ν_l, a point on the boundary of the physical space. If ϵ is small enough, as we have chosen here, then U_ϵ is a double cover of a neighborhood of ν_l in $\mathbb{C}_{\text{slit}}$. A completely analogous identification can be done near $-\nu_l$, also using τ_j, where $j = j(l)$.

The assumption on the spacing of the distinct eigenvalues of H_Y allows us to bound the derivatives of $\chi R \chi$ in a neighborhood of each threshold.

Lemma 4.4. *Suppose the hypotheses of Theorem 4.1 hold, and continue to use the notation $j = j(l)$, $U_{\nu_l}(\epsilon)$ introduced above. Set $N_M = \max((N_Y - 1)/2, N_1)$. There are $\alpha > 0$, $C \in \mathbb{R}$ so that if $\nu_l = \sigma_j \geq \lambda_0 + 1$, then*

$$\left\| \left(\frac{\partial^k}{\partial \tau_j^k} \chi R(\lambda) \chi \right) \restriction_{\lambda=\lambda'} \right\|_{\mathcal{H}\to\mathcal{H}} \leq C k! |\nu_l|^{N_2 + kN_M} \alpha^{-k}, \; k \in \mathbb{N} \tag{4.9}$$

for all $\lambda' \in U_{\pm\nu_l}(\alpha \nu_l^{-N_M}) \subset \hat{Z}$.

Proof. For simplicity, we give the proof only for $U_{\nu_l}(\alpha \nu_l^{-N_M})$.

The assumptions on the spacing of the distinct eigenvalues of H_Y ensure that there is a $\beta > 0$ so that $|\nu_l^2 - \nu_{l\pm1}^2| > \nu_l^{1-N_Y}/\beta$ for all $l > 1$. Moreover, increasing $\beta > 0$ if necessary, our definition of N_M and the hypotheses of Theorem 4.1 ensure that using the coordinate $\tau_{j(l)}$, $\chi R(\lambda)\chi$ is analytic on $U_{\nu_l}(1/(\beta \nu_l^{N_M}))$, again for all l with $\nu_l > \lambda_0 + 1$. Here we use the hypothesis (4.1). Moreover, $\|\chi R(\lambda)\chi\| \leq C(1+\nu_l)^{N_2}$ in this set, with constant C independent of l. Identify $U_{\nu_l}(1/(\beta\nu_l^{N_M}))$ with $B(0; 1/(\beta\nu_l^{N_M}))$. Each point z in $B(0; 1/(2\beta\nu_l^{N_M}))$ has the property that the ball with center z and radius $1/(2\beta\nu_l^{N_M})$ lies in $B(0; 1/(\beta\nu_l^{N_M}))$. Hence, we may prove the lemma by taking $\alpha = 1/(2\beta)$ and by applying the Cauchy estimates on such a ball, recalling that the coordinate is τ_j. □

4.2. **The proof of Theorem 4.1.** We turn more directly to the proof of Theorem 4.1. As in the proof of Theorem 3.2, we shall write $(I - \mathcal{P})u(t)$ as the sum of several integrals. In order to define these, let $\psi \in C_c^\infty(\mathbb{R}; [0,1])$ have its support in a small neighborhood of the origin, and be one in a smaller neighborhood of the origin. For convenience later, choose ψ to be even. Set

$$\tilde{N} = \max(N_Y - 1, 2N_1)$$

and

$$\psi_l(\lambda) = \psi\left(\nu_l^{\tilde{N}}(\lambda^2 - \nu_l^2)\right). \tag{4.10}$$

To prove Proposition 4.2 (rather than Theorem 4.1), the choice of of $\tilde{N}$ does not matter much–we can take $\tilde{N} = 0$. We choose the support of ψ to be small enough that

$$\operatorname{supp}\psi \cap \operatorname{supp}\psi_l = \emptyset, \text{ for } \nu_l^2 \in \operatorname{spec}(H_Y) \setminus \{0\}.$$

To prove Theorem 4.1, by shrinking the support of ψ if necessary, we choose ψ to satisfy

$$\operatorname{supp}\psi_l \cap \operatorname{supp}\psi_{l'} = \emptyset, \text{ if } l \neq l'. \tag{4.11}$$

The assumption on the spacing of the distinct eigenvalues of H_Y and our choice of $\tilde{N} \geq N_Y - 1$ ensure that (4.11) is possible. To prove Proposition 4.2 instead we replace (4.11) by

$$\psi_{sp}(\lambda^2)\psi_l(\lambda)\psi_{l'}(\lambda) \equiv 0 \text{ if } l \neq l'. \tag{4.12}$$

Similarly to (3.10), using the integral representation of Lemma 3.3 we can write

$$(I - \mathcal{P})\begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} = (I_0(t) + I_{thr}(t) + I_r(t))\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.13}$$

where

$$\begin{aligned}
I_0(t) &= \mathrm{PV}\,\frac{1}{2\pi i}\int e^{it\lambda}\psi(\lambda)A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P})d\lambda \\
I_{thr}(t) &= \frac{1}{2\pi i}\int e^{it\lambda}\sum_{l>0}\psi_l(\lambda)A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P})d\lambda \\
I_r(t) &= \frac{1}{2\pi i}\int e^{it\lambda}\left(1 - \psi(\lambda) - \sum_{l>0}\psi_l(\lambda)\right)A(\lambda)(R(\lambda) - R(-\lambda))(I - \mathcal{P})d\lambda.
\end{aligned}$$

Here $A(\lambda)$ is as in (3.8).

