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SMOOTH ERGODIC THEORY OF $\mathbb{Z}^d$-ACTIONS

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ABSTRACT. In the first part of this paper, we formulate a general setting in which to study the smooth ergodic theory of differentiable $\mathbb{Z}^d$-actions preserving a Borel probability measure. This framework includes actions by $C^{1+\text{Hölder}}$ diffeomorphisms of compact manifolds. We construct intermediate unstable manifolds and coarse Lyapunov manifolds for the action as well as establish controls on their local geometry.

In the second part, we consider the relationship between entropy, Lyapunov exponents, and the geometry of conditional measures for rank-1 systems given by a number of generalizations of the Ledrappier–Young entropy formulas.

In the third part, for a smooth action of $\mathbb{Z}^d$ preserving a Borel probability measure, we show that entropy satisfies a certain "product structure" along coarse unstable manifolds. Moreover, given two smooth $\mathbb{Z}^d$-actions—one of which is a measurable factor of the other—we show that all coarse-Lyapunov exponents contributing to the entropy of the factor system are coarse Lyapunov exponents of the total system. As a consequence, we derive an Abramov–Rohlin formula for entropy subordinated to coarse Lyapunov manifolds.

CONTENTS

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1. INTRODUCTION

The primary motivation for this series of papers is to establish a "product structure of entropy" formula as well as a "coarse-Lyapunov Abramov–Rohlin formula" for measure-preserving, non-uniformly hyperbolic $\mathbb{Z}^d$-actions. These formulas appear in Corollary 13.2 and Theorem 13.7 of Part III. While technical to state, these results have been essential for related work, especially in the rigidity theory of actions of higher-rank lattices in [13, 14, 12]. To establish these results, it is necessary to generalize the main results of the seminal papers of Ledrappier and Young [31, 32]. In Part II, we revisit the work of Ledrappier and Young and establish these necessary generalizations. In Part I, we revisit the theory of Lyapunov exponents, non-uniform hyperbolicity, and invariant manifolds in the setting of $\mathbb{Z}^d$-actions.

In the remainder of this introduction, we provide a short outline of each part with brief discussion of related work and our results.

1.1. **Overview of Part I.** In Part I, we introduce a general setting in which to study smooth ergodic theory of $\mathbb{Z}^d$ actions. Our setting includes actions on manifolds which need not be compact and we allow for actions with discontinuities or singularities sufficiently far from a set of full measure. While results in this part may be well known to experts, they do not appear in a comprehensive way in the literature. Among many others, the results here primarily adapt [40, 25]; see also [2, 23].

In Section 2, we formulate the background on Lyapunov exponents for linear cocycles over actions of abelian groups. We address certain integrability hypotheses in equation (6) which seem to be missing from existing literature (see discussion in Section 2.3.)

In Section 3, we formulate a general framework in which to study the smooth ergodic theory of $C^{1+\text{Hölder}}$-actions of $\mathbb{Z}^d$ preserving a Borel probability measure. This framework includes $C^{1+\text{Hölder}}$-actions on compact manifolds. However—as our motivating application includes actions where the underlying manifold is not compact—we introduce certain hypotheses to overcome non-compactness of the underlying space. These hypotheses only require controls on the local $C^{1+\beta}$ dynamics of the action. Thus, it is natural to work in systems where only the dynamics localized to an open set of full measures is assumed to be smooth (or even continuous). Such a setting was introduced in [25] for diffeomorphisms with singularities and discontinuities; we view our standing hypotheses II in Section 3.2 as analogues of the hypotheses introduced in [25] for actions of $\mathbb{Z}^d$.

In Section 4, we introduce unstable manifolds for individual elements of an action and the concept of (tame) invariant measurable foliations for an action of $\mathbb{Z}^d$. We define the notion of coarse Lyapunov exponents, the corresponding coarse Lyapunov foliations, and assert that unstable foliations and coarse Lyapunov foliations provide the primary examples of (tame) invariant measurable foliations for the action. In Section 5, we introduce the technical tool of Lyapunov charts, with various charts adapted to particular objects of study. Assertions made in Section 4 are established using the dynamics inside Lyapunov charts.

1.2. **Overview of Part II.**

For Part II, we recall the results of [31, 32] which established a relationship between the metric entropy of a C^2 measure-preserving diffeomorphism, its Lyapunov exponents, and certain geometric data associated to the measure.

In [31], a certain rigidity of invariant measures is proven extending the main result of [28]. Recall that for a C^1 diffeomorphism $f: M \to M$ of a compact manifold M and an ergodic, f-invariant measure μ with Lyapunov exponents λ_i, the Margulis–Ruelle inequality [45] gives a bound

$$h_\mu(f) \le \sum_{\lambda_i > 0} \lambda_i m_i \tag{1}$$

where m_i is the multiplicity of the exponent. In [31], it is shown that equality holds in (1) if and only if the measure μ has the following geometric property: the conditional measures of μ along unstable manifolds are absolutely continuous (and, in fact, equivalent) to the Riemannian volume along unstable manifolds. See [31, Theorem A]. When the unstable manifolds have an algebraic structure relative to which the dynamics acts by automorphisms, this rigidity can be used to obtain additional invariance of the measure. This idea is used, for example, in [38, 18, 14, 12, 13] to obtain extra invariance of certain measures.

In [32], the defect of equality in (1) is explained in terms of geometric quantities associated with the measure μ. To each positive Lyapunov exponent $\lambda_i > 0$, we define a corresponding contribution to the entropy $h_\mu(f)$. The maximal contribution of each λ_i to the entropy $h_\mu(f)$ is $m_i\lambda_i$. The main result of [32] is that the entropy contribution of each λ_i is given by $\gamma_i\lambda_i$ where $\gamma_i \le m_i$ denotes the "transverse pointwise dimension" of the measure conditioned along the intermediate λ_i-unstable manifold and transverse to the faster unstable foliation. See [32, Theorem C'].

The results of [31, 32] have been generalized to many other settings including the case of i.i.d. random dynamics on compact manifolds [35, 33], more general skew product systems [1, 41], infinite dimensional systems [5], and endomorphisms [46, 34]. In all these settings, the underlying dynamics occurs on compact subsets and the dynamics is assumed C^2. Beyond the compact setting, we point to [43] where finiteness of entropy and a Margulis–Ruelle inequality for diffeomorphisms of non-compact manifolds is studied and [42] for examples where the Margulis–Ruelle inequality fails for C^∞ diffeomorphisms of a non-compact manifolds.

In Part II we reprove the main results of [31, 32] under somewhat more general hypotheses and for slightly more general notions of entropy. These generalizations are needed for the results in Part III which in turn are used in [14] and [12, 13]. In addition to [31, 32], we heavily adapt arguments from [28, 25, 29].

In Section 6, we review basic definitions and facts about the entropy of an invertible measurable transformation. In Section 7, we state our main results, Theorem 7.2 and 7.7. In Section 8, we review the construction of special partitions, namely increasing partitions subordinated to an expanding foliation following [29]; the key technical result is stated as Proposition 8.3. The proof of Theorem 7.7 and Proposition 8.3 occupies Sections 9–11, following essentially the same arguments as in [31, 32]. Finally, in Section 12 we reduce the proof of Theorem 7.2 to the setting considered in [28, Section 3].

The results and proofs in Part II are rather similar to [31, 32] and may not be surprising to experts. We emphasize some specific ways our results extend previous results.

1. The papers [31, 32] and most subsequent results required the dynamics to be C^2. For hyperbolic measures (i.e., measures without zero exponents) the results of [31, 32] still hold for $C^{1+\beta}$ diffeomorphisms of compact manifolds; see [28], and [2] and [3, Appendix]. The main result of [11] allows us to deduce the entropy formulas in the presence of zero Lyapunov exponents for $C^{1+\beta}$ diffeomorphisms.

2. Similarly, the entropy formulas appearing in [31, 32] (and most extensions mentioned above) assumed the manifold M is compact (or the dynamics is concentrated to a compact part of the space). In particular, the derivative and its Lipschitz (or Hölder) norm are assumed to be bounded from above and the injectivity radius of M is assumed to be bounded from below. In the case of random dynamics, one can remove uniform boundedness by assuming some integrability properties of derivative and its Lipschitz variation.

   We replace compactness and the consequent boundedness of local dynamics by assuming log-integrability of the derivative, and slow degeneration of the $C^{1+\text{Hölder}}$ size of the local dynamics and the injectivity radii along orbits. This is formalized through the introduction of dynamical charts $\{\phi_x\}$ in Section 3.2. By the introduction of such charts, we localize our analysis to an open set of full measure restricted to which the the dynamics is a diffeomorphism; no assumptions are made on the dynamics outside of such charts and thus we also consider maps with singularities and discontinuities (assuming the measure is sufficiently concentrated away from the singular set). We obtain analogues of the entropy formulas of [31, 32] in settings similar to those studied in [25].

3. We consider two notions of "relativized" metric entropy: the metric entropy of a transformation *subordinate to a measurable partition* and the metric entropy of a transformation *subordinate to an invariant measurable foliation*. We prove a formula analogous to that of [32] for entropy subordinated to a measurable partition and, as in [31], show that the entropy subordinate to an invariant measurable foliation attains its maximal theoretical value only when the measure is absolutely continuous along the unstable component of the foliation.

1.3. **Overview of Part III.** In Part III, we establish the "product structure of entropy," stated in Corollary 13.2. This result is motivated by two previous results.

First, for commuting C^2 diffeomorphisms of a compact manifold preserving a common invariant probability measure, it was shown in [21, Theorem B] that entropy is subadditive; in particular, given a C^2 action α of $\mathbb{Z}^d$ preserving a probability measure μ, the entropy function $\mathbb{Z}^d \to \mathbb{R}$,

$$n \mapsto h_\mu(\alpha(n)) \tag{2}$$

extends to a semi-norm on $\mathbb{R}^d$. [21, Theorem B] also implies the semi-norm (2) is additive when restricted to each of finitely many connected open sets in the complement of finitely many singular hyperplanes; moreover, the singular hyperplanes correspond to the kernels of a subset of the coarse Lyapunov exponents. From Corollary 13.2, we recover the subadditivity of entropy (see Theorem 13.3). Furthermore, the results of Part II give additional information on the shape of the semi-norm: the singular hyperplanes where the semi-norm stops being linear are in one-to-one correspondence with the kernels of coarse Lyapunov exponents contributing entropy to the system. We note that [21] heavily used the results of [32] and our extension in Corollary 13.2 relies on the results developed in Part II.

Second, in homogeneous settings such as those considered in [15, 16, 17], conditional measures along total unstable manifolds have been shown to exhibit certain product structures (as products of conditional measures along coarse Lyapunov foliations). The "product structure of entropy" we establish follows in such settings from the product structure of conditional measures. However, to show the product structure of conditional measures, it seems essential to work in a homogeneous (and semisimple) setting where the kernel of a Lyapunov exponent acts isometrically on the corresponding coarse Lyapunov foliation. In the setting of smooth $\mathbb{Z}^d$-actions, one can not expect the kernel of a

Lyapunov exponent to act isometrically on the corresponding coarse Lyapunov foliation. Thus, we establish the "product structure of entropy" without establishing any product structure of conditional measures. We note though that the "product structure of entropy" still implies many of the geometric properties of conditional measures along coarse Lyapunov foliations.

Additionally, in Part III, we also consider the setting of two smooth $\mathbb{Z}^d$-actions, one of which is a measurable factor of the other. We show in Theorem 13.5 that any exponent that contributes entropy to the factor system must be positively proportional to an exponent for the extended system; we also establish a "coarse-Lyapunov Abramov–Rohlin formula" in Theorem 13.7 relating the entropy contributions of such pairs of exponents. In Remark 13.6, we observe that this seems to provide finer information distinguishing actions (up to measurable conjugacy) than the coincidence of the entropy functions (2).

We state our main results in Section 13. In Section 14, we define various objects associated to the Lyapunov exponent functionals and various subsets of Lyapunov exponent functionals with good geometric properties that appear in the induction in our proofs. We also establish a number of elementary properties about these objects. In Section 15, we establish our main inductive proposition, Proposition 15.4, and prove Theorem 13.1. We then give the proofs of Theorems 13.5 and 13.7 in Section 16.

Part I. Lyapunov exponents, dynamical charts, and coarse Lyapunov manifolds
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We formulate a general setting in which to study the smooth ergodic theory of differentiable $\mathbb{Z}^d$-actions preserving a Borel probability measure. This framework includes actions by $C^{1+\text{Hölder}}$ diffeomorphisms of compact manifolds. We construct intermediate unstable manifolds and coarse Lyapunov manifolds for the action as well as establish controls on their local geometry.

2. COCYCLES OVER $\mathbb{Z}^d$-ACTIONS

Let (X,μ) be a standard probability space. Consider an action $\alpha\colon \mathbb{Z}^d\times X\to X$ of $\mathbb{Z}^d$ by measurable, invertible, measure-preserving transformations of (X,μ). That is, for every $n,m\in\mathbb{Z}^d$ and $x\in X$

1. $\alpha(n,\alpha(m,x))=\alpha(n+m,x)$;
2. $\alpha(0,x)=x$;
3. $\alpha(n,\cdot)\colon X\to X$ is measurable;
4. if $\phi\in L^1$, then $\int\phi(x)\ d\mu(x)=\int\phi\circ(\alpha(n,x))\ d\mu(x)$.

The action α of $\mathbb{Z}^d$ on (X,μ) is moreover said to be *ergodic* if

5. given $\phi\in L^1(\mu)$, if $\phi(x)=\phi(\alpha(n,x))$ for all $n\in\mathbb{Z}^d$ and a.e. x, then ϕ is constant on a set of full measure.

At times we write $\alpha(n)\colon X \to X$ for the map $\alpha(n)(x) = \alpha(n,x)$. Additional smoothness hypotheses on α and X will be imposed starting in Section 3 below.

Fix a basis for $\mathbb{Z}^d$ and equip $\mathbb{Z}^d$ with the norm $|(n_1,\dots,n_d)| = \max|n_i|$. Note however that all definitions and facts stated below are independent of the choice of basis and norm on $\mathbb{Z}^d$.

2.1. **Slowly increasing functions.**

DEFINITION 2.1. Given a measurable set $X_0 \subset X$, a measurable function $L\colon X \to [1,\infty)$ is *slowly increasing on* X_0 (over the action α) if

$$\lim_{\tau\to\infty} \frac{1}{\tau} \sup_{|n|\le\tau} \log(L(\alpha(n,x))) = 0 \tag{3}$$

for all $x \in X_0$. L is *slowly increasing* if X_0 can be taken with $\mu(X_0) = 1$. L is said to be ε*-slowly increasing on* X_0 if for every $x \in X_0$ and every $n \in \mathbb{Z}^d$ we have

$$L(\alpha(n,x)) \le e^{|n|\varepsilon} L(x)$$

and ε*-slowly increasing* if X_0 can be taken with $\mu(X_0) = 1$.

We have the following claim.

CLAIM 2.2. *Let $X_0 \subset X$, and consider a measurable function $L\colon X \to [1,\infty)$ that is slowly increasing on X_0. Let $Y_0 = \bigcup\{\alpha(n)(X_0) : n \in \mathbb{Z}^d\}$ be the α-orbit of X_0. Then for any $\varepsilon > 0$ there is a measurable function $\hat{L}\colon Y_0 \to [1,\infty)$ that is ε-slowly increasing on Y_0 with*

$$L(x) \le \hat{L}(x)$$

for all $x \in Y_0$.

Proof. Given $x \in Y_0$ define

$$\hat{L}(x) := \sup_{n\in\mathbb{Z}^d} e^{-|n|\varepsilon} L(\alpha(n,x)). \tag{4}$$

Then $\hat{L}$ is defined for $x \in X_0$ by (3). Moreover, for every $k, j \in \mathbb{Z}^d$ we have

$$\begin{aligned}\hat{L}(\alpha(k+j,x)) &:= \sup_{n\in\mathbb{Z}^d} e^{-|n|\varepsilon} L(\alpha(n+k+j,x))\\ &\le \sup_{n\in\mathbb{Z}^d} e^{|k|\varepsilon-|n+k|\varepsilon} L(\alpha(n+k+j,x))\\ &= e^{|k|\varepsilon} \hat{L}(\alpha(j,x)).\end{aligned}$$

Setting $j = 0$ and considering $y = \alpha(k,x) \in Y_0$. it follows that (4) is defined for every $y \in Y_0$. Moreover, given $y \in Y_0$, writing $y = \alpha(j,x)$ for $x \in X_0$ it follows that $\hat{L}$ has the desired properties. □

Applying either the higher-rank pointwise ergodic theorem (see [27, Theorem 2.8]) or adapting the proof of [2, Lemma 2.1.5] (see also [9, Proposition 6]) we have the following claim.

CLAIM 2.3. *Let α be an action of $\mathbb{Z}^d$ on (X,μ) and let $\phi\colon X \to [1,\infty)$ be a measurable function with $\log(\phi) \in L^d(\mu)$. Then ϕ is slowly increasing.*

Claim 2.3 fails for $\log(\phi) \in L^1(\mu)$ and $d > 1$. Indeed, see the discussion in Section 2.3.

2.2. **Multiplicative ergodic theorem and Lyapunov exponents.** Let α be an action of $\mathbb{Z}^d$ on (X,μ). A *k-dimensional linear cocycle* defined over α is measurable function

$$\mathscr{A}\colon \mathbb{Z}^d \times X_0 \to \mathrm{GL}(k,\mathbb{R})$$

satisfying the cocycle relation: $\mathscr{A}(0,x) = \mathrm{Id}$ and

$$\mathscr{A}(m+n,x) = \mathscr{A}(m,\alpha(n,x))\mathscr{A}(n,x) \tag{5}$$

for all $n,m \in \mathbb{Z}^d$ and $x \in X_0$ where $X_0 \subset X$ is an α-invariant subset with $\mu(X_0) = 1$.

Write $L^p(\mu)$ for the standard L^p spaces. Let $L^{p,q}(\mu)$ denote the Lorentz space introduced, for instance, in [36]. We will not provide a definition of the Lorentz spaces but will recall some of their properties: for $1 \le p, q \le \infty$, $\varepsilon > 0$ and $q < q'$ we have that

1. $L^{p,p}(\mu) = L^p(\mu)$;
2. $L^{p+\varepsilon}(\mu) \subset L^{p,1}(\mu)$;
3. $L^{p,q}(\mu) \subset L^{p,q'}(\mu)$.

Given a function $\phi\colon X \to (0,\infty)$, write $\log^+ \phi := \max\{\log\phi, 0\}$. Write $|\cdot|$ for the standard norm on $\mathbb{R}^k$ and $\|\cdot\|$ for induced operator norm.

THEOREM 2.4 (Higher-rank Oseledec's Theorem)**.** *Let α be an ergodic action of $\mathbb{Z}^d$ on (X,μ) and let $\mathscr{A}\colon \mathbb{Z}^d \times X \to \mathrm{GL}(k,\mathbb{R})$ be a measurable cocycle. Assume for every $m \in \mathbb{Z}^d$ that*

$$\big(x \mapsto \log^+ \|\mathscr{A}(m,x)\|\big) \in L^{d,1}(\mu). \tag{6}$$

Then there are

1. *an α-invariant subset $\Lambda_0 \subset X$ with $\mu(\Lambda_0) = 1$;*
2. *linear functionals $\lambda_i\colon \mathbb{Z}^d \to \mathbb{R}$ for $1 \le i \le p$;*
3. *positive integers m_i for $1 \le i \le p$;*
4. *and splittings $\mathbb{R}^k = \bigoplus_{i=1}^p E_{\lambda_i}(x)$ into families of mutually transverse, m_i-dimensional, μ-measurable subbundles $E_{\lambda_i}(x) \subset \mathbb{R}^k$ defined for $x \in \Lambda_0$*

such that for all $x \in \Lambda_0$ and all $v \in E_{\lambda_i}(x) \smallsetminus \{0\}$

(a) $\mathscr{A}(n,x)E_{\lambda_i}(x) = E_{\lambda_i}(\alpha(n,x))$ *and*
(b) $\displaystyle\lim_{n\to\infty} \frac{\log|\mathscr{A}(n,x)(v)| - \lambda_i(n)}{|n|} = 0.$

Moreover, for $x \in \Lambda_0$ we have

(c) $\displaystyle\lim_{n\to\infty} \frac{\log|\det \mathscr{A}(n,x)| - \sum m_i\lambda_i(n)}{|n|} = 0$, *and*
(d) for every λ_i,

$$\lim_{n\to\infty} \frac{1}{|n|} \log\left(\sin\angle\Big(E_{\lambda_i}(\alpha(n,x)), \bigoplus_{\lambda_j \ne \lambda_i} E_{\lambda_j}(\alpha(n,x))\Big)\right) = 0.$$

Here $\angle(V,U)$ denotes the smallest angle between subspaces V,U in $\mathbb{R}^k$ relative to the standard inner product; that is $\angle(V,U) = \inf\{\angle(v,u) : v \in V \smallsetminus \{0\}, u \in U \smallsetminus \{0\}\}$. The modifications of Theorem 2.4 to the case of non-ergodic μ are standard; in particular, the number of exponents $p(x)$, linear functionals $\{\lambda_{i,x}, 1 \le i \le p(x)\}$, and splitting $\mathbb{R}^k = \bigoplus_{i=1}^{p(x)} E_{\lambda_i}(x)$ are invariant or equivariant and depended measurably on x. See for instance [2, Section 3.6.1].

Observe that the limits in (b)–(d) are taken along any sequence $n \to \infty$ in $\mathbb{Z}^d$. The limit in (b) implies the following weaker result: given $x \in \Lambda_0$, $v \in E_{\lambda_i} \smallsetminus \{0\}$, and $n \in \mathbb{Z}^d$

$$\lim_{k\to\infty} \frac{1}{k} \log|\mathscr{A}(kn,x)(v)| = \lambda_i(n). \tag{7}$$

However, convergence along rays in (7) does not imply the limit in (b) holds. It seems the $L^{d,1}(\mu)$ hypothesis is sharp as discussed in [9].

To prove Theorem 2.4, first using only that $\big(x \mapsto \log^+ \|\mathscr{A}(m,x)\|\big) \in L^1(\mu)$, for every $m \in \mathbb{Z}^d$ one can produce the set Λ_0, the splitting $\mathbb{R}^k = \bigoplus_{i=1}^{p} E_{\lambda_i}(x)$ for $x \in \Lambda_0$ and the functionals $\lambda_i : \mathbb{R}^d \to \mathbb{R}$ such that convergence along (the countably many) rays in (7) holds (see [2, Theorem 3.6.6]). Analogues of (c) and (d) also hold along rays.

The convergence (along spheres) in (b) and (c) can be derived from the maximal lemma in [8] following the arguments in [8, Section 3]. Alternatively, the convergence in (b) and (c) can be reinterpreted in terms of random semimetrics modeled on $\mathbb{Z}^d$ in which case the result follows from [4, Theorem 2.4]. The limit in (d) follows from (c) using arguments as in [2, Theorem 1.3.11] and [2, Section 1.3.2].

It seems the integrability hypothesis $L^{d,1}$ is essential in the proof of Theorem 2.4. Indeed, see discussion in [9]. Below, in Section 2.3, we show that Theorem 2.4 fails if we only assume the cocycle is L^1.

DEFINITION 2.5. The linear functionals λ_i in Theorem 2.4 are called *Lyapunov exponent functionals*, or simply *Lyapunov exponents* of the cocycle $\mathscr{A}$. The subspaces $E_{\lambda_i}(x)$ are the *Oseledec's subspaces*, and the set Λ_0 is the set of *regular points*. The integer m_i is the *multiplicity* of λ_i.

We write $\mathscr{L} = \{\lambda_i\}$ for the set of Lyapunov exponents functionals. Note that the exponents λ_i are independent of the choice of norm on $\mathbb{R}^k$ and the choice of generating set and norm on $\mathbb{Z}^d$.

We have the following standard fact.

PROPOSITION 2.6. *Let $\mathscr{A}$ be as in Theorem 2.4. Then for any $\varepsilon > 0$ there are ε-slowly increasing measurable functions $A, K \colon \Lambda_0 \to [1,\infty)$ such that for every $x \in \Lambda_0$ and $v \in E_{\lambda_i}(x)$,*

(a) for all $n \in \mathbb{Z}^d$,

$$A(x)^{-1} e^{\lambda_i(n) - |n|\varepsilon/2}|v| \le |\mathscr{A}(n,x)v| \le A(x) e^{\lambda_i(n)+|n|\varepsilon/2}|v|;$$

(b) for all λ_i,

$$\sin\angle\Big(E_{\lambda_i}(x), \bigoplus_{\lambda_j\neq\lambda_i} E_{\lambda_j}(x)\Big) \geq K(x)^{-1}.$$

Proof. Conclusion (b) follows from Theorem 2.4(d) and Claim 2.2.

