THE PERFECT CONDUCTIVITY PROBLEM WITH ARBITRARY VANISHING ORDERS AND NON-TRIVIAL TOPOLOGY

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Abstract. The perfect conductivity problem concerns optimal bounds for the magnitude of an electric field in the presence of almost touching perfect conductors. This reduces to obtaining gradient estimates for harmonic functions with Dirichlet boundary conditions in the narrow region between the conductors. In this paper we extend estimates of Bao-Li-Yin to deal with the case when the boundaries of the conductors are given by graphs with arbitrary vanishing orders. Our estimates allow us to deal with globally defined narrow regions with possibly non-trivial topology.

We also prove the sharpness of our estimates in terms of the distance between the perfect conductors. The precise optimality statement we give is new even in the setting of Bao-Li-Yin.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary. Let D_1, D_2 be disjoint domains in Ω with smooth boundaries which are a distance $\varepsilon > 0$ apart, and a distance at least $d \gg \varepsilon$ from $\partial\Omega$. We write $\tilde{\Omega} = \Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$.

Fix a smooth function φ on $\partial\Omega$. The setup of the *perfect conductivity problem* is the following PDE:

$$
\begin{aligned}
&\Delta u = 0 \quad \text{in } \tilde{\Omega} \\
&u_+ = u_- \quad \text{on } \partial D_1 \cup \partial D_2 \\
&\nabla u = 0 \quad \text{on } D_1 \cup D_2 \\
&\int_{\partial D_i} \frac{\partial u}{\partial \nu}\Big|_+ = 0 \quad i = 1, 2 \\
&u = \varphi \quad \text{on } \partial\Omega.
\end{aligned}
\tag{1}
$$

Here u_+ and u_- refer to the limits of u from outside and inside (respectively) the sets D_1, D_2. The third equation implies that $u = C_1$ on D_1 and $u = C_2$ on D_2, for constants C_1, C_2. The function $\frac{\partial u}{\partial \nu}|_+$ in the fourth line of (1) is the derivative of u in the direction ν, the unit outward normal vector on ∂D_i. Namely, at $x_0 \in \partial D_i$ it is the limiting value of $\nabla u(x) \cdot \nu(x_0)$ as $x \to x_0$ through values within $\tilde{\Omega}$.

The question is: what happens to $|\nabla u|$ as $\varepsilon \to 0$?

This problem has a physical interpretation in terms of electrical conductivity. The domains D_1 and D_2 represent perfect conductors and u represents the electric potential. The question is then how the magnitude of the electric field ∇u may blow up as the perfect conductors approach each other. When $n = 2$, there is also a physical interpretation in terms of composite materials in which Ω represents the cross-section of a fiber-reinforced composite (here D_1 and D_2 represent the

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embedded fibers). In this case, the electric potential and field are replaced by the out-of-plane elastic displacement and the stress tensor respectively. For background and more details, we refer the reader to [4, 8, 17, 18] and the references therein.

The standard setting in the literature is that D_1 and D_2 are strictly convex sets a distance $\varepsilon > 0$ apart (see figure 1). In this case there is a single "narrow region" of $\tilde{\Omega}$ between the two sets D_1, D_2. After translation, this region is given by points $x = (x', x_n)$ with $f(x') \leq x_n \leq g(x')$ for $|x_1|, \ldots, |x_{n-1}| \leq r$, for a uniform $r > 0$, where f and g are smooth functions with $(g-f)(0') = \varepsilon$, $g - f \geq \varepsilon$ and $D^2 g > 0 > D^2 f$. Here we use the usual notation $x' = (x_1, \ldots, x_{n-1})$. Points outside this narrow region can be characterized by the fact that they are contained in a ball of uniform radius lying completely inside $\tilde{\Omega}$.

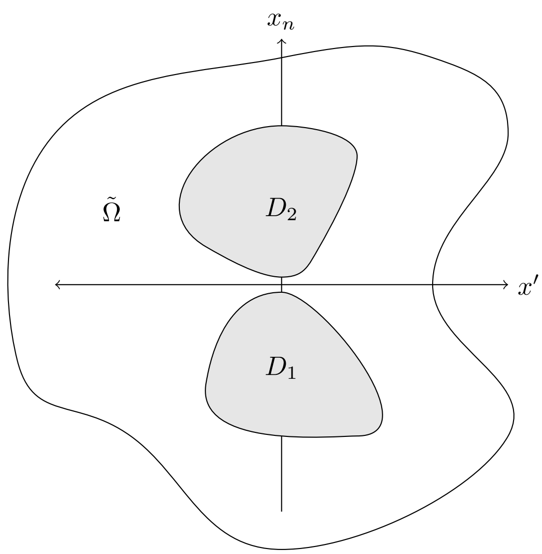

FIGURE 1. The case when D_1 and D_2 are convex sets a distance ε apart.

It has been known for some time that in general the gradient $|\nabla u|$ may blow up as $\varepsilon \to 0$ [9, 20]. It was shown in [9] that $\sup |\nabla u|$ blows up at a rate $\varepsilon^{-1/2}$ for a special solution in $\mathbb{R}^2$. More general solutions in the case when D_1, D_2 are disks with comparable radii in $\mathbb{R}^2$ were dealt with by Ammari-Kang-Lim [3] (who gave the lower bound of $\sup |\nabla u|$) and Ammari-Kang-Lee-Lee-Lim [1] (the upper bound). Yun [21] extended [3] to more general convex subdomains in $\mathbb{R}^2$ which are positively curved at the closest point.

Bao-Li-Yin [5] then proved a much more general result which allows any dimension $n \geq 2$ and the case when the domains do not necessarily have positively curved boundaries. We now describe their main result. If there is a single narrow region as above with

$$\frac{1}{C}(\varepsilon + |x'|^{2\alpha}) \leq g(x') - f(x') \leq C(\varepsilon + |x'|^{2\alpha}), \tag{2}$$

for a uniform C and a constant $\alpha \geq 1$ then $|\nabla u|$ is bounded on $\tilde{\Omega}$ as follows:

$$\sup_{\tilde{\Omega}} |\nabla u| \leq \begin{cases} \dfrac{C}{\varepsilon^{\frac{n-1}{2\alpha}}}, & \text{if } n-1 < 2\alpha \\ \dfrac{C}{\varepsilon|\log \varepsilon|}, & \text{if } n-1 = 2\alpha \\ \dfrac{C}{\varepsilon}, & \text{if } n-1 > 2\alpha. \end{cases} \tag{3}$$

Bao-Li-Yin [5] also proved the optimality of their estimates under some symmetry assumptions on the domains. They made use of a linear functional $Q_\varepsilon(\varphi)$ which we will describe later (see Section 4).

Since then, there have been many further results, refining and extending the estimates (3) and giving detailed asymptotics, see [2, 6, 7, 10, 11, 12, 14, 15, 16, 19, 22], for example (this is far from a complete list).

In this paper we give a broad extension of the Bao-Li-Yin estimates in a different direction, allowing for more complicated geometry and topology of the sets D_1 and D_2. We will allow the vanishing orders of the boundaries of D_1 and D_2 to be different in each of the $n-1$ directions $x_1, \ldots, x_{n-1}$. Namely, we replace the quantity $|x'|^{2\alpha}$ in (2) by

$$\sum_{j=1}^{n-1} x_j^{2\alpha_j}, \quad \text{for constants } \alpha_1, \ldots, \alpha_{n-1} \geq 1.$$

We refer to the constants $\alpha_1, \ldots, \alpha_{n-1}$ as the *vanishing orders* of the boundary, and a key point of this paper is that the α_j need not all be equal. Crucially, we also allow any number of the α_j to take the value $+\infty$, which we take to mean that $x_j^{2\alpha_j}$ doesn't appear in the sum (we may assume that that $|x_j| < 1$).

