Invariant measures and measurable projective factors for actions of higher-rank lattices on manifolds

By Aaron Brown, Federico Rodriguez Hertz, and Zhiren Wang

Abstract

We consider smooth actions of lattices in higher-rank semisimple Lie groups on manifolds. We define two numbers $r(G)$ and $m(G)$ associated with the roots system of the Lie algebra of a Lie group G. If the dimension of the manifold is smaller than $r(G)$, then we show the action preserves a Borel probability measure. If the dimension of the manifold is at most $m(G)$, we show there is a quasi-invariant measure on the manifold such that the action is measurably isomorphic to a relatively measure-preserving action over a standard boundary action.

1. Introduction and statement of results

In this paper we consider lattices Γ in higher-rank Lie groups G acting by $C^{1+\text{Hölder}}$ diffeomorphisms on compact manifolds. The *Zimmer program* refers to a number of questions and conjectures related to such actions. It is expected that all such actions are constructed from algebraic examples or have some algebraic factor. In particular, if the dimension of M is smaller than the dimension of all possible algebraic actions, *Zimmer's conjecture* asserts that all actions factor through the action of a finite group. See [8] and [7] for recent solution to (non-volume-preserving cases of) Zimmer's conjecture by Brown, Fisher and Hurtado for cocompact lattices in split, simple Lie groups as well as for finite-index sublattices of $\mathrm{SL}(m, \mathbb{Z})$.

The main results of this paper concern actions of lattices in low dimensions. Most rigidity results in the literature concerning actions of lattices in low dimensions require additional hypotheses such as the preservation of a

Borel probability measure (see [18], [36]), strong regularity assumptions of the action (see [16]), or extremely low dimensions. (See [37], [11], [19] for actions on the circle and [18], [36] for actions on surfaces.) Our focus in this paper is to establish the existence of an invariant measure for actions in moderately low dimensions and with low differentiability. In particular, in Theorem 1.6 we show that if the dimension of M is sufficiently small relative to algebraic data associated to a simple Lie group G, then for any lattice $\Gamma \subset G$, any $C^{1+\beta}$-action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ preserves a Borel probability measure. The critical dimension below which we are guaranteed an invariant probability is precisely the critical dimension in the non-volume-preserving case of Zimmer's conjecture for split, simple Lie groups. In the case that Γ is cocompact, Theorem 1.6 follows immediately from the main result of [8]; on the other hand, the proof of the main result of [8] uses many of the ideas used to prove Theorem 1.6, particularly our Proposition 5.1 below. Theorem 1.6 moreover holds for actions of non-uniform lattices, which was used in the proof of Zimmer's conjecture [7] for finite-index sublattices of $\mathrm{SL}(m, \mathbb{Z})$

The second main result, Theorem 1.10, concerns actions $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ on manifolds M of certain intermediate dimensions. This range of dimensions includes examples where there exist non-isometric (volume-preserving) actions as well as examples of actions that do not preserve any Borel probability measure. In this case, we show that there exists a quasi-invariant measure μ on M such that the action on (M, μ) is measurably isomorphic to a relatively measure-preserving extension over a standard projective action.

Given an action $\alpha\colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$, the key idea in both theorems is to consider the G-action induced by α on an auxiliary space, which we denote by M^α. We take $P \subset G$ to be a minimal parabolic subgroup and consider P-invariant measures on M^α. This approach should be compared with a number of papers by Nevo and Zimmer, particularly [34], [35]. Nevo and Zimmer consider a manifold with a G-action and G-stationary measure ν. The measure ν decomposes as $\nu_0 * \lambda$, where λ is a P-invariant measure. (See [34, Th. 1.4] for discussion of this decomposition). Assuming that λ satisfies certain technical conditions—namely, that the measure λ is either *P-mixing* in [34] or that every non-trivial element of the maximal split Cartan subgroup $S \subset P$ acts ergodically in [35]—it is shown that the G-action on (M, ν) is a relatively measure-preserving extension over a standard projective action. These technical conditions are typically difficult to verify. In our argument, we exploit the constraints on the dimension of M and verify certain conditions similar to those introduced by Nevo and Zimmer. For instance, the technical condition in [35, Th. 3] that all elements of the maximal split Cartan subgroup $S \subset P$ act ergodically implies our Claim 6.2 below and hence all arguments in Section 6.2 apply. In practice, it is difficult to verify such ergodicity hypotheses.

1.1. *Introduction and reductions.* Throughout we assume that G is a real, connected, semisimple Lie group with $\mathbb{R}$-rank at least 2. By a standard construction, there is a central extension $\tilde{G} \to G$ such that $\tilde{G}$ is the direct product of connected, simple Lie groups:

$$\tilde{G} = \prod G_i.$$

(We abuse terminology and say a connected Lie group is simple if its Lie algebra is simple; in particular, we allow groups with infinite center.) We take $\Gamma \subset G$ to be a lattice subgroup and, writing $\tilde{\Gamma}$ for the lift of Γ to $\tilde{G}$, we assume that for every simple factor $G_i \subset \tilde{G}$ with $\mathbb{R}$-rank 1, the image of of $\tilde{\Gamma}$ to G_i is dense in G_i. Such a lattice will be called a *higher-rank lattice.* This, in particular, includes the cases that

(1) G has no compact factors and $\Gamma \subset G$ is irreducible, or
(2) every non-compact, simple factor of G has $\mathbb{R}$-rank at least 2.

Below, we will study smooth actions of such groups Γ. As we may lift an action of Γ to an action of $\tilde{\Gamma}$, without loss of generality we will assume for the remainder that G is a direct product $G = \prod G_i$ of simple Lie groups.

Note that $G = C \times G'$, where C is the maximal connected compact normal subgroup of G and G' is the maximal connected normal subgroup without compact factors. We remark that our main results—Theorems 1.6 and 1.10— are sharpest when G' is assumed to be simple. Replacing Γ with a subgroup of finite index, one may assume the restriction of the map $G \to G'$ to Γ is injective; in particular, one may assume $G = G'$, and thus that G has no compact factors. So we will assume from now on that G has no compact factors.

Let M be a compact, connected, boundaryless C^∞ manifold, and let $\alpha \colon \Gamma \to \mathrm{Diff}^{1+\beta}(M)$ be an action of Γ on M by $C^{1+\beta}$ diffeomorphisms. For notational convenience later, we assume α is a *right* action; that is, $\alpha(gh)(x) = \alpha(h)(\alpha(g)(x))$. Conjecturally, all such actions are obtained from families of model algebraic actions via standard constructions. In particular, if $\dim(M)$ is sufficiently small so that no model algebraic actions exists, Zimmer's conjecture states that all such actions should factor through actions of finite quotients of Γ; that is, the image $\alpha(\Gamma)$ of Γ in $\mathrm{Diff}^{1+\beta}(M)$ should be finite. Such an action is said to be *trivial.* See [16, Conjs. I, II], [17, Conjs. 4.12, 4.14], or [8, Conj. 2.4] for more precise formulations. See also [8] for recent solution to (the non-volume-preserving case of) Zimmer's conjecture for cocompact lattices in split, simple Lie groups.

We recall that in dimension 1, any lattice in a higher-rank, simple Lie group with finite center acts trivially on the circle [19], [11]. For certain lattices acting on surfaces, we obtain in conjunction with the main results of [18] the following complete results.

THEOREM A ([18, Cor. 1.8] and Theorem 1.6). *Let S be a closed oriented surface, and for $n \geq 4$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{Z})$ be a finite index subgroup. Then every $C^{1+\beta}$ action of Γ on S is trivial.*

THEOREM B ([18, Cor. 1.7] and Theorem 1.6). *Let S be a closed oriented surface of genus at least 1, and for $n \geq 4$, let $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ be a non-uniform lattice. Then every $C^{1+\beta}$ action of Γ on S is trivial.*

More generally, Theorem B holds when $\Gamma \subset G$ is a non-uniform lattice and G is a connected, semisimple Lie group with finite center, no compact factors, and $r(G) \geq 3$ for the integer $r(G)$ defined below ([18, Cor. 1.7]). In particular, the conclusion of Theorem B holds for any non-uniform lattice in a higher-rank, simple Lie group G with finite center such that the restricted root system of the Lie algebra of G is not of type A_2. By the main results of [8], triviality of all actions on surfaces also holds for cocompact lattices in all such groups. A more recent development is the proof of analogues of Theorems A and B for C^1-actions of lattices of higher-rank semisimple Lie groups on manifolds of dimension less than $\mathrm{rank}_{\mathbb{R}}(G)$ (or volume-preserving C^1-actions in dimension $\mathrm{rank}_{\mathbb{R}}(G)$), by Brown, Damjanović, and Zhang [6].

Note that if $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ is any lattice, then there is a model real-analytic action of Γ on a surface S that admits no invariant probability measure— namely, the right projective action of $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ on $\mathbb{R}P^2$ (or S^2). Note that any volume form on $\mathbb{R}P^2$ is quasi-invariant but non-invariant under this action. More generally, consider G a semi-simple Lie group with finite center. Let $Q \subset G$ be a parabolic subgroup, and let $\Gamma \subset G$ be a lattice. Then there is a natural right action of Γ on the quotient $Q \backslash G$ preserving no Borel probability measure but preserving the Lebesgue measure class.

Given the model action discussed above, we have the following conjecture, motivated by Theorems A and B, attributed to Polterovich in [17, Question 4.8], as well as new results from [6].

CONJECTURE 1.1. *Let $\Gamma \subset \mathrm{SL}(3, \mathbb{R})$ be a lattice. Let S be closed, connected a surface and let Γ act on S by C^1 diffeomorphisms. Suppose there is no Γ-invariant Borel probability measure on S. Then S is either $\mathbb{R}P^2$ or S^2; furthermore, any such action is smoothly conjugate to the standard projective action.*

1.2. *Facts from the structure of Lie groups.* To state our main results we recall some facts and definitions from the structure theory of real Lie groups. A standard reference is [23]. Let G be a connected, semisimple Lie group. As usual, write $\mathfrak{g}$ for the Lie algebra of G. Fix a Cartan involution θ of $\mathfrak{g}$, and write $\mathfrak{k}$ and $\mathfrak{p}$, respectively, for the $+1$ and -1 eigenspaces of θ. Denote by $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$ and by $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We let Σ denote the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Note that the

elements of Σ are (non-zero) real linear functionals on $\mathfrak{a}$. Recall that $\dim_{\mathbb{R}}(\mathfrak{a})$ is the $\mathbb{R}$-rank of G.

We choose a family of positive roots $\Sigma_+ \subset \Sigma$ and write Σ_- for the corresponding set of negative roots. For $\beta \in \Sigma$, write $\mathfrak{g}^\beta$ for the associated root space. Then $\mathfrak{n} = \bigoplus_{\beta \in \Sigma_+} \mathfrak{g}^\beta$ is a nilpotent subalgebra. A *standard parabolic subalgebra* (relative to the choice of $\theta, \mathfrak{a}$, and positive roots Σ_+) is any subalgebra of $\mathfrak{g}$ containing $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Recall $\beta \in \Sigma_+$ is a *simple* (positive) *root* if it is not an integer combination of other elements in Σ_+. We denote by $\Pi \subset \Sigma_+$ the set of simple roots in Σ_+. We have that the standard parabolic subalgebras of $\mathfrak{g}$ are parametrized by exclusion of simple (negative) roots: for any sub-collection $\Pi' \subset \Pi$ let

$$(1) \qquad \mathfrak{q}_{\Pi'} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \Sigma_+ \cup \mathrm{Span}(-\Pi')} \mathfrak{g}^\beta.$$

Then $\mathfrak{q}_{\Pi'}$ is a Lie subalgebra of $\mathfrak{g}$ and all standard parabolic subalgebras of $\mathfrak{g}$ are of the form $\mathfrak{q}_{\Pi'}$ for some $\Pi' \subset \Pi$. (See [23, Prop. 7.76] and, in particular, the analysis of corresponding $\mathfrak{sl}(2, \mathbb{R})$-triples, [23, Lemma 7.73].)

Let A, N, and K be the analytic subgroups of G corresponding to $\mathfrak{a}, \mathfrak{n}$ and $\mathfrak{k}$. These are closed subgroups of G and $G = KAN$ is the corresponding Iwasawa decomposition of G. When G has finite center, K is a maximal compact subgroup. Note that the Lie exponential $\exp \colon \mathfrak{g} \to G$ restricts to diffeomorphisms between $\mathfrak{a}$ and A and $\mathfrak{n}$ and N. Fixing a basis for $\mathfrak{a}$, we identify $A = \exp(\mathfrak{a}) = \mathbb{R}^d$. Via this identification we often extend linear functionals on $\mathfrak{a}$ to A. We write $M = C_K(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in K. Then $P = MAN$ is the *standard minimal parabolic subgroup*. Since M is an abelian extension of a compact group, it follows that P is amenable. (See, for example, [2, Prop. G.2.2(ii)].) A *standard parabolic subgroup* (relative to the choice of $\theta, \mathfrak{a}$, and Σ_+ above) is any closed subgroup $Q \subset G$ containing P. The Lie algebra of any standard parabolic subgroup Q is a standard parabolic subalgebra, and the correspondence between standard parabolic subgroups and subalgebras is 1-1.

We say two restricted roots $\beta, \hat\beta \in \Sigma$ are *coarsely equivalent* if there is some $c > 0$ with
$$\hat\beta = c\beta.$$
Note that c takes values only in $\{\frac{1}{2}, 1, 2\}$ and this occurs only if the root system Σ has a factor of type BC_ℓ. Let $\hat\Sigma$ denote the set of *coarse restricted roots*—that is, the set of coarse equivalence classes of Σ. Note that for $\xi \in \hat\Sigma$, $\mathfrak{g}^\xi := \oplus_{\beta \in \xi} \mathfrak{g}^\beta$ is a nilpotent subalgebra and the Lie exponential restricts to a diffeomorphism between $\mathfrak{g}^\xi$ and the corresponding analytic subgroup, which we denote by G^ξ.

Let $\mathfrak{q}$ denote a standard parabolic subalgebra of $\mathfrak{g}$. Observe that if $\mathfrak{g}^\beta \subset \mathfrak{q}$ for some $\beta \in \Sigma$ then, from the structure of parabolic subalgebras, $\mathfrak{g}^\xi \subset \mathfrak{q}$ where

$\xi \in \hat{\Sigma}$ is the coarse restricted root containing β. A standard parabolic (proper) subalgebra $\mathfrak{q}$ is *maximal* if there is no subalgebra $\mathfrak{q}'$ with $\mathfrak{q} \subsetneq \mathfrak{q}' \subsetneq \mathfrak{g}$. Note that maximal standard parabolic subalgebras are of the form $\mathfrak{q}_{\Pi \smallsetminus \beta}$ for some $\beta \in \Pi$.

1.3. *Resonant codimension and related combinatorial numbers.* Given a standard parabolic subalgebra $\mathfrak{q}$, define the *resonant codimension* of $\mathfrak{q}$ to be the cardinality of the set

$$\{\xi \in \hat{\Sigma} \mid \mathfrak{g}^{\xi} \not\subset \mathfrak{q}\}.$$

Given G as above, we define a combinatorial number $r(G)$ as follows.

Definition 1.2. The *minimal resonant codimension* of $\mathfrak{g}$, denoted $r(\mathfrak{g})$, is defined to be the minimal value of the resonant codimension of $\mathfrak{q}$ as $\mathfrak{q}$ varies over all (maximal) proper parabolic subalgebras of $\mathfrak{g}$.

Example 1.3. We compute $r(\mathfrak{g})$ for a number of classical real simple Lie algebras as well as simple real Lie algebras with restricted root systems of exceptional type. Given a simple real Lie algebra $\mathfrak{g}$, the number $r(\mathfrak{g})$ is determined purely by the restricted root system. In particular, we have the following.

Type A_n: $r(\mathfrak{g}) = n$. This includes $\mathfrak{sl}(n + 1, \mathbb{R})$, $\mathfrak{sl}(n + 1, \mathbb{C})$, $\mathfrak{sl}(n + 1, \mathbb{H})$.

Type B_n, C_n, and $(BC)_n$: $r(\mathfrak{g}) = 2n - 1$. This includes $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{so}(n, m)$ for $n < m$, and $\mathfrak{su}(n, m)$ and $\mathfrak{sp}(n, m)$ for $n \leq m$.

Type D_n: $r(\mathfrak{g}) = 2n - 2$ for $n \geq 4$. This includes $\mathfrak{so}(n, n)$ for $n \geq 4$.

