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Hitchin representations of Fuchsian groups

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Abstract. We survey the theory of Hitchin representations of closed surface groups into $\mathsf{PSL}(d,\mathbb{R})$ with a focus on their dynamical and geometric properties. We then describe recent extensions of this work to study Hitchin representations of co-finite area Fuchsian groups. The motivation for this recent work is a conjecture about the geometry of the augmented Hitchin component.

Dedicated to Dennis Sullivan on the occasion of his 80th birthday. Dennis was very kind to me when I was a feckless young mathematician and he continues to be an inspiration now that I am a feckless old mathematician.

1. Introduction

Nigel Hitchin [57] used the theory of Higgs bundles to enumerate the components of the "character variety" of (conjugacy classes of) representations of a closed surface group into $\mathsf{PSL}(d,\mathbb{R})$. He identified a component which is topologically a cell. When $d = 2$, this component is the classical Teichmüller space of a surface, so he called this component the Teichmüller component. This component is now known as the Hitchin component and representations in this component are known as Hitchin representations. In his paper, he makes a comment which served as one of the primary motivations for the new field of higher (rank) Teichmüller theory.

> *"Unfortunately, the analytical point of view used for the proofs gives no indication of the geometrical significance of the Teichmüller component."* Nigel Hitchin [57]

This challenge was taken up by François Labourie [66], from a dynamical viewpoint, and Vladimir Fock and Alexander Goncharov [47], from a more algebraic viewpoint. As one consequence, they were able to show that all Hitchin representations are discrete and faithful, and Labourie shows that they are quasi-isometric embeddings. Subsequently, Andres Sambarino [105] constructed Anosov flows which encoded spectral data associated to a Hitchin representation. This allowed him to invoke results from thermodynamic formalism to obtain counting and equidistribution results for Hitchin representations. Martin Bridgeman, Dick Canary, François Labourie and Andres Sambarino [22, 23], built

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on Sambarino's earlier work and work of Curt McMullen [91] in the classical setting to construct mapping class group invariant analytic Riemannian metrics on the Hitchin component which generalize the Weil–Petersson metric on Teichmüller space. These metrics are called pressure metrics. Nicolas Tholozan [120] described an embedding of the Hitchin component into the Teichmüller space of foliated complex structures on the unit tangent bundle of the surface, first studied by Dennis Sullivan [116], so that the pullback of a "Weil–Petersson metric" on the Teichmüller space is the simple root pressure metric. (This is a highly selective history which reflects the focus of the author and hence of this paper.)

If one pinches a collection of disjoint curves in a hyperbolic surface, one naturally obtains a (possibly disconnected) cusped finite area hyperbolic surface. Bill Abikoff [1] bordified Teichmüller space by appending all such limiting surfaces and the result is known as augmented Teichmüller space. The "strata at infinity" are naturally Teichmüller spaces of (possibly disconnected) finite area hyperbolic surfaces. This space itself is not well behaved topologically, e.g. it is not locally compact, but one may view it as the "universal cover" of the Deligne–Mumford compactification of moduli space. More explicitly, the quotient of augmented Teichmüller space by the action of the mapping class group may be identified with the Deligne–Mumford compactification of Moduli space. Howard Masur [89] showed that the metric completion of Teichmüller space with the Weil–Petersson metric may be identified with augmented Teichmüller space.

In recent years, the author has been fascinated by the goal of developing a theory of an "augmented Hitchin component" and proving that it arises as the metric completion of the Hitchin component with respect to a pressure metric. The analogy is especially compelling when $d = 3$, as Hitchin representations are holonomy maps of convex (real) projective structures (see Choi–Goldman [38]) and points in the augmented Hitchin component conjecturally correspond to (possibly disconnected) finite area convex projective surfaces. John Loftin and Tengren Zhang [83] worked out the topological aspects of the augmented Hitchin component when $d = 3$ and obtain local parametrizations of neighborhoods of the "strata at infinity." In collaboration with Tengren Zhang and Andy Zimmer [33], we developed a theory of Hitchin representations of general geometrically finite Fuchsian groups. In collaboration with Harry Bray, Nyima Kao and Giuseppe Martone [18], we developed a dynamical framework to establish parallels of the counting and equidistribution results of Sambarino. In subsequent work [19], we combine these results to construct pressure metrics on components of Hitchin representations of Fuchsian lattices, which arise as the strata at infinity for the augmented Hitchin component.

The first part of this paper will focus on the developments of the classical Hitchin component, while the second portion will describe a conjectural geometric picture of the augmented Hitchin component and describe the progress made towards this still elusive goal.

2. The dynamical viewpoint of Labourie

François Labourie [66] introduced techniques from Anosov dynamics to study the flow on the flat bundle associated to a Hitchin representation. We recall some notation before describing his work.

Throughout this paper S will be a closed, orientable, connected surface of genus $g \geq 2$. Let $\tau_d \colon \mathsf{PSL}(2,\mathbb{R}) \to \mathsf{PSL}(d,\mathbb{R})$ be the irreducible representation, which is unique up to conjugation. A representation $\rho \colon \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ is said to be *d-Fuchsian* if it is (conjugate to) the result of post-composing a Fuchsian representation $\rho_0 \colon \pi_1(S) \to \mathsf{PSL}(2,\mathbb{R})$ (i.e. a discrete, faithful representation) with the irreducible representation τ_d. A representation $\rho \colon \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ is said to be a *Hitchin representation* if it can be continuously deformed to a d-Fuchsian representation. One then defines the *Hitchin component* $\mathcal{H}_d(S)$ to be the space of $\mathsf{PGL}(d,\mathbb{R})$-conjugacy classes of Hitchin representations, i.e.

$$\mathcal{H}_d(S) \subset \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(d,\mathbb{R}))/\mathsf{PGL}(d,\mathbb{R}).$$

Hitchin proved that $\mathcal{H}_d(S)$ is a cell.

Theorem 2.1 (Hitchin [57]). *The Hitchin component $\mathcal{H}_d(S)$ is a real analytic manifold which is (real analytically) diffeomorphic to $\mathbb{R}^{(d^2-1)(2g-2)}$.*

The *Fuchsian locus* in $\mathcal{H}_d(S)$ consists of (conjugacy classes of) d-Fuchsian representations and is an embedded copy of the Teichmüller space $\mathcal{T}(S)$ of S.

We may identify $S = \mathbb{H}^2/\Gamma$ as a hyperbolic surface, where $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$, so Γ is identified with $\pi_1(S)$. The unit tangent bundle T^1S of S is then the quotient $T^1\mathbb{H}^2/\Gamma$ of the unit tangent bundle of $\mathbb{H}^2$. Hitchin observes that every Hitchin representation $\rho \colon \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ lifts to a representation $\widehat{\rho} \colon \pi_1(S) \to \mathsf{SL}(d,\mathbb{R})$ (see Culler [41] for general criteria guaranteeing lifting which apply in this case). The *flat bundle* associated to a representation $\rho \colon \pi_1(S) \to \mathsf{SL}(d,\mathbb{R})$ is formed as

$$E_\rho = (T^1\mathbb{H}^2 \times \mathbb{R}^d)/\Gamma,$$

where the action on the first factor is the standard action of Γ on $T^1\mathbb{H}^2$ and the action on $\mathbb{R}^d$ is given by $\widehat{\rho}(\Gamma)$.

The geodesic flow

$$\{\phi_t \colon T^1S \to T^1S\}_{t\in\mathbb{R}}$$

lifts to the geodesic flow

$$\{\tilde{\phi}_t \colon T^1\mathbb{H}^2 \to T^1\mathbb{H}^2\}_{t\in\mathbb{R}}$$

and then extends to a flow on $\{\tilde{\psi}_t\}_{t\in\mathbb{R}}$ on $T^1\mathbb{H}^2 \times \mathbb{R}^d$ which acts trivially on the second factor, i.e. $\tilde{\psi}_t(\vec{v},\vec{w}) = (\tilde{\phi}_t(\vec{v}),\vec{w})$. The flow $\{\tilde{\psi}_t\}$ then descends to a flow $\{\psi_t\}$ on E_ρ which "extends" $\{\phi_t\}$. (If you intended to be confusing, you would say that $\{\psi_t\}$ is the flow parallel to the flat connection.)

One way of stating Labourie's fundamental dynamical result is that the flat bundle admits a splitting into line bundles with certain contraction properties. We recall that a

flow $\{\psi_t\}$ on a vector bundle V over a compact base B is *contracting* if given some (any) continuous family $\{\|\cdot\|_b\}_{b\in B}$ of norms on the fibers, there exists $C, c > 0$ so that

$$\|\psi_t(\vec{v})\|_{\phi_t(b)} \leq Ce^{-ct}\|\vec{v}\|_b$$

for all $b \in B$, $t > 0$ and $\vec{v} \in V_b$.

Theorem 2.2 (Labourie [66]). *If $\rho \in \mathcal{H}_d(S)$, then E_ρ admits a flow-invariant splitting*

$$E_\rho = L_1 \oplus L_2 \oplus \cdots \oplus L_d$$

into line bundles so that the flow induced by ψ_t is contracting on $L_i \otimes L_j^*$ if $i > j$.

Notice that this splitting lifts to a flow-invariant, Γ-equivariant splitting

$$T^1\mathbb{H}^2 \times \mathbb{R}^d = \widetilde{L}_1 \oplus \widetilde{L}_2 \oplus \cdots \widetilde{L}_d.$$

If $\gamma \in \Gamma$, let $\vec{v}_i$ be a non-trivial vector in $\widetilde{L}_i$ lying over a tangent vector to the axis of γ. (Notice that, by flow invariance, the line $\widetilde{L}_i$ is independent of which point on the axis you pick.) Then, since the splitting is flow invariant and Γ-equivariant, $\vec{v}_i$ must be an eigenvector for $\widehat{\rho}(\gamma)$. Let $\ell_i(\rho(\gamma))$ denote the associated eigenvalue of $\widehat{\rho}(\gamma)$, i.e.

$$\widehat{\rho}(\gamma)(\vec{v}_i) = \ell_i(\widehat{\rho}(\gamma))\vec{v}_i.$$

The fact that $L_i \otimes L_j^*$ is contracting if $i > j$ implies that $|\ell_i(\widehat{\rho}(\gamma))| > |\ell_j(\widehat{\rho}(\gamma))|$ if $i > j$. In particular, $\rho(\gamma)$ is loxodromic and if $\lambda_i(\rho(\gamma)) = |\ell_i(\widehat{\rho}(\gamma))|$, then

$$\lambda_1(\rho(\gamma)) > \lambda_2(\rho(\gamma)) > \cdots > \lambda_d(\rho(\gamma)).$$

Moreover, since the flow is contracting we see that there exists $C, c > 0$ so that if $\ell(\gamma)$ denotes the translation length of γ on $\mathbb{H}^2$, then

$$\frac{\lambda_i(\rho(\gamma))}{\lambda_{i+1}(\rho(\gamma))} \geq Ce^{c\ell(\gamma)}$$

if $\gamma \in \Gamma$ and $1 \leq i \leq d-1$.