We also deal with the case when the narrow region is defined by a finite union of sets, each of which is given by the set of points between graphs f and g. This allows the possibility that D_1 and D_2 are "close together" in several regions throughout Ω (but far away from $\partial\Omega$). For example there may be a curve (or even higher-dimensional set) of points on ∂D_1 that are at a distance on the order of ε from ∂D_2.

Such situations may arise for example when the conductors have non-trivial topology, such as the case of encircled tori, which may be closely touching along a circle of points (see, for example, figure 5).

Our approach will be to cover the narrow region between D_1, D_2 with small open sets where we do individually get a simple picture, and then to piece this together to a global statement. See figure 2.

Statement of the main results. The region Ω is fixed, independent of ε. It is convenient to regard D_1 and D_2 as elements of a smoothly varying family of domains. We fix a compact set K in Ω, and domains D_1^0, D_2^0 with smooth boundaries contained in K. For a small constant $\varepsilon_0 > 0$ we consider smooth families of domains $\{D_1^\varepsilon\}_{\varepsilon \in [0,\varepsilon_0]}$ and $\{D_2^\varepsilon\}_{\varepsilon \in [0,\varepsilon_0]}$, also contained in K and with smooth boundaries. To make the notion of "smooth family" more precise, for $i = 1, 2$, write $M_i^\varepsilon = \partial D_i^\varepsilon$ for $\varepsilon \in [0, \varepsilon_0]$. These are smooth closed embedded hypersurfaces in $\mathbb{R}^n$. Then there are smooth maps $F_i : M_i^0 \times [0, \varepsilon_0] \to K$ for $i = 1, 2$ such that $x \mapsto F_i(x, \varepsilon)$ is a diffeomorphism from M_i^0 onto M_i^ε and the identity when $\varepsilon = 0$.

We assume that $\overline{D_1^\varepsilon} \cap \overline{D_2^\varepsilon} = \emptyset$ for every $\varepsilon \in (0, \varepsilon_0]$, but the intersection may be nonempty at $\varepsilon = 0$. We also make an additional assumption:

($*$) There is a smooth path from $\partial\Omega$ to ∂D_1^0 that does not intersect $\overline{D_2^0}$, and another smooth path from $\partial\Omega$ to ∂D_2^0 that does not intersect $\overline{D_1^0}$.

This rules out the case when one of the domains D_1^0, D_2^0 completely envelops the other. See the beginning of Section 3 for more discussion of assumption ($*$).

Our goal is to obtain optimal bounds, in terms of ε, for the gradient of u solving (1) for $D_1 = D_1^\varepsilon$ and $D_2 = D_2^\varepsilon$. **For simplicity of notation, in what follows we will drop the superscript ε and write D_1 and D_2 instead of D_1^ε and D_2^ε.**

We assume there is a constant $c_0 > 0$ and an open subset $V \subset \tilde{\Omega}$ such that V satisfies an interior ball condition of radius c_0. Specifically we mean by this that for all $p \in \partial V$ there is a ball B of radius c_0 such that $p \in \partial B$ and $B \subset V$. Thus V consists of points that are "far away" from the narrow region between D_1 and D_2. Next we assume that $\tilde{\Omega} \setminus V$ can be covered by open boxes $U_i = U_i^{(r)}$ for $i = 1, \ldots, k$, of a fixed size $r > 0$, where for each $i = 1, \ldots, k$, after possibly translating and rotating the coordinates, $U_i = \{(x_1, \ldots, x_n) \mid \max_j |x_j| < r\}$. Further, we assume that for each $i = 1, \ldots, k$ there are functions f_i, g_i on the set

$$Q_r := \{x' \mid \max_{j=1,\ldots,n-1} |x_j| < r\} \subset \mathbb{R}^{n-1}$$

such that

$$\tilde{U}_i := U_i \cap \tilde{\Omega} = \{(x, y) \mid x \in Q_r \text{ and } f_i(x') < x_n < g_i(x')\}.$$

See figure 3. Moreover, we assume that the boxes U_i overlap sufficiently so that the union of boxes $U_i^{(r/2)}$ of size $r/2$ still covers the region $\tilde{\Omega} \setminus V$.

Here the graphs of the functions f_i and g_i will describe the portions of the boundaries of D_1 and D_2 which lie within U_i. We assume that the boxes U_i are chosen centered near points where D_1 and D_2 are closest together. Specifically we assume that there is a constant $C > 0$, independent of ε, such that for all $i = 1, \ldots, k$ there are positive numbers (or infinities, see below) $\alpha_1^i, \ldots, \alpha_{n-1}^i$ such that

$$(4) \qquad \frac{1}{C}\big(\varepsilon + \sum_{j=1}^{n-1} x_j^{2\alpha_j^i}\big) < g_i(x') - f_i(x') < C\big(\varepsilon + \sum_{j=1}^{n-1} x_j^{2\alpha_j^i}\big) \quad \text{for all } x' \in Q_r.$$

We interpret the quantity $x_j^{2\alpha_j^i}$ as $(x_j^2)^{\alpha_j^i}$, so that, for example, $x^{2\cdot\frac{1}{2}} = |x|$. We allow the possibility $\alpha_j^i = \infty$, in which case we interpret the above expression containing $x_j^{2\alpha_j^i}$ to mean this term does not appear. We set $\alpha^i := (\alpha_j^i, \ldots, \alpha_{n-1}^i)$.

Here and throughout this article we use C to denote a positive constant that is independent of ε, and it may change from line to line.

We assume that the boxes $U_1, \ldots, U_k$ are fixed independent of ε but that f_i, g_i may depend on ε. Nevertheless, it follows from our assumptions that these functions and their derivatives are uniformly bounded independent of ε, and in particular

$$|f_i(x')|, |g_i(x')|, |\nabla f_i(x')|, |\nabla g_i(x')| \leq C \quad \text{for all } x' \in Q_r.$$

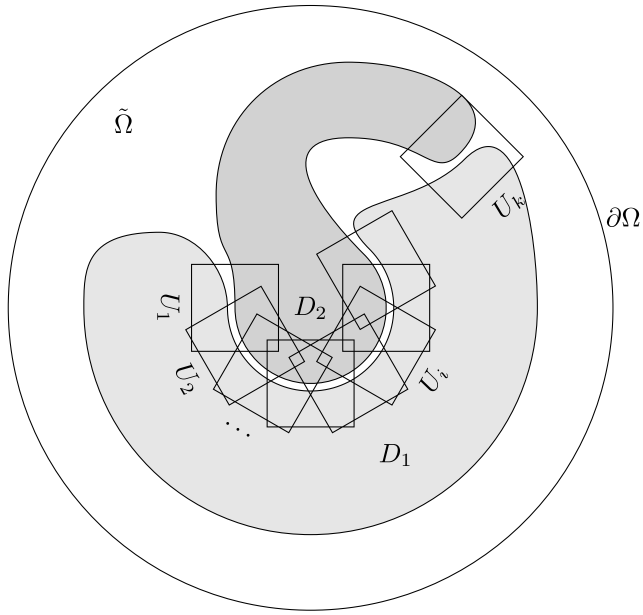

FIGURE 2. A possible configuration of the inclusions D_1 and D_2.