Type E_6: $r(\mathfrak{g}) = 16$.

Type E_7: $r(\mathfrak{g}) = 27$.

Type E_8: $r(\mathfrak{g}) = 57$.

Type F_4: $r(\mathfrak{g}) = 15$.

Type G_2: $r(\mathfrak{g}) = 5$.

In all classical root systems $A_n, B_n, C_n, (BC)_n$ and D_n, the number $r(\mathfrak{g})$ corresponds to the parabolic subalgebra obtained by omitting the left-most root in the standard Dynkin diagrams. Exceptional root systems are checked by hand.

Note that if $\mathfrak{g}$ is non-simple, then $r(\mathfrak{g})$ is $\min\{r(\mathfrak{g}_i) : 1 \leq i \leq n\}$ where $\mathfrak{g}_i$ are the simple (non-compact) factors of $\mathfrak{g}$. We write $r(G) = r(\mathfrak{g})$. Note that inside a fixed family of simple Lie groups, the number $r(G)$ grows with the rank of G but need not coincide with the minimal dimension of a non-trivial algebraic action in the case that G is non-split. In particular, we only obtain the optimal expected dimensions in the case that G is split.

We define a second number $m(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g}$ of G.

Definition 1.4. Given a simple Lie algebra $\mathfrak{g}$ of $\mathbb{R}$-rank at least 2, define $m(\mathfrak{g})$ to be the minimal value of the resonant codimension of $\mathfrak{q}$ as $\mathfrak{q}$ varies over

all proper parabolic subalgebras $\mathfrak{q}$ of the form $\mathfrak{q}_{\Pi \smallsetminus \{\alpha_i, \alpha_j\}}$ where $\alpha_i \neq \alpha_j$ are simple roots in Π. If $\mathfrak{g}$ has rank 1, let $m(\mathfrak{g}) = 1$. If $\mathfrak{g} = \oplus \mathfrak{g}_i$ is semisimple, take $m(\mathfrak{g})$ to be the minimum of $m(\mathfrak{g}_i)$ over all non-compact, simple factors $\mathfrak{g}_i$ of $\mathfrak{g}$.

As before, write $m(G) = m(\mathfrak{g})$.

Example 1.5. Again, we compute the number $m(\mathfrak{g})$ for a number of classical, simple real Lie algebras as well as simple real Lie algebras with restricted root systems of exceptional type. As before, given a simple real Lie algebra $\mathfrak{g}$, the number $m(\mathfrak{g})$ is determined only by the restricted root system.

Type A_n**:** $m(\mathfrak{g}) = 2n - 1$.

Type B_n**,** C_n**, and** $(BC)_n$**:** $m(\mathfrak{g}) = 4n - 4$.

Type D_n**:** $m(\mathfrak{g}) = 9$ for $n = 4$; $m(\mathfrak{g}) = 4n - 6$ for $n \geq 5$.

Type E_6**:** $m(\mathfrak{g}) = 24$.

Type E_7**:** $m(\mathfrak{g}) = 43$.

Type E_8**:** $m(\mathfrak{g}) = 84$.

Type F_4**:** $m(\mathfrak{g}) = 20$.

Type G_2**:** $m(\mathfrak{g}) = 6$.

In all classical root systems except D_4, the number $m(\mathfrak{g})$ corresponds to the parabolic subalgebra obtained by omitting the two left-most roots in the standard Dynkin diagrams. In D_4, the number $m(\mathfrak{g})$ corresponds to omitting two commuting roots. Exceptional root systems are checked by hand.

As before, write $m(G) = m(\mathfrak{g})$.

1.4. *Statement of results.* Let G be a connected semisimple Lie group with real-rank at least 2 and, as introduced above, and let $\Gamma \subset G$ be a higher-rank lattice subgroup.

Recall that α denotes a right action of Γ on a compact, boundaryless manifold M by $C^{1+\beta}$ diffeomorphisms.

1.4.1. *Existence of invariant measures in low dimensions.* Our first main result establishes the existence of an α-invariant measure if the dimension of M is sufficiently small relative to $r(G)$.

THEOREM 1.6. *Let M be a compact manifold with $\dim(M) < r(G)$. Then for any $C^{1+\beta}$ action α of Γ on M, there exists an α-invariant Borel probability measure.*

We remark that in the case that Γ is cocompact, Theorem 1.6 in an immediate corollary of the main result of [8] where Zimmer's conjecture is verified for actions of compact lattices on manifolds of dimension less than $r(G)$.

The proof of the main result of [8] uses the proof of Theorem 1.6, namely the key observation in Proposition 5.1 below. We note also that Theorem 1.6 applies to non-uniform lattices and was used in the recent work [7] on Zimmer's conjecture for $\mathrm{SL}(m, \mathbb{Z})$.

We do not assert any regularity of the measure in Theorem 1.6. In particular, the ergodic components of the measure are expected to be supported on finite sets as such actions are expected to be trivial. Theorems A and B follow directly from the main results in [18] and Theorem 1.6.

1.4.2. *Finite extensions of projective factors in critical dimension.* In the case where $\dim M = r(G)$, we recall as a model the standard right action of $\Gamma \subset \mathrm{SL}(n+1, \mathbb{R})$ on $\mathbb{R}P^n$. Note that $\mathbb{R}P^n$ has the structure of $Q\backslash\mathrm{SL}(n+1, \mathbb{R})$ for a (maximal) parabolic subgroup $Q \subset \mathrm{SL}(n+1, \mathbb{R})$.

THEOREM 1.7. *Let M be a manifold with $\dim(M) = r(G)$. Then given any $C^{1+\beta}$ action α of Γ on M, either*
(a) *there exists an α-invariant Borel probability measure on M; or*
(b) *there exists an α-quasi-invariant Borel probability measure μ on M and a maximal parabolic subgroup $Q \subset G$ such that the action α of Γ on (M, μ) is measurably conjugate to a finite extension of the standard right action of Γ on $(Q\backslash G, m)$ where m is of Lebesgue class.*

Motivated by the above theorem, we extend Conjecture 1.1.

CONJECTURE 1.8. *Let M be a manifold with $\dim(M) = r(G)$. Given any sufficiently smooth action α of Γ on M either*
(a) *there exists an α-invariant Borel probability measure on M; or*
(b) *there is a maximal parabolic subgroup $Q \subset G$ such that M is diffeomorphic to a finite cover of $Q\backslash G$; moreover, the action α is smoothly conjugate to a lift of the standard right-action of Γ on $Q\backslash G$.*

1.4.3. *Projective factors in intermediate dimensions.* Let (X, ν) and (Z, μ) be standard measure spaces, and suppose Γ acts measurably on both X and Z (on the right) and preserves the measure classes of ν and μ respectively. Let (Y, η) be a standard measure space, and write $\mathrm{Aut}(Y, \eta)$ for the group of invertible, measure-preserving transformations of (Y, η). Let α and ρ denote, respectively, the actions of Γ on (Z, μ) and (X, ν).

Definition 1.9. We say that α is a *relatively measure-preserving extension* (modeled on (Y, η)) of ρ if there are
(1) a measurable cocycle $\psi \colon \Gamma \times (X, \nu) \to \mathrm{Aut}(Y, \eta)$ over ρ, and
(2) an isomorphism of measure spaces $\Phi \colon (Z, \mu) \to (X \times Y, \nu \times \eta)$
such that Φ intertwines α and the skew action defined by ψ: if $\Phi(z) = (x, y)$,
$$\Phi(\alpha(\gamma)(z)) = (\rho(\gamma)(x), \psi(\gamma, x)(y)).$$

THEOREM 1.10. *Let M be a manifold with $\dim(M) \leq m(G)$. Then given any $C^{1+\beta}$ action α of Γ on M there is an α-quasi-invariant Borel probability measure μ on M, a standard parabolic subgroup Q, and a Lebesgue space (Y, η) such that the action α on (M, μ) is a relatively measure-preserving extension (modeled on (Y, η)) of the standard right action of Γ on $(Q \backslash G, m)$.*

Note in the above theorem that if $Q = G$, it follows that μ is α-invariant. As discussed above, the result in Theorem 1.10 should be compared to results of Nevo and Zimmer, particularly [34], [35].

2. **Suspension construction and its properties**

We construct an auxiliary space on which the action α of Γ on M embeds as a Poincaré section for an associated G-action. We then equip the tangent space to this G-space with a norm that is well adapted to the geometry of Γ in G and the dynamics of the induced G-action.

2.1. *Suspension construction.* On the product $G \times M$ consider the right Γ-action

$$(g, x) \cdot \gamma = (g\gamma, \alpha(\gamma)(x))$$

and the left G-action

$$a \cdot (g, x) = (ag, x).$$

Define the quotient manifold $M^\alpha := G \times M / \Gamma$. As the G-action on $G \times M$ commutes with the Γ-action, we have an induced left G-action on M^α. We denote this action by $\tilde{\alpha}$. We write $\pi \colon M^\alpha \to G/\Gamma$ for the natural projection map. Note that M^α has the structure of a fiber bundle over G/Γ induced by the map π with fibers diffeomorphic to M. As the action of α is by $C^{1+\beta}$ diffeomorphisms, M^α is naturally a $C^{1+\beta}$ manifold. Equip M^α with a C^∞ structure compatible with the $C^{1+\beta}$-structure.

Note that the action $\tilde{\alpha}$ of G on M^α preserves two transverse distributions E^F and E^G, where $E^F = \ker(D\pi)$ and E^G is tangent to the local G-orbits on M^α. Furthermore, these distributions integrate to $C^{1+\beta}$ foliations of M^α.

We first observe the following.

CLAIM 2.1. *There exists an α-invariant Borel probability measure on M if and only if there exists an $\tilde{\alpha}$-invariant Borel probability measure on M^α.*

That an α-invariant measure on M induces an $\tilde{\alpha}$-invariant measure on M^α is standard. For the reverse implication, see, for instance, [34, Lemma 6.1]. Note that any $\tilde{\alpha}$-invariant measure on M^α projects under π to the Haar measure on G/Γ.

As the suspension space M^α is non-compact in the case that Γ is non-uniform, some care is needed when applying tools from smooth ergodic theory to the G-action on M^α. Indeed, although the non-compactness comes from the homogeneous factor, care is needed in order to control the fiber-wise dynamics

as the corresponding C^1- and $C^{1+\beta}$-norms of the fiberwise dynamics need not be bounded.

Below, we use the quasi-isometry between the Riemannian and word metrics on Γ established in [31] to control the degeneration of the fiber-wise dynamics. We follow the approach of [9] and construct dynamical charts relative to which the tools of classical smooth ergodic theory may be applied. The remainder of this section is devoted to constructing a Riemannian metric on TM^α, corresponding distance function d, and a family of dynamical charts.

The reader interested only in actions of cocompact lattices may skip the remainder of this section.

2.2. *Construction of a fundamental domain and family of fiber metrics.* Recall our standing assumptions on the Lie group G and the lattice Γ.

A set $D \subset G$ is a *fundamental domain* for Γ if $\bigcup_{\gamma \in \Gamma} D\gamma = G$ and if the natural map $G \to G/\Gamma$ is one-to-one on D. A Borel set $D \subset G$ is *almost-open* if the interior of D has full measure in the closure of D. A subset $S \subset G$ is a *fundamental set* if $\bigcup_{\gamma \in \Gamma} S\gamma = G$ and the set $\{\gamma : S\gamma \cap S \neq \emptyset\}$ is finite. The injectivity radius $r^\Gamma(g)$ of Γ at a point $g \in G$ is the largest $0 < r \leq 1$ such that the map $\mathfrak{g} \to G/\Gamma$ given by $X \mapsto \exp_{\mathfrak{g}}(X)g\Gamma$ is injective on

$$\{X \in \mathfrak{g} : \|X\| < r\}.$$

We write

$$V_r(g) := \{\exp_{\mathfrak{g}}(X)g : \|X\| \leq r\}$$

for the remainder.

2.2.1. *Properties of the family of metrics.* Our goal below is to build on TM a continuous family Riemannian metrics $\langle \cdot, \cdot \rangle_g$, parametrized by $g \in G$, and an almost-open, Borel fundamental domain $D \subset G$ for Γ such that

(1) the family of metrics $\langle \cdot, \cdot \rangle_g$ depends continuously on $g \in G$;
(2) the family $\langle \cdot, \cdot \rangle_g$ is Γ-*equivariant*: given $\gamma \in \Gamma$ and $v, w \in T_x M$,

$$\langle v, w \rangle_g = \langle D_x\alpha(\gamma)v, D_x\alpha(\gamma)w \rangle_{g\gamma};$$

(3) writing

$$V = \bigcup_{g \in D} V_{r^\Gamma(g)}(g),$$

the family $\langle \cdot, \cdot \rangle_g$ is uniformly comparable on V: there is a $C > 0$ so that for all $g, \bar{g} \in V, x \in M$, and $v \in T_x M$,

$$\langle v, v \rangle_g \leq C \langle v, v \rangle_{\bar{g}};$$

(4) for every $p \geq 1$, the function $g \mapsto d_G(e, g)$ is L^p on D with respect to the Haar measure where $d_G(\cdot, \cdot)$ is the right-invariant metric on G.

2.2.2. *Reductions.* Note that given a finite-index subgroup $\Gamma' \subset \Gamma$, a fundamental domain D' for Γ' and a Γ'-equivariant family of metrics that satisfy (1)–(4) above for Γ', then we can choose a fundamental domain $D \subset D'$ for Γ

and construct a Γ-equivariant family of metrics satisfying (1)–(4) for Γ by averaging over Γ'-coset representatives in Γ. Below, we will pass to a finite-index subgroup $\Gamma' \subset \Gamma$ and construct such a domain and family of metrics for Γ'.

First, recall that G has no compact factors and has Iwasawa decomposition $G = KAN$. Then K contains the center of G. Since Γ is a lattice, it intersects the center of G in a finite-index subgroup; see [32, Lemma IX.(6.1)]. Passing to a group of finite index we may thus assume $K \cap \Gamma$ is contained in the center of G and has finite index in the center of G.

Let $\mathrm{Ad} \colon G \to \mathrm{Ad}(G)$ be the adjoint representation of G. Since G has no compact factors, it follows that $\mathrm{Ad}(\Gamma)$ is a lattice subgroup in $\mathrm{Ad}(G)$; see [32, Lemma IX.(6.1)].

From the Margulis Arithmeticity Theorem [32], it follows that there is a semisimple linear algebraic group $\mathbf{H}$ such that writing $H = \mathbf{H}(\mathbb{R})^\circ$ for the connected component of the identity in $\mathbf{H}(\mathbb{R})$, there is a surjective homomorphism $\Phi \colon H \to \mathrm{Ad}(G)$ with compact kernel such that

$$\mathrm{Ad}(\Gamma) \cap \Phi(\mathbf{H}(\mathbb{Z}))$$

has finite index in $\mathrm{Ad}(\Gamma)$. Let $\hat{\Gamma} = \Phi^{-1}(\mathrm{Ad}(\Gamma)) \cap \mathbf{H}(\mathbb{Z}) \cap H$. Then $\hat{\Gamma}$ has finite index in $\mathbf{H}(\mathbb{Z})$ and is an arithmetic lattice in H. Replacing Γ, $\mathrm{Ad}(\Gamma)$ and $\hat{\Gamma}$ with finite index subgroups, we may assume that $\hat{\Gamma}$ is torsion-free, that Γ maps surjectively onto $\mathrm{Ad}(\Gamma)$, and that $\Phi \colon \hat{\Gamma} \to \mathrm{Ad}(\Gamma)$ is an isomorphism.

Let $X = K \backslash G$ be the globally symmetric space associated with G. We may select maximal compact subgroups $\overline{K} \subset \mathrm{Ad}(G)$ and $\hat{K} \subset H$ so that X is also identified as

$$X := K \backslash G = \overline{K} \backslash \mathrm{Ad}(G) = \hat{K} \backslash H.$$

Equip G with a right-invariant, left-K-invariant metric, and equip X with the quotient right-invariant metric.

