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Logarithmic Fourier decay for self conformal measures

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Abstract

We prove that the Fourier transform of a self conformal measure on $\mathbb{R}$ decays to 0 at infinity at a logarithmic rate, unless the following holds: The underlying IFS is smoothly conjugated to an IFS that both acts linearly on its attractor and contracts by scales that are not Diophantine. Our key technical result is an effective version of a local limit Theorem for cocycles with moderate deviations due to Benoist-Quint (2016), that is of independent interest.

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1 | INTRODUCTION

Let ν be a Borel probability measure on $\mathbb{R}$. For every $q \in \mathbb{R}$ the Fourier transform of ν at q is defined by

$$\mathcal{F}_q(\nu) := \int \exp(2\pi i q x) d\nu(x).$$

The measure ν is called a *Rajchman measure* if $\lim_{|q|\to\infty} \mathcal{F}_q(\nu) = 0$. It is a consequence of the Riemann-Lebesgue Lemma that if ν is absolutely continuous then it is Rajchman. On the other hand, by Wiener's Lemma if ν has an atom then it is not Rajchman. For measures that are both continuous (no atoms) and singular, determining whether or not ν is a Rajchman measure may be a challenging problem even for well structured measures. The Rajchman property has various geometric consequences on the measure ν and its support, for example, regarding the uniqueness problem [27]. Further information about the rate of decay of $\mathcal{F}_q(\nu)$ has even stronger geometric consequences. For example, by a classical Theorem of Davenport-Erdős-LeVeque [14], establishing a sufficiently fast rate of decay for $F_q(\nu)$ is one means towards finding normal numbers in the support of ν. For some further applications of the Rajchamn property and the rate of decay, see the survey [28].

The goal of this paper is to prove that a wide class of fractal measures enjoy logarithmic Fourier decay, assuming some mild conditions are met: Let $\Phi = \{f_1, \dots, f_n\}$ be a finite set of strict contractions of a compact interval $I \subseteq \mathbb{R}$ (an *IFS* - Iterated Function System), such that every f_i is differentiable. We say that Φ is C^α smooth if every f_i is at least C^α smooth for some $\alpha \geqslant 1$. It is well known that there exists a unique compact set $\emptyset \neq K = K_\Phi \subseteq I$ such that

$$K = \bigcup_{i=1}^{n} f_i(K). \tag{1}$$

The set K is called the *attractor* of the IFS $\{f_1, \dots, f_n\}$. We always assume that there exist i, j such that $x_i \neq x_j$, where x_i is the fixed point of f_i. This ensures that K is infinite. We call Φ *uniformly contracting* if

$$0 < \inf\{|f'(x)| : f \in \Phi, x \in I\} \leqslant \sup\{|f'(x)| : f \in \Phi, x \in I\} < 1.$$

Next, writing $\mathcal{A} = \{1, \dots, n\}$, for every $\omega \in \mathcal{A}^{\mathbb{N}}$ and $m \in \mathbb{N}$ let

$$f_{\omega|_m} := f_{\omega_1} \circ \circ \circ f_{\omega_m}.$$

Fix $x_0 \in I$. Then we have a surjective coding map $\pi : \mathcal{A}^{\mathbb{N}} \to K$ defined by

$$\omega \in \mathcal{A}^{\mathbb{N}} \mapsto x_\omega := \lim_{m\to\infty} f_{\omega|_m}(x_0),$$

which is a well defined map because of uniform contraction (see e.g., [5, Section 2.1]).

Let $\mathbf{p} = (p_1, \dots, p_n)$ be a strictly positive probability vector, that is, $p_i > 0$ for all i and $\sum_i p_i = 1$, and let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be the corresponding Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. We call the measure $\nu = \pi\mathbb{P}$ on K the self conformal measure corresponding to $\mathbf{p}$, and note that our assumptions are known to imply that it is non-atomic. Equivalently, ν is the unique Borel probability measure on K such that

$$\nu = \sum_{i=1}^{n} p_i \cdot f_i\nu, \quad \text{where } f_i\nu \text{ is the push-forward of } \nu \text{ via } f_i.$$

When all the maps in Φ are affine we call Φ a self-similar IFS and ν a self-similar measure.

Next, we say that a C^1 IFS Φ is *Diophantine* if there are $l, C > 0$ such that

$$\inf_{y\in\mathbb{R}} \max_{i\in\{1,\dots n\}} d\big(\log|f_i'(x_i)| \cdot x + y,\, \mathbb{Z}\big) \geqslant \frac{C}{|x|^l}, \text{ for all } x \in \mathbb{R} \text{ large enough in absolute value.} \tag{2}$$

This condition is adopted from the work of Breuillard [8] on effective local limit Theorems for classical random walks on $\mathbb{R}^d$, and serves a similar purpose for us as well. Note that it is invariant under conjugation by C^1 maps with non-vanishing derivative. Next, we say that a C^2 IFS Ψ is *linear* if $g''(x) = 0$ for every $x \in K_\Psi$ and $g \in \Psi$. Note that if Ψ is C^ω and linear then it must be self-similar.

We believe it is possible to construct a linear C^2 IFS that includes maps with non-locally constant derivative on the attractor, and we hope to discuss this in a future work.

Let $\mathcal{L}$ denote the family of all Borel probability measures on $\mathbb{R}$ that have logarithmic Fourier decay. That is, writing $\mathcal{P}(\mathbb{R})$ for the family of Borel probability measures on $\mathbb{R}$,

$$\mathcal{L} := \{\mu : \mu \in \mathcal{P}(\mathbb{R}) \text{ and there exists } \alpha > 0 \text{ such that } \left|\mathcal{F}_q(\nu)\right| \leqslant O\left(\frac{1}{|\log |q||^{\alpha}}\right), \text{ as } |q| \to \infty\}.$$

The following Theorem is the main result of this paper. We say that an IFS Φ is C^r conjugate to an IFS Ψ if there is a C^r diffeomorphism h such that $\Phi = \{h \circ g \circ h^{-1}\}_{g\in\Psi}$.

Theorem 1.1. *Let Φ be an orientation preserving uniformly contracting C^r IFS, where $r \geqslant 2$. If there exists a self conformal measure that is not in $\mathcal{L}$ then Φ is C^r conjugate to a linear non-Diophantine IFS.*

Several remarks are in order: First, Theorem 1.1 improves our previous work [1, Theorem 1.1 and Corollary 1.2] by establishing a rate of decay in many new cases (our previous work was effective only for Diophantine self similar IFSs). Secondly, the orientation preserving assumption is made purely for notational convenience, and can be easily dropped. Finally, we emphasize that no separation conditions are imposed on Φ.

Recent years have seen an explosion of interest and progress regarding the study of Fourier decay for fractal measures. We proceed to give a concise overview of results related to Theorem 1.1, and refer to [1, Section 1] for more details on, for example, the methods involved: Combining the work of Bourgain-Dyatlov [6] with [24], Li [25] proved the Rajchman property for Furstenberg measures for $SL(2,\mathbb{R})$ cocycles under mild assumptions (there are known conditions that ensure that such measures are self-conformal [3, 39]). Sahlsten-Stevens [33, 34] proved the Rajchman property for Gibbs measures on C^{ω} self-conformal sets under some additional assumptions. These include the strong separation condition (i.e., that the union (1) is disjoint), and a stronger non-linearity assumption: The IFS is not conjugate to an IFS where the derivatives of the maps are locally constant on its attractor. Our previous work [1, Corollary 1.2 part (3)] gave a unified proof of the Rajchman property for many of these cases, and Theorem 1.1 further upgrades this result by establishing a logarithmic rate of decay. On the other hand, Bourgain-Dyatlov, Li, and Sahlsten-Stevens, establish a polynomial rate of decay, but these works require various further assumptions. We believe that when Φ is not C^r conjugate to a linear IFS then the assumptions of Theorem 1.1 should ensure that all self conformal measures have polynomial Fourier decay. See the end of this introduction for some more discussion about this issue.

Next, suppose Φ is a C^2 IFS that is smoothly conjugated to a self similar IFS with contractions ratios $\{r_1, \dots, r_n\} \subset \mathbb{R}_+$ such that: There exist $C > 0, l > 2$ with

$$\max_{i\in\{1,\dots n\}} d(\log |r_i| \cdot x, \mathbb{Z}) \geqslant \frac{C}{|x|^l}, \text{ for all } x \in \mathbb{R} \text{ large enough in absolute value.} \tag{3}$$

Then, by [1, Remark 6.7], there exists a C^2 IFS Ψ as in Theorem 1.1 that satisfies (2), and every self conformal measure with respect to Φ is also self-conformal with respect to Ψ. So, in the conjugate-to-self-similar situation, it is enough to assume the self-similar IFS meets condition (3) in order for all self-conformal measures to be in $\mathcal{L}$. This generalizes an effective decay result of Li-Sahlsten [27, Theorem 1.3] for self-similar measures.

In the context of self-similar IFSs, when all contraction ratios are powers of some $r \in (0,1)$, Varjú-Yu [38] proved logarithmic decay as long as r^{-1} is not a Pisot or a Salem number. Kaufman [23] and Mosquera-Shmerkin [30] proved polynomial Fourier decay for C^2 non-linear IFS's that arise by conjugating homogeneous (that are never Diophantine) self-similar IFS's. Solomyak [35, 36] has recently shown that in fact, outside a zero Hausdorff dimension exceptional set of parameters, self-similar measures on $\mathbb{R}$ and certain self-affine measures always have polynomial Fourier decay. Brémont [7] recently resolved the Rajchman problem for self-similar measures on $\mathbb{R}$, and Rapaport [32] extended this to self-similar measures on $\mathbb{R}^d$ for any $d \geqslant 1$ (see also [26]). Finally, we mention the classical work of Erdős [16] and Kahane [22] about polynomial decay being typical for Bernoulli convolutions, and the more recent works [9, 12, 13] about rates in some explicit examples of Bernoulli convolutions.

Let us now outline the proof of Theorem 1.1, and along the way describe the orgnization of this paper. Fix Φ as in Theorem 1.1, and assume it is either Diophantine or not-conjugate-to-linear. We aim to show that all self-conformal measures are in $\mathcal{L}$, which implies Theorem 1.1 since the Diophantine condition (2) is invariant under smooth conjugation. We begin with Section 2, where we define the derivative cocycle of the IFS and the transfer operator corresponding to it and to a fixed probability vector $\mathbf{p}$ as above, and recall some known results about it. We then proceed to prove Theorem 2.5, an estimate on the norm of iterations of the transfer operator, which requires some delicate analysis that is closely related to the work of Dolgopyat [15]. In particular, in the not-conjugate-to-linear case we will make use of the so-called temporal distance function [15, Appendix A.1]. This is, to the best of our information, the first such anylsis to be done in the context of general C^2 IFS's without separation (in the presence of separation there are numreous papers that conduct similar analyses e.g., the work of Naud [31] for separated C^ω IFS's).

Afterwards, in Section 3, we show that certain random walks driven by the derivative cocycle satisfy an effective version of the central limit Theorem. This is Theorem 3.1, that follows from a standard application of the Nagaev-Guivarc'h method as presented in the work of Gouëzel [20]. Thus, all we have to do to this end is to verify that the conditions of [20, Theorem 3.7] are met, which is a consequence of well known results that are discussed in Section 2.2.

Section 4 then contain the most subtle step towards Theorem 1.1, and the main technical result of this paper: We prove an *effective* version of Benoist-Quint's local limit Theorem with moderate deviations [4, Theorem 16.1] for random walks driven by the derivative cocycle. Here we combine our estimates on the contraction properties of the transfer operator obtained in Theorem 2.5 with the work of Breuillard [8], who proved effective local limit Theorems for classical random walks on $\mathbb{R}^d$ under a Diophantine condition similar to (2), and with the work of Benoist-Quint [4, Chapter 16], to derive our local limit Theorem 4.1.

In Section 5 we use these effective limit Theorems to obtain a certain effective conditional local limit Theorem for the derivative cocycle. This is Theorem 5.4, which is an upgraded version of our previous result [1, Theorem 3.7] as it is effective (holds with a polynomial rate). Finally, in Section 6, we show that all self conformal measures belong to $\mathcal{L}$. To this end we combine Theorem 5.4 with a delicate linerization scheme, and a more robust estimation of certain oscillatory integrals as in [1, Section 4.2].

Finally, we remark that in the not-conjugate-to-linear case it might be possible to further upgrade our local limit Theorem 4.1 to hold with an exponential rate of convergence. This would be an important step towards showing that in this case all self-conformal measures have polynomial Fourier decay. Also, it is possible that Theorem 1.1 is optimal in the Diophantine case, since such IFSs may be self-similar, where much less is known regarding polynomial Fourier decay in concrete cases (though "most" self-similar measures do have polynomial decay as shown in the afformentioned work of Solomyak [36]).

2 THE DERIVATIVE COCYCLE AND THE ASSOCIATED TRANSFER OPERATOR

2.1 Preliminaries

Fix an orientation preserving C^2 IFS $\Phi = \{f_1, \dots, f_n\}$ and write $\mathcal{A} = \{1, \dots, n\}$. For every $1 \leqslant a \leqslant n$ let $\iota_a : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ be the map

$$\iota_a(\omega_1, \omega_2, \dots) = (a, \omega_1, \omega_2, \dots).$$

Let G to be the free semigroup generated by the family $\{\iota_a : 1 \leqslant a \leqslant n\}$, which acts on $\mathcal{A}^{\mathbb{N}}$ by composing the corresponding ι_a's. We define the derivative cocycle $c : G \times \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ via

$$c(a, \omega) = -\log f_a'(x_\omega). \tag{4}$$

Let $\rho := \sup_{f \in \Phi} ||f'||_\infty \in (0, 1)$, and define a metric on $\mathcal{A}^{\mathbb{N}}$ via

$$d_\rho(\omega, \omega') := \rho^{\min\{n:\ \omega_n \neq \omega_n'\}}. \tag{5}$$

We record the following standard Claim for future use:

Claim 2.1. For every $a \in \mathcal{A}$ the following statements hold true:

1. The map ι_a is uniformly contracting:

$$d(\iota_a(\omega), \iota_a(\eta)) = \rho d(\omega, \eta).$$

2. The cocycle $c(a, \omega)$ is uniformly bounded, Lipschitz in ω, with a uniformly bounded Lipschitz constant as $a \in \mathcal{A}$ varies.