We recall that we have already studied $I_0(t)$ in Lemma 3.7. Lemma 4.5 shows that $I_r(t)$ does not contribute to the asymptotic expansion of $(I - \mathcal{P})u(t)$. In Lemma 4.7 we evaluate the contribution from any nonzero threshold. Finally we combine these to prove the theorem.

Lemma 4.5. *Under the hypotheses of Theorem 4.1, if the support of ψ is chosen sufficiently small, then for any $k \in \mathbb{N}$, there is an $m \in \mathbb{N}$ depending polynomially on k so that for $t > 0$*

$$\left\|\chi I_r(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\|_{\mathcal{H}\oplus\mathcal{H}} \leq Ct^{-k}(\|(H + M_0)^m f_1\|_{\mathcal{H}} + \|(H + M_0)^m f_2\|_{\mathcal{H}}).$$

Proof. Set

$$\psi_{tot}(\lambda) = \psi(\lambda) + \sum_{l>0}\psi_l(\lambda). \tag{4.14}$$

Note that by our assumptions on ψ, $1 - \psi_{tot}$ vanishes in a neighborhood of $\lambda = 0$ and in a neighborhood of $\lambda = \pm\sigma_j$ for each σ_j. Hence

$$(1 - \psi_{tot}(\lambda))(\chi R(\lambda)(I - \mathcal{P})\chi - \chi R(-\lambda)(I - \mathcal{P})\chi)$$

is a smooth function of λ. Using Lemma 4.3, by choosing $m \in \mathbb{N}$ sufficiently large we can ensure that

$$\left\| \frac{d^{k'}}{d\lambda^{k'}} \left((\lambda^2 + M_0)^{-m} (1 - \psi_{tot}(\lambda)) \chi (R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi \right) \right\|_{\mathcal{H} \to \mathcal{H}} \leq C(1 + |\lambda|)^{-2} \tag{4.15}$$

for all $k' \in \mathbb{N}_0$, $k' \leq k$. The choice of exponent -2 on the right-hand side is somewhat arbitrary, but is made to ensure that the function is integrable. We could replace -2 by $-p$, some other $p > 1$, and such a change may change the value of m which is needed on the left hand side. Now we use (3.12) and integrate by parts k times to prove the lemma. □

By way of comparison, we include following lemma.

Lemma 4.6. *Under the hypotheses of Proposition 4.2, if the support of ψ is chosen sufficiently small, then for any $k \in \mathbb{N}$, there is a $C > 0$ so that*

$$\left\| \chi \psi_{sp}(H) I_r(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_{\mathcal{H} \oplus \mathcal{H}} \leq C t^{-k} (\|f_1\|_{\mathcal{H}} + \|f_2\|_{\mathcal{H}}) \text{ for } t > 0.$$

Proof. We use ψ_{tot} from (4.14). Using the compact support of ψ_{sp} there is a $C > 0$ so that

$$\left\| \frac{d^{k'}}{d\lambda^{k'}} \left(\psi_{sp}(\lambda^2) (1 - \psi_{tot}(\lambda)) \chi (R(\lambda) - R(-\lambda))(I - \mathcal{P})\chi \right) \right\|_{\mathcal{H} \mapsto \mathcal{H}} \leq C(1 + |\lambda|)^{-2}$$

for all $k' \in \mathbb{N}_0$, $k' \leq k$. Then integrating by parts k times proves the lemma. □

Lemma 4.7. *Let H, f_1, and f_2 satisfy the hypotheses of Theorem 4.1. Let ψ_l be as defined in (4.10) for $\nu_l^2 \in \operatorname{spec}(H_Y) \setminus \{0\}$ and let $k_0 \in \mathbb{N}$. Then with the support of the function ψ in (4.10) chosen sufficiently small, there are $b_{l,k,\pm}$, $b^{(')}_{l,k,\pm} \in \langle r \rangle^{1/2+2k+\epsilon} \mathcal{H}$, $k = 0, 1, \ldots, k_0 - 1$ so that*

$$\begin{aligned} \int_{-\infty}^{\infty} & e^{it\lambda} \psi_l(\lambda) A(\lambda) \chi (R(\lambda) - R(-\lambda))(I - \mathcal{P}) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\ &= \sum_{k=0}^{k_0-1} t^{-k-1/2} \begin{pmatrix} \chi b_{l,k,+} e^{it\nu_l} + \chi b_{l,k,-} e^{-it\nu_l} \\ \chi b^{(')}_{l,k,+} e^{it\nu_l} + \chi b^{(')}_{l,k,-} e^{-it\nu_l} \end{pmatrix} + \chi B_{l,k_0}(t). \end{aligned} \tag{4.16}$$

There is an $m \in \mathbb{N}$ depending polynomially on k_0 as well as on N_1, N_2, N_Y and a constant C independent of l so that for $t > 0$

$$\|\chi B_{l,k_0}(t)\|_{\mathcal{H} \oplus \mathcal{H}} \leq C l^{-2} t^{-k_0} (\|(H + M_0)^m f_1\|_{\mathcal{H}} + \|(H + M_0)^m f_2\|_{\mathcal{H}}) \tag{4.17}$$

and

$$\|\chi b_{l,k,\pm}\|_{\mathcal{H}} + \|\chi b^{(')}_{l,k,\pm}\|_{\mathcal{H}} \leq C l^{-2} (\|(H + M_0)^m f_1\|_{\mathcal{H}} + \|(H + M_0)^m f_2\|_{\mathcal{H}}),\ k = 0, \ldots, k_0 - 1. \tag{4.18}$$

The $b_{l,k,\pm}$, $b^{(')}_{l,k,\pm}$ are determined by the initial data f_1, f_2, the value of ν_l, and the derivatives with respect to τ_j of order at most $2k$ of elements of the set $\{\Phi_{j'}\}_{0 \leq \sigma_{j'} \leq \nu_l}$ evaluated at $\pm\nu_l$, where $\sigma_j = \nu_l$.