2.2.3. *Compactification of X and Siegel fundamental sets.* Following [4, Part III, Ch. 9], write $\overline{X}^{\mathrm{BS}}$ for the *Borel-Serre partial compactification* of X. The space $\overline{X}^{\mathrm{BS}}$ has the structure of a real-analytic manifold with corners. The action of $\hat{\Gamma} \cong \mathrm{Ad}(\Gamma)$ on X extends to a continuous, proper action on $\overline{X}^{\mathrm{BS}}$. The quotient $\overline{X}^{\mathrm{BS}}/\hat{\Gamma}$ is a compact, Hausdorff space. Having taken $\hat{\Gamma}$ to be torsion-free, $\hat{\Gamma}$ acts properly-discontinuously on $\overline{X}^{\mathrm{BS}}$ and the quotient $\overline{X}^{\mathrm{BS}}/\hat{\Gamma}$ has the structure of a compact real-analytic manifold with corners. (See [4, Prop. III.5.14] for statement and Propositions III.9.16 and III.9.17 of [4] for further details.)

By the reduction theory of arithmetic groups, we may find a Siegel fundamental set $S \subset X = \hat{K} \backslash H$ (a finite union of rational translates of Siegel sets) for the action of $\hat{\Gamma} \cong \mathrm{Ad}(\Gamma)$ on X. (See, for instance, [32, VIII.1].) Write $\mathrm{int}_X(S)$ for the interior of S in X. Then

(1) $\bigcup_{\hat{\gamma} \in \hat{\Gamma}} \operatorname{int}_X(S) \cdot \hat{\gamma} = X$;
(2) the set $\{\hat{\gamma} \in \hat{\Gamma} : S\hat{\gamma} \cap S \neq \emptyset\}$ is finite;
(3) for any $x_0 \in X$, the function $x \mapsto d_X(x_0, x)$ is L^p on S with respect to the Haar measure for every $1 \leq p < \infty$;
(4) S has compact closure in $\overline{X}^{\mathrm{BS}}$.

Write $\overline{S}$ for the closure of S in $\overline{X}^{\mathrm{BS}}$, and write $\operatorname{int}_{\overline{X}^{\mathrm{BS}}}(\overline{S})$ for the interior of $\overline{S}$ in $\overline{X}^{\mathrm{BS}}$. Then we also have (see discussions in [4, III.9])

(5) $\bigcup_{\hat{\gamma} \in \hat{\Gamma}} \operatorname{int}_{\overline{X}^{\mathrm{BS}}}(\overline{S}) \cdot \hat{\gamma} = \overline{X}^{\mathrm{BS}}$;
(6) the set $\{\hat{\gamma} \in \hat{\Gamma} : \overline{S}\hat{\gamma} \cap \overline{S} \neq \emptyset\}$ is finite;

Moreover, from the construction of Siegel sets, if we denote injectivity radius at $x \in X$ by $r^X(x)$ and write $B_x(x, r)$ for the ball in X centered at x of radius r, we have that

(7) $S' := \bigcup_{x \in S} B(x, r^X(x))$ is contained in a Siegel fundamental set.

2.2.4. *Construction of partition of* 1. Let S be as above. We may cover $\overline{X}^{\mathrm{BS}}/\hat{\Gamma}$ with finitely many open sets U_j, each of which is the injective image of an open set $\hat{U}_j \subset \operatorname{int}_{\overline{X}^{\mathrm{BS}}}(\overline{S})$. Equip $\overline{X}^{\mathrm{BS}}/\hat{\Gamma}$ with a partition of unity $\{\psi_j\}$ subordinate to the cover $\{U_j\}$. For each j, let $\hat{\psi}_j : \overline{X}^{\mathrm{BS}} \to [0, 1]$ be the lift of ψ_j whose support is contained in $\hat{U}_j$.

The map $\Upsilon : G \to K \times X$, $\Upsilon : kan \mapsto (k, Kan)$ is well known to be a bi-Lipschitz map ([2, Lemma 3.6.3]) and is $(K \cap \Gamma)$-equivariant. If $K_0 \subset K$ is a pre-compact open set with $K = K_0 \cdot (\Gamma \cap K)$ and if $S \subset X$ is a fundamental set for the action of $\operatorname{Ad}(\Gamma) \cong \hat{\Gamma}$ on X, then
$$\widetilde{S} := \Upsilon^{-1}(K_0 \times S)$$
is a fundamental set for the Γ-action on G. Moreover, the function $g \mapsto d_G(e, g)$ is L^p on $\widetilde{S}$ with respect to the Haar measure on G for every $1 \leq p < \infty$. Cover $K/(K \cap \Gamma)$ with finitely many open sets $\{V_k\}$, each of which is the injective image of an open set $\hat{V}_k \subset K_0$. Let $\{\eta_k\}$, $\eta_k : K \to [0, 1]$ be a partition of unity of $K/(K \cap \Gamma)$ subordinate to the cover $\{V_k\}$, and for each k, lift η_k to $\hat{\eta}_k : K_0 \to [0, 1]$ supported on $\hat{V}_k \subset K_0$.

Finally, let $\phi_{j,k} : \widetilde{S} \to [0, 1]$ be
$$\phi_{j,k}(g) = \eta_k(\Upsilon_1(g)) \cdot \psi_j(\Upsilon_2(g)),$$
where Υ_1, Υ_2 are the coordinate functions of Υ. Let $\{\phi_i\}$ denote the collection $\{\phi_{j,k}\}$.

For each i and $\gamma \in \Gamma$, write $\phi_{i,\gamma} : G \to [0, 1]$ for the function
$$\phi_{i,\gamma}(g) = \phi_i(g\gamma).$$
We enumerate some properties of $\{\phi_{i,\gamma}\}$.

(1) From the assumptions on the supports of ϕ_i, we have for each i that $\text{supp}(\phi_{i,\gamma}) \cap \text{supp}(\phi_{i,\gamma'}) = \emptyset$ whenever $\gamma \neq \gamma'$.

(2) The set
$$\{\, \gamma \in \Gamma \mid \text{supp}(\phi_{i,\gamma}) \cap \widetilde{S} \neq \emptyset \text{ for some } i \,\}$$
is finite.

(3) The collection $\{\, \phi_{i,\gamma} \mid i \in \{1, \ldots, N\}, \gamma \in \Gamma \,\}$ is a locally-finite, Γ-invariant, partition of unity on G.

2.2.5. *Parametrized families of metrics.* Given $x \in M$, let $\langle \cdot, \cdot \rangle_{0,x}$ denote the inner product on $T_x M$. Given $g \in G$, $x \in M$, and $v, w \in T_x M$, set
$$\langle v, w \rangle_{g,x} := \sum_{i=1}^{N} \sum_{\gamma \in \Gamma} \phi_{i,\gamma}(g) \, \langle D_x \alpha(\gamma)(v), D_x \alpha(\gamma)(w) \rangle_{0,\alpha(\gamma)(x)}.$$

Let $\| \cdot \|_{g,x}$ denote the norm induced by the inner product $\{\langle \cdot, \cdot \rangle_{g,x}\}$, and let $\| \cdot \|'$ denote the induced norm on $G \times TM$.

We collect the following observations.

CLAIM 2.2.

(1) *The norms $\| \cdot \|_{g,x}$ are uniformly comparable on any fundamental set $\widetilde{S}$ in G as above: there is a constant $C > 0$, such that for all $g_1, g_2 \in \widetilde{S}$, $x \in M$, and $v \in T_x M$, we have*
$$\frac{1}{C} \|v\|_{g_1,x} \leq \|v\|_{g_2,x} \leq C \|v\|_{g_1,x}.$$

(2) *For any $g \in G$, $\tilde{\gamma} \in \Gamma$, and $(x, v) \in TM$, we have*
$$\|D_x \alpha(\tilde{\gamma}) v\|_{g\tilde{\gamma}, \alpha(\tilde{\gamma})(x)} = \|v\|_{g,x}.$$

In particular, Γ acts by isometries on the fibers of $G \times TM$ whence $\| \cdot \|'$ descends to a norm $\| \cdot \|$ on the fiberwise tangent bundle E^F over M^α.

Proof. (1) follows from finiteness of the partition of 1. For (2) we verify
$$\|v\|_{g,x}^2 = \sum_{i=1}^{N} \sum_{\gamma \in \Gamma} \phi_{i,\gamma}(g) \|D_x \alpha(\gamma)\|_{0,\alpha(\gamma)(x)}^2$$

$$= \sum_{i=1}^{N} \sum_{\gamma \in \Gamma} \phi_{i,\tilde{\gamma}^{-1}\gamma}(g\tilde{\gamma}) \|D_x \alpha(\tilde{\gamma}\tilde{\gamma}^{-1}\gamma)(v)\|_{0,\alpha(\gamma)(x)}^2$$

$$= \sum_{i=1}^{N} \sum_{\gamma \in \Gamma} \phi_{i,\tilde{\gamma}^{-1}\gamma}(g\tilde{\gamma}) \|D_{\alpha(\tilde{\gamma})(x)} \alpha(\tilde{\gamma}^{-1}\gamma) D_x \alpha(\tilde{\gamma})(v)\|_{0,\alpha(\tilde{\gamma}^{-1}\gamma)\left(\alpha(\tilde{\gamma})(x)\right)}^2$$

$$= \|D_x \alpha(\tilde{\gamma}, x) v\|_{g\tilde{\gamma}, \alpha(\tilde{\gamma})(x)}^2. \qquad \square$$

2.2.6. *Choice of fundamental domain.* Let $D \subset \widetilde{S}$ be a fundamental domain for Γ in G. Since the injectivity radius on G/Γ is comparable with the injectivity radius on $X/\operatorname{Ad}(\Gamma)$, from item (7) of Section 2.2.3 we have that $V = \bigcup_{g \in D} V_{r^\Gamma(g)}(g)$ only intersects the support of finitely many functions $\phi_{i,\gamma}$. The desired properties in Section 2.2.1 follow.

2.2.7. *Induced distance on M^α.* Using the Γ-equivariant family of metrics $\{\langle \cdot, \cdot \rangle_g : g \in G\}$ constructed above and using the right invariant metric on G, we endow the tangent bundle of $G \times M$ with a continuous Riemannian metric such that Γ acts by isometries. This induces a Riemannian metric on TM^α and corresponding distance function $d(\cdot, \cdot)$ on M^α. The restriction of the metric on TM^α to the fiberwise tangent bundle E^F coincides with the metric defined in 2.2.5.

2.3. *Some estimates.* Equip M with any C^∞ Riemannian metric; by compactness, all estimates are independent of the choice of metric. Let $\exp_x \colon T_xM \to M$ be the Riemannian exponential map at x, and fix $r_0 \leq 1$ to be smaller than the injectivity radius of M.

Write $B_x(r) \subset T_xM$ for the norm ball $B_x(r) = \{v \in T_xM : \|v\| < r\}$. Given a diffeomorphism $f \colon M \to M$, let

$$\tilde{f}_x \colon U_{x,f} \subset B_x(r_0) \subset T_xM \to B_{f(x)}(r_0) \subset T_{f(x)}M$$

be the diffeomorphism defined by

$$\tilde{f}_x := \exp_{f(x)}^{-1} \circ f \circ \exp_x$$

on the maximal domain $U_{x,f}$ on which it is defined. Given $U \subset U_{x,f}$, define the local C^1 and Hölder norms of $\tilde{f}_x \restriction_U \colon U \to B_x(r_0) \subset T_{f(x)}M$ to be

$$\|D\tilde{f}_x\|_U = \sup_{v \in U} \|D_v\tilde{f}_x\|, \qquad \operatorname{H\ddot{o}l}^\beta_U(D\tilde{f}_x) := \sup_{v \neq w \in U} \frac{\|D_v\tilde{f}_x - D_w\tilde{f}_x\|}{\|v - w\|^\beta}.$$

If $f \colon M \to M$ is $C^{1+\beta}$, define

(1) $\|Df\| := \sup_{x \in M} \|D\tilde{f}_x\|_{U_{x,f}}$ and
(2) $\operatorname{H\ddot{o}l}^\beta(Df) := \sup_{x \in M} \operatorname{H\ddot{o}l}^\beta_{U_{x,f}}(D\tilde{f}_x)$.

Compactness of M ensures $\|Df\|$ and $\operatorname{H\ddot{o}l}^\beta(Df)$ are finite.

We have the following elementary estimate.

CLAIM 2.3. *Let $f, g \in \operatorname{Diff}^{1+\beta}(M)$. Given $x \in M$ and $U \subset U_{x,g} \subset T_xM$ such that*

$$\tilde{g}_x(U) \subset U_{g(x),f},$$

we have

$$\operatorname{H\ddot{o}l}^\beta_U(D\widetilde{(f \circ g)}_x) \leq \|Df\| \operatorname{H\ddot{o}l}^\beta(Dg) + \|Dg\|^{1+\beta} \operatorname{H\ddot{o}l}^\beta(Df).$$

Proof. For $v, u \in U$ and ξ with $\|\xi\| = 1$,

$$\|D_v \widetilde{(f \circ g)}_x \xi - D_u \widetilde{(f \circ g)}_x \xi\| = \|D_{\tilde{g}_x(v)} \tilde{f}_{g(x)} D_v \tilde{g}_x \xi - D_{\tilde{g}_x(u)} \tilde{f}_{g(x)} D_u \tilde{g}_x \xi\|$$

$$\leq \|D_{\tilde{g}_x(v)} \tilde{f}_{g(x)} D_v \tilde{g}_x \xi - D_{\tilde{g}_x(v)} \tilde{f}_{g(x)} D_u \tilde{g}_x \xi\|$$

$$+ \|D_{\tilde{g}_x(v)} \tilde{f}_{g(x)} D_u \tilde{g}_x \xi - D_{\tilde{g}_x(u)} \tilde{f}_{g(x)} D_u \tilde{g}_x \xi\|$$

$$\leq \|D\tilde{f}_{g(x)}\| \|D_v \tilde{g}_x - D_u \tilde{g}_x\| + \|D\tilde{g}_x\| \|D_{\tilde{g}_x(v)} \tilde{f}_{g(x)} - D_{\tilde{g}_x(u)} \tilde{f}_{g(x)}\|$$

$$\leq \|Df\| \operatorname{H\ddot{o}l}_U^\beta(D\tilde{g}_x) d(u,v)^\beta + \|Dg\| \operatorname{H\ddot{o}l}_{\tilde{g}_x(U)}^\beta(D\tilde{f}_{g(x)}) d(\tilde{g}_x(v), \tilde{g}_x(u))^\beta$$

$$\leq \|Df\| \operatorname{H\ddot{o}l}_U^\beta(D\tilde{g}_x) d(u,v)^\beta + \|Dg\|^{1+\beta} \operatorname{H\ddot{o}l}_{\tilde{g}_x(U)}^\beta(D\tilde{f}_{g(x)}) d(u,v)^\beta. \qquad \square$$

In particular, we have the following.

CLAIM 2.4. *Let* $g_i \in \operatorname{Diff}^{1+\beta}(M)$, $i = \{1, 2, \ldots, \ell\}$, *and fix* C *with* $\|Dg_i\| \leq C$ *and* $\operatorname{H\ddot{o}l}^\beta(Dg_i) \leq C$. *Given* $n \geq 0$ *and*

$$U \subset B_x(C^{-n} r_0) \subset T_x M$$

with $h = g_{i_1} \circ \cdots \circ g_{i_n}$, *we have*

(1) $\|D\tilde{h}_x\|_U \leq C^n$ *and*

(2) $\operatorname{H\ddot{o}l}_U^\beta(D\tilde{h}_x) \leq n C^{n(1+\beta)}$ *for every* x.

2.4. *Construction of dynamical charts.* Let $D \subset G$ be the almost open, fundamental domain for Γ constructed in Section 2.2. In the sequel, we often use the measurable parametrization $D \times M$ of $M^\alpha = (G \times M)/\Gamma$.

Fix a globally defined, Borel family of isometric identifications $\tau_x \colon T_x M \to \mathbb{R}^n$. With respect to any fixed background C^∞ Riemannian metric on M, let $\exp_x \colon T_x M \to M$ denote the Riemannian exponential map at x and let r_0 denote the injectivity radius of M. Let $\mathbb{R}^k = \mathfrak{g} \oplus \mathbb{R}^n$ be equipped with the product Euclidean metric where $k = \dim G + \dim M$.

