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Algebraic independence and linear difference equations

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Abstract. We consider pairs of automorphisms (ϕ, σ) acting on fields of Laurent or Puiseux series: pairs of shift operators $(\phi: x \mapsto x + h_1, \sigma: x \mapsto x + h_2)$, of q-difference operators $(\phi: x \mapsto q_1 x$, $\sigma: x \mapsto q_2 x)$, and of Mahler operators $(\phi: x \mapsto x^{p_1},\ \sigma: x \mapsto x^{p_2})$. Given a solution f to a linear ϕ-equation and a solution g to an algebraic σ-equation, both transcendental, we show that f and g are algebraically independent over the field of rational functions, assuming that the corresponding parameters are sufficiently independent. As a consequence, we settle a conjecture about Mahler functions put forward by Loxton and van der Poorten in 1987. We also give an application to the algebraic independence of q-hypergeometric functions. Our approach provides a general strategy to study this kind of question and is based on a suitable Galois theory: the σ-Galois theory of linear ϕ-equations.

Keywords. Linear difference equations, q-difference, Mahler functions, algebraic independence

1. Introduction

Let K be a field and let F be a field extension endowed with an endomorphism ϕ such that $\phi(K) \subset K$. A *linear ϕ-difference equation* over K is an equation of the form

$$\phi^n(y) + a_{n-1}\phi^{n-1}(y) + \cdots + a_0 y = 0, \tag{1.1}$$

where $a_0, \ldots, a_{n-1} \in K$. The set $\mathrm{Sol}_{\phi,K,F}$, formed by all elements in F that are solution to a linear ϕ-difference equation over K, is a ring containing K. Traditionally, the

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algebraic relations over K between the elements of $\mathrm{Sol}_{\phi,K,F}$ are studied through the difference Galois theory associated with the endomorphism ϕ (see, for instance, [32]). If we assume that K and F are endowed with a second endomorphism σ that is *sufficiently independent* of ϕ, we expect the intersection of $\mathrm{Sol}_{\phi,K,F}$ and $\mathrm{Sol}_{\sigma,K,F}$ to be *small*, even possibly reduced to K. Schäfke and Singer [28] recently confirmed this expectation in several important cases (see Theorem 1.1 below for a precise statement). They consider pairs of shift operators ($\phi: x \mapsto x + h_1,\ \sigma: x \mapsto x + h_2$), of q-difference operators ($\phi: x \mapsto q_1 x,\ \sigma: x \mapsto q_2 x$), and of Mahler operators ($\phi: x \mapsto x^{p_1},\ \sigma: x \mapsto x^{p_2}$). While some special cases were already known, these authors were the first to provide a unified approach to this type of results.

The aim of this paper is to go one step further, promoting the idea that, in the above cases, the elements of $\mathrm{Sol}_{\phi,K,F}$ and $\mathrm{Sol}_{\sigma,K,F}$ should be algebraically independent, unless they belong to the small intersection. Our main result in this direction is Corollary 1.5, which appears to be the first general result supporting this viewpoint. It is obtained as a consequence of the more general Theorem 1.3. Our approach provides a unified strategy for attacking this kind of problem. It rests on a suitable Galois theory: the σ-Galois theory of linear ϕ-difference equations developed in [26]. As a particular instance of Theorem 1.3, we settle a conjecture of Loxton and van der Poorten [31] concerning Mahler functions. The latter, which was itself motivated by its consequence in the theory of finite automata, was our initial motivation for the present work. We also give a second application of Theorem 1.3 to the algebraic independence of q-hypergeometric series. The strategy we follow to prove Theorem 1.3 was initiated recently by the first three authors in [3], where general hypertranscendence results are obtained for solutions of linear difference equations associated with the same three operators (shift, q-difference, and Mahler). However, when working with a parametric operator (here σ) that is an endomorphism, instead of a derivation as in [3], one needs to overcome a number of technical difficulties. Let us also mention that we introduce in Section 4 some new group-theoretic arguments concerning linear algebraic groups that can be used to significantly simplify the proof of the main result of [3].

1.1. Statement of our main result

Throughout this paper, our framework consists of a tower of field extensions $\mathbb{C} \subset K \subset F$ with the following properties:

- The field K is equipped with a pair of automorphisms (ϕ, σ).
- These automorphisms extend over F.

Specifically, we consider the following situations, which we refer to as Cases **2S**, **2Q**, and **2M**, respectively.

Case 2S. In this case, we consider $K = \mathbb{C}(x)$, $F = \mathbb{C}((x^{-1}))$, $\phi(x) = x + h_1$ and $\sigma(x) = x + h_2$, where $h_1, h_2 \in \mathbb{C}$ are $\mathbb{Z}$*-linearly independent*, i.e. $h_1/h_2 \notin \mathbb{Q}$.

Case 2Q. In this case, we let $K = \bigcup_{j\geq 1} \mathbb{C}(x^{1/j})$ denote the field of ramified rational functions. We also use the notation $\mathbb{C}(x^{1/*})$ for this field. We let $F = \bigcup_{j\geq 1} \mathbb{C}((x^{1/j}))$ denote the field of Puiseux series. We also use the notation $\mathbb{C}((x^{1/*}))$ for this field. We let (ϕ, σ) denote the pair of automorphisms of K (and F) defined by

$$\phi(x) = q_1 x \quad \text{and} \quad \sigma(x) = q_2 x,$$

where q_1 and q_2 are two *multiplicatively independent* nonzero complex numbers, i.e. $q_1^{n_1} q_2^{n_2} = 1$ implies $n_1 = n_2 = 0$ for all $n_1, n_2 \in \mathbb{Z}$. Furthermore, in order to apply the result of [28] cited below, we also add the following mild restriction: q_1 and q_2 cannot both be algebraic numbers of modulus 1, whose Galois conjugates all have modulus 1. For instance, one cannot choose $q_1 = (3 + 4i)/5$ and $q_2 = (5 + 12i)/13$. In [28], this condition is imposed in order to ensure that $|q_1|_0 \neq 1$ for some archimedean absolute value $|\cdot|_0$. Note that when q_1 and q_2 are multiplicatively independent, none of them is a root of unity. By Kronecker's theorem (see, for instance, [10, 1.5.9]), if in addition q_1 is algebraic and if $\boldsymbol{k}$ is a number field containing q_1, there exists a valuation ν of $\boldsymbol{k}$ such that $|q_1|_\nu \neq 1$, where $|\cdot|_\nu$ is the absolute value attached to ν. Using this suitable absolute value, it may be possible to get rid of the mild restriction above by adapting the argument of Schäfke and Singer to a nonarchimedean framework (i.e., by replacing the archimedean field $\mathbb{C}$ by an algebraically closed ν-adic completion of K).

Case 2M. In this case, we let $K = \mathbb{C}(x^{1/*})$, $F = \mathbb{C}((x^{1/*}))$, and we let (ϕ, σ) denote the pair of automorphisms of K (and F) defined by

$$\phi(x) = x^{p_1} \quad \text{and} \quad \sigma(x) = x^{p_2},$$

where p_1 and p_2 are two *multiplicatively independent* natural numbers.

The recent result of Schäfke and Singer [28] mentioned before can now be stated as follows.

Theorem 1.1 (Schäfke and Singer). *Let K, F, and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Then an element $f \in F$ cannot satisfy both a linear ϕ-difference equation and a linear σ-difference equation with coefficients in K unless it belongs to K.*

Remark 1.2. Using the same reasoning as in [3, Remark 1.4], Theorem 1.1 may be deduced from [28, Corollaries 14 to 16]. We recall that Case **2Q** was proved, though in less generality, by Bézivin and Boutabaa [9], while Case **2M** is due to the first author and Bell [2]. We refer the reader to [28] and the references therein for more details on the various contributions to the other cases.

Let us recall that an element f in F *satisfies an algebraic σ-equation* over K, or equivalently that f is *σ-algebraic* over K, if there exists $n \in \mathbb{N}$ such that the functions $f, \sigma(f), \ldots, \sigma^n(f)$ are algebraically dependent over K. Otherwise, f is said to be *σ-transcendental* over K.

Our main result is the following generalization of Theorem 1.1. Case **2Q** of Theorem 1.3 also generalizes some results of [14], see Remark 4.2.

Theorem 1.3. *Let K, F, and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Let $f \in F$ be a solution to a linear ϕ-difference equation over K. Then either $f \in K$ or f is σ-transcendental over K.*

As a consequence of Theorem 1.3, we deduce the following result.

Corollary 1.4. *Let K, F, and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Let $f \in F$ be a solution to a linear ϕ-difference equation over K and let $g \in F$ be a solution to an algebraic σ-difference equation over K. Then f and g are algebraically independent over K unless $f \in K$ or g is algebraic over K.*

At first glance, some readers may be puzzled by the lack of symmetry in Corollary 1.4. Indeed, f has to satisfy a *linear* ϕ-difference equation, while g is free to satisfy an *algebraic* σ-difference equation. Thus, it may be tempting to ask for a more general statement in which f could also be allowed to satisfy an algebraic equation. However, the Laurent series J defined by $J(x) := j(e^{2i\pi x}) \in \mathbb{Q}((x))$, where j is the classical modular j-invariant, provides a natural conterexample to such a generalization. Indeed, the function J is known to be transcendental over $\mathbb{Q}(x)$, while, for all positive integers p, it satisfies the algebraic p-Mahler equation

$$\Phi_p(J(x), J(x^p)) = 0,$$

where $\Phi_p(X, Y) \in \mathbb{Z}[X, Y]$ is the pth modular polynomial (see, for instance, [33]). Maybe, the right generalization could be: a Laurent series that is a solution to two algebraic Mahler equations, with respect to two multiplicatively independent parameters, should satisfy an algebraic p-Mahler equation for all positive integers p. In this direction, Theorem 1.1 already shows that a Laurent series satisfying two linear Mahler equations with respect to two multiplicatively independent parameters also satisfies a linear p-Mahler equation for all positive integers p. Of course, the same question arises with the shift and the q-difference operators.

For the sake of symmetry, we also state the following particular case[1] of Corollary 1.4 that was our initial goal.

Corollary 1.5. *Let K, F, and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Let $f \in F$ be a solution to a linear ϕ-difference equation over K and let $g \in F$ be a solution to a linear σ-difference equation over K. Then f and g are algebraically independent over K unless one of them belongs to K.*

Remark 1.6. In the three cases we consider, the independence required for the corresponding pairs (ϕ, σ) could be rephrased and unified as follows: ϕ and σ commute, and if $f \in F$ is not a constant (i.e., $f \notin \mathbb{C}$), then the equation $\phi^n(f) = \sigma^m(f)$ has no non-trivial solution in integers n and m.[2] As suggested by the previous discussion, it would be interesting to formalize a general principle emerging from Theorems 1.1 and 1.3.

[1] This follows from the fact that if g satisfies a linear σ-difference equation over K, then either $g \in K$ or g is transcendental over K (see the proof of Corollary 1.5 in Section 4).

[2] The additional mild condition in Case **2Q** is probably not necessary.

1.2. Application to Mahler functions

Let $\mathbb{K}$ be a field and $p \geq 2$ be an integer. A power series $f(x) \in \mathbb{K}[[x]]$ is a *p-Mahler function* if there exist polynomials $a_0(x), \ldots, a_n(x) \in \mathbb{K}[x]$, not all zero, such that

$$a_0(x) f(x) + a_1(x) f(x^p) + \cdots + a_n(x) f(x^{p^n}) = 0. \tag{1.2}$$

Furthermore, f is a *Mahler function* if it is a p-Mahler function for some integer $p \geq 2$. Regarding (1.2), it is tempting to ask about the significance of the parameter p. In 1976, van der Poorten [30] suggested that Mahler functions associated with multiplicatively independent parameters should behave independently. For instance, he asked whether the two Mahler functions

$$\sum_{n=0}^{\infty} x^{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} x^{3^n}$$

are algebraically independent over $\overline{\mathbb{Q}}(x)$. The transcendence theory of Mahler functions, that is, the study of the transcendence and algebraic independence of values of Mahler functions over $\overline{\mathbb{Q}}$ at algebraic points, was initiated by Mahler in the late 1920s. Mahler only considered (possibly inhomogeneous) order 1 equations, and Mahler's method really restarted in the 1970s in the hands of Kubota, Loxton and van der Poorten, and later Ku. Nishioka (see the recent survey [1] and the references therein). Kubota, Loxton, and van der Poorten expected that Mahler's method could be developed far enough to solve the above question.[3] In 1987, van der Poorten [31] announced a series of results that he expected to prove in his collaboration with Loxton on Mahler's method.[4] Among them, the following two are related to Mahler functions associated with multiplicatively independent parameters.

Conjecture 1.7 (Loxton and van der Poorten). *Let p_1 and p_2 be two multiplicatively independent natural numbers. Then the following hold:*

(i) *An element $f \in \overline{\mathbb{Q}}[[x]]$ cannot be both a p_1- and a p_2-Mahler function unless it is rational.*

(ii) *If $f \in \overline{\mathbb{Q}}[[x]]$ is a p_1-Mahler function and $g \in \overline{\mathbb{Q}}[[x]]$ is a p_2-Mahler function. Then f and g are algebraically independent over $\overline{\mathbb{Q}}(x)$ unless one of them is rational.*

We find that (ii) is a generalization of (i), while (i) itself is a generalization of Cobham's theorem, an important result in the theory of automatic sequences and sets (see, for instance, [2] for more details). As already mentioned in Remark 1.2, part (i) of Conjecture 1.7 has finally been proved by the first author and Bell [2]. A different proof has been given by Schäfke and Singer [28].

[3] In [19], Kubota announced a paper on this problem which never appeared. Also, Loxton and van der Poorten stated some related results in [21], but the corresponding proofs are considered to be incomplete (see the discussion in [25, p. 89]).

[4] However, after this date, there is no new publication on this topic by these authors.

The first result towards part (ii) of Conjecture 1.7 is due to Ku. Nishioka [25] who proved the case where f and g both satisfy an inhomogeneous order 1 equation. Recently, the first author and Faverjon [4, 5] proved the case where f and g are both solutions to a Mahler equation of the form (1.2) with $a_0(0)a_n(0) \neq 0$. The results of [4, 5, 25] are based on Mahler's method. As a direct consequence of Corollary 1.5, we fully prove Conjecture 1.7 (even a more general version where the base field is the field of complex numbers).