For conclusion (a), write $\mathscr{A}_i(n,x) := \mathscr{A}(n,x)\restriction_{E_{\lambda_i}(x)}: E_{\lambda_i}(x) = E_{\lambda_i}(\alpha(n,x))$ and

$$C_i(x) := \sup_{n\in\mathbb{Z}^d} \|\mathscr{A}_i(n,x)\| e^{-\lambda_i(n)-|n|\frac{\epsilon}{2}}, \quad c_i(x) := \inf_{n\in\mathbb{Z}^d} m(\mathscr{A}_i(n,x)) e^{-\lambda_i(n)+|n|\frac{\epsilon}{2}},$$

where $m(B) = \|B^{-1}\|^{-1}$ denotes the conorm of a matrix. From Theorem 2.4(b), we have $0 < c_i(x) \leq C_i(x) < \infty$ for almost every x. Moreover, for $v \in E_{\lambda_i}(x)$ we have

$$c_i(x)e^{\lambda_i(n)-|n|\frac{\epsilon}{2}}|v| \leq |\mathscr{A}(n,x)v| \leq C_i(x)e^{\lambda_i(n)+|n|\frac{\epsilon}{2}}|v|.$$

Observe for all $k,n\in\mathbb{Z}^d$ that

$$\begin{aligned}
&\|\mathscr{A}_i(n,\alpha(k,x))\| e^{-\lambda_i(n)-|n|\frac{\epsilon}{2}-|k|\epsilon}\\
&\quad\leq \|\mathscr{A}_i(n+k,x)\| m(\mathscr{A}_i(k,x))^{-1} e^{-\lambda_i(n)-|n|\frac{\epsilon}{2}-|k|\epsilon}\\
&\quad\leq C_i(x)c_i(x)^{-1} e^{\lambda_i(n+k)+|n+k|\frac{\epsilon}{2}-\lambda_i(k)+|k|\frac{\epsilon}{2}} e^{-\lambda_i(n)-|n|\frac{\epsilon}{2}-|k|\epsilon}\\
&\quad\leq C_i(x)c_i(x)^{-1}
\end{aligned}$$

whence

$$C_i(\alpha(k,x)) \leq e^{|k|\epsilon} C_i(x)c_i(x)^{-1}.$$

Similarly

$$c_i(\alpha(k,x)) \geq e^{-|k|\epsilon} c_i(x)C_i(x)^{-1}.$$

Set

$$A_i(x) := \sup_{n,k\in\mathbb{Z}^d} e^{-|k|\epsilon} C_i(\alpha(k,x)), \qquad a_i(x) := \inf_{n,k\in\mathbb{Z}^d} e^{|k|\epsilon} c_i(\alpha(k,x)).$$

Then $-\infty < a_i(x) \leq c_i(x) \leq C_i(x) \leq A_i(x) < \infty$ and, in particular, for $v \in E_{\lambda_i}(x)$ we have

$$a_i(x)e^{\lambda_i(n)-|n|\frac{\epsilon}{2}}|v| \leq |\mathscr{A}(n,x)v| \leq A_i(x)e^{\lambda_i(n)+|n|\frac{\epsilon}{2}}|v|.$$

Moreover, the inequalities

$$A_i(\alpha(m,x)) \leq e^{|m|\epsilon} A_i(x), \qquad a_i(\alpha(m,x)) \geq e^{-|m|\epsilon} a_i(x)$$

hold by definition. The function $A(x) = \max\{A_i(x), a_i(x)^{-1} : 1 \leq i \leq p\}$ then satisfies the conclusions of the proposition. $\square$

2.3. Failure of Theorem 2.4 in L^1.

The integrability hypothesis

$$(x \mapsto \log^+ \|\mathscr{A}(m,x)\|) \in L^{d,1}(\mu)$$

seems to be sharp in Theorem 2.4 (see discussion in [9]) although there are assertions in the literature that L^1 integrability is sufficient. For instance, both [2, Theorem 3.6.7] and [23, Theorem 1.7.1] are incorrect as stated. For instance, if we only assume $(x \mapsto \log^+ \|\mathscr{A}(m,x)\|) \in L^1(\mu)$, then the limit in (1.7.4) of [23, Theorem 1.7.1] (corresponding to (b) of our Theorem 2.4) need not converge. Similarly, in [2, page 87], the sum defining the Lyapunov metric need not converge and the reduction theorem of [2, Theorem 3.6.7] fails.

Both these defects can be seen in the following example. Take $\beta_1, \beta_2 \in [0,1] \setminus \mathbb{Q}$ to be badly approximable numbers and let $r_\beta \colon S^1 \to S^1$, $r_\beta(x) = x + \beta \mod 1$, denote the rotation on S^1 by β. Let $\alpha \colon \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ be the product action $\alpha((n,m),(x,y)) = (r_{n\beta_1}(x), r_{m\beta_2}(y))$. Given any $\gamma \in (1/2,1)$, let $\phi \colon \mathbb{T}^2 \to \mathbb{R}$ be the function

$$\phi(x,y) = \frac{1}{d(x,0)^\gamma} \frac{1}{d(y,0)^\gamma}$$

where d denotes the distance on S^1. Let $\mathscr{A} \colon \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathrm{GL}(1,\mathbb{R})$ be the (abelian) cocycle

$$\mathscr{A}((n,m),(x,y)) = \exp\left[\phi(\alpha((n,m),(x,y))) - \phi(x,y)\right].$$

Let μ be the Lebesgue measure on $\mathbb{T}^2$. For all $n, m \in \mathbb{Z}^2$ we have

$$(x,y) \mapsto \log^+ \|\mathscr{A}((n,m),(x,y))\| \in L^1(\mu)$$

but

$$(x,y) \mapsto \log^+ \|\mathscr{A}((n,m),(x,y))\| \notin L^2(\mu).$$

As β_1, β_2 are chosen to be badly approximable, there is a C such that, for any $x, y \in S^1$ and any N, there are $1 \le n, m \le N$ with

$$d(r_{n\beta_1}(x), 0) \le \tfrac{C}{N}, \quad d(r_{m\beta_2}(y), 0) \le \tfrac{C}{N},$$

whence $\phi(\alpha((n,m),(x,y))) \ge \frac{N^{2\gamma}}{C^2}$. Then for any $(x,y) \in \mathbb{T}^2$ (outside of the orbits of $\{0\} \times S^1$ and $S^1 \times \{0\}$) one can find a sequence $(n_i, m_i) \to \infty$ with

$$\frac{\log^+ \|\mathscr{A}((n_i,m_i),(x,y))\|}{|(n_i,m_i)|} \to \infty.$$

2.4. **Restricted cocycles.** Let α be an ergodic action of $\mathbb{Z}^d$ on (X,μ). Let $\mathscr{A} \colon \mathbb{Z}^d \times X \to \mathrm{GL}(k,\mathbb{R})$ be a linear cocycle satisfying the hypotheses of Theorem 2.4 and let $\mathscr{L} = \{\lambda_j\}$ be the Lyapunov exponents of α. Let $H \subset \mathbb{Z}^d$ be a subgroup and let $\tilde\alpha$ denote the restriction of α to H. It may be that $\tilde\alpha$ is no longer ergodic. However, for almost every ergodic component $\tilde\mu^e_x$ of the action of $\tilde\alpha$ on (X,μ), the restriction of the cocycle $\mathscr{A}$ to the action $\tilde\alpha$ satisfies the hypotheses of Theorem 2.4. For such an ergodic component $\tilde\mu^e_x$, let $\tilde{\mathscr{L}}_x = \{\tilde\lambda_{i,x}\}$ be the Lyapunov exponents of the restricted cocycle. Clearly, for every i we have $\tilde\lambda_{i,x} = \lambda_j \restriction_H$ for some j. In particular, the collection of linear functionals $\{\tilde\lambda_{i,x}\}$ are independent a.s. of the choice of ergodic component $\tilde\mu^e_x$.

2.5. **Lyapunov metric.** A standard technique which simplifies certain dynamical arguments is to specify a family of inner products and related norms on $\mathbb{R}^k$ relative to which the dynamics of the cocycle $\mathscr{A}$ is uniformly partially hyperbolic. Let $\mathscr{A} \colon \mathbb{Z}^d \times X \to \mathrm{GL}(k,\mathbb{R})$ be a measurable cocycle satisfying (6) over an ergodic $\mathbb{Z}^d$-action α on X. We define a measurable family of inner products, called the *ε-Lyapunov metric* as follows: given any $\varepsilon > 0$, $x \in \Lambda_0$, $\lambda_i \in \mathscr{L}$, and $v, w \in E_{\lambda_i}(x)$

$$\langle\langle\langle v, w \rangle\rangle\rangle_{x,\varepsilon} := \sum_{n \in \mathbb{Z}^d} e^{-2\lambda_i(n) - 2\varepsilon|n|} \langle \mathscr{A}(n,x)v, \mathscr{A}(n,x)w \rangle. \tag{8}$$

The expression in (8) converges for $x \in \Lambda_0$ by Proposition 2.6(a). We extend $\langle\langle\langle \cdot, \cdot \rangle\rangle\rangle_{x,\varepsilon}$ to $\mathbb{R}^k$ by declaring

$$\langle\langle\langle v, w \rangle\rangle\rangle_{x,\varepsilon} = 0$$

for $v \in E_{\lambda_i}(x)$ and $w \in E_{\lambda_j}(x)$ with $\lambda_i \neq \lambda_j$.

For $v_i \in E_{\lambda_i}(x)$, let $|||v_i|||_{x,\varepsilon}$ be the norm on $E_{\lambda_i}(x)$ induced by $\langle\langle\langle \cdot, \cdot \rangle\rangle\rangle_{x,\varepsilon}$. Given $v \in \mathbb{R}^k$, we decompose $v = \sum v_i$ where $v_i \in E_{\lambda_i}(x)$ and define a measurable family of norms $|||\cdot|||_{x,\varepsilon}$, called the *$\varepsilon$-Lyapunov norm*, on $\mathbb{R}^k$ by

$$|||v|||_{x,\varepsilon} = \max\left\{|||v_i|||_{x,\varepsilon}\right\}. \tag{9}$$

We have the following two facts about the family of norms $|||\cdot|||_{x,\varepsilon}$.

PROPOSITION 2.7. *For $x \in \Lambda_0$, $v \in E_{\lambda_i}(x)$, and all $k \in \mathbb{Z}^d$ we have*

$$e^{\lambda_i(k)-\varepsilon|k|}|||v|||_{x,\varepsilon} \le |||\mathscr{A}(k,x)v|||_{\alpha(k,x),\varepsilon} \le e^{\lambda_i(k)+\varepsilon|k|}|||v|||_{x,\varepsilon}. \tag{10}$$

Proof. We have

$$\begin{aligned}
&|||\mathscr{A}(k,x)v|||^2_{\alpha(k,x),\varepsilon} \\
&:= \sum_{n\in\mathbb{Z}^d} e^{-2\lambda_i(n)-2\varepsilon|n|}\langle \mathscr{A}(n,\alpha(k,x))\mathscr{A}(k,x)v, \mathscr{A}(n,\alpha(k,x))\mathscr{A}(k,x)v\rangle \\
&= \sum_{n\in\mathbb{Z}^d} e^{-2\lambda_i(n)-2\varepsilon|n|}\langle \mathscr{A}(n+k)v, \mathscr{A}(n+k)v\rangle \\
&= \sum_{n\in\mathbb{Z}^d} e^{2\lambda_i(k)}e^{-2\lambda_i(n+k)-2\varepsilon|n|}\langle \mathscr{A}(n+k)v, \mathscr{A}(n+k)v\rangle \\
&\le \sum_{n\in\mathbb{Z}^d} e^{2\lambda_i(k)+2\varepsilon|k|}e^{-2\lambda_i(n+k)-2\varepsilon|n+k|}\langle \mathscr{A}(n+k)v, \mathscr{A}(n+k)v\rangle \\
&= \sum_{n\in\mathbb{Z}^d} e^{2\lambda_i(k)+2\varepsilon|k|}|||v|||^2_{x,\varepsilon}
\end{aligned}$$

proving the upper bound. The lower bound is similar. □

LEMMA 2.8. *There are a constant $k_0 > 1$ (depending only on p) and, for every $\varepsilon > 0$, an ε-slowly increasing function $L\colon \Lambda_0 \to [1,\infty)$ such that for $x \in \Lambda_0$ and $v \in \mathbb{R}^k$,*

$$k_0^{-1}|v| \le |||v|||_{x,\varepsilon} \le L(x)|v|. \tag{11}$$

Proof. For $v_i \in E_{\lambda_i}(x)$, we have $|v_i| \le |||v|||_{x,\varepsilon}$. If $v = \sum v_i$ for $v_i \in E_{\lambda_i}(x)$ we have

$$|v| \le p \max\{|v_i| : 1 \le i \le p\}.$$

The lower bound follows with $k_0 = p$.

Let K and A be $(\varepsilon/2)$-slowly increasing functions as in Proposition 2.6. For $v_i \in E_{\lambda_i}(x)$, with $b = \sum_{n\in\mathbb{Z}^d} e^{-\varepsilon|n|}$,

$$|||v|||^2_{x,\varepsilon} \le \sum_{n\in\mathbb{Z}^d} A(x)^2 e^{-\varepsilon|n|}|v_i|^2 = bA(x)^2|v_i|^2.$$

For $v \in \mathbb{R}^k$, write $v = \sum v_i$ where $v_i \in E_{\lambda_i}(x)$. Given any i, the orthogonal projections of v and v_i onto the orthogonal complement of $\bigoplus_{\lambda_j\neq\lambda_i} E_{\lambda_j}(x)$ coincide whence

$$|v| \geq |v_i| \sin\angle\Big(E_{\lambda_i}(x), \bigoplus_{\lambda_j \neq \lambda_i} E_{\lambda_j}(x)\Big) \geq |v_i| K(x)^{-1}$$

and

$$|||v|||_{x,\varepsilon} \leq A(x)\sqrt{b} \max\big\{|v_i| : 1 \leq i \leq p\big\} \leq A(x)\sqrt{b}K(x)|v|. \qquad \square$$

3. SMOOTH ACTIONS OF $\mathbb{Z}^d$ WITH SINGULARITIES AND DISCONTINUITIES

In this section, we establish the notational conventions as well as the standing hypotheses for the remainder of the paper. In particular, we present hypotheses under which generalizations of the entropy formulas of [31, 32] will hold.

3.1. **Notational conventions.** Given a map $f : X \to Y$ between metric spaces, let $\mathrm{Lip}(f)$ denote the Lipschitz constant of f and let $\mathrm{Höl}^\beta(f)$ denote the β-Hölder constant of f. (To avoid awkwardness in definitions, we only ever consider $\mathrm{Höl}^\beta(f)$ when f has bounded range.) Given a norm $\|\cdot\|$ on $\mathbb{R}^m$ and subspace $V \subset (\mathbb{R}^m, \|\cdot\|)$ we write

$$V(r) = V(r, \|\cdot\|) := \{v \in V : \|v\| < r\}.$$

Consider $\mathbb{R}^k$ and $\mathbb{R}^j$ equipped, respectively, with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, an open set $U \subset \mathbb{R}^k$, and a differentiable map $g : U \to \mathbb{R}^j$. Let

$$\|Dg\| = \sup_{u \in U} \|D_u g\|$$

and let $\|g\|_{C^1}$ be the usual C^1 norm of g. Note: we often consider the case $0 \in U \subset \mathbb{R}^k(1, \|\cdot\|_1)$ and $g(0) = 0$ and ignore the C^0 part of g. Considering Dg as map from U to the space of linear maps $\mathbb{R}^k \to \mathbb{R}^j$, we write

$$\mathrm{Höl}^\beta(Dg) = \sup_{u \neq v \in U} \frac{\|D_u g - D_v g\|}{\|u - v\|_1^\beta}$$

for the β-Hölder constant of Dg. Set

$$\|g\|_{C^{1+\beta}} = \max\Big\{\|g\|_{C^1}, \mathrm{Höl}^\beta(Dg)\Big\}.$$

If $\|g\|_{C^{1+\beta}} < \infty$, we say g is uniformly $C^{1+\beta}$.

3.2. **Standing hypotheses.** For many applications[1] it is sufficient to consider the following set of hypotheses.

Fix, once and for all, $0 < \beta \leq 1$.

STANDING HYPOTHESES I. M is a compact manifold and $\alpha : \mathbb{Z}^d \to \mathrm{Diff}^{1+\beta}(M)$ is a homomorphism from $\mathbb{Z}^d$ to the group of uniformly $C^{1+\beta}$ diffeomorphisms of M. μ is an α-invariant Borel probability measure.

[1] Of particular note is the analysis of the dynamics of a maximal $\mathbb{R}$-split torus on the suspension space induced by an action of a cocompact lattice in higher-rank Lie groups considered in [12]

However, there are natural examples of actions where the manifold M may not be compact.[2] We introduce certain dynamical charts $\{\phi_x\}$ that control the non-compactness of M (and the consequent unboundedness of the derivative and its Hölder variation). Such charts allow us to work in a setting similar to that introduced in [25]; following [25], we thus allow for possible discontinuities and singularities of the action.

Let M be a Hausdorff, second countable, k-dimensional, C^∞ manifold without boundary. Let μ be a Borel probability measure on M. Unlike in [25], we do not explicitly assume any properties of any metric on M or the metric completion of M. Let $\alpha\colon \mathbb{Z}^d \times M \to M$ be an action by measurable, invertible, μ-preserving transformations. We do not assume μ to be ergodic.

With $|\cdot|$ the standard norm on $\mathbb{R}^k$, we assume the following hypotheses for the system (M,α,μ). We note that standing hypotheses II subsumes standing hypotheses I, and thus we focus on systems satisfying standing hypotheses II for the remainder.

STANDING HYPOTHESES II. We assume there are

- a measurable, α-invariant subset $\Lambda \subset M$ with $\mu(\Lambda) = 1$;
- an open set $U_0 \supset \Lambda$ equipped with a continuous Riemannian metric and an associated locally-defined distance function d on U_0;
- measurable functions $\rho, D\colon \Lambda \to [1,\infty)$ that are slowly increasing (over α) on Λ (see Definition 2.1); and
- a measurable family of C^1 embeddings

$$\{\phi_x : x \in \Lambda\}, \quad \phi_x\colon \mathbb{R}^k(\rho(x)^{-1}) \to U_0$$

with the following properties:

(H1) $\phi_x\colon \mathbb{R}^k\left(\rho(x)^{-1}\right) \to U_0$ is a C^1 diffeomorphism onto its image with $\phi_x(0) = x$;

(H2) $\left\|D\phi_x\right\| \le D(x)$ and $\left\|D\phi_x^{-1}\right\| \le D(x)$; in particular, each chart

$$\phi_x\colon \mathbb{R}^k(\rho(x)^{-1}) \to (U_0, d)$$

is a bi-Lipschitz embedding with $D(x)^{-1} \le \mathrm{Lip}(\phi_x) \le D(x)$.

Moreover, given any finite, symmetric subset $F \subset \mathbb{Z}^d$ that generates $\mathbb{Z}^d$, we assume there are

- an open subset $\Lambda \subset U \subset U_0$ such that for every $n \in F$, $\alpha(n)(U) \subset U_0$ and the restriction $\alpha(n)\restriction_U\colon U \to U_0$ is a diffeomorphism between U and its range;
- measurable functions $r, C\colon \Lambda \to [1,\infty)$ that are slowly increasing (over α) on Λ with $\rho(x) \le r(x)$

such that

(H3) $\phi_x\left(\mathbb{R}^k\left(r(x)^{-1}\right)\right) \subset U$ and for every $m \in F$,

$$\alpha(m)\left(\phi_x\left(\mathbb{R}^k\left(r(x)^{-1}\right)\right)\right) \subset \phi_{\alpha(m,x)}\left(\mathbb{R}^k\left(\rho\left(\alpha(m,x)^{-1}\right)\right)\right).$$

[2]This happens, for instance, for the actions of maximal $\mathbb{R}$-split tori on the suspension space induced by an action of a non-uniform lattice considered in [14, 13].

Moreover, for each $m \in F$, setting $f(\cdot) = \alpha(m, \cdot)$ and defining

$$\hat{f}_x := \phi_{f(x)}^{-1} \circ f \circ \phi_x \tag{12}$$

for $x \in \Lambda$, we assume

(H4) $\hat{f}_x : \mathbb{R}^k(r(x)^{-1}) \to \mathbb{R}^k(\rho(f(x))^{-1})$ is uniformly $C^{1+\beta}$ with

$$\|\hat{f}_x\|_{1+\beta} \le C(x).$$

As F is a generating set for $\mathbb{Z}^d$, it follows that given $x \in \Lambda$ and $n \in \mathbb{Z}^d$, the map

$$\hat{\alpha}(n, x) := \phi_{\alpha(n,x)}^{-1} \circ \alpha(n, x) \circ \phi_x$$

is defined and is a uniformly $C^{1+\beta}$ diffeomorphism on some neighborhood of 0. This induces a measurable cocycle $\mathscr{A} : \mathbb{Z}^d \times \Lambda \to \mathrm{GL}(k, \mathbb{R})$ given by the derivative

$$\mathscr{A}(n, x) = D_0 \hat{\alpha}(n, x). \tag{13}$$

We moreover assume that

(H5) $\left(x \mapsto \log^+ \|\mathscr{A}(n, x))\|\right) \in L^{d,1}(\mu)$ for every $n \in \mathbb{Z}^d$.

Applying Theorem 2.4, we write $\Lambda_0 \subset \Lambda$ in the remainder for the set of *regular points* of the cocycle $\mathscr{A}$ over the action of α on (M, μ).

REMARK 3.1. Given local diffeomorphisms $g_1, g_2 : \mathbb{R}^k(1) \to \mathbb{R}^k$ preserving 0, define $h = g_2 \circ g_1$ on the maximal domain of definition. Then we have

1. $\|Dh\| \le \|Dg_2\| \|Dg_1\|$;
2. $\mathrm{Höl}^\beta(Dh) \le \|Dg_2\| \, \mathrm{Höl}^\beta(Dg_1) + \mathrm{Höl}^\beta(Dg_2) \|Dg_1\|^{1+\beta}$.

Suppose that a family of charts $\{\phi_x\}$, open set U_0, and functions ρ and D satisfying (H1)–(H2) exist such that for some finite symmetric generating set $F \subset \mathbb{Z}^d$ there are U, r, and C such that (H3)–(H4) hold. Then, given any other finite symmetric generating set F', we may modify the functions r and C and the open set U so that (H3)–(H4) hold for F'.

Condition (H5) is independent of F, U, r, and C and hence remains valid passing to F'. Moreover, by the cocycle property (5), it is enough to verify $\left(x \mapsto \log^+ \|\mathscr{A}(n, x))\|\right) \in L^{d,1}(\mu)$ only for n in some finite symmetric generating subset.

REMARK 3.2. Given an invertible, measurable, measure-preserving transformation

$$f : (M, \mu) \to (M, \mu)$$

we say that f *satisfies standing hypotheses I* or *II* if the $\mathbb{Z}$-action generated by f does. In this of standing hypotheses II, we may take $F = \{1, -1\}$ so that (H3) and (H4) hold for both f and f^{-1}.

REMARK 3.3. As $L^{d,1}(\mu) \subset L^{d-1,1}(\mu)$ it follows that if the action $\alpha : \mathbb{Z}^d \times (M, \mu) \to (M, \mu)$ satisfies Standing Hypotheses I or II, then for any subgroup $H \subset \mathbb{Z}^d$, setting $\overline{\alpha} : H \times M \to M$ to be the restriction of α to H, we have that $(M, \overline{\alpha}, \mu)$ also satisfies our standing hypotheses. Moreover, for almost every $\overline{\alpha}\restriction_H$-ergodic component μ_x^e of μ, $(M, \overline{\alpha}\restriction_H, \mu_x^e)$ satisfies standing hypotheses I or II.

4. UNSTABLE MANIFOLDS AND $C^{1+\beta}$-TAME FOLIATIONS

We introduce one of our primary dynamical objects of study. Recall $U_0 \subset M$ introduced in standing hypotheses II is equipped with a locally-defined distance d; in the case of standing hypotheses I, we may take $U_0 = M$. For $x \in U_0$, let $B(x,r) \subset U_0$ denote the open ball centered at x of radius r.

4.1. $C^{1+\beta}$-tame foliations.