Given $p = (g, x) \in D \times M$, let $\rho(g) = \frac{1}{2} \min\{r^\Gamma(g), r_0\}$, and let

$$\phi_p \colon \mathbb{R}^k(\rho(g)) \to M^\alpha$$

be the natural embedding

$$\phi_p \colon (X, v) \mapsto \left(\exp(X)g, \exp_x(\tau_x^{-1} v)\right)/\Gamma,$$

where we write $\mathbb{R}^k(r) := \{v \in \mathbb{R}^k : \|v\| < r\}$. We immediately verify that, relative to the induced metric in 2.2.7, the charts $\phi_{(g,x)}$ are C^1 with $\|D\phi_{(g,x)}\|$ and $\|D\phi_{(g,x)}^{-1}\|$ uniformly bounded; in particular, relative to the distance function d in 2.2.7 the charts are uniformly bi-Lipschitz. As the injectivity radius $r^\Gamma(g\Gamma)$ is comparable to the distance from $g\Gamma$ to a fixed base point $x_0 \in G/\Gamma$, we have that $g \mapsto -\log(\rho(g))$ is L^q with respect to the Haar measure for all $1 \leq q < \infty$.

Recall that we let A be the analytic subgroup of G corresponding to $\mathfrak{a}$. Fixing a basis for $\mathfrak{a}$, via the parametrization $\exp\colon \mathfrak{a} \to A$ we identify A with $\mathbb{R}^d$, where $d \geq 2$ is the rank of G. Below, we consider an arbitrary lattice subgroup $\mathbb{Z}^d \subset A$ and fix a finite, symmetric, generating set $F = \{s_j : 1 \leq j \leq m\}$ for $\mathbb{Z}^d$.

Following the notation of [9], we let $U = U_0 = \Lambda = D \times M = M^\alpha$ for any such $\mathbb{Z}^d$ and F.

In the sequel, we will be concerned with A-invariant measures μ on M^α that project to the Haar measure on G/Γ.

PROPOSITION 2.5. *Let μ be an A-invariant probability measure on M^α projecting to Haar measure on G/Γ. Then for any lattice $\mathbb{Z}^d \subset A \simeq \mathbb{R}^d$ and any finite, symmetric, generating set $F = \{s_j : 1 \leq j \leq \ell\}$ for $\mathbb{Z}^d$, the standing hypotheses of [9, §3.2] hold relative to the charts $\{\phi_p : p \in M^\alpha\}$ above.*

That is, there are measurable functions $r\colon D \to (0,1]$ and $C\colon D \to [1,\infty)$ and a constant L with

$$r(g) \leq \rho(g), \; -\log r(g) \in L^q(D), \; and \; \log C(g) \in L^q(D) \; for \; all \; 1 \leq q < \infty$$

such that, writing

$$r(p) = r(g), \rho(p) = \rho(g), C(p) = C(g)$$

for $p = (g,x) \in M^\alpha = D \times M$, we have

(H1) *$\phi_p \colon \mathbb{R}^k(\rho(p)) \to M^\alpha$ is a C^1 diffeomorphism onto its image with $\phi_p(0) = p$;*
(H2) *$\|D\phi_p\| \leq L$ and $\|D\phi_p^{-1}\| \leq L$; in particular, $\phi_p \colon \mathbb{R}^k(\rho(p)) \to (U,d)$ is a bi-Lipschitz embedding with $\mathrm{Lip}(\phi_p) \leq L$ and $\mathrm{Lip}(\phi_p^{-1}) \leq L$.*

Moreover, for each $m \in F$, setting $f(\cdot) = \tilde{\alpha}(m,\cdot)$, we have for $p \in M^\alpha$ that

(H1) *the map*

(2) $$\tilde{f}_p := \phi_{f(p)}^{-1} \circ f \circ \phi_p$$

is well defined on $\mathbb{R}^k(r(p))$ with range contained in $\mathbb{R}^k(\rho(f(p)))$;
(H2) *$\tilde{f}_p \colon \mathbb{R}^k(r(p)) \to \mathbb{R}^k(\rho(f(p)))$ is uniformly $C^{1+\beta}$ with*

$$\|\tilde{f}_p\|_{1+\beta} \leq C(p)$$

(H3) *for every $n \in \mathbb{Z}^d$, $\big(p \mapsto \log^+ \|D_p\tilde{\alpha}(n)\|\big) \in L^q(\mu)$ for any $1 \leq q < \infty$; in particular, $\big(p \mapsto \log^+ \|D_p\tilde{\alpha}(n)\|\big) \in L^{d,1}(\mu)$.*

Here $L^{d,1}(\mu)$ is the Lorentz integrability space (see [30]). We have $L^q(\mu) \subset L^{d,1}(\mu)$ for any $q > d$. The assertion that $\big(p \mapsto \log^+ \|D_p\tilde{\alpha}(n)\|\big) \in L^{d,1}(\mu)$ guarantees the cocycle satisfies the hypotheses of the higher-rank multiplicative ergodic theorem. As $-\log\rho, -\log r$ and $\log C$ are L^d on the domain D, it follows that, in the terminology of [9], they are slowly growing functions over the action of $\mathbb{Z}^d$.

Proof. Fix a finite, symmetric generating set $S = \{\gamma_i : 1 \le i \le \ell\}$ for Γ. For each $1 \le i \le \ell$, take $g_i = \alpha(\gamma_i) \colon M \to M$ and set $\hat{C} > 1$ with

(1) $\|Dg_i\| \le \hat{C}$ and

(2) $\mathrm{H\ddot{o}l}^\beta(Dg_i) \le \hat{C}$.

Let d_{word} denote the corresponding word metric on Γ. Let d_G denote the distance on G induced by the right-invariant metric on G. Note that d_G restricts to a metric on $\Gamma \subset G$. It follows from [31] that if Γ is a higher-rank lattice as introduced in Section 1, the metrics d_{word} and d_G are quasi-isometrically equivalent: there are $A > 1$ and $B > 0$ such that for all $\gamma, \hat{\gamma} \in \Gamma$, we have

(3) $\qquad A^{-1} d_G(\gamma, \hat{\gamma}) - B \le d_{\mathrm{word}}(\gamma, \hat{\gamma}) \le A d_G(\gamma, \hat{\gamma}) + B.$

Now consider any lattice subgroup $\mathbb{Z}^d$ in $A \simeq \mathbb{R}^d$ and finite symmetric generating set F in $\mathbb{Z}^d$. Given $g \in D$ and $s_j \in F$, let $\gamma_j(g)$ be such that $s_j g \in D\gamma_j(g)$. Define

$$N(g) = \max_{s_j \in F}\{d_{\mathrm{word}}(e, \gamma_j(g))\}.$$

We have

$$d_{\mathrm{word}}(e, \gamma_j(g)) \le A\left[d(e,g) + d(e,s_j) + d(e, s_j g(\gamma_j(g))^{-1})\right] + B.$$

We denote with dg the Haar measure on G/Γ and naturally identify it with dg on D. Remember that $g \mapsto d(e,g)$ is in $L^q(D, dg)$ for all $1 \le q < \infty$ by the choice of the Siegel domain. Also, as the map $D \to D$ given by $g \mapsto s_j g(\gamma_j(g))^{-1}$ preserves the Haar measure, it follows that $g \mapsto N(g)$ is in $L^q(D, dg)$ for all $1 \le q < \infty$.

We set $r(g,x) = r(g) := \hat{C}^{-N(g)}\rho(g)$. We have that $0 < r(g,x) \le \rho(g)$ for every $(g,x) \in D \times M$. Moreover, we have that

$$\int (-\log(r(g,x)))^q \, d\mu(g,x) = \int_D (-\log(r(g)))^q \, dg < \infty.$$

Given $s_j \in F$, let $f = \tilde{\alpha}(s_j)$. Write $\tilde{f}_{(g,x)} \colon \mathbb{R}^k(r(g)) \to \mathbb{R}^k(\rho(f(g))$ for

$$\tilde{f}_{(g,x)} := \phi_{f(g,x)}^{-1} \circ f \circ \phi_{(g,x)}.$$

(H1) then follows. From Claim 2.4, we have

$$\|D\tilde{f}_{(g,x)}\| \le \hat{C}^{N(g)}, \quad \mathrm{H\ddot{o}l}^\beta(D\tilde{f}_{(g,x)}) \le N(g)\hat{C}^{N(g)(1+\beta)}$$

whence (H2) follows. Moreover, we have that the function

(4) $\qquad\qquad\qquad (g,x) \mapsto \log\|D_0\tilde{f}_{(g,x)}\|$

is $L^q(\mu)$ for every $1 \le q < \infty$. From the cocycle property, (H3) follows for all elements of the action. $\qquad\square$

We have the following stronger version of (H3), which follows from the uniform comparability of the norms on the fundamental domain D, inequality (3), and that $g \mapsto d_G(e,g)$ is L^q on D with respect to the Haar measure on G for all $1 \le q < \infty$. Let $\|\cdot\|$ denote the norm on TM^α constructed in 2.2.5.

CLAIM 2.6. *Let μ be Borel probability measure on M^α projecting to Haar measure on G/Γ. Given any compact set $B \subset G$, the function*

$$p \mapsto \sup_{g \in B} \log^+ \|D_p \tilde{\alpha}(g)\|$$

is $L^q(\mu)$ for any $1 \le q < \infty$.

3. Lyapunov exponents, coarse foliations, and conditional entropy

This section is mostly a summary of results we need from [9], [5], [10]. We consider the restriction of the action $\tilde{\alpha}$ on M^α to the subgroup A in G. Take μ to be an A-invariant probability measure on M^α. Let $\nu = \pi_*(\mu)$ be the projection of μ to G/Γ. In the case that Γ is not cocompact, assume the projection ν is G-invariant so that the charts in Section 2.4 satisfy properties (H1)–(H3) of Proposition 2.5 for an arbitrarily fixed lattice $\mathbb{Z}^d \subset A$.

3.1. *Lyapunov exponent functionals.* From the $L^{d,1}$ integrability of (H3) of Proposition 2.5 it follows that the restriction to A of the derivative cocycle $D\tilde{\alpha}$ on (M^α, μ) satisfies the hypotheses of the Oseledec's multiplicative ergodic theorem in every direction $s \in \mathbb{R}^d$ (see (6) below). Moreover, we have uniform convergence along spheres guaranteed by the stronger conclusions of the higher-rank Oseledec's multiplicative ergodic theorem.

Equip $A \simeq \mathbb{R}^d$ with any norm $|\cdot|$.

THEOREM 3.1 (Higher-rank multiplicative ergodic theorem; [9, Th. 2.4]). *Let μ be any A-invariant Borel probability measure on M^α satisfying $(H3)$ of Proposition 2.5. Then there exist*

(1) *a full measure, A-invariant subset $\Lambda_0 \subset M^\alpha$;*
(2) *an A-invariant measurable function $r\colon \Lambda_0 \to \mathbb{N}$;*
(3) *an A-invariant measurable family of linear functionals $\lambda_i(p)\colon A \to \mathbb{R}$ for $1 \le i \le r(p)$;*
(4) *and a family of mutually transverse, $D\tilde{\alpha}\!\restriction_A$-invariant, measurable subbundles $E_{\lambda_i}(p) \subset TM^\alpha$ with $T_pM^\alpha = \bigoplus_{i=1}^{r(p)} E_{\lambda_i}(p)$ for $p \in \Lambda_0$*

such that

$$(5) \qquad \lim_{s \to \infty} \frac{\log \|D_p\tilde{\alpha}(s)(v)\| - \lambda_i(p)(s)}{|s|} = 0$$

for all $v \in E_{\lambda_i}(p) \smallsetminus \{0\}$.

We note that (5) is stated in [9, Th. 2.4] only for integer vectors $n \in \mathbb{Z}^d$ but easily generalizes to all $s \in A$. Indeed, it suffices to approximate s by an integer vector n with uniformly bounded error. The resulting error between $\lambda_i(p)(s)$ and $\lambda_i(p)(n)$ is uniformly bounded. To control the error between $\log \|D_p\tilde{\alpha}(s)(v)\|$ and $\log \|D_p\tilde{\alpha}(n)(v)\|$, fix a compact fundamental set $\mathcal{K}$ for the

lattice subgroup $\mathbb{Z}^d$ in A, and let

$$\phi(p) = \max\left\{\sup_{s\in\mathcal{K}}\log^+ \|D_p\tilde{\alpha}(s)\|, \sup_{s\in\mathcal{K}}\log^+ \|D_p\tilde{\alpha}(s)^{-1}\|, 1\right\}.$$

We have

$$\left|\log\|D_p\tilde{\alpha}(s)(v)\| - \log\|D_p\tilde{\alpha}(n)(v)\|\right| \le \phi(\tilde{\alpha}(n)(p)).$$

By Claim 2.6, we have $\phi \in L^d(\mu)$; by the pointwise ergodic theorem for actions of $\mathbb{Z}^d$, for μ-almost every p, the sum

$$\frac{1}{N^d}\sum_{|n|\le N}\left(\phi(\tilde{\alpha}(n)(p))\right)^d$$

converges. It follows for almost every p that

$$\max_{|n|=N}\frac{\left(\phi(\tilde{\alpha}(n)(p))\right)^d}{N^d} \to 0$$

as $N \to \infty$ whence

$$\max_{|n|=N}\frac{\phi(\tilde{\alpha}(n)(p))}{N} \to 0$$

as $N \to \infty$.

From (5), for almost every $p \in M^\alpha$ and every $s \in A$, we have convergence along rays

$$(6) \qquad \lim_{k\to\infty}\frac{1}{k}\log\|D_p\tilde{\alpha}(ks)(v)\| = \lambda_i(p)(s)$$

for all $v \in E_{\lambda_i}(p) \smallsetminus \{0\}$. The linear functionals $\lambda_i(p)\colon A \to \mathbb{R}$ are the *Lyapunov exponent functionals*. The dimension of the corresponding $E_{\lambda_i}(p)$ is the *multiplicity of $\lambda_i(p)$*.

Recall the two $D\tilde{\alpha}$-invariant subbundles E^F and E^G of TM^α. We may restrict the derivative cocycle $\{D\tilde{\alpha}(s) : s \in A\}$ to either of the two A-invariant distributions E^F or E^G. These restrictions satisfy the hypotheses of the higher-rank multiplicative ergodic theorem. For the restricted cocycles, we obtain Lyapunov exponent functionals $\{\lambda_i^F(p)\}$ and $\{\lambda_j^G(p)\}$ and splittings $E^F(p) = \oplus E_{\lambda_i^F(p)}^F(p)$, $1 \le i \le r^F(p)$ and $E^G(p) = \oplus E_{\lambda_j^G(p)}^G(p)$ for $1 \le j \le r^G(p)$ defined on full measure A-invariant subsets. By a direct computation, we have that the linear functionals $\{\lambda_j^G(p)\}$ coincide with Σ, the restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. In particular, the number $r^G(p)$, the functions $\{\lambda_j^G(p)\}$, and the subspaces $E_{\lambda_j^G(p)}^G(p)$ are defined at every point $p \in M^\alpha$ and are independent of p.

Below, we write $\mathcal{L}(p)$, $\mathcal{L}^F(p)$ and $\mathcal{L}^G(p) = \Sigma$, respectively, for the corresponding collections of Lyapunov exponent functionals at the point p for the derivative cocycle and its restrictions to E^F and E^G. If μ is A-ergodic, we write $\mathcal{L}(\mu)$, $\mathcal{L}^F(\mu)$ and $\mathcal{L}^G(\mu)$ or simply $\mathcal{L}$, $\mathcal{L}^F$ and $\mathcal{L}^G$ if the measure is understood.

3.2. *Coarse Lyapunov exponents and coarse Lyapunov manifolds.* For this section assume that μ is A-ergodic and that the charts in Section 2.4 satisfy properties (H1)–(H3) of Proposition 2.5. Note that Lyapunov exponents and dimension of the corresponding subspaces are independent of the point almost surely.

As with restricted roots, we group Lyapunov exponent functionals into *coarse equivalence classes* by declaring that two exponents are equivalent if they are positively proportional. We write $\hat{\mathcal{L}}$ for the set of coarse exponents; that is $\hat{\mathcal{L}}$ is the set of equivalence classes of Lyapunov exponents. For $\chi \in \hat{\mathcal{L}}$, we write $E_\chi(p) = \bigoplus_{\lambda \in \chi} E_\lambda(p)$.

Recall that we equipped M^α with a Riemannian metric which, in turn, induces a distance d on M^α. Given $s \in A$ and $p \in M^\alpha$ we write

$$W^u_s(p) := \left\{ y \in M^\alpha : \limsup_{n \to -\infty} \frac{1}{n} \log d\left(\tilde{\alpha}(ns)(p), \tilde{\alpha}(ns)(y)\right) < 0 \right\}$$

for the unstable manifold through p for the action of $s \in A$ on M^α. For μ-almost every $p \in M^\alpha$, we have that $W^u_s(p)$ is a connected, injectively immersed, $C^{1+\beta}$ manifold with $T_p W^u_s(p) = \bigoplus_{\lambda \in \mathcal{L}: \lambda(s) > 0} E_\lambda(p)$. Observe that given $s \in A$, the collection of global unstable manifolds $\{W^u_s(p) : p \in M^\alpha\}$ forms a (generally non-measurable) partition of (M^α, μ).