Before proving the lemma, we note that as in (4.15) the choice of exponent -2 for l in (4.17) and (4.18) is somewhat arbitrary. We choose it to ensure the sum over l converges. As in (4.15), l^{-2} may be replaced by l^{-p}, $p > 1$, if m is chosen sufficiently large, depending on p.

Proof. Let $j \in \mathbb{N}$ be such that $\nu_l = \sigma_j$. We apply Lemma A.1.

By our choice of ψ_l, the support of ψ_l contains no thresholds other than $\pm\nu_l$. Then with $\sigma_j = \nu_l$, $\tau_j(\lambda)\chi R(\lambda)(I - \mathcal{P})\chi$ is a smooth function of $\tau_j(\lambda)$ on the support of $\psi_l(\lambda)$, $\lambda \in \mathbb{R}$, but $\chi\tau_j(\lambda)R(-\lambda)(I - \mathcal{P})\chi$ is not in general. Hence we shall rewrite the integral to avoid the use of $R(-\lambda)$. At the same time, for $m \in \mathbb{N}_0$ we write $f_1 = (H + M_0)^{-m}(H + M_0)^m f_1$, and similarly for f_2. Thus we rewrite

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{it\lambda}\psi_l(\lambda)A(\lambda)\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\
&= \int_{-\infty}^{\infty} (e^{it\lambda}A(\lambda) - e^{-it\lambda}A(-\lambda))\psi_l(\lambda)\chi R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda. \quad (4.19)
\end{aligned}$$

This integral has an asymptotic expansion, with contributions arising from the neighborhoods of $\lambda = \sigma_j = \nu_l$ and $\lambda = -\sigma_j = -\nu_l$; each will contribute both to the $b's$ and to B_{l,k_0}.

Now we recall from the proof of Lemma 4.4 that there are neighborhoods of $\nu_l = \sigma_j$ and $-\nu_l = -\sigma_j$ in $\hat{Z}$ on which we may use τ_j as a coordinate and on which $\chi\tau_j(\lambda)R(\lambda)(I - \mathcal{P})\chi$ is a smooth function of τ_j. Under the hypotheses of Theorem 4.1, the radii of the balls about $\pm\nu_l$ can be taken proportional to $\min(\sigma_j^{(1-N_Y)/2}, \sigma_j^{-N_1}) = \min(\nu_l^{(1-N_Y)/2}, \nu_l^{-N_1})$ in the τ_j coordinate. Hence by our choice of $\tilde{N}$ we can choose our original function ψ, with $\psi_l(\lambda) = \psi(\nu_l^{\tilde{N}}(\lambda^2 - \nu_l^2))$, so that we can extend $\psi_l(\lambda)A(\pm\lambda)\chi\tau_j(\lambda)R(\lambda)(I - \mathcal{P})\chi$ to be a smooth, compactly supported function of τ_j in this complex ball. In fact, with ψ chosen to be even, $\psi(\nu_l^{\tilde{N}}|\lambda^2 - \nu_l^2|)A(\pm\lambda)\chi\tau_j(\lambda)R(\lambda)(I - \mathcal{P})\chi$ provides such an extension (where we understand λ to be the locally well-defined function of τ_j). With the support of ψ chosen sufficiently small, this will hold for all $l \in \mathbb{N}$.

Thus we may apply Lemma A.1 in order to find an expansion for the portion of the integral in (4.19) over $(0, \infty)$. From the second part of Lemma A.1 we obtain immediately that

$$\begin{aligned}
&\int_{0}^{\infty} (e^{it\lambda}A(\lambda) - e^{-it\lambda}A(-\lambda))\psi_l(\lambda)\chi R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\
&= -\int_{0}^{\infty} e^{-it\lambda}A(-\lambda)\psi_l(\lambda)\chi R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m}(H + M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda + O(t^{-k_0}). \quad (4.20)
\end{aligned}$$

Now by applying the first part of Lemma A.1 and (A.1), we have an expansion of (4.20) of the form in (4.16) with exponential $e^{-it\nu_l}$, and the coefficients in the expansion; that is, the $b_{l,k.-}$ and $b^{(\prime)}_{l,k,-}$; are determined by $\sigma_j = \nu_l$ and derivatives with respect to τ_j of

$$\psi_l(\lambda)A(-\lambda)\tau_j(\lambda)\chi R(\lambda)(I - \mathcal{P})(\lambda^2 + M_0)^{-m} \begin{pmatrix} (H + M_0)^m f_1 \\ (H + M_0)^m f_2 \end{pmatrix} \quad (4.21)$$

of order at most $2k$ evaluated at $\lambda = \sigma_j = \nu_l$.