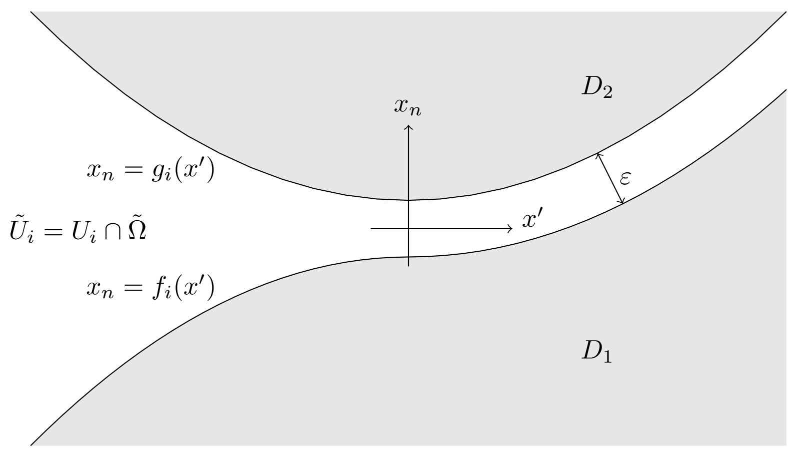

FIGURE 3. The neighborhood $\tilde{U}_i$ in the narrow region

To state our main theorem we define, for each $i = 1, \dots, k$,

$$\gamma_i := \sum_{j=1}^{n-1} \frac{1}{2\alpha_j^i}, \; i = 1, \dots, k$$

(where, if $\alpha_j^i = \infty$, then $\frac{1}{2\alpha_j^i} = 0$), and set

$$\gamma := \min_{i=1,\dots,k} \gamma_i. \tag{5}$$

The main result is:

Theorem 1.1. *With assumptions as above, let u solve (1) for $D_1 = D_1^\varepsilon$ and $D_2 = D_2^\varepsilon$. Given any smooth boundary data φ on $\partial\Omega$ there exists a constant C independent of $\varepsilon > 0$ such that*

$$\sup_{\tilde{\Omega}} |\nabla u| \leq \begin{cases} \dfrac{C}{\varepsilon^\gamma}, & \text{if } \gamma < 1 \\ \dfrac{C}{\varepsilon |\log \varepsilon|}, & \text{if } \gamma = 1 \\ \dfrac{C}{\varepsilon}, & \text{if } \gamma > 1. \end{cases}$$

One can check that the exponents match with the Bao-Li-Yin estimate (3) in the case of a single coordinate patch with $\alpha := \alpha_1^1 = \cdots = \alpha_{n-1}^1$.

Our next result shows that these estimates are optimal in terms of ε.

Theorem 1.2. *With the assumptions as above, there exists smooth boundary data φ on $\partial\Omega$ such that the following holds. If u solves (1) for $D_1 = D_1^\varepsilon$ and $D_2 = D_2^\varepsilon$ then there exists a constant $C > 0$ such that for $\varepsilon > 0$ sufficiently small,*

$$\sup_{\tilde{\Omega}} |\nabla u| \geq \begin{cases} \dfrac{1}{C\varepsilon^\gamma}, & \text{if } \gamma < 1 \\ \dfrac{1}{C\varepsilon |\log \varepsilon|}, & \text{if } \gamma = 1 \\ \dfrac{1}{C\varepsilon}, & \text{if } \gamma > 1. \end{cases}$$

The statement and proof of Theorem 1.2 appear to be new even in the case when D_1 and D_2 are strictly convex, since the optimality results of [6] made symmetry assumptions on the sets D_1, D_2 and Ω.

We have assumed smoothness of $\partial\Omega$, ∂D_1, ∂D_2 and φ for the sake of simplicity. As in [6] the regularity can be relaxed to $C^{2+\beta}$ for the boundaries (for some $0 < \beta < 1$), and C^2 for φ.

The outline of this paper is as follows. In Section 2 we recall some preliminary results from [6] which we will need to make use of. In Sections 3 and 4 we prove Theorems 1.1 and 1.2 respectively. Finally, in Section 5 we illustrate our results with some examples.

2. Preliminaries

In this section, we gather some preliminary results whose proofs follow from the corresponding arguments of Bao-Li-Yin [6]. First, the bound on $|\nabla u|$ reduces to an estimate on $|C_1 - C_2|$, where we recall that $u = C_1$ on D_1 and $u = C_2$ on D_2.

Lemma 2.1. *There exists a constant C independent of ε such that*

$$\frac{|C_1 - C_2|}{\varepsilon} \leq \sup_{\tilde{\Omega}} |\nabla u| \leq \frac{C}{\varepsilon}|C_1 - C_2| + C.$$

Proof. See the proof of [6, Proposition 2.1]. For the reader's convenience, we sketch the idea here.

The lower bound of $\sup|\nabla u|$ follows from the Mean Value Theorem. For the upper bound, we write $u = v + w + C_2$ where v and w are the unique solutions of

$$\Delta v = 0 \text{ in } \tilde{\Omega}, \quad v = C_1 - C_2 \text{ on } \partial D_1, \quad v = 0 \text{ on } \partial D_2 \cup \partial\Omega$$

$$\Delta w = 0 \text{ in } \tilde{\Omega}, \quad w = 0 \text{ on } \partial D_1 \cup \partial D_2, \quad w = \varphi - C_2 \text{ on } \partial\Omega.$$

Since $|v| \leq |C_1 - C_2|$ we can apply standard gradient estimates for harmonic functions to obtain $|\nabla v| \leq C|C_1 - C_2|/\varepsilon$. On the other hand $|\nabla w| \leq C$ on ∂D_1 by comparison with a harmonic function on $\Omega \setminus \overline{D}_1$ which vanishes on ∂D_1 and has uniformly large absolute value on $\partial\Omega$. Similarly $|\nabla w|$ is bounded on ∂D_2, and hence $|\nabla w| \leq C$ on $\tilde{\Omega}$. The upper bound of $|\nabla u|$ follows. □

For the second result of this section, we need some definitions. Define functions v_1 and v_2 by

$$\Delta v_1 = 0 \text{ in } \tilde{\Omega}, \quad v_1 = 0 \text{ on } \partial D_2 \cup \partial\Omega, \quad v_1 = 1 \text{ on } \partial D_1$$

$$\Delta v_2 = 0 \text{ in } \tilde{\Omega}, \quad v_2 = 0 \text{ on } \partial D_1 \cup \partial\Omega, \quad v_2 = 1 \text{ on } \partial D_2$$

As in [6], define the linear functional

$$Q_\varepsilon(\varphi) := \int_{\partial\Omega} \varphi \frac{\partial v_1}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial\Omega} \varphi \frac{\partial v_2}{\partial \nu} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}.$$

The following gives an estimate for $|C_1 - C_2|$.