Let $\mathscr{F}$ be a partition of (M,μ). We do not assume $\mathscr{F}$ to be measurable. Let $\mathscr{F}(x)$ denote the atom of $\mathscr{F}$ containing x.

DEFINITION 4.1. A *measurable foliation* is a partition $\mathscr{F}$, a set $B = B(\mathscr{F}) \subset M$ with $\mu(B) = 0$, and a measurable function $r\colon M \to (0,\infty)$ such that

1. for almost every $x \in M$, $\mathscr{F}(x) \smallsetminus B$ is a C^1 injectively immersed manifold in M of constant dimension (over connected components)

and, writing $\mathscr{F}(x, r(x))$ for the path-connected component of

$$(\mathscr{F}(x) \smallsetminus B) \cap B(x, r(x))$$

(relative to the immersed manifold topology in $\mathscr{F}(x) \smallsetminus B$) containing x,

2. the family $\{\mathscr{F}(x, r(x))\}$ is a measurable family of C^1 embedded discs.

REMARK 4.2. Above, the set B is a negligible singular set on which atoms of $\mathscr{F}$ may fail to have any manifold structure (arising from construction or definition of $\mathscr{F}$.) In particular, while $\mathscr{F}(x)$ may not have a manifold structure, $\mathscr{F}(x) \smallsetminus B$ is a (possibly disconnected) manifold for a.e. x.

As a primary example, if we consider systems with singularities and discontinuities satisfying standing hypotheses II in Section 3.2, one should expect that global unstable sets defined by Definition 4.2 below do not have any manifold structure. However, for almost every x, the locally-defined unstable sets have the structure of a connected manifold. Furthermore, when we restrict the global unstable set through almost every x to the co-null subset of points $y \in M$ whose backwards orbit (1) never leaves a dynamically good set $\hat{U}$ and (2) eventually enters some local unstable manifold, the restricted global unstable set has the structure of an immersed submanifold; see the proof of Proposition 4.6 on page 473.

If $\mathscr{F}$ is a measurable foliation, then the measurability of $x \mapsto \mathscr{F}(x, r(x))$ implies, by Lusin's theorem, that after removing the set B and a set of arbitrarily small measure, for almost every x there is neighborhood of x on which $\mathscr{F}$ locally restricts to a lamination with uniformly C^1 leaves and such that the local leaves vary continuously in the C^1 topology. We write $\mathscr{F}'(x) := \mathscr{F}(x) \smallsetminus B$ and refer to $\mathscr{F}'(x)$ as the *leaf of $\mathscr{F}$ through x*. Note that $\mathscr{F}'(x)$ need not be connected (though we do assume $\mathscr{F}'(x)$ is second countable).

Consider a μ-preserving action $\alpha\colon \mathbb{Z}^d \times M \to M$ satisfying hypotheses II of Section 3.2.

DEFINITION 4.3. A measurable foliation $\mathscr{F}$ is *α-invariant* if, for every finite symmetric generating set $F \subset \mathbb{Z}^d$, the set $U \subset U_0$ in Section 3.2 and the null set B in Definition 4.1 can be taken so that for all $m \in F$,

$$\alpha(m)\left(\mathscr{F}'(x) \cap U\right) \subset \mathscr{F}'(\alpha(m,x)).$$

Recall that $\alpha(m)\restriction_U : U \to U_0$ is a diffeomorphism onto its image for $m \in F$. It follows that the dimension of $\mathscr{F}'(x)$ is constant along orbits whence for $m \in F$, $\alpha(m)$ is a diffeomorphism between $\mathscr{F}'(x) \cap U$ and an open subset of $\mathscr{F}'(\alpha(m,x))$. In particular, if μ is α-ergodic and $\mathscr{F}$ is α-invariant, then the leaves of $\mathscr{F}$ have constant dimension a.s.

Note that the geometry of the leaves of $\mathscr{F}$ as embedded in M may degrade along orbits of α arbitrarily fast. We restrict ourselves to foliations for which this degradation is subexponential. We also impose additional regularity on the local geometry of leaves of the foliation.

Let $\alpha : \mathbb{Z}^d \times (M,\mu) \to (M,\mu)$ be an action satisfying the standing hypotheses I or II of Section 3.2. Recall the family of charts ϕ_x introduced in standing hypotheses II; in the case of standing hypotheses I, we may take the charts ϕ_x to be exponential charts relative to some Borel trivialization of the TM.

DEFINITION 4.4. A measurable foliation $\mathscr{F}$ is *$C^{1+\beta}$-tame* (for the action α and relative to the charts ϕ_x) if there is a set $\Lambda_{\mathscr{F}} \subset \Lambda$ with $\mu(\Lambda_{\mathscr{F}}) = 1$ such that for every $\varepsilon > 0$ there is a measurable function $\ell_{\mathscr{F}} : \Lambda_{\mathscr{F}} \to [1,\infty)$ that is ε-slowly increasing (over α) on Λ_F and such that for $x \in \Lambda_{\mathscr{F}}$, writing $\widehat{\mathscr{F}}(x)$ for the the path component (relative to the immersed topology) of

$$\phi_x^{-1}\left(\mathscr{F}'(x)\right) \cap \mathbb{R}^k\left(\ell_{\mathscr{F}}^{-1}(x)\right)$$

containing 0, $\widehat{\mathscr{F}}(x)$ is the graph of a $C^{1+\beta}$ function

$$h_x^{\mathscr{F}} : U_x \subset T_x\widehat{\mathscr{F}}(x) \to T_x\widehat{\mathscr{F}}(x)^{\perp}$$

for some connected open subset $U_x \subset T_x\widehat{\mathscr{F}}(x)$ with

1. $h_x^{\mathscr{F}}(0) = 0$ and $D_0 h_x^{\mathscr{F}} = 0$;
2. $\mathrm{H\ddot{o}l}^{\beta}\left(Dh_x^{\mathscr{F}}\right) \le (\ell_{\mathscr{F}}(x))^{\beta}$ whence $\left\|Dh_x^{\mathscr{F}}\right\| \le 1$.

Note that the family of functions $\{h_x^{\mathscr{F}} : x \in \Lambda_{\mathscr{F}}\}$ depends measurably on x.

The primary examples of α-invariant, $C^{1+\beta}$-tame, measurable foliations are the partitions into strong unstable manifolds and coarse and intermediate Lyapunov foliations arising in higher-rank, non-uniformly hyperbolic dynamics (defined below in Proposition 4.6).

4.2. **Global unstable manifolds.** Let $\alpha : \mathbb{Z}^d \times (M,\mu) \to (M,\mu)$ be an action satisfying the standing hypotheses I or II of Section 3.2. To simplify notation, we moreover assume the action α is ergodic with respect to μ. All definitions here may be generalized to the non-ergodic case by considering ergodic components.

Let $\mathscr{L} = \{\lambda_i : 1 \le i \le p\}$ denote the Lyapunov exponent functionals of the derivative cocycle (13) equipped with a choice of enumeration. Given $n \in \mathbb{Z}^d$

we choose a permutation $\sigma(n)$ of $\{1,2,\dots,p\}$ and $1 \le u(n) \le p$ such that

$$\lambda_{\sigma(n)(1)}(n) \ge \lambda_{\sigma(n)(2)}(n) \ge \cdots \ge \lambda_{\sigma(n)(u(n))}(n) > 0 \ge \cdots \ge \lambda_{\sigma(n)(p)}(n).$$

Note that by choosing $n \in \mathbb{Z}^d$ in general position (in particular, outside of finitely many hyperplanes) we can ensure all inequalities above are strict.

DEFINITION 4.5. Given $x \in M$, $n \in \mathbb{Z}^d$, and $1 \le i \le u(n)$, define the *ith unstable manifold through x for $\alpha(n)$* to be the set

$$W_n^{u,i}(x) = \left\{ y \in M \,\middle|\, \limsup_{k\to-\infty} \frac{1}{k} \log d(\alpha(kn,x),\alpha(kn,y)) \le -\lambda_{\sigma(n)(i)}(n) \right\}.$$

The *unstable manifold through x for $\alpha(n)$* is the set

$$W_n^{u}(x) := \left\{ y \in M \,\middle|\, \limsup_{k\to-\infty} \frac{1}{k} \log d(\alpha(kn,x),\alpha(kn,y)) < 0 \right\}.$$

Although $W_n^{u,i}(x)$ may be quite pathological for certain x, it will follow from Proposition 4.6 below that—after possibly removing an ambient singular set of measure zero if our dynamics has singularities or discontinuities—$W_n^{u,i}(x)$ is an injectively immersed manifold for almost every x, hence the terminology. Moreover, we have $W_n^u(x) = W_n^{u,u(n)}(x)$ for almost every x. Note that if n is not in general position so that $\lambda_{\sigma(n)(i)}(n) = \lambda_{\sigma(n)(i+1)}(n)$ for some i with $\lambda_{\sigma(n)(i+1)}(n) > 0$ then the above definition implies

$$W_n^{u,i}(x) = W_n^{u,i+1}(x).$$

Implicit in the above definition is that $d(\alpha(kn,x),\alpha(kn,y))$ is defined for all but finitely many $k<0$; in particular, we have $\alpha(kn,x) \in U_0$ and $\alpha(kn,y) \in U_0$ and are sufficiently close for all but finitely many $k \le 0$. If $\alpha(kn,x) \in M \smallsetminus U_0$ for infinitely many $k<0$, declare $W_n^{u,i}(x) = \{x\}$. From the above definition, it follows for each $n \in \mathbb{Z}^d$ that the collection of ith unstable manifolds forms a partition $\mathscr{W}_n^{u,i}$ of M. We also write $\mathscr{W}_n^u$ for the partition of M into unstable manifolds for $\alpha(n)$. It follows from Proposition 4.6 below that $\mathscr{W}_n^u$ and $\mathscr{W}_n^{u,u(n)}$ coincide off a null set.

As we do not assume the action α to be by diffeomorphisms, $W_n^{u,i}(x)$ is, in general, not a submanifold. However, from Proposition 4.6 below, under our standing hypotheses, the partition $\mathscr{W}_n^{u,i}$ has the structure of a $C^{1+\beta}$-tame, measurable foliation tangent a.e. to $D_0\phi_x\left(\bigoplus_{\lambda_j(n)\ge\lambda_{\sigma_n(i)}(n)} E_{\lambda_j}(x)\right)$. Then, with the notation introduced above, $\left(\mathscr{W}_n^{u,i}\right)'(x)$ is an injectively immersed manifold of dimension $\dim\left(\bigoplus_{\lambda_j(n)\ge\lambda_{\sigma_n(i)}(n)} E_{\lambda_j}(x)\right)$.

Given an α-invariant measurable foliation $\mathscr{F}$ and $x \in M$, let $d_{\mathscr{F}}$ denote the distance on $\mathscr{F}'(x) \cap U_0$ induced by restriction of the continuous Riemannian metric on U_0. In particular, $d_{\mathscr{F}}(x,y)$ is defined if and only if $y \in \mathscr{F}'(x)$ and there is a C^1 path in $\mathscr{F}'(x) \cap U_0$ from x to y. Given a measurable foliation $\mathscr{F}$ and

$n \in \mathbb{Z}^d$ define

$$\left(\mathscr{F} \vee \mathscr{W}_n^{u,i}\right)(x) := \left\{ y \in M \,\middle|\, \limsup_{k\to-\infty} \frac{1}{k} \log d_{\mathscr{F}}(\alpha(kn,x),\alpha(kn,y)) \leq -\lambda_{\sigma(n)(i)}(n) \right\}$$
$$= \left\{ y \in \mathscr{F}'(x) \,\middle|\, \limsup_{k\to-\infty} \frac{1}{k} \log d_{\mathscr{F}}(\alpha(kn,x),\alpha(kn,y)) \leq -\lambda_{\sigma(n)(i)}(n) \right\}.$$

This defines a partition $\mathscr{F} \vee \mathscr{W}_n^{u,i}$ of (M,μ). Similarly define $\mathscr{F} \vee \mathscr{W}_n^{u}$.

The following proposition will be shown in the next section.

PROPOSITION 4.6. *Assume (M,μ) and $\alpha\colon \mathbb{Z}^d \times (M,\mu) \to (M,\mu)$ satisfy the standing hypotheses II with μ ergodic. Given any $n \in \mathbb{Z}^d$,*

(a) $\mathscr{W}_n^{u,i}$ is a $C^{1+\beta}$-tame, α-invariant measurable foliation with

$$T_x\left(\mathscr{W}_n^{u,i}\right)'(x) = D_0\phi_x\left(\bigoplus_{\lambda_j(n)\geq\lambda_{\sigma_n(i)}(n)} E_{\lambda_j}(x)\right)$$

for almost every x;

(b) given any $C^{1+\beta}$-tame, α-invariant measurable foliation $\mathscr{F}$, the partition $\mathscr{F} \vee \mathscr{W}_n^{i}$ is a $C^{1+\beta}$-tame, α-invariant measurable foliation with

$$T_x\left(\left(\mathscr{F} \vee \mathscr{W}_n^{u,i}\right)'(x)\right) = T_x\left(\mathscr{W}_n^{u,i}\right)'(x) \cap T_x\mathscr{F}'(x)$$

for almost every x.

REMARK 4.7. Note that the term *measurable foliation* indicates that the local transverse structure to plaques of the foliation is measurable. In general, a measurable foliation is *not* a measurable partition. In particular, the partition of (M,μ) into global unstable manifolds $\mathscr{W}_n^u$ is never a measurable partition if $\alpha(n)$ has positive metric entropy.

As a corollary of Proposition 4.6 we have the following uniqueness property.

LEMMA 4.8. *Let $n_1, n_2 \in \mathbb{Z}^d$ have the following property: for some $1 \leq i \leq$* $\min\{u(n_1), u(n_2)\}$

1. $\{\sigma(n_1)(j) : 1 \leq j \leq i\} = \{\sigma(n_2)(j) : 1 \leq j \leq i\}$,
2. $\lambda_{\sigma(n_1)(i)}(n_1) > \lambda_{\sigma(n_1)(i+1)}(n_1)$, *and*
3. $\lambda_{\sigma(n_2)(i)}(n_2) > \lambda_{\sigma(n_2)(i+1)}(n_2)$.

Then $\mathscr{W}_{n_1}^{u,i} \mathrel{\mathring{=}} \mathscr{W}_{n_2}^{u,i}$. In particular, if $\operatorname{sgn}(\lambda_i(n_1)) = \operatorname{sgn}(\lambda_i(n_2))$ *for all $\lambda_i \in \mathscr{L}$, then $\mathscr{W}_{n_1}^{u} \mathrel{\mathring{=}} \mathscr{W}_{n_2}^{u}$.*

Indeed, the lemma follows from Proposition 4.6 as for almost every x,

$$T_x\left(\mathscr{W}_{n_1}^{u,i}\right)'(x) = T_x\left(\mathscr{W}_{n_2}^{u,i}\right)'(x)$$

whence $\mathscr{W}_{n_1}^{u,i} \vee \mathscr{W}_{n_2}^{u,i} \mathrel{\mathring{=}} \mathscr{W}_{n_1}^{u,i}$.

4.3. **Coarse Lyapunov exponents, manifolds, and foliations.** We continue to assume $\alpha\colon \mathbb{Z}^d \times (M,\mu) \to (M,\mu)$ satisfies standing hypotheses II with μ ergodic. Let $\mathcal{L} = \{\lambda_i : 1 \le i \le p\}$ denote the Lyapunov exponent functionals of the derivative cocycle (13).

DEFINITION 4.9. Two Lyapunov exponents λ_i and $\lambda_j \in \mathcal{L}$ are *equivalent* if they are positively proportional; that is, if there is a $c > 0$ such that $\lambda_i = c\lambda_j$. A *coarse Lyapunov exponent* is an equivalence class in $\mathcal{L}$.

We write $\widehat{\mathcal{L}}$ for the set of coarse Lyapunov exponents. Note that for $\chi \in \widehat{\mathcal{L}}$, the sign (positive, negative, or zero) of $\chi(n)$ is well-defined.

DEFINITION 4.10. Given $\chi \in \widehat{\mathcal{L}}$ with $\chi \ne 0$, the *coarse Lyapunov foliation* corresponding to χ is

$$\mathscr{W}^\chi := \bigvee_{\{n\in\mathbb{Z}^d : \chi(n)>0\}} \mathscr{W}^u_n. \tag{14}$$

The *coarse Lyapunov manifold* corresponding to χ through x is the corresponding leaf $W^\chi(x) := (\mathscr{W}^\chi)'(x)$.

From Lemma 4.8, the intersection in (14) above is equivalent to an intersection taken over a finite subset of $\mathbb{Z}^d$. It then follows from Proposition 4.6 that $\mathscr{W}^\chi$ is a $C^{1+\beta}$-tame, α-invariant measurable foliation.

5. LYAPUNOV CHARTS; PROPERTIES OF TAME AND UNSTABLE FOLIATIONS

Let (M,μ) and $\alpha\colon \mathbb{Z}^d \times (M,\mu) \to (M,\mu)$ satisfy the standing hypotheses II of Section 3.2. To simplify notation, we assume that α acts ergodically on (M,μ). We present here a standard construction which, via a local change of coordinates, locally converts the non-uniformly partially hyperbolic dynamics into uniformly partially hyperbolic dynamics.

In Section 5.1 we fix some standing notation. In Section 5.2 we construct "standard" Lyapunov charts $\{\Phi_x\}$ relative to which the dynamics along orbits becomes uniformly partially hyperbolic in a neighborhood of the orbit (relative to our original fixed charts $\{\phi_x\}$. In Section 5.3, given a $C^{1+\beta}$-tame measurable foliation $\mathscr{F}$, we show in Proposition 5.3 that Lyapunov charts $\{\Phi_x\}$ can be chosen so that relative to these charts, the local leaves of $\mathscr{F}$ are uniformly $C^{1+\beta}$. In Section 5.4, we then construct charts $\{\Psi_x\}$ on local leaves of $\mathscr{F}$ (built from such family $\{\Phi_x\}$ as in Section 5.3) relative to which the dynamics along the local leaves of $\mathscr{F}$ becomes uniformly partially hyperbolic. In Section 5.5, we use the dynamics in such charts to construct local unstable manifolds and justify assertions made in Section 4.

5.1. **Standing notation.** Recall the Lyapunov exponent functionals $\mathcal{L} = \{\lambda_i : 1 \le i \le p\}$. For the remainder of this section, fix $F \subset \mathbb{Z}^d$ to be a finite, symmetric generating set as in standing hypotheses II of Section 3.2. We recall the sets Λ, U_0 and U, functions r,ρ and C, and all other notation from Section 3.2. All constructions below are relative to this choice of F and corresponding U,r,ρ and C.

Write

$$\lambda_0 = \max\{|\lambda_i(n)| : n \in F, 1 \le i \le p\} \tag{15}$$

and

$$\varepsilon_0 := \frac{1}{100} \min\big\{1, |\lambda_i(n)|, |\lambda_i(n) - \lambda_j(n)| : n \in F, 1 \le i, j \le p, \\ \lambda_i(n) \ne \lambda_j(n), \lambda_i(n) \ne 0\big\}. \tag{16}$$

Note that when $\lambda_i(n) > 0$, for any $0 \le \varepsilon \le \varepsilon_0$ we have

$$e^{\lambda_i(n)-2\varepsilon} \le e^{\lambda_i(n)-\varepsilon} - \varepsilon \le e^{\lambda_i(n)+\varepsilon} + \varepsilon \le e^{\lambda_i(n)+2\varepsilon};$$

if $\lambda_i(n) = 0$, then $e^{\lambda_i(n)+\varepsilon} + \varepsilon \le e^{\lambda_i(n)+2\varepsilon}$.

5.2. **Lyapunov charts.** Recall that $\Lambda_0 \subset \Lambda$ denotes the set of regular points (for the measure μ) in Theorem 2.4. We specify an alternative norm on $\mathbb{R}^k$. Fix an orthogonal (with respect to the standard inner product) decomposition $\mathbb{R}^k = \bigoplus \mathbb{R}_i$ where $\dim \mathbb{R}_i = m_i$ is the dimension of $E_{\lambda_i}(x)$ for $x \in \Lambda_0$. Define the norm $\|\cdot\|$ on $\mathbb{R}^k$ as follows: writing $v = \sum v_i$ for $v_i \in \mathbb{R}_i$ set $\|v\| = \max\{|v_i|\}$ where $|v_i|$ restricts to the norm induced by the standard inner product on each $\mathbb{R}_i$.

PROPOSITION 5.1 (Lyapunov charts)**.** *For every $0 < \varepsilon < \varepsilon_0$ there is an ε-slowly increasing function $\ell \colon \Lambda_0 \to [1, \infty)$ and a measurable family of invertible linear maps*

$$\left\{L_x \colon (\mathbb{R}^k, \|\cdot\|) \to (\mathbb{R}^k, |\cdot|) : x \in \Lambda\right\}$$

such that

(a) $L_x \mathbb{R}_i = E_{\lambda_i}(x)$;
(b) $L_x \colon \mathbb{R}^k(2, \|\cdot\|) \subset \mathbb{R}^k\left(\rho(x)^{-1}, |\cdot|\right)$ and $L_x \colon \mathbb{R}^k\left(e^{-\lambda_0-2\varepsilon}, \|\cdot\|\right) \subset \mathbb{R}^k\left(r(x)^{-1}, |\cdot|\right)$.

Define a measurable family of C^1 embeddings $\{\Phi_x : x \in M\}$ by

$$\Phi_x = \phi_x \circ L_x \colon \mathbb{R}^k(2, \|\cdot\|) \to M.$$

For each $n \in F$, write $f = \alpha(n)$. Then

(c) for each x, $\Phi_x(0) = x$ and Φ_x is diffeomorphism between $\mathbb{R}^k(2, \|\cdot\|)$ and an open subset of U;
(d) the map $\tilde f_x \colon \mathbb{R}^k\left(e^{-\lambda_0-2\varepsilon}, \|\cdot\|\right) \to \mathbb{R}^k(1, \|\cdot\|)$ given by

$$\tilde f_x(v) := \Phi^{-1}_{f(x)} \circ f \circ \Phi_x(v) = L^{-1}_{f(x)} \circ \hat f_x \circ L_x \tag{17}$$

is well-defined (where $\hat f_x$ is as in (12));
(e) $D_0 \tilde f_x \mathbb{R}_i = \mathbb{R}_i$ for every i and for $v \in \mathbb{R}_i$

$$e^{\lambda_i(n)-\varepsilon}\|v\| \le \|D_0 \tilde f_x v\| \le e^{\lambda_i(n)+\varepsilon}\|v\|;$$

(f) $\mathrm{Höl}^\beta(D\tilde f_x) \le \varepsilon$ hence $\mathrm{Lip}(\tilde f_x - D_0 \tilde f_x) \le \varepsilon$;
(g) $\|L_x\| \le 1$, $\left\|L_x^{-1}\right\| \le \ell(x)$, $\mathrm{Lip}(\Phi_x) \le 1$, and $\mathrm{Lip}(\Phi_x^{-1}) \le \ell(x)$.

We call a family of embeddings $\{\Phi_x\}$ satisfying the above properties a family of *ε-Lyapunov charts*. Given $0<\varepsilon'<\varepsilon<\epsilon_0$, a family of ε'-Lyapunov charts is automatically a family ε-Lyapunov charts. While we often restrict dynamics to smaller balls, it is convenient in Proposition 5.2 below to allow each Φ_x to have a larger domain $\mathbb{R}^k(2,\|\cdot\|)$.

As our construction is slightly different than others in the literature, we include an outline of the construction.

Proof outline of Proposition 5.1. Take $0<\varepsilon'<\varepsilon/4$ sufficiently small. Recall the ε'-Lyapunov metric $\langle\!\langle\!\langle\cdot,\cdot\rangle\!\rangle\!\rangle_{x,\varepsilon'}$ defined by (8) and the corresponding family of norms $|\!|\!|\cdot|\!|\!|_{x,\varepsilon'}$. Relative to the inner products $\langle\!\langle\!\langle\cdot,\cdot\rangle\!\rangle\!\rangle_{x,\varepsilon'}$ on $\mathbb{R}^k$, choose a measurable orthonormal basis for each $E_{\lambda_i}(x)$ which, in turn, defines a measurable family of linear isometries $\hat\tau_x\colon \left(\mathbb{R}^k,|\!|\!|\cdot|\!|\!|_{x,\varepsilon'}\right)\to\left(\mathbb{R}^k,\|\cdot\|\right)$ with $\hat\tau_x E_{\lambda_i}(x)=\mathbb{R}_i$ for every i.