Next we turn to the question of uniformity in l, as in (4.17) and (4.18). This is immediate if H_Y is bounded, so we consider only the case of unbounded H_Y. Note that in this case our assumption

(4.3) ensures that $\nu_l \geq (C_Y l)^{1/(N_Y+1)} + O(1)$. By Lemma 4.4 the derivatives of fixed order of $\chi R(\lambda)(I-\mathcal{P})\chi$ with respect to τ_j near $\pm\sigma_j$ grow at worst polynomially in $\sigma_j = \nu_l$, as do the derivatives of $\psi_l(\lambda)$ by definition. Thus by taking m sufficiently large, depending polynomially on k_0, we can guarantee that the analog of (4.18) holds. Similarly, using the remainder estimate of Lemma A.1, we find that the analog of (4.17) holds.

The integral in (4.19) over $(-\infty, 0)$ can be handled in an analogous manner, and gives us the $b_{l,k,+}$ and $b^{(')}_{l,k,+}$.

Thus far we have proved that the coefficients $b_{l,k,-}$ (and $b^{(')}_{l,k,-}$) are determined by the value of $\nu_l = \sigma_j$ and the derivatives with respect to τ_j of (4.21) evaluated at $\lambda = \nu_l = \sigma_j$. We can say a bit more. Here we concentrate on describing the origin of the $b_{l,k,-}$ and do not worry about bounding them uniformly in l, as we have already done so. Rather than using (4.20) we return to the original expression

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{it\lambda}\psi_l(\lambda)A(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\
&= \int_{-\infty}^{0} e^{it\lambda}\psi_l(\lambda)A(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})(\lambda^2+M_0)^{-m}(H+M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\
&\quad + \int_{0}^{\infty} e^{it\lambda}\psi_l(\lambda)A(\lambda)\chi(R(\lambda)-R(-\lambda))(I-\mathcal{P})(\lambda^2+M_0)^{-m}(H+M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda.
\end{aligned}$$

We concentrate on the integral over $(-\infty, 0)$, which gives us the $b_{l,k,-}$ and $b^{(')}_{l,k,-}$ (the integral over $(0,\infty)$ leads to the $b_{l,k,+}$). Using Lemma 2.5,

$$\begin{aligned}
&\int_{-\infty}^{0} e^{it\lambda}A(\lambda)\psi_l(\lambda)\chi[R(\lambda)-R(-\lambda)](I-\mathcal{P})(\lambda^2+M_0)^{-m}(H+M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\
&= \frac{-i}{2}\int_{0}^{\sigma_j} e^{-it\lambda}A(-\lambda)\psi_l(\lambda)\chi \sum_{0\leq \sigma_{j'}<\sigma_j} \frac{\Phi_{j'}(\lambda)\otimes\Phi_{j'}(\lambda)}{\tau_{j'}(\lambda)}(\lambda^2+M_0)^{-m}(H+M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda \\
&\quad - \frac{i}{2}\int_{\sigma_j}^{\infty} e^{-it\lambda}A(-\lambda)\psi_l(\lambda)\chi \sum_{0\leq \sigma_{j'}\leq\sigma_j} \frac{\Phi_{j'}(\lambda)\otimes\Phi_{j'}(\lambda)}{\tau_{j'}(\lambda)}(\lambda^2+M_0)^{-m}(H+M_0)^m \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda.
\end{aligned} \tag{4.22}$$

We can apply Lemma A.2 to each of these two integrals. We know from our previous discussion that the sum will have an asymptotic expansion in powers of $t^{-k-1/2}$, but we learn some more specific information about the coefficients this way. This shows us that the $b_{l,k,-}$ and $b^{(')}_{l,k,-}$ are actually determined by the value $\nu_l = \sigma_j$, the initial conditions f_1 and f_2, and elements of the set

$$\left\{ (\partial_{\tau_j}^{k'}\Phi_{j'})\restriction_{\lambda=\sigma_j} \mid 0\leq \sigma_{j'}\leq\sigma_j,\ 0\leq k'\leq 2k \right\}.$$

This also provides a natural way to see that $b_{l,k-} \in \langle r\rangle^{1/2+2k+\epsilon}\mathcal{H}$, since $\partial_{\tau_j}^{k'}\Phi_j\restriction_{\lambda=\sigma_j} \in \langle r\rangle^{1/2+k'+\epsilon}\mathcal{H}$ if $\sigma_{j'}\leq\sigma_j$. □

The next lemma is almost parallel to Lemma 4.7. It differs in that it assumes only the hypotheses of Proposition 4.2, and only achieves uniformity in l because of the multiplication by the compactly supported function $\psi_{sp}(\lambda^2)$. We omit the proof because it is so similar.