Let $\mathbb{Z}^d$ be any lattice in $A \simeq \mathbb{R}^d$. Given a coarse Lyapunov exponent $\chi \in \hat{\mathcal{L}}$ we write $W^\chi(p)$ for the path connected (relative to the immersed topologies) component of

$$\bigcap_{\{s \in \mathbb{Z}^d : \chi(s) > 0\}} W^u_s(p)$$

containing p. The set $W^\chi(p)$ is called the *coarse Lyapunov manifold* corresponding to χ through p. For almost every p, $W^\chi(p)$ is a $C^{1+\beta}$ injectively immersed manifold with $T_p W^\chi(p) = E_\chi(p)$ (see [9]). We let $\mathcal{W}^\chi$ denote the partition of (M^α, μ) into coarse Lyapunov manifolds $W^\chi(p)$. In the terminology of [9], $\mathcal{W}^\chi$ is a $C^{1+\beta}$-tame, $\tilde{\alpha}{\restriction}_A$-invariant, measurable foliation. Note that the partition $\mathcal{W}^\chi$ is defined independently of the choice of lattice $\mathbb{Z}^d \subset A$ in that for any two choices of lattice, the corresponding partitions coincide modulo μ.

Similarly, in the terminology of [9], the partition $\mathcal{G}$ of M^α into G-orbits and the partition $\mathcal{F}$ of M^α into fibers of $\pi \colon M^\alpha \to G/\Gamma$ form $C^{1+\beta}$-tame, $\tilde{\alpha}$-invariant, measurable foliations. We similarly define $W^{\chi^F}(p)$ and $W^\xi(p)$ for the coarse Lyapunov manifolds associated to coarse fiberwise Lyapunov exponents $\chi^F \in \hat{\mathcal{L}}^F$ and coarse roots $\xi \in \hat{\Sigma}$. Note that if $\xi \in \hat{\Sigma}$, then $W^\xi(p)$ is simply the orbit $\tilde{\alpha}(G^\xi)(p)$ of p by the unipotent subgroup $G^\xi = \exp \mathfrak{g}^\xi$ of G. We similarly define measurable foliations $\mathcal{W}^{\chi^F}$ and $\mathcal{W}^\xi$ given by the partitions into fiberwise coarse Lyapunov manifolds and orbits of coarse root groups.

3.3. *Conditional entropy, entropy product structure, and coarse-Lyapunov Abramov–Rohlin formula.* We continue to assume that μ is an A-invariant, A-ergodic probability measure on M^α. Recall that in the case that Γ is not cocompact, we assume the projection ν of μ to G/Γ is G-invariant so that the charts in Section 2.4 satisfy properties (H1)–(H3) of Proposition 2.5.

Recall that for $s \in A$, the μ-metric entropy of $\tilde\alpha(s)$ is
$$h_\mu(\tilde\alpha(s)) := \sup\{h_\mu(\tilde\alpha(s), \eta)\}.$$
Above, the supremum is taken over all measurable partitions η of (M^α, μ). For each η, $h_\mu(\tilde\alpha(s), \eta)$ is given by the mean conditional entropy
$$h_\mu(\tilde\alpha(s), \eta) = H_\mu(\eta^+ \mid \tilde\alpha(s)\eta^+),$$
where $\eta^+ = \bigvee_{i=0}^\infty \tilde\alpha(s^i)\eta$.

Given the partition $\mathcal{W}^\chi$ into coarse Lyapunov manifolds for $\chi \in \hat{\mathcal{L}}$, for $s \in A$ with $\chi(s) > 0$, we define the *conditional metric entropy* of $\tilde\alpha(s)$ *relative to* $\mathcal{W}^\chi$ as follows: A measurable partition ξ of (M^α, μ) is said to be *subordinate to* $\mathcal{W}^\chi$ if, for almost every p,

(1) the atom $\xi(p)$ is contained in $\mathcal{W}^\chi(p)$,

(2) the atom $\xi(p)$ contains a neighborhood of p in $\mathcal{W}^\chi(p)$, and

(3) the atom $\xi(p)$ is precompact in $\mathcal{W}^\chi(p)$.

The conditional metric entropy of $\tilde\alpha(s)$ relative to $\mathcal{W}^\chi$ is
$$h_\mu(\tilde\alpha(s) \mid \mathcal{W}^\chi) := \sup\{h_\mu(\tilde\alpha(s), \eta \vee \xi)\},$$
where the supremum is over all partitions ξ subordinate to $\mathcal{W}^\chi$ and all measurable partitions η.

From [10] we have the following result, which states that entropy behaves like a product along coarse Lyapunov manifolds.

PROPOSITION 3.2 ([10, Cor. 13.2]). *For $s \in A$,*
$$h_\mu(\tilde\alpha(s)) = \sum_{\chi(s)>0} h_\mu(\tilde\alpha(s) \mid \mathcal{W}^\chi).$$

Fix a coarse exponent $\chi \in \hat{\mathcal{L}}$. Write $\chi(F) \in \hat{\mathcal{L}}^F$ for the unique fiberwise coarse exponent with $\chi(F)$ with $\chi(F) \subset \chi$ if such a fiberwise coarse exponent exists; that is, $\chi(F)$ is positively propositional to χ. Otherwise let $\chi(F)$ denote the 0 functional. Similarly, define $\chi(G)$ to be the unique coarse restricted root $\hat\xi \in \hat\Sigma$ that is positively proportional to χ and 0 otherwise. Note that given a non-zero coarse Lyapunov exponent $\chi \in \hat{\mathcal{L}}$, at least one of $\chi(F)$ or $\chi(G)$ is non-zero.

Let ν denote the image of μ under $\pi \colon M^\alpha \to G/\Gamma$. From the Abramov-Rohlin formula (cf. [29], [3]), we may decompose entropy of μ into the sum of

the entropy along fibers and the entropy of the factor: for any $s \in A$,

$$(7) \qquad h_\mu(\tilde\alpha(s)) = h_\nu(s) + h_\mu(\tilde\alpha(s) \mid \mathcal{F}).$$

Here $\mathcal{F}$ is the partition into preimages of the projection $\pi \colon M^\alpha \to G/\Gamma$ and $h_\nu(s)$ is the ν-metric entropy of the translation by s on $(G/\Gamma, \nu)$. From [10], we have a similar decomposition into fiber and factor entropy along coarse manifolds.

PROPOSITION 3.3 ([10, Th. 13.7]). *Let $s \in A$ be such that $\chi(s) > 0$. Then*

$$(8) \qquad h_\mu(\tilde\alpha(s) \mid \mathcal{W}^\chi) = h_\nu(s \mid \chi(G)) + h_\mu(\tilde\alpha(s) \mid \mathcal{W}^{\chi(F)}).$$

Above, $h_\nu(s \mid \chi(G))$ denotes the metric entropy of translation by s on $(G/\Gamma, \nu)$ conditioned on the partition of $(G/\Gamma, \nu)$ into orbits of $G^{\chi(G)}$. Note that for our applications below, if $\chi(F) = 0$, then $h_\mu(\tilde\alpha(s) \mid \mathcal{W}^{\chi(F)}) = 0$.

Proposition 3.3 is a special case of [10, Th. 13.7] that establishes an Abramov-Rohlin formula for entropy subordinate to coarse Lyapunov manifolds for two smooth $\mathbb{Z}^d$-actions, one of which is a measurable factor of the other. In the current setting, our factor map $\pi \colon M^\alpha \to G/\Gamma$ is smooth and we obtain Proposition 3.3 directly from Proposition 3.2. We include a proof of Proposition 3.3 in our current setting.

Proof of Proposition 3.3. Note that, as the map $\pi \colon M^\alpha \to G/\Gamma$ is smooth, every coarse restricted root $\hat\xi \in \hat\Sigma$ for the action of A on G/Γ coincides with some coarse Lyapunov exponent $\chi \in \hat{\mathcal{L}}$ for the action of A on (M^α, μ); in particular, every $\hat\xi \in \hat\Sigma$ is of the form $\hat\xi = \chi(G)$ for some $\chi \in \hat{\mathcal{L}}$.

Given $\chi \in \hat{\mathcal{L}}$, set $\overline\chi = \chi(G)$ and take $s \in A$ with $\chi(s) > 0$. If $\overline\chi = 0$, take $\overline\eta$ to be the point partition on G/Γ. Otherwise, take $\overline\eta$ to be a measurable partition of G/Γ such that

(1) $s^{-1} \cdot \overline\eta \geq \overline\eta$;
(2) the atom $\overline\eta(x)$ of $\overline\eta$ containing x is contained in the $G^{\overline\chi}$-orbit of x and contains an open neighborhood of x in the $G^{\overline\chi}$-orbit;
(3) $\bigvee_{n \in \mathbb{N}} s^{-n} \cdot \overline\eta$ is the point partition.

Let $\eta = \pi^{-1}(\overline\eta)$. Take ζ to be a measurable partition of M^α such that

(1) $\tilde\alpha(s^{-1})(\zeta) \geq \zeta$;
(2) the atom $\zeta(x)$ of ζ containing x is contained in $\mathcal{W}^\chi(x)$ and contains an open neighborhood of x in $\mathcal{W}^\chi(x)$ for almost every x;
(3) $\bigvee_{n \in \mathbb{N}} \tilde\alpha(s^{-n})(\zeta)$ is the point partition.

The existence of the partitions $\overline\eta$ and ζ follows from a standard argument. See [28, Prop. 3.1] or [27, Lemma 9.1.1].

The partitions $\overline\eta$ and ζ satisfy

$$h_\nu(s, \overline\eta) = h_\nu(s \mid \overline\chi) \text{ and } h_\mu(\tilde\alpha(s), \zeta \vee \mathcal{F}) = h_\mu(\tilde\alpha(s) \mid \mathcal{W}^{\chi(F)}).$$

We have the following standard computation (cf. [22, Lemma 6.1]):

$$
\begin{aligned}
h_\mu(\tilde\alpha(s) \mid \mathcal{W}^\chi) &:= h_\mu(\tilde\alpha(s), \eta \vee \zeta) \\
&\leq h_\mu(\tilde\alpha(s), \eta) + h_\mu\left(\tilde\alpha(s), \zeta \vee \bigvee_{n\in\mathbb{Z}} \alpha(s^n)(\eta)\right) \\
&= h_\nu(s, \overline\eta) + h_\mu(\tilde\alpha(s), \zeta \vee \mathcal{F}) \\
&= h_\nu(s \mid \overline\chi) + h_\mu(\tilde\alpha(s) \mid \mathcal{W}^{\chi(F)}).
\end{aligned}
$$

Now, fix $\chi_0 \in \hat{\mathcal{L}}$. Given any $s \in A$ with $\chi_0(s) > 0$, we have from (7) and the analogue of Proposition 3.2 applied to the total, fiber, and base entropies (see full formulation in [10, Th. 13.1]) that

$$
\begin{aligned}
h_\mu(\tilde\alpha(s)) &= \sum_{\chi(s)>0} h_\mu(\tilde\alpha(s) \mid \mathcal{W}^\chi) \\
&\leq \sum_{\chi(s)>0} h_\nu(s \mid \chi(G)) + \sum_{\chi(s)>0} h_\mu(\tilde\alpha(s) \mid \mathcal{W}^{\chi(F)}) \\
&= h_\nu(s) + h_\mu(\tilde\alpha(s) \mid \mathcal{F}) \\
&= h_\mu(\tilde\alpha(s)).
\end{aligned}
$$

Since entropies are non-negative quantities, it follows that

$$
h_\mu(\tilde\alpha(s) \mid \mathcal{W}^\chi) = h_\nu(s \mid \chi(G)) + h_\mu(\tilde\alpha(s) \mid \mathcal{W}^{\chi(F)})
$$

for all $\chi \in \hat{\mathcal{L}}$ with $\chi(s) > 0$. $\qquad\square$

We note that Proposition 3.3 will be applied only in the special case where $\mathcal{W}^{\chi(F)}$ is trivial, in Proposition 5.1. Since the proof in this special case would not be much simpler than the one provided above, we give the general version of Proposition 3.3.

4. Conditional measures and criteria for invariance

Let G be as in the introduction. That is, $G = G_1 \times \cdots \times G_\ell$ is the direct product of non-compact simple Lie groups. Consider X any locally compact, second countable metric space, and suppose that X admits a continuous left G-action $x \mapsto g \cdot x$. We moreover assume the action is *locally free*; that is, for every $x \in X$, there is a neighborhood $e \in U_x \subset G$ such that the map $U_x \to X, g \mapsto g \cdot x$ is injective. (The size of U_x does not have to be uniform.) It follows that for every x, we have a canonical identification of G with a covering space of the orbit $G \cdot x$ given by $g \mapsto g \cdot x$.

4.1. *Leafwise measures along orbits of subgroups.* Consider any Borel probability measure μ on X. Let $V \subset G$ be a connected Lie subgroup, and let η be a measurable partition of (X, μ) such that for μ-almost every $x \in X$, the atom $\eta(x)$ is contained in the V-orbit $V \cdot x$ and contains an open neighborhood of x in the V-orbit $V \cdot x$. Such a partition is said to be *subordinate to V-orbits.*

As above, we identify $V \subset G$ with a cover of the V-orbit through x. Fix a decreasing sequence of measurable partitions $\eta_0 \geq \eta_1 \geq \eta_2 \cdots$ subordinate to V-orbits such that for any compact set $K \subset V$, for almost every x, there is a j with $K \cdot x \subset \eta_j(x)$. For each j, let $\{\mu_x^{\eta_j}\}$ be an associated family of conditional probability measures on X. By fixing a choice of normalization, a standard construction gives for almost every $x \in X$ a locally finite Radon measure μ_x^V, supported on V, which is canonical up to the choice of normalization. Indeed, it is enough to define μ_x^V on compact sets $K \subset V$ on which the function $v \mapsto v \cdot x$ is injective. For any such K, let j be sufficiently large so that $K \cdot x \subset \eta_j(x)$, and set

$$\mu_x^V(K) = \frac{\mu_x^{\eta_j}(K \cdot x)}{\mu_x^{\eta_j}(\eta_0(x))}.$$

Because the sequence η_j is decreasing, we have $\mu_x^{\eta_j}(\eta_0(x)) > 0$ for μ-almost every x and every $j \geq 0$, and we also obtain that $\mu_x^V(K)$ is independent of the choice of j as long as j is sufficiently large. However, μ_x^V is unique only up to renormalization determined by the choice of η_0. To emphasize the lack of uniqueness, we write $[\mu_x^V]$ for the equivalence class of the measure μ_x^V up to normalization of the measure. See [12, §6] for further details.

Using the family of measures $[\mu_x^V]$ on V, we recover the conditional measures for any partition η subordinate to V-orbits. For simplicity, suppose that almost every V-orbit is free. We then have the following: for any partition η subordinate to V-orbits, there is a function $c^\eta \colon X \to (0, \infty)$ such that if $\{\mu_x^\eta : x \in X\}$ is a family of conditional measures on (X, μ) associated with the measurable partition η, then

$$\mu_x^\eta = c^\eta(x)(v \mapsto v \cdot x)_* (\mu_x^V)\restriction_{\eta(x)}.$$

Note that the subgroups V above need not be unimodular. We have the following claim, which follows from local disintegration and the definition of the left Haar measure.

CLAIM 4.1. *Let $V \subset G$ be a connected Lie subgroup. Then the measure $[\mu_x^V]$ coincides with the (positive proportionally class of the) left Haar measure on V for μ-almost every $x \in M$ if and only if the measure μ is invariant under the action of V.*

The remainder of this section is devoted to a number of criteria that will guarantee that $[\mu_x^V]$ is the left Haar measure.

4.2. *Invariance from the structure of parabolic subgroups.* Recall that we write $P = MAN$ for the minimal parabolic subgroup of G. Suppose μ is a P-invariant, Borel probability measure on X. Given a coarse negative root $\xi \in \hat{\Sigma}_-$ and a non-trivial subgroup $V \subset G^\xi$ such that μ is V-invariant then, as the stabilizer of a measure is a closed subgroup of G, it follows from the

structure theory of parabolic subgroups that μ is invariant by the full coarse root subgroup G^ξ. In the case that the subgroup V above varies with the point $x \in X$, we have the following lemma. Note that G^ξ is nilpotent so subgroups of G^ξ are unimodular.