Labourie's splitting also gives rise to a ρ-equivariant, Hölder continuous limit map $\xi_\rho\colon \partial\mathbb{H}^2 \to \mathcal{F}_d$ where $\mathcal{F}_d$ is the space of d-dimensional flags. If $x \neq y \in \mathbb{H}^2$, consider a flow line (a.k.a. geodesic) joining y to x and choose non-trivial vectors $\vec{v}_i \in \widetilde{L}_i$ lying over a vector tangent to the flow line. One then defines

$$\xi_\rho(x) = \big(\langle\vec{v}_1\rangle, \langle\vec{v}_1, \vec{v}_2\rangle, \ldots, \langle\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{d-1}\rangle\big),$$
$$\xi_\rho(y) = \big(\langle\vec{v}_d\rangle, \langle\vec{v}_d, \vec{v}_{d-1}\rangle, \ldots, \langle\vec{v}_d, \vec{v}_{d-1}, \ldots, \vec{v}_2\rangle\big).$$

In particular, $\xi_\rho(x)$ is transverse to $\xi_\rho(y)$. The contraction properties of the flow imply that $\xi_\rho(x)$ does not depend on y and that ξ_ρ is Hölder.

We summarize these properties below.

Theorem 2.3 (Labourie [66]). *If $\rho \in \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(d,\mathbb{R}))$ is a Hitchin representation, then*

(1) *there exists a Hölder continuous ρ-equivariant map $\xi_\rho\colon \partial\mathbb{H}^2 \to \mathcal{F}_d$ so that if $x \neq y$, then $\xi_\rho(x)$ is transverse to $\xi_\rho(y)$;*

(2) *if $\gamma \in \pi_1(S)$, then $\rho(\gamma)$ is loxodromic and $\xi_\rho(\gamma^+)$ is the attracting flag of $\rho(\gamma)$, where γ^+ is the attracting fixed point of γ;*

(3) *there exists $C, c > 0$ so that*

$$\frac{\lambda_i(\rho(\gamma))}{\lambda_{i+1}(\rho(\gamma))} \geq Ce^{c\ell(\gamma)}$$

if $\gamma \in \Gamma$ and $1 \leq i \leq d-1$.

The properties above do not characterize Hitchin representations. For example, if one takes the direct product of a Fuchsian representation and the trivial one-dimensional representation, it will satisfy all the properties above. Such representations are known as Barbot representations, since they were first studied by Thierry Barbot [4]. However, Oliver Guichard [52] extended Labourie's work to provide the following characterization.

Theorem 2.4 (Guichard [52]). *If $\rho \in \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(d,\mathbb{R}))$, then ρ is a Hitchin representation if and only if there exists a continuous ρ-equivariant map $\xi\colon \partial\mathbb{H}^2 \to \mathbb{P}(\mathbb{R}^d)$ so that if $\{x_1, \ldots, x_d\}$ are distinct points in $\partial\mathbb{H}^2$, then*

$$\xi(x_1) \oplus \cdots \oplus \xi(x_d) = \mathbb{R}^d.$$

Notice that it is clear that Barbot representations do not satisfy this characterization, since the image of any such map cannot span $\mathbb{R}^3$ in this case. (In fact, small deformations of Barbot representations also admit limit maps which fail to satisfy Guichard's criterion.)

Labourie's work also allows him to extend Fricke's theorem to the setting of Hitchin components. We recall that the mapping class group $\mathrm{Mod}(S)$ of a closed orientable surface is the group of (isotopy classes of) orientation-preserving self-homeomorphisms of S.

Theorem 2.5 (Labourie [69]). *The mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously on $\mathcal{H}_d(S)$.*

Remarks. (1) Hitchin's work [57] establishes analogous results for representations into all split real Lie groups, but we will only discuss the case of $\mathsf{PSL}(d,\mathbb{R})$, which is the most studied case. Hitchin's work uses the theory of Higgs bundles. Due to the author's woeful ignorance, we will not discuss any of the subsequent work on Hitchin representations from this more analytic viewpoint.

(2) In Labourie's seminal paper, he more generally defines Anosov representations of a hyperbolic group into any semi-simple Lie group. This definition was explored more fully by Guichard–Wienhard [54], and later by Guéritaud–Guichard–Kassel–Wienhard [51], Kapovich–Leeb–Porti [61], Bochi–Potrie–Sambarino [9], Kassel–Potrie [62], Tsouvalas [121] and others. In particular, analogues of all the basic properties discussed above exist

in this setting. For generalizations of Fricke's theorem to this setting, see Guichard–Wienhard [54, Corollary 5.4] or Canary [31, Theorem 6.4].

The theory of Anosov representations has emerged as a central language in higher Teichmüller theory. For those interested in me blathering on endlessly about Anosov representations, lecture notes are available on my webpage.[1]

3. The positive viewpoint of Fock–Goncharov

Vladimir Fock and Alexander Goncharov [47] characterize Hitchin representations as those representations which admit a positive limit map.

We begin by recalling the definition of a positive map of a subset of the circle into $\mathcal{F}_d$. This definition relies on the work of Lusztig [84] and others on positivity in semi-simple Lie groups. Given an ordered basis $\mathcal{B}$ for $\mathbb{R}^d$, we say that a unipotent element $A \in \mathsf{SL}(d,\mathbb{R})$ is *totally positive* with respect to $\mathcal{B}$, if its matrix with respect to $\mathcal{B}$ is upper triangular and all its minors (which are not forced to be 0 by the fact that the matrix is upper triangular) are strictly positive. The set $\mathcal{U}_{>0}(\mathcal{B})$ of unipotent, totally positive, upper triangular matrices with respect to $\mathcal{B}$ is a semi-group. An ordered k-tuple $(F_1, F_2, \dots, F_k)$ of distinct flags in $\mathcal{F}_d$ is positive with respect to an ordered basis $\mathcal{B} = (b_1, \dots, b_d)$ for $\mathbb{R}^d$ if $b_i \in F_1^{(i)} \cap F_k^{(d-i+1)}$ for all i, and there exists $u_2, \dots, u_{k-1} \in \mathcal{U}_{>0}(\mathcal{B})$ so that $F_i = u_{k-1} \cdots u_i F_k$ for all $i = 2, \dots, k-1$. Here $F^{(k)}$ denotes the k-dimensional component of a flag F. If X is a subset of S^1 then a map $\xi\colon X \to \mathcal{F}_d$ is *positive* if whenever $(x_1, \dots, x_n)$ is a cyclically ordered subset of distinct points in X, then $(\xi(x_1), \dots, \xi(x_n))$ is positive with respect to some ordered basis. (In fact, it suffices to only consider 4-tuples of points.)

If $\mathcal{B} = \{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$, then

$$\mathcal{U}_{>0}(\mathcal{B}) = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a > 0 \right\}$$

and one may check that a n-tuple of distinct points in $\mathcal{F}_2 = \mathbb{P}(\mathbb{R}^2) \cong S^1$ is positive with respect to $\mathcal{B}$ if and only if $x_1 = [e_1]$, $x_n = [e_2]$ and the other points proceed monotonically in the counter-clockwise direction. More generally, one may check that a map $\xi\colon X \to \mathcal{F}_2$ is positive if and only if it is monotonic.

Vladimir Fock and Alexander Goncharov obtain the following characterization of Hitchin representations.

Theorem 3.1 (Fock–Goncharov [47]). *A representation $\rho\colon \pi_1(S) \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation if and only if there exists a positive ρ-equivariant map $\xi\colon \partial\mathbb{H}^2 \to \mathbb{P}(\mathbb{R}^d)$.*

They develop further structure which allows them to explicitly parametrize Hitchin components using natural algebraic data. This viewpoint was further developed by Francis Bonahon and Guillaume Dreyer [11, 12].

[1] http://www.math.lsa.umich.edu/~canary

Remark. Olivier Guichard and Anna Wienhard [55] developed a more general notion of a Θ-positivity for real semi-simple Lie groups and characterize exactly which Lie groups admit such structures. Guichard, Labourie and Wienhard [53] recently proved that Θ-positive representations are Anosov and that the space of Θ-positive representations into a given Lie group contains entire components of the space of reductive representations. This theory encompasses both Hitchin representations into split real Lie groups and maximal representations into Hermitian Lie groups of tube type, as well as certain representations into $\mathsf{SO}(p,q)$ (when $p \neq q$) and four other exceptional Lie groups. In the case of $\mathsf{SO}(p,q)$, Jonas Beyrer and Beatrice Pozzetti [6] further show that the set of Θ-positive representations is exactly a collection of components of the representation variety. The fundamental conjecture is that Θ-positive representations account for all components of representation varieties of surface groups into simple Lie groups which consist entirely of discrete faithful representations. (For a Higgs bundle-theoretic perspective, see Bradlow–Collier–García–Prada–Gothen–Oliveira [16].)

4. Sambarino's geodesic flows

Andres Sambarino [105] defined a family of Anosov flows associated to a Hitchin representation which record the spectral data of the representation. We first recall some linear algebra so that we can state his results.

Let

$$\mathfrak{a} = \{\vec{a} \in \mathbb{R}^d : a_1 + \cdots + a_d = 0\}$$

be the standard Cartan algebra for $\mathsf{PSL}(d,\mathbb{R})$. The space $\mathfrak{a}^*$ of linear functionals on $\mathfrak{a}$ is generated by the simple roots $\{\alpha_1, \ldots, \alpha_{d-1}\}$ where

$$\alpha_i(\vec{a}) = a_i - a_{i+1}.$$

The standard positive Weyl Chamber $\mathfrak{a}^+$ is then the set where all the α_i are non-negative, i.e.

$$\mathfrak{a}^+ = \{\vec{a} \in \mathfrak{a} : a_1 \geq \cdots \geq a_d\} = \{\vec{a} \in \mathfrak{a} : \alpha_i(\vec{a}) \geq 0 \ \forall\ i\}.$$

We will also be interested in the linear functionals given by the fundamental weights ω_k and the Hilbert length ω_H, where

$$\omega_k(\vec{a}) = a_1 + \cdots + a_k \quad \text{and} \quad \omega_H(\vec{a}) = a_1 - a_d = \omega_1(\vec{a}) + \omega_{d-1}(\vec{a}).$$

The *Jordan projection* $\nu\colon \mathsf{PSL}(d,\mathbb{R}) \to \mathfrak{a}^+$ records the spectral data associated to an element of $\mathsf{PSL}(d,\mathbb{R})$. If $A \in \mathsf{PSL}(d,\mathbb{R})$ has generalized eigenvalues with moduli

$$\lambda_1(A) \geq \cdots \geq \lambda_d(A),$$

then

$$\nu(A) = \big(\log \lambda_1(A), \ldots, \log \lambda_d(A)\big).$$

If $\rho \in \mathcal{H}_d(S)$, the *Benoist limit cone* $\mathcal{B}(\rho)$ encodes the spectral data of $\rho(\Gamma)$. Explicitly,

$$\mathcal{B}(\rho) = \overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_+ \nu(\rho(\gamma))} \subset \mathfrak{a}^+.$$

We will be interested in the collection $\mathcal{B}^+(\rho)$ of linear functionals which are positive on $\mathcal{B}(\rho) \setminus \{\vec{0}\}$, given by

$$\mathcal{B}^+(\rho) = \{\phi \in \mathfrak{a}^* \mid \phi(\vec{a}) > 0 \ \forall\ \vec{a} \in \mathcal{B}(\rho) \setminus \{\vec{0}\}\}.$$

Theorem 2.3 implies that $\mathcal{B}(\rho) \setminus \{\vec{0}\}$ is contained in the interior of $\mathfrak{a}^+$, so

$$\Delta = \left\{\phi \in \mathfrak{a}^* \mid \phi = \sum c_i\alpha_i,\ c_i \geq 0\ \forall\ i,\ \text{and}\ \sum c_i > 0\right\} \subset \mathcal{B}(\rho)^+.$$

Notice that, in particular, each fundamental weight $\omega_k \in \Delta$ and $\omega_H \in \Delta$.