Let k_0 be the constant and $L(x)$ the ε'-slowly increasing function in Lemma 2.8. Let $\hat C(x)$ be an ε'-slowly increasing function such that the functions $D(x)$, $C(x)$, $r(x)$, $\rho(x)$ appearing in our standing hypotheses of Section 3.2 are bounded above by $\hat C(x)$ for all $x\in\Lambda_0$. Take $\tau_x\colon(\mathbb{R}^k,\|\cdot\|)\to(\mathbb{R}^k,|\cdot|)$ defined by $\tau_x(v)=\hat\tau_x^{-1}(v)$. We have

$$\|\tau_x\|\le k_0,\qquad \|\tau_x^{-1}\|\le L(x).$$

Take

$$\hat\ell(x)=\left((\varepsilon/2)^{-1}k_0^{1+\beta}L(x)\hat C(x)\right)^{\frac{1}{\beta}}$$

and take $L_x\colon\left(\mathbb{R}^k,\|\cdot\|\right)\to\left(\mathbb{R}^k,|\cdot|\right)$ to be the linear map defined by

$$L_x(v)=\tau_x\left(\hat\ell^{-1}(x)v\right).$$

Having taken ε' sufficiently small, $\hat\ell$ is $(\varepsilon/2)$-slowly increasing and we verify properties (a), (b), and (c).

Given $n\in F$ write $f=\alpha(n)$. Given $x\in\Lambda_0$, write $\hat f_x$ as in (12). Consider first $\overline{f}_x\colon\mathbb{R}^k\left(r(x)^{-1}k_0^{-1},\|\cdot\|\right)\to\left(\mathbb{R}^k,\|\cdot\|\right)$ given by $\overline{f}_x=\tau_{f(x)}^{-1}\circ\hat f_x\circ\tau_x$. We have

$$D_v\overline{f}_x(\xi)=\tau_{f(x)}^{-1}\left(D_{\tau_x(v)}\hat f_x(\tau_x(\xi))\right)$$

hence for ξ with $\|\xi\|=1$

$$\begin{aligned}\left\|D_v\overline{f}_x(\xi)-D_u\overline{f}_x(\xi)\right\|&\le k_0L(x)\,\mathrm{H\ddot ol}^\beta(Df_x)\,|\tau_x(v)-\tau_x(u)|^\beta\\&\le k_0L(x)\hat C(x)k_0^\beta\|u-v\|^\beta\end{aligned}$$

and

$$\mathrm{H\ddot ol}^\beta\left(D\overline{f}_x\right)\le k_0L(x)\hat C(x)k_0^\beta\le\hat\ell(x)^\beta\varepsilon/2.$$

Moreover for $v\in\mathbb{R}_i$ we have

$$e^{\lambda_i(n)-\varepsilon'}\|v\|\le\|D_0\overline{f}_xv\|\le e^{\lambda_i(n)+\varepsilon'}\|v\|.$$

By (b), $\tilde f_x\colon\mathbb{R}^k(e^{-\lambda_0-2\varepsilon},\|\cdot\|)\to\mathbb{R}^k$ defined by

$$\tilde f_x(v)=\hat\ell(f(x))\overline{f}_x\left(\hat\ell(x)^{-1}v\right)$$

is well-defined. As $e^{\lambda_0+\varepsilon}+\varepsilon \le e^{\lambda_0+2\varepsilon}$, the Lipschitz constant of $\tilde{f}_x$ is bounded above by $e^{\lambda_0+2\varepsilon}$ and (d) follows. We have

$$D_v\tilde{f}_x(\xi) = \hat{\ell}(f(x))\hat{\ell}(x)^{-1}\left(D_{\hat{\ell}(x)^{-1}v}\overline{f}_x(\xi)\right)$$

hence

$$\begin{aligned}\left\|D_v\tilde{f}_x(\xi)-D_u\tilde{f}_x(\xi)\right\| &\le \hat{\ell}(f(x))\hat{\ell}(x)^{-1}\,\mathrm{H\ddot{o}l}^\beta\left(D\overline{f}_x\right)\left\|\hat{\ell}(x)^{-1}v-\hat{\ell}(x)^{-1}u\right\|^\beta\\ &\le \hat{\ell}(f(x))\hat{\ell}(x)^{-1}\varepsilon/2\,\|u-v\|^\beta \le \varepsilon\,\|u-v\|^\beta\end{aligned}$$

whence (f) follows. Also,

$$e^{-\varepsilon/2}\left\|D_0\overline{f}_x(v)\right\| \le \left\|D_0\tilde{f}_x(v)\right\| = \hat{\ell}(f(x))\hat{\ell}(x)^{-1}\left\|D_0\overline{f}_x(v)\right\| \le e^{\varepsilon/2}\left\|D_0\overline{f}_x(v)\right\|$$

and (e) follows.

Finally, with $\ell(x) = k_0L(x)\hat{C}(x)\hat{\ell}(x)$, we have that $\ell(x)$ is ε-slowly increasing and satisfies the bounds in (g). □

5.3. **Lyapunov charts adapted to $C^{1+\beta}$-tame foliations.** Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame measurable foliation. The following proposition guarantees that Lyapunov charts above may be chosen so that, relative to the charts Φ_x, the local leaves of $\mathscr{F}$ are uniformly $C^{1+\beta}$ embedded.

PROPOSITION 5.2. *An α-invariant measurable foliation $\mathscr{F}$ of (M,μ) is $C^{1+\beta}$-tame if and only if for every $0<\varepsilon<\varepsilon_0$, the charts Φ_x in Proposition 5.1 can be chosen so that, in addition to the properties in Proposition 5.1, there are*

- *a set $\Lambda' \subset \Lambda_0$ with $\mu(\Lambda') = 1$;*
- *a subspace $V \subset \mathbb{R}^k$ with orthogonal complement W;*
- *a measurable family of $C^{1+\beta}$ functions*

$$\tilde{h}^{\mathscr{F}}_x : V(1,\|\cdot\|) \to W(1,\|\cdot\|)$$

defined for $x \in \Lambda'$

such that, writing $\tilde{\mathscr{F}}_x$ for the path component (relative to the immersed topology) of $\Phi_x^{-1}(\mathscr{F}(x)') \cap \mathbb{R}^k(2)$ containing 0, $\tilde{\mathscr{F}}_x$ contains $\operatorname{graph}\left(\tilde{h}^{\mathscr{F}}_x\right)$ as an open submanifold. Moreover

(i) $\tilde{h}^{\mathscr{F}}_x(0) = 0$; $D_0\tilde{h}^{\mathscr{F}}_x = 0$;
(j) $\mathrm{H\ddot{o}l}^\beta\left(D\tilde{h}^{\mathscr{F}}_x\right) \le \epsilon$ and hence $\|D\tilde{h}^{\mathscr{F}}_x\| \le \epsilon$;
(k) for all $n \in F$, writing $\tilde{f}_x$ as in (17), $\tilde{f}_x$ is a diffeomorphism between

$$\operatorname{graph}\left(\tilde{h}^{\mathscr{F}}_x\right) \cap \mathbb{R}^n\left(e^{-\lambda_0-2\varepsilon}\right)$$

and an open subset of $\operatorname{graph}\left(\tilde{h}^{\mathscr{F}}_{\alpha(n)(x)}\right)$.

We note for each $n \in F$, writing $f = \alpha(n)$ and $\tilde{f}_x$ as in (17), we have that V is $D_0\tilde{f}_x$-invariant.

The proof of Proposition 5.2 relies on the following lemma which, in turn, follows from the Implicit Function Theorem with Hölder estimates, [40, Lemma 2.1.1] or [2, Lemma 7.5.2]. Consider $\mathbb{R}^k$ equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Let $V^1, V^2 \subset \mathbb{R}^k$ be subspaces and for $j \in \{1,2\}$, let W^j be a subspace of complementary dimension transverse to V^j. We assume the decompositions $V^j \oplus W^j$ and norms $\|\cdot\|_j$ have the property that, for any vector $u \in \mathbb{R}^k$, writing $u = v + w$ for $v \in V^j$ and $w \in W^j$ we have

$$\|u\|_j \geq \max\left\{\|v\|_j, \|w\|_j\right\}. \tag{18}$$

Our applications below of the above setup are: $\|\cdot\|_j$ is the Euclidean norm and W^j is the orthogonal complement of V^j; or $\|\cdot\|_j$ is the norm on $\mathbb{R}^k$ specified in Section 5.2, V^j satisfies $V^j = \bigoplus_i (V \cap \mathbb{R}_i)$, and W^j is the orthogonal complement of V^j.

With the above setup we have the following.

LEMMA 5.3. *For $0 < r < 1$, let $h\colon V^1(r, \|\cdot\|_1) \to W^1$ be a $C^{1+\beta}$ function such that (relative to $\|\cdot\|_1$) we have*

- $h(0) = 0$ *and* $D_0 h = 0$;
- $\mathrm{H\ddot{o}l}^\beta(Dh) \leq a$;
- $\|Dh\| \leq 1$.

Let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map with $L(V^1) = V^2$. Take

- $a_0 = a\|L^{-1}\|^{1+\beta}$;
- $b_0 = 2\|L^{-1}\|$;
- $c_0 = \|L\|$.

Then with

$$r_0 = \min\left\{\frac{r/2}{\|L^{-1}\|}, \frac{r/2}{2b_0c_0\|L^{-1}\|}, \frac{1}{(1+2b_0c_0)(2a_0c_0)^{1/\beta}}, \frac{1}{\left((1+b_0c_0)^2(8a_0c_0)\right)^{1/\beta}}\right\}$$

we have that $L(\mathrm{graph}(h))$ contains as an open set the graph of a $C^{1+\beta}$-function $\hat{h}\colon V^2(r_0) \to W^2$ such that (relative to $\|\cdot\|_2$) we have

1. $\hat{h}(0) = 0$ *and* $D_0\hat{h} = 0$;
2. $\mathrm{H\ddot{o}l}^\beta\left(D\hat{h}\right) \leq 8a_0c_0(1+b_0c_0)^2$;

whence

3. $\|D\hat{h}\| \leq 1$.

Proof. We have that $\mathrm{graph}(h)$ in $(\mathbb{R}^k, \|\cdot\|_1)$ is the solution set to $\psi(v, w_1) \equiv 0$ where

$$\psi(v_1, w_1) = w_1 - h(v_1) \in W_1$$

is defined for all (v_1, w_1) with $\|v_1\|_1 < r$. Then $L(\mathrm{graph}(h))$ contains as an open set the solution set $\overline{\psi}(v_2, w_2) \equiv 0$ of the function

$$\overline{\psi}\colon V^2\left(\frac{r}{2\|L^{-1}\|}, \|\cdot\|_2\right) \times W^2\left(\frac{r}{2\|L^{-1}\|}, \|\cdot\|_2\right) \to W^1$$

given by

$$\overline{\psi}(v_2, w_2) = \psi \circ L^{-1}(v_2, w_2).$$

Decomposing the domains $\mathbb{R}^k = V^2 \oplus W^2$ and $\mathbb{R}^k = V^1 \oplus W^1$ of L^{-1} and ψ, respectively, we have

$$D_{(v,w_2)}\overline{\psi}(\xi) = \left[\begin{array}{c|c} -D_{L^{-1}v}h & I \end{array}\right] L^{-1}(\xi).$$

We check the following.

1. The partial derivative $D_{2,(0,0)}\overline{\psi}\colon W^2 \to W^1$ is of the form
$$D_{2,(0,0)}\overline{\psi}(\xi) = \left[\begin{array}{c|c} 0 & I \end{array}\right] L^{-1}(\xi)$$
and is a linear isomorphism with
$$\left\| \left(D_{2,(0,0)}\overline{\psi}\right)^{-1} \right\| \le \|L\|.$$
Indeed, given $w_2 \in W^2$ write $L^{-1}(w_2) = v_1 + w_1$. Then $D_{2,(0,0)}\overline{\psi}(w_2) = w_1$, and writing $L(w_1) = w_2 - L(v_1)$ we have from (18) that
$$\|w_2\|_2 \le \|Lw_1\|_2 \le \|L\|\|w_1\|_1.$$
2. The partial derivative $D_{1,(v,0)}\overline{\psi}\colon V^2 \to W^1$ is of the form
$$D_{1,(v,0)}\overline{\psi}(\xi) = \left[\begin{array}{c|c} -D_{L^{-1}v}h & I \end{array}\right] L^{-1}(\xi)$$
hence
$$\max \|D_{1,(v,0)}\overline{\psi}\| \le \|L^{-1}\|.$$
3. $D_{(v,w_2)}\overline{\psi}(\xi) - D_{(\hat v,\hat w_2)}\overline{\psi}(\xi) = \left[\begin{array}{c|c} -D_{L^{-1}v}h + D_{L^{-1}\hat v}h & 0 \end{array}\right] L^{-1}(\xi)$, so
$$\text{Höl}^\beta\left(D\overline{\psi}\right) \le \|L^{-1}\|^{1+\beta}a.$$

The conclusion of the lemma then follows from [40, Lemma 2.1.1] and the fact that $L(\text{graph}(h))$ is tangent to V^2. □

Proof of Proposition 5.2. First, suppose that $\mathscr{F}$ is uniformly $C^{1+\beta}$ relative to Lyapunov charts as in Proposition 5.2. We may apply Lemma 5.3 with the linear map $L = L_x\colon (\mathbb{R}^k, \|\cdot\|) \to (\mathbb{R}^k, |\cdot|)$ the maps guaranteed from Proposition 5.1 and deduce that $\mathscr{F}$ is $C^{1+\beta}$-tame.

Suppose now that $\mathscr{F}$ is an α-invariant $C^{1+\beta}$-tame, measurable foliation. We retain all notation from Definitions 4.3 and 4.4. In particular, we assume for $n \in F$ that $\alpha(n)\left(\mathscr{F}'(x)\cap U\right) \subset \mathscr{F}'(\alpha(n,x))$ for almost all x. Take $\Lambda' = \Lambda_0 \cap \Lambda_{\mathscr{F}}$. Note that the α-invariance of $\mathscr{F}$ implies that $T_0\widehat{\mathscr{F}}(x) = \bigoplus\left(E_{\lambda_i} \cap T_0\widehat{\mathscr{F}}(x)\right)$ for almost every x (where $\widehat{\mathscr{F}}(x)$ is as in Definition 4.4.) In the proof of Proposition 5.1 we may select $V \subset \mathbb{R}^k$ and construct the maps $\tau_x\colon \left(\mathbb{R}^k, \|\cdot\|\right) \to \left(\mathbb{R}^k, |\cdot|\right)$ so that $\tau_x(V) = T_0\widehat{\mathscr{F}}(x)$ for all $x \in \Lambda'$.

Taking $\varepsilon' > 0$ sufficiently small in the proof of Proposition 5.1 and an ε'-slowly increasing function $\ell_{\mathscr{F}}$ as in Definition 4.4, applying Lemma 5.3 to the maps $\tau_x^{-1}\colon \left(\mathbb{R}^k, |\cdot|\right) \to \left(\mathbb{R}^k, \|\cdot\|\right)$ appearing the proof of Proposition 5.1 we may find an $(\varepsilon/2)$-slowly increasing function $\overline{\ell}(x)$ so that
$$\tau_x^{-1}\left(\widehat{\mathscr{F}}(x)\right) \cap \mathbb{R}^k\left(2\overline{\ell}(x)^{-1}, \|\cdot\|\right)$$
is the graph of a $C^{1+\beta}$ function
$$\hat h_x\colon \hat U_x \to V^\perp\left(2\overline{\ell}(x)^{-1}, \|\cdot\|\right),$$
where

1. $V\left(\overline{\ell}(x)^{-1}, \|\cdot\|\right) \subset \hat U_x \subset V$;

2. $\hat{h}_x(0) = 0$ and $D_0 \hat{h}_x(0) = 0$;
3. $\text{Höl}^\beta \left(D\hat{h}_x\right) \le \varepsilon \overline{\ell}(x)^\beta$.

Then taking

$$\hat{\ell}(x) = \max\left\{\overline{\ell}(x), \left((\varepsilon/2)^{-1} k_0^{1+\beta} L(x)^p \hat{C}(x)^2\right)^{\frac{1}{\beta}}\right\}$$

in the proof of Proposition 5.1, the results of Proposition 5.1 remain valid.

Let $\tilde{h}_x \colon V(1, \|\cdot\|) \to V^\perp(1, \|\cdot\|)$ be given by

$$\tilde{h}_x(v) = \hat{\ell}(x)\hat{h}_x\left(\hat{\ell}(x)^{-1} v\right).$$

Then the set

$$L_x^{-1}\left(\widehat{\mathscr{F}}(x)\right) \cap \mathbb{R}^k(2, \|\cdot\|)$$

contains the graph of a $C^{1+\beta}$ function

$$\tilde{h}_x \colon V(1, \|\cdot\|) \to V^\perp(1, \|\cdot\|),$$

where

1. $\tilde{h}_x(0) = 0$ and $D_0 \tilde{h}_x(0) = 0$;
2. $\text{Höl}^\beta \left(D\tilde{h}_x\right) \le \varepsilon$ whence $\|D\tilde{h}_x\| \le \varepsilon$.

The proposition then follows. □

5.4. **Lyapunov charts adapted to dynamics restricted to leaves.** Note that if $\mathscr{F}$ is an α-invariant, $C^{1+\beta}$-tame, measurable foliation, then, with V as in the notation of Proposition 5.2, we have

$$V = \bigoplus_{i=1}^{p} (V \cap \mathbb{R}_i).$$

As in Propositions 5.1 and 5.2, given an α-invariant, $C^{1+\beta}$-tame, measurable foliation $\mathscr{F}$ we construct a family of charts for the restriction of the dynamics to leaves of $\mathscr{F}$. Given $0 < \varepsilon' < \varepsilon_0$ sufficiently small, let $\{\Phi_x\}$ a family of ε'-Lyapunov charts satisfying Proposition 5.2. Take Λ' and $\tilde{h}_x^{\mathscr{F}}(v)$ as in Proposition 5.2. Given $x \in \Lambda'$ define

$$\tilde{H}_x^{\mathscr{F}} \colon \mathbb{R}^k(1, \|\cdot\|) \to \mathbb{R}^k(\|\cdot\|)$$

relative to the orthogonal decomposition $\mathbb{R}^k = V \oplus W$ by

$$\tilde{H}_x^{\mathscr{F}}(v, w) = \left(v, w + \tilde{h}_x^{\mathscr{F}}(v)\right).$$

We then define a measurable family of embeddings $\Psi_x^{\mathscr{F}} \colon V(1, \|\cdot\|) \to \mathscr{F}(x)$ by

$$\Psi_x^{\mathscr{F}}(v) = \Phi_x \circ \tilde{H}_x^{\mathscr{F}}(v).$$

PROPOSITION 5.4. *For every $0 < \varepsilon < \varepsilon_0$ there is a family of ε-Lyapunov charts $\{\Phi_x : x \in \Lambda_0\}$, an ε-slowly increasing function $\overline{\ell} \colon \Lambda' \to [1, \infty)$, and a measurable family of C^1 embeddings $\{\Psi_x^{\mathscr{F}} : x \in \Lambda'\}$ defined as above with*

(a) $\Psi_x^{\mathscr{F}}(0) = x$ and $\Psi_x^{\mathscr{F}}(V(1, \|\cdot\|)) \subset \mathscr{F}'(x)$ for almost every x.

Furthermore, for every $n \in F$, writing $f = \alpha(n)$ we have

(b) *the function* $\overline{f}_x\colon V\left(e^{-\lambda_0-2\varepsilon},\|\cdot\|\right)\to V(1,\|\cdot\|)$ *given by*

$$\overline{f}_x(v) := \left(\Psi^{\mathscr{F}}_{f(x)}\right)^{-1}\circ f\circ \Psi^{\mathscr{F}}_x(v)$$

is well-defined;

(c) $D_0\overline{f}_x(\mathbb{R}_i\cap V) = (\mathbb{R}_i\cap V)$ *for every* i *and for* $v\in(\mathbb{R}_i\cap V)$

$$e^{\lambda_i(n)-\varepsilon}\|v\|\le \|D_0\overline{f}_x v\|\le e^{\lambda_i(n)+\varepsilon}\|v\|;$$

(d) $\text{Höl}^\beta\left(D\overline{f}_x\right)\le\varepsilon$ *hence* $\text{Lip}(\overline{f}_x - D_0\overline{f}_x)\le\varepsilon$*;*

(e) $\text{Lip}(\Psi^{\mathscr{F}}_x)\le 1$ *and* $\text{Lip}((\Psi^{\mathscr{F}}_x)^{-1})\le\overline{\ell}(x)^{-1}$.

Proof. For $0<\varepsilon'<\varepsilon_0$ sufficiently small, let $\{\Phi_x\}$ be a family of ε'-Lyapunov charts satisfying Proposition 5.2. Then with $\tilde H^{\mathscr{F}}_x$ as above we have

1. $\|D\tilde H^{\mathscr{F}}_x\|\le 1+\varepsilon'$ and $\|D(\tilde H^{\mathscr{F}}_x)^{-1}\|\le 1+\varepsilon'$;
2. $D_0(\tilde H^{\mathscr{F}}_x)=\text{Id}$;
3. $\text{Höl}^\beta\left(D\tilde H^{\mathscr{F}}_x\right)\le\varepsilon'$ and $\text{Höl}^\beta\left(D\left(\tilde H^{\mathscr{F}}_x\right)^{-1}\right)\le\varepsilon'$.

Given $n\in F$, write $f=\alpha(n)$ and define $\tilde f_x$ as in (17). Define

$$\overline{F}_x\colon \mathbb{R}^k\left(\frac{1}{(1+\varepsilon')^2}e^{-\lambda_0-2\varepsilon'},\|\cdot\|\right)\to\mathbb{R}^k(1,\|\cdot\|)$$

by

$$\overline{F}_x=\left(\tilde H^{\mathscr{F}}_{f(x)}\right)^{-1}\circ\tilde f_x\circ\tilde H^{\mathscr{F}}_x.$$

Note $\overline{F}_x$ is well-defined.

We have

1. $D_0\overline{F}_x = D_0\tilde f_x$;

and check that

2. $\text{Höl}^\beta\left(D\overline{F}_x\right)\le\left(1+\varepsilon'\right)e^{\lambda_0+2\varepsilon'}\varepsilon'+\left(1+\varepsilon'\right)^{2+\beta}\varepsilon'+\left(\left(1+\varepsilon'\right)e^{\lambda_0+2\varepsilon'}\right)^{1+\beta}\varepsilon'$.

Having taken $0<\varepsilon'<\varepsilon$ sufficiently small, the charts $\{\Phi_x\}$ are a family of ε-charts, and we can ensure $\overline{f}_x=\overline{F}_x\restriction_{V(e^{-\lambda_0-2\varepsilon})}$ is well defined and has the desired properties. Moreover we can construct a function $\overline{\ell}$ with the desired properties. □

5.5. **Local unstable manifolds and Proof of Proposition 4.6.** Relative to either the charts Φ_x in Proposition 5.1 or $\Psi^{\mathscr{F}}_x$ in Proposition 5.4 we may perform either the Perron–Irwin method or Hadamard graph transform method to construct (un)stable manifolds. See for instance [20], [19], or [2] for more details.

Fix $n\in\mathbb{Z}^d$. Let $F\subset\mathbb{Z}^d$ be a finite, symmetric, generating set containing n. Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame measurable foliation. Let U be as in Section 3.2 and Definition 4.3 and take ε_0 and λ_0 as in (15) and (16). Let $f=\alpha(n)$ and fix and $0<\varepsilon<\varepsilon_0$. Take $\Lambda=\Lambda_0$ or Λ', $E=\mathbb{R}^k$ or V, and $f_x=\tilde f_x$ or $f_x=\overline{f}_x$, respectively, with the notation of either Proposition 5.1 or Proposition 5.4.