Lemma 2.2. *Assume there is a uniform constant $c > 0$ such that*

$$-\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \geq c, \quad -\int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} \geq c. \tag{6}$$

Then

$$c|Q_\varepsilon(\varphi)| \left(\int_{\tilde{\Omega}} |\nabla v_1|^2 \right)^{-1} \leq |C_1 - C_2| \leq C|Q_\varepsilon(\varphi)| \left(\int_{\tilde{\Omega}} |\nabla v_1|^2 \right)^{-1}.$$

Proof. The proof is contained in [6, Section 2], but again for the sake of convenience we include here a brief outline of the argument. Define

$$a_{11} := -\int_{\partial D_1} \frac{\partial v_1}{\partial \nu} = \int_{\tilde{\Omega}} |\nabla v_1|^2, \quad a_{22} := -\int_{\partial D_2} \frac{\partial v_2}{\partial \nu} = \int_{\tilde{\Omega}} |\nabla v_2|^2,$$

$$a_{12} := -\int_{\partial D_1} \frac{\partial v_2}{\partial \nu} = \int_{\tilde{\Omega}} \nabla v_1 \cdot \nabla v_2 = -\int_{\partial D_2} \frac{\partial v_1}{\partial \nu} =: a_{21},$$

where we have used integration by parts. Define another function v_3 by

$$\Delta v_3 = 0 \text{ in } \tilde{\Omega}, \quad v_3 = 0 \text{ on } \partial D_1 \cup \partial D_2, \quad v_3 = \varphi \text{ on } \partial\Omega,$$

and for $i = 1, 2$,

$$b_i := -\int_{\partial D_i} \frac{\partial v_3}{\partial \nu} = \int_{\tilde{\Omega}} \nabla v_i \cdot \nabla v_3 = \int_{\partial\Omega} \varphi \frac{\partial v_i}{\partial \nu}.$$

Since $u = C_1 v_1 + C_2 v_2 + v_3$, the fourth line of (1) gives

$$a_{11} C_1 + a_{12} C_2 + b_1 = 0, \quad a_{21} C_1 + a_{22} C_2 + b_2 = 0.$$

Hence

$$C_1 - C_2 = \frac{(a_{11} + a_{21}) b_2 - (a_{22} + a_{12}) b_1}{a_{11} a_{22} - a_{12}^2} = \frac{Q_\varepsilon(\varphi)}{a_{11} a_{22} - a_{12}^2}, \tag{7}$$

where we have used the fact that

$$a_{11} + a_{21} = -\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}, \quad a_{22} + a_{12} = -\int_{\partial\Omega} \frac{\partial v_2}{\partial \nu},$$

and assuming that $a_{11}a_{22} - a_{12}^2 \neq 0$, which we will shortly prove.

From the assumption (6) and standard derivative estimates for harmonic functions v_1, v_2 in a neighborhood of the boundary Ω, we have

$$c \leq a_{11} + a_{21} \leq C, \quad c \leq a_{22} + a_{12} \leq C.$$

Next

$$a_{11}a_{22} - a_{12}^2 = a_{11}(a_{22} + a_{12}) - a_{12}(a_{11} + a_{21}),$$

and hence, since $a_{12} \leq 0$, so in particular, $|a_{12}| < a_{11}$ (using $a_{11} + a_{12} > 0$),

$$ca_{11} \leq a_{11}a_{22} - a_{12}^2 \leq Ca_{11}. \tag{8}$$

The result follows from (7) and (8). □

3. Proof of Theorem 1.1

In this section we complete the proof of the main theorem.

First we note that the assumption ($*$) in the introduction implies that there exists a connected domain W_1 in $\Omega \setminus (D_1^0 \cup D_2^0)$ with smooth boundary ∂W_1 such that ∂W_1 has an open portion on $\partial\Omega$ and another open portion on ∂D_1^0. Moreover, we may assume that $\overline{W}_1$ and $\overline{D_2^0}$ are disjoint. Similarly there exists another domain W_2, interchanging the roles of D_1^0 and D_2^0. See figure 4 for an example illustrating this.

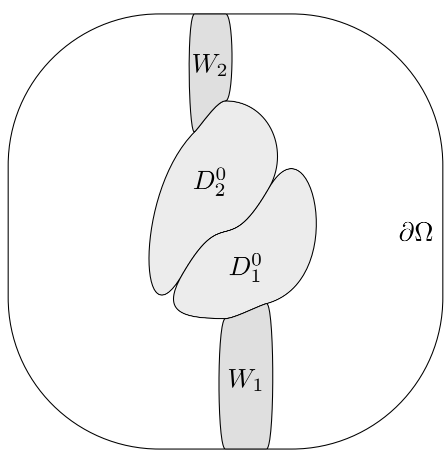

Figure 4. Example configuration satisfying ($*$).

In order to apply Lemma 2.2 we need to establish the estimates (6).

Lemma 3.1. *There is a uniform constant $c > 0$ such that*

$$-\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu} \geq c, \quad -\int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} \geq c. \tag{9}$$

Proof. Note that by the definitions of v_1 and v_2 we have $\frac{\partial v_1}{\partial \nu}, \frac{\partial v_2}{\partial \nu} \leq 0$ on $\partial\Omega$. We will show that $\left|\frac{\partial v_1}{\partial \nu}\right|$ and $\left|\frac{\partial v_2}{\partial \nu}\right|$ must be bounded uniformly away from zero on a portion of the boundary $\partial\Omega$.

We make use of a **basic fact** (see for example [13, Ex. 2.2]) that if a function w is harmonic on a connected domain D and has $w = 0 = \frac{\partial w}{\partial \nu}$ on an open smooth portion of the boundary ∂D then w vanishes identically on D.

Let Π be an open portion of $\partial\Omega$ which is contained in ∂W_1. We recall that $v_1 = v_1^\varepsilon$ depends on ε. We claim that for $\varepsilon > 0$ sufficiently small, $\sup_{p\in\Pi} |\frac{\partial v_1^\varepsilon}{\partial \nu}(p)| \geq c$ for some constant $c > 0$. Indeed if not then we can find a sequence $\varepsilon_j \to 0$ with

$$\sup_{p\in\Pi} \left|\frac{\partial v_1^{\varepsilon_j}}{\partial \nu}(p)\right| \leq 1/j \to 0, \text{ as } j \to \infty. \tag{10}$$

The functions $v_1^{\varepsilon_j}$ are not necessarily defined on W_1 since D_1^ε is changing with ε. However, after composing with diffeomorphisms which converge to the identity, and are equal to the identity in a neighborhood of $\partial W_1 \cap \partial\Omega$, and after passing to a subsequence, the functions $v_1^{\varepsilon_j}$ converge smoothly on W_1 to a harmonic function v_1^0 on W_1 which is equal to 0 on $\partial\Omega \cap \partial W_1$ and equal to 1 on $\partial D_1^0 \cap \partial W_1$.

In particular the function v_1^0 cannot be identically zero on W_1. But from (10) we obtain $\frac{\partial v_1^0}{\partial \nu} = 0$ on an open portion of the boundary of ∂W_1, a contradiction by the basic fact.

Hence we have shown that $\sup_{p\in\Pi} |\frac{\partial v_1^\varepsilon}{\partial \nu}(p)| \geq c$ for $\varepsilon > 0$ sufficiently small and it follows by standard derivative estimates for v_1^ε that $|\frac{\partial v_1^\varepsilon}{\partial \nu}(p)| \geq c/2$ on a small open portion of $\partial\Omega$. This establishes the required estimate for v_1. The argument for v_2 is similar. □

Note that without assumption $(*)$ the estimates (9) may fail. Indeed, if D_2 completely surrounds D_1 (for example if D_1 is a solid ball and D_2 is a solid spherical shell enclosing D_1) then $v_1 \equiv 0$ on the region outside the outer boundary of D_2 and hence $\frac{\partial v_1}{\partial \nu} = 0$ on $\partial\Omega$. We wish to exclude this case, which is not very interesting from the point of view of estimating $|\nabla u|$. Physically, if u represents an electric potential and D_1, D_2 are perfect conductors then u will take a constant value $C_2 = C_1$ throughout D_1, D_2 and the region between the two. In this case $\sup |\nabla u|$ does not blow up as $\varepsilon \to 0$.