If $\phi \in \mathcal{B}^+(\rho)$, then one obtains a natural associated length function, given by

$$\ell^\phi(\rho(\gamma)) = \phi(\nu(\gamma)).$$

If $d = 2$ and $\phi = \alpha_1 = \omega_H = 2\omega_1$, then $\ell^\phi(\rho(\gamma))$ is just the usual translation length $\ell(\rho(\gamma))$ of $\rho(\gamma)$. One may then consider an associated *ϕ-topological entropy* $h^\phi(\rho)$ which records the exponential growth rate of the number of (conjugacy classes of) elements of ϕ-length at most T. Concretely, let

$$R_T^\phi(\rho) = \{[\gamma] \in [\Gamma] : \ell^\phi(\rho(\gamma)) \leq T\} \quad \text{and} \quad h^\phi(\rho) = \lim_{T\to\infty} \frac{\log \# R_T^\phi(\rho)}{T},$$

where $[\Gamma]$ is the set of conjugacy classes of elements of Γ.

We can now summarize some of Sambarino's work. We recall that a flow space $\mathbf{U}_1$ is said to be Hölder orbit equivalent to a flow space $\mathbf{U}_2$ if there is a Hölder homeomorphism $f\colon \mathbf{U}_1 \to \mathbf{U}_2$ which takes flow lines to flow lines (but does not necessarily preserve the time parameter).

Theorem 4.1 (Sambarino [104,105]). *If $\rho \in \mathcal{H}_d(S)$ and $\phi \in \mathcal{B}^+(\rho)$, then there exists an Anosov flow $\mathbf{U}^\phi(\rho)$ which is Hölder orbit equivalent to the geodesic flow on $T^1(S)$ so that the period of the orbit of $\mathbf{U}_\rho^\phi$ associated to $[\gamma] \in [\Gamma]$ is given by $\ell^\phi(\rho(\gamma))$. Moreover, the topological entropy of $\mathbf{U}^\phi(\rho)$ is exactly $h^\phi(\rho)$ and*

$$\#R_T(\phi) \sim \frac{e^{h^\phi(\rho)T}}{h^\phi(\rho)T}, \quad \textit{i.e.} \lim_{T\to\infty} \frac{\#(R_T(\phi))h^\phi(\rho)T}{e^{h^\phi(\rho)T}} = 1.$$

If $\alpha_{i,j} \in \mathfrak{a}^*$ is given by $\alpha_{i,j}(\vec{a}) = a_i - a_j$ (so $\alpha_i = \alpha_{i,i+1}$) and $i > j$, then one may obtain $\mathbf{U}^{\alpha_{i,j}}(\rho)$ from the contracting line bundle $L_i \otimes L_j^*$. The flows $\mathbf{U}^{\alpha_i}(\rho)$ are known as the *simple root flows*. Similarly, if $\omega_1 \in \mathfrak{a}^*$ is the first fundamental weight, i.e. $\omega_1(\vec{a}) = a_1$, then one may observe that L_1 is contracting and obtain $\mathbf{U}^{\omega_1}(\rho)$, which we call the *spectral radius flow*, in the same manner.

We now explain how to obtain a flow space $\mathbf{U}_L$, Hölder orbit equivalent to T^1S, from a contracting (Hölder) line bundle L over T^1S. One first lifts L to the contracting line bundle $\widetilde{L}$ over $T^1\mathbb{H}^2$ and considers the associate principal $\mathbb{R}$-bundle $\widehat{L}$ over $T^1\mathbb{H}^2$ so that the fiber over $\vec{v} \in T^1\mathbb{H}^2$ is given by $(\widetilde{L}|_{\vec{v}} \setminus \{\vec{0}\})/\pm 1$ and the action of $t \in \mathbb{R}$ is given by $[\vec{v}] \to [e^t\vec{v}]$. Notice that there is a projection map $\pi: T^1\mathbb{H}^2 \to \partial^2\mathbb{H}^2$ (where $\partial^2\mathbb{H}^2 = \{(x, y) : x, y \in \partial\mathbb{H}^2,\ x \neq y\}$) and all the vectors tangent to the geodesic joining x to y are taken to (x, y). Then, $\widetilde{\mathbf{U}}_L = \pi_*\widehat{L}$ is a principal $\mathbb{R}$-bundle over $\partial^2\mathbb{H}^2$, so admits a natural geodesic flow. The group Γ acts on $\widetilde{\mathbf{U}}_L$ with quotient $\mathbf{U}_L$ (see [23, Proposition 2.4] for more details). If $\gamma \in \Gamma$, then the closed orbit of $\mathbf{U}_L$ associated to $[\gamma]$ has period

$$-\log \frac{\|\phi_{\ell(\gamma)}(\vec{v})\|}{\|\vec{v}\|}$$

for any vector $\vec{v} \in T^1\mathbb{H}^2$ tangent to the axis of γ. For example, if $L = L_1$, this period is the spectral radius $\omega_1(\rho(\gamma)) = \log \lambda_1(\gamma)$ of $\rho(\gamma)$, while if $L = L_i \otimes L_j^*$, the period is given by

$$\alpha_{i,j}(\rho(\gamma)) = \log \frac{\lambda_i(\rho(\gamma))}{\lambda_j(\rho(\gamma))}.$$

It is not difficult to write down an explicit Hölder orbit equivalence between $T^1\mathbb{H}^2$ and $\mathbf{U}_L$ in general (see [22, Proposition 4.2] for details).

For more general $\phi \in \mathcal{B}^+(\rho)$, Sambarino [105] makes use of the Iwasawa cocycle. Quint [102] defines the *Iwasawa cocycle* $B: \mathsf{PSL}(d, \mathbb{R}) \times \mathcal{F}_d \to \mathfrak{a}$ in terms of the Iwasawa decomposition. Specifically, if $F \in \mathcal{F}_d$, then there exists $K \in \mathsf{PO}(d)$, so that $F = K(F_0)$ where F_0 is the flag determined by the standard basis and if $A \in \mathsf{SL}(d, \mathbb{R})$, then $B(A, F)$ is the unique element of $\mathfrak{a}$ satisfying

$$AK = Le^{B(A,F)}U$$

for some $L \in \mathsf{PO}(d)$ and upper triangular, unipotent element U. More geometrically, if $\vec{v}_1 \in F^{(1)}$ is non-trivial, then

$$\omega_1(B(A, F)) = \log \frac{\|A(\vec{v}_1)\|}{\|\vec{v}_1\|}$$

and if $\vec{v}_k$ is a non-trivial vector in $E^k(F^{(k)}) \in \mathbb{P}(E^k\mathbb{R}^d)$ is non-trivial, where E^k is the kth exterior power, then

$$\omega_k(B(A, F)) = \log \frac{\|E^kA(\vec{v}_k)\|}{\|\vec{v}_k\|}.$$

Notice that if A is loxodromic and F_A is the attracting flag of A, then $B(A, F_A) = \nu(A)$. The Iwasawa cocycle satisfies the cocycle relation, $B(CD, F) = B(C, D(F)) + B(D, F)$ for all $C, D \in \mathsf{PSL}(d, \mathbb{R})$.

Given $\rho \in \mathcal{H}_d(S)$ and $\phi \in \mathcal{B}^+(\rho)$, Sambarino defines the Hölder cocycle

$$\beta^\phi_\rho: \Gamma \times \partial\mathbb{H}^2 \to \mathfrak{a} \quad \text{given by } \beta^\phi_\rho(\gamma, x) = \phi\big(B(\rho(\gamma), \xi_\rho(x))\big).$$

The period of a Hölder cocycle β associated to γ is given by $\beta(\gamma, \gamma^+)$, so the period of β^ϕ_ρ associated to γ is $\phi(v(\rho(\gamma)))$. Using the theory of Hölder cocycle developed by Ledrappier [75], Sambarino shows that the action of Γ on $\partial^2\mathbb{H}^2 \times \mathbb{R}$ defined by

$$\gamma(x, y, t) = \big(\gamma(x), \gamma(y), t + \beta^\phi_\rho(\gamma, y)\big)$$

is properly discontinuous and cocompact. One may then define the quotient flow space $\mathbf{U}^\phi(\rho) = \partial^2\mathbb{H}^2 \times \mathbb{R}/\Gamma$.

Rafael Potrie and Andres Sambarino later showed that simple root entropy is constant on the Hitchin component and used this to establish a remarkable entropy rigidity theorem for Hitchin representations.

Theorem 4.2 (Potrie–Sambarino [99]). *If $\rho \in \mathcal{H}_d(S)$ and $1 \le i \le d-1$, then $h^{\alpha_i}(\rho) = 1$. Moreover, if $\phi = \sum c_i\alpha_i \in \Delta$, then*

$$h^\rho(\phi) \le \frac{1}{c_1 + \cdots + c_d}$$

and if $c_i > 0$ for all i, then equality holds if and only if ρ is d-Fuchsian.

Remark. Sambarino [105, Corollary 7.15] also obtains analogous results for the growth rate of translation length on the symmetric space. In subsequent work, Sambarino [106] establishes a mixing property for the Weyl chamber flow which allowed him to establish equidistribution results for $\mathbf{U}^\phi(\rho)$, see also Chow–Sarkar [40]. For more recent developments, see Burger–Landesberg–Lee–Oh [29], Carvajales [36, 37], Edwards–Lee–Oh [45], Landesberg–Lee–Lindenstraus–Oh [74], Lee–Oh [78, 79], Pozzetti–Sambarino–Wienhard [100, 101], and Sambarino [108].

5. Pressure metrics for the Hitchin component

In the 1970's, Bill Thurston proposed that one could construct a new Riemannian metric on Teichmüller space, by considering the "Hessian of the length of a random geodesic." Scott Wolpert [125] (see also Fathi–Flaminio [46]) proved that Thurston's metric was a scalar multiple of the classical Weil–Petersson metric, which is defined using quadratic differentials and Beltrami differentials. Bonahon [10] later re-interpreted Thurston's metric in terms of geodesic currents. McMullen [91] showed that one could use the thermodynamic formalism to construct a pressure form on the space $\mathcal{H}_0(T^1S)$ of all pressure zero Hölder functions on T^1S, embed Teichmüller space in $\mathcal{H}_0(T^1S)$ and obtain Thurston's metric as the pullback of the pressure form. Bridgeman [20] extended McMullen's analysis to quasifuchsian space, obtaining a path metric which is an analytic Riemannian metric away from the Fuchsian locus. By construction, it is mapping class group invariant and agrees with the Weil–Petersson metric (up to scalar multiplication) on the Fuchsian locus.

Martin Bridgeman, Dick Canary, François Labourie and Andres Sambarino [22, 23] showed that one can use McMullen's procedure to produce analytic pressure forms on the

Hitchin component associated to any linear functional in Δ. One key technical ingredient here is to show that the limit map varies analytically over the space of Hitchin representations, see [22, Theorem 6.1]. Since the Busemann cocycle β_ρ^ϕ is defined in terms of the limit map, the thermodynamic formalism then implies that most natural dynamical quantities vary analytically. In particular, the entropy varies analytically over the Hitchin component (see [22, Theorem 1.3] and Pollicott–Sharp [97, Theorem 3]). The foundational texts of thermodynamic formalism are books by Bowen [13], Parry–Pollicott [95] and Ruelle [103]. A fuller description of the use of thermodynamic formalism to construct pressure metrics is given in the survey article by Bridgeman, Canary and Sambarino [24].