This is standard, since all the maps in Φ are C^2 smooth, and since by uniform contraction

$$0 < D := \min\{-\log|f'(x)| : f \in \Phi, x \in I\}, \quad D' := \max\{-\log|f'(x)| : f \in \Phi, x \in I\} < \infty. \tag{6}$$

Next, let $H^1 = H^1(\rho)$ denote the space of Lipschitz functions $\mathcal{A}^{\mathbb{N}} \to \mathbb{C}$ in the metric d_ρ, and equip H^1 with the norm

$$|\varphi|_1 = ||\varphi||_\infty + c_1(\varphi), \text{ where } c_1(\varphi) = \sup_{\omega \neq \omega'} \frac{|\varphi(\omega) - \varphi(\omega')|}{d_\rho(\omega, \omega')} = \text{ the Lipschitz constant of } \varphi. \tag{7}$$

Following Dolgopyat [15, Section 6], for every $\theta \neq 0$ we define yet another norm on H^1 via

$$||\varphi||_{(\theta)} = \max\left\{ ||\varphi||_\infty, \frac{c_1(\varphi)}{2C_6|\theta|} \right\} \tag{8}$$

for a constant $C_6 > 0$ whose exact choice will be explained soon.

Next, let $\mathbf{p} = (p_1, \dots, p_n)$ be a strictly positive probability vector on $\mathcal{A}$, and let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be the corresponding product measure on $\mathcal{A}^{\mathbb{N}}$. Note that $\mathbb{P}$ is the unique stationary measure corresponding to the measure $\mu := \sum_{a\in\mathcal{A}} p_a \cdot \delta_{\{\iota_a\}}$ on G.

Definition 2.2. For every $\theta \in \mathbb{R}$ let $P_{i\theta} : H^1 \to H^1$ denote the transfer operator defined by, for $\phi \in H^1$ and $\omega \in \mathcal{A}^{\mathbb{N}}$,

$$P_{i\theta}(\phi)(\omega) = \int e^{2\pi i\theta c(a,\omega)}\phi(\iota_a(\omega))d\mathbf{p}(a).$$

We can now remark that the constant $C_6 > 0$ is chosen so that $||P^n_{i\theta}||_{(\theta)} \leqslant 1$ for all n - see [15, Proposition 2] for more details.

2.2 | Some properties of the transfer operator

In this Section we recall some properties of the family of operators $\{P_{i\theta}\}_{\theta\in\mathbb{R}}$, working with the norm (7) on H^1. We begin with following standard results:

Theorem 2.3. *Suppose Φ satisfies the conditions of Theorem 1.1. Let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. Then the following properties hold true:*

1. *[4, Lemma 11.17] $P_{i\theta}$ is an analytic function of θ.*
2. *[4, Lemma 15.1 and Lemma 15.3] The constant function $\mathbf{1} \in H^1$ is an isolated and simple eigenvalue of P_{i0}. All other eigenvalues of P_{i0} have absolute value less than 1, and its essential spectrum is strictly contained inside the unit disc.*

Let us take a moment to explain how our setup fits into the more general one outlined in the work of Benoist-Quint [4]: With the notations of [4, Chapter 11], our acting semigroup is G as in the beginning of Section 2.1, F is the trivial group (and so is the morphism s), and E is simply taken to be $\mathbb{R}$. Recalling the definition of the measure μ on G from before Definition 2.2, the compact metric G-space on which G is $(\mu, 1)$-contracting is taken to be $\mathcal{A}^{\mathbb{N}}$ (this follows from Claim 2.1 part (1)), and recall that $\mathbb{P}$ is the unique stationary measure. Since μ is finitely supported, via Claim 2.1 part (2) our cocycle c trivially has both finite exponential moment and its Lipschitz constant has finite moment [4, Eq. (11.14) and (11.15)]. Thus, in our setup [4, Lemmas 11.17, 15.1, 15.3] can all be applied. Now, [4, Lemma 11.17] immediately implies Theorem 2.3 part (1). Since, by [4, Equation (15.3)] the only eigenfunction of modulus 1 of P_{i0} is $\mathbf{1}$, [4, Lemma 15.1 and Lemma 15.3] imply Theorem 2.3 part (2).

We proceed to recall some results proved by Benoist-Quint [4] regarding certain contraction properties of $P_{i\theta}$: For every small enough θ the operator $N_{i\theta} : H^1 \to H^1$ is defined in [4, Lemma 11.18] as an analytic continuation of the operator $N_0(\varphi) = N(\varphi) = \mathbb{P}(\varphi)$. Furthermore, $N_{i\theta}$ is the projection onto the one dimensional eigenspace spanned by the eigenvector with the leading eigenvalue $\lambda_{i\theta}$ of $P_{i\theta}$. The local behaviour of $\lambda_{i\theta}$ near 0 plays a crucial role in the analysis of Benoist-Quint [4, Parts *iii* and *iv*], and also in our work.

In the following Proposition we use the standard re-centring trick [4, Equation (3.9)] and assume

$$\chi = \chi_{\mathbf{p}} = \int c(a,\omega)d\mathbf{p}(a)d\mathbb{P}(\omega) = 0. \tag{9}$$

Notice that this amounts to changing the cocycle c to a re-centred version $c - \chi$, which only adds a constant phase to $P_{i\theta}$, so it does not affect its norm (note that χ equals $\sigma_{\mathbf{p}}$ in the notations of [4, Equation (3.9)]).

Proposition 2.4 (Benoist-Quint). *Assume the conditions of Theorem 2.3 hold, and suppose in addition that Φ is either Diophantine or not-conjugate-to-linear. Then we have:*

1. *[4, Corollary 15.2] Let $J \subset \mathbb{R}_+$ be a compact set such that $0 \notin J$. Then there are $n_0 \in \mathbb{N}$ and $C' \in (0,1)$ such that*

$$\sup_{\theta\in J} ||P_{i\theta}^{n_0}||_1 < C' < 1.$$

2. *[4, Lemmas 11.18 and 11.19] For every $\epsilon > 0$ small enough there is some constant $C'' \in (0,1)$ such that*

$$\sup_{|\theta|\in[0,\epsilon]} ||P_{i\theta}^{n}||_1 \leqslant 2e^{-C''\cdot\theta^2\cdot n}.$$

Here we are using the norm from (7) for the operator norm, as in [4, Chapter 11.3]. Now, part (1) follows from [4, Corollary 15.2] since our assumptions are known to imply that $P_{i\theta}$ does not have an eigenfunction of modulus 1 for $\theta \neq 0$: This follows from, for example, [1, Section 6.1] in the Diophantine case, and from [1, Section 6.4] in the not-conjugate-to-linear case. To derive part (2) from [4, Lemmas 11.18 and 11.19] we need to explain why here the variance $r_0 = r_0(\mathbf{p})$ of the associated Gaussian as in the central limit Theorem [4, Theorem 12.1 part (i)] (see also Section 3) satisfies that $r_0 > 0$: Recall that I is an interval such that every $f \in \Phi$ is a self map of I. We can define a derivative cocycle c' directly on $\mathcal{A} \times I$ via

$$c'(i,x) = -\log f_i'(x).$$

It is well known that having $r_0 = 0$ implies that the cocycle c' is C^1 co-homologous to a constant (see e.g., [1, Section 6.4] for a very similar argument). This is clearly impossible if Φ is Diophantine. In addition, if c' is C^1 co-homologous to a constant, then a standard argument shows that Φ is conjugate to linear (in fact, this argument is included in the proof of Claim 2.13 below). Thus, in our setting $r_0 > 0$, and so one may use the Taylor-Young formula for $\lambda_{i\theta}$ obtained via [4, Lemmas 11.18 and 11.19] similarly to, for example, [20, third paragraph in the proof of Theorem 3.7] to derive part (2) (Note: in that proof $\lambda_{i\theta}$ is denoted by $\lambda(t)$).

2.3 | Contraction properties of $P_{i\theta}$ for large θ

As in the work of Dolgopyat [15], for every $\beta > 0$ and $\theta \in \mathbb{R}$ let

$$n(\beta, \theta) = [\beta \cdot \log |\theta|].$$

The following Theorem is the key behind the proof of our effective local limit Theorem with moderate deviations, Theorem 4.1:

Theorem 2.5. *Suppose Φ satisfies the conditions of Theorem 1.1 and is either Diophantine or not-conjugate-to-linear. Then there are $\alpha, \beta, C > 0$ such that for every $|\theta| > 1$ we have*

$$||P_{i\theta}^{n(\beta,\theta)}||_{(\theta)} \leqslant 1 - \frac{C}{|\theta|^{\alpha}}$$

where the operator norm is taken with respect to the norm $|| \cdot ||_{(\theta)}$.

The proof of Theorem 2.5 relies on some ideas going back to the work of Dolgopyat [15]. First, we will need:

Lemma 2.6 [15, Lemma 3]. *Let $\alpha > 0$. If there is some $\beta > 0$ such that for every θ with $|\theta| > 1$ and for every $\varphi \in H^1$ with $||\varphi||_{(\theta)} \leqslant 1$ there exists some $\omega_0 \in \mathcal{A}^{\mathbb{N}}$ and $0 \leqslant n \leqslant 3n(\beta, \theta)$ such that*

$$\left| P_{i\theta}^{n}(\varphi)(\omega_0) \right| \leqslant 1 - \frac{1}{|\theta|^{\alpha}},$$

then there exist $\tilde{\beta}, C_{15}, \alpha_9 > 0$ such that for every $|\theta| > 1$

$$||P_{i\theta}^{n(\tilde{\beta},\theta)}||_{(\theta)} \leqslant 1 - \frac{C_{15}}{4|\theta|^{\alpha_9}}.$$

We remark that α_9 is related to α and to the entropy of $\mathbb{P}$. Notice that the formal conclusion of [15, Lemma 3] is different from that of Lemma 2.6. Nonetheless, the conclusion of Lemma 2.6 follows from the proof of [15, Lemma 3] - which is explicitly stated in the argument (for the readers' convenience we use the same notation α_9, C_{15} as in [15]).

Next, we recall what happens if Φ fails the conditions of Lemma 2.6. First, we require the following Definition:

Definition 2.7. We say Φ has the approximate eigenfunctions (*AAE*) property if for every $\alpha_0 > 0$ there are $\alpha, \beta > \alpha_0$ such that one can find arbitrarily large θ satisfying:

There are $\Theta = \Theta(\theta) \in \mathbb{R}$ and $H = H_\theta \in H^1$ with $|H(\omega)| = 1$ for all $\omega \in \mathcal{A}^{\mathbb{N}}$, such that:

$$\left| e^{i\theta c\left(\omega|_{n(\beta,\theta)},\, \sigma^{n(\beta,\theta)}(\omega)\right)} H\left(\sigma^{n(\beta,\theta)}(\omega)\right) - e^{i\Theta} H(\omega) \right| \leqslant \frac{1}{|\theta|^{\alpha}} \tag{10}$$

and the Lipschitz norm of H satisfies

$$\max\{||H||_{\infty}, c_1(H)\} \leqslant O(|\theta|).$$

We remark that the terminology *AAE* is adopted from [19, Section 4.3.2]. The following Lemma is proved in [15, Section 8]:

Lemma 2.8 [15, Lemma 4]. *If $\alpha > 0$ fails the conditions of Lemma 2.6 for every $\beta > 0$, then there is some $\beta = \beta(\alpha) > 0$ and a sequence $|\theta_k| \to \infty$ with associated sequences of $\Theta_k \in \mathbb{R}$, $H_k \in H^1$ with $|H_k| = 1$, such that (10) holds true for all k. Furthermore, β can be taken to be arbitrarily large, so if Φ fails the conditions of Lemma 2.6 for every $\alpha, \beta > 0$ then it has the AAE property.*

Notice that [15, Lemma 4] is stated in terms of approximate eigenfunctions of iterations of a certain operator defined in [15, Page 2] - our statement avoids this notation, and follows by unwinding Dolgopyat's definitions. Next, a-priori [15, Lemma 4] makes a different assumption, about the norm of the resolvent operator, but for the proof of [15, Lemma 5] (which is the crucial step in the proof) only [15, Equation (3)] is required - and this is precisely the assumption made in Lemma 2.8. We remark that Dolgopyat's extra assumption on the norm of the resolvent operator is required for his analysis in [15, Section 9], which allows him to upgrade the conclusion of Lemma 2.8 into having $\Theta \equiv 0$. We do not know if in our setting such a bound on the norm of the resolvent operator holds true.