Theorem 1.8. *Let p_1 and p_2 be two multiplicatively independent natural numbers. Let $f \in \mathbb{C}[[x]]$ be a p_1-Mahler function and let $g \in \mathbb{C}[[x]]$ be a p_2-Mahler function. Then f and g are algebraically independent over $\mathbb{C}(x)$ unless one of them is rational.*

Let us also mention that, in an ongoing work, Medvedev, Nguyen, and Scanlon [23] study a similar problem where linear Mahler equations of arbitrary order are replaced by nonlinear order 1 equations of the form $f(x^p) = P(x, f(x))$, where P is a polynomial in two variables.

After completion of this work, the first author and Faverjon [6] have strongly improved the results of [4,5]. This allows them to remove the unnecessary condition $a_0(0)a_n(0)\neq 0$. They also generalize Theorem 1.8 (over $\overline{\mathbb{Q}}$) to the case of finitely many q_i-Mahler functions $f_i(x)$, $1 \leq i \leq r$, such that the parameters q_i are pairwise multiplicatively independent (see [6, Corollary A.4]).

1.3. Application to q-hypergeometric series

Let q be a nonzero complex number that is not a root of unity. For every nonnegative integer n, we let

$$[n]!_q = \prod_{i=1}^{n} \frac{1-q^i}{1-q}$$

denote the q-analog of the factorial $n!$. The *q-Pochhammer symbol*, also called q-shifted factorial, is defined as

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i).$$

The most classical q-analog of hypergeometric functions is given by the so-called (generalized) *basic hypergeometric functions* or *q-hypergeometric functions*:

$${}_{r+1}\phi_s\left(\begin{matrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_s\end{matrix}\middle| q, x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1;q)_n\cdots(\alpha_r;q)_n}{(\beta_1;q)_n\cdots(\beta_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{s-r}\frac{x^n}{[n]!_q}.$$

Here, r and s are two nonnegative natural numbers and $\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_s$ are complex numbers. For the coefficients to exist we must assume that $\beta_i \notin q^{\mathbb{Z}_{\leq 0}}$ for all i. By definition, a q-hypergeometric function belongs to $\mathbb{C}[[x]]$. It is also well-known that a q-hypergeometric function satisfies a q-linear difference equation (see, for instance, [16]). As a direct consequence of Case **2Q** of Corollary 1.5, we obtain the following result.

Theorem 1.9. *Let q_1 and q_2 be two multiplicatively independent complex numbers satisfying the conditions given in Case **2Q**. Let f be a q_1-hypergeometric function and let g be a q_2-hypergeometric function. Then f and g are algebraically independent over $\mathbb{C}(x)$ unless one of them is rational.*

1.4. Organization of the paper

This article is organized as follows. In Section 2, we give a short introduction to the Galois theory and the σ-Galois theory of linear ϕ-difference equations, following [26,32]. Several auxiliary results are gathered in Section 3. Section 4 is devoted to the proof of Theorem 1.3. Finally, Appendix A provides the reader with some complementary results on difference algebraic groups and their representations that are needed for the proof of Theorem 1.3.

2. Galois theories of linear difference equations

In this section, we provide a short introduction to the Galois theory of linear ϕ-equations and to the σ-Galois theory of linear ϕ-equations.

2.1. Galois theory of linear difference equations

We first recall some notation, as well as classical results, concerning the Galois theory of linear difference equations. We refer the reader to [32] for more details.

2.1.1. Notation in operator algebra. In what follows, all rings are commutative, with identity, and contain $\mathbb{Q}$. In particular, all fields are of characteristic zero. Given a ring R, we let $\mathrm{Quot}(R)$ denote the total quotient ring of R, that is, the localization of R at the multiplicatively closed subset of all nonzero divisors. Given a field K, a K-algebra R, and a subset S of R, we let $K(S)$ denote the total quotient ring of the ring generated by S over K.

A *ϕ-difference ring*, or *ϕ-ring* for short, is a pair (R,ϕ) where R is a ring and ϕ is a ring endomorphism of R. When R is a field, R is called a *ϕ-field*. An ideal I of R such that $\phi(I)\subset I$ is called a *difference ideal* or a *ϕ-ideal*. The difference ring (R,ϕ) is *simple* if the only ϕ-ideals of R are $\{0\}$ and R. Two difference rings (R_1,ϕ_1) and (R_2,ϕ_2) are *isomorphic* if there exists a ring isomorphism φ between R_1 and R_2 such that $\varphi\circ\phi_1=\phi_2\circ\varphi$. A difference ring (S,ϕ') is a *difference ring extension* of (R,ϕ) if S is a ring extension of R and $\phi'_{|R}=\phi$. In this case, we usually keep on denoting ϕ' by ϕ. When R is a ϕ-field, we say that S is an *R-ϕ-algebra*. Two difference ring extensions (R_1,ϕ) and (R_2,ϕ) of the difference ring (R,ϕ) are *isomorphic over* (R,ϕ) if there exists a difference ring isomorphism φ from (R_1,ϕ) to (R_2,ϕ) such that $\varphi_{|R}=\mathrm{Id}_R$. The *ring of constants* of the difference ring (R,ϕ) is defined by

$$R^{\phi}:=\{r\in R:\phi(r)=r\}.$$

If R^ϕ is a field, it is called the *field of constants*. If there is no risk of confusion, we usually simply say that R, instead of (R, ϕ), is a difference ring (or a difference field, or a difference ring extension).

2.1.2. Difference equations and linear difference systems. A *linear ϕ-equation* of order n over a ϕ-field K is an equation of the form

$$\mathscr{L}(y) := \phi^n(y) + a_{n-1}\phi^{n-1}(y) + \cdots + a_0 y = 0, \tag{2.1}$$

with $a_0, \ldots, a_{n-1} \in K$. If $a_0 \neq 0$, this relation can be written in matrix form as

$$\phi(Y) = A_{\mathscr{L}} Y \tag{2.2}$$

where

$$A_{\mathscr{L}} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{pmatrix} \in \mathrm{GL}_n(K).$$

The matrix $A_{\mathscr{L}}$ is called the *companion matrix* associated with (2.1). It is often more convenient to use the notion of *linear difference system*, that is, of system of the form

$$\phi(Y) = AY \quad \text{with } A \in \mathrm{GL}_n(K). \tag{2.3}$$

We recall that two difference systems $\phi(Y) = AY$ and $\phi(Y) = \widetilde{A}Y$ with $A, \widetilde{A} \in \mathrm{GL}_n(K)$ are said to be *equivalent over K* if there exists a gauge transformation $T \in \mathrm{GL}_n(K)$ such that $\widetilde{A} = \phi(T)AT^{-1}$. In that case, $\phi(Y) = AY$ if and only if $\phi(TY) = \widetilde{A}(TY)$.

Remark 2.1. Let $L|K$ be an extension of ϕ-fields and assume that $f = (f_1, \ldots, f_n)^\top \in L^n$ is a solution to $\phi(Y) = AY$ with $A \in \mathrm{GL}_n(K)$. As the K-subspace of L generated by $f_1, \ldots, f_n$ is closed under ϕ, it follows that each coordinate f_i of f satisfies a nontrivial linear ϕ-equation over K of order at most n.

2.1.3. Galois theory of linear difference equations. In this section, we give a brief summary of the Galois theory of linear difference equations.

Definition 2.2 ([26, Definition 2.2]). A *ϕ-pseudofield* is a ϕ-simple, Noetherian ϕ-ring K such that every non-zero-divisor of K is invertible in K. If K is a ϕ-pseudofield then there exist orthogonal idempotents $e_1, \ldots, e_d$ of K such that

- $K = e_1.K \oplus \cdots \oplus e_d.K$,
- $\phi(e_1) = e_2, \phi(e_2) = e_3, \ldots, \phi(e_d) = e_1$,
- $e_i.K$ is a field for $i = 1, \ldots, d$ (so $e_i.K$ is a ϕ^d-field).

Let K be a ϕ-field, R be a K-ϕ-algebra and $S \subset R$ be a subset of R. We let $K\{S\}_\phi$ denote the smallest K-ϕ-subalgebra of R that contains S. If R is a ϕ-pseudofield, we let $K\langle S\rangle_\phi$ denote the smallest ϕ-pseudofield of R that contains S.

A *Picard–Vessiot ring* for (2.3) over a difference field (K, ϕ) is a K-ϕ-algebra R_A satisfying the following three properties.

(1) There exists $U \in \mathrm{GL}_n(R_A)$ such that $\phi(U) = AU$. Such a matrix U is called a *fundamental matrix*.

(2) R_A is generated as a K-algebra by the coordinates of U and by $(\det U)^{-1}$, that is, $R_A = K[U, (\det U)^{-1}]$.

(3) R_A is a simple ϕ-ring.

A *Picard–Vessiot extension* K_A for (2.3) over the difference field (K, ϕ) is a ϕ-pseudofield extension of K with the following properties:

- $K_A = K(U)$ where $U \in \mathrm{GL}_n(K_A)$ is a fundamental matrix for (2.3).
- $K_A^\phi = K^\phi$.

Note that $K^\phi = \mathbb{C}$ in our main cases of interest, i.e. in Cases **2S**, **2Q**, and **2M**. By [32, Section 1.1] and [26, Proposition 2.14, Corollary 2.15], we obtain the following proposition that connects the two definitions.

Proposition 2.3. *If $C = K^\phi$ is an algebraically closed field, then there exists a unique (up to isomorphism of K-ϕ-algebras) Picard–Vessiot extension for (2.3). Moreover, the following properties hold:*

- *Given a Picard–Vessiot extension K_A with fundamental matrix $U \in \mathrm{GL}_n(K_A)$, the K-ϕ-algebra $R_A := K[U, (\det U)^{-1}] \subset K_A$ is a Picard–Vessiot ring.*
- *Given a Picard–Vessiot ring R_A, the total quotient ring $\mathrm{Quot}(R_A)$ is a Picard–Vessiot extension.*

From now on, we assume that K is a ϕ-field with $C = K^\phi$ an algebraically closed field of characteristic zero. Let $A \in \mathrm{GL}_n(K)$ and let K_A be a Picard–Vessiot extension for $\phi(Y) = AY$. Let $R_A \subset K_A$ denote the Picard–Vessiot ring defined in Proposition 2.3. The *Galois group* $\mathrm{Gal}(K_A|K)$ of $\phi(Y) = AY$ or of $K_A|K$ is the functor from the category of C-algebras to the category of groups that associates to any C-algebra B the group of all $K \otimes_C B$-ϕ-automorphisms of $R_A \otimes_C B$. Here ϕ acts as the identity on B. The functor $\mathcal{G} = \mathrm{Gal}(K_A|K)$ is representable. In fact, $\mathcal{G}$ is represented by $C[\mathcal{G}] = (R_A \otimes_K R_A)^\phi$ (see [22, Theorem 2.8] or [32, Section 1.2]).

Let $U \in \mathrm{GL}_n(R_A)$ be a fundamental matrix for $\phi(Y) = AY$ and let B be a C-algebra. For any $\tau \in \mathcal{G}(B)$, there exists $[\tau]_U \in \mathrm{GL}_n(B)$ such that $\tau(U \otimes 1) = (U \otimes 1)[\tau]_U$. Let $\mathrm{GL}_{n,C}$ be the functor defined by $\mathrm{GL}_{n,C}(B) = \mathrm{GL}_n(B)$ for every C-algebra B. The morphism of functors $\mathcal{G} \to \mathrm{GL}_{n,C}$, given by $\mathcal{G}(B) \to \mathrm{GL}_n(B)$, $\tau \mapsto [\tau]_B$, for any C-algebra B, identifies $\mathcal{G}$ with a closed subgroup of $\mathrm{GL}_{n,C}$. Since C is algebraically closed and of characteristic zero, an algebraic group $\mathcal{G}$ over C can be identified with $\mathcal{G}(C)$. Therefore one often identifies $\mathrm{Gal}(K_A|K)$ with $\mathrm{Gal}(K_A|K)(C)$.

In that setting, there is a Galois correspondence. We will only need the following special case.

Proposition 2.4 ([32, Lemma 1.28]). *With the above notation, let $\mathcal{H} \subset \mathrm{Gal}(K_A|K)$ be a closed subgroup. Then the following two statement are equivalent:*

- $K_A^{\mathcal{H}} := \{f \in K_A : \tau(f) = f, \forall\, \tau \in \mathcal{H}\} = K$.
- $\mathcal{H} = \mathrm{Gal}(K_A|K)$.

We will also need the following result.

Proposition 2.5 ([15, Theorem 3], [7, Theorem 12.6]). *If moreover K_A is a field, then:*

- $\mathrm{Gal}(K_A|K)$ *is connected if and only if K is relatively algebraically closed in K_A.*
- *If $\mathcal{N}$ is a normal closed subgroup of* $\mathrm{Gal}(K_A|K)$*, then $K_A^{\mathcal{N}}$ is a Picard–Vessiot extension for some system $\phi(Y) = A'Y$, where $A' \in \mathrm{GL}_{n'}(K)$ and n' is a positive integer.*

2.2. σ-Galois theory of linear difference equations

In this section, we consider rings endowed with two endomorphisms that commute. More formally, a $\phi\sigma$-ring is a triple (R, ϕ, σ) where R is a ring and (ϕ, σ) is a pair of endomorphisms of R that commutes. The notions of Section 2.1.1 extend to $\phi\sigma$-rings in a straightforward fashion. Given a $\phi\sigma$-field K and $A \in \mathrm{GL}_n(K)$, the σ-Galois theory developed in [26] aims to understand the algebraic relations between the solutions of $\phi(Y) = AY$ and their successive transforms with respect to σ from a Galoisian point of view. In this section, we assume that K is a $\phi\sigma$-field of characteristic zero such that $C = K^{\phi}$ is algebraically closed. Note that C is a σ-field because σ and ϕ commute.

A *ϕ-pseudo-σ-field* L is a $\phi\sigma$-ring that is a ϕ-pseudofield. The following definition is concerned with the notion of *minimal ring of solutions* in the context of parametrized difference equations. It summarizes in our context [26, Definition 2.18 and Proposition 2.21].

Definition 2.6. Let $A \in \mathrm{GL}_n(K)$. A ϕ-pseudo-σ-field extension L_A of K is a *σ-Picard–Vessiot extension* for $\phi(Y) = AY$ over K if there exists $U \in \mathrm{GL}_n(L_A)$ such that $\phi(U) = AU$, $L_A = K\langle U\rangle_\sigma$, and $L_A^{\phi} = K^{\phi}$. The K-$\phi\sigma$-algebra $S_A = K\{U, (\det U)^{-1}\}_\sigma$ is called a *σ-Picard–Vessiot ring* for $\phi(Y) = AY$ (over K). The ϕ-ring S_A is ϕ-simple and L_A is the total quotient ring of S_A.

The following definition introduces the σ-Galois group.