Given two Hitchin representation $\rho\colon \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ and $\eta\colon \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ and $\phi \in \Delta$, we define their *pressure intersection*

$$I^\phi(\rho,\eta) = \lim_{T\to\infty} \frac{1}{\#(R_T^\phi(\rho))} \sum_{[\gamma]\in R_T(\rho)} \frac{\ell^\phi(\eta(\gamma))}{\ell^\phi(\rho(\gamma))},$$

which one may think of as the ϕ-length (in η) of a random geodesic (with respect to ϕ-length in ρ). One then considers the *renormalized pressure intersection* given by

$$J^\phi(\rho,\eta) = \frac{h^\phi(\eta)}{h^\phi(\rho)} I^\phi(\rho,\eta).$$

(Thurston and McMullen did not need to renormalize the pressure intersection since there is a single projective class of linear functionals and each entropy is constant on the Teichmüller space of a closed surface.) The functions I^ϕ and J^ϕ are analytic on the Hitchin component (see [22, Theorem 1.3]) and J^ϕ achieves its global minimum of 1 along the diagonal (see [22, Corollary 8.2]). (The results in [22] referenced in the last sentence are stated only for $\phi = \omega_1$, but the proofs easily generalize to all linear functionals in Δ, see [19].)

Theorem 5.1 (Bridgeman–Canary–Labourie–Sambarino [22, Corollary 6.2]). *If S is a closed orientable surface, $d \geq 3$ and $\phi \in \Delta$, then J^ϕ is an analytic function on $\mathcal{H}_d(S) \times \mathcal{H}_d(S)$. Moreover, if $\rho, \eta \in \mathcal{H}_d(S)$, then*

$$J^\phi(\rho,\rho) = 1 \quad \textit{and} \quad J^\phi(\rho,\eta) \geq 1.$$

If $\phi \in \Delta$, then one may define the ϕ-pressure form on $\mathcal{H}_d(S)$ by

$$\mathbb{P}^\phi|_{T_\rho \mathcal{H}_d(S)} = \operatorname{Hess} J^\phi(\rho,\cdot).$$

Since J^ϕ is analytic, $\mathbb{P}^\phi$ is analytic, and since J^ϕ achieves its minimum along the diagonal, $\mathbb{P}^\phi$ is non-negative at every point. Notice that, by construction, $\mathbb{P}^\phi$ is mapping class group invariant and, by Wolpert's result [125], agrees with (a scalar multiple of) the Weil–Petersson metric on the Fuchsian locus. The most difficult portion of the analysis then involves determining if $\mathbb{P}^\phi$ is non-degenerate, and hence gives rise to an analytic

Riemannian metric. (Mark Pollicott and Richard Sharp [98] provide an alternate formulation of the pressure form $\mathbb{P}^{\phi}$ when $\phi = \omega_1$.)

It is important to notice that the ϕ-pressure form will not be non-degenerate for all ϕ. For example, since ω_H is invariant under the contragredient involution (which takes A to $(A^T)^{-1}$), it is easy to see that $\mathbb{P}^{\omega_H}$ will be degenerate on the self-dual locus (i.e. the fixed point set of the contragredient involution) in $\mathcal{H}_d(S)$ (see [24, Lemma 5.22]), which always includes the Fuchsian locus. The same analysis applies to any linear functional which is invariant under the contragredient involution. For example, $\mathbb{P}^{\alpha_n}$ is degenerate on the self-dual locus of $\mathcal{H}_{2n}(S)$, see [23, Proposition 8.1]. Similarly, the pressure metric on quasifuchsian space is degenerate along the Fuchsian locus, which is the fixed point set of the involution of quasifuchsian space induced by complex conjugation, see [20].

However, in the case that ϕ is either the first fundamental weight ω_1 or first simple root α_1, $\mathbb{P}^{\phi}$ is non-degenerate. One hopes that Potrie and Sambarino's result that simple root entropy is constant on the Hitchin component, will make the simple root pressure metric more tractable to study. No other cases are fully understood at this point.

Theorem 5.2 ([22, Corollary 1.6] and [23, Theorem 1.6]). *If S is a closed orientable surface and $d \geq 3$, then the pressure forms $\mathbb{P}^{\omega_1}$ and $\mathbb{P}^{\alpha_1}$ are analytic Riemannian metrics on the Hitchin component $\mathcal{H}_d(S)$ which are invariant under the action of the mapping class group* $\mathrm{Mod}(S)$. *Moreover, the restrictions of both metrics to the Fuchsian locus are scalar multiples of the Weil–Petersson metric.*

The following elementary conjecture illustrates how little is known about the pressure metric. See [24, Section 7] for a further discussion of questions about the pressure metric, all of which remain wide open. However, since the time that survey was written, François Labourie and Richard Wentworth [72] and Xian Dai [42] have made significant progress in describing the pressure metric at the Fuchsian locus.

Conjecture. *There exists a sequence of points in $\mathcal{H}_d(S)$ whose $\mathbb{P}^{\omega_1}$-distance to the Fuchsian locus diverges to ∞.*

One natural place to start would be to study Hitchin components of triangle groups when $d = 3$, which are often one-dimensional, see Choi–Goldman [39]. (Nie [93] explicitly computes the deformation spaces in some cases.) All the results discussed so far go through immediately for Hitchin components of cocompact triangle groups (see, for example, Alessandrini–Lee–Schaffhauser [3]). It is still unknown whether or not these one-dimensional Hitchin components have finite diameter. One would also like to investigate this conjecture and all the questions in [24] for the simple root pressure metric $\mathbb{P}^{\alpha_1}$.

When $\phi = \omega_H$ and $d = 3$, one can give a complete analysis of the degeneracy of $\mathbb{P}^{\omega_H}$. We say that a non-zero vector $\vec{v} \in T_\rho \mathcal{H}_d(S)$ is *self-dual* if $dC(\vec{v}) = -\vec{v}$, where

$$C\colon \mathcal{H}_d(S) \to \mathcal{H}_d(S)$$

is the contragredient involution, i.e. $C(\rho) \in \mathcal{H}_d(S)$ is given by

$$C(\rho)(\gamma) = \rho(\gamma^{-1})^T.$$

Notice that self-dual vectors are based at the self-dual locus and that when $d = 3$ the self-dual locus is exactly the Fuchsian locus.

Theorem 5.3 (Bridgeman–Canary–Sambarino [24, Section 5.8] and Bray–Canary–Kao–Martone [19]). *If S is a closed orientable surface and $\vec{v} \in T\mathcal{H}_3(S)$ is non-zero, then*

$$\mathbb{P}^{\omega_H}(\vec{v}, \vec{v}) = 0$$

if and only if $\vec{v}$ is a self-dual vector. Therefore, the pressure form $\mathbb{P}^{\omega_H}$ gives rise to a mapping class group invariant path metric which is an analytic Riemannian metric away from the Fuchsian locus and agrees with (a scalar multiple of) the Weil–Petersson metric on the Fuchsian locus.

Remarks. (1) Qiongling Li [80] produced another mapping class group invariant Riemannian metric on $\mathcal{H}_3(S)$, which she calls the Loftin metric. The Loftin metric also restricts to a scalar multiple of the Weil–Petersson metric on the Fuchsian locus. One expects that her metric differs from our pressure metrics, but that is unknown so far.

Inkang Kim and Genkai Zhang [64] constructed a mapping class group invariant Kähler metric on $\mathcal{H}_3(S)$ in which the Fuchsian locus is a totally geodesic complex submanifold whose intrinsic metric agrees with the Weil–Petersson metric, see also Labourie [70, Corollary 1.3.2]. The relationship of this metric to the pressure metrics and Li's metric is also not understood.

(2) Marc Burger [26] was the first one to consider the pressure intersection, in the context of convex cocompact rank one representations. His work was motivated by rigidity result for pairs of Fuchsian representations due to Chris Bishop and Tim Steger [8]. The pressure intersection can also be interpreted in terms of Gerhard Knieper's geodesic stretch [65], see the discussion in Schapira–Tapie [111].

(3) Bridgeman, Canary, Labourie and Sambarino [22] more generally define a pressure form $\mathbb{P}^{\omega_1}$ associated to the first fundamental weight at smooth points of deformation spaces of projective Anosov representations into $\mathsf{SL}(d, \mathbb{R})$ which is non-degenerate at all "generic" representations.

6. Geodesic currents and collar lemmas for Hitchin representations

In this section, we discuss some of the work which further explores the analogy between the Hitchin component and Teichmüller space. The choice of topics reflects our personal tastes and the focus of this article.

Francis Bonahon [10] exhibited a geodesic current μ_ρ, known as the Liouville current, associated to a Fuchsian representation $\rho\colon \pi_1(S) \to \mathsf{PSL}(2, \mathbb{R})$ such that if $\gamma \in \pi_1(S) - \{\mathrm{id}\}$,

then

$$i(\mu_\rho, \gamma) = \ell(\rho(\gamma))$$

(where $\ell(\rho(\gamma))$ is the hyperbolic translation length of $\rho(\gamma)$). He used his theory of Liouville currents to reinterpret both Thurston's compactification of Teichmüller space and Thurston's definition of the Weil–Petersson metric. Recall that a geodesic current on S is a locally finite, $\pi_1(S)$-invariant Radon measure on $\partial\pi_1(S) \times \partial\pi_1(S) - \Delta$ (where Δ is the diagonal). Moreover, currents associated to (weighted) closed curves are dense in the space $\mathcal{C}(S)$ of geodesic currents and the intersection function $i\colon \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}$ agrees with geometric intersection number on pairs of closed curves.

If $\rho \in \mathcal{H}_d(S)$, François Labourie [67] (see also [23] and Martone–Zhang [88]) exhibit a geodesic current μ_ρ^H, again called the *Liouville current*, so that

$$i(\mu_\rho^H, \gamma) = \ell^{\omega_H}(\rho(\gamma)) = \log\left(\frac{\lambda_1(\rho(\gamma))}{\lambda_d(\rho(\gamma))}\right)$$

if $\gamma \in \pi_1(S) - \{\mathrm{id}\}$. Bridgeman, Canary, Labourie, and Sambarino [23] define the *Liouville volume*

$$\mathrm{vol}_L(\rho) = i(\mu_\rho^H, \mu_\rho^H)$$

of a Hitchin representation. They also show that μ_ρ^H is a multiple of the Bowen–Margulis current for the simple root flow $U_\rho^{\alpha_1}$ (see [23, Theorem 1.3]) and use this fact and a result of Nicolas Tholozan [119] to establish a rigidity result when $d = 3$. It is natural to ask whether a similar rigidity holds in higher dimensions.

Theorem 6.1 (Bridgeman–Canary–Labourie–Sambarino [23, Corollary 1.5]). *If S is a closed orientable surface and $\rho \in \mathcal{H}_3(S)$, then $\mathrm{vol}_L(\rho) \geq 4\pi^2|\chi(S)|$ with equality if and only if ρ lies in the Fuchsian locus.*

If μ is a geodesic current on S and $\mathbf{U}$ is a geodesic flow orbit equivalent to T^1S, then one may use the Hopf parametrization of $T^1\mathbb{H}^2$ to obtain a (possibly degenerate) volume form $\mu \otimes dt$ on $\mathbf{U}$ by considering the local product of μ and the element dt of path length, see [23] for details. Bridgeman, Canary, Labourie and Sambarino show that one may re-interpret $I^{\alpha_1} = J^{\alpha_1}$ in terms of the Liouville current, see [23, Section 6]. Specifically,

$$I^{\alpha_1}(\rho, \eta) = \frac{\int_{\mathbf{U}^{\alpha_1}(\eta)} \mu_\rho^H \otimes dt}{\int_{\mathbf{U}^{\alpha_1}(\rho)} \mu_\rho^H \otimes dt}.$$

More generally, Giuseppe Martone and Tengren Zhang [88] produce, for each $k \in \{1, \ldots, d-1\}$ and Hitchin representation $\rho\colon \pi_1(S) \to \mathsf{PSL}(d, \mathbb{R})$, a geodesic current μ_ρ^k so that if γ is a closed curve on S, then

$$i(\mu_\rho^k, \gamma) = \ell^{\omega_k + \omega_{d-k}}(\rho(\gamma)).$$

In their construction, μ_ρ^1 is exactly the symmetrization of μ_ρ^H (i.e. the geodesic current with the same periods which is invariant under the involution exchanging the first and

second coordinate). Martone and Zhang also introduce the more general class of positively ratioed representations and produce geodesic currents with analogous properties in this more general setting.