The proof of Theorem 1.1 will now be an almost immediate consequence of the following lemma, whose proof uses the same basic strategy as in [6]. A key difference from [6] is that we use a patching argument to deal with multiple coordinate boxes, rather than working in a single one. Also, the fact that the exponents $\alpha_1^i, \ldots, \alpha_n^i$ are not necessarily equal gives rise to a more complicated integral.

Lemma 3.2. *Let v_1 satisfy*

$$\Delta v_1 = 0 \text{ in } \tilde{\Omega}, \quad v_1 = 0 \text{ on } \partial D_2 \cup \partial\Omega, \quad v_1 = 1 \text{ on } \partial D_1.$$

Then

$$\begin{aligned} \frac{1}{C} &< \int_{\tilde{\Omega}} |\nabla v_1|^2 < C, && \text{if } \gamma > 1 \\ \frac{1}{C}\log\frac{1}{\varepsilon} &< \int_{\tilde{\Omega}} |\nabla v_1|^2 < C\log\frac{1}{\varepsilon}, && \text{if } \gamma = 1 \\ \frac{1}{C}\frac{1}{\varepsilon^{1-\gamma}} &< \int_{\tilde{\Omega}} |\nabla v_1|^2 < C\frac{1}{\varepsilon^{1-\gamma}}, && \text{if } \gamma < 1, \end{aligned}$$

where we recall that γ is defined by (5).

Proof. Recall that for each $i = 1, \ldots, k$ the boundaries of D_1, D_2 are described in the box U_i as the graphs of the functions $f_i(x')$ and $g_i(x')$, with $f_i < g_i$.

For each i define the function

$$w_i(x) = \frac{g_i(x') - x_n}{g_i(x') - f_i(x')} \tag{11}$$

which is defined on all of $U_i \cap \tilde{\Omega}$.

For the lower bound of $\int_{\tilde{\Omega}} |\nabla v_1|^2$, we fix an index i and work in U_i. For $x' \in Q_r$ the function $x_n \mapsto w_i(x', x_n)$ is of the form $x_n \mapsto a(x')x_n + b(x')$ and has the property that $w_i|_{x_n = f_i(x')} = 1 = v_1|_{x_n = f_i(x')}$ and $w_i|_{x_n = g_i(x')} = 0 = v_1|_{x_n = g_i(x')}$. In particular, for any fixed $x' \in Q_r$,

$$\int_{f_i(x')}^{g_i(x')} |\partial_{x_n} w_i|^2 \, dx_n \leq \int_{f_i(x')}^{g_i(x')} |\partial_{x_n} v_1|^2 \, dx_n \leq \int_{f_i(x')}^{g_i(x')} |\nabla v_1|^2 \, dx_n,$$

where the first inequality follows from the fact that linear functions of one variable minimize the Dirichlet energy among functions with the same endpoints. Therefore

$$\begin{aligned}
\int_{\tilde{\Omega}} |\nabla v_1|^2 &\geq \int_{U_i \cap \tilde{\Omega}} |\nabla v_1|^2 \\
&= \int_{x' \in Q_r} \int_{f_i(x')}^{g_i(x')} |\nabla v_1|^2 \, dx_n \, dx' \\
&\geq \int_{x' \in Q_r} \int_{f_i(x')}^{g_i(x')} |\partial_{x_n} w_i|^2 \, dx_n \, dx' \\
&= \int_{x' \in Q_r} \frac{dx'}{g_i(x') - f_i(x')} \\
&\geq \frac{1}{C} \int_{x' \in Q_r} \frac{dx'}{\varepsilon + \sum_{j=1}^{n-1} x_j^{2\alpha_j^i}},
\end{aligned} \tag{12}$$

recalling (4). Define

$$I_i(\varepsilon) := \int_{x' \in Q_r} \frac{dx'}{\varepsilon + \sum_{j=1}^{n-1} x_j^{2\alpha_j^i}}. \tag{13}$$

Then we see that $\int_{\tilde{\Omega}} |\nabla v_1|^2 \geq \frac{1}{C} \max_i I_i(\varepsilon)$. Below we will bound the term $I_i(\varepsilon)$. Before then we consider the upper bound.

Note that by standard estimates and the definition of $V \subset \tilde{\Omega}$ we may assume that $|\nabla v_1|^2 \leq C$ at all points of $V \subset \tilde{\Omega}$. In particular $\int_V |\nabla v_1|^2 < C$.

Recall that the region $\tilde{\Omega} \setminus V$ is covered by the "half sized" boxes $U_i^{(r/2)}$ of size $r/2$. We denote by $U_i^{(3r/4)}$ the "three-quarter sized" boxes of size $3r/4$. Let $\{\sigma_i\}_{i=1}^k$ be a partition of unity such that: (1) each $\sigma_i : \bigcup_j U_j \to [0,1]$ is a smooth function with compact support in $U_i = U_i^{(r)}$; and (2) $\sum_i \sigma_i = 1$ on $\bigcup_i U_i^{(3r/4)}$. The function

$$w := \sum_{i=1}^k \sigma_i w_i,$$

is a well-defined smooth function on $\bigcup_{i=1}^k U_i$ which is equal to 1 on $\partial D_1 \cap \bigcup_i U_i^{(3r/4)}$ and equal to 0 on $\partial D_2 \cap \bigcup_i U_i^{(3r/4)}$.

Next let $\rho : \Omega \to [0,1]$ be a smooth cut-off function which is identically equal to 1 on the union of half-sized boxes $\bigcup_i U_i^{(r/2)}$ and is supported on the union of three-quarter sized boxes $\bigcup_i U_i^{(3r/4)}$.

Then the function $W = \rho w + (1-\rho)v_1$ has the following properties.

(i) W is a well-defined continuous function on $\overline{\tilde{\Omega}}$, smooth on $\tilde{\Omega}$.
(ii) W is equal to 1 on ∂D_1 and equal to 0 on $\partial D_2 \cup \partial\Omega$.

For (ii), we observe that at points in $\bigcup_i U_i^{(3r/4)}$ the functions w and v_1 are both equal to 1 on ∂D_1 and equal to 0 on ∂D_2, whereas outside this union, $W = v_1$ which is equal to 1 on ∂D_1 and vanishes on $\partial D_2 \cup \partial\Omega$.

Note also that $0 \le v_1, w, w_i, \sigma_i, \rho \le 1$ at all points of $\tilde{\Omega}$ that each is defined, and that $|\nabla\sigma_i|, |\nabla\rho| \le C$.