Definition 2.7 ([26, Definition 2.50]). Let $A \in \mathrm{GL}_n(K)$ and let $L_A = K\langle U\rangle_\sigma$ be a σ-Picard–Vessiot extension for $\phi(Y) = AY$. Set $S_A = K\{U, (\det U)^{-1}\}_\sigma$. The *$\sigma$-Galois group* $\mathrm{Gal}^\sigma(L_A|K)$ of L_A over K is the functor from the category of C-σ-algebras to the category of groups that associates to any C-σ-algebra B the group of all $K \otimes_C B$-$\phi\sigma$-automorphism of $S_A \otimes_C B$. Here ϕ acts as the identity on B.

It is proved in [26, Lemma 2.51] that this functor is represented by a finitely σ-generated C-σ-algebra. Therefore, $\mathrm{Gal}^\sigma(L_A|K)$ is a σ-algebraic group over C in the sense of Definition A.2. For a brief introduction to σ-algebraic groups we refer to Section A.1 of the appendix. The C-σ-algebra $C\{G\}$ representing $G = \mathrm{Gal}^\sigma(L_A|K)$ can be

explicitly described as $C\{G\} = (S_A \otimes_K S_A)^\phi$. Moreover, by [26, Lemma 2.41],

$$S_A \otimes_K S_A = S_A \otimes_C C\{G\}. \tag{2.4}$$

If $U \in \mathrm{GL}_n(S_A)$ is a fundamental matrix for $\phi(Y) = AY$, then $C\{G\} = (S_A \otimes_K S_A)^\phi = C\{Z, (\det Z)^{-1}\}_\sigma$ where $Z = (U \otimes 1)^{-1}(1 \otimes U) \in \mathrm{GL}_n(S_A \otimes_K S_A)$.

For a C-σ-algebra B, the action of $\tau \in \mathrm{Gal}^\sigma(L_A|K)(B)$ on $S_A \otimes_C B$ is determined by $\tau(U \otimes 1) = (U \otimes 1)[\tau]_U$, where $[\tau]_U$ denotes the image of Z in $\mathrm{GL}_n(B)$ under the morphism $C\{Z, (\det Z)^{-1}\}_\sigma \to B$ corresponding to τ under $G(B) \simeq \mathrm{Hom}(C\{Z, (\det Z)^{-1}\}_\sigma, B)$. The assignment $G(B) \to \mathrm{GL}_n(B),\ \tau \mapsto [\tau]_U$, identifies G with a σ-closed subgroup of $\mathrm{GL}_{n,C}$, in the sense of Definition A.4.

In this setting, there is a complete Galois correspondence. However, we will only need the following special case of [26, Theorem 2.52].

Proposition 2.8. *With notation as in Definitions 2.6 and 2.7, we have*

$$\{s \in S_A : \tau(s \otimes 1) = s \otimes 1,\ \forall\ \tau \in \mathrm{Gal}^\sigma(L_A|K)(B),\ \forall\ C\text{-}\sigma\text{-algebras } B\} = K.$$

The following definition introduces the difference analog of the transcendence degree of field extensions; see [20, Definition 4.1.7].

Definition 2.9. Let $L|K$ be a σ-field extension. Then $a_1, \ldots, a_d \in L$ are *σ-algebraically independent* over K if the elements $\sigma^i(a_j)$ $(i \in \mathbb{N},\ 1 \le j \le d)$ are algebraically independent over K. A single element $a \in L$ is also called *σ-transcendental* over K if it is σ-algebraically independent over K. Otherwise, a is called *σ-algebraic* over K.

A σ-transcendence basis of L over K is a maximal subset of L formed by σ-algebraically independent elements over K. Any two σ-transcendence bases of $L|K$ have the same cardinality [20, Proposition 4.1.6] and so we can define the *σ-transcendence degree* $\sigma\text{-tr.deg}(L|K)$ of $L|K$ as the cardinality of any σ-transcendence basis of L over K.

The following lemma will be used several times in what follows.

Lemma 2.10. *Let $L|K$ be a σ-field extension and let $s \ge 1$ be an integer. Then $f \in L$ is σ-algebraic over K if and only if f is σ^s-algebraic over K.*

Proof. If f is σ-algebraic over K then the transcendence degree of $K\langle f\rangle_\sigma$ over K is finite. Since $K \subset K\langle f\rangle_{\sigma^s} \subset K\langle f\rangle_\sigma$, the transcendence degree of $K\langle f\rangle_{\sigma^s}$ over K is finite and f is σ^s-algebraic over K. The converse is obviously true: if f is σ^s-algebraic over K, it is σ-algebraic over K. ∎

The σ-dimension $\sigma\text{-dim}(G)$ of a σ-algebraic group is defined in Definition A.8.

Proposition 2.11. *In addition to the notation of Definitions 2.6 and 2.7, assume that L_A is a field. Then $\sigma\text{-dim}(\mathrm{Gal}^\sigma(L_A|K)) = \sigma\text{-tr.deg}(L_A|K)$.*

Proof. It is shown in [26, Lemma 2.53] that $\sigma\text{-dim}(\mathrm{Gal}^\sigma(L_A|K)) = \sigma\text{-dim}(S_A)$. By [36, Proposition 3.1], the σ-dimension of a finitely σ-generated K-σ-algebra, which is an integral domain with σ injective, agrees with the σ-transcendence degree of its field of fractions. Therefore $\sigma\text{-dim}(S_A) = \sigma\text{-tr.deg}(L_A/K)$. ∎

The following proposition explains the connection between the σ-Galois theory of linear difference equations and the (usual) Galois theory of linear difference equations. The meaning of "Zariski dense σ-closed subgroup" is explained in Definitions A.4 and A.5.

Proposition 2.12. *In addition to the notation of Definitions 2.6 and 2.7, assume that L_A is a field. Then $K_A = K(U) \subset L_A$ is a Picard–Vessiot extension for $\phi(Y) = AY$ and the σ-Galois group $G = \mathrm{Gal}^\sigma(L_A|K)$ is a Zariski dense σ-closed subgroup of the Galois group $\mathscr{G} = \mathrm{Gal}(K_A|K)$.*

Proof. Because $L_A^\phi = K$, it is clear that K_A is a Picard–Vessiot extension for $\phi(Y) = AY$. For every C-σ-algebra B, every $K \otimes_C B$-$\phi\sigma$-automorphism of $S_A \otimes_C B$ restricts to a $K \otimes_C B$-ϕ-automorphism of $R_A \otimes_C B$. So $G(B) \subset \mathscr{G}(B)$ and we obtain an inclusion $G \subset [\sigma]_C\mathscr{G}$, where $[\sigma]_C\mathscr{G}$ is defined in Example A.3. As we can see in the paragraph after Definition A.5, to prove that G is Zariski dense in $\mathscr{G}$, it suffices to prove that the corresponding map $C[\mathscr{G}] \to C\{G\}$ of coordinate rings is injective. But this map is the inclusion $(R_A \otimes_K R_A)^\phi \to (S_A \otimes_K S_A)^\phi$, which is clearly injective. ∎

3. Auxiliary results

In this section, we gather some auxiliary results needed for the proof of Theorem 1.3.

3.1. Extending the constants

We will need the following elementary lemma about solutions of linear difference equations.

Lemma 3.1. *Let K be a ϕ-field, $C = K^\phi$, R a K-ϕ-algebra, and $A \in \mathrm{GL}_n(K)$. Then, for every C-algebra B (considered as a ϕ-constant ϕ-ring), one has*

$$\{y \in R^n : \phi(y) = Ay\} \otimes_C B = \{y \in (R \otimes_C B)^n : \phi(y) = Ay\}.$$

Proof. Clearly $\{y \in R^n : \phi(y) = Ay\} \otimes_C B \subset \{y \in (R \otimes_C B)^n : \phi(y) = Ay\}$. Let y be contained in the right-hand side and fix a C-basis $(b_i)_{i\in I}$ of B. Then $y = \sum y_i \otimes b_i$ for some uniquely determined $y_i \in R^n$. We have

$$\sum \phi(y_i) \otimes b_i = \phi(y) = Ay = \sum Ay_i \otimes b_i.$$

Therefore $\phi(y_i) = Ay_i$ and so y is contained in the left-hand side. ∎

3.2. Compatible difference systems

Two difference systems $\phi(Y) = AY$ and $\sigma(Y) = \widetilde{A}Y$ over a $\phi\sigma$-field K are *compatible* if

$$\phi(\widetilde{A})A = \sigma(A)\widetilde{A}. \tag{3.1}$$

This condition is equivalent to the existence of a K-$\phi\sigma$-algebra S and $U \in \mathrm{GL}_n(S)$ such that $\phi(U) = AU$ and $\sigma(U) = \widetilde{A}U$. Compatible difference systems over the projective line have been studied by Schäfke and Singer [28]. The following result shows that compatibility is a strong constraint for discrete systems over the projective line.

Proposition 3.2 ([28, Theorem 13]). *Let K be one of the $\phi\sigma$-fields defined in Cases **2S**, **2Q**, and **2M**. Let $A, \widetilde{A} \in \mathrm{GL}_n(K)$ be such that the systems $\phi(Y) = AY$ and $\sigma(Y) = \widetilde{A}Y$ are compatible. Then there exists $T \in \mathrm{GL}_n(K)$ such that $\phi(T)AT^{-1} \in \mathrm{GL}_n(\mathbb{C})$ and $\sigma(T)\widetilde{A}T^{-1} \in \mathrm{GL}_n(\mathbb{C})$. In other words, the systems $\phi(Y) = AY$ and $\sigma(Y) = \widetilde{A}Y$ are simultaneously equivalent over K to difference systems with constant coefficients.*

3.3. Reducible Galois groups

For a finite-dimensional C-vector space V, a closed subgroup $\mathcal{G}$ of GL_V is *irreducible* if V is an irreducible representation of $\mathcal{G}$. If this is not the case, $\mathcal{G}$ is called *reducible*.

The following lemma characterizes linear difference equations with a reducible Galois group.

Lemma 3.3 ([3, Lemma 4.4]). *Let K be a ϕ-field with $C = K^\phi$ algebraically closed. Let $A \in \mathrm{GL}_n(K)$ and let $\mathcal{G} \subset \mathrm{GL}_{n,C}$ be the Galois group of $\phi(Y) = AY$. For an integer r with $0 < r < n$ the following statements are equivalent:*

- *There exists a $\mathcal{G}$-subrepresentation of C^n of dimension r over C.*
- *There exists $T \in \mathrm{GL}_n(K)$ such that*

$$\phi(T)AT^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$$

with $B_1 \in \mathrm{GL}_r(K)$.

In particular, $\mathcal{G}$ is reducible if and only if the above statements hold for some r, with $0 < r < n$.

3.4. Condition $\mathcal{H}$ and consequences

Similarly to [3], we need to consider the following setting. We say that a $\phi\sigma$-field K satisfies *Condition $\mathcal{H}$* if the following properties hold:

- $\sigma\colon K \to K$ is an automorphism.
- $C = K^\phi$ is algebraically closed.
- For every positive integer r, K has no finite nontrivial ϕ^r-field extension.

Remark 3.4. Note that if $K^\phi = C$ is an algebraically closed field then by [3, Lemma 4.8], $K^{\phi^s} = C$ for all positive integers s. Thus, any $\phi\sigma$-field satisfying Condition $\mathcal{H}$, also satisfies Condition $\mathcal{H}$ as a (ϕ^r, σ^s)-field, for all positive integers r and s.

Lemma 3.5 ([3, Lemma 4.9]). *Let K be one of the $\phi\sigma$-fields defined in Cases **2S**, **2Q**, and **2M**. Then (K, ϕ) satisfies Condition $\mathcal{H}$.*

The following lemma is a straightforward generalization of [14, Lemma 4.5].

Lemma 3.6. *Let K be a $\phi\sigma$-field satisfying Condition $\mathcal{H}$. Let $A \in \mathrm{GL}_n(K)$ and let L_A be a σ-Picard–Vessiot extension for $\phi(Y) = AY$ over K. If L_A is a field, then K is relatively algebraically closed in L_A.*

Proof. Let M denote the relative algebraic closure of K in L_A. We have to show that $M = K$. Given a fundamental matrix $U \in \mathrm{GL}_n(L_A)$ for $\phi(Y) = AY$ and an integer $i \geq 0$, the intersection $M \cap K(U, \ldots, \sigma^i(U))$ is a finite ϕ-field extension of K and therefore trivial. Thus $M = K$. ∎

We are now interested in iterating difference systems. Given $A \in \mathrm{GL}_n(K)$ and a positive integer r, we set $A_{[r]} := \phi^{r-1}(A) \cdots \phi(A)A$. Note that $\phi(Y) = AY$ implies that $\phi^r(Y) = A_{[r]}Y$. The following proposition shows that, considering some iterates of the operators σ and ϕ if necessary, one can always reduce the situation to the case where the σ-Picard–Vessiot extension L_A is a field.

Proposition 3.7 ([14, Proposition 4.6]). *Let K be a $\phi\sigma$-field such that $C = K^\phi$ is algebraically closed. Let $A \in \mathrm{GL}_n(K)$, F a $\phi\sigma$-field extension of K such that $F^\phi = K^\phi$, and let $u_1, \ldots, u_n \in F^*$ be such that $(u_1, \ldots, u_n)^\top$ is a solution to $\phi(Y) = AY$. Then the following properties hold:*

(1) *There exist some positive integers r and s and a σ^s-Picard–Vessiot extension L_A for the system $\phi^r(Y) = A_{[r]}Y$ over (K, ϕ^r) such that L_A is a field and $K\{u_1, \ldots, u_n\}_{\sigma^s,\phi^r}$ embeds into L_A.*

(2) *If K satisfies Condition $\mathcal{H}$, then the Galois group of $\phi^r(Y) = A_{[r]}Y$ over (K, ϕ^r) coincides with the connected component of the Galois group of the system $\phi(Y) = AY$ over (K, ϕ).*

Proof. The paper [14] is concerned with Case **2Q** but the proof works the same at this level of generality. ∎

Remark 3.8. As a straightforward consequence of (2) in Proposition 3.7, we obtain that if the Galois group $\mathcal{G}$ of the system $\phi(Y) = AY$ over (K, ϕ) is connected, then, for every positive integer r, the Galois group of $\phi^r(Y) = A_{[r]}Y$ over (K, ϕ^r) coincides with $\mathcal{G}$.

A σ-algebraic group G over a σ-field C is *σ-integral* if its coordinate ring $C\{G\}$ is a *σ-domain*, i.e. $C\{G\}$ is an integral domain and $\sigma\colon C\{G\} \to C\{G\}$ is injective.