Let $E^i=\mathbb{R}_i\cap E$. Fix $1\le k\le p$ such that $\lambda_k(n)>0$ and let

$$E_{\ge k}:=\bigoplus_{\lambda_j(n)\ge\lambda_k(n)}E^j\quad\text{and}\quad E_{<k}:=\bigoplus_{\lambda_j(n)<\lambda_k(n)}E^j.$$

LEMMA 5.5. *Suppose $\lambda_k(n) > 0$. Then for every $x \in \Lambda$ there is a $C^{1+\beta}$ function*

$$h_x \colon E_{\geq k}\left(e^{-\lambda_0-2\varepsilon}, \|\cdot\|\right) \to E_{<k}(1, \|\cdot\|)$$

with

(a) $h_x(0) = 0$, *and* $D_0 h_x(0) = 0$;
(b) $\mathrm{H\ddot{o}l}^\beta(Dh_x) < c$ *for some* c *independent of* x;
(c) $\|Dh_x\| \leq 1/3$;
(d) *for* $\delta \leq e^{-\lambda_0-2\varepsilon}$, *writing* $W_{x,\delta} := \mathrm{graph}\left(h_x\restriction_{E_{\geq k}(\delta,\|\cdot\|)}\right)$ *we have*

$$W_{f(x),\delta} \subset f_x(W_{x,\delta});$$

(e) *if* $u, v \in W_{x,\delta}$, *then*

$$e^{\lambda_k(n)-2\varepsilon}\|v-u\| \leq \|f_x(v) - f_x(u)\| \leq e^{\lambda_0+2\varepsilon}\|v-u\|.$$

Moreover, the family $\{h_x\}$ *depends measurably on* $x \in \Lambda$. *Write* $f_x^{-j} = f^{-1}_{f^{-j}(x)} \circ \cdots \circ f^{-1}_{f^{-1}(x)}$ *where defined. Then*

(f) *for* $u \in W_{x,\delta}$ *and* $n \geq 0$, $f_x^{-n}(u)$ *is defined and*

$$W_{x,\delta} := \left\{u \in E(\delta, \|\cdot\|) : \limsup_{j\to\infty} \frac{1}{j}\log\left\|f_x^{-j}(u)\right\| \leq -\lambda_k(n) + 10\varepsilon\right\}. \tag{19}$$

Continue to write $f = \alpha(n)$. Let $V_{\mathrm{loc},x,\varepsilon}$ be the image of

$$W_{x,e^{-2\lambda_0-4\varepsilon}}$$

in Lemma 5.5 under either Φ_x or $\Psi_x^{\mathscr{F}}$. With $f = \alpha(n)$ we still have

$$f(V_{\mathrm{loc},x,\varepsilon}) \supset V_{\mathrm{loc},f(x),\varepsilon}.$$

Moreover, for $m \in F$ and a.e. x we have

1. $\alpha(m)(V_{\mathrm{loc},x,\varepsilon})$ is contained in the image of $W_{\alpha(m)(x),e^{-\lambda_0-2\varepsilon}}$ under either $\Phi_{\alpha(m,x)}$ or $\Psi^{\mathscr{F}}_{\alpha(m,x)}$;
2. $f^{-j}(\alpha(m)(V_{\mathrm{loc},x,\varepsilon})) \subset U$ for all $j \geq 0$;
3. for $y \in V_{\mathrm{loc},x,\varepsilon}$ and $j \geq 0$,

$$d\left(f^{-j}(\alpha(m,y)), f^{-j}(\alpha(m,x))\right) \leq e^{-j(\lambda_k(n)+2\varepsilon)}.$$

Proof of Proposition 4.6. Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame measurable foliation. In the case of (a) of Proposition 4.6, take $\mathscr{F} = \{M\}$. Recall we write $f = \alpha(n)$ for our distinguished n. Also recall we fix a finite symmetric generating set F containing n and set $U \subset U_0$ with $\alpha(m) \colon U \to U_0$ a diffeomorphism onto its image for each $m \in F$. Take $U \subset \hat{U} \subset U_0$ open such that $f\restriction_{\hat{U}} \to U_0$ is a diffeomorphism onto its image. We may then replace U in the hypotheses of Section 3.2 with a smaller open set so that

$$\alpha(m)(U) \subset \hat{U}$$

for all $m \in F$.

Recall our permutation $\sigma(n)$ such that

$$\lambda_{\sigma(n)(1)}(n) \geq \lambda_{\sigma(n)(2)}(n) \geq \cdots \geq \lambda_{\sigma(n)(u(n))}(n) > 0 \geq \cdots \geq \lambda_{\sigma(n)(p)}(n).$$

For distinguished i in Proposition 4.6, fix $1 \le k \le p$ with $\sigma(n)(i) = k$. Take Λ as in Lemma 5.5. Let $V_{\mathrm{loc},x,\varepsilon}$ be as constructed from Lemma 5.5 above.

For $x \in \Lambda$ take

$$\begin{aligned} V_x := & \bigcup_{j\ge 0} (f\restriction_{\hat U})^j \left(V_{\mathrm{loc},f^{-j}(x),\varepsilon}\right) \\ = & \left\{y : f^{-j}(y) \in \hat U \text{ for all } j \ge 0 \text{ and } f^{-j}(y) \in V_{\mathrm{loc},f^{-j}(x),\varepsilon} \text{ for some } j \ge 0\right\}. \end{aligned}$$

In particular, $V_x \subset \hat U$.

We have

CLAIM 5.6. *For $x, y \in \Lambda$ and $m \in F$*

1. *if $y \in V_x \cap U$, then $\alpha(m)(y) \in V_{\alpha(m)(x)}$;*
2. *if $y \in V_x$, then*
$$\limsup_{j\to\infty} \frac{1}{n} \log d\left(f^{-j}(y), f^{-j}(x)\right) \le -\lambda_{\sigma(n)(i)}(n) = -\lambda_k(n);$$
3. *if $y \notin V_x$, then $V_x \cap V_y = \varnothing$;*
4. *if $y \in V_x$, then $V_x = V_y$.*

Take $B(n) = M \smallsetminus (\bigcup_{x\in\Lambda} V_x)$. From the above discussion, for $x \in \Lambda$ we have

1. V_x is a C^1 injectively immersed manifold and is defined independently of ε;
2. $\left(\mathscr{F} \vee \mathscr{W}_n^{u,i}\right)(x) \smallsetminus B(n) = V_x$.

In particular,

3. $\mathscr{F} \vee \mathscr{W}_n^{u,i}$ is an α-invariant, $C^{1+\beta}$-tame, measurable foliation.

The proposition follows. □

REMARK 5.7. Given an α-invariant, $C^{1+\beta}$-tame, measurable foliation $\mathscr{F}$ and sufficiently small $\varepsilon > 0$, let Φ_x be a family of ε-charts; let $\Psi_x^{\mathscr{F}}$ be the family constructed from Φ_x in Proposition 5.4.

Let h_x^u be as in Lemma 5.5 for the charts Φ_x and $i = u(n)$. Let $\tilde h_x^{\mathscr{F}}$ be as in Proposition 5.2 and let $h_x^{\mathscr{F},u}$ be as in Lemma 5.5 for the charts Ψ_x with $i = u(n)$. Let $W_{x,\delta}^u := \operatorname{graph}\left(h_x^u\restriction_{\mathbb{R}_{\le r}(\delta,\|\cdot\|)}\right)$ and let $\tilde{\mathscr{F}}_x$ be the graph of $\tilde h_x^{\mathscr{F}}$.

Let $W_x^{\mathscr{F},u}$ be the graph of $\tilde h_x^{\mathscr{F}} \circ h_x^{\mathscr{F},u}$ and let $W_x^{\mathscr{F},u}(\delta) = W_x^{\mathscr{F},u} \cap \mathbb{R}^k(\delta)$.

It follows from the characterization (19), the Lipschitzness of H_x in the proof of Proposition 5.4, and the local dynamics and invariance of manifolds in charts that for all $\delta < e^{-\lambda_0-2\varepsilon}$, $W_x^{\mathscr{F},u}(\delta) = \tilde{\mathscr{F}}_x \cap W_{x,\delta}^u$.

Part II. Entropy formulas for rank-1 systems by Aaron Brown

In this part, we extend the main result of [31] and the entropy formulas from [32] to the setting of diffeomorphisms satisfying our standing hypotheses. As a corollary of the proof, we obtain the finiteness of entropy for systems satisfying the standing hypotheses II of Section 3.2. Although we provide most details here, the arguments in this section are heavily adapted from the original papers [31, 32]

6. DEFINITIONS AND FACTS ABOUT METRIC ENTROPY

Before stating our main results in Section 7, we recall some standard definitions and facts about the metric entropy of measure-preserving transformations that will be used throughout.

For this section, let (X,μ) be a standard probability space. Given a measurable partition ξ of (X,μ), we indicate by $\left\{\mu_x^\xi\right\}$ a family of conditional probability measures relative to the partition ξ. In particular, the assignment $x\mapsto\mu_x^\xi$ is $\sigma(\xi)$-measurable (where $\sigma(\xi)$ denotes the σ-algebra of ξ-saturated sets) and, given a measurable $A\subset X$, $\mu(A)=\int\mu_x^\xi(A)\;d\mu(x)$.

6.1. **Metric entropy of an invertible transformation.**

Given measurable partitions η,ξ of (X,μ), the *conditional information* of η relative to ξ is $I_\mu(\eta\mid\xi)(x)=-\log\left(\mu_x^\xi(\eta(x))\right)$ and the *conditional entropy* of η relative to ξ is $H_\mu(\eta\mid\xi)=\int I_\mu(\eta\mid\xi)(x)\;d\mu(x)$. The *entropy of* η is $H_\mu(\eta)=H_\mu(\eta\mid\{\varnothing,X\})$. If $H_\mu(\eta)<\infty$, then η is countable and $H_\mu(\eta)=-\sum_{C\in\eta}\mu(C)\log\mu(C)$.

6.1.1. *Entropy of an invertible transformation.* Let $f\colon(X,\mu)\to(X,\mu)$ be an invertible, measurable, measure-preserving transformation. Let η be a measurable partition of (X,μ). We define

$$\eta^+ := \bigvee_{i=0}^{\infty} f^i\eta, \qquad \eta^f := \bigvee_{i\in\mathbb{Z}} f^i\eta.$$

We define the ("unstable" or "future[3]") *entropy of f given the partition* η to be

$$h_\mu(f,\eta) := H_\mu\left(\eta\mid f\eta^+\right) = H_\mu\left(\eta^+\mid f\eta^+\right) = H_\mu\left(f^{-1}\eta^+\mid\eta^+\right).$$

We define the *metric entropy of f* relative to μ to be

$$h_\mu(f)=\sup\{h_\mu(f,\eta)\}$$

w here the supremum is taken over all measurable partitions of (X,μ).

The quantity $h_\mu(f)$ is the main object studied in [31, 32] in the setting of C^2 diffeomorphisms of compact manifolds. In addition to relaxing the C^2 regularity and compactness, we will need analogues of the main results of [31, 32] for related quantities, the entropy subordinated to a measurable partition or an invariant foliation, which we define below.

[3] It is perhaps more standard to define the entropy $h_\mu(f,\eta)$ as $H_\mu\left(\eta\mid f^{-1}\eta^-\right)$ as in standard references such as [44]. Note that we typically expect asymmetry of these definitions: $H_\mu\left(\eta\mid f\eta^+\right)\neq H_\mu\left(\eta\mid f^{-1}\eta^-\right)$. However, if η satisfies $H_\mu(\eta)<\infty$, then the symmetry of the two definitions holds. We choose to define $h_\mu(f,\eta)=H_\mu\left(\eta\mid f\eta^+\right)$ to have results most consistent with statements in [31, 32] and related work.

6.1.2. *Entropy subordinate to a partition.* Given two partitions ξ and η write $\eta \prec \xi$ if ξ refines η.

DEFINITION 6.1. Given a measurable partition η of (M,μ) we define the *entropy of f subordinate to* η to be the quantity

$$h_\mu(f \mid \eta) := \sup\left\{h_\mu(f,\xi) : \eta \prec \xi\right\} = \sup\left\{h_\mu(f,\eta \vee \zeta)\right\}.$$

In general, $h_\mu(f \mid \eta) \neq h_\mu\left(f^{-1} \mid \eta\right)$. However, if η is an f-invariant partition, then

$$h_\mu(f \mid \eta) = h_\mu\left(f^{-1} \mid \eta\right).$$

REMARK 6.2. When $f\eta = \eta$ or $f\eta \prec \eta$, then the above definition coincides with the usual definition of entropy of a transformation conditioned on an invariant partition or invariant σ-algebra. See for example [26, Definition II.1.3].

6.1.3. *Entropy subordinate to a measurable foliation.* We now take $X = M$ to be a C^∞ manifold equipped with a Borel probability measure μ. Consider a measurable foliation $\mathscr{F}$ of M (see Definition 4.1.) Note that the partition into leaves of $\mathscr{F}$ is generally not a measurable partition of (M,μ).

DEFINITION 6.3. A measurable partition ξ of (M,μ) is *subordinate to* $\mathscr{F}$ if for a.e. $x \in M$

1. $\xi(x) \subset \mathscr{F}(x)$;
2. $\xi(x)$ contains an open neighborhood (in the immersed topology) of x in $\mathscr{F}'(x)$ (where $\mathscr{F}'(x)$ is as in Section 4.1).

DEFINITION 6.4. Let $\mathscr{F}$ be an f-invariant, measurable foliation of (M,μ). We define the *entropy of f subordinate to* $\mathscr{F}$ to be

$$h_\mu(f \mid \mathscr{F}) := \sup\left\{h_\mu(f \mid \xi) : \xi \text{ is subordinate to } \mathscr{F}\right\}.$$

More generally, if η is any measurable partition of (M,μ) we define

$$h_\mu(f \mid \eta \vee \mathscr{F}) := \sup\left\{h_\mu(f \mid \xi \vee \eta) : \xi \text{ is subordinate to } \mathscr{F}\right\}.$$

REMARK 6.5. In most constructions of partitions subordinate to a foliation, one may further assume each atom $\xi(x)$ is precompact in the immersed topology of $\mathscr{F}'(x)$ for almost every x or at least for a positive measure subset of x.

When $h_\mu(f) < \infty$, it is with no loss of generality to assume in Definition 6.4 that all partitions ξ have the additional property that

3. $\xi(x)$ is precompact in the immersed topology of $\mathscr{F}'(x)$ for a positive measure set of x.

Indeed, let ξ be as in Definition 6.3. We may measurably select a representative x_ξ in each atom of ξ (see [6, Theorem 9.1.3].) For each x_ξ, we may select (in a measurable way) a precompact (in $\mathscr{F}'(x_\xi)$) neighborhood of x_ξ with positive conditional measure so that, taken together with its complement, the resulting partition $\tilde{\xi}$ is measurable. Since $\xi \prec \tilde{\xi}$, $H_\mu(\tilde{\xi} \mid \xi) \leq \log 2$, and $h_\mu(f,\xi) < \infty$, we have (see Section 6.2 below)

$$\begin{aligned} H_\mu\left(\tilde{\xi} \mid f\xi^+\right) &= H_\mu\left(\tilde{\xi} \vee \xi \mid f\xi^+\right) \\ &= H_\mu\left(\xi \mid f\xi^+\right) + H_\mu\left(\tilde{\xi} \mid \xi^+\right) \\ &\le H_\mu\left(\xi \mid f\xi^+\right) + H_\mu\left(\tilde{\xi} \mid \xi\right) < \infty \end{aligned}$$

and it follows from property (9) of Section 6.2 below that $h_\mu\left(f \mid \tilde{\xi}\right) \ge h_\mu(f \mid \xi)$.

6.2. **Properties of metric entropy.** We recall some properties of the above definitions. A standard reference for proofs and details is [44]. Consider a standard probability space (X,μ), an invertible, measurable, measure-preserving transformation $f\colon (X,\mu) \to (X,\mu)$, and measurable partitions η, ξ, and ζ of (X,μ).

1. $h_\mu(f,\xi) \le H_\mu(\xi)$.
2. $I_\mu(\eta \vee \zeta \mid \xi)(x) = I_\mu(\eta \mid \xi)(x) + I_\mu(\zeta \mid \eta \vee \xi)(x)$ a.e. whence
$$H_\mu(\eta \vee \zeta \mid \xi) = H_\mu(\eta \mid \xi) + H_\mu(\zeta \mid \eta \vee \xi).$$
3. If $\xi \prec \eta$, then $H_\mu(\eta \mid \zeta) \ge H_\mu(\xi \mid \zeta)$ and $H_\mu(\zeta \mid \eta) \le H_\mu(\zeta \mid \xi)$.
4. $h_\mu(f,\eta \vee \zeta) \le h_\mu(f,\eta) + h_\mu(f,\zeta)$.
5. If $\zeta_n \nearrow \zeta$ and if $H_\mu(\eta \mid \zeta_1) < \infty$, then $H_\mu(\eta \mid \zeta_n) \searrow H_\mu(\eta \mid \zeta)$.
6. $h_\mu(f) = \sup\left\{h_\mu(f,\mathscr{P}) : H_\mu(\mathscr{P}) < \infty\right\}$ and $h_\mu(f \mid \eta) = \sup\left\{h_\mu(f,\eta \vee \mathscr{P}) : H_\mu(\mathscr{P}) < \infty\right\}$.
7. $h_\mu(f,\eta \vee \xi) = h_\mu(f,\eta \vee f^k(\xi))$ for $k \in \mathbb{Z}$.
8. If $\eta \prec \xi$ and $H_\mu\left(\xi \mid f\eta^+\right) < \infty$, then $\frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} f^i \xi \mid f^n \eta^+\right) \searrow h_\mu(f,\xi)$.
9. If $\eta \prec \xi$ and $H_\mu\left(\xi \mid f\eta^+\right) < \infty$, then $h_\mu(f,\eta) \le h_\mu(f,\xi)$. In particular, if $H_\mu(\mathscr{P}) < \infty$, then $h_\mu(f,\eta) \le h_\mu(f,\eta \vee \mathscr{P})$.
10. If either $h_\mu(f,\xi) < \infty$ or $H_\mu(\xi \mid \eta) < \infty$, then
$$h_\mu(f,\xi \vee \eta) \le h_\mu(f,\eta) + h_\mu\left(f,\xi \vee \eta^f\right).$$

We also have a more precise version of (9) and (10).

11. If $H_\mu\left(\xi \vee \eta \mid f\eta^+\right) < \infty$, then
$$h_\mu(f,\eta \vee \xi) = h_\mu(f,\eta) + h_\mu\left(f,\xi \vee \eta^f\right).$$

We note that all inequalities hold for ∞-valued quantities.

Conclusion (8) is [44, 7.3]. (9) is [44, 8.7]. (10) holds as

$$\begin{aligned} h_\mu(f,\xi \vee \eta) &= H_\mu\left(\eta \vee f^n(\xi) \mid f\left(\eta^+\right) \vee f^{n+1}\left(\xi^+\right)\right) \\ &= H_\mu\left(\eta \mid f\left(\eta^+\right) \vee f^{n+1}\left(\xi^+\right)\right) + H_\mu\left(f^n(\xi) \mid \eta^+ \vee f^{n+1}\left(\xi^+\right)\right) \\ &= H_\mu\left(\eta \mid f\left(\eta^+\right) \vee f^{n+1}\left(\xi^+\right)\right) + H_\mu\left(\xi \mid f^{-n}\left(\eta^+\right) \vee f\left(\xi^+\right)\right). \end{aligned}$$

We have $H_\mu\left(\eta \mid f\left(\eta^+\right) \vee f^{n+1}\left(\xi^+\right)\right) \le h_\mu(f,\eta)$ and, assuming either that $h_\mu(f,\xi) < \infty$ or that $H_\mu(\xi \mid \eta) < \infty$, we obtain that $H_\mu\left(\xi \mid f^{-n}\left(\eta^+\right) \vee f\left(\xi^+\right)\right)$ decreases to $H_\mu\left(\xi \mid \eta^f \vee f\left(\xi^+\right)\right)$ as $n \to \infty$. As η^f is f-invariant, we have $H_\mu\left(\xi \mid \eta^f \vee f\left(\xi^+\right)\right) = h_\mu\left(f,\xi \vee \eta^f\right)$.

The assertion in (11) is [44, 7.7] which can be derived from (8) and (10).

6.3. **Abramov–Rohlin-type formulas.** We say a partition η of (X,μ) is f*-increasing* if $f\eta \prec \eta$. Note in this case that $\eta^+ = \eta$ and $h_\mu(f,\eta) = H_\mu(\eta \mid f\eta)$. We say η *generates* if η^f is the point partition.

Consider a second measure-preserving transformation $g\colon (Y,\nu) \to (Y,\nu)$ of a standard probability space (Y,ν). Suppose there is a measurable $\psi\colon X \to Y$ with $\psi_*\mu = \nu$ and $\psi \circ f = g \circ \psi$. We say that g is a *factor of f induced by ψ*. Write $\mathscr{A}^\psi$ for the partition of (X,μ) into preimages of ψ. Note that $\mathscr{A}^\psi$ is f-invariant.

We have the following equalities which include the classical Abramov–Rohlin formula, (21) below (see [30, 7]).

COROLLARY 6.6. *Let $g\colon (Y,\nu) \to (Y,\nu)$ be a measurable factor of $f\colon (X,\mu) \to (X,\mu)$ induced by ψ. Let $\hat\eta$ be a measurable partition of (Y,ν) that is increasing for g. Then*

$$h_\mu\left(f \mid \psi^{-1}(\hat\eta)\right) = h_\nu\left(g,\hat\eta\right) + h_\mu\left(f \mid \psi^{-1}\left(\hat\eta^g\right)\right)$$

and if $\hat\eta$ generates for g, then

$$h_\mu\left(f \mid \psi^{-1}(\hat\eta)\right) = h_\nu\left(g,\hat\eta\right) + h_\mu\left(f \mid \mathscr{A}^\psi\right). \tag{20}$$

In particular,

$$h_\mu(f) = h_\nu(g) + h_\mu\left(f \mid \mathscr{A}^\psi\right). \tag{21}$$

Indeed, if $h_\nu\left(g,\hat\eta\right) = \infty$ there is nothing to prove. If $h_\nu\left(g,\hat\eta\right) < \infty$ we may take the supremum of $h_\mu\left(f, \mathscr{P} \vee \psi^{-1}(\hat\eta)\right)$ over all partitions $\mathscr{P}$ of (X,μ) with $H_\mu(\mathscr{P}) < \infty$ and obtain the first conclusion from (11) of Section 6.2. (21) follows by taking supremums over finite entropy partitions of (X,μ) and (Y,ν).

7. STATEMENT OF RESULTS

We extend the main results of [31, 32] to the setting of rank-1 systems (i.e., diffeomorphisms) satisfying the hypotheses introduced in Section 3.2. Beyond establishing formulas as in [31, 32] for the total entropy $h_\mu(f)$, we also establish results analogous to those of [31, 32] for entropy subordinated to a foliation, $h_\mu(f \mid \mathscr{F})$, or a measurable partition, $h_\mu(f \mid \eta)$, following the definitions in Section 6.

Let M be a k-dimensional, C^∞ manifold equipped with a Borel probability measure μ. Let $f\colon M \to M$ be an invertible, measurable, μ-preserving transformation. Then f generates an action of $\mathbb{Z}$ on M. We assume the induced $\mathbb{Z}$-action satisfies hypotheses II of Section 3.2. In particular, we fix the generating set $F = \{-1,1\}$ and consider all constructions from Part I including the set $U \subset U_0$ from Section 3.2 and the Lyapunov charts from Section 5 to be relative to F.

For the remainder of Part II, we further assume that μ is f-ergodic as the generalizations of all results stated here to the non-ergodic case are standard.

As f induces an action of $\mathbb{Z}$, we may identify the Lyapunov exponent functionals $\lambda_i\colon \mathbb{Z} \to \mathbb{R}$ for the derivative cocycle (13) with their coefficients $\lambda_i := \lambda_i(1) \in \mathbb{R}$.

For the remainder of Part II, Lyapunov exponents are assumed to be real numbers, listed without multiplicity, and ordered so that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0 \geq \lambda_{r+1} > \cdots > \lambda_p.$$

We write $E^i(x) = E_{\lambda_i}(x)$ and write m_i for the almost-surely constant value of $\dim E^i(x)$. Also write $E^u(x) = \bigoplus_{\lambda_i>0} E^i(x)$, $E^0(x) = E_0(x)$, and $E^s(x) = \bigoplus_{\lambda_i<0} E^i(x)$. Recall we view each $E^i(x)$ as a subspace of $\mathbb{R}^k$ via the charts in ϕ_x in standing hypotheses II in Section 3.2; we may push forward each $E^i(x)$ to T_xM under $D_0\phi_x$.

Given $1 \leq i \leq r$, we write $W^i(x) = W^i_1(x)$ and $\mathscr{W}^i = \mathscr{W}^i_1$ for the ith unstable manifold and ith unstable foliation corresponding to the generator $f = \alpha(1)$. Similarly write $\mathscr{W}^u = \mathscr{W}^u_1$. Given any $C^{1+\beta}$-tame, f-invariant, measurable foliation $\mathscr{F}$ we write $\mathscr{F}^u := \mathscr{F} \vee \mathscr{W}^u$. We similarly define stable manifolds $W^s(x)$ and the stable foliation $\mathscr{W}^s$ to be the unstable manifold and foliation relative to f^{-1}.