Since v is harmonic on $\tilde{\Omega}$, conditions (i) and (ii) imply that $\int_{\tilde{\Omega}} |\nabla v_1|^2 \le \int_{\tilde{\Omega}} |\nabla W|^2$ and hence

$$\begin{aligned}
\int_{\tilde{\Omega}} |\nabla v_1|^2 &\le \int_{\tilde{\Omega}} |\nabla(\rho w + (1-\rho)v_1)|^2 \\
&= \int_{\tilde{\Omega}} |w\nabla\rho + \rho\nabla w - v_1\nabla\rho + (1-\rho)\nabla v_1|^2 \\
&\le C\big(1 + \int_{\cup_{i=1}^k U_i \cap \tilde{\Omega}} |\nabla w|^2\big) \\
&\le C\big(1 + \sum_{i=1}^k \int_{U_i\cap\tilde{\Omega}} |\nabla w_i|^2\big),
\end{aligned} \tag{14}$$

where for the third line we used the fact that $|\nabla v_1|$ is uniformly bounded on the set V and hence on $\tilde{\Omega} \setminus \bigcup_i U_i^{(r/2)}$. From (11) we estimate on $U_i \cap \tilde{\Omega}$,

$$|\nabla w_i|^2(x) \le \frac{C}{(g_i(x') - f_i(x'))^2},$$

and hence for each $i = 1, \dots, k$,

$$\begin{aligned}
\int_{U_i\cap\tilde{\Omega}} |\nabla w_i|^2 &\le \int_{x'\in Q_r} \int_{f_i(x')}^{g_i(x')} \frac{C\,dx_n}{(g_i(x') - f_i(x'))^2}\,dx' \\
&= C\int_{x'\in Q_r} \frac{dx'}{g_i(x') - f_i(x')} \\
&\le C I_i(\varepsilon).
\end{aligned} \tag{15}$$

Combining (14) and (15) we find

$$\int_{\tilde{\Omega}} |\nabla v_1|^2 \le C\left(1 + \max_{i=1,\dots,k} I_i(\varepsilon)\right). \tag{16}$$

The lemma is then a consequence of (12), (16) and the following elementary claim.

Claim. For $i = 1, \ldots, k$, writing $\gamma_i = \displaystyle\sum_{j=1}^{n-1} \frac{1}{2\alpha_j^i}$, we have

$$
\begin{aligned}
\frac{1}{C} &< I_i(\varepsilon) < C, && \text{if } \gamma_i > 1 \\
\frac{1}{C}\log\frac{1}{\varepsilon} &< I_i(\varepsilon) < C\log\frac{1}{\varepsilon}, && \text{if } \gamma_i = 1 \\
\frac{1}{C}\frac{1}{\varepsilon^{1-\gamma_i}} &< I_i(\varepsilon) < C\frac{1}{\varepsilon^{1-\gamma_i}}, && \text{if } \gamma_i < 1,
\end{aligned}
\tag{17}
$$

where we recall that $I_i(\varepsilon)$ is defined by (13).

To prove the claim, we drop the index i and write $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. Rearranging the components if necessary we assume that $1 \leq \alpha_1, \ldots, \alpha_\ell < \infty$ and $\alpha_{\ell+1} = \ldots = \alpha_{n-1} = \infty$. Then we need to compute the integral

$$
I(\varepsilon) = \int_{x' \in Q_r} \frac{dx'}{\varepsilon + \sum_{j=1}^{n-1} x_j^{2\alpha_j}} = 2^{n-1} r^{n-1-\ell} \underbrace{\int_0^r \cdots \int_0^r}_{\ell} \frac{dx_1 \cdots dx_\ell}{\varepsilon + \sum_{j=1}^{\ell} x_j^{2\alpha_j}}. \tag{18}
$$

Note that $\ell = 0$ if and only if $\alpha = (\infty, \ldots, \infty)$. In this case the above integral evaluates exactly to $2^{n-1} r^{n-1}/\varepsilon$, giving (17) in the case $\gamma = \gamma_i = 0$. Henceforth we assume $\ell \geq 1$.

We first reduce the claim to estimating an integral of one variable. Namely, we will show that

$$
\frac{1}{C}\int_0^{R_0} \frac{\rho^{2\gamma-1}\, d\rho}{\varepsilon + \rho^2} \leq I(\varepsilon) \leq C \int_0^{R_1} \frac{\rho^{2\gamma-1}\, d\rho}{\varepsilon + \rho^2}, \tag{19}
$$

where $R_0 := \min_j r^{\alpha_j}$, $R_1 := \sqrt{\ell}\max_j r^{\alpha_j}$ (for j ranging from 1 to ℓ) and $C > 0$ depends only on α, r and n.

Making the substitution $u_j = x_j^{\alpha_j}$ we find

$$
\begin{aligned}
\int_0^r \cdots \int_0^r \frac{dx_1 \cdots dx_\ell}{\varepsilon + \sum_{j=1}^{\ell} x_j^{2\alpha_j}} &= \frac{1}{\alpha_1 \cdots \alpha_\ell} \int_0^{r^{\alpha_1}} \cdots \int_0^{r^{\alpha_\ell}} \frac{\prod_j u_j^{\frac{1}{\alpha_j}-1} du_1 \cdots du_\ell}{\varepsilon + \sum_{j=1}^{\ell} u_j^2} \\
&< \frac{1}{\alpha_1 \cdots \alpha_\ell} \int_{B_{R_1}^+} \frac{\prod_j u_j^{\frac{1}{\alpha_j}-1} du_1 \cdots du_\ell}{\varepsilon + \sum_{j=1}^{\ell} u_j^2},
\end{aligned}
\tag{20}
$$

where $B_{R_1}^+$ denotes the portion of the ball of radius $R_1 = \sqrt{\ell}\max_j r^{\alpha_j}$, centered at the origin in $\mathbb{R}^\ell$, where all the coordinates are positive. Similarly we find

$$
\int_0^r \cdots \int_0^r \frac{dx_1 \cdots dx_\ell}{\varepsilon + \sum_{j=1}^{\ell} x_j^{2\alpha_j}} > \frac{1}{\alpha_1 \cdots \alpha_\ell} \int_{B_{R_0}^+} \frac{\prod_j u_j^{\frac{1}{\alpha_j}-1} du_1 \cdots du_\ell}{\varepsilon + \sum_{j=1}^{\ell} u_j^2}, \tag{21}
$$

where $R_0 = \min_j r^{\alpha_j}$. In spherical coordinates $(\rho, \varphi_1, \ldots, \varphi_{\ell-1})$ we have

$$u_j = \rho(\cos\varphi_j)\big(\prod_{q<j} \sin\varphi_q\big) \quad \text{for } j = 1, \ldots, \ell-1$$
$$u_\ell = \rho\big(\prod_{q<\ell} \sin\varphi_q\big)$$
$$du = \rho^{\ell-1} \prod_{j=1}^{\ell-2} (\sin\varphi_j)^{\ell-1-j} d\rho\, d\varphi_1 \cdots d\varphi_{\ell-1},$$

and we find for any $R > 0$ that

$$\int_{B_R^+} \frac{\prod_j u_j^{\frac{1}{\alpha_j}-1} du_1 \cdots du_\ell}{\varepsilon + \sum_{j=1}^{\ell} u_j^2} = A \int_0^R \frac{\rho^{2\gamma-1}\, d\rho}{\varepsilon + \rho^2}, \tag{22}$$

where $\gamma = \sum_j \frac{1}{2\alpha_j}$ and

$$A = \prod_{q=1}^{\ell-1} \int_0^{\frac{\pi}{2}} (\sin\varphi_q)^{-1+\sum_{j=q+1}^{\ell} \frac{1}{\alpha_j}} (\cos\varphi_q)^{-1+\frac{1}{\alpha_q}}\, d\varphi_q. \tag{23}$$

Combining (18), (20), (21), (22) and (23) proves (19) as required.