Lemma 3.9. *Let K be a $\phi\sigma$-field satisfying Condition $\mathcal{H}$ and let $L_A|K$ be a σ-Picard–Vessiot extension for $\phi(Y) = AY$. If L_A is a field, then $\mathrm{Gal}^\sigma(L_A|K)$ is σ-integral and the Galois group of $\phi(Y) = AY$ is connected.*

Proof. Since K is an inversive σ-field, i.e. $\sigma\colon K \to K$ is surjective, by [29, Proposition 1.2 (iii)], the K-σ-algebra L_A satisfies all the equivalent conditions of [29, Proposition 1.2]. In particular, point (vii) of that proposition implies that $\sigma\colon L_A \otimes_K S' \to L_A \otimes_K S'$ is injective for any K-σ-algebra S' with $\sigma\colon S' \to S'$ injective.

Thus the map $\sigma\colon L_A \otimes_K L_A \to L_A \otimes_K L_A$ is injective. Since K is relatively algebraically closed in L_A by Lemma 3.6, we see that L_A is a regular field extension of K. So by [11, Chapter V, §17, No. 3, Proposition 2], $L_A \otimes_K L_A$ is an integral domain.

Let $S_A \subset L_A$ denote the corresponding σ-Picard–Vessiot ring. Because $L_A \otimes_K L_A$ is a σ-domain we see that also $S_A \otimes_L S_A \subset L_A \otimes_K L_A$ is a σ-domain. By (2.4) we have $S_A \otimes_K S_A = S_A \otimes_C C\{G\}$, where $G = \mathrm{Gal}^\sigma(L_A|K)$. Therefore also $C\{G\}$ is a σ-domain and so G is σ-integral. Because the coordinate ring $C[\mathcal{G}]$ of the Galois group $\mathcal{G}$ of $\phi(Y) = AY$ is contained in $C\{G\}$ (cf. the proof of Proposition 2.12), it follows that $C[\mathcal{G}]$ is an integral domain. Therefore $\mathcal{G}$ is connected (cf. [24, Summary 1.36]). ∎

4. Proofs of Theorem 1.3 and Corollaries 1.4 and 1.5

Throughout this section the $\phi\sigma$-fields K and F are as in one of Cases **2S**, **2Q**, and **2M**. Our main goal is to establish Theorem 1.3 that is restated here as Theorem 4.1.

Theorem 4.1. *Let $f \in F$ be a solution of the linear ϕ-equation of order n*

$$\phi^n(y) + a_{n-1}\phi^{n-1}(y) + \cdots + a_1\phi(y) + a_0 y = 0, \tag{4.1}$$

where $a_0, \ldots, a_{n-1} \in K$. Then either $f \in K$ or f is σ-transcendental over K.

Remark 4.2. Let us consider Case **2Q**, and let us make two more additional assumptions. First, let us assume $|q_1|, |q_2| \neq 1$. The second more restrictive assumption is on the Galois group. More precisely, let G be the corresponding Galois group and let $G^{\mathrm{der},\circ}$ be the derived subgroup of the connected component of the identity of G. By [14, Corollary 8.2], if one of the following conditions is satisfied:

- $n \geq 2$ and $G^{\mathrm{der},\circ} = \mathrm{SL}_n(\mathbb{C})$;
- $n \geq 3$ and $G^{\mathrm{der},\circ} = \mathrm{SO}_n(\mathbb{C})$;
- n is even and $G^{\mathrm{der},\circ} = \mathrm{SP}_n(\mathbb{C})$,

then either $f = 0$ or f is σ-transcendental over K. Note that the assumption on the Galois group implies that there are no nonzero solutions of (4.1) in K, so Theorem 4.1 generalizes [14, Corollary 8.2].

The following simple lemma shows that Theorem 4.1 implies Corollary 1.4 and thus Corollary 1.5.

Lemma 4.3. *Let $L|K$ be a σ-field extension. Let $f, g \in L$ be such that g is σ-algebraic over K. If f is σ-algebraic over $K\langle g\rangle_\sigma$, then f is σ-algebraic over K.*

Proof. By the transitivity of being σ-algebraic [20, Theorem 4.1.2 (i)], f is σ-algebraic over K. ∎

Example 4.4. Let us equip $\mathbb{C}(x)$ with two automorphisms $\phi(x) = x + 1$ and $\sigma(x) = x + h$, with $h \in \mathbb{C}$. When $h \notin \mathbb{Q}$, the σ-transcendence of the Gamma function has been proved in [26, Example 3.7]. By Lemma 4.3, we then deduce that for $h \notin \mathbb{Q}$, $\Gamma(x)$ and

$\Gamma(xh^{-1})$ are algebraically independent over $\mathbb{C}(x)$. More generally, $\Gamma(x)$ is σ-transcendental over $\mathbb{C}(x)\langle\Gamma(xh^{-1})\rangle_\sigma$.

Proof of Corollary 1.4. Since f does not belong to K, f is σ-transcendental over K by Theorem 4.1. By Lemma 4.3, f is also σ-transcendental over $K\langle g\rangle_\sigma$. In particular, f is transcendental over $K(g)$. Since g is assumed to be transcendental over K, we deduce that f and g are algebraically independent over K. ∎

Proof of Corollary 1.5. Let us assume that f and g do not belong to K. By Lemma 3.5, an element of F that is algebraic over K and satisfies a linear difference equation over K (with respect to ϕ or σ) belongs to K. Hence f and g are both transcendental. We thus deduce from Corollary 1.4 that f and g are algebraically independent over K, as wanted. ∎

Similarly to [3], Theorem 4.1 is proved by induction on the order n of equation (4.1). Before proving Theorem 4.1 in full generality, we first consider the following special cases:

- The function f is a solution of an inhomogeneous equation of order 1.
- The function f is a solution of a difference equation whose corresponding Galois group is both connected and irreducible.

4.1. Affine order 1 equations

We begin this subsection by studying homogeneous order 1 equations $\phi(y) = ay$. The following proposition characterizes the coefficients $a \in K$ for which this equation has a nontrivial σ-algebraic solution. Case **2Q** is already proved in [14, Proposition 5.3]. The following proposition summarizes all three cases.

Proposition 4.5. *Let K and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Let $a \in K^*$. Let L be a σ-Picard–Vessiot extension for $\phi(y) = ay$ and assume that L is a field. Let $u \in L^*$ be such that $\phi(u) = au$. Then the following statements are equivalent.*

(1) *The element $u \in L$ is σ-algebraic over K.*

(2) *There exist $c \in \mathbb{C}^*$, $n \in \mathbb{Q}$ ($n = 0$ in Cases **2S** and **2M**) and $b \in K^*$ such that $a = cx^n \frac{\phi(b)}{b}$.*

Proof. If (2) holds, a straightforward computation, similar to the one in the proof of [14, Proposition 5.3], shows that when $n = 0$, $\frac{\sigma(u/b)}{u/b} \in L^\phi = \mathbb{C}$. In general, we have $\sigma\big(\frac{\sigma(u/b)}{u/b}\big)/\big(\frac{\sigma(u/b)}{u/b}\big) \in L^\phi = \mathbb{C}$. This shows that u is σ-algebraic over K. So (2) implies (1).

Let us assume that u is σ-algebraic over K. The Galois-theoretic arguments used in [26, Theorem 3.1] for Cases **2Q** and **2S** also apply to Case **2M** and we find that, since u is σ-algebraic over K, there exist $r \in \mathbb{Z}_{\geq 0}$, $k_0, \dots, k_r \in \mathbb{Z}$ with $k_0 k_r \neq 0$, and $b \in K^*$ such that

$$\prod_{\ell=0}^{r} \sigma^\ell(a^{k_\ell}) = \frac{\phi(b)}{b}. \tag{4.2}$$

Consider first Case **2S**. Following [17, §4.1], we define the h_1-divisor $\mathrm{div}_{h_1}(f)$ of $f \in \mathbb{C}(x)$ as the formal sum $\mathrm{div}_{h_1}(f) = \sum_{[\alpha]\in\mathbb{C}/h_1\mathbb{Z}} n_\alpha[\alpha]$, where n_α is the sum of the valuations of f at the points $[\alpha] = \alpha + h_1\mathbb{Z}$. By [17, Lemme 4.3], there exist $c \in \mathbb{C}^*$ and $0 \neq b \in \mathbb{C}(x)$ such that $f = c\frac{\phi(b)}{b}$ if and only if $\mathrm{div}_{h_1}(f) = 0$. Thanks to this characterization, we only need to prove that $\mathrm{div}_{h_1}(a) = 0$. Suppose to the contrary that this is not the case and that $\mathrm{div}_{h_1}(a) = \sum_{i=1}^m n_i[\zeta_i]$ for some pairwise distinct elements $[\zeta_i] \in \mathbb{C}/h_1\mathbb{Z}$ and nonzero integers n_i. From (4.2), we deduce

$$0 = \mathrm{div}_{h_1}\left(\frac{\phi(b)}{b}\right) = \sum_{l=0}^{r} k_l \sum_{i=1}^{m} n_i[\zeta_i - h_2 l]. \tag{4.3}$$

Let

$$I = \big\{i \in \{1,\dots,m\} : [\zeta_i] = [\zeta_1 + h_2 n] \text{ for some integer } n\big\}.$$

Let $i_1,\dots,i_s$ be pairwise distinct integers such that $I = \{i_1,\dots,i_s\}$. Up to renumbering, we can assume that

$$[\zeta_{i_1}] \prec \cdots \prec [\zeta_{i_s}]$$

where, for any $[x],[y] \in \mathbb{C}/h_1\mathbb{Z}$, $[x] \prec [y]$ means that $[y] = [x] + kh_2$ $(= [x + kh_2])$ for some $k \in \mathbb{N}^*$. Note that the binary relation $\prec$ is well defined because $h_1/h_2 \notin \mathbb{Q}$. It is clear that $[\zeta_{i_1} - rh_2] \prec [\zeta_{i_k} - jh_2]$ for all $j \in \{0,\dots,r\}$ and $k \in \{1,\dots,s\}$ such that $(j,k) \neq (r,1)$. Moreover, for $j \in \{0,\dots,r\}$ and $i \in \{1,\dots,m\} \setminus I$, we have $[\zeta_{i_1} - rh_2] \neq [\zeta_i - jh_2]$ by definition of I. Therefore, the coefficient of $[\zeta_{i_1} - rh_2]$ in equation (4.3) is equal to 0, i.e. $k_r n_{i_1} = 0$. This provides a contradiction.

Consider now Case **2M**. For $f \in F^*$, let $v_f \in \mathbb{Q}$ be its valuation and $t_f = fx^{-v_f}|_{x=0} \in \mathbb{C}^*$. By [27, Lemma 20], the equation $\phi(y) = \frac{ax^{-v_a}}{t_a} y$ has a solution $f \in F^*$. This can also be verified directly by recursively solving for the coefficients of f.

Let $\widetilde{f} := \prod_{\ell=0}^r \sigma^\ell(f^{k_\ell}) \in F^*$ and $\tilde{c} := \prod_{\ell=0}^r t_a^{k_\ell} \in \mathbb{C}^*$. With (4.2) and $\phi\sigma = \sigma\phi$, we find that there exists $N \in \mathbb{Q}$ such that

$$\phi(\widetilde{f}) = x^N \prod_{\ell=0}^{r} \frac{\sigma^\ell(a^{k_\ell})}{t_a^{k_\ell}} \widetilde{f} = x^N \frac{\phi(b)}{b} \frac{\widetilde{f}}{\tilde{c}} = \frac{\phi(x^{N/(p_1-1)}b)}{x^{N/(p_1-1)}b} \frac{\widetilde{f}}{\tilde{c}}. \tag{4.4}$$

For $g \in F^*$, we have $t_{\phi(g)} = t_g$. By (4.4), we obtain $\tilde{c} = 1$. Hence $\frac{\widetilde{f}}{x^{N/(p_1-1)}b} \in F^\phi = \mathbb{C}$ and there exists $d \in \mathbb{C}^*$ such that

$$\prod_{\ell=0}^{r} \sigma^\ell(f^{k_\ell}) = dx^{N/(p_1-1)}b. \tag{4.5}$$

Let ∂ be the derivation $x\frac{d}{dx}$. Since $\partial\phi = p_1\phi\partial$ and $\partial\sigma^\ell = p_2^\ell\sigma^\ell\partial$, we can easily compute the logarithmic derivatives of $\phi(f) = \frac{ax^{-v_a}}{t_a} f$ and of (4.5). We find that $\partial f/f \in F$ satisfies the following linear σ-difference and ϕ-difference equations over K:

$$p_1\phi(y) = \frac{\partial\,(ax^{-v_a})}{ax^{-v_a}} + y \quad \text{and} \quad \sum_{\ell=0}^{r} p_2^\ell k_\ell \sigma^\ell(y) = \frac{\partial(x^{N/(p_1-1)}b)}{x^{N/(p_1-1)}b}.$$

By Theorem 1.1, $\partial f/f \in K$. So $f \in F$ satisfies a linear ϕ-difference equation and a linear differential equation with coefficients in K. By [3, Theorem 1.2] (see also [28]), $f \in K$. Therefore

$$a = t_a \frac{\phi(f x^{v_a/(p_1-1)})}{f x^{v_a/(p_1-1)}}$$

has the desired form. ∎

In [13, §3.3], the authors study the difference algebraic relations satisfied by the solutions of a differential equation of the form $y' = ay + b$ with $a, b \in K$. Here, we adapt and generalize some of the arguments from [13] to our context in order to establish the following special case of Theorem 4.1

Proposition 4.6. *Let K, F, and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Let $f \in F$ satisfy $\phi(f) = af + b$ with $a, b \in K$. Then either f is σ-transcendental over K, or $f \in K$.*

Recall that ϕ is an automorphism of K. Therefore, the result is trivial if $a = 0$. So let us assume that $a \neq 0$ and let us consider the ϕ-system

$$\phi(Y) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} Y = AY. \tag{4.6}$$

Then, $\binom{f}{1}$ is a vector solution of (4.6). Recall that $F^\phi = K^\phi$. By Proposition 3.7, there exist $r, s \in \mathbb{N}$ and a σ^s-Picard–Vessiot extension L_A over (K, ϕ^r) for the system $\phi^r(Y) = A_{[r]}Y$ with $A_{[r]} = \phi^{r-1}(A) \cdots A$ such that the following properties hold:

- L_A is a field.
- $f \in L_A$.
- Let $K_A \subset L_A$ denote the Picard–Vessiot extension as in Proposition 2.12. Then the classical Galois group $\mathrm{Gal}(K_A|K)$ of $\phi^r(Y) = A_{[r]}Y$ over (K, ϕ^r) coincides with the identity component of the classical Galois group of $\phi(Y) = AY$ over (K, ϕ).