2.3.1 | Proof of Theorem 2.5 under the Diophantine condition (2)

We show that if Φ satisfies the Diophantine condition (2) then there is some $\alpha > 0$ that satisfies the conditions of Lemma 2.6. Thus, via the conclusion of Lemma 2.6, Theorem 2.5 will follow. Suppose that $\alpha > 0$ fails the conditions of Lemma 2.6 for every $\beta > 0$. Then by Lemma 2.8 there is some $\beta = \beta(\alpha) > 0$ such that we can find a sequence $|\theta_k| \to \infty$ with associated sequences of $\Theta_k \in \mathbb{R}$, $H_k \in H^1$ with $|H_k| = 1$, such that

$$\left| e^{i\theta_k c\left(\omega|_{n(\beta,\theta_k)},\, \sigma^{n(\beta,\theta_k)}(\omega)\right)} H_k\left(\sigma^{n(\beta,\theta)}(\omega)\right) - e^{i\Theta_k} H(\omega) \right| \leqslant \frac{1}{|\theta_k|^{\alpha}}. \tag{11}$$

Now, for every $a \in \mathcal{A}$ let $\overline{a} \in \mathcal{A}^{\mathbb{N}}$ be the constant sequence a. Let x_a be the fixed point of f_a. It follows from (11) by plugging in $\omega = \overline{a}$, $a \in \mathcal{A}$, that for every k there is some $y_k \in \mathbb{R}$ (that corresponds to Θ_k), such that for $m_a \in \mathbb{Z}$ that may differ between the a's,

$$\theta_k \cdot n(\beta, \theta_k) \cdot \log|f_a'(x_a)| + y_k = 2\pi m_a + O(|\theta_k|^{-\alpha}).$$

Therefore, for all k we get

$$\inf_{y\in\mathbb{R}} \max_{a\in\mathcal{A}} d\left(\frac{1}{2\pi} \cdot \theta_k \cdot n(\beta,\theta_k) \cdot \log|f_a'(x_a)| + y,\, \mathbb{Z}\right) = O(|\theta_k|^{-\alpha}).$$

On the other hand, by the Diophantine condition there are $\ell, C > 0$ such that for every $s \in \mathbb{R}$ large enough in absolute value,

$$\inf_{y\in\mathbb{R}} \max_{a\in\mathcal{A}} d\left(s \cdot \log|f_a'(x_a)| + y,\, \mathbb{Z}\right) \geqslant \frac{C}{|s|^{\ell}}.$$

Combining the last two displayed equations and using that as $k \to \infty$ we have $|\theta_k| \to \infty$, we see that $\alpha \leqslant \ell$. Therefore, for every $\alpha > \ell$ there exists some $\beta > 0$ such that the conditions of Lemma 2.6 hold true. This completes the proof of Theorem 2.5 in this case.

2.3.2 | Proof of Theorem 2.5 assuming Φ is not conjugate to linear

We now prove Theorem 2.5 assuming Φ is not conjugate to linear. First, we require the following definition, that is originally due to Chernov [10].

Definition 2.9 [15, Appendix A.1]. The symbolic temporal distance function $D : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$D(\xi,\zeta,\omega,\eta) := \lim_n \left(\left(\log f'_{\xi|_n}(x_\omega) - \log f'_{\xi|_n}(x_\eta) \right) - \left(\log f'_{\zeta|_n}(x_\omega) - \log f'_{\zeta|_n}(x_\eta) \right) \right).$$

The Euclidean temporal distance function $E : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \times I \times I \to \mathbb{R}$, where I is the interval Φ is acting on, is defined by

$$D(\xi,\zeta,x,y) := \lim_n \left(\left(\log f'_{\xi|_n}(x) - \log f'_{\xi|_n}(y) \right) - \left(\log f'_{\zeta|_n}(x) - \log f'_{\zeta|_n}(y) \right) \right).$$

Notice that D and E are well defined since Φ is uniformly contracting and C^2. The following Theorem is essentially [15, Theorem 6], with some variations similar to [29, Theorem 5.6]. For a bounded set $X \subset \mathbb{R}$ we denote its lower box dimension by $\underline{\dim}_B X$.

Theorem 2.10. *If Φ has the AAE property then* $\underline{\dim}_B D(\mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}}) = 0$.

Proof. First, for every $n \in \mathbb{N}$ and $(\xi,\zeta,\omega,\eta) \in (\mathcal{A}^{\mathbb{N}})^4$ we define

$$D_n(\xi,\zeta,\omega,\eta) := \left(\log f'_{\xi|_n}(x_\omega) - \log f'_{\xi|_n}(x_\eta) \right) - \left(\log f'_{\zeta|_n}(x_\omega) - \log f'_{\zeta|_n}(x_\eta) \right)$$

and notice that, since $\rho = \sup_{f\in\Phi} ||f'||_\infty < 1$, we have

$$D(\xi,\zeta,\omega,\eta) = D_n(\xi,\zeta,\omega,\eta) + O(\rho^n).$$

Combining this with the definition of c,

$$\begin{aligned}
\exp(i\theta D(\xi,\zeta,\omega,\eta)) &= \exp(i\theta D_n(\xi,\zeta,\omega,\eta)) + O(|\theta|\cdot\rho^n)\\
&= \frac{\exp\left(i\theta \log f'_{\xi|_n}(x_\omega)\right)}{\exp\left(i\theta \log f'_{\xi|_n}(x_\eta)\right)} \cdot \frac{\exp\left(i\theta \log f'_{\zeta|_n}(x_\eta)\right)}{\exp\left(i\theta \log f'_{\zeta|_n}(x_\omega)\right)} + O(|\theta|\cdot\rho^n)\\
&= \frac{\exp(i\theta c((\xi|_n.\omega)|_n, \sigma^n(\xi|_n.\omega)))}{\exp(i\theta c((\xi|_n.\eta)|_n, \sigma^n(\xi|_n.\eta)))} \cdot \frac{\exp(i\theta c((\zeta|_n.\eta)|_n, \sigma^n(\zeta|_n.\eta)))}{\exp(i\theta c((\zeta|_n.\omega)|_n, \sigma^n(\zeta|_n.\omega)))}\\
&\quad + O(|\theta|\cdot\rho^n).
\end{aligned}$$

Let $\alpha_0 > 0$ be fixed, and let $\alpha, \beta > \alpha_0$. Using the AAE property and the equation above, we can find arbitrarily large θ and $H = H_\theta \in H^1$ as in Definition 2.7 such that we have

$$\exp(i\theta D(\xi,\zeta,\omega,\eta)) = \frac{H(\xi|_{n(\beta,\theta)}.\omega)}{H(\xi|_{n(\beta,\theta)}.\eta)} \cdot \frac{H(\zeta|_{n(\beta,\theta)}.\eta)}{H(\zeta|_{n(\beta,\theta)}.\omega)} + O(|\theta|^{-\alpha}) + O(|\theta|\cdot\rho^{n(\beta,\theta)}).$$

Since $n(\beta,\theta) = [\beta \log|\theta|]$, via Lemma 2.8 we may assume β is large enough so that we have

$$|\theta|\rho^{n(\beta,\theta)} \leqslant \rho^{-1} \cdot |\theta|^{-\alpha}.$$

Therefore, since $|H| \equiv 1$ we have

$$\left|\frac{H(\xi|_{n(\beta,\theta)}.\omega)}{H(\xi|_{n(\beta,\theta)}.\eta)} - 1\right| = \left|H(\xi|_{n(\beta,\theta)}.\omega) - H(\xi|_{n(\beta,\theta)}.\eta)\right| \leqslant c_1(H)\cdot d_\rho(\xi|_{n(\beta,\theta)}.\eta,\, \xi|_{n(\beta,\theta)}.\omega)$$

$$\leqslant O(|\theta|\rho^{n(\beta,\theta)}) \leqslant O(|\theta|^{-\alpha}).$$

Since the same is true for the term corresponding to ζ, it follows that

$$\left|\exp\left(i\theta D(\xi,\zeta,\omega,\eta)\right) - 1\right| = O(|\theta|^{-\alpha})$$

for arbitrarily large θ and every (ξ,η,ω,η). Thus, there is some $C = C(\alpha)$ such that for arbitrarily large θ,

$$D(\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}) \subseteq \bigcup_{j\in\mathbb{Z}}\left(\frac{2\pi j}{|\theta|} - \frac{C}{|\theta|^{\alpha+1}}, \frac{2\pi j}{|\theta|} + \frac{C}{|\theta|^{\alpha+1}}\right).$$

So,

$$\underline{\dim}_B(D(\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}})) \leqslant \frac{1}{\alpha+1}.$$

The result follows since α can be made arbitrarily large. □

Thus, it is our main task to verify that in the not-conjugate-to-linear setting, the box dimension appearing in Theorem 2.10 cannot vanish. To this end, we adopt a variant of Naud's non local integrability condition [31, Definitions 2.1-2.2]:

Lemma 2.11. *If there exist $(\xi,\zeta,\omega,\eta) \in (\mathcal{A}^{\mathbb{N}})^4$ such that the function*

$$g : I \to \mathbb{R}, \quad g(x) = E(\xi,\zeta,x,x_\eta)$$

satisfies that $g'(x_\omega) \neq 0$, then $\underline{\dim}_B D(\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}\times\mathcal{A}^{\mathbb{N}}) > 0$.

In particular, Φ fails the AAE property.

Proof. The assumption means that there is some $n \in \mathbb{N}$ such that g' does not vanish on $f_{\omega|_n}(K)$. This means that g' is bi-Lipschitz on $f_{\omega|_n}(K)$. So,

$$\dim_H D(\xi,\zeta,\mathcal{A}^{\mathbb{N}},\eta) = \dim_H E(\xi,\zeta,K,x_\eta) \geqslant \dim_H E(\xi,\zeta,f_{\omega|_n}(K),x_\eta)$$

$$= g(f_{\omega|_n}(K)) = \dim_H K > 0.$$

The last part of the Lemma now follows via Theorem 2.10. □

We proceed to prove two Claims that together will allow us to verify the conditions of Lemma 2.11 in our setting. We follow the general strategy of Avila-Gouëzel-Yoccoz [2, Proposition 7.4], with some modifications due to the possible lack of separation in the IFS.

Claim 2.12. If there exists some $c > 0$ such that for infinitely many n there are $\xi = \xi(n), \zeta = \zeta(n) \in \mathcal{A}^{\mathbb{N}}$ such that for some $x_0 = x_0(n) \in K$

$$\left|\frac{d}{dx}\left(\log f'_{\xi|_n} - \log f'_{\zeta|_n}\right)(x_0)\right| \geqslant c$$

then the condition of Lemma 2.11 holds.

Proof. Suppose the condition of Lemma 2.11 fails. Then for every $(\xi, \zeta, \eta) \in (\mathcal{A}^{\mathbb{N}})^3$ the corresponding function g as in Lemma 2.11 satisfies $g'(x) = 0$ for every $x \in K$. So, for all $x \in K$ and every n

$$\begin{aligned}
0 &= g'(x) \\
&= \lim_k \frac{d}{dx}\left(\log f'_{\xi|_k}(x) - \log f'_{\zeta|_k}(x)\right) \\
&= \sum_{k=1} \frac{f''_{\xi_k} \circ f_{\xi|_{k-1}}(x) \cdot f'_{\xi|_{k-1}}(x)}{f'_{\xi_k} \circ f_{\xi|_{k-1}}(x)} - \sum_{k=1} \frac{f''_{\zeta_k} \circ f_{\zeta|_{k-1}}(x) \cdot f'_{\zeta|_{k-1}}(x)}{f'_{\zeta_k} \circ f_{\zeta|_{k-1}}(x)} \\
&= \frac{d}{dx}\left(\log f'_{\xi|_n}(x) - \log f'_{\zeta|_n}(x)\right) \\
&\quad + \sum_{k\geqslant n} \frac{f''_{\xi_k} \circ f_{\xi|_{k-1}}(x) \cdot f'_{\xi|_{k-1}}(x)}{f'_{\xi_k} \circ f_{\xi|_{k-1}}(x)} - \sum_{k\geqslant n} \frac{f''_{\zeta_k} \circ f_{\zeta|_{k-1}}(x) \cdot f'_{\zeta|_{k-1}}(x)}{f'_{\zeta_k} \circ f_{\zeta|_{k-1}}(x)}.
\end{aligned}$$

Since $\sup_{x\in I, f\in\Phi} |\frac{f''(x)}{f'(x)}| = O(1)$, we obtain

$$\left\|\frac{d}{dx}\left(\log f'_{\xi|_n} - \log f'_{\zeta|_n}\right)\right\|_{\infty,K} = O(\sup_{f\in\Phi} ||f'||_\infty^{n-1}).$$

This contradicts our assumptions. □

Here is the final ingredient in our proof:

Claim 2.13. If the condition in Claim 2.12 fails then Φ is C^r conjugate to a an IFS Ψ such that

$$g''(x) = 0 \text{ for every } x \in K_\Psi \text{ and } g \in \Psi.$$

Proof. Suppose the condition in Claim 2.12 fails. Then for any $\xi, \zeta \in \mathcal{A}^{\mathbb{N}}$ and any $x \in K$ we have

$$\lim_n \frac{d}{dx} \log f'_{\xi|_n}(x) = \lim_n \frac{d}{dx} \log f'_{\zeta|_n}(x). \tag{12}$$

Now, fix $i \in \mathcal{A}$, and let $\bar{i} \in \mathcal{A}^{\mathbb{N}}$ be the corresponding σ-periodic point. Fix $x_0 \in I$. Define a function $\varphi_i : I \to \mathbb{R}$ via

$$\varphi_i(x) := \lim_n \log f'_{\bar{i}|_n}(x) - \log f'_{\bar{i}|_n}(x_0)$$

It is standard that φ_i is C^{r-1}. Now, for every $x \in I$ we have

$$\begin{aligned}\varphi_i(f_i(x)) = \lim_n \log f'_{\bar{i}|_n} \circ f_i(x) - \log f'_{\bar{i}|_n}(x_0) &= \lim_n \sum_{j \leqslant n} \log f'_i \circ f_{\bar{i}|_{j+1}}(x) - \log f'_{\bar{i}|_n}(x_0) \\ &= \varphi_i(x) - \log f'_i(x) + \log f'_i(x_{\bar{i}}).\end{aligned}$$