Martone and Zhang use their theory to get a deep understanding of how the Hilbert length entropy of a sequence of Hitchin representations can converge to 0. One particularly easy stated consequence of their work is the following relationship between the entropy and the systole of a Hitchin representation. If $\phi \in \mathcal{B}^+(\rho)$, we can define the *$\phi$-systole* of ρ to be

$$\mathrm{sys}^{\phi}(\rho) = \min\left\{\ell^{\phi}(\gamma) \mid \gamma \in \Gamma \setminus \{\mathrm{id}\}\right\}.$$

Theorem 6.2 (Martone–Zhang [88, Corollary 7.6]). *Let S be a closed orientable surface. Given $d \geq 3$ and $k \in \{1, \ldots, d-1\}$, there exists $L = L(S, d, k) > 0$ so that if $\rho \in \mathcal{H}_d(S)$, then*

$$h^{\omega_k + \omega_{d-k}}(\rho)\, \mathrm{sys}^{\omega_k + \omega_{d-k}}(\rho) \leq L.$$

In particular, $h^{\omega_H}(\rho)\, \mathrm{sys}^{\omega_H}(\rho) \leq L$.

Tengren Zhang [126, 127] (and Xin Nie [92, 93] when $d = 3$) produce sequences $\{\rho_n\}$ in $\mathcal{H}_d(S)$ of representations so that $\mathrm{sys}^{\omega_H}(\rho_n) \to \infty$ and $h^{\omega_H}(\rho_n) \to 0$. If $d = 3$,

$$\omega^H(\rho_n(\gamma)) = \ell^{\alpha_1}(\rho_n(\gamma)) + \ell^{\alpha_1}(\rho_n(\gamma^{-1})),$$

so if γ is any element of $\pi_1(S) - \{1\}$, then

$$\max\{\ell^{\alpha_1}(\rho_n(\gamma)), \ell^{\alpha_1}(\rho_n(\gamma^{-1}))\} \to \infty.$$

One hopes that such sequences would have their distance to the Fuchsian locus diverge to infinity.

One might naively think that Zhang's sequences would have $h^{\alpha_1}(\rho_n) \to 0$. However, we know, from Potrie–Sambarino [99], that $h^{\alpha_1}(\rho_n) = 1$ for all n. One possible explanation for this, in the simple case where $d = 3$, would be that for many elements in $\pi_1(S)$, the middle eigenvalue $\lambda_2(\rho_n(\gamma))$ remains "near" to either $\lambda_1(\rho_n(\gamma))$ or $\lambda_3(\rho_n(\gamma))$, so that one of $\{\ell^{\alpha_1}(\rho_n(\gamma)), \ell^{\alpha_1}(\rho_n)(\gamma^{-1})\}$ is growing quickly while the other remains moderate or even bounded. This phenomenon was observed in explicit computations done by Martin Bridgeman and the author for Hitchin components of certain triangle groups. This suggests the surprising possibility that the simple root systole is bounded above on $\mathcal{H}_d(S)$.

Question. Given a closed surface S and d, does there exist $L > 0$ so that $\mathrm{sys}^{\alpha_1}(\rho) \leq L$ for all $\rho \in \mathcal{H}_d(S)$?

Another deep analogy with the traditional theory of Fuchsian groups was established when Gye-Seon Lee and Tengren Zhang [77] proved an analogue of the collar lemma for Fuchsian groups for Hitchin representations. We will not state the precise version of their results, but we recall the following consequences which hold for all Hitchin representations.

Theorem 6.3 (Lee–Zhang [77, Corollary 1.2]). *If S is a closed, orientable surface of genus at least two, α and γ are homotopically non-trivial closed curves on S, and $\rho \in \mathcal{H}_d(S)$, then*

(1) *if $i(\alpha, \gamma) \neq 0$, then*

$$\left(e^{\ell^{\omega_H}(\rho(\alpha))} - 1\right)\left(e^{\ell^{\omega_H}(\rho(\alpha))} - 1\right) > 1,$$

(2) *and if α is not simple, then $\ell^{\omega_H}(\rho(\alpha)) \geq \log(2)$.*

One consequence of their work is that sufficiently complicated curve systems determine proper multi-length functions.

Corollary 6.4 (Lee–Zhang [77, Corollary 1.4]). *If S is a closed orientable surface of genus at least two and $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ is a collection of homotopically non-trivial closed curves which contains a pants decomposition of S and so that if α is any homotopically non-trivial curve α on S, then $i(\alpha, \Gamma) \neq 0$, then the map*

$$L_\Gamma \colon \mathcal{H}_d(S) \to \mathbb{R}^k \quad \text{given by } L_\Gamma(\rho) = \left(\ell^{\omega_H}(\rho(\gamma_i))\right)_{i=1}^k$$

is proper.

Marc Burger and Beatrice Pozzetti [30] subsequently established collar lemmas for maximal representations into $\mathsf{Sp}(2n, \mathbb{R})$ and Jonas Beyrer and Beatrice Pozzetti [5] for partially hyperconvex representations of surface groups.

We briefly mention some other important work which explore analogies between Hitchin components and Teichmüller spaces.

(1) Richard Skora [114] showed that one may describe Thurston's compactification of Teichmüller space in terms of actions of $\pi_1(S)$ on $\mathbb{R}$-trees. Anne Parreau [94] constructed a similar compactification of the Hitchin component by actions of $\pi_1(S)$ on $\mathbb{R}$-buildings. (Her work applies to compactification of much more general character varieties.) Marc Burger, Alessandra Iozzi, Anne Parreau and Beatrice Pozzetti [27] analyzed the Parreau compactification and showed that there is a non-empty open domain of discontinuity for the action of the mapping class group on the boundary. In subsequent work, they study the real spectrum compactification of the Hitchin component which admits a continuous, surjective map to the Parreau compactification which is mapping class group invariant, see their survey article [28]. Their work depends crucially on the theory of geodesic currents.

(2) Zhe Sun, Anna Wienhard and Tengren Zhang [117, 118, 124] studied Goldman's symplectic form on the Hitchin component. They prove an analogue of Wolpert's magic formula in this setting and construct Darboux coordinates for the symplectic structure. They also construct a half-dimensional space of Hamiltonian flows which generalize the twist flows on Teichmüller space.

(3) Nigel Hitchin [57] offers a parametrization of $\mathcal{H}_d(S)$ as $\prod_{d=2}^k Q^k(X)$ where $Q^k(X)$ is the space of holomorphic k-differentials on a Riemann surface X homeomorphic to S. Unfortunately, this parametrization is not invariant with respect to the mapping

class group. François Labourie [68] and John Loftin [81] showed that there is a homeomorphism from $\mathcal{H}_3(S)$ to the bundle of cubic holomorphic differentials over $\mathcal{T}(S)$ which is equivariant with respect to the action of the mapping class group. Recent work of Vlad Markovic [85] indicates that Labourie's approach in [70] will not generalize to produce similar mapping class group invariant parametrizations when $d > 3$.

(4) Bridgeman, Pozzetti, Sambarino and Wienhard [25, Corollary A] showed that $\mathcal{H}_d(S)$ is an isolated minimum for the α_1-entropy functional for the space of quasi-Hitchin representations into $\mathsf{PSL}(d,\mathbb{C})$. (A representation $\rho\colon \pi_1(S) \to \mathsf{PSL}(d,\mathbb{C})$ is quasi-Hitchin if it is Borel Anosov and can be deformed to a Hitchin representation through Borel Anosov representations.) Moreover, they describe the Hessian of the α_1-entropy functional at the Hitchin locus. Their results generalize work of Bowen [14], Bridgeman [20] and McMullen [91] in the quasifuchsian setting.

(5) François Labourie and Greg McShane [71] proved an analogue of McShane's identity for Hitchin representations, while Nick Vlamis and Andrew Yarmola [123] proved an analogue of the Basmajian identity for Hitchin representations. Yi Huang and Zhe Sun [58] proved versions of McShane's identity for holonomy maps of finite area convex projective surfaces (and for positive representations of Fuchsian lattices).

(6) Richard Schwartz and Richard Sharp [113] proved an explicit correlation result for lengths on hyperbolic surfaces. Specifically, they prove that given $\rho_1, \rho_2 \in \mathcal{T}(S) = \mathcal{H}_2(S)$ and $\varepsilon > 0$, there exist $C, M > 0$ so that

$$\#\big\{[\gamma] \in [\Gamma] : \ell(\rho_1(\gamma)) \in (x, x+\varepsilon) \text{ and } \ell(\rho_2(\gamma)) \in (x, x+\varepsilon)\big\} \sim C\frac{e^{Mx}}{x^{3/2}}.$$

Xian Dai and Giuseppe Martone [43, Theorem 1.7] generalize this result to arbitrary linear functions in Δ, by showing that given $\rho_1, \rho_2 \in \mathcal{H}_d(S)$, $\phi \in \Delta$ and $\varepsilon > 0$, there exist $C, M > 0$ so that

$$\#\big\{[\gamma] \in [\Gamma] : h^\phi(\rho_1)\ell^\phi(\rho_1(\gamma)) \in (x, x + h^\phi(\rho_1)\varepsilon) \text{ and } h^\phi(\rho_2)\ell^\phi(\rho_2(\gamma)) \in (x, x + h^\phi(\rho_2)\varepsilon)\big\} \sim C\frac{e^{Mx}}{x^{3/2}}.$$

Schwartz and Sharp [113] asked whether or not M can be arbitrarily close to 0. Dai and Martone [43, Theorem 1.3] show that this can happen even in the Fuchsian setting.

(7) It is a classical result, that one can find finitely many simple closed curves whose lengths determine a point in Teichmüller space. Ursula Hamenstadt [56] and Paul Schmutz [112] showed that $6g-5$ curves suffice, but that no set of $6g-6$ curves completely determine a point in Teichmüller space. Bridgeman, Canary, Labourie and Sambarino [22, Theorem 1.2] show that the ω_1-lengths of all curves on S determine a point in $\mathcal{H}_d(S)$. Bridgeman, Canary and Labourie [21] showed that the ω_1-length of all simple closed curves on S determine a points in $\mathcal{H}_d(S)$ if S has genus at least 3. It would be interesting to know whether or not finitely many curves suffice. (Sourav Ghosh [49] recently showed

that there are finitely many elements of $\pi_1(S)$ whose full Jordan projections determine a point in $\mathcal{H}_d(S)$.)

(8) Vladimir Fock and Alexander Thomas [48] have defined the notion of a higher complex structure on a closed surface. They conjecture that the space of higher complex structures of order d on a surface S is canonically isomorphic to the Hitchin component $\mathcal{H}_d(S)$.