Note that the equation corresponding to $\phi^r(Y) = A_{[r]}Y$ is still an affine order 1 equation. Without loss of generality, we may assume that $r = 1$. By Lemma 2.10, without loss of generality, up to replacing σ by some power of σ, we may also reduce to the case where $s = 1$.

We may assume that the fundamental matrix is of the form $U = \left(\begin{smallmatrix} u & f \\ 0 & 1 \end{smallmatrix}\right)$, where $0 \neq u \in L_A$ is a solution of $\phi(u) = au$, since the lower left entry of U can always be eliminated by subtracting a multiple of $\binom{f}{1}$ from the first column. Let $\mathrm{Gal}^\sigma(L_A|K)$ be the σ-Galois group. Via its action on the fundamental matrix U, for any $\mathbb{C}$-σ-algebra B, $\mathrm{Gal}^\sigma(L_A|K)(B)$ can be represented as a subgroup of

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha \in B^*, \beta \in B \right\}.$$

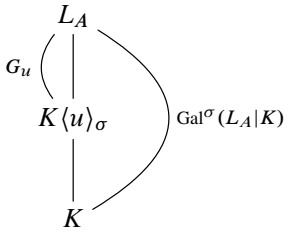
Let $\mathbb{G}_u$ denote the algebraic subgroup of $\mathrm{GL}_{2,\mathbb{C}}$ given by

$$\mathbb{G}_u(B) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \beta \in B \right\} \quad \text{for any } \mathbb{C}\text{-}\sigma\text{-algebra } B,$$

and set $G_u = \mathrm{Gal}^\sigma(L_A|K) \cap \mathbb{G}_u$. Since $\tau(u \otimes 1) = u \otimes \alpha$ for $\tau = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gal}^\sigma(L_A|K)(B)$, the Galois correspondence implies that $G_u = \mathrm{Gal}^\sigma(L_A|K\langle u\rangle_\sigma)$. Moreover, we have $\phi(f/u) = f/u + b/(au)$, so that $L_A|K\langle u\rangle_\sigma$ is a σ-Picard–Vessiot extension for $\phi(y) = y + b/(au)$. The action of an element $\tau = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) \in G_u(B) = \mathrm{Gal}^\sigma(L_A|K\langle u\rangle_\sigma)(B)$ on L_A is given by

$$\tau\left(\frac{f}{u} \otimes 1\right) = \frac{f}{u} \otimes 1 + 1 \otimes \beta. \tag{4.7}$$

The situation is summarized in the following picture:



The proof of [13, Proposition 3.16] for nonhomogeneous differential equations of order 1 with discrete parameters passes word for word to our context and yields the following lemma.

Lemma 4.7. *Let $L|K$ be a $\phi\sigma$-field extension such that $C = K^\phi = L^\phi$. Fix $u, f \in L$, $u \neq 0$, such that $\phi(f) = af + b$ and $\phi(u) = au$ with $a, b \in K$. Assume that $f \notin K$ and that $\sigma\colon K \to K$ is surjective. If u is σ-transcendental over K, then f is σ-transcendental over K.*

We will also need the following auxiliary result that generalizes [18, Lemma 6.5].

Lemma 4.8. *For $\tilde{a}, \tilde{b} \in K$, assume that there exists a σ-Picard–Vessiot extension L_A for $\phi(Y) = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ 0 & 1 \end{smallmatrix}\right)Y = AY$ over K that is a field. Consider an intermediate ϕ-field L with $L_A|L|K$ and let $x \in L_A^*$ be such that $\phi(x)/x = \alpha \in L^*$. If $\phi(y) = \tilde{a}y + \tilde{b}$ has a solution in $L(x)$, then it has a solution in L.*

Proof. Let $g \in L(x)$ be a solution of the equation $\phi(y) = \tilde{a}y + \tilde{b}$. Let us first assume that x is algebraic over L. Then g is algebraic over L and the orbit of g under the Galois group $\mathcal{G}$ of $\phi(Y) = AY$ over L is finite. Let $g_1, \dots, g_\ell \in L_A$ denote the elements in the $\mathcal{G}$-orbit of g and set $\tilde{g} := g_1 + \cdots + g_\ell$. Since $\tilde{g}$ is fixed by $\mathcal{G}$, it belongs to L by Proposition 2.4. As $\tilde{a}, \tilde{b} \in K \subset L$, we have $\phi(g_k) = \tilde{a}g_k + \tilde{b}$ for $1 \leq k \leq \ell$. Therefore, $\phi(\tilde{g}/\ell) = \tilde{a}\tilde{g}/\ell + \tilde{b}$ with $\tilde{g}/\ell \in L$. This proves the claim in that case.

Now, let us assume that x is transcendental over L. We can extend ϕ to the field $L((x))$ of Laurent series by setting $\phi(\sum_{k\geq r} a_k x^k) = \sum_{k\geq r} \phi(a_k)\alpha^k x^k$. Writing $g = \sum_{k\geq r} a_k x^k$ with $a_k \in L$ and $r \in \mathbb{Z}$, we get $\phi(g) = \sum_{k\geq r} \phi(a_k)\alpha^k x^k$. Using the uniqueness of the Laurent series expansion and the fact that $\phi(g) = \tilde{a}g + \tilde{b}$, we deduce that $\phi(a_0) = \tilde{a}a_0 + \tilde{b}$. Since $a_0 \in L$, this completes the proof of the lemma. ∎

After these preliminaries, we are ready to prove Proposition 4.6.

Proof of Proposition 4.6. Let us argue by contradiction and suppose that f is σ-algebraic and does not belong to K. By Lemma 4.7, the element u is σ-algebraic over K. By Proposition 4.5, there exist $c \in \mathbb{C}^*, \widetilde{b} \in K^*$ and $n \in \mathbb{Q}$, which is zero in Cases **2S** and **2M**, such that $a = cx^n \phi(\widetilde{b})/\widetilde{b}$. Replacing f by $f/\widetilde{b}$ and b by $b/\phi(\widetilde{b})$, we can assume that $\widetilde{b} = 1$. By Proposition 2.11, $\sigma\text{-tr.deg}(L_A|K\langle u\rangle_\sigma) = 0 = \sigma\text{-dim}(G_u)$, and the σ-algebraic group G_u must be a proper subgroup of $\mathbb{G}_u$ whose σ-dimension is 1. Let $\mathbb{C}[\sigma]$ denote the set of linear σ-operators with coefficients in $\mathbb{C}$. By [13, Corollary A.3], there exists $\mathcal{L} \in \mathbb{C}[\sigma]$ such that for B a $\mathbb{C}$-σ-algebra,

$$G_u(B) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(B) : \mathcal{L}(\beta) = 0 \right\}.$$

Let first $n = 0$. As a consequence of (4.7), for $\tau \in \mathrm{Gal}^\sigma(L_A|K\langle u\rangle_\sigma)(B) = G_u(B)$,

$$\tau\left(\mathcal{L}\left(\frac{f}{u}\right) \otimes 1\right) = \mathcal{L}\left(\tau\left(\frac{f}{u} \otimes 1\right)\right) = \mathcal{L}\left(\frac{f}{u}\right) \otimes 1 + 1 \otimes \mathcal{L}(\beta) = \mathcal{L}\left(\frac{f}{u}\right) \otimes 1.$$

So the Galois correspondence implies that $\mathcal{L}\left(\frac{f}{u}\right) \in K\langle u\rangle_\sigma$. Since $a \in \mathbb{C}^*$, we find that

$$\phi\left(\frac{\sigma(u)}{u}\right) = \frac{\sigma(\phi(u))}{\phi(u)} = \frac{\sigma(a)\sigma(u)}{au} = \frac{\sigma(u)}{u}.$$

Then $d := \frac{\sigma(u)}{u} \in K^\phi = \mathbb{C}$, so that $K\langle u\rangle_\sigma = K(u)$. Since $\sigma(u) = du$ with $d \neq 0$ and $\mathcal{L}(\frac{f}{u}) \in K(u)$, there exists a nonzero $\widetilde{\mathcal{L}} \in \mathbb{C}[\sigma]$ such that $\widetilde{\mathcal{L}}(f) \in K(u)$. Recall that $a \in \mathbb{C}^*$ so that $\widetilde{\mathcal{L}}(f)$ is a solution of $\phi(y) = ay + \widetilde{\mathcal{L}}(b)$. By Lemma 4.8, there exists $g \in K$ such that $\phi(g) = ag + \widetilde{\mathcal{L}}(b)$. Then $\frac{\widetilde{\mathcal{L}}(f)-g}{u} \in K(u)$ is fixed by ϕ, and since $K(u)^\phi \subset L_A^\phi = \mathbb{C}$, there exists $d' \in \mathbb{C}$ such that $\widetilde{\mathcal{L}}(f) = d'u + g$. Since u and g both satisfy some linear σ-difference equations over K, the same holds for $\widetilde{\mathcal{L}}(f)$. Therefore, f satisfies a nontrivial linear σ-difference equation over K. By Theorem 1.1, we find $f \in K$. This contradiction concludes the proof in the case $n = 0$ (and hence in Cases **2S** and **2M**).

Now, let us consider Case **2Q** with $n \neq 0$. By [13, Theorem A.9], we only have to consider the following two cases.

(1) There exist integers $\nu > m \geq 0$ such that $\sigma^\nu(\alpha) = \sigma^m(\alpha)$ and $\mathcal{L} \in \mathbb{C}[\sigma]$ such that $\mathcal{L}(\beta) = 0$ for all $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gal}^\sigma(L_A|K)(B)$ and all $\mathbb{C}$-σ-algebras B.

(2) There exists an integer $\nu \geq 0$ such that $\sigma^\nu(\beta) = 0$ for all $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) \in G_u(B)$ and all $\mathbb{C}$-σ-algebras B.

Consider the first case. For all $\tau \in \mathrm{Gal}^\sigma(L_A|K)(B)$, we have $\tau(u \otimes 1) = u \otimes \alpha$ with $\sigma^\nu(\alpha) = \sigma^m(\alpha)$. Then $g = \sigma^\nu(u)/\sigma^m(u)$ satisfies $\tau(g \otimes 1) = g \otimes 1$ and we deduce from the Galois correspondence that $g \in K$. Since g is a nonzero solution of $\phi(y) = q_2^{n(\nu-m)} y$ and a nonzero Puiseux series of the form $\sum_{\ell \in \frac{1}{r}\mathbb{Z}} g_\ell x^\ell$, we find that $q_1^\ell g_\ell = q_2^{n(\nu-m)} g_\ell$ for any $\ell \in \frac{1}{r}\mathbb{Z}$. Since q_1 and q_2 are multiplicatively independent, the coefficients g_ℓ must all vanish, which contradicts the fact that g is nonzero.

In the second case, a similar computation and the Galois correspondence ensure that $\sigma^\nu(f) \in K\langle u\rangle_\sigma$. Recall that u is σ-algebraic over K. Let $m \in \mathbb{N}$ be such that $\sigma^\nu(f) \in K(u, \sigma(u), \dots, \sigma^m(u))$. Since $\phi(\sigma^k(u)) = \sigma^k(a)\sigma^k(u)$ and $\phi(\sigma^\nu(f)) = \sigma^\nu(a)\sigma^\nu(f) + \sigma^\nu(b)$, one can apply recursively Lemma 4.8 to find a K-rational solution of $\phi(y) = \sigma^\nu(a)y + \sigma^\nu(b)$. Since K is inversive, there exist $g \in K$ such that $\phi(g) = ag + b$ and as in the $n = 0$ case, a complex number $d \in \mathbb{C}$ such that $f = g + du$. Since $f \notin K$, the constant d is nonzero. Then $u = (f - g)/d$ belongs to F^*. Let $k \in \mathbb{Q}$ be its valuation. Since $\phi(u) = au = cx^n u$, we find $k = k + n$, which is a contradiction. This ends the proof. ∎

4.2. Connected and irreducible Galois groups

As a next step, we consider the case where the Galois group is both connected and irreducible. The goal of this subsection is to prove the following proposition.

Proposition 4.9. *Let K, F, and (ϕ, σ) be defined as in Cases **2S**, **2Q**, and **2M**. Let us assume that $n \geq 2$, and let $f \in F$ be a nonzero solution to equation (1.1), which we are going to consider as a ϕ-system $\phi(Y) = AY$, where A is the corresponding companion matrix. If the Galois group of $\phi(Y) = AY$ over K is connected and irreducible, then f is σ-transcendental over K.*

Throughout this subsection we adopt the following conventions. As previously, K is one of the $\phi\sigma$-fields of Cases **2S**, **2Q**, or **2M**. We let L_A denote a σ-Picard–Vessiot extension for $\phi(Y) = AY$, where $A \in \mathrm{GL}_n(K)$. We define K_A as in Proposition 2.12. As we will see, Proposition 4.9 will be deduced from the following more general statement.

Proposition 4.10. *Let $A \in \mathrm{GL}_n(K)$ with $n \geq 2$. Assume that L_A is a field and the Galois group $\mathrm{Gal}(K_A|K)$ is irreducible. Then every nonzero solution $u \in L_A^n$ of the system $\phi(Y) = AY$ contains at least one coordinate that is σ-transcendental over K.*

Proof of Proposition 4.9. We argue by contradiction, assuming that f is σ-algebraic over K. With $\phi\sigma = \sigma\phi$, we deduce that all coordinates of the vector $(f, \dots, \phi^{n-1}(f))^\top$ are also σ-algebraic over K. Since $(f, \dots, \phi^{n-1}(f))^\top \in F^n$ is nonzero, Proposition 3.7 ensures the existence of positive integers r, s and a σ^s-Picard–Vessiot extension $L_{A_{[r]}}$ for $\phi^r(Y) = A_{[r]}Y$ over K that is a field, such that the vector $(f, \dots, \phi^{n-1}(f))^\top$ is the first column of a fundamental matrix U. Furthermore, Remark 3.8 ensures that the Galois group of $\phi^r(Y) = A_{[r]}(Y)$ over K is equal to $\mathrm{Gal}(K_A|K)$, for the latter is connected. By Lemma 2.10, all coordinates of the vector $(f, \dots, \phi^{n-1}(f))^\top$ are also σ^s-algebraic over K. Thus, Proposition 4.10 applies with ϕ replaced by ϕ^r, and provides a contradiction. ∎

Before proving Proposition 4.10, let us first recall some terminology from the theory of linear algebraic groups. The *radical* of a linear algebraic group (over an algebraically closed field of characteristic zero) is the largest connected solvable normal closed subgroup. A linear algebraic group is *semisimple* if it is connected and its radical is trivial. It

is *almost-simple* if it is nontrivial, semisimple and every proper normal closed subgroup is finite. A linear algebraic group is *simple* if it is nontrivial, semisimple and every proper normal closed subgroup is trivial. In particular, a simple group cannot be abelian.