7.1. **Finiteness of entropy.** Our first result is the following version of the Margulis–Ruelle inequality [45] for systems satisfying our standing hypotheses.

PROPOSITION 7.1. *Let (M,μ) and f be as above. Then*

$$h_\mu(f) \leq \sum_{\lambda_i>0} \lambda_i m_i.$$

In particular, $h_\mu(f) < \infty$.

7.2. **Geometric rigidity of measures satisfying the entropy formula.** Let $\mathscr{F}$ be an f-invariant, measurable foliation. Observe that the distribution $x \mapsto T_x\mathscr{F}'(x)$ is Df-invariant and thus for a.e. x,

$$T_x\mathscr{F}'(x) = \bigoplus_{i=1}^{p} \big(D_0\phi_x E^i(x) \cap T_x\mathscr{F}'(x)\big).$$

(Here ϕ_x are the dynamical charts in standing hypotheses II in Section 3.2.) Define the multiplicity of each λ_i relative to $\mathscr{F}$ to be (the almost-surely constant value of)

$$m_i(\mathscr{F}) := \dim\Big(D_0\phi_x E^i(x) \cap T_x\mathscr{F}'(x)\Big).$$

We say a measure μ is *absolutely continuous along* $\mathscr{F}$ if μ^η_x is absolutely continuous with respect to the Riemannian volume on $\eta(x) \subset \mathscr{F}(x)'$ for any measurable partition η subordinate to $\mathscr{F}$ and almost every x.

We have the following extension of [31, Theorem A].

THEOREM 7.2. *Let $\mathscr{F}$ be an f-invariant, $C^{1+\beta}$-tame, measurable foliation. Then*

$$h_\mu(f \mid \mathscr{F}) \leq \sum_{1\leq i\leq r} \lambda_i m_i(\mathscr{F}). \tag{22}$$

Moreover, equality holds if and only if μ is absolutely continuous along $\mathscr{F}^u$.

Furthermore, in the case of equality in (22), the measures μ_x^η are *equivalent* to the Riemannian volume on almost every $\eta_x \subset \mathscr{F}^u(x)$ for any measurable partition η subordinate to $\mathscr{F}^u$. (See [31, Corollary 6.1.4]).

Note that Proposition 7.1 follows from the statement of Theorem 7.2 by taking $\mathscr{F} = M$. However, we establish Proposition 7.1 separately from Theorem 7.2.

As in the main result of [28], Theorem 7.2 follows from Jensen's inequality after we establish that all entropy of the system is carried by unstable manifolds (or the unstable part of an invariant foliation). (Compare with [31, Corollary 5.3].)

PROPOSITION 7.3. *For an f-invariant, $C^{1+\beta}$-tame, measurable foliation $\mathscr{F}$ we have*

$$h_\mu(f \mid \mathscr{F}) = h_\mu(f \mid \mathscr{F}^u).$$

7.3. **Pointwise transverse dimensions.** In Part III we will want a geometric description of the entropy $h_\mu(f \mid \eta)$ where η is an invariant measurable partition. We thus prove a generalization of the classical Ledrappier–Young entropy formula for the quantity $h_\mu(f \mid \eta)$.

Take η to be an arbitrary measurable partition of (M, μ). Given $1 \le i \le r$, let ξ^i be a measurable partition subordinate to $\mathscr{W}^i$. Let $\left\{\mu_x^{\xi^i \vee \eta}\right\}$ be a family of conditional measures relative to the measurable partition $\xi^i \vee \eta$. We define the ith upper and lower pointwise dimensions of μ relative to η at x to be

$$\overline{\dim}^i(\mu, x|\eta) := \limsup_{r\to 0} \frac{\log\left(\mu_x^{\xi^i\vee\eta}(B(x,r))\right)}{\log r},$$

$$\underline{\dim}^i(\mu, x|\eta) := \liminf_{r\to 0} \frac{\log\left(\mu_x^{\xi^i\vee\eta}(B(x,r))\right)}{\log r}$$

and the ith pointwise dimension of μ relative to η at x to be

$$\dim^i(\mu, x|\eta) := \lim_{r\to 0} \frac{\log\left(\mu_x^{\xi^i\vee\eta}(B(x,r))\right)}{\log r}$$

when the limit exists. One verifies that the functions $\overline{\dim}^i(\mu, x|\eta)$ and $\underline{\dim}^i(\mu, x|\eta)$ are measurable and independent of the choice of ξ^i. Moreover, if $f\eta \prec \eta$ and $h_\mu(f, \eta) < \infty$, it follows that $\overline{\dim}^i(\mu, x|\eta)$ and $\underline{\dim}^i(\mu, x|\eta)$ are constant along orbits of f and hence, by ergodicity of μ, constant a.s. Let $\overline{\dim}^i(\mu|\eta)$, $\underline{\dim}^i(\mu|\eta)$ denote the a.s. constant values.

We first claim the following (whose proof follows from the inequalities in Section 10.1).

PROPOSITION 7.4. *Let η be a measurable partition of (M, μ). Then*

$$\overline{\dim}^i(\mu|\eta^+) = \underline{\dim}^i(\mu|\eta^+).$$

Write $\dim^i(\mu|\eta^+)$ for the common value guaranteed by Proposition 7.4. Set $\dim^0(\mu|\eta^+) = 0$. For $1 \le i \le r$ define the *ith transverse dimension of μ relative to η^+* to be

$$\gamma^i\left(\mu|\eta^+\right) = \dim^i\left(\mu|\eta^+\right) - \dim^{i-1}\left(\mu|\eta^+\right).$$

CLAIM 7.5. *For $1 \le i \le r$, $\gamma^i\left(\mu|\eta^+\right) \le m_i := \dim E^i$.*

Note that if $\hat\eta$ is a measurable partition of (M,μ) with $\eta \prec \hat\eta$, then, by definition, we have

$$h_\mu(f \mid \eta) \ge h_\mu\left(f \mid \hat\eta\right).$$

We have a similar result for the transverse dimensions above.

PROPOSITION 7.6. *If $\eta^+ \prec \hat\eta^+$, then for every $1 \le i \le r$*

$$\gamma^i\left(\mu|\eta^+\right) \ge \gamma^i\left(\mu|\hat\eta^+\right).$$

Proposition 7.6 follows from discussion at the end of Section 11.3.

7.4. **Geometric characterization of the defect in the entropy formula.** As in [32] we have an explicit geometric description of the defect of equality in (22) as well as its generalization to the entropy subordinate to a measurable partition.

THEOREM 7.7. *Let η be a measurable partition of (M,μ). Then*

$$h_\mu(f \mid \eta) = \sum_{1\le i\le r} \lambda_i \gamma^i\left(\mu|\eta^+\right).$$

Taking η to be the trivial partition $\{X,\varnothing\}$, we obtain an extension of the entropy formula from [32] to $C^{1+\beta}$ diffeomorphisms of noncompact manifolds satisfying our standing hypotheses.

Suppose $g\colon (Y,\nu) \to (Y,\nu)$ is a measurable factor of f induced by $\psi\colon (M,\mu) \to (Y,\nu)$. As a primary application of Theorem 7.7, we obtain a Ledrappier–Young entropy formula for the fiber entropy $h_\mu\left(f \mid \mathscr{A}^\psi\right)$ of smooth systems when the elements of the fiber partition $\mathscr{A}^\psi$ are only measurable. In particular, from (21) the entropy of the factor system $g\colon (Y,\nu) \to (Y,\nu)$ can be computed in terms of the Lyapunov exponents of the total system f and the geometry of the conditional measures of μ and of $\left\{\mu_x^{\mathscr{A}^\psi}\right\}$ along unstable manifolds. In the case that the fibers are smooth manifolds, a Ledrappier–Young formula for fiber entropy follows from [41].

7.5. **A characterization of the Pinsker partition.** Let π denote the Pinsker partition for the action of f on (M,μ). Also, let $\mathscr{B}^u$ and $\mathscr{B}^s$ be the measurable hulls of the (typically non-measurable) partitions $\mathscr{W}^u$ and $\mathscr{W}^s$.

THEOREM 7.8. *Under the above assumptions*

$$\pi \mathrel{\mathring{=}} \mathscr{B}^u \mathrel{\mathring{=}} \mathscr{B}^s.$$

Theorem 7.8 follows from the discussion in [44, (12.4)] exactly as in [31, (6.3)] and [28, (2.3)] from Proposition 8.3 and Remark 8.1 below.

8. PREPARATIONS: SPECIAL PARTITIONS AND THEIR ENTROPY PROPERTIES

8.1. **Standing notation.** We write Λ_0 for the set of regular points of μ. Let

$$\varepsilon_0 = \frac{1}{100}\min\left\{1, |\lambda_i|, |\lambda_i - \lambda_j| : i \neq j, \lambda_i \neq 0\right\}.$$

For Sections 8–11, fix $0 < \varepsilon < \varepsilon_0$. Let $\{\Phi_x\}$ be a fixed family of ε-Lyapunov charts as in Proposition 5.1 with corresponding function ℓ. Given $0 \leq i \leq r$ and $x \in \Lambda_0$, let h^i_x be a function as in Lemma 5.5 (relative to the charts Φ_x). Let $\lambda_0 = \max\{|\lambda_i|\}$. For $0 < \delta \leq e^{-\lambda_0 - 2\varepsilon}$, let

$$W^i_{x,\delta} = \operatorname{graph}\left(h^i_x\restriction_{\bigoplus_{j\leq i}\mathbb{R}^j(\delta)}\right)$$

and write

$$V^i_{\mathrm{loc},x,\varepsilon} := \Phi_x\left(\operatorname{graph}\left(h^i_x\restriction_{\bigoplus_{j\leq i}\mathbb{R}^j(-2\lambda_0-4\varepsilon)}\right)\right) = \Phi_x(W^i_{x,e^{-2\lambda_0-4\varepsilon}})$$

for the *ith local unstable manifold* relative to the charts $\{\Phi_x\}$.

Consider a $C^{1+\beta}$-tame, f-invariant, measurable foliation $\mathscr{F}$. We may furthermore choose charts $\{\Phi_x\}$ above as in Proposition 5.4 and build an associated family of charts $\Psi^{\mathscr{F}}_x$. Let $h^{\mathscr{F},u}_x = h^r_x$ be as in Lemma 5.5 (relative to the charts $\{\Psi^{\mathscr{F}}_x\}$) and write

$$V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon} := \Psi^{\mathscr{F}}_x\left(\operatorname{graph}\left(h^{\mathscr{F},u}_x\restriction_{\bigoplus_{j\leq r}(V\cap\mathbb{R}^j)(e^{-2\lambda_0-4\varepsilon})}\right)\right)$$

for the local manifold of $\mathscr{F}^u := \mathscr{F} \vee \mathscr{W}^u$ through x relative to the charts $\{\Psi^{\mathscr{F}}_x\}$. Also write

$$V^{\mathscr{F}}_{\mathrm{loc},x,\varepsilon} := \Psi^{\mathscr{F}}_x(V(1)) = \Phi_x(\operatorname{graph}(\tilde{h}^{\mathscr{F}}_x\restriction_{V(1)})),$$

where $\tilde{h}^{\mathscr{F}}_x$ is as in Proposition 5.2.

Recall the ambient (locally-defined) metric d on U_0. Write $d_{V^i_{\mathrm{loc},x,\varepsilon}}$ for the restriction of d to the embedded manifold $V^i_{\mathrm{loc},x,\varepsilon}$ where $d_{V^i_{\mathrm{loc},x,\varepsilon}}(x,y) = \infty$ if $y \notin V^i_{\mathrm{loc},x,\varepsilon}$. Denote by $B(x,\delta) := \{y \in U_0 : d(x,y) < \delta\}$ and

$$B^i(x,\delta) := \left\{y \in V^i_{\mathrm{loc},x,\varepsilon} : d_{V^i_{\mathrm{loc},x,\varepsilon}}(x,y) < \delta\right\}. \tag{23}$$

8.2. **Expanding partitions subordinate to unstable manifolds.** Following the constructions in [29, Proposition 3.1], [31, Lemma 3.1.1], and [32, Lemma 9.1.1], there exists measurable partitions ξ^i for $1 \leq i \leq r$ with the following properties:

(1) ξ^i is subordinate to $\mathscr{W}^i$ (see Definition 6.3); moreover for a positive measure set of x we have $\xi^i(x) \subset V^i_{\mathrm{loc},x,\varepsilon}$;
(2) $\xi^i \prec f^{-1}\xi^i$;
(3) $\bigvee_{n=0}^{\infty} f^{-n}\xi^i$ is the point partition;
(4) for $1 \leq i \leq r-1$, we have $\xi^{i+1} \prec \xi^i$.

A partition satisfying (1)–(3) is said to be an *increasing generator subordinate to* $\mathscr{W}^i$.

REMARK 8.1. Take $i = r$. For any partition $\xi^u := \xi^r$ satisfying (1) and (2) we have

$$\bigwedge_{n\geq 0} f^n \xi^u = \mathscr{B}^u.$$

The construction of partitions ξ^i with the above properties outlined in [32, Lemma 9.1.1] requires one to first subfoliate each $V^u_{\mathrm{loc},x,\varepsilon}$ by a continuous foliation by fast unstable manifolds. The characterization of atoms of these partitions given on [31, p. 520] and [32, p. 555] is then given in terms of distances inside global fast unstable manifolds. Although this may be carried out word-for-word in our setting, we prefer a construction that only uses the measurable family of local fast unstable manifolds $V^i_{\mathrm{loc},x,\varepsilon}$. In particular, we first follow [29, Proposition 3.1] and build partitions $\tilde\xi^i$ satisfying (1)–(3); we then refine these to obtain partitions ξ^i satisfying (4). Moreover, as we allow our system to have discontinuities and singularities, we prefer a (partial) characterization of atoms of each partition in terms of the dynamics inside local fast unstable manifolds in Lemma 8.2 below. Since our construction is somewhat different from [32, Lemma 9.1.1] and since we will refer back to sets used in the construction, we outline the construction here.

8.2.1. *Outline of construction.* Recall our fixed $0 < \varepsilon < \varepsilon_0$ and family of ε-Lyapunov charts Φ_x with corresponding function ℓ. Also recall the charts $\Psi^{\mathscr{F}}_x$ built from Φ_x as in Proposition 5.4. Fix $\ell_0 > 0$ sufficiently large. Let $\Lambda_1 \subset \Lambda_0$ be a compact set of positive measure such that:

- $\ell(x) \leq \ell_0$ for $x \in \Lambda_1$;
- the charts $x \mapsto \Phi_x$ and $x \mapsto \Psi^{\mathscr{F}}_x$ vary continuously in the C^1 topology on Λ_1;
- for $1 \leq i \leq r$ the family of functions $h^{u,i}_x$ vary continuously in the C^1 topology on Λ_1;
- the familes of functions $\tilde h^{\mathscr{F}}_x$ and $h^{\mathscr{F},u}_x$ vary continuously in the C^1 topology on Λ_1.

It follows that the families of embedded manifolds $x \mapsto V^i_{\mathrm{loc},x,\varepsilon}$, $x \mapsto V^{\mathscr{F}}_{\mathrm{loc},x,\varepsilon}$, and $x \mapsto V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}$ vary continuously in the C^1 topology on Λ_1. Let x_0 be a density point of Λ_1. Given $\rho > 0$ sufficiently small and $y \in B(x_0, \rho) \cap \Lambda_1$, let

$$V^{u,i}(y,\rho) = V^i_{\mathrm{loc},y,\varepsilon} \cap B(x_0,\rho).$$

Write $V^u(y,\rho) = V^{u,r}(y,\rho)$. Similarly, given $y \in B(x_0,\rho)\cap\Lambda_1$, write $V^{\mathscr{F},u}(y,\rho) = V^{\mathscr{F},u}_{\mathrm{loc},y,\varepsilon} \cap B(x_0,\rho)$ and $V^{\mathscr{F}}(y,\rho) = V^{\mathscr{F}}_{\mathrm{loc},y,\varepsilon} \cap B(x_0,\rho)$. Set

$$S^0_\rho := B(x_0,\rho)\cap\Lambda_1.$$

Taking $0 < \rho_0$ sufficiently small, we may assume for $0 < \rho < \rho_0$ that $N = B(x_0,\rho)$ is an embedded open submanifold with $\overline{N} \subset U$ an embedded submanifold with boundary and that for all and $y_1, y_2 \in S^0_\rho$ the following hold:

(a) $V^{u,i}(y_1,\rho)$ is a connected open neighborhood of y_1 with compact closure in $V^i_{\mathrm{loc},y_1,\varepsilon}$;

(b) if $y_1 \in V^i_{\mathrm{loc},y_2,\varepsilon}$, then $V^{u,i}(y_1,\rho) = V^{u,i}(y_2,\rho)$;

(c) $V^{\mathscr{F}}(y_1,\rho)$ is a connected open neighborhood of y_1 with compact closure in $V^{\mathscr{F}}_{\mathrm{loc},y_1,\varepsilon}$;

(d) $V^{\mathscr{F},u}(y_1,\rho)$ is a connected open neighborhood of y_1 with compact closure in $V^{\mathscr{F},u}_{\mathrm{loc},y_1,\varepsilon}$ and

$$V^{\mathscr{F},u}(y_1,\rho) = V^u(y_1,\rho) \cap V^{\mathscr{F}}(y_1,\rho);$$

(e) if $y_1 \in V^{\mathscr{F}}_{\mathrm{loc},y_2,\varepsilon}$, then $V^{\mathscr{F}}(y_1,\rho) = V^{\mathscr{F}}(y_2,\rho)$, and if $y_1 \in V^{\mathscr{F},u}_{\mathrm{loc},y_2,\varepsilon}$, then $V^{\mathscr{F},u}(y_1,\rho) = V^{\mathscr{F},u}(y_2,\rho)$.

(Properties (c)—(e) will be used later in Section 12.)

For $1 \le i \le r$, set

$$S^i_\rho = \bigcup_{x \in S^0_\rho} V^{u,i}(x,\rho).$$

Consider $y \in S^i_\rho$. There exists $x \in S^0_\rho$ with $y \in V^{u,i}(x,\rho)$; moreover, if $y \in V^{u,i}(x',\rho)$ for some $x' \in S^0_\rho$, then

$$V^{u,i}(x,\rho) = V^{u,i}(x',\rho).$$

For such $y \in S^i_\rho$ and x, set $D^i(y) = V^{u,i}(x,\rho)$. It follows that

$$\hat{\xi}^i(y) = \begin{cases} D^i(y) & y \in S^i_\rho, \\ M \smallsetminus S^i_\rho & y \notin S^i_\rho \end{cases}$$

defines a partition of a full measure subset of M. Take $\tilde{\xi}^i := (\hat{\xi}^i)^+$.

By ergodicity, for any $0 < \rho < \rho_0$ and a.e. y, we have that $f^{-n}(y) \in S^0_\rho$ for some $n \ge 0$. It follows that $\tilde{\xi}^i(y) \subset \mathscr{W}^i(y)$ for a.e. y. We check that $\tilde{\xi}^i$ satisfy properties (2) and (3) enumerated at the beginning of this section. Indeed, property (2) follows from construction and property (3) follows from the dynamics inside local unstable manifolds. Property (1) follows for Lebesgue-almost every choice of $\rho < \rho_0$ (ensuring the boundary of a.e. $V^{u,i}(x,\rho)$ has zero measure and that backwards orbits do not accumulate too quickly near the boundaries of $V^{u,i}(x,\rho)$) by the same arguments as in [29, Section 3]. Fix such a choice of $\rho < \rho_0$.

We have the following partial characterization atoms of $\tilde{\xi}^i$ which will be used in the sequel.

LEMMA 8.2. *Set $\delta = e^{-2\lambda_0 - 4\varepsilon}$ and suppose $\rho < \frac{1}{2}\delta\ell_0^{-1}$. Suppose z and y satisfy the following: for all $m \ge 0$,*

1. $f^{-m}(y) \in S^i_\rho$ if and only if $f^{-m}(z) \in S^i_\rho$;
2. $d\left(f^{-m}(y), f^{-m}(z)\right) \le \frac{1}{2}\delta\ell_0^{-1} e^{m(-\lambda_i + 2\varepsilon)}$.

Then $\tilde{\xi}^i(z) = \tilde{\xi}^i(y)$.

Proof. Given any $m \ge 0$, we claim that $f^{-m}(z) \in \hat{\xi}^i(f^{-m}(y))$; the result then follows. For m such that $f^{-m}(y) \notin S^i_\rho$, the claim follows by hypothesis.

Consider any $m \geq 0$ such that $f^{-m}(y) \in S^i_\rho$. There is some $x \in S^0_\rho$ with $y \in V^{u,i}(x,\rho)$. Then $d\left(x, f^{-m}(z)\right) \leq \ell_0^{-1}$ and $f^{-m}(z)$ is in the domain of Φ_x^{-1}. For $k \geq 0$ we have

$$\left\|\Phi^{-1}_{f^{-k}(x)}\left(f^{-m-k}(y)\right)\right\| \leq e^{(-\lambda_i+2\varepsilon)k}\ell_0\rho \leq \tfrac{1}{2}\delta e^{(-\lambda_i+2\varepsilon)k}.$$

Since

$$\ell\left(f^{-k}(x)\right) d\left(f^{-m-k}(z), f^{-m-k}(y)\right) \leq \ell_0 e^{\varepsilon k} d\left(f^{-m-k}(z), f^{-m-k}(y)\right) \leq \frac{1}{2}\delta$$

it follows that $f^{-m-k}(z)$ is in the domain of $\Phi^{-1}_{f^{-k}(x)}$ for all $k \geq 0$. Then for $k \geq 0$

$$\left\|\Phi^{-1}_{f^{-k}(x)}\left(f^{-m-k}(z)\right)\right\| \leq \ell\left(f^{-k}(x)\right)\cdot\left(\ell_0^{-1}\delta e^{k(-\lambda_i+2\varepsilon)}\right) \leq \delta e^{k(-\lambda_i+3\varepsilon)}$$

which, from (19) of Lemma 5.5, implies $f^{-m}(z) \in V^i_{\mathrm{loc},x,\varepsilon}$. Since $f^{-m}(z) \in S^i_\rho$, we have

$$f^{-m}(z) \in V^{u,i}(x,\rho) = D^i\left(f^{-m}(y)\right) = \hat{\xi}^i\left(f^{-m}(y)\right)$$

and the lemma follows. □

For $1 \leq i \leq r$, note that $V^{u,i}(x,\rho)$ is defined only for $x \in S^0_\rho$. In particular, while $V^{u,i}(x,\rho) \subset V^{u,i+1}(x,\rho)$, the collection $\{V^{u,i}(y,\rho) : y \in V^{u,i+1}(x,\rho)\}$ may not subfoliate $V^{u,i+1}(x,\rho)$. In particular, while $S^i_\rho \subset S^{i+1}_\rho$, we do not automatically have $\hat{\xi}^{i+1}(y) < \hat{\xi}^i(y)$ nor $\tilde{\xi}^{i+1}(y) < \tilde{\xi}^i(y)$ (as $\tilde{\xi}^i(y)$ is refined on return times to S^i_ρ rather than return times to S^{i+1}_ρ.) Thus, having built $\tilde{\xi}^1, \ldots, \tilde{\xi}^r$ satsifying (1)–(3), for $1 \leq i \leq r$ set

$$\xi^i = \bigvee_{j=i}^{r} \tilde{\xi}^j. \tag{24}$$

Property (4) now clearly holds for the family of partitions ξ^i for each $1 \leq i \leq r$. Moreover, properties (2) and (3) are inherited by the partition ξ^i and, since $V^i_{\mathrm{loc},x,\varepsilon}$ is an embedded submanifold of $V^j_{\mathrm{loc},x,\varepsilon}$ for all $j \geq i$, property (1) continues to hold for ξ^i.

We fix a sufficiently small $\rho > 0$ and write $S = S^0_\rho$ and $S^i = S^i_\rho$ for the remainder. Also write $\xi^u = \xi^r$.

The following proposition is the analogue of the main results of [31]; see [31, Corollary 5.3]. Roughly, the proposition says that all entropy is "carried by unstable manifolds," the most difficult step in establishing the SRB property in [31]. The proof will follow directly from Proposition 9.1 and the inequalities in 10.1 below.

PROPOSITION 8.3. *Let $\xi^u = \xi^r$ be a partition as above. Then for any measurable partition η,*

$$h_\mu(f \mid \eta) = h_\mu\left(f, \eta \vee \xi^u\right) = H_\mu\left(\eta^+ \vee \xi^u \mid f\left(\eta^+ \vee \xi^u\right)\right).$$

In particular, $h_\mu(f) = h_\mu\left(f, \xi^u\right)$.