Therefore, for any $i \in \mathcal{A}$ and any $x \in I$,

$$\varphi_i \circ f_i(x) = -\log f'_i(x) + \varphi_i(x) + \log f'_i(x_{\bar{i}}). \tag{13}$$

Note that we can produce such a function φ_j for every $j \in \mathcal{A}$. So, for every $j \in \mathcal{A}$ we define a function $d_j : I \to \mathbb{R}$ via

$$d_j(x) = \varphi_1(x) - \varphi_j(x).$$

By (12) for every $x \in K$ we have that $\varphi'_1(x) = \varphi'_j(x)$ so $d'_j(x) = 0$ for all $x \in K$. Also, using (13), for any $x \in I$ and $i \in \mathcal{A}$,

$$\begin{aligned}\varphi_1 \circ f_i(x) = \varphi_i \circ f_i(x) + d_i \circ f_i(x) &= -\log f'_i(x) + \varphi_i(x) + d_i \circ f_i(x) + \log f'_i(x_{\bar{i}}) \\ &= -\log f'_i(x) + \varphi_1(x) - d_i(x) + d_i \circ f_i(x) + \log f'_i(x_{\bar{i}}).\end{aligned}$$

To conclude, for every $i \in \mathcal{A}$ the function $F_i : I \to \mathbb{R}$ defined by

$$F_i(x) := d_i \circ f_i(x) - d_i(x) + \log f'_i(x_{\bar{i}})$$

satisfies that

$$\varphi_1 \circ f_i(x) = -\log f'_i(x) + \varphi_1(x) + F_i(x) \text{ for every } x \in I, \text{ and } F'_i(x) = 0 \text{ for all } x \in K. \tag{14}$$

Finally, let $h : I \to \mathbb{R}$ be a C^r smooth function that is a primitive of $\exp(\varphi_1(x))$ on I. For every $i \in \mathcal{A}$ define a function $g_i : h(I) \to h(I)$ via

$$g_i(x) := h \circ f_i \circ h^{-1} : h(I) \to h(I)$$

and let Ψ be the IFS consisting of the maps g_i. Then Ψ is C^r conjugate to Φ.

We claim that Ψ is a linear IFS. Indeed, by (14), for every $i \in \mathcal{A}$ and every $y \in h(I)$

$$\begin{aligned}g'_i(y) &= \left(h \circ f_i \circ h^{-1}\right)'(y) \\ &= \frac{h'\left(f_i \circ h^{-1}(y)\right) \cdot f'_i(h^{-1}(y))}{h'(h^{-1}(y))}\end{aligned}$$

$$= \exp\left(\varphi_1 \circ f_i \circ h^{-1}(y) + \log\left(f_i' \circ h^{-1}(y)\right) - \varphi_1 \circ h^{-1}(y)\right)$$
$$= \exp\left(F_i \circ h^{-1}(y)\right).$$

Therefore, for every $y \in h(K)$ we have

$$g_i''(y) = F_i'(h^{-1}(y)) \cdot \left(h^{-1}\right)'(y) \cdot \exp\left(F_i \circ h^{-1}(y)\right) = 0$$

as F_i' vanishes on K by (14). Since $h(K)$ is the attractor of Ψ, the proof is complete. □

Proof of Theorem 2.5. We show that in the not-conjugate-to-linear setting there is some $\alpha > 0$ that satisfies the conditions of Lemma 2.6. Thus, via the conclusion of Lemma 2.6, Theorem 2.5 will follow. Indeed, if this is not the case then by Lemma 2.8 Φ has the AAE property. However, since Φ assumed not to be conjugate to linear, by Claim 2.13 the condition in Claim 2.12 holds true. This in turn implies that the the condition of Lemma 2.11 holds true. But by Lemma 2.11 Φ cannot have the AAE property. This is a contradiction. The Theorem is proved. □

3 | AN EFFECTIVE CENTRAL LIMIT THEOREM FOR THE DERIVATIVE COCYCLE

Let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$, and keep the notations and assumptions as in Section 2. In this Section we discuss an effective version of the central limit Theorem for a certain random walk driven by the derivative cocycle (4). This random walk is defined as follows: Denoting by $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ the left shift, for every $n \in \mathbb{N}$ we define a function on $\mathcal{A}^{\mathbb{N}}$ via

$$S_n(\omega) = -\log f'_{\omega|_n}(x_{\sigma^n(\omega)}). \tag{15}$$

Let $X_1 : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ be the random variable

$$X_1(\omega) := c(\omega_1, \sigma(\omega)) = -\log f'_{\omega_1}(x_{\sigma(\omega)}), \tag{16}$$

and note that our assumptions on Φ imply that $X_1 \in H^1$. Next, for every integer $n > 1$ we define

$$X_n(\omega) = -\log f'_{\omega_n}\left(x_{\sigma^n(x_\omega)}\right) = X_1 \circ \sigma^{n-1}.$$

Let κ be the law of the random variable X_1. Then for every n, $X_n \sim \kappa$. By uniform contraction there exists $D, D' \in \mathbb{R}$ as in (6), so $\kappa \in \mathcal{P}([D, D'])$. In particular, the support of κ is bounded away from 0. It is easy to see that for every $n \in \mathbb{N}$ and $\omega \in A^{\mathbb{N}}$ we have

$$S_n(\omega) = \sum_{i=1}^{n} X_i(\omega).$$

Thus, in this sense S_n is a random walk.

We proceed to state a version of the central limit Theorem for the random walk S_n: For $r > 0$ let $N(0, r^2)$ be the distribution of a Gaussian random variable with 0 mean and variance r^2. Also, for any Bernoulli measure $\mathbb{P}$ on $\mathcal{A}^{\mathbb{N}}$ recall that we write $\chi = \chi_{\mathbf{p}} = \int c(a,\omega)d\mathbf{p}(a)d\mathbb{P}(\omega)$. The Berry-Esseen type central limit Theorem we now state follows from a standard application of the Nagaev-Guivarc'h method as presented in the work of Gouëzel [20]:

Theorem 3.1 [20, Theorem 3.7]. *Suppose Φ satisfies the conditions of Theorem 1.1 and is either Diophantine or not-conjugate-to-linear. Let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. Then there exists some $r_0 = r_0(\mathbf{p}) > 0$ such that*

$$\sup_z \left| \mathbb{P}\left(\frac{S_n - n\chi}{\sqrt{n}} \leqslant z \right) - \left(N(0, r_0^2) \leqslant z\right) \right| = O\left(\frac{1}{n^{\frac{1}{2}}} \right)$$

where $(N(0, r_0^2) \leqslant z)$ stands for the probability that $N(0, r_0^2)$ is less than z.

To explain how the setup of Theorem 3.1 fits into the conditions of [20, Theorem 3.7], we note that since c is a cocycle, for every θ the constant function $\mathbf{1} \in H^1$ satisfies

$$\mathbb{E}\left(e^{2\pi i\theta S_n}\right) = \mathbb{E}\left(P_{i\theta}^n(\mathbf{1})\right).$$

This confirms the coding assumption in [20, Theorem 2.4]. The other assumptions of [20, Theorem 2.4] and [20, Theorem 3.7] follow directly from Theorem 2.3 and Proposition 2.4 part (2) (where it is explained why here $r_0 > 0$).

Finally, we remark that the very recent works of Fernando-Liverani [18] and Cuny-Dedecker-Merlevède [11] are closely related to this. We refer the reader to [18, 19] for an exhaustive bibliography of some further related results.

4 | AN EFFECTIVE LOCAL LIMIT THEOREM WITH MODERATE DEVIATIONS

Let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$, and keep the notations and assumptions as in Section 2. For every $n \in \mathbb{N}$ and $\omega \in \mathcal{A}^{\mathbb{N}}$ consider the distribution of the centred n-step random walk driven by c that starts from ω. This distribution is given by a measure $\mu_{n,\omega}$ on $\mathbb{R}$ such that, for $X \subseteq \mathbb{R}$

$$\mu_{n,\omega}(X) = \int 1_X(c(a,\omega) - n\chi)d\mathbf{p}^n(a)$$

where, as in Section 3, χ is the Lyapunov exponent. Let G_n be the density of the n-fold convolution of the Gaussian $N^{*n}(0, r_0^2)$ with r_0 as in Theorem 3.1. That is,

$$G_n(v) = \frac{e^{-\frac{v^2 \cdot r_0^2}{2n}}}{\sqrt{2\pi n}}, \text{ for } v \in \mathbb{R}.$$

The following local limit Theorem is one of the main keys behind the proof of Theorem 1.1. It is an effective version of a local limit Theorem with moderate deviations due to Benoist-Quint [4, Theorem 16.1]. Recall that λ denotes the Lebesgue measure on $\mathbb{R}$.

Theorem 4.1. *Suppose Φ satisfies the conditions of Theorem 1.1 and is either Diophantine or not-conjugate-to-linear. Then for every $R > 0$ there is some $\delta = \delta(\mathbf{p}, R) > 0$ such that for every bounded interval $C \subseteq \mathbb{R}$*

$$\sup\left\{\left|\frac{\mu_{n,\omega}(C+v_n)}{G_n(v_n)} - \lambda(C)\right| \,:\, \omega \in \mathcal{A}^{\mathbb{N}},\ |v_n| \leqslant \sqrt{Rn\log n}\right\} = O_{\lambda(C)}\left(\frac{1}{n^{\delta}}\right),\ \text{as } n \to \infty.$$

Here by $O_{\lambda(C)}(\frac{1}{n^{\delta}})$ we mean that the multiplicative constant inside the big-O depends on $\lambda(C)$, but we do note that it also depends on other universal multiplicative factors and on $\mathbf{p}$. We do not attempt to give more specific quantitative estimates of the rate, although this is possible. This result may be extended to other cocycles taking values in vector spaces over $\mathbb{R}$ subject to certain contraction and moment conditions, along with conditions ensuring that the transfer operator contracts fast enough for large frequencies (as in Theorem 2.5). Also, similarly to, for example, [4, Proposition 16.6] Theorem 4.1 may be adapted to work with a target. However, having the Fourier decay result Theorem 1.1 in our sights, we do not study these more general situations here.

The scheme of proof of Theorem 4.1 is modelled after the proof of Benoist-Quint's local limit Theorem [4, Theorem 16.1], which is essentially the same as Theorem 4.1 but without an explicit rate of decay. The proof of Benoist-Quint roughly follows three main steps: First, they prove a version with the interval C replaced by certain smooth functions on $\mathbb{R}$ [4, Lemma 16.11]. Secondly, they prove that the indicator function 1_C admits "good" approximations via such smooth functions [4, Lemma 16.13]. The third and final step is an estimation of $\frac{\mu_{n,\omega}(C+v_n)}{G_n(v_n)}$ for moderately large $v_n \in \mathbb{R}$ using the previous two steps.

Thus, we will show that the conditions of Theorem 4.1 yield an effective version of [4, Lemma 16.11], the local limit Theorem for smooth functions. Section 4.1, that contains this result, critically relies on Theorem 2.5 to derive certain estimates on an integral that arises from Fourier inversion. This is inspired by the work of Breuillard [8, Lemme 3.1], and is related to the analysis of Fernando-Liverani [18, Theorem 2.4]. Then, in Section 4.2, we show that the proof of [4, Lemma 16.13] actually yields a polynomial error term. We then combine these into a proof of Theorem 4.1 in Section 4.3, following along the lines of [4, Eq. (16.21) and (16.23)].

From this point forward, we use the standard re-centring trick as in (9) and assume $\chi = 0$. This will make our computation a bit simpler. Notice that this amounts to changing the cocycle c to a re-centred version $c - n\chi$, which is precisely how the distributions $\mu_{n,\omega}$ are defined. From now on, this will be our cocycle.

4.1 | Effective local limit theorem for smooth functions

We proceed to prove a version of our effective local limit Theorem for certain smooth functions. This is in accordance with the strategy of Benoist-Quint [4, Section 16.2], but via Theorem 2.5 and ideas going back to Breuillard [8] and Stone [37] we make this Theorem effective.

Fix a non-negative Schwartz function α on $\mathbb{R}$ such that $\lambda(\alpha) = 1$, $||\alpha||_\infty \leqslant 1$, and $\hat{\alpha}$ has compact support, as in [4, Definition 16.8 and Remark 16.9]. For every $\epsilon > 0$ we define

$$\alpha_\epsilon(v) := \frac{1}{\epsilon}\alpha(\frac{v}{\epsilon}).$$

Fix a bounded interval C and define

$$\psi_{\epsilon,C}(v) := \int \alpha_\epsilon(w)1_C(v-w)d\lambda(w) = (\alpha_\epsilon\lambda) * 1_C$$

which is still a non-negative Schwartz function [4, Page 268]. For $f \in C^k(\mathbb{R})$ let

$$C^k(f) = \max_{0\leqslant j\leqslant k} \| f^{(j)} \|_{L^1}, \text{ and (even for more general functions) } \hat{f}(\theta) = \int e^{-i\theta x} f(x)dx$$

and note that for every integer $k \geqslant 1$

$$||\widehat{\psi_{\epsilon,C}}||_\infty \leqslant \lambda(C), \quad ||\psi_{\epsilon,C}||_\infty \leqslant \frac{\lambda(C)}{\epsilon}, \quad C^k(\psi_{\epsilon,C}) \leqslant \lambda(C)\cdot C^k(\alpha_\epsilon) \leqslant O_\alpha\left(\frac{\lambda(C)}{\epsilon^k}\right). \tag{17}$$

Recall the notations $\mu_{n,\omega}, G_n$ introduced before Theorem 4.1. The following is an effective version of [4, Lemma 16.11]:

Theorem 4.2. *Let Φ be as in Theorem 4.1 and let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure. Let $\ell = \alpha + 1$, where α is as in Theorem 2.5.*

Then for every $r \geqslant 2$ there exists $\delta = \delta(r) > 0$ such that, setting $k = \lceil \ell \cdot r + 2\rceil$, for every $\epsilon > 0$ we have

$$\sup_{\omega\in\mathcal{A}^{\mathbb{N}}} \left|\mu_{n,\omega}(\psi_{\epsilon,C}) - \lambda(\psi_{\epsilon,C}\cdot G_n)\right| \leqslant O_{\lambda(C)}\left(\frac{1}{n^\delta}\right)\cdot\lambda(\psi_{\epsilon,C}\cdot G_n) + \frac{1}{\epsilon^k}\cdot O_{\lambda(C)}\left(\frac{1}{n^{\frac{r}{2}}}\right)$$

Notice that in Theorem 4.2 the dependence on ϵ is explicit in the second error term - this will be important later on. Also, here the sequence ϵ_n as in [4, Lemma 16.11] is the polynomially decaying sequence $O_{\lambda(C)}(\frac{1}{n^\delta})$. We can, in fact, indicate a more precise rate - see (21) below.