Let $A \in \mathrm{GL}_n(K)$ and $d \in \mathbb{N}^*$. We say that the system $\phi(Y) = AY$ is σ^d-*isomonodromic* if there exists $\widetilde{A} \in \mathrm{GL}_n(K)$ such that $\phi(Y) = AY$ and $\sigma^d(Y) = \widetilde{A}Y$ are compatible, i.e. $\phi(\widetilde{A})A = \sigma^d(A)\widetilde{A}$. The following proposition is analogous to [13, Theorem 6.4] since, by Lemma 3.6, the base field K is relatively algebraically closed in L_A.

Proposition 4.11. *Let $A \in \mathrm{GL}_n(K)$. Assume that L_A is a field and $\mathrm{Gal}(K_A|K)$ is a simple linear algebraic group. If $\mathrm{Gal}^\sigma(L_A|K)$ is a proper σ-closed subgroup of $\mathrm{Gal}(K_A|K)$, then $\phi(Y) = AY$ is σ^d-isomonodromic for some $d \geq 1$.*

Proof. Let us fix a fundamental matrix $U \in \mathrm{GL}_n(L_A)$ to obtain embeddings of $G = \mathrm{Gal}^\sigma(L_A|K)$ and $\mathcal{G} = \mathrm{Gal}(K_A|K)$ into GL_n. We know from [26, Theorem 2.55] that $\phi(Y) = AY$ is σ^d-isomonodromic if and only if there exists $h \in \mathrm{GL}_n(\mathbb{C})$ such that $\sigma^d(\tau) = h\tau h^{-1}$ for all $\tau \in G(B)$ and all $\mathbb{C}$-σ-algebras B. It therefore suffices to find such d and h. According to the proof of [13, Theorem A.20] (see, in particular, the first displayed formula on page 114) such d and h exist for every σ-reduced Zariski dense σ-closed proper subgroup G of a simple algebraic group $\mathcal{G} \leq \mathrm{GL}_n$ defined over an inversive algebraically closed σ-field. Since G is Zariski dense in $\mathcal{G}$ by Proposition 2.12 and G is σ-reduced, i.e., $\sigma\colon \mathbb{C}\{G\} \to \mathbb{C}\{G\}$ is injective, by Lemma 3.9, this completes the proof. ∎

We refer to Example A.3 for the definition of $[\sigma]_{\mathbb{C}}\,\mathrm{Gal}(K_A|K)$ that is used in the following proposition.

Proposition 4.12. *Let $A \in \mathrm{GL}_n(K)$. Assume that L_A is a field. If $\mathrm{Gal}(K_A|K)$ is a simple linear algebraic group, then*

$$\mathrm{Gal}^\sigma(L_A|K) = [\sigma]_{\mathbb{C}}\,\mathrm{Gal}(K_A|K).$$

In particular, $\sigma\text{-tr.deg}(L_A|K) > 0$.

Proof. We argue by contradiction, assuming that $G = \mathrm{Gal}^\sigma(L_A|K)$ is properly contained in $[\sigma]_{\mathbb{C}}\mathcal{G}$, where $\mathcal{G} = \mathrm{Gal}(K_A|K)$. By Proposition 4.11, we obtain the existence of a matrix $\widetilde{A} \in \mathrm{GL}_n(K)$ and an integer $d \geq 1$ such that the systems $\phi(Y) = AY$ and $\sigma^d(Y) = \widetilde{A}Y$ are compatible. By Proposition 3.2, we deduce that the system $\phi(Y) = AY$ is equivalent over K to a system $\phi(Y) = A_1Y$, where $A_1 \in \mathrm{GL}_n(\mathbb{C})$. By [8, Lemma 2.1 and Remark 2.2], the Galois group of such a system is always abelian, providing a contradiction with the assumption that $\mathcal{G}$ is simple. Thus $G = [\sigma]_{\mathbb{C}}\mathcal{G}$.

Now, since $\mathcal{G}$ is a simple linear algebraic group, it cannot be finite and $\dim(\mathcal{G}) > 0$. Using Proposition 2.11 and Example A.9, we thus conclude that $\sigma\text{-tr.deg}(L_A|K) = \sigma\text{-dim}(G) = \dim(\mathcal{G}) > 0$. ∎

Proposition 4.13. *Let $A \in \mathrm{GL}_n(K)$. Assume that L_A is a field. Then either $\mathrm{Gal}(K_A|K)$ is a connected solvable linear algebraic group or $\sigma\text{-tr.deg}(L_A|K) > 0$.*

Proof. Lemma 3.9 ensures that $\mathcal{G} = \mathrm{Gal}(K_A|K)$ is connected. Let us assume that $\mathcal{G}$ is not solvable and let us show that $\sigma\text{-tr.deg}(L_A|K) > 0$. As $\mathcal{G}$ is not solvable, the radical $R(\mathcal{G})$ of $\mathcal{G}$ is a proper closed normal subgroup of $\mathcal{G}$ and $\mathcal{G}/R(\mathcal{G})$ is a nontrivial semisimple linear algebraic group. It follows from the structure theory of semisimple linear algebraic groups that any nontrivial semisimple linear algebraic groups is an almost-direct product of (finitely many) almost-simple groups [24, Theorem 21.51]. It follows that such a group has a simple quotient. Thus, there exists a normal closed subgroup $\mathcal{N}$ of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{N}$ is a simple linear algebraic group. By the second fundamental theorem of Galois theory (Proposition 2.5) there exist a positive integer n' and $A' \in \mathrm{GL}_{n'}(K)$ such that $(K_A)^{\mathcal{N}}$ is a Picard–Vessiot extension for $\phi(Y) = A'Y$ over K, with Galois group $\mathcal{G}/\mathcal{N}$. Given a fundamental matrix $U' \in \mathrm{GL}_{n'}((K_A)^{\mathcal{N}})$ of $\phi(Y) = A'Y$, we find that $L_{A'} = K\langle U'\rangle_\sigma \subseteq L_A$ is a σ-Picard–Vessiot extension for $\phi(Y) = A'Y$ over K. Since the Galois group of $\phi(Y) = A'Y$ is simple, it follows from Proposition 4.12 that $\sigma\text{-tr.deg}(L_{A'}|K) > 0$. Hence also $\sigma\text{-tr.deg}(L_A|K) > 0$, as desired. ∎

Proof of Proposition 4.10. Let $S_A \subset L_A$ denote the σ-Picard–Vessiot ring. As in Appendix A.2, we let $\pi(S_A) = \pi(S_A|K)$ denote the set of all elements of S_A that are σ-algebraic over K. Since L_A is a field, S_A is a domain. By Corollary A.18, $\pi(S_A)$ is a K-$\phi\sigma$-subalgebra of S_A. Let us assume by contradiction that all coordinates of u are σ-algebraic over K. Then $u \in \pi(S_A)^n$. Let $V \subseteq (S_A)^n$ denote the n-dimensional $\mathbb{C}$-vector space of all solutions of $\phi(Y) = AY$ in S_A^n. Since, by assumption, $u \in V \cap \pi(S_A)^n =: W$, we see that W is nonzero subspace of V. By Corollary A.18, $\pi(S_A)$ is stable under the action of $\mathrm{Gal}^\sigma(L_A|K)$ and so the same holds for W. Since the representation V of $\mathrm{Gal}(K_A|K)$ is irreducible, Lemma A.7 implies that the representation V of $\mathrm{Gal}^\sigma(L_A|K)$ is irreducible too. Thus $W = V$ and so $V \subseteq \pi(S_A)^n$. Hence, all entries of all elements of V are σ-algebraic over K. Since L_A is σ-generated over K by all these entries, it follows that $\sigma\text{-tr.deg}(L_A|K) = 0$. Thus, by Proposition 4.13, $\mathrm{Gal}(K_A|K)$ is a connected solvable linear algebraic group. However, by the Lie–Kolchin Theorem [24, Theorem 16.30], such a group stabilizes a line in any nonzero representation. Since $n \geq 2$, it cannot act irreducibly, providing a contradiction. ∎

4.3. The general case

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. We argue by induction on n. More precisely, our induction assumption reads as follows.

(H_n) For any positive integer k, if $f \in F$ is a solution to a linear ϕ^k-difference equation of order at most n with coefficients in K, then either f is σ-transcendental over K or $f \in K$.

Proposition 4.6 with $b = 0$ implies that (H_1) holds true. Let $n \geq 2$ and assume (H_{n-1}). Let $f \in F$ be a solution to a linear ϕ^k-difference equation of order n with coefficients in K.

Without any loss of generality, we can assume that $f \neq 0$ and $k = 1$. Considering the companion matrix A associated with this equation, Proposition 3.7 ensures the existence of positive integers r and s such that the following properties hold:

(a) The vector $(f, \phi(f), \dots, \phi^{n-1}(f))^\top$ is a solution to the system $\phi^r(Y) = A_{[r]}Y$.

(b) There exists a (ϕ^r, σ^s)-Picard–Vessiot extension L_A for $\phi^r(Y) = A_{[r]}Y$ that is a field and such that the vector $(f, \phi(f), \dots, \phi^{n-1}(f))^\top$ is the first column of a fundamental matrix $U \in \mathrm{GL}_n(L_A)$.

(c) The Galois group $\mathcal{G}$ of the system $\phi^r(Y) = A_{[r]}Y$ over K is connected.

By Lemma 2.10, without loss of generality, up to replacing σ by some power of σ, we may reduce to the case where $s = 1$. Let us first assume that $\mathcal{G}$ is irreducible. Since L_A is a field, Proposition 4.9 shows that f is σ-transcendental. Hence (H_n) holds. From now on, we assume that $\mathcal{G}$ is reducible and that f is σ-algebraic over K. Thus, it remains to prove that $f \in K$. Without loss of generality, we can assume that $r = 1$.

By Lemma 3.3, there exists a gauge transformation $T = (t_{i,j}) \in \mathrm{GL}_n(K)$ such that

$$\phi(T)AT^{-1} = \begin{pmatrix} A_1 & A_{1,2} \\ 0 & A_2 \end{pmatrix},$$

where $A_i \in \mathrm{GL}_{n_i}(K)$, $n_1 + n_2 = n$, and $n_2 < n$. Furthermore, let us assume that n_1 is minimal with this property. Set

$$g = \begin{pmatrix} g' \\ g'' \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = T \begin{pmatrix} f \\ \vdots \\ \phi^{n-1}(f) \end{pmatrix} \in F^n, \tag{4.8}$$

where

$$g' = \begin{pmatrix} g_1 \\ \vdots \\ g_{n_1} \end{pmatrix} \in F^{n_1} \quad \text{and} \quad g'' = \begin{pmatrix} g_{n_1+1} \\ \vdots \\ g_n \end{pmatrix} \in F^{n_2}.$$

Since the coefficients of T belong to K, it follows from (4.8) that $g \in K^n$ if and only if $f \in K$. From now on, we assume for contradiction that $f \notin K$. Hence, at least one coordinate of g does not belong to K.

We have

$$\phi \begin{pmatrix} g' \\ g'' \end{pmatrix} = \begin{pmatrix} A_1 & A_{1,2} \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} g' \\ g'' \end{pmatrix}. \tag{4.9}$$

Thus, g'' is a solution to the system $\phi(Y) = A_2Y$. Furthermore, since f is σ-algebraic over K, the g_i are also σ-algebraic over K. By (H_{n_2}) and Remark 2.1, we find that $g'' \in K^{n_2}$.

Let $\mathcal{G}_1$ denote the Galois group of the system $\phi(Y) = A_1Y$ over K. Let us prove that $\mathcal{G}_1$ is irreducible. Indeed, if $\mathcal{G}_1$ were reducible, by Lemma 3.3, there would exist a gauge transformation changing A_1 into a block upper triangular matrix, contradicting the minimality of n_1. Thus the Galois group of $\phi(Y) = A_1Y$ is irreducible.

Claim. *One has* $n_1 = 1$.

Proof of the claim. Let $S_A = K\{U, \frac{1}{\det(U)}\}_\sigma \subseteq L_A$ denote the σ-Picard–Vessiot ring for the system $\phi(Y) = AY$. Note that, by assumption, f belongs to S_A and thus all the g_i's also belong to S_A. We know that $g'' \in K^{n_2}$. Since, by assumption, $g \notin K^n$, we see that $g' \notin K^{n_1}$. It follows from the σ-Galois correspondence (Proposition 2.8) that there exist a $\mathbb{C}$-σ-algebra B and $\tau \in \mathrm{Gal}^\sigma(L_A/K)(B)$ (i.e. $\tau\colon S_A \otimes_{\mathbb{C}} B \to S_A \otimes_{\mathbb{C}} B$ is a $K \otimes_{\mathbb{C}} B$-$\phi\sigma$-automorphism) with $\tau(g') \neq g'$. By (4.9), $\phi(g') = A_1 g' + A_{1,2} g''$. As τ commutes with ϕ and fixes the elements of K, we see that $\tau(g')$ is also a solution to $\phi(Y) = A_1 Y + A_{1,2} g''$. Therefore $g' - \tau(g') \in (S_A \otimes_{\mathbb{C}} B)^n$ is a nonzero solution of $\phi(Y) = A_1 Y$. Since L_A is a field, S_A is an integral domain. By Corollary A.18, $\pi(S_A/K)$, the set of elements that are σ-algebraic over K and in S_A is invariant under the action $\mathrm{Gal}^\sigma(L_A|K)$ and is a σ-ring. Since $g' \in \pi(S_A/K)^{n_1}$, we find successively $\tau(g') \in (\pi(S_A/K) \otimes_{\mathbb{C}} B)^{n_1}$ and $g' - \tau(g') \in (\pi(S_A/K) \otimes_{\mathbb{C}} B)^{n_1}$. The latter is a nonzero solution of $\phi(Y) = A_1 Y$. It follows from Lemma 3.1 that $\phi(Y) = A_1 Y$ has a nonzero solution in $\pi(S_A/K)^{n_1}$. As the Galois group of $\phi(Y) = A_1 Y$ is irreducible and L_A is a field, Proposition 4.10 implies that $n_1 = 1$. ∎

As $n_1 = 1$, we see that $g' = g_1 \in F$ is a solution to the inhomogeneous linear order 1 equation

$$\phi(g_1) = A_1 g_1 + A_{1,2} g''.$$

Since $g_1 \in F$ is σ-algebraic over K and since A_1 and $A_{1,2} g''$ belong to K, Proposition 4.6 implies that $g_1 \in K$. It follows that $g = \binom{g_1}{g''} \in K^n$, and thus $f \in K$, providing a contradiction. ∎

Appendix A. Difference algebraic groups and their actions

In this appendix we collect some basic definitions and results concerning difference algebraic groups and their actions that are needed for the proof of Theorem 1.3. For a more detailed introduction to difference algebraic groups see [12, Appendix A] and [35, 37].