7. The Teichmüller theoretic viewpoint of Sullivan and Tholozan

In August 2017, Nicolas Tholozan gave an inspirational talk describing how to use the work of Dennis Sullivan to give a Teichmüller-theoretic interpretation of the simple root pressure metric $\mathbb{P}^{\alpha_1}$. We will give a brief description of his work, which unfortunately is not fully available yet. If you want more details I suggest you view his talk on YouTube[2], consult his lecture notes[3] for a mini-course given at the University of Michigan in December 2019 and/or read Sullivan's beautiful paper [116].

One may consider the unit tangent bundle $T^1\mathbb{H}^2$ of the hyperbolic plane as a foliated space where the leaves are the central stable leaves of the geodesic flow. More prosaically, each leaf of the foliation consists of tangent vectors to all geodesic ending at a fixed point in $\partial\mathbb{H}^2$. Each leaf is canonically identified with $\mathbb{H}^2$ (up to isometry) and hence admits a well-defined complex structure. One may thus view $T^1\mathbb{H}^2$ as admitting a complex foliation. If $S = \mathbb{H}^2/\Gamma$, then Γ acts as a group of holomorphic automorphisms of this complex foliation, so T^1S is also a complex foliated manifold.

Sullivan [116] developed a theory of the Teichmüller space $\mathcal{T}(L)$ of complex structures on a complex laminated space L (i.e. spaces which admit a local product structure in which the horizontal leaves admit a complex structure). Notice that a complex structure on a leaf induces a smooth structure on the leaf. An element of $\mathcal{T}(L)$ may be viewed as a complex foliated laminated space with a homeomorphism to L which preserves the lamination structure and is smooth on each leaf (up to appropriate marked equivalence). Sullivan's Teichmüller theory involves generalizations of both quadratic differentials and Beltrami differentials, so he is able to construct both a Teichmüller metric and a Weil–Petersson metric on $\mathcal{T}(L)$.

Tholozan [120] constructs a continuous bijection CF between the space of (conjugacy classes of) entropy one geodesic flows which are Hölder orbit equivalent to T^1S and the space $\mathcal{T}^h(T^1S)$ of elements of $\mathcal{T}(T^1S)$ where the homeomorphism to $T^1(S)$ is transversely Hölder.[4] Roughly, one maps the stable leafs of flow space to horocycles. The fact that the entropy is 1 allows one to see that this gives an identification of (the cover of) each

[2] https://www.youtube.com/watch?v=8BdSaum7OeI&t=8s
[3] http://www.math.ens.fr/~tholozan/Annexes/CocyclesReparametrizations2.pdf
[4] See Theorem 0.4 of Tholozan's lecture notes.

leaf with $\mathbb{H}^2$. A result of Candel [35] then shows that this gives rise to a foliated complex structure on T^1S.

From CF one obtain an embedding

$$R\colon \mathcal{H}_d(S) \to \mathcal{T}(T^1S),$$

where $R(\rho) = CF(\mathbf{U}^{\alpha_1}(\rho))$. Tholozan's main result is that $\mathbb{P}^{\alpha_1}$ is the pullback of a "Weil–Petersson metric" on $\mathcal{T}(T^1S)$.

Theorem 7.1 (Tholozan [120]). *If S is a closed orientable hyperbolic surface of genus $g \geq 2$, then the simple root pressure metric $\mathbb{P}^{\alpha_1}$ is the pullback, via the embedding R, of a scalar multiple of a "Weil–Petersson metric" on $\mathcal{T}(T^1S)$.*

This opens up the possibility of using classical Teichmüller theoretic techniques to study the, so far mysterious, simple root pressure metric. One can also pull back the Teichmüller metric on $\mathcal{T}(T^1S)$ to obtain a metric $\mathbb{Q}^{\alpha_1}$ on $\mathcal{H}_d(S)$, which one might call the *simple root Teichmüller metric*. As in the classical setting, the simple root Teichmüller metric should be less regular, but easier to control. One might first study the properties of the simple root Teichmüller metric, and then study its relationship with the simple root pressure metric.

Remark. Labourie [67] also constructs a candidate for a highest Teichmüller space in which all Hitchin components embed. He uses cross ratios to embed every Hitchin component $\mathcal{H}_d(S)$ in the space of (conjugacy classes of) representations of $\pi_1(S)$ into the space of Hölder self-homeomorphisms of the space J of 1-jets of real-valued functions on the circle. This embedding records the Hilbert length functional and is closely related to Labourie's Liouville current. There is a relationship between Labourie's work and that of Tholozan coming from the fact that the Liouville current is the Bowen–Margulis current of the (first) simple root flow.

8. Hitchin representations of Fuchsian groups

If Γ is a Fuchsian group, i.e. a discrete subgroup of $\mathsf{PSL}(2,\mathbb{R})$, we say that a representation $\rho\colon \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation if there exists a ρ-equivariant positive map $\xi\colon \Lambda(\Gamma) \to \mathcal{F}_d$, where $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$ is the limit set of Γ. If Γ is convex cocompact (i.e. finitely generated and without parabolic elements) and torsion-free, Hitchin representations of Γ were studied by Labourie and McShane [71]. If $\mathbb{H}^2/\Gamma$ has finite volume and $d = 3$, then $\rho(\Gamma)$ preserves and acts properly discontinuously on a strictly convex domain $\Omega_\rho \subset \mathbb{RP}^2$ and $\Omega_\rho/\rho(\Gamma)$ is a finite area real projective surface (or orbifold), see Choi–Goldman [38] or Marquis [86, 87].

Dick Canary, Tengren Zhang and Andy Zimmer [33] prove that Hitchin representations of finitely generated Fuchsian groups have many of the same geometric properties as Hitchin representations of closed surface groups. (More generally, they study Anosov representations of finitely generated Fuchsian groups.)

Theorem 8.1 (Canary–Zhang–Zimmer [33]). *If Γ is a geometrically finite Fuchsian group, b_0 is a basepoint for $\mathbb{H}^2$ and $\rho\colon \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation, then*

(1) *There exists $B, b > 0$ so that if $\gamma \in \Gamma$, then*

$$Be^{b\ell(\gamma)} \geq \frac{\lambda_k(\rho(\gamma))}{\lambda_{k+1}(\rho(\gamma))} \geq \frac{1}{B}e^{\frac{\ell(\gamma)}{b}}$$

for all $k \in \{1,\dots,d-1\}$.

(2) *If $\alpha \in \Gamma$ is parabolic, then $\rho(\alpha)$ is unipotent and its Jordan normal form has only one block.*

(3) *The orbit map $\tau_\rho\colon \Gamma(b_0) \to X_d(\mathbb{R})$ given by $\tau_\rho(\gamma(b_0)) = [\rho(\gamma)]$ is a quasi-isometric embedding.*

(4) *The limit map ξ_ρ is Hölder.*

(5) *If $z \in \Lambda(\Gamma)$, then $\xi_\rho(z)$ varies analytically over the space of Hitchin representations.*

Harry Bray, Dick Canary, Nyima Kao and Giuseppe Martone [18] developed analogues of the dynamical results of Sambarino in the setting of general Hitchin representations of a torsion-free, finitely generated Fuchsian group Γ. If Γ is convex cocompact, then Sambarino's original theory applies. The key new difficulty in the presence of parabolic elements is that one can no longer model the (recurrent portion of the) geodesic flow on T^1X, where $X = \mathbb{H}^2/\Gamma$ by a finite Markov coding. However, Françoise Dal'bo and Marc Peigné [44] (in the case where X has infinite area) and Manuel Stadlbauer, Francois Ledrappier and Omri Sarig [76, 115] (in the case where X has finite area) have developed well-behaved countable Markov codings. These codings are natural generalizations of the Bowen–Series [15] (finite) Markov codings of convex cocompact Fuchsian groups. Kao [59] first used these codings to construct a pressure metric on Teichmüller space, and Bray, Canary and Kao [17] used them to construct pressure metrics on deformation spaces of cusped quasifuchsian groups.

We recall that a one-sided countable Markov shift (Σ^+,σ) is determined by a countable alphabet $\mathcal{A}$ and a transition matrix $T \in \{0,1\}^{\mathcal{A}\times\mathcal{A}}$. An element $x \in \Sigma^+$ is a one-sided infinite string $x = (x_i)_{i\in\mathbb{N}}$ of letters in $\mathcal{A}$ so that $T(x_i, x_{i+1}) = 1$ for all $i \in \mathbb{N}$. The shift σ simply removes the first letter and shifts every other letter one place to the left, i.e. $\sigma(x) = (x_{i+1})_{i\in\mathbb{N}}$. Let Fix^n denote the set of periodic words in Σ^+ with period n.

Given a torsion-free finitely generated group Γ, then the associated countable Markov shift (Σ^+,σ) constructed by Dal'bo–Peigné or Stadlbauer–Ledrappier–Sarig has the following crucial properties.

(1) There exists a finite-to-one Hölder map $\omega\colon \Sigma^+ \to \Lambda(\Gamma)$ which surjects onto the complement $\Lambda_c(\Gamma)$ of the set of fixed points of parabolic elements of Γ.

(2) There exists a map $G\colon \mathcal{A} \to \Gamma$ so that if $x \in \mathrm{Fix}^n$, then $\omega(x)$ is the attracting fixed point of $G(x_1)\cdots G(x_n)$.

(3) If $\gamma \in \Gamma$ is hyperbolic, then there exists $x \in \mathrm{Fix}^n$ (for some n) so that γ is conjugate to $G(x_1)\cdots G(x_n)$. Moreover, x is unique up to cyclic permutation.

In addition, Σ^+ is well behaved in the sense that it satisfies the assumptions needed to make use of the powerful thermodynamic formalism for countable Markov shifts developed by Daniel Mauldin and Mariusz Urbanski [90] and Omri Sarig (see his lecture notes[5] and [109]).

Bray, Canary, Kao and Martone [18] show that given $\rho \in \mathcal{H}_d(\Gamma)$ there is a vector-valued function $\tau_\rho\colon \Sigma^+ \to \mathfrak{a}$ which records all the spectral data of $\rho(\Gamma)$. Specifically, they define

$$\tau_\rho(x) = B\big(\rho(G(x_1)), \rho(G(x_1))^{-1}(\xi_\rho(\omega(x)))\big)$$

and prove that it has the following property.

Theorem 8.2 (Bray–Canary–Kao–Martone [18]). *Suppose that Γ is a torsion-free, finitely generated Fuchsian group, with associated one-sided Markov shift (Σ^+, σ) and $\rho\colon \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation. There is a locally Hölder continuous function $\tau_\rho\colon \Sigma^+ \to \mathfrak{a}$ so that if $\phi \in \Delta$, $\tau_\rho^\phi = \phi \circ \tau_\rho$, and $x \in \mathrm{Fix}^n$, then*

$$S_n\tau_\rho^\phi(x) = \sum_{i=0}^{n-1} \tau_\rho^\phi(\sigma^i(x)) = \ell^\phi(G(x_1)\cdots G(x_n)).$$

Much as in the classical case one may define the ϕ-topological entropy of a Hitchin representations by letting

$$R_T^\phi(\rho) = \big\{[\gamma] \in [\Gamma_{\mathrm{hyp}}] : \ell^\phi(\rho(\gamma)) \le T\big\} \quad \text{and} \quad h^\phi(\rho) = \lim \frac{\log \# R_T^\phi(\rho)}{T},$$

where $[\Gamma_{\mathrm{hyp}}]$ is the collection of conjugacy classes of hyperbolic elements of Γ. The only difference here is that we omit consideration of parabolic elements, which are not present in the case of closed surface groups. One may then also generalize the definitions of pressure intersection and renormalized intersection. If $\rho, \eta \in \mathcal{H}_d(\Gamma)$ and $\phi \in \Delta$, then

$$I^\phi(\rho,\eta) = \lim_{T\to\infty} \frac{1}{\#(R_T^\phi(\rho))} \sum_{[\gamma]\in R_T(\rho)} \frac{\ell^\phi(\eta(\gamma))}{\ell^\phi(\rho(\gamma))}$$

and

$$J^\phi(\rho,\eta) = \frac{h^\phi(\eta)}{h^\phi(\rho)} I^\phi(\rho,\eta).$$

Bray, Canary, Kao and Martone use the renewal theorem of Marc Kesseböhmer and Sabrina Kombrink [63] for countable Markov shifts to establish counting and equidistribution results in the setting of countable Markov shifts in the spirit of the work of Steven Lalley [73] in the setting of finite Markov shifts. In the case of Hitchin representations, their counting result has the following form. (When $d = 3$, then our results are a special case of more general results of Feng Zhu [128] when $\phi = \omega_H$.)