To prove Theorem 4.2 we utilize Theorem 2.5 and Proposition 2.4 to establish Theorem 4.4, a Breuillard [8, Lemme 3.1] type estimate on large frequencies of an integral that arises via Fourier inversion on $\mu_{n,\omega}(\psi_{\epsilon,C})$. In Section 4.1.2 we show how to derive Theorem 4.2 from this estimate.

4.1.1 | A Breuillard type estimate

We begin by deriving the following Corollary from Theorem 2.5:

Corollary 4.3. *Let $\alpha, \beta, C > 0$ be as in Theorem 2.5. If $|\theta| > 1$ and $n \in \mathbb{N}$ satisfy*

$$n > \log|\theta|\cdot\beta\cdot 2.$$

Then

$$||P_{i\theta}^n(\mathbf{1})||_\infty \leqslant e^{-\frac{n\cdot C}{|\theta|^{\alpha+1}}}.$$

Proof. Let $n_0 = [\beta \cdot \log|\theta|]$. First, for every $k \in \mathbb{N}$ and $|\theta| > 1$ we have by Theorem 2.5

$$||P_{i\theta}^{n_0\cdot k}(\mathbf{1})||_{(\theta)} \leqslant ||P_{i\theta}^{n_0}||_{(\theta)}^k \cdot ||\mathbf{1}||_{(\theta)} \leqslant \left(1 - \frac{C}{|\theta|^\alpha}\right)^k \cdot 1 \leqslant e^{-\frac{Ck}{|\theta|^\alpha}}.$$

Note that in [15, Section 6] the choice of $C_6 > 0$ as in (8) is made so that for every $r \in \mathbb{N}$,

$$||P_{i\theta}^r||_{(\theta)} \leqslant 1.$$

Now, write $n = k \cdot n_0 + r$ where $k = [\frac{n}{n_0}]$. Via the last two displayed equations we see that

$$||P_{i\theta}^n(\mathbf{1})||_\infty \leqslant ||P_{i\theta}^n(\mathbf{1})||_{(\theta)} \leqslant ||P_{i\theta}^{n_0\cdot k}(\mathbf{1})||_{(\theta)} \leqslant e^{\frac{-k}{|\theta|^\alpha}} \leqslant e^{\frac{-n}{|\theta|^{\alpha+1}}}$$

where in the last inequality we use that $|\theta| > 1$. □

Our analysis now allows to estimate one crucial quantity that will come up in the proof of Theorem 4.2. We retain the assumption that the cocycle c is already re-centred, so that (9) holds. The following Theorem is inspired by the work of Breuillard [8, Lemme 3.1] - in fact, *it is* essentially [8, Lemme 3.1] put in our setting.

Theorem 4.4. *Let $r > 0$ and $\ell' = \alpha + 1$ where α is as in Theorem 2.5. Then there is a constant $D(r, \mathbf{p}) > 0$ such that for every $D > D(r, \mathbf{p})$ we have*

$$\int_{|\theta| \geqslant \sqrt{\frac{D\log n}{n}}} \hat{f}(\theta) \cdot P_{i\theta}^n(\mathbf{1})(\omega)d\theta = C^k(f) \cdot o_{\mathbf{p}}(\frac{1}{n^r}) \tag{18}$$

uniformly in $\omega \in \{1, .., n\}^{\mathbb{N}}$ and $f \in C^k$ such that $C^k(f) < \infty$ and $k > \ell' \cdot r + 1$.

The proof of Theorem 4.4 follows by mimicking the proof of [8, Lemme 3.1]: First, the role of $\hat{\mu}(x)$ in [8] is replaced by $P_{i\theta}(\omega)$ here. Secondly, the estimate of Corollary 4.3 replaces the estimate $|\hat{\mu}(x)| \leqslant \exp(-C/|x|^l)$ as in [8]. Finally, we remark that the estimates of Proposition 2.4 are used to obtain an analogue of [8, first equation in the proof of Lemme 3.1]. With these observations in hand, Theorem 4.4 follows readily from the proof of [8, Lemme 3.1].

4.1.2 | Proof of Theorem 4.2

Recall the family of operators $\{N_{i\theta}\}$ discussed in and before Proposition 2.4. We will require the following asymptotic expansion result from [4].

Lemma 4.5 [4, Lemma 16.12]. *Let $r \geqslant 2$. There are polynomial functions A_i on $\mathbb{R}$, $0 \leqslant i \leqslant r-1$ with degree at most $3i$ and no constant term for $i > 0$, with values in the space $\mathcal{L}(H^1)$ of bounded endomorphisms of H^1 such that:*

For any $M > 0$, uniformly in $\theta \in \mathbb{R}$ with $|\theta| \leqslant \sqrt{M \log n}$ and $\varphi \in H^1$ we have $A_0(\theta)\varphi = N\varphi$ and in H^1

$$e^{\frac{\theta^2 r_0^2}{2}} \cdot e^{-i\sqrt{n}\theta\cdot\chi} \cdot \lambda^n_{\frac{i\theta}{\sqrt{n}}} \cdot N_{\frac{i\theta}{\sqrt{n}}}\varphi = \sum_{i=0}^{r-1} \frac{A_i(\theta)\varphi}{n^{\frac{i}{2}}} + O\left(\frac{(\log n)^{\frac{3r}{2}}|\varphi|_1}{n^{\frac{r}{2}}}\right).$$

Proof of Theorem 4.2. Recall that we are using the re-centred cocycle, so that

$$\chi = \int c(a,\omega)d\mathbf{p}(a)d\mathbb{P}(\omega) = 0.$$

Fix $\omega \in \mathcal{A}^{\mathbb{N}}$. By Fourier inversion [4, Equation (16.13)] we have

$$I_n := 2\pi\mu_{n,\omega}(\psi_{\epsilon,C}) = \int \widehat{\psi_{\epsilon,C}}(\theta)\mathcal{F}_\theta(\mu_{n,\omega})(\theta)d\theta = \int \widehat{\psi_{\epsilon,C}}(\theta)P^n_{i\theta}(\mathbf{1})(\omega)d\theta.$$

We will decompose I_n as $I_n = I_n^2 + I_n^3 + I_n^4$. Note that unlike [4, Page 266] we have no need for I_n^1, but we keep the notation to make the comparison with [4] easier for the reader.

First, let $T > 0$ and define

$$I_n^2 := \int_{|\theta|^2 \geqslant T\frac{\log n}{n}} \widehat{\psi_{\epsilon,C}}(\theta)P^n_{i\theta}(\mathbf{1})(\omega)d\theta.$$

If $T \gg D$ as in Theorem 4.4 then by (17)

$$|I_n^2| = C^k(\psi_{\epsilon,C}) \cdot o(\frac{1}{n^r}) \leqslant O\left(\frac{\lambda(C)}{\epsilon^k}\right)\frac{1}{n^r}.$$

Next, appealing to [4, Lemma 11.18] there is some $\delta \in (0,1)$ such that in a small neighbourhood of 0, $P_{i\theta} - \lambda_{i\theta}N_{i\theta}$ has spectral radius $< \delta$. So, via (17), as long as n is large enough,

$$I_n^3 := \int_{|\theta|^2 \leqslant T\frac{\log n}{n}} \widehat{\psi_{\epsilon,C}}(\theta)\big(P^n_{i\theta} - \lambda^n_{i\theta}N_{i\theta}\big)(\mathbf{1})(\omega)d\theta = O_{\lambda(C)}(\delta^n).$$

It remains to control

$$I_n^4 := \int_{|\theta|^2 \leqslant T\frac{\log n}{n}} \widehat{\psi_{\epsilon,C}}(\theta)\lambda^n_{i\theta}N_{i\theta}(\mathbf{1})(\omega)d\theta.$$

By Lemma 4.5, since $\chi = 0$ we have via (17)

$$I_n^4 = \int_{|\theta|^2 \leqslant T\frac{\log n}{n}} \widehat{\psi_{\epsilon,C}}(\theta) \cdot \sum_{i=0}^{r-1} \frac{\widehat{G_n}(\theta) \cdot A_i(\sqrt{n}\theta)\mathbf{1}(\omega)}{n^{\frac{i}{2}}}d\theta + O_{\lambda(C)}\left(\left(\frac{\log^3 n}{n}\right)^{\frac{r+1}{2}}\right)$$

where

$$\widehat{G_n}(\theta) = e^{-\frac{nr_0^2\cdot\theta^2}{2}} = \text{The Fourier transform of the Gaussian function } G_n.$$

Since for every $0 \leqslant i \leqslant r-1$ A_k has degree $\leqslant 3i$ we get

$$\int_{|\theta|^2 \geqslant T\frac{\log n}{n}} \widehat{\psi_{\epsilon,C}}(\theta) \cdot \frac{\widehat{G_n}(\theta) \cdot A_i(\sqrt{n}\theta)\mathbf{1}(\omega)}{n^{\frac{i}{2}}} d\theta = O_{\lambda(C)}\left(\frac{\log n^{\frac{3i+1}{2}}}{n^{\frac{T+i+1}{2}}} \right).$$

So, choosing $T \gg 1$ we find that

$$I_n = \int_{\mathbb{R}} \widehat{\psi_{\epsilon,C}}(\theta) \cdot \sum_{i=0}^{r-1} \frac{\widehat{G_n}(\theta) \cdot A_i(\sqrt{n}\theta)\mathbf{1}(\omega)}{n^{\frac{i}{2}}} d\theta + \frac{1}{\epsilon^k} \cdot O_{\lambda(C)}\left(\frac{1}{n^{\frac{r}{2}}} \right). \tag{19}$$

Now, for every $0 \leqslant i \leqslant r-1$ there exists a polynomial function B_i on $\mathbb{R}$ with values in H^1 such that $\deg(B_i) \leqslant 3i$ and for every $\omega \in \mathcal{A}^{\mathbb{N}}$ the function on $\mathbb{R}$ given by

$$\theta \mapsto e^{-\frac{r_0^2\theta^2}{2}} A_i(\theta)\mathbf{1}(\omega) = e^{-\frac{r_0^2\theta^2}{2}} A_i(\theta)$$

is the Fourier transform of the function on $\mathbb{R}$ given by

$$v \mapsto G_1(v)B_i(v)(\omega).$$

So, by (19) and Fourier inversion

$$I_n = 2\pi \int_{\mathbb{R}} \psi_{\epsilon,C}(v) \cdot G_n(v) \sum_{i=0}^{r-1} \frac{B_i(\frac{v}{\sqrt{n}})(\omega)}{n^{\frac{i}{2}}} d\lambda(v) + \frac{1}{\epsilon^k} \cdot O_{\lambda(C)}\left(\frac{1}{n^{\frac{r}{2}}} \right). \tag{20}$$

Next, for every $0 \leqslant i \leqslant r-1$,

$$\int_{|v|^2 \geqslant Tn\log n} \psi_{\epsilon,C}(v) \cdot G_n(v) \frac{B_i(\frac{v}{\sqrt{n}})(\omega)}{n^{\frac{i}{2}}} d\lambda(v) = O\left(\frac{\log n^{\frac{3i+1}{2}}}{n^{\frac{T-i}{2}}} \right) ||\psi_{\epsilon,C}||_\infty$$
$$\leqslant \frac{1}{\epsilon} \cdot O\left(\frac{\log n^{\frac{3i+1}{2}}}{n^{\frac{T-i}{2}}} \right) \leqslant \frac{1}{\epsilon^k} \cdot O\left(\frac{\log n^{\frac{3i+1}{2}}}{n^{\frac{T-i}{2}}} \right)$$

and since $\psi_{\epsilon,C}$ is non-negative,

$$\int_{|v|^2 \leqslant Tn\log n} \psi_{\epsilon,C}(v) \cdot G_n(v) \frac{B_i(\frac{v}{\sqrt{n}})(\omega)}{n^{\frac{i}{2}}} d\lambda(v) = O\left(\frac{\log n^{\frac{3i}{2}}}{n^{\frac{i}{2}}} \right) \lambda(\psi_{\epsilon,C} \cdot G_n).$$

So, choosing $T \gg 1$ the leading term in (20) is the one with $i = 0$. Since $A_0(\theta) = N$ and $N\varphi = \mathbb{P}(\varphi)$ for all φ, we get $B_0(v)(\omega) = \mathbb{P}(\mathbf{1}) = 1$, so that

$$I_n = 2\pi\lambda(\psi_{\epsilon,C} \cdot G_n) + \lambda(\psi_{\epsilon,C} \cdot G_n) O\left(\frac{\log n^{\frac{3(r-1)}{2}}}{n^{\frac{1}{2}}} \right) + \frac{1}{\epsilon^k} O_{\lambda(C)}\left(\frac{1}{n^{\frac{r}{2}}} \right). \tag{21}$$

Recalling the definition of I_n, the Theorem follows. □

4.2 | Approximation of C by a smooth function

The following Lemma allows us to relate the quantities as in Theorem 4.2 to those appearing in Theorem 4.1. Both the statement and the proof are not much different than [4, Lemma 16.13]; our modest contribution is to notice that the proof given in [4] can be made effective. We keep the notation $\psi_{\epsilon,C}$ as in the previous Sections.