8.3. **Entropy properties of the partitions ξ^i.** Following (the proof of) [32, (9.2)] we have the following.

CLAIM 8.4. *Let ξ^i be any partition as in Section 8.2. Then $h_\mu(f,\xi^i) < \infty$.*

Proof. Given $\delta > 0$, let

$$A_\delta := \left\{x : B^i(f(x),\delta) \subset \xi^i(f(x))\right\},$$

where $B^i(\cdot,\delta)$ is as in (23). Then $\mu(A_\delta) \to 1$ as $\delta \to 0$. Let

$$g(x) = -\log \mu_x^{\xi^i}\left(f^{-1}\xi^i(x)\right).$$

Given any $M < h_\mu(f,\xi^i) = H_\mu(f^{-1}\xi^i \mid \xi^i)$, take $\delta > 0$ so that

$$\int_{A_\delta} g \, d\mu \ge M.$$

Write

$$U^i(x,n,\delta) := \bigcap_{\{j:0\le j\le n-1, f^j(x)\in A_\delta\}} \left(f^{-(j+1)}\xi^i\right)(x).$$

Note that if $k = \max\{0 \le j \le n-1 : f^j(x) \in A_\delta\}$, then

$$U^i(x,n,\delta) = f^{-(k+1)}\xi^i(x).$$

Relative to the family of ε-Lyapunov charts $\{\Phi_x\}$, we observe for every sufficiently large $n \ge 1$ (such that $f^j(x) \in A_\delta$ for some $0 \le j \le n-1$) that

$$B^i\left(x, \delta\ell(x)^{-1}e^{(-\lambda_1-2\varepsilon)n}\right) \subset U^i(x,n,\delta).$$

Moreover

$$-\log\mu_x^{\xi^i}\left(B^i\left(x,\delta\ell(x)^{-1}e^{(-\lambda_1-2\varepsilon)n}\right)\right) \ge -\log\mu_x^{\xi^i}\left(U^i(x,n,\delta)\right) \ge \sum_{j=0}^{n-1}(\mathbb{1}_{A_\delta}\cdot g)(f^j(x)).$$

Then, by the pointwise ergodic theorem, for μ-a.e. x we have

$$\liminf_{n\to\infty} -\frac{1}{n}\log\mu_x^{\xi^i}\left(B^i\left(x,\delta\ell(x)^{-1}e^{(-\lambda_1-2\varepsilon)n}\right)\right) \ge \int_{A_\delta} g \ge M.$$

On the other hand (see [31, Lemma 4.1.4]), fixing a bi-Lipschitz identification of the embedded manifold $V^i_{\mathrm{loc},x,\varepsilon}$ with $\mathbb{R}^{m_i+\cdots+m_1}$ we have for a.e. x that

$$\limsup_{n\to\infty} -\frac{1}{n}\log\mu_x^{\xi^i}\left(B^i\left(x,\delta\ell(x)^{-1}e^{(-\lambda_1-2\varepsilon)n}\right)\right) \le (m_i+\cdots+m_1)(\lambda_1+2\varepsilon)$$

whence $M \le (m_i+\cdots+m_1)(\lambda_1+2\varepsilon)$. □

We frequently use the following fact that partitions satisfying properties (2) and (3) of Section 8.2 locally maximize entropy.

CLAIM 8.5. *Let ξ be any measurable partition of (M,μ) satisfying properties (2) and (3) of Section 8.2. Let η and ζ be measurable partitions with $h_\mu(f,\zeta) < \infty$. Then*

$$h_\mu(f,\zeta\vee\xi\vee\eta) \le h_\mu(f,\xi\vee\eta).$$

Indeed, this follows from (10) of Section 6.2 as $(\xi \vee \eta)^f = \bigvee_{n\in\mathbb{Z}} f^n(\xi \vee \eta)$ is the point partition.

Combined with (9) of Section 6.2, we have the following (compare to [31, Lemma 3.2.1]).

COROLLARY 8.6. *Let ξ be any measurable partition of (M,μ) satisfying (2) and (3) of Section 8.2. Let η and $\mathscr{P}$ be measurable partitions with $H_\mu(\mathscr{P}) < \infty$. Then*

$$h_\mu(f, \xi \vee \eta \vee \mathscr{P}) = h_\mu(f, \xi \vee \eta).$$

As in [31, Lemma 3.1.2] we have the following.

LEMMA 8.7. *For each $1 \le i \le r$, let ξ_1^i and ξ_2^i be two partitions as in Section 8.2 and let η be an arbitrary measurable partition. Then*

$$h_\mu(f, \xi_1^i \vee \eta) = h_\mu(f, \xi_2^i \vee \eta).$$

In particular, $h_\mu(f, \xi_1^i) = h_\mu(f, \xi_2^i)$.

8.4. **Finite entropy partitions adapted to Lyapunov charts.** Recall our family of ε-Lyapunov charts $\{\Phi_x\}$. Fix $0 < \delta < 1$. For $x \in \Lambda_0$, define the corresponding *center-unstable sets*:

$$S^{cu}_{\delta,x} = \left\{ y \in \mathbb{R}^k : \left\| \Phi^{-1}_{f^{-m}(x)} \circ f^{-m} \circ \Phi_x(y) \right\| < \delta \text{ for all } m \ge 0 \right\}.$$

DEFINITION 8.8. We say a measurable partition $\mathscr{P}$ of (M,μ) is adapted to $(\{\Phi_x\}, \delta)$ if, for almost every x,

$$\mathscr{P}^+(x) \subset \Phi_x\left(S^{cu}_{\delta,x}\right).$$

LEMMA 8.9. *For every $0 < \delta < 1$, there exists a measurable partition $\mathscr{P}$ adapted to $(\{\Phi_x\}, \delta)$ with $H_\mu(\mathscr{P}) < \infty$.*

Proof. Let $\ell\colon \Lambda \to [1,\infty)$ be the function associated with the charts $\{\Phi_x\}$. Recall in Section 8.2 we fixed an open ball N with $\overline{N} \subset U$ a closed ball and $S := S^0_\rho \subset N$ with $\mu(S) > 0$ and $\ell(x) \le \ell_0$ for $x \in S$.

Let $n\colon S \to \mathbb{N}$ be the first return function: $n(x) = \min\left\{ j \ge 1 : f^j(x) \in S \right\}$. We have $\int n(x)\, d\mu\restriction_S = 1$.

Let $\rho\colon S \to (0,1)$ be

$$\rho(x) = \delta \ell_0^{-1} e^{-n(x)(\lambda_0 + 2\varepsilon)}.$$

Let $\hat\mu = \frac{1}{\mu(S)}\mu\restriction_S$. We have $\int -\log(\rho)\, d\hat\mu < \infty$ hence, adapting [37, Lemma 2] to $(\overline{N}, \hat\mu)$, there is a partition $\hat P$ of $(S, \hat\mu)$ with $H_{\hat\mu}(\hat{\mathscr{P}}) < \infty$ and $\operatorname{diam}(\hat{\mathscr{P}}(x)) < \rho(x)$ for almost every $x \in S$. Let $\mathscr{P} = \hat{\mathscr{P}} \cup \{M \smallsetminus S\}$. Then $H_\mu(\mathscr{P}) < \infty$. Moreover, from the choice of ρ and the properties of the charts $\{\Phi_x\}_{x\in S}$, the same computations as in [31, Lemma 2.4.2] show that $\mathscr{P}$ is adapted to $(\{\Phi_x\}, \delta)$. □

CLAIM 8.10. *Let $\delta < e^{-2\lambda_0 - 4\varepsilon}$ and let $\mathscr{P}$ be adapted to $(\{\Phi_x\}, \delta)$. Then for $1 \le i \le r$ and almost every x,*

$$(\mathscr{P}^+ \vee \xi^i)(x) \subset V^i_{\mathrm{loc},x,\varepsilon}.$$

Proof. Let $\eta = \mathscr{P}^+ \vee \xi^i$. For $x \in S$ we have

$$\eta(x) \subset \xi^i(x) \subset V^i_{\mathrm{loc},x,\varepsilon}.$$

In particular, $\Phi_x^{-1}(\eta(x)) \subset W^i_{x,\delta}$.

Now consider arbitrary x and $y \in \eta(x)$. Suppose $\Phi_x^{-1}(y) \in W^i_{x,\delta}$ and $f(y) \in \eta(f(x))$. By the choice of δ,

$$\Phi^{-1}_{f(x)}(f(y)) \in W^i_{f(x),e^{\lambda_0+2\varepsilon}\delta} \cap \mathbb{R}^k(\delta) = W^i_{f(x),\delta}$$

whence $f(y) \in V^i_{\mathrm{loc},f(x),\varepsilon}$.

For $x \notin S$, let $n(x) = \max\left\{k \geq 1 : f^{-k}(x) \in S\right\}$. For $k \geq 0$ we have $f^{-k}(\eta(x)) \subset \eta\left(f^{-k}(x)\right)$. Then for each $0 \leq k \leq n(x)$ and $y \in \eta(x)$ we have $f^{-k}(y) \in \eta\left(f^{-k}(x)\right)$. Moreover, we have $\Phi^{-1}_{f^{-n(x)}(x)}\left(f^{-n(x)}(y)\right) \in W^i_{f^{-n(x)}(x),\delta}$. By repeated application of the preceding paragraph,

$$\Phi_{f^{-k+1}(x)}(f^{-k+1}(y)) \in W^i_{f^{-k+1}(x),\delta} \quad \text{and} \quad f^{-k+1}(y) \in V^i_{\mathrm{loc},f^{-k+1}(x),\varepsilon}$$

for all $1 \leq k \leq n(x)$ and the claim follows. □

9. LOCAL ENTROPIES ALONG FAST UNSTABLE FOLIATIONS

As in [32, Proposition 7.2.1, (9.2), and (9.3)], we establish a version of the Brin–Katok entropy formula (c.f. [10]) for entropy conditioned along fast unstable foliations $\mathscr{W}^i$ and relate it to the entropy given by the partitions ξ^i constructed in Section 8.2.

Let η be a measurable partition of (M,μ). We do not assume $h_\mu(f \mid \eta) < \infty$. Given $1 \leq i \leq r$, let ξ^i be any measurable partition subordinate to $\mathscr{W}^i$.

For $1 \leq i \leq r$ define the *ith unstable Bowen ball*

$$V^i(x,n,\delta) := \left\{y : d_{V^i_{\mathrm{loc},f^k(x),\varepsilon}}(f^k(x), f^k(y)) < \delta \text{ for } 0 \leq k \leq n\right\}.$$

Define for $1 \leq i \leq r$

- $\overline{h}_i(x,\delta,\eta) := \limsup_{n\to\infty} -\frac{1}{n}\log\left(\mu_x^{\eta^+\vee\xi^i} V^i(x,n,\delta)\right)$;
- $\underline{h}_i(x,\delta,\eta) := \liminf_{n\to\infty} -\frac{1}{n}\log\left(\mu_x^{\eta^+\vee\xi^i} V^i(x,n,\delta)\right)$;
- $\overline{h}_i(x,\eta) := \lim_{\delta\to 0} \overline{h}_i(x,\delta,\eta)$;
- $\underline{h}_i(x,\eta) := \lim_{\delta\to 0} \underline{h}_i(x,\delta,\eta)$.

The last two limits exist by monotonicity. It is clear that the definitions are independent of the choice of partitions ξ^i subordinate to $\mathscr{W}^i$.

To unify formulas later, let ξ^0 denote the point partition on (M,μ). Then $\mu_x^{\xi^0\vee\eta^+} = \delta_x$ is atomic and, with the same notations as above, we have

- $\underline{h}_0(x,\eta) = \overline{h}_0(x,\eta) = 0$.

PROPOSITION 9.1. *For $1 \leq i \leq r$, let ξ_i be a partition as in Section 8.2. Then for μ-a.e. x,*

$$\underline{h}_i(x,\eta) = \overline{h}_i(x,\eta) =: h_i(x,\eta) = h_\mu(f,\xi^i \vee \eta) = H_\mu(f^{-1}(\eta^+ \vee \xi^i) \mid \eta^+ \vee \xi^i).$$

Moreover, $h_\mu(f,\xi^i\vee\eta)<\infty$.

We remark that finiteness of $h_\mu(f,\xi^i\vee\eta)$ follows from 9.1.1 below and the inequality (see the proof of Claim 8.4 and [31, Lemma 4.1.4])

$$\overline{h}_i(x,\eta)<\infty.$$

Write $h_i(\eta)=h_\mu(f,\xi^i\vee\eta)$ for the almost-surely constant value of $h_i(x,\eta)$. Note that $h_i(\eta)$ is independent of the choice of $\varepsilon<\varepsilon_0$ and the family of Lyapunov charts $\{\Phi_x\}$ and, by Lemma 8.7, independent of the choice of ξ^i.

9.1. **Proof of Proposition 9.1.** We prove Proposition 9.1 in two steps.

9.1.1. *Proof that* $\underline{h}_i(x,\eta)\geq H_\mu\left(f^{-1}\left(\eta^+\vee\xi^i\right)\mid\eta^+\vee\xi^i\right)$. For fixed integers $1\leq i\leq r$ and $k\geq 0$, write

$$\xi^{i,k}(x):=(f^{-k}\xi^i)(x).$$

For almost every x, the set $\xi^{i,k}(x)$ contains an open neighborhood of x in $\xi^i(x)$. Thus

$$\mu_x^{\xi^{i,k}\vee\eta^+}=\frac{1}{\mu_x^{\xi^i\vee\eta^+}\left(\left(\xi^{i,k}\vee\eta^+\right)(x)\right)}\,\mu_x^{\xi^i\vee\eta^+}\restriction_{(\xi^{i,k}\vee\eta^+)(x)}.$$

Then,

$$\begin{aligned}\frac{\mu_x^{\xi^{i,k}\vee\eta^+}\left(f^{-1}\left(\xi^{i,k}\vee\eta^+\right)(x)\right)}{\mu_x^{\xi^{i,k}\vee\eta^+}\left(f^{-1}\xi^{i,k}(x)\right)}&=\frac{\mu_x^{\xi^i\vee\eta^+}\left(f^{-1}\left(\xi^{i,k}\vee\eta^+\right)(x)\right)}{\mu_x^{\xi^i\vee\eta^+}\left(f^{-1}\xi^{i,k}(x)\right)}\\&=\mathbb{E}_{\mu_x^{\xi^i\vee\eta^+}}\left(\mathbb{1}_{f^{-1}\eta^+(x)}\mid\sigma\left(f^{-1}\xi^{i,k}\right)\right)(x).\end{aligned}$$

Given $b>0$, set

$$A_{b,k}:=\left\{x:\frac{\mu_x^{\xi^{i,n}\vee\eta^+}\left(f^{-1}\left(\xi^{i,n}\vee\eta^+\right)(x)\right)}{\mu_x^{\xi^{i,n}\vee\eta^+}\left(f^{-1}\xi^{i,n}(x)\right)}\geq e^{-b}\text{ for all }n\geq k\right\}.$$

By pointwise convergence for martingales, we have $\mu(A_{b,k})\to 1$ as $k\to\infty$.

Given $b>0,\delta>0$ and $k\in\mathbb{N}$, set

$$A_{b,k,\delta}:=\left\{x\in A_{b,k}:B^i(f(x),\delta)\subset\xi^{i,k}(f(x))\right\}.$$

We have $\mu(A_{b,k,\delta})\to\mu(A_{b,k})$ as $\delta\to 0$. Given any

$$M<H_\mu\left(f^{-1}\left(\eta^+\vee\xi^{i,k}\right)\mid\eta^+\vee\xi^{i,k}\right)=H_\mu\left(f^{-1}\left(\eta^+\vee\xi^i\right)\mid\eta^+\vee\xi^i\right)$$

and

$$0<b<H_\mu(f^{-1}(\eta^+\vee\xi^{i,k})\mid\eta^+\vee\xi^{i,k})-M$$

choose first k, then δ_0 so that for any $0<\delta\leq\delta_0$

$$M+b\leq\int_{A_{b,k,\delta}}-\log\left(\mu_x^{\eta^+\vee\xi^{i,k}}\left(f^{-1}\left(\eta^+\vee\xi^{i,k}\right)(x)\right)\right)\,d\mu(x).$$

Then

$$M\leq\int_{A_{b,k,\delta}}-\log\left(\mu_x^{\eta^+\vee\xi^{i,k}}\left(\left(f^{-1}\xi^{i,k}\right)(x)\right)\right)\,d\mu(x).$$

Define

$$U_i(x,n,\delta) := \bigcap_{0\le j\le n-1:f^j(x)\in A_{b,k,\delta}} \left(f^{-(j+1)}\xi^{i,k}\right)(x).$$

Then for all sufficiently large $n \ge 1$ (such that $f^j(x) \in A_{b,k,\delta}$ for some $0 \le j \le n-1$) we have

$$V^i(x,n,\delta) \subset U_i(x,n,\delta)$$

and with

$$g(x) := -\log \mu_x^{\eta^+\vee\xi^{i,k}}(f^{-1}\xi^{i,k})(x)$$

we have

$$-\log \mu_x^{\eta^+\vee\xi^{i,k}}(U_i(x,n,\delta)) \ge \sum_{j=0}^{n-1} (\mathbb{1}_{A_{b,k,\delta}}\cdot g)\left(f^j(x)\right).$$

Then for a.e. x,

$$\begin{aligned}\underline{h}_i(x,\delta,\eta) &:= \liminf_{n\to\infty} -\frac{1}{n}\log \mu_x^{\eta^+\vee\xi^i}(V^i(x,n,\delta))\\ &\ge \liminf_{n\to\infty} -\frac{1}{n}\log \mu_x^{\eta^+\vee\xi^i}(U_i(x,n,\delta))\\ &\ge \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}(\mathbb{1}_{A_{b,k,\delta}}\cdot g)(f^j(x))\\ &= \int_{A_{b,k,\delta}} g\, d\mu \ge M\end{aligned}$$

and the inequality follows. □

9.1.2. *Proof that* $\overline{h}_i(x,\eta) \le H_\mu\left(f^{-1}\left(\eta^+\vee\xi^i\right) \mid \eta^+\vee\xi^i\right)$. Given $\ell \le k$ and a partition η of (M,μ) define

$$\eta_\ell^k := \bigvee_{j=\ell}^{k} f^{-j}\eta. \tag{25}$$

Exactly as in [32, Lemma 9.3.1], the following follows from the Chung–Neveu lemma (see [39, Lemma 2.1].)

LEMMA 9.2. *Let ζ be a partition of M with $H_\mu(\zeta \mid \xi^i\vee\eta^+) < \infty$. Then for μ-a.e. x*

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu_x^{\xi^i\vee\eta^+}\left((\zeta\vee\xi^i\vee\eta)_0^n(x)\right) = H_\mu\left(\xi^i\vee\eta \mid f(\xi^i\vee\eta^+)\right).$$

Recall the unstable manifolds $W^i_{x,\delta} = \operatorname{graph}\left(h^i_x\restriction_{\mathbb{R}_{j\le i}(\delta)}\right)$ defined inside charts the Φ_x in Section 8.1. With the notation of (25), we have the following.

LEMMA 9.3. *Given $0<\varepsilon'$ sufficiently small, there exists a partition $\mathscr{P}$ of (M,μ) with $H_\mu(\mathscr{P}) < \infty$ and a measurable $n_0\colon M \to \mathbb{N}$ such that for every $1 \le i \le r$, almost every $x \in M$, and $n \ge n_0(x)$,*

$$\Phi_x^{-1}\left(\left(\mathscr{P}_0^n\vee\xi^i\right)(x)\right) \subset W^i_{x,\varepsilon' e^{-(\lambda_i-2\varepsilon)n}}.$$

In particular, for $n \ge n_0(x)$,

(a) $\left(\mathscr{P}_0^n\vee\xi^i\right)(x) \subset V^i\left(x,n,\varepsilon'\right)$;

(b) $\left(\mathscr{P}_0^n \vee \xi^i\right)(x) \subset B^i\left(x, e^{-(\lambda_i - 2\varepsilon)n}\right)$.

The final two conclusions follow from the fact each chart Φ_x is 1-Lipschitz for almost every x. The proof is nearly identical to that of [32, Lemmas 9.3.3, 9.3.2]. We include it for completeness.

We note in the statement and proof of Lemma 9.3 that the same partition $\mathscr{P}$ works for all indices i.

Proof of Lemma 9.3. Let $S = S^0_\rho$ be as in Section 8.2. In particular, $\mu(S) > 0$, $\ell(x) \le \ell_0$ for $x \in S$, and there is an open ball N with $\overline{N}$ a closed ball such that $S \subset N$.

Given $x \in S$, define

$$n_+(x) = \min\left\{n > 0 : f^n(x) \in S\right\}, \qquad n_-(x) = \min\left\{n > 0 : f^{-n}(x) \in S\right\}.$$

Let $\delta = e^{-2\lambda_0 - 4\varepsilon}$ and consider any $0 < \varepsilon' < \delta$. Let $\psi, \psi_+ \colon S \to (0,1)$ be

$$\psi(x) = \varepsilon' {\ell_0}^{-1} e^{-(\lambda_0 + 2\varepsilon)\max\{n_+(x), n_-(x)\}}, \quad \psi_+(x) = \varepsilon' {\ell_0}^{-1} e^{-(\lambda_0+2\varepsilon)n_+(x)}.$$

Let $\mu_S = \frac{1}{\mu(S)}\mu\restriction_S$. Then $\int \log(\psi)\, d\mu_S < \infty$. Adapting [37, Lemma 2] (to $(\overline{N}, \mu_S)$), we may find a measurable partition $\mathscr{P}'$ of S with

1. $H_{\mu_S}\left(\mathscr{P}'\right) < \infty$;
2. $\mathscr{P}'(x) \subset B(x, \psi(x))$ for all $x \in S$.

Let $\mathscr{P} = \{S, M \setminus S\} \vee \mathscr{P}'$. Then we still have $H_\mu(\mathscr{P}) < \infty$ and $\mathscr{P}(x) = \mathscr{P}'(x) \subset B(x, \psi(x))$ for all $x \in S$.

For $x \in M$ define

$$n_0(x) := \min\{n \ge 0 : f^n(x) \in S\}$$

and

$$r_0(x) := \max\left\{n < 0 : f^n(x) \in S\right\} = n_0(x) - n_-(f^{n_0(x)}(x)).$$

We claim the lemma holds with $\mathscr{P}$ and n_0 defined above.