We can now finish the proof of the claim. If $\gamma > 1$ then

$$\int_0^{R_1} \frac{\rho^{2\gamma-1}}{\varepsilon + \rho^2}\, d\rho < \int_0^{R_1} \rho^{2\gamma-3}\, d\rho = C,$$

and

$$\int_0^{R_0} \frac{\rho^{2\gamma-1}}{\varepsilon + \rho^2}\, d\rho > \int_{R_0/2}^{R_0} \frac{\rho^{2\gamma-1}}{1 + \rho^2} d\rho = \frac{1}{C}$$

for all $\varepsilon < 1$, which establishes this case.

Next, if $\gamma = 1$, then

$$\int_0^{R_1} \frac{\rho^{2\gamma-1}}{\varepsilon + \rho^2}\, d\rho = \frac{1}{2} \log\big(1 + \frac{R_1^2}{\varepsilon}\big) < C \log\frac{1}{\varepsilon}$$

and

$$\int_0^{R_0} \frac{\rho^{2\gamma-1}}{\varepsilon + \rho^2}\, d\rho = \frac{1}{2} \log\big(1 + \frac{R_0^2}{\varepsilon}\big) > \frac{1}{C} \log\frac{1}{\varepsilon}.$$

Finally if $0 < \gamma < 1$ let $\beta = 2\gamma - 1$ and note that $-1 < \beta < 1$. Then by setting $v = \rho/\varepsilon^{1/2}$ we see that for all $R > 0$ we have

$$\int_0^R \frac{\rho^\beta\, d\rho}{\varepsilon + \rho^2} = \frac{1}{\varepsilon^{1-\gamma}} \int_0^{R/\varepsilon^{1/2}} \frac{v^\beta\, dv}{1 + v^2}.$$

But since $-1 < \beta < 1$ we see that

$$\int_0^{R_1/\varepsilon^{1/2}} \frac{v^\beta\, dv}{1 + v^2} < \int_0^1 v^\beta\, dv + \int_1^\infty v^{\beta-2}\, dv = C$$

and

$$\int_0^{R_0/\varepsilon^{1/2}} \frac{v^\beta\, dv}{1 + v^2} > \frac{1}{2} \int_0^1 v^\beta\, dv = \frac{1}{C},$$

for all $\varepsilon < R_0^2$. This completes the proof of the claim and hence the lemma. □

Finally we complete the proof of the main theorem.

Proof of Theorem 1.1. This is an immediate consequence of Lemmas 2.1, 2.2, 3.1 and 3.2 and the fact that $|Q_\varepsilon(\varphi)| \leq C$. □

4. Optimality of the bounds

In this section we prove Theorem 1.2 on the optimality of the bounds of Theorem 1.1. From Lemmas 2.1, 2.2, 3.1 and 3.2 we have

$$\sup_{\tilde{\Omega}} |\nabla u| \geq \begin{cases} \dfrac{|Q_\varphi(\varphi)|}{C\varepsilon^\gamma}, & \text{if } \gamma < 1 \\ \dfrac{|Q_\varepsilon(\varphi)|}{C\varepsilon|\log \varepsilon|}, & \text{if } \gamma = 1 \\ \dfrac{|Q_\varepsilon(\varphi)|}{C\varepsilon}, & \text{if } \gamma > 1. \end{cases}$$

Hence to prove Theorem 1.2 it suffices to show the existence of φ so that $|Q_\varepsilon(\varphi)| \geq c$ for a constant $c > 0$ independent of ε.

Note that in general, it is not the case that $Q_\varepsilon(\varphi)$ will be nonzero for all choices of φ (one could take $\varphi = 0$, for example).

Proof of Theorem 1.2. First we note that there exists an open portion P of $\partial\Omega$ such that

$$\left| \frac{\partial v_1^\varepsilon}{\partial \nu} - \frac{\partial v_2^\varepsilon}{\partial \nu} \right| \geq c, \tag{24}$$

for all $\varepsilon > 0$ sufficiently small.

Indeed, to see this define $v^\varepsilon = v_1^\varepsilon - v_2^\varepsilon$, which is harmonic on $\Omega \setminus (D_1^\varepsilon \cup D_2^\varepsilon)$, vanishes on $\partial\Omega$, is equal to 1 on D_1^ε and equal to -1 on D_2^ε. We apply the same argument as in Lemma 3.1 above. Let W_1 be as defined there, and let Π be an open portion of $\partial\Omega$ which is contained in ∂W_1. For $\varepsilon > 0$ sufficiently small, we claim that $\sup_{p \in \Pi} |\frac{\partial v^\varepsilon}{\partial \nu}(p)| \geq c$ for some constant $c > 0$. If not then we can find a sequence $\varepsilon_j \to 0$ such that v^{ε_j} converges (after diffeomorphisms) to a harmonic function v^0 on W_1 which is equal to 0 on $\partial\Omega \cap \partial W_1$, equal to 1 on $\partial D_1^0 \cap \partial W_1$ and $\frac{\partial v^0}{\partial \nu} = 0$ on an open portion of the boundary of ∂W_1. This is a contradiction which proves (24).

To complete the proof of the theorem, we note that for any sequence $\varepsilon_j \to 0$, the harmonic functions $v_1^{\varepsilon_j}$, $v_2^{\varepsilon_j}$ and their derivatives are uniformly bounded in a neighborhood of the boundary $\partial\Omega$. In particular, we can pass to a subsequence ε_{j_k} such that

$$-\int_{\partial\Omega} \frac{\partial v_1^{\varepsilon_{j_k}}}{\partial \nu} \to a_1, \quad -\int_{\partial\Omega} \frac{\partial v_2^{\varepsilon_{j_k}}}{\partial \nu} \to a_2, \text{ as } k \to \infty, \tag{25}$$

for bounded constants a_1, a_2 with $a_1, a_2 > 0$ (by Lemma 3.1).

Recalling (24), we may assume without loss of generality that on the open portion P of $\partial\Omega$ we have:

$$-\frac{\partial v_1^\varepsilon}{\partial \nu} + \frac{\partial v_2^\varepsilon}{\partial \nu} \geq c > 0.$$

If $a_2 \geq a_1$, let φ be a smooth nonnegative function supported on P with $\varphi \geq 1$ on an open set $S \subset P$. Then

$$-a_2 \varphi \frac{\partial v_1^\varepsilon}{\partial \nu} \geq -a_1 \varphi \frac{\partial v_2^\varepsilon}{\partial \nu} + c a_2, \quad \text{on } S,$$

and

$$-a_2\varphi\frac{\partial v_1^\varepsilon}{\partial \nu} \geq -a_1\varphi\frac{\partial v_2^\varepsilon}{\partial \nu}, \quad \text{on } \partial\Omega,$$

and using (25) it follows that for k sufficiently large, $Q_{\varepsilon_{j_k}}(\varphi) \leq -c'$ for a uniform constant $c' > 0$.

The argument in the case when $a_2 < a_1$ is similar except that we take φ to have the opposite sign and obtain $Q_{\varepsilon_{j_k}}(\varphi) \geq c'$.

We have shown that for any sequence $\varepsilon_j \to 0$ there is a subsequence ε_{j_k} such that $|Q_{\varepsilon_{j_k}}(\varphi)| \geq c > 0$. Arguing by contradiction, this implies that $|Q_\varepsilon(\varphi)| \geq c > 0$ for all $\varepsilon > 0$ sufficiently small, after possibly shrinking $c > 0$. □

We end with a remark on the argument for optimality of estimates in [6, Section 3]. Bao-Li-Yin assume that Ω has a reflective symmetry and that the strictly convex set D_2^0 is a reflection of D_1^0, and, using a different argument, obtain examples for a large class of boundary data φ. They also consider the case of $\Omega = \mathbb{R}^n$ (see [6, Proposition 3.2]).