Lemma 4.6. *Let $R \geqslant 0$ be fixed. Then there is some $\delta'' = \delta''(R) > 0$ such that*

$$\sup\left\{\left|\frac{\lambda(\psi_{\epsilon,C+v}\cdot G_n)}{G_n(v)} - \lambda(C)\right| : \omega \in \mathcal{A}^{\mathbb{N}},\ |v| \leqslant \sqrt{Rn\log n},\ \epsilon \in (0,1)\right\} = O(\frac{1}{n^{\delta''}}).$$

Proof. Fix $n \geqslant 1, v \in \mathbb{R}$ with $|v| \leqslant \sqrt{Rn\log n}$ and $\epsilon \in (0,1)$. Let

$$J_n := \frac{\lambda(\psi_{\epsilon,C+v}\cdot G_n)}{G_n(v)} - \lambda(C) = \frac{\lambda(\psi_{\epsilon,C+v}\cdot G_n)}{G_n(v)} - \lambda(C+v).$$

Since λ is translation invariant and $\lambda(\alpha_\epsilon) = 1$ we get

$$J_n = \int_{\mathbb{R}\times\mathbb{R}} \alpha_\epsilon(w)1_{C+v}(w'-w)(\frac{G_n(w')}{G_n(v)} - 1)\,d\lambda(w)d\lambda(w').$$

We decompose this as a sum $J_n = J_n^1 + J_n^2$ with

$$J_n^1 = \int_{|w|\leqslant n^{\frac14}} \alpha_\epsilon(w)1_{C+v}(w'-w)(\frac{G_n(w')}{G_n(v)} - 1)\,d\lambda(w)d\lambda(w').$$

$$J_n^2 = \int_{|w|\geqslant n^{\frac14}} \alpha_\epsilon(w)1_{C+v}(w'-w)(\frac{G_n(w')}{G_n(v)} - 1)\,d\lambda(w)d\lambda(w').$$

Bounding J_n^1: Here, for w, w' such that $w' - v \in C + w$ we have,

$$w' - v = c + w, \text{ where } ||w|| \leqslant n^{1/4}, c \in C, \text{ and } |v| \leqslant \sqrt{Rn\log n}$$

so,

$$|w' - v| = O(n^{\frac14}), \quad \text{and } |v + w'| = O\Big((n\log n)^{1/2}\Big).$$

Therefore,

$$|(v+w')(v-w')| = o(n^{\frac56}).$$

Finally,

$$|\frac{G_n(w')}{G_n(v)} - 1| = |e^{\frac{r_0^2\cdot(v+w')(v-w')}{2n}} - 1| = O(n^{-\frac16})$$

and so $J_n^1 = O(n^{-\frac16})$. Here, we use that $\int \alpha_\epsilon(x)\,dx = 1$. Notice: the norm of α_ϵ does not affect this term.

Bounding J_n^2: First, since $|v| \leqslant \sqrt{Rn\log n}$ we obtain

$$\frac{G_n(w')}{G_n(v)} \leqslant e^{\frac{r_0^2 v^2}{2n}} \leqslant n^{\frac{R}{2}}$$

Next, since α is a Schwartz function there is some $C_9 = C_9(\alpha)$ such that

$$\sup_{x\in\mathbb{R}} |\alpha(x)| \leqslant \frac{C_9}{(1+|x|)^{4R+1}}.$$

So,

$$J_n^2 \leqslant n^{\frac{R}{2}}\lambda(C+v)\int_{|w|\geqslant n^{\frac{1}{4}}} \alpha_\epsilon(w)d\lambda(w) \leqslant n^{\frac{R}{2}}\lambda(C)\int_{|x|\geqslant n^{\frac{1}{4}}} \alpha(x)\,dx$$
$$\leqslant O_{\lambda(C)}\left(n^{\frac{R}{2}}\right)\cdot \frac{C_9}{n^{(4R-1+1)/4}} = O_{\lambda(C)}\left(\frac{1}{n^{R/2}}\right),$$

this decays polynomially.

Combining the bounds for J_n^1 and J_n^2, we are done. □

Notice that the implicit constant in the $O_{\lambda(C)}(\cdot)$ above also depends on α. However, in practice we will always use the same α, so this is indeed a universal bound.

4.3 | Proof of Theorem 4.1

Fix $R > 0$ and let $r > 3 + R$ and let $k = k(r, \ell)$ be as in Theorem 4.2. From now on we fix a Schwartz function α that satisfies

$$\int_{|w|\geqslant n^{\frac{1}{2k}}} \alpha(w)d\lambda(w) \leqslant n^{-\frac{1}{k}}.$$

For example, this holds for $\alpha(x) = e^{-x^2/2}$. We proceed to combine our previous work to obtain Theorem 4.1.

Lemma 4.7 (Upper bound). *There is some $\delta_0 > 0$ such that*

$$\sup\left\{\frac{\mu_{n,\omega}(C+v)}{G_n(v)} \,:\, \omega\in\mathcal{A}^{\mathbb{N}},\ |v|\leqslant\sqrt{Rn\log n}\right\} \leqslant \lambda(C+v) + O(\frac{1}{n^{\delta_0}}).$$

The proof is based on [4, Proof of Eq. (16.21)], which with our previous analysis is made effective:

Proof. Let $n \in \mathbb{N}$ and notice that for every $w \in \mathbb{R}$ with $|w| \leqslant n^{-\frac{1}{2k}}$ we have

$$C \subseteq C + B_0(n^{-\frac{1}{2k}}) + w, \text{ where } B_0(e) \text{ is the open ball about 0 of radius } e > 0.$$

Denote $C^{(n^{-\frac{1}{2k}})} := C + B_0(n^{-\frac{1}{2k}})$. So, since we also have for every $\epsilon > 0$

$$\mu_{n,\omega}(\psi_{\epsilon,C}) = \int_{\mathbb{R}} \alpha_\epsilon(w)\mu_{n,\omega}(C+w)d\lambda(w). \tag{22}$$

Plugging in $\epsilon = n^{-\frac{1}{k}}$ we obtain

$$(1 - n^{-\frac{1}{k}}) \cdot \mu_{n,\omega}(C+v) \leqslant \mu_{n,\omega}\left(\psi_{n^{-\frac{1}{k}}, C^{(n^{-\frac{1}{2k}})}+v}\right). \tag{23}$$

We also recall that

$$G_n(v)^{-1} \leqslant (2\pi)^{\frac{1}{2}} n^{\frac{1+R}{2}}.$$

Applying successively (23), Theorem 4.2, Lemma 4.6 we see that:

$$\begin{aligned}
\frac{\mu_{n,\omega}(C+v)}{G_n(v)} &\leqslant \frac{1}{1-n^{-\frac{1}{k}}} \cdot \frac{\mu_{n,\omega}\left(\psi_{n^{-\frac{1}{k}}, C^{(n^{-\frac{1}{2k}})}+v}\right)}{G_n(v)} \\
&\leqslant \frac{\lambda(\psi_{n^{-\frac{1}{k}}, C^{(n^{-\frac{1}{2k}})}+v} \cdot G_n) + O\left(\frac{1}{n^\delta}\right) \cdot \lambda(\psi_{n^{-\frac{1}{k}}, C^{(n^{(-\frac{1}{2k})}}+v} \cdot G_n) + n^{\frac{k}{k}} \cdot O(\frac{1}{n^{\frac{r}{2}}})}{G_n(v)} \\
&\times \frac{1}{1-n^{-\frac{1}{k}}} \\
&\leqslant \frac{1}{1-n^{-\frac{1}{k}}} \cdot \left(\frac{\lambda(\psi_{n^{-\frac{1}{k}}, C^{(n^{-\frac{1}{2k}})}+v} \cdot G_n)}{G_n(v)} + O\left(\frac{1}{n^\delta}\right) + O\left(n^{1+\frac{1+R}{2}} \cdot \frac{1}{n^{\frac{r}{2}}}\right)\right) \\
&\leqslant \frac{1}{1-n^{-\frac{1}{k}}} \cdot \left(\lambda\left(C^{(n^{-\frac{1}{2k}})} + v\right) + O\left(\frac{1}{n^{\delta''}}\right) + O\left(\frac{1}{n^\delta}\right) + O\left(n^{\frac{3+R-r}{2}}\right)\right) \\
&\leqslant \frac{1}{1-n^{-\frac{1}{k}}} \cdot \left(\lambda(C) + O\left(\frac{1}{n^{\frac{1}{2k}}}\right) + O\left(\frac{1}{n^{\delta''}}\right) + O\left(\frac{1}{n^\delta}\right) + O\left(n^{\frac{3+R-r}{2}}\right)\right) \\
&= \left(1 + O\left(\frac{1}{n^{\frac{1}{k}}}\right)\right) \cdot \left(\lambda(C) + O\left(\frac{1}{n^{\frac{1}{2k}}}\right) + O\left(\frac{1}{n^{\delta''}}\right) + O\left(\frac{1}{n^\delta}\right) + O\left(n^{\frac{3+R-r}{2}}\right)\right)
\end{aligned}$$

By the choice of r we are done. □

The lower bound is an effective analogue of [4, Proof of Eq. (16.23)]:

Lemma 4.8 (Lower bound). *There is some $\delta_1 > 0$ such that*

$$\inf\left\{\frac{\mu_{n,\omega}(C+v)}{G_n(v)} \,:\, \omega \in \mathcal{A}^{\mathbb{N}},\, |v| \leqslant \sqrt{Rn\log n}\right\} \geqslant \lambda(C+v) - O\left(\frac{1}{n^{\delta_1}}\right).$$

Proof. Let $n \in \mathbb{N}$ and notice that for every $w \in \mathbb{R}$ with $|w| \leqslant n^{-\frac{1}{2k}}$ we have

$$\bigcap_{u \in B_0(n^{-\frac{1}{2k}})} (C-u) + w \subseteq C.$$

Let $C_{(n^{-\frac{1}{2k}})} := \bigcap_{u \in B_0(n^{-\frac{1}{2k}})} (C-u)$. Plugging $\epsilon = n^{-\frac{1}{k}}$ into (22) we have

$$\mu_{n,\omega}(C+v) \geqslant \int_{|w| \leqslant n^{-\frac{1}{2k}}} \alpha_{n^{-\frac{1}{k}}}(w) \mu_{n,\omega}(C_{(n^{-\frac{1}{2k}})} + v + w) d\lambda(w) \tag{24}$$

$$\geqslant \mu_{n,\omega}\left(\psi_{n^{-\frac{1}{k}}, C_{(n^{-\frac{1}{2k}})}+v}\right) - K_n^1 - K_n^2 \tag{25}$$

where

$$K_n^1 = \int_{n^{-\frac{1}{2k}} \leqslant |w| \leqslant n^{\frac{1}{4}}} \alpha_{n^{-\frac{1}{k}}}(w) \mu_{n,\omega}(C_{(n^{-\frac{1}{2k}})} + v + w) d\lambda(w)$$

$$K_n^2 = \int_{|w| \geqslant n^{\frac{1}{4}}} \alpha_{n^{-\frac{1}{k}}}(w) \mu_{n,\omega}(C_{(n^{-\frac{1}{2k}})} + v + w) d\lambda(w).$$

First, via the upper bound from Lemma 4.7 and the proof of Lemma 4.6 we get

$$\begin{aligned}
\frac{K_n^1}{G_n(v)} &\leqslant \int_{n^{-\frac{1}{2k}} \leqslant |w| \leqslant n^{\frac{1}{4}}} \alpha_{n^{-\frac{1}{k}}}(w) \frac{G_n(v+w)}{G_n(v)} \left(\lambda(C+v+w) + O\left(\frac{1}{n^{\delta_0}}\right)\right) d\lambda(w) \\
&\leqslant \int_{n^{-\frac{1}{2k}} \leqslant |w| \leqslant n^{\frac{1}{4}}} \alpha_{n^{-\frac{1}{k}}}(w) \left(1 + O\left(\frac{1}{n^{\frac{1}{6}}}\right)\right) \cdot \left(\lambda(C+v+w) + O\left(\frac{1}{n^{\delta_0}}\right)\right) d\lambda(w) \\
&\leqslant \left(1 + O\left(\frac{1}{n^{\frac{1}{6}}}\right)\right) \cdot \left(\lambda(C) + O\left(\frac{1}{n^{\delta_0}}\right)\right) \cdot n^{-\frac{1}{k}}.
\end{aligned}$$

Secondly, since $|v| \leqslant \sqrt{Rn \log n}$ then as in the second part of the proof of Lemma 4.6

$$\frac{K_n^2}{G_n(v)} \leqslant n^{\frac{R}{2}} \int_{|w| \geqslant n^{\frac{1}{4}}} \alpha_{n^{-\frac{1}{k}}}(w) d\lambda(w) = n^{\frac{R}{2}} \int_{|w| \geqslant n^{\frac{1}{4}+\frac{1}{k}}} \alpha(w) d\lambda(w) = O\left(\frac{1}{n^{R/2}}\right).$$

Applying successively (24), Theorem 4.2, Lemma 4.6 we get

$$\begin{aligned}
\frac{\mu_{n,\omega}(C+v)}{G_n(v)} &\geqslant \frac{\mu_{n,\omega}\left(\psi_{n^{-\frac{1}{k}}, C_{(n^{-\frac{1}{2k}})}+v}\right) - K_n^1 - K_n^2}{G_n(v)} \\
&\geqslant \frac{\lambda(\psi_{n^{-\frac{1}{k}}, C_{(n^{-\frac{1}{2k}})}+v} \cdot G_n) - O\left(\frac{1}{n^{\delta}}\right) \cdot \lambda(\psi_{n^{-\frac{1}{k}}, C_{(n^{-\frac{1}{2k}})}+v} \cdot G_n) - n^{\frac{k}{k}} \cdot O\left(\frac{1}{n^{\frac{r}{2}}}\right)}{G_n(v)}
\end{aligned}$$

$$
\begin{aligned}
&- O\left(\frac{1}{n^{R/2}}\right) - O(n^{-\frac{1}{k}}) \\
&\geqslant \left(\frac{\lambda(\psi_{n^{-\frac{1}{k}}, C_{(n^{-\frac{1}{2k}})} + v} \cdot G_n)}{G_n(v)} - O\left(\frac{1}{n^{\delta}}\right) - O\left(n^{\frac{3+R}{2}} \cdot \frac{1}{n^{\frac{r}{2}}}\right) \right) \\
&- O\left(\frac{1}{n^{R/2}}\right) - O(n^{-\frac{1}{k}}) \\
&\geqslant \lambda(C) + O\left(\frac{1}{n^{\delta''}}\right) + O\left(\frac{1}{n^{\delta}}\right) + O\left(n^{\frac{3+R-r}{2}}\right) - O\left(\frac{1}{n^{R/2}}\right) - O(n^{-\frac{1}{k}}).
\end{aligned}
$$

By the choice of r we are done. □

Via Lemma 4.8 and Lemma 4.7 the proof of Theorem 4.1 is complete.