LEMMA 4.2. *Let μ be a P-invariant measure on X, and suppose for some $\xi \in \hat{\Sigma}_-$ that for μ-almost every $x \in X$, there is a non-trivial, connected Lie subgroup $V(x) \subset G^\xi$ such that $[\mu_x^{G^\xi}]$ coincides with the Haar measure on $V(x)$. Moreover, assume the assignment $x \mapsto V(x)$ is measurable and A-invariant. Then the measure μ is G^ξ-invariant.*

Proof. Let $\{\mu_x^e\}$ denote the A-ergodic decomposition of μ. It is enough to verify that the measure μ_x^e is G^ξ-invariant for almost every x.

Take $s \in A$ such that $\| \operatorname{Ad}(s)|\mathfrak{n}\| < 1$. It follows by Lemma 4.3 below that the partition into A-ergodic components is refined by the partition into N-orbits. In particular, for μ-almost every x, the measure μ_x^e is N-invariant by Claim 4.1.

Fix a generic $x \in X$. Let V be the μ_x^e-almost surely constant value of $x \mapsto V(x)$. Let $H(x)$ be the closed subgroup of G under which μ_x^e is invariant, and let $\mathfrak{h}_x = \operatorname{Lie}(H(x))$.

As $-\xi$ is a positive coarse restricted root, we have $\mathfrak{g}^{-\xi} \subset \mathfrak{h}_x$. Moreover, given a non-zero $Y \in \operatorname{Lie}(V)$, from the analysis of $\mathfrak{sl}(2, \mathbb{R})$ triples in $\mathfrak{g}$ (see [23, Lemma 7.73]), we have that $(\operatorname{ad}(Y))^2$ maps $\mathfrak{g}^{-\xi}$ onto $\mathfrak{g}^\xi$. In particular $\mathfrak{g}^\xi \subset \mathfrak{h}_x$ whence μ_x^e is G^ξ-invariant. $\qquad\square$

LEMMA 4.3. *Let U be a closed connected subgroup of G normalized by an element $s \in A$ such that $\|\operatorname{Ad}(s)\!\restriction_\mathfrak{u}\| < 1$, where $\mathfrak{u}$ is the Lie algebra of U. If $\mathcal{E}_s$ is the partition into ergodic components of μ with respect to s, then $\mathcal{E}_s$ is refined by the partition into U-orbits mod μ.*

Proof. We say a point x is generic with respect to μ if for every compactly supported continuous function ϕ, $\frac{1}{N} \sum_{n=0}^{N-1} \phi(s^k \cdot x) \to \int \phi \, d\mu_x^{\mathcal{E}_s}$, where $\mu_x^{\mathcal{E}_s}$ is the ergodic component of μ with respect to s at x. Note that if $u \in U$, then $s^k u s^{-k} \cdot x \to x$ as $k \to +\infty$ uniformly on compact sets in X and hence for every compactly supported continuous function ϕ,

$$|\phi(s^k \cdot x) - \phi(s^k \cdot u \cdot x)| \to 0$$

uniformly in x as $k \to +\infty$. Finally if x and y are generic points with $y = u \cdot x$ and if ϕ is a compactly supported continuous function, we have

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{k=0}^{N-1} \phi(s^k \cdot x) = \lim_{N \to +\infty} \frac{1}{N} \sum_{k=0}^{N-1} \phi(s^k \cdot y).$$

It follows that $\mu_x^{\mathcal{E}_s} = \mu_y^{\mathcal{E}_s}$. $\qquad\square$

4.3. *High-entropy method.* We have the following theorem of Einsiedler and Katok from which we deduce additional invariance of an A-invariant probability measure along unipotent subgroups by considering its support along coarse root spaces. We say a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *contracting* if it is invariant under the adjoint action of A and if there is some $s \in A$ with

$$\mathfrak{h} = \bigoplus_{\xi \in \hat{\Sigma}:\xi(s)<0} (\mathfrak{g}^\xi \cap \mathfrak{h}).$$

Note that any such $\mathfrak{h}$ is nilpotent, hence unimodular. We state a simplified version of the *High Entropy Theorem* from [14].

THEOREM 4.4 (High Entropy Theorem, [14, Th. 8.5]). *Let μ be an A-invariant probability measure on X, and let $\mathfrak{h} \subset \mathfrak{g}$ be a contracting Lie algebra with corresponding analytic subgroup H. Then for μ-almost every x, there are Lie subgroups*

$$H_x \subset S_x \subset H$$

with

(1) μ_x^H *is supported on S_x;*

(2) μ_x^H *is invariant under left and right multiplication by H_x;*

(3) H_x *and S_x are connected and their Lie algebras are direct sums of subspaces of root spaces;*

(4) H_x *is normal in S_x, and if $\xi, \xi' \in \hat{\Sigma}$ with $\xi \neq \xi'$ are distinct coarse roots, then for $g \in S_x \cap G^\xi$ and $h \in S_x \cap G^{\xi'}$, the cosets gH_x and hH_x commute in S_x/H_x;*

(5) $\mu_x^{G^\xi}$ *is left- and right- invariant under multiplication by elements of $H_x \cap G^\xi$.*

It follows that the groups S_x and H_x are equivariant under conjugation by A; that is $S_{s \cdot x} = sS_xs^{-1}$. Unlike in [14], here we only consider the adjoint action of A on $\mathfrak{g}$. As this action is semisimple with real roots, it follows that the groups S_x and H_x are normalized by A. In particular, the maps $x \mapsto S_x$ and $s \mapsto H_x$ are constant along A-orbits.

4.4. *Invariance from entropy considerations.* Let μ be an A-invariant, A-ergodic measure. Given a coarse root $\xi \in \hat{\Sigma}$, let $\mathcal{W}^\xi$ be the partition of X into orbits of G^ξ. We have a standard fact (see, for example, [28]) that if μ is G^ξ-invariant, then for $s \in A$ with $\xi(s) > 0$, the entropy of the action of s on (X, μ) conditioned along orbits of G^ξ is given by

$$h_\mu(s \mid \mathcal{W}^\xi) = \sum_{\beta \in \xi} \beta(s) \dim(\mathfrak{g}^\beta).$$

The converse also holds.

LEMMA 4.5. *Let $\xi \in \hat{\Sigma}$ be such that*

$$h_\mu(s \mid \mathcal{W}^\xi) = \sum_{\beta \in \xi} \beta(s) \dim(\mathfrak{g}^\beta)$$

for some $s \in A$ with $\xi(s) > 0$. Then μ is G^ξ-invariant.

Indeed, Ledrappier shows in [24, Th. 3.4] that μ has absolutely continuous conditional measures along G^ξ-orbits. In our setting, this is [5, Th. 7.2]. Moreover, from the explicit computation of the density function in the proof of [24, Th. 3.4] it follows that the conditional measures of μ along G^ξ-orbits coincide with the image of the Haar measure on G^ξ. See also [26, (6.1)] for the argument in English. In our setting, see Claim 12.2 and the computation preceding it in [5]. From Claim 4.1 it follows that μ is G^ξ-invariant.

We remark that deriving extra invariance of a measure by verifying that conditional entropy is maximized also underlies the proof of the so-called "invariance principle" for fiber-wise conditional measures invariant under a skew product, developed by Ledrappier for projective and linear cocycles in [25] and extended to non-linear cocycles in [1].

5. **Main propositions and proofs of Theorems 1.6, 1.7, and 1.10**

5.1. *Non-resonance implies invariance.* We return to the setting introduced in Section 2. Fix any lattice $\mathbb{Z}^d \subset A$, and consider an A-invariant, A-ergodic measure μ on M^α satisfying (H3) of Proposition 2.5. We say a restricted root $\beta \in \Sigma$ of $\mathfrak{g}$ is *resonant* (with the fiber exponents $\mathcal{L}^F(\mu)$ of μ) if there exist a $c > 0$ and a fiberwise Lyapunov exponent $\lambda \in \mathcal{L}^F(\mu)$ with

$$\beta = c\lambda.$$

If no such c and λ exist, we say β is *non-resonant*. We similarly say that a fiberwise Lyapunov exponent $\lambda \in \mathcal{L}^F(\mu)$ is *resonant* (with $\mathfrak{g}$) if there are a $c > 0$ and a $\beta \in \Sigma$ with

$$\lambda = c\beta.$$

Note that resonance and non-resonance are well defined on the set of coarse restricted roots $\hat{\Sigma}$ and coarse fiberwise exponents $\hat{\mathcal{L}}^F(\mu)$.

The proof of Theorem 1.6 follows directly from the following key proposition.

PROPOSITION 5.1. *Let μ be an A-invariant, A-ergodic Borel probability measure on M^α such that the image of μ in G/Γ is G-invariant. Then, given a coarse restricted root $\xi \in \hat{\Sigma}$ that is non-resonant with the fiberwise Lyapunov exponents of μ, the measure μ is G^ξ-invariant for the action $\tilde{\alpha}$.*

Proof. Indeed if ξ is a non-resonant coarse restricted root, then $\xi = \chi(G)$ for some coarse exponent $\chi \in \hat{\mathcal{L}}$ with $\mathcal{W}^{\chi(F)}$ being trivial. Since the image of

μ in G/Γ is the Haar measure, it follows for $s \in A$ with $\xi(s) > 0$ that

$$h_{\text{Haar}}(s \mid \xi) = \sum_{\beta \in \xi} \beta(s) \dim(\mathfrak{g}^\beta).$$

From Proposition 3.3 and the fact that the partitions $\mathcal{W}^\chi = \mathcal{W}^\xi$ coincide in M^α, it follows that $h_\mu(\tilde{\alpha}(s) \mid \mathcal{W}^{\chi(F)}) = 0$ whence

$$h_\mu(\tilde{\alpha}(s) \mid \mathcal{W}^\xi) = \sum_{\beta \in \xi} \beta(s) \dim(\mathfrak{g}^\beta).$$

The G^ξ-invariance of μ then follows from Lemma 4.5. $\square$

We remark that the proof of Proposition 5.1 is similar to key steps in [33] and [15] where one deduces extra invariance of a measure by computing conditional entropy, verifying that the entropy is the maximal value permitted by the Margulis–Ruelle inequality, and applying Lemma 4.5 to obtain invariance.

5.2. *P-invariant measures and the proof of Theorem 1.6.* Recall that P is the minimal standard parabolic subgroup and is hence amenable. It follows (even in the case that Γ is non-uniform) that there exists an ergodic, P-invariant probability measure μ for the restriction of the action $\tilde{\alpha}$ on M^α to P. Indeed, take any measure μ_0 on M^α that projects to the Haar measure on G/Γ, and let $\{F_k\}$ be a Følner sequence in the amenable group P. Then the average of μ_0 over each F_k, which we denote by μ_k, projects to the Haar measure. In consequence, the family $\{\mu_k\}$ is precompact and any weak-$*$ limit μ_∞ of μ_k is a P-invariant probability measure. (In other words, there is no escape of mass.) One my then take an ergodic component μ of μ_∞. We remark also that any P-invariant probability measure on M^α projects to the Haar measure on G/Γ, since the latter is the only P-invariant probability measure on G/Γ.

Fix a P-invariant, P-ergodic measure μ on M^α. Recall that $A \subset P$ and recall the data $r(\cdot)$, $\lambda_i(\cdot)$, $E_{\lambda_i}(\cdot)$ defined in Theorem 3.1 for the action of A on (M, μ) as well as the corresponding data $r^F(\cdot)$, $\lambda_i^F(\cdot)$, and $E_{\lambda_i^F}(\cdot)$ and $r^G(\cdot)$, $\lambda_i^G(\cdot)$, and $E_{\lambda_i^G}(\cdot)$ for the fiberwise and orbit cocycles. As observed earlier, the data $r^G(\cdot)$, $\lambda_i^G(\cdot)$, and $E_{\lambda_i^G}(\cdot)$ are independent of the measure μ and the point. We show that for μ as above, the remaining data is independent of the point.

CLAIM 5.2. *Suppose that μ is a P-invariant, P-ergodic measure. Then the functions $r(\cdot), r^F(\cdot),\ \lambda_i(\cdot),\ \lambda_i^F(\cdot)$ and the dimensions of the corresponding subspaces $E_{\lambda_i}(\cdot),\ E_{\lambda_i^F}(\cdot)$ are constant almost surely.*

Proof. Note that μ is P-ergodic but need not be A-ergodic. Let $\{\mu_p^e\}_{p \in M^\alpha}$ denote the A-ergodic decomposition of μ. We may select $s \in A$ so that $\beta(s) < 0$ for every $\beta \in \Sigma_+$. By the pointwise ergodic theorem (see Lemma 4.3), it follows that ergodic components are refined by the measurable hull of the partition into N-orbits. Then μ_p^e is N-invariant for almost every $p \in M^\alpha$. It follows that the data in the claim is constant along AN-orbits.

Finally, recall that $P = MAN$ with M contained in the centralizer of A. It follows that the data is constant along M orbits. By the P-ergodicity of μ, the result follows. $\qquad\square$

From Claim 5.2 it follows that for any P-invariant, P-ergodic measure μ on M^α, the set of resonant roots depends only on the measure μ and not the decomposition of μ into A-ergodic components.

Theorem 1.6 now follows immediately from Proposition 5.1.

Proof of Theorem 1.6. Let μ be any P-invariant, P-ergodic measure on M^α. Let Q be the stabilizer of G in μ. Note that Q is a closed subgroup with $P \subset Q \subset G$. If $\dim M < r(G)$, then there at most $r(G) - 1$ fiberwise Lyapunov exponent functionals in $\mathcal{L}^F$, hence at most $r(G) - 1$ coarse fiberwise Lyapunov exponent functionals in $\hat{\mathcal{L}}^F$. It follows that there are at most $r(G) - 1$ resonant coarse restricted roots $\xi \in \hat{\Sigma}$. From Proposition 5.1, it follows that Q is a standard parabolic subgroup with resonant codimension strictly smaller than $r(G)$. But then $Q = G$ by definition of $r(G)$.

It follows that μ is a G-invariant, Borel probability measure on M^α. From Claim 2.1, it follows that there exists a Γ-invariant Borel probability measure on M. $\qquad\square$

5.3. *Parabolic subgroups associated to conditional measures.* We continue to assume μ is a P-invariant, P-ergodic measure on M^α. The proof of Theorems 1.7 and 1.10 follow from an analysis of the geometry of the measures $[\mu_p^G]$ constructed in the previous section.

We define subgroups $Q_{\mathrm{In}}(\mu) \subset Q_{\mathrm{Out}}(\mu)$ of G as follows: Given $p \in M^\alpha$, let

(1) $Q_{\mathrm{In}}(\mu)$ denote the largest subgroup of G for which μ is invariant for the action $\tilde{\alpha}$;
(2) $Q_{\mathrm{Out}}(\mu; p)$ denote the smallest, closed, $[\mu_p^G]$-co-null subgroup of G.

Note that both $Q_{\mathrm{In}}(\mu)$ and $Q_{\mathrm{Out}}(\mu; p)$ are standard parabolic subgroups. As $P \subset Q_{\mathrm{Out}}(\mu; p)$, it follows that $Q_{\mathrm{Out}}(\mu; p)$ is constant along P-orbits. By P-ergodicity of μ, we write $Q_{\mathrm{Out}}(\mu)$ for the almost-surely constant value of $Q_{\mathrm{Out}}(\mu; p)$.

Theorems 1.7 and 1.10 will follow from verifying that $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$. We use the criteria in the previous section to verify this condition. First, consider the case that every fiberwise Lyapunov exponent λ_i^F of μ is resonant with a negative root. In this setting we immediately obtain that $Q_{\mathrm{In}}(\mu)$ and $Q_{\mathrm{Out}}(\mu)$ coincide.

PROPOSITION 5.3. *Suppose that for every* $\lambda_i^F \in \mathcal{L}^F$, *there are a* $\beta \in \Sigma_-$ *and* $c > 0$ *with* $\lambda_i^F = c\beta$. *Then* $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$.

We also verify that $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$ given the combinatorics of the number $m(G)$.

PROPOSITION 5.4. *Suppose* $\mathfrak{g}$ *has no rank-1 simple ideals and that* $Q_{\mathrm{In}}(\mu)$ *is a maximal parabolic subgroup. Then* $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$.