Lemma 4.8. *Let H, f_1, and f_2 satisfy the hypotheses of Proposition 4.2. Let ψ_l be as defined in (4.10) and let $k_0 \in \mathbb{N}$. Then with the support of the function ψ in (4.10) chosen sufficiently small, for $l \in \mathbb{N}$ there are $b_{l,k,\pm}$, $b^{(\prime)}_{l,k,\pm} \in \langle r\rangle^{1/2+2k+\epsilon}\mathcal{H}$, $k = 0, 1, ..., k_0 - 1$ so that*

$$\int_{-\infty}^{\infty} e^{it\lambda}\psi_l(\lambda)\psi_{sp}(\lambda^2)A(\lambda)\chi(R(\lambda) - R(-\lambda))(I - \mathcal{P})\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} d\lambda$$

$$= \sum_{k=0}^{k_0-1} t^{-k-1/2}\begin{pmatrix} \chi b_{l,k,+}e^{it\nu_l} + \chi b_{l,k,-}e^{-it\nu_l} \\ \chi b^{(\prime)}_{l,k,+}e^{it\nu_l} + \chi b^{(\prime)}_{l,k,-}e^{-it\nu_l} \end{pmatrix} + \chi B_{l,k_0}(t) \quad (4.23)$$

with

$$\|\chi B_{l,k_0}(t)\|_{\mathcal{H}} \le C_{l,k}t^{-k_0}(\|f_1\|_{\mathcal{H}} + \|f_2\|_{\mathcal{H}}).$$

The $b_{l,k,\pm}$, $b^{(\prime)}_{l,k,\pm}$ are determined by the initial data f_1, f_2, the value of ν_l, and the derivatives with respect to τ_j of order at most $2k$ of elements of the set $\{\Phi_{j'}(\lambda),\ \psi_{sp}(\lambda^2)\Phi_{j'}(\lambda)\}_{0\le\sigma_{j'}\le\nu_l}$ evaluated at $\pm\nu_l$, where $\sigma_j = \nu_l$.

Of course, the $b_{l,k,\pm}$, $b^{(\prime)}_{l,k,\pm}$ in Lemma 4.8 are 0 if $\pm\nu_l$ are not in the support of $\psi_{sp}(\lambda^2)$.

Proof of Theorem 4.1. We write $u(t) = \mathcal{P}u(t) + (I - \mathcal{P})u(t)$. The expansion of $u_e(t) = \mathcal{P}u(t)$ is given in Lemma 3.1. From equation (4.13) we see that $(I - \mathcal{P})u(t)$ is given by a sum of contributions from I_0, I_{thr}, and I_r. By Lemma 4.5, the contribution of I_r is of order t^{-k} for any k. Note that using Lemma 4.7 and summing over $l \in \mathbb{N}$ evaluates the contribution of I_{thr}; the estimates (4.17) and (4.18) ensure the convergence of the sums over l to bound the remainder and the absolute convergence of the sum over l in (4.4), respectively. Lemma 3.7 gives the contribution of I_0. Summing the contributions of the terms from I_0 and I_{thr} gives u_{thr,k_0}. □

Proof of Proposition 4.2. We use (4.13), multiplying on the left hand side by $\chi\psi_{sp}(H) = \chi\psi_{sp}(H)\psi_{sp,1}(H)$. By Lemma 4.6, $\|\chi\psi_{sp}(H)I_r(t)\chi\|_{\mathcal{H}\oplus\mathcal{H}\to\mathcal{H}\oplus\mathcal{H}} \le Ct^{-k}$ for any k. Then by Lemma 3.7 and Lemma 4.8, summing over the finite number of l with $\nu_l^2 \in \operatorname{supp}\psi_{sp}$, we see that the sum of the contributions of $\chi\psi_{sp}(H)I_0(t)$ and $\chi\psi_{sp}(H)I_{thr}(t)$ gives an expansion as claimed. □

Appendix A. Asymptotic expansions of some integrals

In this section we prove two lemmas which are used in evaluating the contribution of the thresholds to the asymptotics of the solutions of the wave equation. The proofs of these lemmas use a change of variables and stationary phase to find asymptotic expansions of two types of integrals.

Lemma A.1. *Fix $c_0 \in (0, 1)$ and set $B_0 = \{z \in \mathbb{C} \mid |z| < c_0/2\}$. Let $\mathcal{X}$ be a Banach space. For each $k_0 \in \mathbb{N}$ there is a $C > 0$ so that if $\sigma_j > c_0$ and $F \in C_c^\infty(B_0; \mathcal{X})$ then there are*

$b_k = b_k(F, \sigma_j) \in \mathcal{X}$ so that for $t > 0$

$$\left\| \int_0^\infty e^{-i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - (\sigma_j t)^{-1/2} e^{-i\sigma_j t} \sum_{k=0}^{k_0-1} b_k (t\sigma_j)^{-k} \right\|_{\mathcal{X}}$$
$$\leq C \left[(\sigma_j t)^{-k_0} \sum_{k \leq 2k_0+1} \sup_\tau \left\| \sigma_j^k F^{(k)(\tau)} \right\|_{\mathcal{X}} + t^{-k_0} (1+\sigma_j)^{k_0} \sum_{k \leq 2k_0+1} \sup_\tau \|F^{(k)}(\tau)\|_{\mathcal{X}} \right]. \quad \text{(A.1)}$$

Here $b_0 = e^{-i\pi/4} F(0)\sqrt{2\pi}$, and the coefficients b_k are determined by the derivatives with respect to τ of $F(\sigma_j \tau)/\sqrt{\tau^2+1}$ of order at most $2k$, evaluated at $\tau = 0$. Moreover, under the same assumptions, for $t > 0$

$$\left\| \int_0^\infty e^{i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \right\|_{\mathcal{X}}$$
$$\leq C \left[(\sigma_j t)^{-k_0} \sum_{k \leq 2k_0+1} \sup_\tau \left\| \sigma_j^k F^{(k)}(\tau) \right\|_{\mathcal{X}} + t^{-k_0} (1+\sigma_j)^{k_0} \sum_{k \leq 2k_0+1} \sup_\tau \left\| F^{(k)}(\tau) \right\|_{\mathcal{X}} \right]. \quad \text{(A.2)}$$

Proof. In order to simplify notation, we give the proof for $\mathcal{X} = \mathbb{C}$, with the notation $|\alpha| = \|\alpha\|_{\mathbb{C}}$. The proof for a general Banach space $\mathcal{X}$ is essentially identical, though notationally more complicated.