5 | AN EFFECTIVE CONDITIONAL LOCAL LIMIT THEOREM FOR SMOOTH FUNCTIONS

Let Φ be an IFS as in Theorem 1.1 that is either Diophantine or not-conjugate-to-linear. Let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. In this Section we prove Theorem 5.4, a conditional local limit Theorem which will be the key behind the proof of Theorem 1.1. This is an effective version of [1, Theorem 3.7], and is proved via the effective local limit Theorem 4.1 and the effective central limit Theorem 3.1 that we previously discussed.

We first define the following function on $\mathcal{A}^{\mathbb{N}}$. Though it resembles one, it is *not* a stopping time: Recalling (15), we let

$$\tau_k(\omega) := \min\{n : S_n(\omega) \geqslant k\chi\}.$$

Note that we allow k to take positive non-integer values. We also recall that χ is the corresponding Lyapunov exponent.

Recalling (6), it is clear that for every $k > 0$ and $\omega \in A^{\mathbb{N}}$ we have

$$-\log |f'_{\omega|_{\tau_k(\omega)}}(x_{\sigma^{\tau_k(\omega)}(\omega)})| = S_{\tau_k(\omega)}(\omega) \in [k\chi, k\chi + D'].$$

Next, we introduce some partitions of the space $\mathcal{A}^{\mathbb{N}}$, that are modelled after [1, Definition 3.3]:

Definition 5.1. Given a finite word $\eta' = (\eta'_1, \dots, \eta'_\ell) \in \mathcal{A}^\ell$:

1. Denote by $A_{\eta'} \subseteq \mathcal{A}^{\mathbb{N}}$ the set of infinite words that begin with η',

$$A_{\eta'} := \{\omega : \quad (\omega_1, \dots, \omega_\ell) = \eta'\}.$$

2. We define the event

$$A_{k,\eta'} := \{\omega : \quad \sigma^{\tau_k(\omega)-1}(\omega) \in A_{\eta'}\}.$$

3. Given $k, h' \geqslant 0$ we denote by $\mathcal{A}_k^{h'}$ the finite partition of $\mathcal{A}^{\mathbb{N}}$ according to the map

$$\iota_k^{h'}(\omega) = \iota^{h'}\left(\sigma^{\tau_k(\omega)-1}(\omega)\right)$$

where

$$\iota^h(\eta) = \eta|_{\tilde{\tau}_h(\eta)}, \text{ where } \tilde{\tau}_h(\eta) = \min\{n : -\log \max_{x\in I} |f'_{\eta|_n}(x)| \geqslant h \cdot \chi\}.$$

Note that every cell of the partition $\mathcal{A}_k^{h'}$ is of the form $A_{k,\eta'}$. Given $k, h' \geqslant 0$ and $\omega \in \mathcal{A}^{\mathbb{N}}$ we write $\mathcal{A}_k^{h'}(\omega)$ for the unique $\mathcal{A}_k^{h'}$ cell that contains ω. For $\mathbb{P}$-a.e. ω, we denote the conditional measure of $\mathbb{P}$ on the corresponding cell by $\mathbb{P}_{\mathcal{A}_k^{h'}(\omega)}$. Recall that λ is the Lebesgue measure on $\mathbb{R}$, and that X_1 is defined in (16).

Definition 5.2. Let $k \in \mathbb{N}$ and let $\eta' \in \mathcal{A}^*$ be a finite word. Assuming $\mathbb{P}(A_{k,\eta'}) > 0$, we define a probability measure $\Gamma_{A_{k,\eta'}}$ on $[k\chi, k\chi + D']$ by

$$\Gamma_{A_{k,\eta'}} := \frac{\int_{A_{\eta'}} \lambda|_{[k\chi,\, k\chi+X_1(\omega')]}\, d\mathbb{P}(\omega')}{\int_{A_{\eta'}} X_1(\omega')\, d\mathbb{P}(\omega')}.$$

The following Lemma is straightforward:

Lemma 5.3 [1, Lemma 3.5]. *If $\mathbb{P}(A_{k,\eta'}) > 0$ then $\Gamma_{A_{k,\eta'}} \ll \lambda_{[k\chi,k\chi+D']}$ with a density that depends only on $\mathbb{P}$, such that its norm is bounded above by $\frac{1}{D}$ independently of k and η'*

We now state an effective version of our previous conditional local limit Theorem [1, Theorem 3.7]. Its effectiveness is what will ultimately allow us to obtain the rate of decay for $\mathcal{F}_q(\nu)$, where ν is the projection of $\mathbb{P}$ to the fractal (the corresponding self-conformal measure). The idea behind it is to describe some local limit like phenomenon for the random variable S_{τ_k}, and this is achieved subordinate to the partitions $\mathcal{A}_k^{h'}$. We also note that the h' part in $\mathcal{A}_k^{h'}$ is useful for the linearization argument outlined in Section 6.2.

Theorem 5.4. *If Φ satisfies the conditions of Theorem 1.1 and is either Diophantine or not-conjugate-to-linear, then there exists some $\delta_0 = \delta_0(\mathbf{p}) > 0$ such that:*

For every $k, h' > 0$ there exists a subset $\widetilde{A}_k^{h'} \subseteq \mathcal{A}^{\mathbb{N}}$ such that:

(i) $\mathbb{P}(\widetilde{A}_k^{h'}) \geqslant 1 - O(\frac{1}{k^{\frac{1}{4}}})$.

(ii) *for all $\xi \in \widetilde{A}_k^{h'}$, $\mathbb{P}(\mathcal{A}_k^{h'}(\xi)) > 0$.*

(iii) *for all $\xi \in \widetilde{A}_k^{h'}$ and for any sub-interval $J \subseteq [k\chi, k\chi + D']$,*

$$\mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}(S_{\tau_k} \in J) = \Gamma_{\mathcal{A}_k^{h'}(\xi)}(J) + O\left(\frac{1}{k^{\delta_0}}\right)$$

All big-O terms should be understood to depend on $\mathbf{p}$. There are two main differences between Theorem 5.4 and [1, Theorem 3.7]: The most substantial one is that the error terms are explicit and polynomial in k. The second one is that in [1, Theorem 3.7] we work inside cylinders to get pointwise normality (which requires more parameters), but here we only care about Fourier decay which allows us to make the statement simpler.

The proof of Theorem 5.4 is similar to that of Theorem [1, Theorem 3.7]. Let us now explain how the quantitative estimates we previously obtained can be used to make [1, Proof of Theorem 3.7] effective:

For every k we define the interval

$$I_k = [k - \sqrt{k \log k}, k + \sqrt{k \log k}]. \tag{26}$$

The proof of Theorem 5.4 relies on a decomposition of the left hand side in (iii) as

$$\mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}\left(S_{\tau_k} \in J\right) = \sum_{m \notin I_k} \mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}\left(S_{\tau_k} \in J,\, \tau_k = m+1\right) + \sum_{m \in I_k} \mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}\left(S_{\tau_k} \in J,\, \tau_k = m+1\right) \tag{27}$$

Both terms are respectively treated by Proposition 5.5 and Proposition 5.6 below, and the Theorem follows. First, we have:

Proposition 5.5. *There exists a set $\widetilde{A}$ such that claims (i) and (ii) hold and for all $\xi \in \widetilde{A}$,*

$$\mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}(\tau_k - 1 \notin I_k) = O\left(\frac{1}{k^{\frac{1}{4}}}\right).$$

Notice that we are using the abbreviated notation $\widetilde{A}$ instead of $\widetilde{A}_k^{h'}$. Proposition 5.6 is an effective version of [1, Proposition 3.12]. The key to the proof is showing that

$$\mathbb{P}(\tau_k - 1 \notin I_k) = O\left(\frac{1}{\sqrt{k}}\right). \tag{28}$$

Indeed, once (28) is established, the result follows by an application of Markov's inequality, similarly to the end of the proof of [1, Proposition 3.12]. The proof of (28) is rather straightforward and is essentially the same as the proof of [1, Eq. (15)]. The latter proof can now be made effective by replacing the use of the non-effective central limit Theorem [4, Theorem 12.1] with the effective Theorem 3.1.

The second term in (27) is treated in the following Proposition:

Proposition 5.6. *Under the assumptions of Theorem 5.4, there is some $\delta_0 > 0$ such that for all ξ in the set $\widetilde{A}$ from Proposition 5.5,*

$$\sum_{m \in I_k} \mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}\left(S_{\tau_k} \in J,\, \tau_k = m+1\right) = \Gamma_{\mathcal{A}_k^{h'}(\xi)}(J) + O\left(\frac{1}{k^{\delta_0}}\right).$$

This is an effective version of [1, Proposition 3.13]. Let us explain how, via Theorem 4.1, [1, Proof of Proposition 3.13] can be made effective. Let η' be the finite word such that $\mathcal{A}_k^{h'}(\xi) = A_{k,\eta'}$.

Write

$$\sum_{m\in I_k} \mathbb{P}_{\mathcal{A}_k^{h'}(\xi)}\left(S_{\tau_k} \in J,\, \tau_k = m+1\right) = \frac{\sum_{m\in I_k} \mathbb{P}\left(S_{\tau_k} \in J,\, \tau_k = m+1,\, \omega \in A_{k,\eta'}\right)}{\mathbb{P}(A_{k,\eta'})}.$$

As shown in [1, Proof of Proposition 3.13], each summand in the numerator can be written as

$$\mathbb{P}\left(S_{\tau_k} \in J,\, \tau_k = m+1,\, \omega \in A_{k,\eta'}\right) = \int_{A_{\eta'}} \mathbb{P}_{\sigma^{-m}(\{\omega'\})}\left(S_m \in J^{\omega'}\right) d\mathbb{P}(\omega'),$$

where the interval $J^{\omega'}$ is defined by

$$J^{\omega'} := [k\chi - X_1(\omega'),\, k\chi) \cap \left(J - X_1(\omega')\right).$$

Note that $J^{\omega'}$ also depends on k, but we suppress this in our notation. Let $a_{\omega',k}$ be the left endpoint of $J^{\omega'}$. We now slightly change our notation:

We write $G_s(\cdot)$ for the density of the Gaussian random variable $N(0, s^2)$.

By Theorem 4.1, we have for every ω', using that $m \in I_k$ so $|a_{\omega',k} - m\chi| \leqslant 8\chi\sqrt{m\log m}$,

$$\mathbb{P}_{\sigma^{-m}(\{\omega'\})}\left(S_m \in J^{\omega'}\right) = G_{\sqrt{mr}}(a_{\omega',k} - m\chi) \cdot \left(\lambda(J^{\omega'}) + O\left(\frac{1}{m^{\delta}}\right)\right).$$

Using that $m \in I_k$ and applying [1, Lemma 3.14], there is some δ' such that

$$\mathbb{P}_{\sigma^{-m}(\{\omega'\})}\left(S_m \in J^{\omega'}\right) = G_{\sqrt{kr}}((m-k+\beta)\chi) \cdot \left(\lambda(J^{\omega'}) + O(\frac{1}{k^{\delta'}})\right) \tag{29}$$

for all $\omega' \in A_{\eta'}$, $m \in I_k$ and $\beta \in [0,1)$ as $k \to \infty$.

From here, one simply swaps [1, Equation (20)] for (29), and essentially the same proof given for [1, Proposition 3.13]) yields Proposition 5.6.

6 | PROOF OF THEOREM 1.1

In this Section we prove Theorem 1.1. Fix an IFS Φ as in Theorem 1.1 that is either Diophantine or not-conjugate-to-linear, and let $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ be a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. As in [1], we first require a preliminary step - an adaptation of Theorem 5.4 to Fourier modes. This is the content of the following Section.