A.1. Difference algebraic groups

Difference algebraic groups are the group objects in the category of difference varieties. So we first introduce difference varieties. Throughout this appendix, C denotes an arbitrary σ-field (not necessarily of characteristic zero). The *σ-polynomial ring* $C\{y_1, \dots, y_n\}$ over C is the polynomial ring over C in the variables $y_1, \dots, y_n, \sigma(y_1), \dots, \sigma(y_n), \dots$ with action of σ extended from C as suggested by the names of the variables. If B is a C-σ-algebra and $x = (x_1, \dots, x_n) \in B^n$, then a σ-polynomial $f \in C\{y_1, \dots, y_n\}$ can be evaluated at x by replacing $\sigma^i(y_j)$ with $\sigma^i(x_j)$. For a subset F of $C\{y_1, \dots, y_n\}$ we denote the set of solutions of F in B^n with $\mathbb{V}_B(F)$. Note that $B \rightsquigarrow \mathbb{V}_B(F)$ is naturally a functor from the category of C-σ-algebras to the category of sets.

A difference variety is in essence the set of solutions of a set of σ-polynomials. There are two reasons why we cannot simply consider the solutions in some fixed large σ-field extension of C. Firstly, there is no suitable notion of a σ-closure of a σ-field, similar to the algebraic closure of a field or the differential closure of a differential field. Secondly, there are many systems of algebraic difference equations where the solutions in σ-field extensions reflect very poorly the content of the equations. For example, the system $y + \sigma(y) = 1$, $y\sigma(y) = 0$ has no solution in a σ-field. However, $x = (1, 0, 1, 0, \ldots) \in \mathbb{C}^{\mathbb{N}}$ is a solution in the σ-ring of sequences. (Here the action of σ on $\mathbb{C}^{\mathbb{N}}$ is given by $\sigma((x_i)_{i\in\mathbb{N}}) = (x_{i+1})_{i\in\mathbb{N}}$.) We therefore define difference varieties as functors, rather than subsets of some $\widetilde{C}^n$ for some large σ-field extension $\widetilde{C}$ of C.

Definition A.1. A *σ-variety* X (over C) is a functor from the category of C-σ-algebras to the category of sets that is isomorphic to a functor of the form $B \rightsquigarrow \mathbb{V}_B(F)$ for some $n \geq 1$ and $F \subset C\{y_1, \ldots, y_n\}$. A *morphism* of σ-varieties is a morphism of functors.

The functor X given by $B \rightsquigarrow \mathbb{V}_B(F)$ is representable, i.e. there exists a C-σ-algebra $C\{X\}$ such that X is isomorphic to $\mathrm{Hom}(C\{X\}, -)$. Indeed, to specify a solution of F in B^n is equivalent to specifying a morphism $C\{y_1, \ldots, y_n\} \to B$ of C-σ-algebras that sends all elements in F to zero. The latter is equivalent to specifying a morphism $C\{y_1, \ldots, y_n\}/[F] \to B$, where $[F] = (\sigma^i(f) : f \in F,\ i \in \mathbb{N}) \subseteq C\{y_1, \ldots, y_n\}$ denotes the σ-ideal generated by F. We can therefore choose $C\{X\} = C\{y_1, \ldots, y_n\}/[F]$. We will usually identify X and $\mathrm{Hom}(C\{X\}, -)$.

A C-σ-algebra B is *finitely σ-generated* if there exists a finite subset M of B such that $B = C\{M\}_\sigma$. Clearly, the C-σ-algebra $C\{X\} = C\{y_1, \ldots, y_n\}/[F]$ is finitely σ-generated. Conversely, every finitely σ-generated C-σ-algebra is isomorphic to one of the form $C\{y_1, \ldots, y_n\}/[F]$. It follows that a functor X from the category of C-σ-algebras to the category of sets is a σ-variety if and only if it is representable by a finitely σ-generated C-σ-algebra. By the Yoneda lemma, this finitely σ-generated C-σ-algebra is uniquely determined by X up to isomorphism of C-σ-algebras. We denote it by $C\{X\}$ and call the *coordinate ring* of X. Moreover, a morphism $\eta\colon X \to Y$ of σ-varieties corresponds to a morphism $\eta^*\colon C\{Y\} \to C\{X\}$ of C-σ-algebras. Note that η can be recovered from η^* via $\eta_B\colon X(B) = \mathrm{Hom}(C\{X\}, B) \to \mathrm{Hom}(C\{Y\}, B) = Y(B),\ \psi \mapsto \eta^* \circ \psi$, for any C-σ-algebra B. In summary, we see that the category of σ-varieties over C is anti-equivalent to the category of finitely σ-generated C-σ-algebras. We will usually identify a σ-variety X with $\mathrm{Hom}(C\{X\}, -)$.

The category of σ-varieties has products. Indeed, if X and Y are σ-varieties, then the functor $X \times Y$ given by $B \rightsquigarrow X(B) \times Y(B)$ is represented by $C\{X\} \otimes_C C\{Y\}$. Therefore we can make the following definition.

Definition A.2. A *σ-algebraic group* G (over C) is a group object in the category of σ-varieties (over C), i.e. a σ-variety G together with morphisms of σ-varieties $G \times G \to G$ (multiplication), $G \to G$ (inversion), $1 \to G$ (identity) satisfying the group axioms.

Here 1 is the functor that associates the trivial group to any C-σ-algebra B. In particular, $G(B)$ is a group for any C-σ-algebra B. A morphism $\eta\colon G \to H$ of σ-algebraic groups

is a morphism of σ-varieties that respects the group structure, i.e. $\eta_B\colon G(B) \to H(B)$ is a morphism of groups for every C-σ-algebra B.

A C-σ-*Hopf algebra* is a C-σ-algebra with the structure of a C-Hopf algebra such that the Hopf algebra structure maps are morphisms of C-σ-algebras. Since the category of σ-varieties over C is anti-equivalent to the category of finitely σ-generated C-σ-algebras, it follows that the category of σ-algebraic groups over C is equivalent to the category of C-σ-Hopf algebras that are finitely σ-generated over C.

Example A.3. If $\mathcal{G}$ is an affine algebraic group over C, then the functor $[\sigma]_C\mathcal{G}$ given by $B \rightsquigarrow \mathcal{G}(B^\sharp)$ is a σ-algebraic group. Here $B^\sharp$ denotes the C-algebra obtained from the C-σ-algebra by forgetting σ. For example, if V is a finite-dimensional vector space and $\mathcal{G} = \mathrm{GL}_V$, then $[\sigma]_C\mathrm{GL}_V$ is the functor that associates to a C-σ-algebra B the group of B-linear automorphisms of $V \otimes_C B$. Fixing a basis of V, we see that $[\sigma]_C\mathrm{GL}_V$ is represented by

$$C\{[\sigma]_C\mathrm{GL}_{n,C}\} = C\{T_{ij}, (\det T)^{-1}\}_\sigma,$$

the σ-polynomial ring in the variables T_{ij} $(1 \le i, j \le n)$ localized at the multiplicatively closed subset generated by $\det T$, $\sigma(\det T), \ldots$. See [37, Example 2.6 and Section 1.3] for a description of $C\{[\sigma]_C\mathcal{G}\}$ in the general case. In particular, we have an injective morphism $C[\mathcal{G}] \to C\{[\sigma]_C\mathcal{G}\}$ of C-Hopf algebras.

See [37, Section 2] for a list of further examples of σ-algebraic groups.

If X is a σ-variety and I is a σ-ideal in $C\{X\}$, we can define a subfunctor Y of X by $Y(B) = \{\psi \in \mathrm{Hom}(C\{X\}, B) : \psi(I) = 0\} \subset \mathrm{Hom}(C\{X\}, B) = X(B)$ for any C-σ-algebra B. Note that Y is a σ-variety since it is represented by $C\{X\}/I$. We call Y the *σ-closed σ-subvariety* of X defined by I. For example, if $F \subset F' \subset C\{y_1, \ldots, y_n\}$, then the functor Y given by $B \rightsquigarrow \mathbb{V}_B(F')$ is a σ-closed σ-subvariety of the functor X given by $B \rightsquigarrow \mathbb{V}_B(F)$, since it corresponds to all morphisms on $C\{X\} = C\{y_1, \ldots, y_n\}/[F]$ that annihilate the image of $[F']$ in $C\{X\}$. For a given σ-variety X, the σ-closed σ-subvarieties of X are in bijection with the σ-ideals of $C\{X\}$ (see [37, Lemma 1.4]).

Definition A.4. A *σ-closed subgroup* H of a σ-algebraic group G is a σ-closed σ-subvariety H of G such that $H(B)$ is a subgroup of $G(B)$ for any C-σ-algebra B. A σ-closed subgroup of an affine algebraic group $\mathcal{G}$ is a σ-closed subgroup of $[\sigma]_C\mathcal{G}$.

Definition A.5. Let $\mathcal{G}$ be an affine algebraic group and let G be a σ-closed subgroup of $\mathcal{G}$ (in particular, $G(B) \le \mathcal{G}(B)$ for all C-σ-algebras B). Then G is *Zariski dense in* $\mathcal{G}$ if every regular function on $\mathcal{G}$ that vanishes on G is zero, i.e. for $f \in C[\mathcal{G}]$ with $f(g) = 0$ for all $g \in G(B)$ and all C-σ-algebras B, we have $f = 0$.

In the above definition, we can identify $\mathcal{G}(B)$ with the set of C-algebra morphisms from $C[\mathcal{G}]$ to B. For $f \in C[\mathcal{G}]$ and $g\colon C[\mathcal{G}] \to B$ a C-algebra morphism, we then have $g(f) = f(g)$.

A σ-closed subgroup G of $\mathcal{G}$ gives rise to the composition of two morphisms $\psi\colon C[\mathcal{G}] \to C\{[\sigma]_C\mathcal{G}\} \to C\{G\}$ of C-algebras. Note that for a C-σ-algebra B, the inclusion $G(B) = \mathrm{Hom}(C\{G\}, B) \to \mathcal{G}(B^\sharp) = \mathrm{Hom}(C[\mathcal{G}], B^\sharp)$ is given by precomposing

with ψ. The morphism ψ is injective if and only if G is Zariski dense in $\mathcal{G}$. (This follows by choosing $B = C\{G\}$ and $g = \mathrm{id} \in G(B) = \mathrm{Hom}(C\{G\}, C\{G\})$.)

Definition A.6. Let G be a σ-algebraic group over C. A *representation* of G is a finite-dimensional C-vector space V together with a morphism $G \to [\sigma]_C \mathrm{GL}_V$ of σ-algebraic groups. In particular, for a C-σ-algebra B, every $g \in G(B)$ acts on $V \otimes_C B$ through a B-linear automorphism.

A C-subspace W of V is a *subrepresentation* of V if $g\colon V \otimes_C B \to V \otimes_C B$ maps $W \otimes_C B$ into $W \otimes_C B$ for every $g \in G(B)$ and every C-σ-algebra B.

The representation V of G is *irreducible* if the only subrepresentations of V are the zero subspace and V itself.

A morphism $f\colon V \to W$ of representations of G is a C-linear map such that the diagram

$$\begin{array}{ccc} V \otimes_C B & \xrightarrow{f \otimes B} & W \otimes_C B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V \otimes_C B & \xrightarrow{f \otimes B} & W \otimes_C B \end{array}$$

commutes for every $g \in G(B)$ and every C-σ-algebra B. As for affine algebraic groups (see, for instance, [34, Section 3.2]), the category of representations of G is equivalent to the category of finite-dimensional comodules for the Hopf algebra $C\{G\}$.

Lemma A.7. *Let V be a finite-dimensional C-vector space and let $\mathcal{G}$ be a closed subgroup of GL_V. If G is a σ-closed Zariski dense subgroup of $\mathcal{G}$, then any G-subrepresentation of V is a $\mathcal{G}$-subrepresentation of V. In particular, if V is irreducible as a representation of $\mathcal{G}$, then V is irreducible as a representation of G.*

Proof. Since G is Zariski dense in $\mathcal{G}$, we have $C[\mathcal{G}] \subset C\{G\}$. The comodule structure $\rho\colon V \to V \otimes_C C\{G\}$ corresponding to the representation of G is obtained from the comodule structure $\overline{\rho}\colon V \to V \otimes_C C[\mathcal{G}]$ corresponding to the representation of $\mathcal{G}$ by composing with the inclusion $V \otimes_C C[\mathcal{G}] \to V \otimes_C C\{G\}$. A C-subspace W of V is a G-subrepresentation if and only if $\rho(W) \subset W \otimes_C C\{G\}$. Since ρ factors through $V \otimes_C C[\mathcal{G}]$, this implies $\overline{\rho}(W) \subset W \otimes_C C[\mathcal{G}]$, i.e. W is a $\mathcal{G}$-subrepresentation. ∎

The following definition introduces the σ-dimension of a σ-algebraic group.

Definition A.8. Let B be a finitely σ-generated C-σ-algebra and let $M \subseteq B$ be a finite set such that $B = C\{M\}_\sigma$. For $i \in \mathbb{N}$ let $d_i \in \mathbb{N}$ denote the Krull dimension of the C-algebra $C[M, \sigma(M), \ldots, \sigma^i(M)]$. One can show (see [36, Theorem 2.2]) that $\sigma\text{-dim}(B) = \lim_{i\to\infty} \frac{d_i}{i+1}$ exists and does not depend on the choice of M. The *σ-dimension* $\sigma\text{-dim}(G)$ of a σ-algebraic group G is $\sigma\text{-dim}(C\{G\})$.

In general, $\sigma\text{-dim}(B)$ need not be an integer. However, one can show [36, Theorem 5.1] that $\sigma\text{-dim}(G)$ is always an integer. From this it follows that our definition of the σ-dimension of a σ-algebraic group agrees with the somewhat more complicated definition given in [12, Definition A.25].

Example A.9. If $\mathcal{G}$ is a linear algebraic group over C, then $\sigma\text{-dim}([\sigma]_C\mathcal{G}) = \dim(\mathcal{G})$ by [37, Example 3.10].