We may write $F(\tau_j(\lambda)) = F_e(\tau_j(\lambda)) + F_o(\tau_j(\lambda))$, where

$$F_e(\tau_j(\lambda)) = \frac{1}{2}\big(F(\tau_j(\lambda)) + F(-\tau_j(\lambda))\big)$$
$$F_o(\tau_j(\lambda)) = \frac{1}{2}\big(F(\tau_j(\lambda)) - F(-\tau_j(\lambda))\big).$$

Now $F_o(\tau_j(\lambda))/\tau_j(\lambda)$ is in fact a smooth function of $\tau_j^2 = \lambda^2 - \sigma_j^2$, and hence a smooth function of λ. Then, integrating by parts,

$$\left| \int_0^\infty e^{\pm i\lambda t} \frac{F_o(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda \right| \leq t^{-k_0} \left\| \frac{d^{k_0}}{d\lambda^{k_0}} \frac{F_o(\tau_j(\lambda))}{\tau_j(\lambda)} \right\|_{L^1}$$
$$\leq C t^{-k_0} (1+\sigma_j)^{k_0} \sum_{k \leq 2k_0+1} \sup_\tau |D_\tau^k F(\tau)|.$$

To evaluate the integral $\int_0^\infty e^{\pm it\lambda} \frac{F_e(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda$, we make a change of variables. For $\lambda \in [\sigma_j, \infty)$, $\tau_j(\lambda) \in [0, \infty)$ and we use the variable $\tau' = \tau_j$; for $\lambda \in [0, \sigma_j]$, $\tau_j(\lambda) \in i[0, \infty)$ and we use the variable $\tau' = -i\tau_j$. Hence

$$\int_0^\infty e^{\pm it\lambda} \frac{F_e(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda$$
$$= \int_0^\infty e^{\pm it\sqrt{(\tau')^2+\sigma_j^2}} \frac{F_e(\tau')}{\sqrt{(\tau')^2+\sigma_j^2}} d\tau' - i \int_0^{\sigma_j} e^{\pm it\sqrt{\sigma_j^2-(\tau')^2}} \frac{F_e(i\tau')}{\sqrt{\sigma_j^2-(\tau')^2}} d\tau'. \quad \text{(A.3)}$$

For the first integral on the right-hand side of (A.3) we perform a change of variable in order to be able to track dependence on σ_j. Using $\tau' = \sigma_j\tau$, we have

$$\begin{aligned}\int_0^\infty e^{\pm it\sqrt{(\tau')^2+\sigma_j^2}}\frac{F_e(\tau')}{\sqrt{(\tau')^2+\sigma_j^2}}d\tau' &= \int_0^\infty e^{\pm it\sigma_j\sqrt{\tau^2+1}}\frac{F_e(\sigma_j\tau)}{\sqrt{\tau^2+1}}d\tau \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{\pm it\sigma_j\sqrt{\tau^2+1}}\frac{F_e(\sigma_j\tau)}{\sqrt{\tau^2+1}}d\tau \qquad \text{(A.4)}\end{aligned}$$

where for the second equality we have used that the integrand is even in τ. For this integral, we may use the method of stationary phase. Note that the only stationary point is at $\tau = 0$. By [Hör90, Theorem 7.7.5], we have that there are constants $\tilde{b}_{k\pm}$, depending on F_e and σ_j, so that

$$\begin{aligned}&\left|\int_0^\infty e^{\pm it\sigma_j\sqrt{\tau^2+1}}\frac{F_e(\sigma_j\tau)}{\sqrt{\tau^2+1}}d\tau - (\sigma_j t)^{-1/2}\, e^{\pm i\sigma_j t}\sum_{k=0}^{k_0-1}\tilde{b}_{k\pm}(\sigma_j t)^{-k}\right| \\ &\qquad\qquad \le C(\sigma_j t)^{-k_0}\sum_{|\alpha|\le 2k_0}\sup_\tau\left|D_\tau^\alpha\left(\frac{F_e(\sigma_j\tau)}{\sqrt{\tau^2+1}}\right)\right|. \qquad \text{(A.5)}\end{aligned}$$

Moreover, the $\tilde{b}_{k\pm}$ are determined by derivatives with respect to τ of $F_e(\sigma_j\tau)/\sqrt{\tau^2+1}$ of order less than or equal to $2k$, evaluated at $\tau = 0$. The coefficient $\tilde{b}_{0\pm} = F(0)\sqrt{\pi/2}e^{\pm i\pi/4}$. By allowing the constant to depend on k_0, we may bound

$$\sum_{|\alpha|\le 2k_0}\sup_\tau\left|D_\tau^\alpha\left(\frac{F_e(\sigma_j\tau)}{\sqrt{\tau^2+1}}\right)\right| \le C_{k_0}\sum_{k\le 2k_0}\sup_\tau\left|\sigma_j^k F^{(k)}(\tau)\right|.$$

A similar computation gives a similar expansion for the second integral on the right-hand side of (A.3) since the support of $F_e(i\tau')$ is small enough that $1/\sqrt{\sigma_j^2-(\tau')^2}$ is smooth on the support of F_e. We note that $k=0$ coefficient for the expansion of the second term on the right-hand side of (A.3) (including the factor of $-i$ in front) is $-i\sqrt{\pi/2}F(0)e^{\mp i\pi/4}$. This finishes the proof of (A.1), and shows that the integral on the left in (A.2) has a similar expansion.