5. Examples

We illustrate the above results in some special configurations in $\mathbb{R}^3$, using coordinates x, y, z. We indicate briefly how Theorem 1.1 can be applied in each case.

5.1. **A parabolic cylinder and a quartic cylinder.** Consider an example in $\mathbb{R}^3$ where D_1 and D_2 are only close to each other near the origin, and near there ∂D_2 is given locally as the graph of the parabolic cylinder $z = g := \varepsilon + x^2$ and ∂D_1 is the graph of the quartic cylinder $z = f := -y^4$. Then $g - f = \varepsilon + x^2 - y^4$, so that $\alpha = (1, 2)$ and $\gamma = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Then in this configuration we find

$$\frac{|Q_\varepsilon(\varphi)|}{C\varepsilon^{3/4}} \leq \sup_{\tilde{\Omega}} |\nabla u| \leq \frac{C}{\varepsilon^{3/4}}.$$

5.2. **Encircled tori.** Consider two tori, with one tightly ringed around the other. See figure 5.

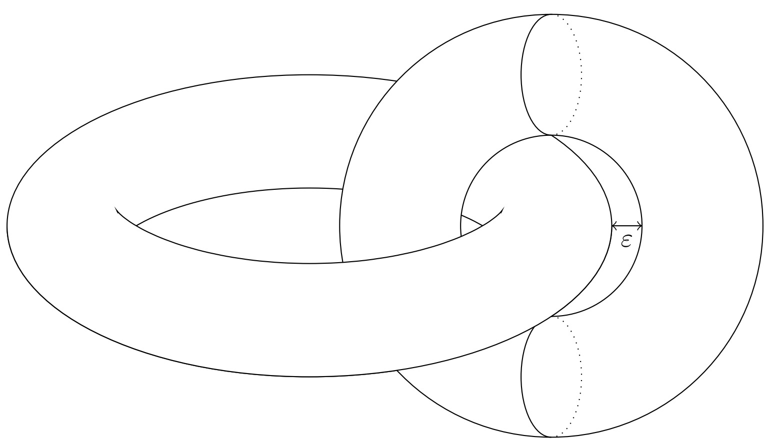

Figure 5. Encircled tori.

Specifically, let D_2 be the open solid torus in $\mathbb{R}^3$ bounded by

$$\left(\sqrt{x^2+y^2}-a\right)^2+z^2=A^2,$$

for constants $a > A > 0$. The boundary torus is the result of revolving the circle $(x-a)^2+z^2=A^2$ in the xz-plane about the z-axis.

Let D_1 be the region bounded by

$$\left(\sqrt{(x-a)^2+z^2}-b\right)^2+y^2=B^2,$$

for constants $b > B > 0$, which is the circle $y^2+(z-b)^2=B^2$ in the $x=a$ plane, revolved about the line through $(a,0,0)$ which is parallel to the y-axis.

If we assume that $a \gg b$ and that $b = A+\varepsilon+B$, then D_1 is wrapped around D_2 with only ε of distance between the two.

As $\varepsilon \to 0$ the two boundaries intersect in a circle. The narrow region, which is a neighborhood of this circle, can be covered with open boxes U_i with $\alpha^i = (1,\infty)$ and $\gamma_i = 1/2$. We illustrate this in the case of a neighborhood of the point $(a-A,0,0)$.

Near the point $(a-A,0,0)$, we can describe the boundaries of D_1, D_2 as functions of y, z. Solving for x in the equation for D_2 we find that the boundary of D_2 is given locally as a graph

$$x = g(y,z) = a - A - \frac{y^2}{2(a-A)} + \frac{z^2}{2A} + \cdots$$

D_1 is given locally as a graph

$$x = f(y,z) = a - b + B - \frac{y^2}{2B} + \frac{z^2}{2(b-B)} + \cdots$$

Compute

$$g - f = \varepsilon + \frac{a-A-B}{2B(a-A)}y^2 + O(\varepsilon|z|^2) + O(|y|^3),$$

so that

$$\frac{1}{C}(\varepsilon+y^2) < g-f < C(\varepsilon+y^2).$$

Hence the α^i and γ_i corresponding to the box are given by $(1,\infty)$ and $\frac{1}{2}$ respectively.

The estimate we obtain for $\sup|\nabla u|$ is

$$\frac{|Q_\varepsilon(\varphi)|}{C\sqrt{\varepsilon}} \le \sup_{\tilde\Omega}|\nabla u| \le \frac{C}{\sqrt{\varepsilon}}.$$

5.3. **A torus and a sphere.** Consider the region in $\mathbb{R}^3$ inside the torus obtained by revolving a planar disk of radius A about an axis a distance $a > A$ from the disk's center. For example let D_1 be the region

$$D_1 = \{(x,y,z) \in \mathbb{R}^n \mid (\sqrt{x^2+y^2}-a)^2+z^2 < A^2\}.$$

Now suppose D_2 is the region inside a sphere of radius R in $\mathbb{R}^3$, which is a distance ε from D_1. The optimal estimate for $|\nabla u|$ will depend on where the sphere is centered.

Most configurations are already covered by the result of Bao-Li-Yin [5]. For example, if the sphere were centered at $(a+A+R+\varepsilon,0,0)$, then near the point $(a+A,0,0)$ the boundary of D_1 can be described to second order by

$$x \approx a + A - C_1y^2 - C_2z^2$$

and the boundary of D_2 is given, again to second order, by

$$x \approx a + A + \varepsilon + C_3 y^2 + C_4 z^2$$

where $C_1, \ldots, C_4$ are positive constants. Hence we find $\gamma = 1$ in this configuration, giving

$$\frac{|Q_\varepsilon(\varphi)|}{C\varepsilon|\log\varepsilon|} \leq \sup_{\tilde{\Omega}} |\nabla u| \leq \frac{C}{\varepsilon|\log\varepsilon|}$$

On the other hand, a special configuration is when the sphere has radius $R = a - A - \varepsilon$, centered at the origin (namely, the sphere is "in the donut hole"). See figure 6. Then we have an entire circle of close proximity between the two regions. In this case one can calculate $\gamma = \frac{1}{2}$ and thus

$$\frac{|Q_\varepsilon(\varphi)|}{C\sqrt{\varepsilon}} \leq \sup_{\tilde{\Omega}} |\nabla u| \leq \frac{C}{\sqrt{\varepsilon}},$$

as in the case of encircled tori above.

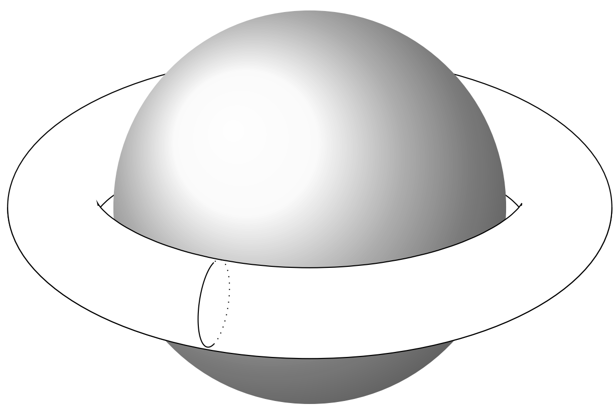

FIGURE 6. Sphere closely surrounded by a torus.

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