[5] https://www.weizmann.ac.il/math/sarigo/sites/math.sarigo/files/uploads/tdfnotes.pdf

Theorem 8.3 (Bray–Canary–Kao–Martone [18]). *Suppose that Γ is a torsion-free, finitely generated Fuchsian group and $\rho\colon \Gamma \to \mathsf{PSL}(d,\mathbb{R})$ is a Hitchin representation. If $\phi \in \mathcal{B}^+(\rho)$, then there exists $h^\phi(\rho) > 0$ so that*

$$\#R_T^\phi(\rho) \sim \frac{e^{t h^\phi(\rho)}}{t h^\phi(\rho)}.$$

Our equidistribution result [18, Corollary 1.6] expresses the geometrically defined pressure intersection function in terms of equilibrium states, which allows one to employ the machinery of thermodynamical formalism to verify analyticity and construct analytic pressure forms. A more precise statement would take us further into a discussion of thermodynamic formalism than time allows for in a brief survey paper.

The results in [18] and [33] combine to show that entropies and pressure intersection vary analytically.

Corollary 8.4 ([19]). *If Γ is a finitely generated Fuchsian group and $\phi \in \Delta$, then h^ϕ is an analytic function on $\mathcal{H}_d(\Gamma)$ and I^ϕ and J^ϕ are analytic functions on $\mathcal{H}_d(\Gamma) \times \mathcal{H}_d(\Gamma)$. Moreover, $J^\phi(\rho,\rho) = 1$ and $J^\phi(\rho,\eta) \geq 1$ for all $\rho, \eta \in \mathcal{H}_d(\Gamma)$.*

If $\phi \in \Delta$, then we can again define a ϕ-pressure form $\mathbb{P}^\phi$ on $\mathcal{H}_d(\Gamma)$ by considering the Hessian of the renormalized pressure intersection J^ϕ. Corollary 8.4 allows one to use the thermodynamic formalism for countable Markov shifts developed by Mauldin–Urbanski and Sarig to produce pressure metrics on Hitchin components of general finitely generated, torsion-free Fuchsian groups. (One can embed Hitchin components of finitely generated Fuchsian groups with torsion into Hitchin components of finitely generated, torsion-free Fuchsian groups and thus obtain pressure metrics on them as well.) Recall that $\mathrm{Mod}(\Gamma)$ is the group of (isotopy classes of) orientation-preserving self-homeomorphisms of $X = \mathbb{H}^2/\Gamma$.

Theorem 8.5 (Bray–Canary–Kao–Martone [19]). *If Γ is a finitely generated torsion-free Fuchsian group and $d \geq 3$, then the pressure forms $\mathbb{P}^{\omega_1}$ is an analytic Riemannian metric on the Hitchin component $\mathcal{H}_d(\Gamma)$ which is invariant under the action of the mapping class group* $\mathrm{Mod}(\Gamma)$.

Canary, Zhang and Zimmer [34] showed that if Γ is a lattice (i.e. $\mathbb{H}^2/\Gamma$ has finite area), then the simple root entropies are constant over $\mathcal{H}_d(\Gamma)$, which generalizes a result of Potrie–Sambarino [99] from the cocompact case.

Theorem 8.6 (Canary–Zhang–Zimmer [34]). *If Γ is a torsion-free Fuchsian lattice and $\rho \in \mathcal{H}_d(\Gamma)$, then $h^{\alpha_k}(\rho) = 1$ for all $k \in \{1,\dots,d-1\}$.*

This allows Bray, Canary, Kao and Martone to establish the non-degeneracy of the simple root pressure metric.

Theorem 8.7 (Bray–Canary–Kao–Martone [19]). *If Γ is a torsion-free Fuchsian lattice and $d \geq 3$, then the pressure form $\mathbb{P}^{\alpha_1}$ is an analytic Riemannian metric on the*

Hitchin component $\mathcal{H}_d(\Gamma)$ *which is invariant under the action of the mapping class group* $\mathrm{Mod}(\Gamma)$.

The analysis of the Hilbert length pressure metric was carried out for all finitely generated Fuchsian groups, yielding the following general result.

Theorem 8.8 (Bray–Canary–Kao–Martone [19]). *If* Γ *is a finitely generated torsion-free Fuchsian group and* $\vec{v} \in T_\rho \mathcal{H}_3(\Gamma)$ *is non-zero, then* $\mathbb{P}^{\omega_H}(\vec{v}, \vec{v}) = 0$ *if and only if* $\vec{v}$ *is a self-dual vector. Therefore, the pressure form* $\mathbb{P}^{\omega_H}$ *gives rise to a mapping class group invariant path metric which is an analytic Riemannian metric away from the Fuchsian locus.*

Canary, Zhang and Zimmer [34] are also able to generalize the entropy rigidity theorem of Rafael Potrie and Andres Sambarino [99]. (In the process they obtain a result for the Hausdorff dimension of $(1, 1, 2)$-hypertransverse groups which is a common generalization of the results of Beatrice Pozzetti, Andres Sambarino and Anna Wienhard [100] for $(1, 1, 2)$-hyperconvex Anosov representations and those of Chris Bishop and Peter Jones [7] for discrete subgroups of $\mathsf{SO}(d, 1)$.) We recall that Sambarino [107], generalizing earlier unpublished work of Olivier Guichard, showed that if $\rho\colon \Gamma \to \mathsf{PSL}(d, \mathbb{R})$ is Hitchin, then either $\rho(\Gamma)$ is Zariski dense or its Zariski closure is conjugate to either $\tau_d(\mathsf{PSL}(2, \mathbb{R}))$ (in which case ρ is d-Fuchsian), G_2 (in which case $d = 7$), $\mathsf{PSO}(n, n-1)$ (in which case $d = 2n-1$) or $\mathsf{PSp}(2n, \mathbb{R})$ (in which case $d = 2n$).

Theorem 8.9 (Canary–Zhang–Zimmer [34]). *If* Γ *is a finitely generated Fuchsian group,* $\rho \in \mathcal{H}_d(\Gamma)$ *and* $\phi = \sum c_i \alpha_i \in \Delta$, *then*

$$h^\phi(\rho) \leq \frac{1}{c_1 + \cdots + c_{d-1}}.$$

Moreover, equality occurs exactly when Γ *is a lattice and either*

(1) ρ *is* d*-Fuchsian;*

(2) $\phi = c_k \alpha_k$ *for some* k*;*

(3) $d = 2n$, *the Zariski closure of* $\rho(\Gamma)$ *is conjugate into* $\mathsf{PSp}(2n, \mathbb{R})$ *and*

$$\phi = c_k \alpha_k + c_{d-k} \alpha_{d-k}$$

for some k*;*

(4) $d = 2n - 1$, *the Zariski closure of* $\rho(\Gamma)$ *is conjugate into* $\mathsf{PSO}(n, n-1)$ *and*

$$\phi = c_k \alpha_k + c_{d-k} \alpha_{d-k}$$

for some k*;*

(5) $d = 7$, *the Zariski closure of* $\rho(\Gamma)$ *is conjugate to* G_2, *and*

$$\phi = c_1 \alpha_k + c_3 \alpha_3 + c_4 \alpha_4 + c_6 \alpha_6.$$

Remarks. (1) Hitchin representations of finitely generated groups are relatively dominated, in the sense developed by Feng Zhu [129, 130], and relatively Anosov from various viewpoints developed in the work of Misha Kapovich and Bernhard Leeb [60]. Many of the properties detailed in Theorem 8.1 can be derived in their frameworks. Our main motivation for developing our viewpoint was to establish the analytic variation of the limit map, which was not yet available from either of the previous viewpoints.

(2) Our work with Bray, Kao and Martone, was partially inspired by the work of Barbara Schapira and Samuel Tapie [111]. In particular, their work develops the notion of an entropy gap at infinity for a geodesic flow on a negatively curved manifold, which we adapt in our setting of Hölder potentials on "well-behaved" countable Markov shifts, see also Velozo [122]. One may obtain related counting and equidistribution results for cusped Hitchin representations using the theory in Paulin-Pollicott-Schapira [96] and/or Schapira–Tapie [110].

9. The augmented Hitchin component

In this section, we recall the theory of the augmented Teichmüller space of a closed surface from classical Teichmüller theory and describe an analogous conjectural geometric picture of the augmented Hitchin component. The Hitchin component $\mathcal{H}_3(S)$ can be viewed as the space of (marked) real projective surfaces homeomorphic to S, so the analogy is easiest to discuss when $d = 3$. Moreover, John Loftin and Tengren Zhang [83] have explored the topological picture when $d = 3$. We will restrict our discussion to this case, although we hope that there is an analogous picture for all d.

Augmented Teichmüller space

The augmented Teichmüller space $\widehat{\mathcal{T}}(S)$ of a closed orientable surface S of genus $g \geq 2$, is obtained from Teichmüller space by appending all finite area hyperbolic surfaces obtained by pinching a collection of disjoint simple closed curves on S. It was introduced by Bill Abikoff [1] who proved that the mapping class group $\mathrm{Mod}(S)$ acts properly discontinuously on $\widehat{\mathcal{T}}(S)$ and that its quotient is homeomorphic to the Deligne–Mumford compactification of the Moduli space of S. (Recall that the Moduli space of S is the quotient of $\mathcal{T}(S)$ by the action of $\mathrm{Mod}(S)$.) As such, one may view the augmented Teichmüller space as the "orbifold universal cover" of the Deligne–Mumford compactification of Moduli space.

The most concrete way to describe the augmented Teichmüller space is to look at the local coordinates given by extending the Fenchel–Nielsen coordinates on Teichmüller space. Suppose that $P = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ is a pants decomposition, i.e. a collection of disjoint simple closed curves decomposing S into $2g - 2$ pairs of pants (subsurfaces homeomorphic to a twice-punctured disk). The Fenchel–Nielsen coordinates associate to each hyperbolic surface in $\mathcal{T}(S)$ and each curve a positive real co-ordinate which is the length of the associated geodesic in the surface and another real coordinate which

records the "twist" about the curve. (The choice of twist co-ordinate involves additional choices, but is canonical once appropriate choices are made.) This results in a real analytic coordinate system for Teichmüller space as

$$\mathcal{T}(S) \cong (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}.$$

If C is any non-empty sub-collection of P, then there is a "stratum at infinity" consisting of (marked) finite area hyperbolic surfaces homeomorphic to $S \setminus C$. One naturally, obtains Fenchel–Nielsen coordinates on this strata by looking at the curves in $P \setminus C$, so

$$\mathcal{T}(S \setminus C) \cong (\mathbb{R}_{>0} \times \mathbb{R})^{\#(P \setminus C)} \cong \prod_{R \in S \setminus C} \mathcal{T}(R),$$

where R is a component of $S \setminus C$.