To complete the proof of (A.2) it suffices to show that the coefficients in the expansion are 0. We shall give two proofs of this. For the first, we return to (A.3), but with the "+" sign, writing

$$\begin{aligned}&\int_0^\infty e^{it\lambda}\frac{F_e(\tau_j(\lambda))}{\tau_j(\lambda)}d\lambda \\ &\quad = \int_0^\infty e^{it\sqrt{(\tau')^2+\sigma_j^2}}\frac{F_e(\tau')}{\sqrt{(\tau')^2+\sigma_j^2}}d\tau' - i\int_0^{\sigma_j} e^{it\sqrt{\sigma_j^2-(\tau')^2}}\frac{F_e(i\tau')}{\sqrt{\sigma_j^2-(\tau')^2}}d\tau' \\ &\quad = \frac{1}{2}\left(\int_{-\infty}^\infty e^{it\sqrt{(\tau')^2+\sigma_j^2}}\frac{F_e(\tau')}{\sqrt{(\tau')^2+\sigma_j^2}}d\tau' - i\int_{-\sigma_j}^{\sigma_j} e^{it\sqrt{\sigma_j^2-(\tau')^2}}\frac{F_e(i\tau')}{\sqrt{\sigma_j^2-(\tau')^2}}d\tau'\right). \qquad \text{(A.6)}\end{aligned}$$

As previously for the second equality we have used that the integrands are even in τ'. Setting $g(\tau') = ((\tau')^2+\sigma_j^2))^{1/2} - (\sigma_j + (\tau')^2/2))$ we find from using the explicit expression for the stationary phase coefficients (see, for example, [Hör90, Theorem 7.7.5]) and summing the contributions from the two integrals that the coefficients in the asymptotic expansion of (A.6) are linear combinations of

$$e^{i\pi/4} D^{2\nu}_{\tau'} \left(\frac{g^{\mu}(\tau') F_e(\tau')}{\sqrt{(\tau')^2+\sigma_j^2}} \right)_{\restriction_{\tau'=0}} - ie^{-i\pi/4}(-1)^{\nu} D^{2\nu}_{\tau'} \left(\frac{g^{\mu}(i\tau') F_e(i\tau')}{\sqrt{(i\tau')^2+\sigma_j^2}} \right)_{\restriction_{\tau'=0}} \tag{A.7}$$

for $\nu, \mu \in \mathbb{N}_0$. Since if h is a smooth function in a complex neighborhood of the origin, then $D^{2\nu}_{\tau'} h(i\tau')\restriction_{\tau'=0} = (i)^{2\nu} D^{2\nu}_{\tau'} h(\tau')\restriction_{\tau'=0}$, we see that the quantity in (A.7) is 0, and the sum of the terms coming from the stationary phase expansions in (A.6) is 0.

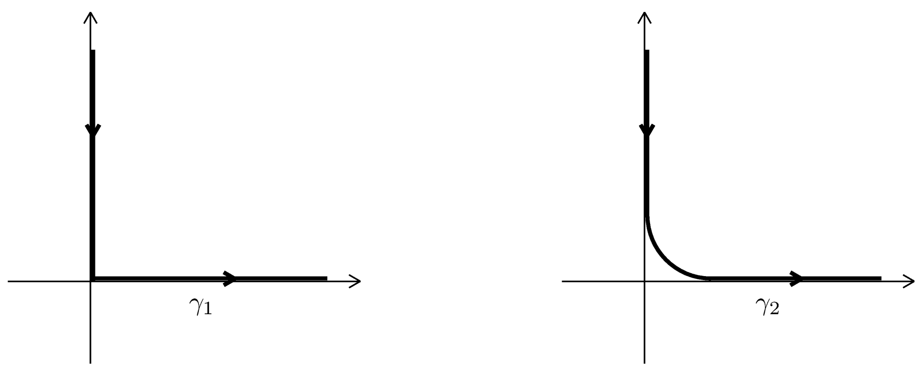

FIGURE 7. The contours of integration γ_1 and γ_2.

Now we outline an alternate, perhaps more intuitive, proof that the sum of the stationary phase coefficients in arising from the right-hand side of (A.6) is 0. The middle expression in (A.6) may be written

$$\int_{\gamma_1} e^{it\sqrt{z^2+\sigma_j^2}} \frac{F_e(z)}{\sqrt{z^2+\sigma_j^2}} dz \tag{A.8}$$

where γ_1 is as in Figure A: the path that goes down the positive imaginary axis to the origin, and then to infinity along the positive real axis. We understand the square root to be analytic in the closed first quadrant away from $i\sigma_j$ and to be positive on the positive real axis; this ensures $\operatorname{Im}\sqrt{z^2+\sigma_j^2} > 0$ in the open first quadrant. *If* F_e were analytic in a neighborhood of the origin, then by Cauchy's Theorem we could write