A.2. Difference algebraic elements

The aim of this second part of the appendix is to show that, at least under some mild technical assumptions, the subring $\pi(S|K)$ of a σ-Picard–Vessiot ring S over K consisting of all elements of S that are σ-algebraic over K, is stable under the action of the σ-Galois group G. It is obvious that every K-$\phi\sigma$-automorphism of S maps $\pi(S|K)$ into itself, i.e. $\pi(S|K)$ is stable under $G(C)$, where $C = K^{\phi}$. It is much less obvious that $\pi(S|K)$ is stable under the action of G in the strong sense that for every C-σ-algebra B, every element of $G(B)$ maps $\pi(S|K) \otimes_C B$ into $\pi(S|K) \otimes_C B$.

Let C be a σ-field and B a C-σ-algebra. Recall that $a \in B$ is called *σ-algebraic over C* if a satisfies a nonzero σ-polynomial equation over C. We define

$$\pi(B|C) = \{b \in B : b \text{ is } \sigma\text{-algebraic over } C\}.$$

The following example shows that $\pi(B|C)$ is in general not a subring of B.

Example A.10. Let $C = \mathbb{Q}$ (with $\sigma = \mathrm{id}$) and let $K = \mathbb{Q}(x_1, y_1, x_2, y_2, \ldots)$ be a rational function field in infinitely many variables. Let $B = K^{\mathbb{N}}$ be the ring of sequences in K that we equip with a structure of σ-ring with $\sigma((a_n)_{n\in\mathbb{N}}) = (a_{n+1})_{n\in\mathbb{N}}$. Then $x = (x_1, 0, x_2, 0, \ldots) \in B$ and $y = (0, y_1, 0, y_2, \ldots) \in B$ are σ-algebraic over C because $x\sigma(x) = 0$ and $y\sigma(y) = 0$. However, $x + y = (x_1, y_1, x_2, y_2, \ldots) \in B$ is in general not σ-algebraic over C.

However, it follows from [20, Theorem 4.1.2 (iii)] that if B is a σ-domain, (i.e. B is an integral domain and $\sigma\colon B \to B$ is injective), then $\pi(B|C)$ is a C-σ-subalgebra of B. Because of the pathology exhibited in Example A.10, we modify the definition of $\pi(B|C)$, but in such a way that it remains unchanged for σ-domains.

Definition A.11. Let B be a C-σ-algebra. An element $b \in B$ is *σ-bounded* (over C) if the sequence $(\dim(C[b, \sigma(b), \ldots, \sigma^i(b)]))_{i\in\mathbb{N}}$ is bounded. We set

$$\mu(B|C) = \{b \in B : b \text{ is } \sigma\text{-bounded}\}.$$

The following lemma relates the Krull dimension to the maximal number of algebraically independent elements. It will be implicitly used several times in what follows.

Lemma A.12 ([36, Proposition 2.1]). *Let us consider a C-algebra B and let us assume the existence of a finite set $A \subset B$ such that $B = C[A]$. Then*

$$\dim(B) = \max\{|D| : D \subset A,\ D \text{ is algebraically independent over } C\}.$$

The following lemma explains the connection between $\pi(B|C)$ and $\mu(B|C)$.

Lemma A.13. *Let B be a C-σ-algebra. Then $\mu(B|C) \subset \pi(B|C)$. Moreover, if B is a σ-domain, then $\mu(B|C) = \pi(B|C)$.*

Proof. If $b \in B$ is σ-bounded, then $\dim(C[b, \sigma(b), \dots, \sigma^i(b)]) < i + 1$ for some $i \geq 1$. Thus $b, \sigma(b), \dots, \sigma^i(b)$ are algebraically dependent over C and so b is σ-algebraic. Therefore $\mu(B|C) \subset \pi(B|C)$.

Let us now assume that B is a σ-domain and let us prove the reverse inclusion. Let $b \in B$ be σ-algebraic over C. Because B is a σ-domain, its fraction field L is naturally a σ-field extension of C and we can consider the σ-subfield $C\langle b\rangle_\sigma$ of L. According to [20, Corollary 4.1.18], the transcendence degree $m = \mathrm{tr.deg}(C\langle b\rangle_\sigma|C)$ is finite. Thus $\dim(C[b, \sigma(b), \dots, \sigma^i(b)]) \leq m$ for all $i \geq 0$. ∎

We now establish some properties of $\mu(B|C)$ that will permit us later on to deduce properties of $\pi(B|C)$ when B is a σ-domain.

Lemma A.14. *Let B be a C-σ-algebra. Then $\mu(B|C)$ is a C-σ-subalgebra of B.*

Proof. Let $a, b \in B$ be σ-bounded, say

$$\dim(C[a, \dots, \sigma^i(a)]) \leq m \quad \text{and} \quad \dim(C[b, \dots, \sigma^i(b)]) \leq n \quad \text{for all } i \geq 0.$$

Then $\dim(C[a, b, \dots, \sigma^i(a), \sigma^i(b)]) \leq mn$ for all $i \geq 0$. Since $C[a + b, \dots, \sigma^i(a + b)]$ and $C[ab, \dots, \sigma^i(ab)]$ are both contained in the latter C-algebra, it follows that we have $\dim(C[a + b, \dots, \sigma^i(a + b)]) \leq mn$ and $\dim(C[ab, \dots, \sigma^i(ab)]) \leq mn$. Thus $\mu(B|C)$ is a subring of B. If $b \in B$ is σ-bounded, then clearly $\sigma(b)$ is also σ-bounded. Thus $\mu(B|C)$ is a C-σ-subalgebra of B. ∎

The following lemma is a slight generalization of [35, Lemma 6.27]. Recall from [11, Chapter V, §17, No. 3, Definition 1] that an algebra B over a field C is called *regular* if $B \otimes_C D$ is an integral domain for every field extension D of C.

Lemma A.15. *Let C be an inversive σ-field (i.e. $\sigma\colon C \to C$ is surjective) and let D be a σ-field extension of C such that D is a regular field extension of C. Then there exists a σ-field extension E of D such that every element of E that is fixed by all C-σ-field automorphisms of E lies in C, i.e. $E^{\mathrm{Aut}^\sigma(E|C)} = C$.*

Proof. We first establish the following claim. There exists a σ-field E of D satisfying the following properties:

- For every $d \in D \setminus C$ there exists a C-σ-automorphism τ of E with $\tau(d) \neq d$.
- Every C-σ-automorphism of D extends to a C-σ-automorphism of E.
- The field extension $E|C$ is regular.

Because D is a regular extension of C, the ring $D \otimes_C D$ is an integral domain [11, Chapter V, §17, No. 3, Proposition 2]. Since C is inversive, it follows from [29, Corollary 1.6] that $\sigma\colon D \otimes_C D \to D \otimes_C D$ is injective. Therefore the field E of fractions of $D \otimes_C D$ is naturally a σ-field. We consider E as a σ-field extension of D via the embedding $d \mapsto d \otimes 1$. The C-σ-automorphism τ of E determined by $\tau(d_1 \otimes d_2) = d_2 \otimes d_1$ satisfies $\tau(d) \neq d$ for every $d \in D \setminus C$. Moreover, every C-σ-automorphism ψ of D extends to a C-σ-automorphism of E by $\psi(d_1 \otimes d_2) = \psi(d_1) \otimes d_2$. The tensor product of two

regular algebras is regular [11, Chapter V, §17, No. 3, Prop. 3] and the field of fractions of a regular algebra is regular [11, Chapter V, §17, No. 4, Cor. to Prop. 4]. It thus follows that E is a regular extension of C and the claim is established.

Let us now prove the lemma. By the above claim there exists a σ-field extension E_1 of D such that every $d \in D \setminus C$ can be moved by a C-σ-field automorphism of E_1, every C-σ-automorphism of D extends over E_1 and $E_1|C$ is regular. Applying the claim again (with $E_1|C$ in place of $D|C$) yields a σ-field extension E_2 of E_1 such that every element of $E_1 \setminus C$ can be moved by a C-σ-field automorphism of E_2, every C-σ-automorphism of E_1 extends over E_2 and $E_2|C$ is regular. Continuing, we obtain an ascending chain $C \subset D \subset E_1 \subset E_2 \subset \cdots$ of σ-field extensions. The union $E = \bigcup_{i=1}^{\infty} E_i$ has the desired property. ∎

Lemma A.16. *Let C be an inversive σ-field and let B be a C-σ-algebra. Let $D|C$ be an extension of σ-fields such that D is a regular field extension of C. Then*

$$\mu(B \otimes_C D|D) = \mu(B|C) \otimes_C D.$$

Proof. If $b \in B$ is σ-bounded over C, then $b \otimes 1 \in B \otimes_C D$ is σ-bounded over D. Thus $\mu(B|C) \subseteq \mu(B \otimes_C D|D)$. Since $\mu(B \otimes_C D|D)$ is a D-σ-subalgebra of $B \otimes_C D$ (Lemma A.14), it follows that $\mu(B|C) \otimes_C D \subset \mu(B \otimes_C D|D)$. The crucial step to prove the reverse inclusion is to show that $\mu(B \otimes_C D|D)$ descends to C.

By Lemma A.15, there exists a σ-field extension E of D such that the fixed field of the group $\mathrm{Aut}^{\sigma}(E|C)$ of all σ-field automorphisms of $E|C$ is C. For $\gamma \in \mathrm{Aut}^{\sigma}(E|C)$ we also denote the induced automorphism $B \otimes_C E \to B \otimes_C E$, $b \otimes \lambda \to b \otimes \gamma(\lambda)$, with γ. Note that $\gamma\colon B \otimes_C E \to B \otimes_C E$ is an automorphism of C-σ-algebras that maps E isomorphically onto E. Assume that $a \in B \otimes_C E$ is σ-bounded over E. Then γ maps $E\{a\}_{\sigma}$ isomorphically onto $E\{\gamma(a)\}_{\sigma}$ and $E[a, \ldots, \sigma^i(a)]$ isomorphically onto $E[\gamma(a), \ldots, \sigma^i(\gamma(a))]$ for every $i \geq 0$. It follows that $\gamma(a)$ is σ-bounded over E. Consequently, $\mu(B \otimes_C E|E) \subseteq B \otimes_C E$ is stable under the action of $\mathrm{Aut}^{\sigma}(E|C)$.

Let B' be the subset of all elements of $\mu(B \otimes_C E|E)$ fixed under the $\mathrm{Aut}^{\sigma}(E|C)$-action. It is a C-σ-subalgebra of B. Therefore (cf. [11, Cor. to Prop. 6, Chapter V, §10.4, A.V. 63]) $\mu(B \otimes_C E|E) = B' \otimes_C E$. If $b \otimes 1 \in B \otimes_C E$ is σ-bounded over E, then b has to be σ-bounded over C. Therefore $B' \subset \mu(B|C)$ and it follows that $\mu(B \otimes_C E|E) \subset \mu(B|C) \otimes_C E$ and so $\mu(B \otimes_C E|E) = \mu(B|C) \otimes_C E$. Thus

$$\mu(B \otimes_C D|D) \subset \mu(B \otimes_D E|E) = \mu(B|C) \otimes_C E.$$

But as $\mu(B \otimes_C D|D) \subset B \otimes_C D$, we indeed have the desired inclusion $\mu(B \otimes_C D|D) \subset \mu(B|C) \otimes_C D$. ∎

Proposition A.17. *Let K be a $\phi\sigma$-field of characteristic zero such that $\sigma\colon K \to K$ is surjective and $C = K^{\phi}$ is algebraically closed. Let S be a σ-Picard–Vessiot ring over K and let G be the σ-Galois group. Assume that G is σ-integral, i.e. $C\{G\}$ is an integral domain and $\sigma\colon C\{G\} \to C\{G\}$ is injective. Then $\mu(S|K) \subset S$ is stable under the G-action, i.e. for every C-σ-algebra B, every $\tau \in G(B)$ maps $\mu(S|K) \otimes_C B$ into $\mu(S|K) \otimes_C B$.*

Proof. It suffices to show that the co-action $\rho\colon S \to S \otimes_C C\{G\}$ maps $\mu(S|K)$ into $\mu(S|K) \otimes_C C\{G\}$. Since $\rho(\mu(S|K)) \subset \mu(S \otimes_C C\{G\}|K)$, it suffices to show that $\mu(S \otimes_C C\{G\}|K) \subseteq \mu(S|K) \otimes_C C\{G\}$. Because $\sigma\colon K \to K$ is surjective, also $\sigma\colon C \to C$ is surjective. As C is algebraically closed and inversive, also $K\{G\} = K \otimes_C C\{G\}$ is a σ-domain [12, Cor. A.14 (ii)]. Let D denote the field of fractions of $K\{G\}$. Every integral domain over an algebraically closed field is regular [11, Chapter V, §17, No. 5, Cor. 2]. Thus $C\{G\}$ is regular over C. As regularity is preserved under base change [11, Chapter V, §17, No. 3, Prop. 3], we see that $K \otimes_C C\{G\}$ is regular over K. Finally, since the field of fractions of a regular algebra is regular [11, Chapter V, §17, No. 4, Cor. to Prop. 4], we can conclude that D is a regular field extension of K. We can therefore apply Lemma A.16 to deduce that $\mu(S \otimes_K D|D) = \mu(S|K) \otimes_K D$. So

$$\mu(S \otimes_C C\{G\}|K) = \mu(S \otimes_K K\{G\}|K) \subseteq \mu(S \otimes_K D|D) = \mu(S|K) \otimes_K D.$$

Therefore

$$\begin{aligned}\mu(S \otimes_K K\{G\}|K) &\subseteq (\mu(S|K) \otimes_K D) \cap (S \otimes_K K\{G\}) \\ &= \mu(S|K) \otimes_K K\{G\} = \mu(S|K) \otimes_C C\{G\}\end{aligned}$$

as desired. ∎

Corollary A.18. *Let K be one of the $\phi\sigma$-fields in Cases **2S**, **2Q**, and **2M**. Let L_A be a σ-Picard–Vessiot extension for $\phi(Y) = AY$ with $A \in \mathrm{GL}_n(K)$ and let $S_A \subset L_A$ be the σ-Picard–Vessiot ring. Assume that L_A is a field. Then $\pi(S_A|K)$ is a K-$\phi\sigma$-subalgebra of S_A and is stable under the action of the σ-Galois group* $\mathrm{Gal}^\sigma(L_A|K)$.

Proof. In all cases $\sigma\colon K \to K$ is surjective and $C = K^\phi$ is algebraically closed and of characteristic zero. By Lemma 3.9, the σ-Galois group is σ-integral. We can therefore apply Proposition A.17 to deduce that $\mu(S_A|K)$ is stable under the action of the σ-Galois group. By Lemma A.13, $\pi(S_A|K) = \mu(S_A|K)$ and $\mu(S_A|K)$ is a σ-subring by Lemma A.14. Because ϕ and σ commute, $\pi(S_A|K)$ is also stable under ϕ. ∎

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