5.4. *Proofs of Theorems 1.7 and 1.10.* Given a P-invariant, P-ergodic measure μ as above, let $\tilde{\mu}$ denote the locally finite measure on $G \times M$ obtained from lifting μ on fundamental domains of Γ. Given $g \in G$, let μ_g denote the conditional probability measure on M defined by disintegrating $\tilde{\mu}$ along fibers and identifying each fiber $\{g\} \times M$ with M.

As $\tilde{\mu}$ lifts μ, we have that $\{\mu_g : g \in G\}$ is Γ-equivariant:

$$\mu_{g\gamma} = \alpha(\gamma)_*\mu_g.$$

Moreover, as μ is $Q_{\mathrm{In}}(\mu)$-invariant, for almost every $g \in G$, we have that $\mu_g = \mu_{qg}$ for every $q \in Q_{\mathrm{In}}(\mu)$. Let $Q = Q_{\mathrm{In}}(\mu)$. We equip $Q\backslash G$ with any measure m in the Lebesgue class. Let $\overline{\mu}$ be the measure on $Q\backslash G \times M$ given by

$$\overline{\mu}(B) = \int \mu_g(\{x : (Qg, x) \in B\}) \, dm(Qg),$$

and let $\hat{\mu}$ be the measure on M given by

$$\hat{\mu}(B) = \int \mu_g(B) \, dm(Qg).$$

Note that $\hat{\mu}$ is image of $\overline{\mu}$ under the natural projection $\pi \colon Q\backslash G \times M \to M$.

Consider the $\overline{\mu}$-measurable partition ζ^π on $Q\backslash G \times M$ into level sets of the map π. We have that ζ^π is measurably equivalent to the partition

$$\{Q_{\mathrm{In}}(\mu)\backslash Q_{\mathrm{Out}}(\mu) \times \{x\} : x \in M\}.$$

In particular, in the case $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$, the following claim follows immediately.

CLAIM 5.5. *If* $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$, *then the projection* $(Q\backslash G \times M, \overline{\mu}) \to (M, \hat{\mu})$ *is a measurable isomorphism.*

Theorems 1.7 and 1.10 follow from Γ-equivariance of the family $\{\mu_g\}$ and Claim 5.5.

Proof of Theorems 1.7 and 1.10. Let μ be a P-invariant, P-ergodic measure on M^α.

First consider the setting of Theorem 1.7 where $\dim(M) = r(G)$. If there exists a non-resonant, fiberwise Lyapunov exponent λ_i^F for μ then, by dimension counting, there are at most $r(G)-1$ coarse resonant roots $\xi \in \hat{\Sigma}$. However, as μ is P-invariant and as there are no proper parabolic subalgebras of resonant codimension smaller than $r(G)$, it follows that μ is necessarily G-invariant. It then follows that if $\tilde{\alpha}$ has no invariant probability measure on M^α, then every

fiberwise Lyapunov exponent of μ is resonant with a root of $\mathfrak{g}$. We claim in this case that every fiberwise exponent for μ is in fact resonant with a negative root $\beta \in \Sigma_-$. Indeed, if there existed a fiberwise exponent that was resonant with a positive root, then there would be at most $r(G) - 1$ resonant negative roots. As we assume μ is P-invariant, we again generate a parabolic subgroup that preserves μ and with resonant codimension smaller than $r(G)$. This again implies the existence of an $\tilde{\alpha}$ invariant probability on M^α.

Thus, in the case that $\dim(M) = r(G)$ it follows that if there is no α-invariant probability measure on M, then there exists $s \in A$ such that $\lambda_i^F(s) < 0$ for every fiberwise Lyapunov exponent λ_i^F of μ.

Proposition 5.3 then holds, and a standard argument shows in this case that the fiberwise conditional measures μ_g are supported on a finite set for almost every g. It then follows by ergodicity that the number of atoms is constant almost surely. This argument could have been due to R. Mañé, but we were unable to find an explicit reference in the literature; we include it as the following lemma. In our case we view M^α as $(G/\Gamma) \times M$ by choosing a measurable trivialization of the bundle.

LEMMA 5.6. *Let* $f : (\Omega, \nu) \to (\Omega, \nu)$ *and* $F : (\Omega \times M, \mu) \to (\Omega \times M, \mu)$ *be invertible, ergodic, measure-preserving transformations of standard probability spaces with* F *ergodic. Let* $p \colon \Omega \times M \to \Omega$ *be projection into first coordinate, and assume that* $p_* \mu = \nu$. *Let* $\omega \to \mu_\omega$ *be the disintegration into conditional measure along the partition into preimages of* p. *Assume* M *is a Riemannian manifold* (*not necessarily compact*) *and that*

$$F(\omega, x) = (f(\omega), g_\omega(x)),$$

where g_ω *is a* C^1 *diffeomorphism for every* ω *and* $\log \|g_\omega\|_{C^1}$ *is* ν-*integrable. If the Lyapunov exponents of* F *along the* M *direction are all negative, then there is a* $k \in \mathbb{N}$ *such that for* ν-*almost every* $\omega \in \Omega$, μ_ω *is an atomic measure with* k *atoms of mass* $\frac{1}{k}$.

Proof of Lemma 5.6. Let $g_\omega^{(n)} \colon M \to M$ be the diffeomorphism such that $F^n(\omega, x) = \left(f^n(\omega), g_\omega^{(n)}(x) \right)$. The F-invariance of μ implies for ν-almost every ω that $(g_\omega^{(n)})_* \mu_\omega = \mu_{f^n(\omega)}$ for all n.

We shall use the following consequence of ergodicity and the negativity of the Lyapunov exponents: there exist $\lambda > 0$ and a measurable function $r \colon \Omega \times M \to (0, \infty)$, defined μ-almost everywhere, such that for μ-almost every (ω, x),

$$\operatorname{diam} g_\omega^{(n)} \left(B_{r(\omega,x)}(x) \right) < e^{-\lambda n}.$$

Given $\delta > 0$, set $G_\delta := \{(\omega, x) : \mu_\omega(B_{r(\omega,x)}(x)) \geq \delta\}$. Fix some $\delta > 0$ with $\mu(G_\delta) > 0$. For μ-almost every (ω, x), there are $n_j \to \infty$ with $F^{-n_j}(\omega, x) \in G_\delta$.

Then for all such (ω, x) and n_j,

$$\mu_\omega(B_{e^{-\lambda n_j}}(x)) \geq \mu_\omega \left(g^{(n_j)}_{f^{-n_j}(\omega)} \left(B_{r(F^{-n_j}(\omega,x))}(g_\omega^{-n_j}(x)) \right) \right)$$

$$= \mu_{f^{-n_j}(\omega)} \left(B_{r(F^{-n_j}(\omega,x))}(g_\omega^{-n_j}(x)) \right) \geq \delta.$$

Taking $n_j \to \infty$, it follows for μ-almost every (ω, x) that $\mu_\omega(\{x\}) \geq \delta$. Ergodicity of μ gives the remainder of the claims. □

We now continue the proof of Theorems 1.7 and 1.10. In the case that $\dim(M) \leq m(G)$ and every fiberwise Lyapunov exponent is resonant with a negative root, the same analysis as above holds. In particular, the hypotheses of Proposition 5.3 hold. Note that this holds even if $\mathfrak{g}$ has rank-1 simple ideals (so $m(G) = 1$ and M is a circle.) If $\dim(M) \leq m(G)$ and not every fiberwise Lyapunov exponent is resonant with a negative root, then there are at most $m(G) - 1$ resonant, negative coarse restricted roots. Note that if $\mathfrak{g}$ has rank-1 simple ideals then, as $m(G) - 1 = 0$, this implies $Q_{\text{In}}(\mu) = Q_{\text{Out}}(\mu) = G$. Thus we may assume $\mathfrak{g}$ has no rank-1 simple ideals. From the definition of $m(G)$, it follows that either $Q_{\text{In}}(\mu) = G$ or that $Q_{\text{In}}(\mu)$ is a maximal parabolic subgroup and, from Proposition 5.4, we have that $Q_{\text{In}}(\mu) = Q_{\text{Out}}(\mu)$.

In particular, under the hypotheses of either Theorem 1.7 or 1.10, we have $Q := Q_{\text{In}}(\mu) = Q_{\text{Out}}(\mu)$.

In the setting of either theorem, the spaces (M, μ_g) are Lebesgue probability spaces. As there are at most countably many isomorphism types of Lebesgue probability spaces, by P-ergodicity it follows that the spaces (M, μ_g) are all measurably isomorphic to a fixed abstract Lebesgue probability space (Y, η). In particular, we may select a measurable family of measurable isomorphisms $\phi_g \colon (M, \mu_g) \to (Y, \eta)$. Since we have $\mu_g = \mu_{qg}$ for all $q \in Q$, we may moreover assume $\phi_g = \phi_{qg}$ for almost every g and every $q \in Q$. The family of isomorphisms ϕ_g translate the Γ-equivariance of the family μ_g to a family of automorphisms of the measure space (Y, η) parametrized by $Q \backslash G$:

$$\psi(\gamma, Qg) := \phi_{g\gamma} \circ \alpha(\gamma)_* \circ \phi_g^{-1} \in \text{Aut}(Y, \eta).$$

One verifies that ψ is a cocycle over the right Γ-action on $Q \backslash G$.

It now follows from Claim 5.5 that $(M, \hat{\mu})$ is measurably isomorphic to $(Q \backslash G, \nu) \times (Y, \eta)$. Moreover, the action α of Γ on $(M, \hat{\mu})$ is measurably conjugate via this isomorphism to the skew action defined by ψ over the standard right action of Γ on $Q \backslash G$. Theorems 1.7 and 1.10 now follow. □

6. **Proof of Propositions 5.3 and 5.4**

We recall the notation of Section 5.3. In particular, we take μ to be a P-invariant, P-ergodic measure on M^α. Recall also the definitions of $Q_{\text{In}}(\mu)$ and $Q_{\text{Out}}(\mu)$ in Section 5.3. We verify under the hypotheses of Propositions 5.3 and 5.4 that $Q_{\text{In}}(\mu) = Q_{\text{Out}}(\mu)$.

6.1. *Conditional measures along coarse root spaces.* Under the assumption that $Q_{\mathrm{In}}(\mu) \neq Q_{\mathrm{Out}}(\mu)$, the following claim guarantees the existence of a coarse restricted root $\xi \in \hat{\Sigma}$ with $G^\xi \not\subset Q_{\mathrm{In}}(\mu)$ and such that the measure $[\mu_p^{G^\xi}]$ is non-trivial. Write $Q = Q_{\mathrm{In}}(\mu)$ and $\mathfrak{q} = \mathrm{Lie}(Q)$ for the remainder.

CLAIM 6.1. *Suppose $Q_{\mathrm{In}}(\mu) \neq Q_{\mathrm{Out}}(\mu)$. Then there is a coarse restricted root $\xi \in \hat{\Sigma}$ with $\mathfrak{g}^\xi \not\subset \mathfrak{q}$ such that $\mu_p^{G^\xi}$ is non-atomic for μ-almost every $p \in M^\alpha$.*

The claim follows from the local product structure of A-invariant measures on G-spaces demonstrated in [13, Prop. 8.3] and further developed in [14, Ths. 7.5, 8.4]. We sketch a short proof here for completeness.

Given standard parabolic subgroup Q with Lie algebra $\mathfrak{q}$, let

$$\Sigma_Q = \{\beta \in \Sigma : \mathfrak{g}^\beta \subset \mathfrak{q}\}, \quad \Sigma_Q^\perp = \{\beta \in \Sigma : \mathfrak{g}^\beta \not\subset \mathfrak{q}\}.$$

We have that $\Sigma = \Sigma_Q \cup \Sigma_Q^\perp$ and Σ_Q and $\Sigma_Q^\perp$ are saturated by coarse equivalence classes of restricted roots.

Proof. Recall that we write $Q = Q_{\mathrm{In}}(\mu)$ and the measure μ_p^G is a Q- and hence A-invariant measure on G. Let $\mathfrak{g}^\perp := \bigoplus_{\beta \in \Sigma_Q^\perp} \mathfrak{g}^\beta$. Note that $\Sigma_Q^\perp$ consists of negative roots. Let V be the analytic subgroup corresponding to $\mathfrak{g}^\perp$. Let C_s denote conjugation by $s \in A$. We have $C_s(V) = V$ for $s \in A$. As μ is A-invariant, we have for $s \in A$ that $[(C_s)_* \mu_p^V] = [\mu_{\tilde{\alpha}(s)(p)}^V]$ for almost every p.

As $\Sigma_Q^\perp \subset \Sigma_-$, we may find an $s_0 \in A$ and a coarse restricted root $\xi \subset \Sigma^\perp$ with
- $\beta(s_0) = 0$ for $\beta \in \xi$;
- $\beta(s_0) < 0$ for all $\beta \in \Sigma_Q^\perp \smallsetminus \xi$.

Let V' be the analytic subgroups of V corresponding to $\bigoplus_{\beta \in \Sigma_Q^\perp \smallsetminus \xi} \mathfrak{g}^\beta$.

Suppose first that μ_p^V is not supported V' for a positive measure of set of $p \in M^\alpha$. As $\tilde{\alpha}(s_0)$ commutes with the action of G^ξ, we have $[\mu_{\tilde{\alpha}(s_0)(p)}^{G^\xi}] = [\mu_p^{G^\xi}]$ for almost every p. Moreover, as $\tilde{\alpha}(s_0)$ contracts V'-orbits, it follows from Poincaré recurrence to compact sets of sufficiently large measure where the measures $[\mu_q^{G^\xi}]$ are well defined, vary continuously in q, and contain the identity in their supports, that for μ_p^V-almost every $g \in V$, writing $g = vu$ for $u \in G^\xi$ and $v \in V'$, u is in the support of $[\mu_p^{G^\xi}]$. Thus, $\mu_p^{G^\xi}$ is atomic on a positive measure set of p only if μ_p^V is supported on V' for a positive measure of set of $p \in M^\alpha$. Thus, $\mu_p^{G^\xi}$ is non-atomic on a positive measure set of p. Note that the actions by A and M preserve the coarse root subgroups G^ξ and also preserve the measure μ. Also, as the A-ergodic components of μ are N-invariant, it follows from P-ergodicity of μ that $\mu_p^{G^\xi}$ is non-atomic for almost every p.

If μ_p^V is supported on a single V'-orbit for almost every $p \in M^\alpha$, we may recursively repeat the above argument with V replaced V'.   □

6.2. *Recurrence and the proof of Proposition 5.3.* We show under the assumption that every fiberwise Lyapunov exponent is resonant with a negative

root, that $Q_{\text{In}}(\mu) = Q_{\text{Out}}(\mu)$. Suppose that $Q_{\text{In}}(\mu) \neq Q_{\text{Out}}(\mu)$, and let ξ be a coarse restricted root as in Claim 6.1. We will show below that μ is G^ξ-invariant. The contradiction completes the proof of Proposition 5.3.

Let $A' \subset A$ denote the kernel of ξ; that is, $s \in A'$ if $\beta(s) = 0$ for all $\beta \in \xi$. As we assume Γ has dense image in every rank-1 simple subgroup of G, it follows from Moore's ergodicity theorem (applied to each irreducible factor) that A' acts ergodically on G/Γ (see, for example, [38, Th. 2.2.6]).

As discussed in the proof of Theorem 1.7, the assumption that every fiber-wise Lyapunov exponent is resonant with a negative root implies that μ_g has finite support for almost every $g\Gamma \in G/\Gamma$. Moreover, P-ergodicity of μ ensures that the number of atoms is constant in $g\Gamma$.

The discussion below will always be in the sense of modulo μ-null sets. Throughout the argument we will use only invariance of μ under P and, in particular, its subgroup A and therefore the null sets are preserved under these actions.

Note that (as we assume $\mu_p^{G^\xi}$ is non-atomic) the partition of (M^α, μ) into full G^ξ-orbits is non-measurable. Let η^ξ denote the measurable hull of this partition—that is, the finest measurable partition of (M^α, μ) containing full G^ξ-orbits modulo μ. We remark that this construction does not require μ to be G^ξ-invariant. However, as the partition into G^ξ-orbits and μ are both invariant under A', the partition η^ξ is A'-invariant modulo μ-null sets.

Consider the action of A' on (M^α, μ). Note that the action need not be ergodic. Let $\mathcal{E}_{A'}$ denote the partition into ergodic components of μ with respect to the action of A'. We have the following claim, which will provide the necessary recurrence to complete the proof of Proposition 5.3.