$$\int_{\gamma_1} e^{it\sqrt{z^2+\sigma_j^2}} \frac{F_e(z)}{\sqrt{z^2+\sigma_j^2}} dz = \int_{\gamma_2} e^{it\sqrt{z^2+\sigma_j^2}} \frac{F_e(z)}{\sqrt{z^2+\sigma_j^2}} dz$$

where γ_2 is smooth, differs from γ_1 only in a suitably small neighborhood of the origin, and is contained in the closure of the first quadrant; see Figure A. Since for $t > 0$, $|e^{i\sqrt{z^2+\sigma_j^2}t}| \leq 1$ on γ_2 and (with suitably parameterized γ_2) the phase has no stationary points on γ_2, by repeated integration by parts we can see that the integral in (A.6) is $O(t^{-k})$, any $k \in \mathbb{N}$, as $t \to \infty$.

If F_e is only smooth, not complex analytic, in a neighborhood of the origin, write $F_e = \tilde{\psi} T_{2k} + (F_e - \tilde{\psi} T_{2k})$ where T_{2k} is the $2k$th Taylor polynomial of F_e at 0 and $\tilde{\psi} \in C_c^{\infty}(\mathbb{C})$ is 1 in a neighborhood of the origin. Then the argument outlined above may be applied to the integral with $\tilde{\psi}T_{2k}$ if γ_2 differs from γ_1 only on the set where $\tilde{\psi}$ is 1. Since $(F_e - \tilde{\psi}T_{2k})$ vanishes to order $2k+1$ at the origin, we may integrate by parts k times to see that

$$\int_{\gamma_1} e^{it\sqrt{z^2+\sigma_j^2}} \frac{F_e(z) - \tilde{\psi}(z)T_{2k}(z)}{\sqrt{z^2+\sigma_j^2}} dz = O(t^{-k}).$$

□

The second argument for (A.2) also gives an intuitive reason for the difference between (A.1) and (A.2). In place of (A.8) we have instead for (A.1) the integral $\int_{\gamma_1} e^{-it\sqrt{z^2+\sigma_j^2}} F_e(z)(z^2+\sigma_j^2)^{-1/2} dz$. If F_e is analytic near the origin, we can, as in the argument above, use a contour deformation argument to deform the contour of integration to γ_2. But for z in the open first quadrant, $e^{-it\sqrt{z^2+\sigma_j^2}}$ is exponentially increasing as $t \to \infty$. If we instead deform γ_1 to avoid the origin and the first quadrant, the deformed path must have portions in each of quadrants 2, 3, and 4. But $e^{-it\sqrt{z^2+\sigma_j^2}}$ is exponentially increasing as $t \to \infty$ if z is in the open third quadrant.

We state another lemma, with results similar to those of Lemma A.1. Note that this differs from Lemma A.1 in the domain of integration, the assumptions on where F and G are smooth, and the less explicit bound on the error. We remark that the powers $t^{-k/2}$, rather than t^{-k} of Lemma A.1 are a consequence of the fact that after changing variable $\tau = \sqrt{\lambda^2 - \sigma_j^2}$, $\tau = 0$ is both a stationary point of the phase and an endpoint of integration.

Lemma A.2. *Let $\mathcal{X}$ be a Banach space and let $\sigma_j > 0$. Let $F \in C_c^\infty([0, \sigma_j/2); \mathcal{X})$ and $G \in C_c^\infty(i[0, \sigma_j/2); \mathcal{X})$. Then given $k_0 \in \mathbb{N}$ there are $\alpha_{k,\pm}$, $\beta_{k,\pm} \in \mathcal{X}$, $k \in 0, 1, ..., 2k_0 - 2$, $C = C(F, G, \sigma_j, k_0) > 0$ so that*

$$\left\| \int_{\sigma_j}^\infty e^{\pm i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - t^{-1/2} e^{\pm it\sigma_j} \sum_{k=0}^{2k_0-2} \alpha_{k,\pm} t^{-k/2} \right\|_{\mathcal{X}} \leq Ct^{-k_0},\ t > 0 \tag{A.9}$$

and

$$\left\| \int_0^{\sigma_j} e^{\pm i\lambda t} \frac{G(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda - t^{-1/2} e^{\pm it\sigma_j} \sum_{k=0}^{2k_0-2} \beta_{k,\pm} t^{-k/2} \right\|_{\mathcal{X}} \leq Ct^{-k_0},\ t > 0. \tag{A.10}$$

Here the $\alpha_{k,\pm}$ (respectively $\beta_{k,\pm}$) are determined by σ_j and the derivatives of F (respectively G) of order at most k, evaluated at 0.

Proof. We prove only (A.9), as the proof of (A.10) is almost identical. By introducing $\tau = \tau_j(\lambda)$ as the variable of integration,

$$\int_{\sigma_j}^\infty e^{\pm i\lambda t} \frac{F(\tau_j(\lambda))}{\tau_j(\lambda)} d\lambda = \int_0^\infty e^{\pm it\sqrt{\tau^2+\sigma_j^2}} \frac{F(\tau)}{\sqrt{\tau^2+\sigma_j^2}} d\tau. \tag{A.11}$$

Then an application of [Erd56, Section 2.9] proves (A.9), with coefficients $\alpha_{k,\pm}$ determined by σ_j and derivatives of F, evaluated at 0, of order at most k. □

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