6.1 | Application of Theorem 5.4 to Fourier modes

Fix a Borel probability measure $\rho \in \mathcal{P}(\mathbb{R})$. For every $q \in \mathbb{R}$ we define a function $g_{q,\rho} : \mathbb{R} \to \mathbb{R}$ via

$$g_{q,\rho}(t) = \left|\mathcal{F}_q(M_{e^{-t}}\rho)\right|^2, \text{ where for any } s, x \in \mathbb{R},\ M_s(x) := s \cdot x.$$

The following is a version of Theorem 5.4 for Fourier modes instead of intervals:

Theorem 6.1. *Fix parameters q, k, h' with $k, h', |q| > 0$, and let $\rho \in \mathcal{P}(\mathbb{R})$ be a measure such that*

$$diam(supp(\rho)) = O(e^{-h'\chi}).$$

Then for every $\xi \in \widetilde{A}_k^{h'} \subseteq \mathcal{A}^{\mathbb{N}}$, for $\delta_0 > 0$ as in Theorem 5.4, we have

$$\left| \mathbb{E}_{\mathcal{A}_k^{h'}(\xi)}\left[g_{q,\rho}(S_{\tau_k(\omega)}) \right] - \int_{k\chi}^{k\chi + D'} g_{q,\rho}(x) d\Gamma_{\mathcal{A}_k^{h'}(\xi)}(x) \right| \leqslant O\left(\frac{2}{qe^{-(k+h')\chi}} + (qe^{-(k+h')\chi})^2 \frac{1}{k^{\delta_0}} \right).$$

This is an analogue of [1, Theorem 4.1]. The main difference is that we swap the o_k term in [1, Theorem 4.1], which is the non-effective rate at which [1, Theorem 3.7] holds, for a more explicit bound in terms of our parameters and the effective rate at which Theorem 5.4 holds. To sketch the proof, note that the Lipschitz norm of the function

$$t \in [k\chi, k\chi + D'] \mapsto g_{q,\rho}(t)$$

is $4\pi q e^{-\chi k} \cdot \mathrm{diam}(\mathrm{supp}(\rho))$. This allows for the construction of a $O(\frac{1}{qe^{-(k+h')\chi}})$-approximating step-function on this interval in the sup-norm, with $(qe^{-(k+h')\chi})^2$-steps. Each step corresponds to an indicator function of some sub-interval of $[k\chi, k\chi + D']$, where Theorem 5.4 holds with a uniform rate of $O(k^{-\delta_0})$. This implies the Theorem. For a detailed outline of this sketch, see [1, Proof of Theorem 4.1].

6.2 | Collecting error terms

Let ν be the self-conformal measure that arises by projecting $\mathbb{P}$ to the fractal K. Fix parameters q, k, h' where q will be the frequency of the Fourier transform of ν, and k, h' positive numbers that will depend on q. In this Section we will bound $\mathcal{F}_q(\nu)$ via a sum of certain error terms that depend variously on $|q|, k, h'$. These error terms will arise from three main sources: Linearization, the local limit Theorem, and an oscillatory integral. This is analogues to [1, Section 4.2], so we exclude some of the proofs (but we will indicate exactly what we are using from [1]). For brevity, let $\tilde{A}$ be the set $\widetilde{A}_k^{h'}$ as in Theorem 5.4 for our parameters. The most technically involved estimate arises from a linearization scheme, whose outcome is summarized in the following Theorem. Here, and throughout this Section, all big-O terms should be understood to depend on $\mathbb{P}$ and Φ. For every $s, x \in \mathbb{R}$ we denote the scaling map by $M_s(x) := s \cdot x$, and recall that Φ is C^r smooth, $r \geqslant 2$.

Theorem 6.2 (Linearization). *There is some integer $P > 1$ such that for any $\beta \in (0, 1)$,*

$$\left| \mathcal{F}_q(\nu) \right|^2 \leqslant \sum_{|\rho| \leqslant 2P} \int_{\xi \in \tilde{A}} \mathbb{E}_{\mathcal{A}_k^{h'}(\xi)} \left| \mathcal{F}_q\left(M_{e^{-S_{\tau_k(\omega)}(\omega)}} \circ f_{\bar{\eta}'} \circ f_\rho \nu \right) \right|^2 d\mathbb{P}(\xi)$$
$$+ O\left(\frac{1}{k^{\frac{1}{4}}} \right) + O\left(|q| \cdot e^{-(k+h')\chi - \beta \cdot h'\chi} \right),$$

where:

1. *For every* $\xi \in \tilde{A}$ *the* η' *inside the integral corresponds to the cell* $\mathcal{A}_k^{h'}(\xi) = A_{k,\eta'}$.
2. *For every* η' *we define* $\bar{\eta'} := \eta'|_{|\eta'|-P}$, *the prefix of* η' *of length* $|\eta'| - P$.
3. *There is a global constant* $C' > 1$ *such that for all* $\bar{\eta'}$ *and* ρ *as above,*

$$|\left(f_{\bar{\eta'}} \circ f_\rho\right)'(x)| = \Theta_{C'}\left(e^{-h'\chi}\right), \quad \forall x \in I.$$

Proof. This is a combination of [1, Lemma 4.3, Lemma 4.4, Claim 4.5, and Corollary 4.6], and since Φ is assumed to be orientation preserving. □

Now, fix some ρ with $|\rho| \leqslant 2P$ and consider the corresponding term in Theorem 6.2,

$$\int_{\xi\in\tilde{A}_\eta} \mathbb{E}_{\mathcal{A}_k^{h'}(\xi)} \left|\mathcal{F}_q\left(M_{e^{-S_{\tau_k(\omega)}(\omega)}} \circ f_{\bar{\eta'}} \circ f_\rho \nu\right)\right|^2 d\mathbb{P}(\xi).$$

We now appeal to the local limit Theorem 6.1 for every event $\mathcal{A}_k^{h'}(\xi)$ separately. To do this, we notice that by Theorem 6.2, for every $f_{\bar{\eta'}} \circ f_\rho$ involved

$$\text{diam}\left(\text{supp}\left(f_{\bar{\eta'}} \circ f_\rho \nu\right)\right) = O(e^{-h'\chi}).$$

Notice that the error term in Theorem 6.1 is $O(\frac{2}{|q|e^{-(k+h')\chi}} + (|q|e^{-(k+h')\chi})^2 \frac{1}{k^{\delta_0}})$ independently of the event $\mathcal{A}_k^{h'}(\xi)$. So,

$$\begin{aligned}
&\int_{\xi\in\tilde{A}_\eta} \mathbb{E}_{\mathcal{A}_k^{h'}(\xi)} \left|\mathcal{F}_q\left(M_{e^{-S_{\tau_k(\omega)}(\omega)}} \circ f_{\bar{\eta'}} \circ f_\rho \nu\right)\right|^2 d\mathbb{P}(\xi) \\
&\leqslant \int_{\xi\in\tilde{A}_\eta} \int_{k\chi}^{k\chi+D'} \left|\mathcal{F}_q\left(M_{e^{-x}} \circ f_{\bar{\eta'}} \circ f_\rho \nu\right)\right|^2 d\Gamma_{\mathcal{A}_k^{h'}(\xi)}(x)\, d\mathbb{P}(\xi) \\
&\quad + O\left(\frac{2}{|q|e^{-(k+h')\chi}} + (|q|e^{-(k+h')\chi})^2 \frac{1}{k^{\delta_0}}\right)
\end{aligned}$$

Since this is true for every ρ with $|\rho| \leqslant 2P$, combining with Theorem 6.2 we see that

$$\begin{aligned}
\left|\mathcal{F}_q(\nu)\right|^2 \leqslant& \sum_{|\rho|\leqslant 2P} \int_{\xi\in\tilde{A}} \int_{k\chi}^{k\chi+D'} \left|\mathcal{F}_q\left(M_{e^{-x}} \circ f_{\bar{\eta'}} \circ f_\rho \nu\right)\right|^2 d\Gamma_{\mathcal{A}_k^{h'}(\xi)}(x) d\mathbb{P}(\xi) \\
&+ O\left(\frac{2}{|q|e^{-(k+h')\chi}} + (|q|e^{-(k+h')\chi})^2 \frac{1}{k^{\delta_0}}\right) + O\left(\frac{1}{k^{\frac{1}{4}}}\right) + O\left(|q| \cdot e^{-(k+h')\chi - \beta\cdot h'\chi}\right).
\end{aligned}$$

Recall that by Lemma 5.3, the probability measure $\Gamma_{\mathcal{A}_k^{h'}(\xi)}$ is absolutely continuous with respect to the Lebesgue measure on $[k\chi, k\chi + D']$, such that the norm of its density function is uniformly bounded by $\frac{1}{D} > 0$ independently of all parameters. Using this fact, we obtain

$$\left|\mathcal{F}_q(\nu)\right|^2 \leqslant \sum_{|\rho|\leqslant 2P} \int_{\xi\in\tilde{A}_\eta} \left(\int_{k\chi}^{k\chi+D'} \left|\mathcal{F}_q\left(M_{e^{-z}} \circ f_{\bar{\eta}'} \circ f_\rho \nu\right)\right|^2 \cdot \frac{1}{D} dz \right) d\mathbb{P}(\xi)$$

$$+ O\left(\frac{2}{|q|e^{-(k+h')\chi}} + (|q|e^{-(k+h')\chi})^2 \frac{1}{k^{\delta_0}} \right) + O\left(\frac{1}{k^{\frac{1}{4}}} \right) + O\left(|q| \cdot e^{-(k+h')\chi - \beta\cdot h'\chi} \right).$$

This leads us to the last error term, that comes from the sum of the oscillatory integrals as in the equation above:

Proposition 6.3 (Oscillatory integral). *For every $|\rho| \leqslant 2P$, $\xi \in \tilde{A}$, and every $r > 0$ we have*

$$\int_{k\chi}^{k\chi+D'} \left|\mathcal{F}_q\left(M_{e^{-z}} \circ f_{\bar{\eta}'} \circ f_\rho \nu\right)\right|^2 \cdot \frac{1}{D} dz \leqslant O\left(\frac{1}{r|q|e^{-(k+h')\chi}} + \sup_y \nu(B_r(y)) \right).$$

Proof. This follows from Hochman's Lemma [21, Lemma 3.2] about the average of the Fourier transform of scaled measures. Here we are using it in the form [1, Lemma 2.6], and are also making use of the fact that there is a global constant $C' > 1$ such that for all $\bar{\eta}'$ and ρ as above,

$$|\left(f_{\bar{\eta}'} \circ f_\rho\right)'(x)| = \Theta_{C'}\left(e^{-h'\chi}\right), \quad \forall x \in I$$

which follows from Theorem 6.2. See [1, Pages 41-42] for more details. □

6.3 | Conclusion of the proof

Following the argument in Section 6.2, we bounded $|\mathcal{F}_q(\nu)|^2$ by the sum of the following terms. Every term is bounded with dependence on the Bernoulli measure $\mathbb{P} = \mathbf{p}^{\mathbb{N}}$ and some fixed $\beta \in (0, 1)$. For simplicity, we ignore global multiplicative constants, so we omit the big-O notation. Recall that $\delta_0 > 0$ is as in Theorem 5.4:

Linearization - Theorem 6.2:

$$|q|e^{-(k+h')\chi} e^{-\beta h'\chi};$$

Local limit Theorem - The discussion in between Theorem 6.2 and Proposition 6.3:

$$\frac{2}{|q|e^{-(k+h')\chi}} + (|q|e^{-(k+h')\chi})^2 \frac{1}{k^{\delta_0}};$$

Oscillatory integral: Via Proposition 6.3, for every $r > 0$,

$$\frac{1}{r|q|e^{-(k+h')\chi}} + \sup_y \nu(B_r(y)).$$

Choice of parameters: For $|q|$ large we choose $k = k(|q|)$ and $h' = \sqrt{k}$ such that

$$|q| = k^{\frac{\delta_0}{4}} \cdot e^{(k+h')\chi}.$$

Fix $r = k^{\frac{-\delta_0}{8}}$. Then we get:

Linearization:

$$|q|e^{-(k+h')\chi}e^{-\beta h'\chi} = k^{\frac{\delta_0}{4}} \cdot e^{-\beta\sqrt{k}\chi}, \quad \text{This decays exponentially fast in } k.$$

Local limit Theorem:

$$\frac{2}{|q|e^{-(k+h')\chi}} + \left(|q|e^{-(k+h')\chi}\right)^2 \frac{1}{k^{\delta_0}} = \frac{2}{k^{\frac{\delta_0}{4}}} + k^{\frac{\delta_0}{2}} \cdot \frac{1}{k^{\delta_0}}, \quad \text{This decays polynomially fast in } k.$$

Oscillatory integral: There is some $d = d(\nu)$ such that

$$\frac{1}{r|q|e^{-(k+h')\chi}} + \sup_y \nu(B_r(y)) \leqslant \frac{k^{\frac{\delta_0}{8}}}{k^{\frac{\delta_0}{4}}} + k^{\frac{-d\cdot\delta_0}{8}}, \quad \text{This decays polynomially fast in } k.$$

Here we made use[†] of [17, Proposition 2.2], where it is shown that there is some $C > 0$ such that for every $r > 0$ small enough

$$\sup_y \nu(B_r(y)) \leqslant Cr^d.$$

Finally, by summing these error terms we see that for some $\alpha = \alpha(\nu) > 0$ we have $|\mathcal{F}_q(\nu)| = O(\frac{1}{k^\alpha})$. Since as $|q| \to \infty$ we have $k \geqslant O(\log |q|)$ our claim follows.

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[†] We note that although [17, Proposition 2.2] is stated for self-similar IFSs, essentially the same proof works in the case of general C^r smooth IFS's.

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