CLAIM 6.2. *The partition η^ξ refines $\mathcal{E}_{A'}$.*

Proof. Let $\mathcal{E}_A$ denote the partition into ergodic components of μ with respect to the action of A on M^α. Taking $s \in A$ such that $\xi(s) < 0$, by Lemma 4.3 we get that η^ξ refines $\mathcal{E}_s$. Because $\mathcal{E}_s$ refines $\mathcal{E}_A$, it follows that η^ξ refines $\mathcal{E}_A$.

To complete the proof of the claim we show $\mathcal{E}_A = \mathcal{E}_{A'}$.

Fix $p \in M^\alpha$, and let $\mu_p^{\mathcal{E}_A}$ be the A-ergodic component of μ containing p. Let $\tilde{\mathcal{E}}(p)$ denote the partition of $(M^\alpha, \mu_p^{\mathcal{E}_A})$ into A'-ergodic components. We claim that $\tilde{\mathcal{E}}(p)$ is finite for almost every p. Indeed first note that, as both A and A' act ergodically on G/Γ, the A and A'-ergodic components of (M^α, μ) project to the Haar measure on G/Γ. Furthermore, as the fiber conditional measures $\mu_{g\Gamma}$ are purely atomic and as the ergodic components of the A'-action on $(M^\alpha, \mu_p^{\mathcal{E}_A})$ are mutually singular, it follows that the partition $\tilde{\mathcal{E}}(p)$ is finite.

As $A' \subset A$ with A abelian, it follows that A permutes elements of the partition $\tilde{\mathcal{E}}(p)$ of $(M^\alpha, \mu_p^{\mathcal{E}_A})$. Note that the partition $\tilde{\mathcal{E}}(p)$ is finite, A acts

ergodically on $(M^\alpha, \mu_p^{\mathcal{E}_A})$, and A is a connected group. In particular, A acts ergodically on the (finite) factor measure space $(M^\alpha, \mu_p^{\mathcal{E}_A})/\tilde{\mathcal{E}}(p)$. This yields a contradiction unless the partition $\tilde{\mathcal{E}}(p)$ contains only one element. It follows that $\mathcal{E}_A = \mathcal{E}_{A'}$. $\qquad\square$

From standard measure rigidity arguments for actions of abelian groups we obtain the following.

CLAIM 6.3. μ is invariant under the action G^ξ.

Proof. Fix $U \subset G^\xi$ to be a pre-compact, open neighborhood of the identity in G^ξ. Given almost every $p \in M^\alpha$, the measure $\mu_p^{G^\xi}$ gives positive mass to U. For such p, normalize $\mu_p^{G^\xi}$ on U.

Let A' be as above. Then any $s \in A'$ commutes with G^ξ whence $sUs^{-1} = U$ and thus $\mu_{\tilde{\alpha}(s)(p)}^{G^\xi} = \mu_p^{G^\xi}$. Let $K \subset M^\alpha$ be a compact set on which the assignment $p \mapsto \mu_p^{G^\xi}$ is continuous (where locally finite measure on G^ξ are endowed with the topology dual to compactly supported continuous functions).

Since η^ξ refines $\mathcal{E}_{A'}$, we have the following: for μ-almost every $p \in M^\alpha$ and for $\mu_p^{G^\xi}$-almost every $u \in G^\xi$, setting $p' = \tilde{\alpha}(u)(p)$ we have $p' \in \mathcal{E}_{A'}(p)$. Consider such p and p' such that $p' \in K$ is a density point of K with respect to the A'-ergodic component of μ containing p. It follows that there is a sequence $s_k \in A'$ with

(1) $\tilde{\alpha}(s_k)(p) \in K$ for every $k \in \mathbb{N}$;
(2) $\tilde{\alpha}(s_k)(p) \to p'$ as $k \to \infty$;
(3) $\mu_p^{G^\xi} = \mu_{\tilde{\alpha}(s_k)(p)}^{G^\xi}$ for every $k \in \mathbb{N}$.

It follows that $\mu_p^{G^\xi} = \mu_{p'}^{G^\xi}$. Taking sets K_j as above of measure arbitrarily close to 1, for typical points p, it follows that $\mu_p^{G^\xi} = \mu_{p'}^{G^\xi}$ for all $p' = \tilde{\alpha}(v)(p)$ for a $\mu_p^{G^\xi}$-conull set of v. It follows that for almost every p, the group of isometries of G^ξ preserving $\mu_p^{G^\xi}$ up to normalization acts transitively on the support of $\mu_p^{G^\xi}$ in G^ξ. In fact, the group of right-translations of G^ξ preserving $\mu_p^{G^\xi}$ up to normalization acts transitively on the support of $\mu_p^{G^\xi}$ in G^ξ.

It now follows from arguments developed in [21, §5] that $\mu_p^{G^\xi}$ coincides with the Haar measure on a non-trivial subgroup $V(p) \subset G^\xi$. See also [20, §6.1] for an argument in the framework described here. Moreover, the assignment $p \mapsto V(p)$ is measurable and constant on A-orbits. From Lemma 4.2 it follows that μ is G^ξ-invariant. $\qquad\square$

Recall that our initial choice of ξ was such that $G^\xi \not\subset Q = Q_{\mathrm{In}}(\mu)$. From this contradiction we conclude that $Q_{\mathrm{In}}(\mu) = Q_{\mathrm{Out}}(\mu)$. This completes the proof of Proposition 5.3

6.3. *Proof of Proposition* 5.4. The proof of Proposition 5.4 is a direct application of the Theorem 4.4. Recall the definitions of Σ_Q and $\Sigma_Q^\perp$ above. Suppose that $Q_{\text{In}}(\mu) \neq Q_{\text{Out}}(\mu)$. Let $\xi \in \hat{\Sigma}$ be as in Claim 6.1. Then $\xi \subset \Sigma_{Q_{\text{In}}(\mu)}^\perp$. Write $Q = Q_{\text{In}}(\mu)$.

If ξ contains two elements, we have $\xi = \{\beta', 2\beta'\}$ for some root $\beta' \in \Sigma_-$. In this case, take $\beta = 2\beta'$ if $\mu_p^{G^\xi}$ is supported on $G^{2\beta'}$ for almost every p and $\beta = \beta'$ otherwise. If ξ is a single root, take β with $\xi = \{\beta\}$.

We claim the following.

CLAIM 6.4. *If $Q_{\text{In}}(\mu)$ is a maximal parabolic subgroup then, with β as above, there is a root $\gamma \in \Sigma_{Q_{\text{In}}(\mu)}$ with*
(1) $\gamma \neq -c\beta$, *for any $c > 0$;*
(2) $\gamma + \beta \in \Sigma$;
(3) $\gamma + \beta \in \Sigma_{Q_{\text{In}}(\mu)}^\perp$.

Proof. Indeed let $\mathfrak{q} = \text{Lie}(Q_{\text{In}}(\mu))$. Then $\mathfrak{q} = \mathfrak{q}_{\Pi \smallsetminus \{\alpha\}}$ for some simple root α. If $\beta = -\alpha$ then, as we assume there are no rank-1 simple ideals, there is a simple positive root $\hat{\alpha} \neq -\alpha$ adjacent to α in the Dynkin diagram corresponding to the simple factor containing α. Then $\hat{\alpha} - \beta = \hat{\alpha} + \alpha$ is a root. Take $\gamma = -\hat{\alpha}$. Then (since $\mathfrak{q}$ is of the form $\mathfrak{q}_{\Pi \smallsetminus \{\alpha\}}$) $\gamma = -\hat{\alpha} \in \Sigma_{Q_{\text{In}}(\mu)}$ and $\gamma + \beta \in \Sigma_{Q_{\text{In}}(\mu)}^\perp$. Similarly, if $\beta = -2\alpha$ (so that β is a root in factor of type BC_n), then α is the right-most root in the Dynkin diagram; with $\hat{\alpha}$ the root adjacent to (that is, to the left of) α, since $\hat{\alpha} + 2\alpha$ is a root, $\gamma = -\hat{\alpha}$ satisfies the conclusions of the claim.

If $\beta \neq -\alpha$ and $\beta \neq -2\alpha$, then β is of the form
$$\beta = c_\alpha \alpha + \sum_{\hat{\alpha} \neq \alpha \in \Pi} c_{\hat{\alpha}} \hat{\alpha},$$
where $c_\alpha < 0$, $c_{\hat{\alpha}} \leq 0$, and $\sum_{\hat{\alpha} \neq \alpha \in \Pi} c_{\hat{\alpha}} \leq -1$. Since β is not a simple negative root, there is a simple (positive) root $\alpha' \in \Pi$ such that $\beta + \alpha'$ is a negative root. If $\alpha' \neq \alpha$ then, since $\beta = (\beta + \alpha') - \alpha'$ and $-\alpha' \in \Sigma_{Q_{\text{In}}(\mu)}$, it follows that
$$(\beta + \alpha') \notin \Sigma_{Q_{\text{In}}(\mu)}$$
since $Q_{\text{In}}(\mu)$ is a subgroup. Then $\gamma = \alpha'$ satisfies the conclusion of the claim. On the other hand, if $\alpha' = \alpha$ then, since $\beta + \alpha$ is a negative root,
$$-(\beta + \alpha) \in \Sigma_+ \subset \Sigma_{Q_{\text{In}}(\mu)}$$
and
$$\beta + -(\beta + \alpha) = -\alpha \notin \Sigma_{Q_{\text{In}}(\mu)}.$$
Since β and α are linearly independent, the root $\gamma = -(\beta + \alpha)$ satisfies the conclusion of the claim. $\qquad\square$

As we assume that $\gamma \neq -c\beta$ for $c > 0$, it follows that we may find $s \in A$ with $\beta(s) < 0$ and $\gamma(s) < 0$. Let $\mathfrak{h}$ be the Lie subalgebra generated by $\mathfrak{g}^\xi \oplus \mathfrak{g}^{[\gamma]}$ where $[\gamma]$ is the coarse equivalence class of γ. Then $H = \exp(\mathfrak{h})$ is the minimal

subgroup containing G^ξ and $G^{[\gamma]}$ that is contracted by all s with $\beta(s) < 0$ and $\gamma(s) < 0$. For a generic $p \in M^\alpha$, let $H_p \subset S_p \subset H$ be the subgroups guaranteed by Theorem 4.4.

By Claim 6.1 and choice of β in the coarse Lyapunov class ξ, $G^\xi \cap S_p$ is non-empty. Moreover, either $G^\xi = G^\beta$, or $G^\xi = G^{2\beta} \oplus G^\beta$ and $G^\xi \cap S_p$ is not contained in $G^{2\beta}$. By Theorem 4.4(3), in the latter case, $G^\xi \cap S_p$ contains a non-trivial connected subgroup $\{\exp(tY) : t \in \mathbb{R}\}$ of G^β for some $Y \in \mathfrak{g}^\beta$.

Let $\overline{\beta} = \beta + \gamma$. By Claim 6.4, $\overline{\beta} \in \Sigma$, which implies that there is some $Z \in \mathfrak{g}^\gamma$ such that $\mathrm{ad}_Z Y \in \mathfrak{g}^{\beta+\gamma}$ is non-trivial. Theorem 4.4 implies that $\mathrm{Lie}(H_p)$ contains the line spanned by $[Z, Y] \in [\mathfrak{g}^\gamma, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\overline{\beta}}$ for μ-almost every p.
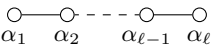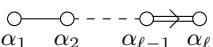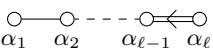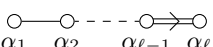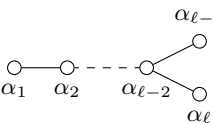
Denote by $V(p)$ the largest connected Lie subgroup of $G^{\overline{\beta}} \cap H_p$, which is non-trivial by the above discussion. This is a measurable family. By the discussion after Theorem 4.4, $V(p)$ is equivariant under the action by A.

In particular, the family $V(p)$ satisfies the hypotheses of Lemma 4.2. From Lemma 4.2, it follows that the measure μ is $G^{[\overline{\beta}]}$-invariant contradicting the choice of γ. This completes the proof of Proposition 5.4.

Appendix A. Tables of root data for classical root systems

A table of simple roots and all positive roots is given in Table 1. We express the roots in terms of a standard presentation (cf. [23, App. C].) In all cases, the parabolic subalgebra $\mathfrak{q}$ of minimal resonant codimension is $\mathfrak{q} = \mathfrak{q}_{\Pi \smallsetminus \{\alpha_1\}}$ from which we immediately verify $r(\mathfrak{g})$ in Example 1.3 from Table 1. We also verify that $m(\mathfrak{g})$ is the resonant codimension of $\mathfrak{q} = \mathfrak{q}_{\Pi \smallsetminus \{\alpha_1, \alpha_2\}}$ except for D_4 from which we verify $m(\mathfrak{g})$ in Examples 1.5.

Table 1. Roots systems and positive roots for classical root systems

	Simple roots and Dynkin diagram	Positive roots	
A_ℓ	$\alpha_i = e_i - e_{i+1}$ $1 \le i \le \ell$	$\alpha_i + \cdots + \alpha_k = e_i - e_{k+1}$	$1 \le i < k \le \ell$
B_ℓ	$\alpha_i = e_i - e_{i+1},$ $1 \le i \le \ell - 1;$ $\alpha_\ell = e_\ell$	$\alpha_i + \cdots + \alpha_k = e_i - e_{k+1}$ $\alpha_i + \cdots + \alpha_\ell = e_i$ $\alpha_i + \cdots + \alpha_k + 2\alpha_{k+1} + \ldots$ $\qquad + 2\alpha_\ell = e_i + e_{k+1}$	$1 \le i \le k \le \ell - 1$ $1 \le i \le \ell$ $1 \le i \le k < \ell$
C_ℓ	$\alpha_i = e_i - e_{i+1},$ $1 \le i \le \ell - 1;$ $\alpha_\ell = 2e_\ell$	$\alpha_i + \cdots + \alpha_k = e_i - e_{k+1}$ $\alpha_i + \cdots + \alpha_k + 2\alpha_{k+1} + \ldots$ $\qquad + 2\alpha_{\ell-1} + \alpha_\ell = e_i + e_{k+1}$ $2\alpha_i + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = 2e_i$	$1 \le i \le k \le \ell - 1$ $1 \le i \le k < \ell$ $1 \le i \le \ell$
BC_ℓ	$\alpha_i = e_i - e_{i+1},$ $1 \le i \le \ell - 1;$ $\alpha_\ell = e_\ell$	$\alpha_i + \cdots + \alpha_k = e_i - e_{k+1}$ $\alpha_i + \cdots + \alpha_\ell = e_i$ $\alpha_i + \cdots + \alpha_k + 2\alpha_{k+1} + \ldots$ $\qquad + 2\alpha_\ell = e_i + e_{k+1}$ $2\alpha_i + \cdots + 2\alpha_{\ell-1} + 2\alpha_\ell = 2e_i$	$1 \le i \le k \le \ell - 1$ $1 \le i \le \ell$ $1 \le i \le k < \ell$ $1 \le i \le \ell$
D_ℓ	$\alpha_i = e_i - e_{i+1},$ $1 \le i \le \ell - 1;$ $\alpha_\ell = e_{\ell-1} + e_\ell$	α_i $\alpha_i + \cdots + \alpha_k = e_i - e_{k+1}$ $\alpha_i + \cdots + \alpha_{\ell-2} + \alpha_{\ell-1} = e_i - e_\ell$ $\alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell = e_i + e_\ell$ $\alpha_i + \cdots + \alpha_{\ell-1} + \alpha_\ell = e_i + e_{\ell-1}$ $\alpha_i + \cdots + \alpha_k + 2\alpha_{k+1} + \ldots$ $\qquad + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ $\qquad = e_i + e_{k+1}$	$1 \le i \le \ell$ $1 \le i < k \le \ell - 2$ $1 \le i \le \ell - 2$ $1 \le i \le \ell - 2$ $1 \le i \le \ell - 2$ $1 \le i \le k < \ell - 2$

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NORTHWESTERN UNIVERSITY, EVANSTON, IL
E-mail: awb@northwestern.edu

PENNSYLVANIA STATE UNIVERSITY, STATE COLLEGE, PA
E-mail: hertz@math.psu.edu

PENNSYLVANIA STATE UNIVERSITY, STATE COLLEGE, PA
E-mail: zhirenw@psu.edu