If $C = \{\gamma_1\}$, then one may append the stratum $\mathcal{T}(S \setminus \{\gamma_1\})$ by allowing the length coordinate associated to γ_1 to be 0 and forgetting the twist co-ordinate when this occurs. So,

$$\mathcal{T}(S) \cup \mathcal{T}(S \setminus \{\gamma_1\}) \cong (\mathbb{R}_{>0} \times \mathbb{R})^{3g-4} \times (\mathbb{R}_{\geq 0} \times \mathbb{R})/\sim,$$

where the equivalence relation is given by letting $(0, s) \sim (0, t)$ for all $s, t \in \mathbb{R}$. Notice that the resulting space has the unfortunate property of failing to be locally compact. More generally, one may append all the strata at infinity which are Teichmüller spaces of surfaces pinched along sub-collections of P at the same time, to obtain

$$\widehat{\mathcal{T}}^P(S) = \left((\mathbb{R}_{\geq 0} \times \mathbb{R})/\sim\right)^{3g-3}.$$

Then, one defines the augmented Teichmüller space $\widehat{\mathcal{T}}(S)$ to be the union of the $\widehat{\mathcal{T}}^P(S)$ over the collection $\mathcal{P}$ of all pants decompositions of S, with the obvious identification of (marked) isometric surfaces and the topology induced by regarding the $\widehat{\mathcal{T}}^P(S)$ as local coordinate systems, so

$$\widehat{\mathcal{T}}(S) = \bigcup_{P \in \mathcal{P}} \widehat{\mathcal{T}}^P(S).$$

We regard $\mathcal{T}(S) = \mathcal{T}(S \setminus \emptyset)$ as the central stratum and each $\mathcal{T}(S \setminus C)$ as a stratum at infinity. This description is discussed more fully in Bill Abikoff's book [2, Section II.3.4].

One may also give a more representation-theoretic viewpoint on the augmented Teichmüller space. One may regard an element of $\mathcal{T}(S \setminus C) \cong \prod_{R \in S \setminus C}$ as a collection $\rho = \{\rho_R\}_{R \in S \setminus C}$ of (conjugacy classes of) representations of $\pi_1(R)$ into $\mathsf{PSL}(2, \mathbb{R})$ where R is a component of $S \setminus C$ and ρ_R takes each curve freely homotopic into the boundary of R to a parabolic element. If each $\rho_i = \{(\rho_i)_{R_i}\}_{R_i \in S \setminus C_i}$ is a collection of elements of $\mathcal{T}(S \setminus C_i)$, where each C_i is a collection of disjoint simple closed curves on S, then we say that $\{\rho_i\}$ converges to $\{(\rho_\infty)_{R_\infty}\}_{R_\infty \in S \setminus C_\infty}$ if C_i is contained in C_∞ for all large enough i, and for each component R_∞ of $S \setminus C_\infty$, the sequence $\{\rho_i|_{R_\infty}\}$ (which makes sense for large enough i) converges (up to conjugation) to $(\rho_\infty)_{R_\infty}$. This viewpoint is explored more fully in the larger setting of representations into $\mathsf{PSL}(2, \mathbb{C})$ by Dick Canary and Pete Storm [32].

Howard Masur [89] showed that the augmented Teichmüller space is homeomorphic to the metric completion of Teichmüller space with the Weil–Petersson metric. Moreover, the induced metric on each stratum at infinity is its Weil–Petersson metric. We will formulate his result in a manner which is convenient for our later description of the conjectural geometric picture of the augmented Hitchin component.

Theorem 9.1 (Masur [89]). *Let d_C^{WP} denote the Weil–Petersson metric on $\mathcal{T}(S \setminus C)$ where C is a (possibly empty) collection of disjoint non-parallel simple closed curves. There exists a complete metric $\widehat{d}$ on $\widehat{\mathcal{T}}(S)$ such that if $\mathcal{T}(S \setminus C)$ is any stratum of $\widehat{\mathcal{T}}(S)$ then the restriction of $\widehat{d}$ to $\mathcal{T}(S \setminus C)$ agrees with d_C^{WP}.*

Augmented Hitchin components

Suhyoung Choi and Bill Goldman [38] showed that if $\rho \in \mathcal{H}_3(S)$, then there exists a strictly convex domain Ω_ρ in the projective plane $\mathbb{RP}^2$ so that $\rho(\pi_1(S))$ acts freely and properly discontinuously on Ω_ρ, so $X_\rho = \Omega_\rho/\rho(\pi_1(S))$ is a strictly convex projective surface. (A domain in $\mathbb{RP}^2$ is strictly convex if its closure lies in an affine chart for $\mathbb{RP}^2$ and is strictly convex in that chart.) Bill Goldman [50] produced coordinates on $\mathcal{H}_3(S)$ which generalize the classical Fenchel–Nielsen coordinates.

John Loftin and Tengren Zhang [83] use a modification of Goldman's coordinates for $\mathcal{H}_3(S)$ which was developed by Tengren Zhang [127]. One again begins with a pants decomposition $P = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of S. For each curve γ_i there are two length coordinates, given by $\alpha_1(\rho(\gamma_i))$ and $\alpha_2(\rho(\gamma_i))$, which determine $\rho(\gamma_i)$ up to conjugacy, and two real-valued coordinates, one of which extends the twist coordinate from the Fuchsian setting and the other of which is called the bulge coordinate. Associated to every pair of pants in $S - P$ there are two real-valued coordinates, which are called internal coordinates. The projective structure on each pair of pants in $S - P$ is determined by the length coordinates of its boundary curves together with its two internal coordinates. In these coordinates,

$$\mathcal{H}_3(S) \cong \left((\mathbb{R}_{>0})^2 \times \mathbb{R}^2\right)^{3g-3} \times (\mathbb{R}^2)^{2g-2}.$$

If C is a non-empty subcollection of the pants decomposition P, then we let $\mathcal{H}_3(S \setminus C)$ denote the space of marked finite area projective structures on $S \setminus C$. Then, both length coordinates associated to any curve in C are 0 and we have no twist or bulge co-ordinate associated to any curve in C, so

$$\mathcal{H}_3(S \setminus C) \cong \left((\mathbb{R}_{>0})^2 \times \mathbb{R}^2\right)^{\#(P \setminus C)} \times (\mathbb{R}^2)^{2g-2} \cong \prod_{R \in S \setminus C} \mathcal{H}_3(R),$$

where $\mathcal{H}_3(R) = \mathcal{H}_3(\Gamma_R)$ where $\mathbb{H}^2/\Gamma_R$ is a finite area hyperbolic surface homeomorphic to R. (Ludovic Marquis [86] also describes a parametrization of $\mathcal{H}_3(S \setminus C)$.) If $C = \{\gamma_1\}$, we append $\mathcal{H}_3(S \setminus \{\gamma_1\})$ to $\mathcal{H}_3(S)$ as a "stratum at infinity" much as we did in the Teichmüller setting by allowing the additional length coordinate pair $(0, 0)$ for the curve γ_1 but ignoring the twist and bulge coordinates whenever this occurs, so

$$\mathcal{H}_3(S) \cup \mathcal{H}_3\big(S \setminus \{\gamma_1\}\big) \cong \left((\mathbb{R}_{>0})^2 \times \mathbb{R}^2\right)^{3g-4} \times \left((\mathbb{R}_{\geq 0})^2 \times \mathbb{R}^2\right)/\sim \times (\mathbb{R}^2)^{2g-2}$$

where $(0,0,a,b) \sim (0,0,c,d)$ for all $a,b,c,d \in \mathbb{R}$. Again, the resulting space fails to be locally compact. If we let $\widehat{\mathcal{H}}_3^P(S)$ be the result of appending $\mathcal{H}_3(S \setminus C)$ for all non-empty subcollections C of P, then the resulting space is parametrized as

$$\widehat{\mathcal{H}}_3^P(S) \cong \left(((\mathbb{R}_{\geq 0})^2 \times \mathbb{R}^2)/\sim\right)^{3g-3} \times (\mathbb{R}^2)^{2g-2}.$$

Then, one defines the augmented Hitchin component $\widehat{\mathcal{H}}_3(S)$ to be the union of the $\widehat{\mathcal{H}}_3^P(S)$ over the collection $\mathcal{P}$ of all pants decompositions of S, with the obvious identification of (marked) convex projective surfaces and the topology induced by regarding the $\widehat{\mathcal{H}}_3^P(S)$ as local coordinate systems, so

$$\widehat{\mathcal{H}}_3(S) = \bigcup_{P \in \mathcal{P}} \widehat{\mathcal{H}}_3^P(S).$$

Notice that if $\mathcal{C}$ is the set of (isotopy classes of) non-empty collections of disjoint non-parallel curves on S, then

$$\widehat{\mathcal{H}}_3(S) = \mathcal{H}_3(S) \bigsqcup_{C \in \mathcal{C}} \mathcal{H}_3(S \setminus C)$$

and we regard $\mathcal{H}_3(S)$ as the central stratum and each $\mathcal{H}_3(S \setminus C)$ as a stratum at infinity.

Loftin and Zhang consider a larger augmented Hitchin component where the "pinched" curves are not required to map to unipotent elements. This space is natural from various viewpoints. However, we expect, with very little evidence, that our smaller augmented Hitchin component arises as the metric completion of the Hitchin component with a pressure metric. John Loftin [82] earlier gave a more analytic description of the augmented Hitchin component from the point of view of the parametrization of $\mathcal{H}_3(S)$ as the bundle of cubic differentials over Teichmüller space. This more analytic viewpoint is also useful in the study of the augmented Hitchin component, but we will not discuss it further here.

The geometric picture

We are now ready to give a description of our conjectural geometric picture. If $\phi \in \Delta$, then there is a pressure form $\mathbb{P}^\phi$ on each "stratum at infinity" of $\widehat{\mathcal{H}}_3(S)$ and on the central stratum $\mathcal{H}_3(S)$. This gives rise to a path pseudo-metric d^ϕ on each stratum. If ϕ is ω_1 or α_1, then d_ϕ is an analytic Riemannian metric on each stratum, while if $\phi = \omega_H$ then it is a path metric on each stratum. One way to state our conjecture is the following.

Conjecture. *If ϕ is ω_1, α_1 or ω_H, then d^ϕ extends to a complete metric $\widehat{d}^\phi$ on $\widehat{\mathcal{H}}_3(S)$. In particular, $\widehat{\mathcal{H}}_3(S)$ is the metric completion of $\mathcal{H}_3(S)$ with the pressure metric d^ϕ.*

Even though the metric d^{ω_H} is only a path metric on each stratum, it may be the most natural metric to investigate. If $\phi \in \mathcal{H}_3(S)$, then the Hilbert metric on Ω_ϕ is Gromov hyperbolic and induces a Finsler metric on $X_\rho = \Omega_\rho/\rho(\pi_1(S))$ and $\ell^{\omega_H}(\rho(\gamma))$ is the length of the unique closed geodesic on X_ρ in the free homotopy class of γ, see Marquis [87].

One expects that the conjectural picture described above should generalize to all values of d. In this case, one should be able to replace the Zhang-modified Goldman coordinates, with the coordinates described by Fock–Goncharov [47] or Bonahon–Dreyer [11]. The representation theoretic viewpoint also generalizes, where one replaces the assumption that the images of boundary components of complementary surfaces map to parabolic elements with the assumption that the restriction to each complementary surface R is a Hitchin representation of a finite area uniformization of R.

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