First consider for $x \in S$. From the dynamics inside Lyapunov charts we have, exactly as in [32, Lemma 9.3.2(1)], that if $\Phi_x^{-1}(y) \in W^i_{x,\delta}$ and $d(x,y) \le \psi_+(x)$, then

$$\left\|\Phi^{-1}_{f^j(x)}\left(f^j(y)\right)\right\| < \varepsilon', \quad \text{for } 0 \le j \le n_+(x) \tag{26}$$

and

$$\Phi^{-1}_{f^{n_+(x)}(x)}\left(f^{n_+(x)}(y)\right) \in W^i_{f^{n_+(x)}(x),\delta}. \tag{27}$$

Consider x with $f^n(x) \in S$ infinitely often as $n \to \pm\infty$. Consider any

$$y \in \mathscr{P}_0^{n_0(x)}(x) \cap \xi^i(x) := \bigvee_{j=0}^{n_0(x)} f^{-j}\mathscr{P}(x) \cap \xi^i(x).$$

Note first that $f^{n_0(x)}(y) \in \mathscr{P}\left(f^{n_0(x)}(x)\right)$. From the choice of $\psi(x)$, for $0 \le j \le n_0(x) - r_0(x)$ we have

$$\Phi^{-1}_{f^{n_0(x)-j}(x)}\left(f^{n_0(x)-j}\left(\mathscr{P}\left(f^{n_0(x)}(x)\right)\right)\right) \subset \mathbb{R}^k(\delta). \tag{28}$$

Since ξ^i is increasing we have $f^{r_0(x)}(y) \in \xi^i(f^{r_0(x)}(x))$ and since $f^{r_0(x)}(x) \in S$ we have

$$\Phi^{-1}_{f^{r_0(x)}(x)}\left(f^{r_0(x)}(y)\right) \in W^i_{f^{r_0(x)}(x),\delta}.$$

It follows from the choice of δ, (28), and the dynamics inside Lyapunov charts that for $0 \le j \le n_0(x) - r_0(x)$,

$$\Phi^{-1}_{f^{r_0(x)+j}(x)}\left(f^{r_0(x)+j}(y)\right) \in W^i_{f^{r_0(x)+j}(x),\delta}.$$

In particular,

$$\Phi^{-1}_{f^{n_0(x)}(x)}\left(f^{n_0(x)}(y)\right) \in W^i_{f^{n_0(x)}(x),\delta}$$

and

$$d\left(f^{n_0(x)}(y), f^{n_0(x)}(x)\right) \le \psi_+\left(f^{n_0(x)}(x)\right).$$

Now, let $n_k(x)$ denote the subsequent returns of x to S. For $n \ge n_0(x)$, take k with $n_k(x) \le n < n_{k+1}(x)$. If $y \in (\mathscr{P}_0^n \vee \xi^i)(x)$, then $f^{n_j(x)}(y) \in \mathscr{P}(f^{n_j(x)}(x))$ for all $0 \le j \le k$ whence

$$d\left(f^{n_k(x)}(y), f^{n_k(x)}(x)\right) \le \psi_+\left(f^{n_k(x)}(x)\right)$$

and we recursively verify as in (27) that

$$\Phi^{-1}_{f^{n_{k+1}(x)}(x)} f^{n_{k+1}(x)}(y) \in W^i_{f^{n_{k+1}(x)}(x),\delta}.$$

Then for our $n_k(x) \le n < n_{k+1}(x)$, we have

$$\Phi^{-1}_{f^n(x)}\left(f^n(y)\right) \in W^i_{f^n(x),\delta}$$

and as in (26),

$$\left\|\Phi^{-1}_{f^n(x)}(f^n(y))\right\| < \varepsilon'.$$

The results follows applying the dynamics along W^i manifolds inside Lyapunov charts. □

Proof that $\overline{h}_i(x,\eta) \le H_\mu\left(f^{-1}\left(\eta^+ \vee \xi^i\right) \mid \eta^+ \vee \xi^i\right)$. Given $\varepsilon' > 0$ sufficiently small,

$$\begin{aligned}
\overline{h}_i(x,\varepsilon',\eta) &:= \limsup_{n\to\infty} -\frac{1}{n}\log\left(\mu_x^{\eta^+\vee\xi^i} V^i(x,n,\varepsilon')\right) \\
&\le \limsup_{n\to\infty} -\frac{1}{n}\log\left(\mu_x^{\eta^+\vee\xi^i} \mathscr{P}_0^n(x)\right) \\
&\le \limsup_{n\to\infty} -\frac{1}{n}\log\left(\mu_x^{\eta^+\vee\xi^i} (\eta^+ \vee \xi^i \vee \mathscr{P})_0^n(x)\right) \\
&= H_\mu\left(f^{-1}\left(\eta^+ \vee \xi^i\right) \mid \eta^+ \vee \xi^i\right).
\end{aligned}$$

where the first inequality follows from Lemma 9.3 and the final equality follows from Lemma 9.2. □

10. BOUNDS ON LOCAL ENTROPIES

10.1. **Proof of Propositions 7.1 and 7.4, Claim 7.5, Theorem 7.7, and Proposition 8.3.** Consider a measurable partition η of (M,μ). As we have not yet established Proposition 7.1, we will moreover assume that $h_\mu(f,\eta)<\infty$. (This is sufficient to prove Proposition 7.1 from which it follows that $h_\mu(f,\eta)<\infty$ for all η.) Propositions 7.1 and 7.4, Claim 7.5, Theorem 7.7, and Proposition 8.3 then follow directly from the following four inequalities whose proofs occupy the rest of this and the following section.

Recall the local entropies $h_i(\eta)$ defined in Section 9 (and independent of $\varepsilon>0$). We claim for $1\le i\le r$ that the following inequalities hold:

(I) $h_i(\eta)-h_{i-1}(\eta)\ge\lambda_i\left(\overline{\dim}^i(\mu|\eta^+)-\overline{\dim}^{i-1}(\mu|\eta^+)\right)$;
(II) $h_i(\eta)-h_{i-1}(\eta)\le\lambda_i\left(\underline{\dim}^i(\mu|\eta^+)-\underline{\dim}^{i-1}(\mu|\eta^+)\right)$;
(III) $h_i(\eta)-h_{i-1}(\eta)\le\lambda_i m_i$;
(IV) $h_r(\eta)=h_\mu(f\mid\eta)$.

10.2. **Proof of (I).** This is identical to [32, (10.2)]; we include it for completeness. Recall our fixed $0<\varepsilon<\varepsilon_0$ and family of ε-Lyapunov charts $\{\Phi_x\}$ with corresponding function ℓ. For $0\le i\le r$, let ξ^i be a measurable partition as in Section 8.2.

LEMMA 10.1. *For each $1\le i\le r$, there exists a partition $\mathscr{P}$ with $H(\mathscr{P})<\infty$ and a measurable function $n_0\colon M\to\mathbb{N}$ such that for μ-a.e. x, the following hold for all $n\ge n_0(x)$:*

(a) $$\frac{\log\mu_x^{\xi^{i-1}\vee\eta^+}B^{i-1}\left(x,e^{-n(\lambda_i-2\varepsilon)}\right)}{-n(\lambda_i-2\varepsilon)}\le\overline{\dim}^{i-1}(\mu\mid\eta^+)+\varepsilon;$$
(b) $-\frac{1}{n}\log\mu_x^{\xi^{i-1}\vee\eta^+}\mathscr{P}_0^n(x)\ge h_{i-1}(\eta)-\varepsilon$;
(c) $\xi^i(x)\cap\mathscr{P}_0^n(x)\subset B^i\left(x,e^{-n(\lambda_i-2\varepsilon)}\right)$;
(d) $-\frac{1}{n}\log\mu_x^{\xi^{i}\vee\eta^+}\mathscr{P}_0^n(x)\le h_{i}(\eta)+\varepsilon$;
(e) $B^{i-1}\left(x,e^{-n(\lambda_i-2\varepsilon)}\right)\subset\xi^{i-1}(x)$.

Moreover, for infinitely many $n\ge n_0(x)$

(f) $$\frac{\log\mu_x^{\xi^{i}\vee\eta^+}B^{i}\left(x,2e^{-n(\lambda_i-2\varepsilon)}\right)}{-n(\lambda_i-2\varepsilon)}\ge\overline{\dim}^{i}(\mu\mid\eta^+)-\varepsilon.$$

Proof. (a), (e), and (f) follow from definition. (b) and (c) follows from Lemma 9.3 taking sufficiently small $\varepsilon'>0$. (d) follows from Lemma 9.2 as $\mathscr{P}_0^n(x)\supset(\mathscr{P}\vee\xi^i\vee\eta^+)_0^n(x)$. Note also that (a), (b), (d) and (e) hold trivially when $i-1=0$. □

We now prove the first inequality of Section 10.1.

Proof of (I). We retain all notation from Lemma 10.1.

With $\Gamma:=\{x:n_0(x)\le N_1\}$, select N_1 large enough so that for some $x\in\Gamma$ satisfying Lemma 10.1 we have for all $n\ge N_1$

$$\mu_x^{\xi^{i-1}\vee\eta^+}\left(\Gamma\cap B^{i-1}\left(x,e^{-n(\lambda_i-2\varepsilon)}\right)\right)\geq\frac{1}{2}\mu_x^{\xi^{i-1}\vee\eta^+}\left(B^{i-1}\left(x,e^{-n(\lambda_i-2\varepsilon)}\right)\right)$$
$$\geq\frac{1}{2}e^{(-n(\lambda_i-2\varepsilon))\left(\overline{\dim}^{i-1}(\mu|\eta^+)+\varepsilon\right)}.$$

Fix such an x.

For $n\geq N_1$, take $L_n:=B^{i-1}\left(x,e^{-n(\lambda_i-2\varepsilon)}\right)$. For $y\in\Gamma\cap L_n\cap\eta^+(x)$ and $n\geq N_1$, using that $\xi^{i-1}(y)=\xi^{i-1}(x)$ and Lemma 10.1(e), we have $B^{i-1}\left(y,e^{-n(\lambda_i-2\varepsilon)}\right)\subset\xi^{i-1}(x)$ and hence from Lemma 10.1(b),

$$\mu_x^{\xi^{i-1}\vee\eta^+}\mathscr{P}_0^n(y)=\mu_y^{\xi^{i-1}\vee\eta^+}\mathscr{P}_0^n(y)\leq e^{-n(h_{i-1}(\eta)-\varepsilon)}.$$

For $n\geq N_1$, we a obtain lower bound on the cardinality of the number of distinct $\mathscr{P}_0^n\vee\xi^i$-atoms meeting $\Gamma\cap L_n\cap\eta^+(x)$ by

$$\#\left\{\left(\mathscr{P}_0^n\vee\xi^i\right)(y):y\in\Gamma\cap L_n\cap\eta^+(x)\right\}\geq\mu_x^{\xi^{i-1}\vee\eta^+}(\Gamma\cap L_n)/e^{-n(h_{i-1}(\eta)-\varepsilon)}$$
$$\geq\frac{1}{2}e^{(-n(\lambda_i-2\varepsilon))\left(\overline{\dim}^{i-1}(\mu|\eta^+)+\varepsilon\right)}e^{n(h_{i-1}(\eta)-\varepsilon)}.$$

For $y\in\Gamma\cap L_n\cap\eta^+(x)$, we have by Lemma 10.1(c) that

$$\left(\mathscr{P}_0^n\vee\xi^i\right)(y)\subset B^i\left(y,e^{-n(\lambda_i-2\varepsilon)}\right)$$

whence $(\mathscr{P}_0^n\vee\xi^i)(y)\subset B^i\left(x,2e^{-n(\lambda_i-2\varepsilon)}\right)$. From Lemma 10.1(d), we have

$$\mu_y^{\xi^i\vee\eta^+}\mathscr{P}_0^n(y)\geq e^{-n(h_i(\eta)+\varepsilon)}$$

and obtain inequalities

$$\mu_x^{\xi^i\vee\eta^+}B^i\left(x,2e^{-n(\lambda_i-2\varepsilon)}\right)$$
$$\geq\#\left\{(\mathscr{P}_0^n\vee\xi^i)(y):y\in\Gamma\cap L_n\cap\eta^+(x)\right\}\cdot e^{-n(h_i(\eta)+\varepsilon)}$$
$$\geq\frac{1}{2}e^{(-n(\lambda_i-2\varepsilon))\left(\overline{\dim}^{i-1}(\mu|\eta^+)+\varepsilon\right)}e^{n(h_{i-1}(\eta)-\varepsilon)}e^{-n(h_i(\eta)+\varepsilon)}.$$

Comparing to Lemma 10.1(f) we have for infinitely many n that

$$(\lambda_i-2\varepsilon)\left(\overline{\dim}^i(\mu\mid\eta^+)-\varepsilon\right)\leq\frac{\log 2}{n}+(\lambda_i-2\varepsilon)\left(\overline{\dim}^{i-1}(\mu\mid\eta^+)+\varepsilon\right)$$
$$+h_i(\eta)-h_{i-1}(\eta)+2\varepsilon.$$

Choosing n sufficiently large we have

$$h_i(\eta)-h_{i-1}(\eta)\geq(\lambda_i-2\varepsilon)\left(\overline{\dim}^i(\mu\mid\eta^+)-\overline{\dim}^{i-1}(\mu\mid\eta^+)-2\varepsilon\right)-3\varepsilon.$$

Inequality (I) follows by the arbitrariness of $\varepsilon<\varepsilon_0$. □

11. PROOFS OF (II), (III), AND (IV)

11.1. **Properties of fake unstable manifolds inside charts.** Recall we fixed $0 < \varepsilon \le \varepsilon_0$, a family of ε-Lyapunov charts $\{\Phi_x\}$, and for $x \in \Lambda_0$, define

$$\tilde{f}_x \colon \mathbb{R}^k\left(e^{-\lambda_0-2\varepsilon}, \|\cdot\|\right) \to \mathbb{R}^k(1, \|\cdot\|)$$

as in Proposition 5.1. It is convenient to extend each locally-defined $\tilde{f}_x$ to a diffeomorphism $F_x \colon \mathbb{R}^k \to \mathbb{R}^k$. Fix a C^∞ bump function $\Theta \colon \mathbb{R}^k \to [0,1]$ with $\Theta(v) = 0$ for $\|v\| \ge 1$ and $\Theta(v) = 1$ for $\|v\| \le \frac{1}{2}$. Given $0 < \hat{\delta} < 1$, define $F_x \colon \mathbb{R}^k \to \mathbb{R}^k$ by

$$F_x(v) = D_0\tilde{f}_x(v) + \Theta(\hat{\delta}^{-1}v)\left(\tilde{f}_x(v) - D_0\tilde{f}_x(v)\right). \tag{29}$$

Assume now that $\hat{\delta} < e^{-2\lambda_0-4\varepsilon}$. We have that F_x is well-defined. Moreover, if $\hat{\delta}$ is sufficiently small, then $\|F_x - D_0\tilde{f}_x\|_{C^1} < \varepsilon$. Moreover, we have a uniform bound $\mathrm{H\ddot{o}l}^\beta(DF_x) \le C$ which will be used later to obtain Lipschitz control on certain holonomies. Fix such sufficiently small $\hat{\delta} > 0$.

As $\|F_x - D_0\tilde{f}_x\|_{C^1} < \varepsilon$, for each $1 \le i \le r+1$ and $z \in \mathbb{R}^k$ that there is a C^1 function

$$g^i_{x,z} \colon \bigoplus_{j\le i}\mathbb{R}^j \to \bigoplus_{j>i}\mathbb{R}^j$$

with $\|Dg^i_{x,z}\| \le \frac{1}{3}$ such that, writing $\widetilde{W}^i_x(z)$ for the graph of $g^i_{x,z}$, we have $z \in \widetilde{W}^i_x(z)$, $F_x(\widetilde{W}^i_x(z)) = \widetilde{W}^i_{f(x)}(F_x(z))$, and $\widetilde{W}^{i-1}_x(z) \subset \widetilde{W}^i_x(z)$. Moreover if $\hat{z} \in \widetilde{W}^i_x(z)$, then $\widetilde{W}^i_x(z) = \widetilde{W}^i_x(\hat{z})$ and

$$e^{\lambda_i-2\varepsilon}\|z-\hat{z}\| \le \|F_x(z) - F_x(\hat{z})\| \le e^{\lambda_1+2\varepsilon}\|z-\hat{z}\|. \tag{30}$$

Moreover, $\widetilde{W}^i_x(z)$ consists of all points $\hat{z} \in \mathbb{R}^k$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\|F^{-1}_{f^{-n}(x)}\circ\cdots\circ F^{-1}_{f^{-1}(x)}(z) - F^{-1}_{f^{-n}(x)}\circ\cdots\circ F^{-1}_{f^{-1}(x)}(\hat{z})\| \le -\lambda_i + 10\varepsilon.$$

Write

$$\widetilde{W}^i_{x,r}(z) := \left\{\hat{z} \in \widetilde{W}^i_x(z) : \|\hat{z}\| < r\right\}. \tag{31}$$

Note we use $\widetilde{W}$ to denote the "fake" unstable manifolds. These depend on the choice of globalized dynamics $F_x \colon \mathbb{R}^k \to \mathbb{R}^k$ above. We note, in particular, that when $i = r+1$, "genuine" center-unstable manifolds through 0 do not exist and one can only define "fake" center-unstable manifolds relative to some choice of globalized dynamics. Claim 11.1 below relates these to "genuine" dynamically-defined objects.

Recall the center-unstable sets $S^{cu}_{\delta,x}$ defined in Section 8.4. We collect a number of properties of the above objects.

CLAIM 11.1. *For $1 \le i \le r$, $\delta < \hat{\delta}/4$, and the notation of (31) the following hold:*

(a) $S^{cu}_{2\delta,x} = \left\{z \in \mathbb{R}^k(2\delta) : F^{-1}_{f^{-n}(x)}\circ\cdots\circ F^{-1}_{f^{-1}(x)}(z) \in \mathbb{R}^k(2\delta) \text{ for all } n \ge 0\right\}$;

(b) $\widetilde{W}^i_{x,2\delta}(0) = W^i_{x,2\delta}$;

(c) $S^{cu}_{2\delta,x} \subset \widetilde{W}^r_{x,2\delta}(0)$ *if* $\lambda_{r+1} < 0$ *and* $S^{cu}_{2\delta,x} \subset \widetilde{W}^{r+1}_{x,2\delta}(0)$ *if* $\lambda_{r+1} = 0$.

Moreover, for $z \in S^{cu}_{\delta,x}$

(d) $\widetilde{W}^i_{x,2\delta}(z) \subset S^{cu}_{2\delta,x}$;
(e) $\widetilde{W}^i_{x,2\delta}(z) = \tilde{f}_{f^{-1}(x)}\left(\widetilde{W}^i_{f^{-1}(x),2\delta}\left(\tilde{f}_x^{-1}(z)\right)\right) \cap \mathbb{R}^k(2\delta)$;
(f) *if* $\Phi_x(z) \in \Lambda_0$, *then*

$$\Phi_x^{-1}\left(V^i_{\mathrm{loc},\Phi_x(z),\varepsilon}\right) \cap S^{cu}_{2\delta,x} \subset \widetilde{W}^i_{x,2\delta}(z);$$

(g) *if* $\Phi_x(z) \in \Lambda_0$ *and* $\ell(\Phi_x(z)) \le e^{-2\lambda_0-4\varepsilon}(4\delta)^{-1}$, *then*

$$\Phi_x\left(\widetilde{W}^i_{x,2\delta}(z)\right) \subset V^i_{\mathrm{loc},\Phi_x(z),\varepsilon}.$$

Proof. Conclusions (a)–(c) follow using that F_x and $\tilde{f}_x$ coincide on $\mathbb{R}^k(2\delta)$ and the dynamics inside charts. Using that F_x is an ε-Lipschitz small perturbation of the linear map $D_0\tilde{f}_x$, for any $\|z\| \le \delta$ we have

$$\widetilde{W}^i_{f(x),2\delta}(F_x(z)) \subset F_x\left(\widetilde{W}^i_{x,2\delta}(z)\right).$$

Indeed, $D_0\tilde{f}_x\left(\widetilde{W}^i_{x,2\delta}(z)\right)$ is the graph of a function defined on an open ball around 0 in $\bigoplus_{j\le i}\mathbb{R}^j$ of radius $e^{\lambda_i-\varepsilon}(2\delta)$; it follows that $F_x\left(\widetilde{W}^i_{x,2\delta}(z)\right)$ is the graph of a function defined on an open ball around 0 of $\bigoplus_{j\le i}\mathbb{R}^j$ of radius $(e^{\lambda_i-\varepsilon}-\varepsilon)2\delta \ge e^{\lambda_i-2\varepsilon}2\delta \ge 2\delta$. Conclusions (d) and (e) follow.

Conclusions (f) and (g) follow from the dynamics of F_x along $\widetilde{W}^i$-leaves and the fact that F_x and the dynamics $\tilde{f}_x$ in the charts Φ_x coincide on $\mathbb{R}^k(2\delta)$. □

11.2. **Construction of auxiliary partitions.** Consider a fixed family of measurable partitions $\{\xi^i\}_{0\le i\le r}$ as in Section 8.2. Recall in the construction of ξ^i in Section 8.2 we fix $x_0 \in \Lambda_1$ to be a density point of Λ_1. Recall the choice of S^0_ρ, S^i_ρ, and ℓ_0 in Section 8.2. Write $S^i = S^i_\rho$. With $\delta_0 = \min\left\{\hat{\delta}/4, e^{-2\lambda_0-4\varepsilon}/(4\ell_0)\right\}$, let $E \subset S^0$ be a set with $\mu(E) > 0$ and

$$E \subset B\left(x_0, \frac{\delta_0\ell_0^{-1}}{2}\right).$$

Then for $x \in E$ we have $E \subset \Phi_x(\mathbb{R}^k(\delta_0))$. Let $\mathscr{P}$ be a measurable partition of (X,μ) with the following properties:

1. $\mathscr{P}$ is adapted to $(\{\Phi_x\},\delta_0)$ (see Definition 8.8);
2. $\mathscr{P}$ refines $\{S^i, M\smallsetminus S^i\}$ for each $0 \le i \le r$;
3. $\mathscr{P}$ refines $\{E, M\smallsetminus E\}$;
4. $\mathscr{P}$ refines $\{E', M\smallsetminus E'\}$ where $E' \subset E$ is a subset with $\mu(E') > 0$ to be specified below;
5. $H_\mu(\mathscr{P}) < \infty$.

Recall that η is an arbitrary measurable partition of (M,μ) (with $h_\mu(f,\eta) < \infty$). Take $\eta_* = (\eta \vee \mathscr{P})^+$ and for $0 \le i \le r$ take

$$\eta_i = \eta_* \vee \xi^i.$$

Note that η_0 is the point partition. For notational convention, we also write $\eta_{r+1} := \eta_*$. Note that if no exponent is zero, it follows as in [28] that $\eta_* = \eta_{r+1} = \eta_r$. In the presence of zero exponents we may have $\eta_r \neq \eta_{r+1}$. From Claim 11.1(c) and Claim 8.10, for almost every x we have

1. $\Phi_x^{-1}(\eta_{r+1}(x)) \subset S^{cu}_{\delta_0,x} \subset \widetilde{W}^{r+1}_{x,\delta_0}(0)$;
2. $\eta_i(x) \subset V^i_{\mathrm{loc},x,\varepsilon}$ for $1 \le i \le r$.

As in [31, Lemmas 3.3.1–3.3.2] and [32, 11.1.2–11.1.3], we have the following.

LEMMA 11.2. *For each $1 \le i \le r$, almost every x, and every $y \in \Lambda_0 \cap \eta_{i+1}(x)$, writing $\hat{y} = \Phi_x^{-1}(y)$, we have*

(a) $\Phi_x\left(\widetilde{W}^i_{x,2\delta_0}(\hat{y})\right) \cap \eta_{i+1}(x) = \eta_i(y)$;
(b) $f^{-1}(\eta_i(y)) = \eta_i(f^{-1}(y)) \cap f^{-1}(\eta_{i+1}(x))$.

Proof. For (a), first consider $z \in \Phi_x\left(\widetilde{W}^i_{x,2\delta_0}(\hat{y})\right) \cap \eta_{i+1}(x)$. We follow the notation in Section 8.2 and show the following claim: for each $i \le j \le r$ we have $z \in \tilde{\xi}^j(y)$. It then follows from definition (see (24)) that $z \in \xi^i(y)$ whence $z \in \eta_i(y)$. For the proof of the claim, recall that $\mathscr{P}$ refines $\{S^j, M \setminus S^j\}$ and thus if $z \in \mathscr{P}^+(y)$, it follows for all $m \le 0$ that $f^m(z) \in S^j$ if and only if $f^m(y) \in S^j$. Moreover, if $z \in \Phi_x\left(\widetilde{W}^i_{x,2\delta_0}(\hat{y})\right)$, using that Lyapunov charts are 1-Lipschitz and (30) we have

$$d\left(f^{-m}(y), f^{-m}(z)\right) \le 2\delta_0 e^{m(-\lambda_i+2\varepsilon)} \le 2\delta_0 e^{m(-\lambda_j+2\varepsilon)},$$

which, by Lemma 8.2 and our choice of δ_0, implies $z \in \tilde{\xi}^j(y)$. For the reverse inclusion, Claim 8.10 implies $\eta_i(y) \subset V^i_{\mathrm{loc},y,\varepsilon} \cap \eta_*(x)$, whence

$$\Phi_x^{-1}(\eta_i(y)) \subset \Phi_x^{-1}\left(V^i_{\mathrm{loc},y,\varepsilon}\right) \cap S^{cu}_{\delta_0,x} \subset \widetilde{W}^i_{x,2\delta_0}\left(\Phi_x^{-1}(y)\right)$$

follows from Claim 11.1(f).

For (b), first note that we have

$$f^{-1}(\eta_i(y)) \subset \eta_i(f^{-1}(y)) \cap f^{-1}(\eta_{i+1}(x))$$

because $f^{-1}(\eta_i(y)) \subset \eta_i(f^{-1}(y))$ and $\eta_i(y) \subset \eta_{i+1}(x)$. For the reverse inequality, set $\hat{y} = \Phi_x^{-1}(y)$ and $\hat{y}_{-1} = \Phi_{f^{-1}(x)}(f^{-1}(y))$. From part (a) and Claim 11.1(e), we have

$$\begin{aligned}
&\Phi_x^{-1}\left(f(\eta_i(f^{-1}(y))) \cap (\eta_{i+1}(x))\right)\\
&\quad = \tilde{f}_{f^{-1}(x)}\left(\widetilde{W}^i_{f^{-1}(x),2\delta_0}(\hat{y}_{-1}) \cap \Phi^{-1}_{f^{-1}(x)}(\eta_{i+1}(f^{-1}(x)))\right) \cap \Phi_x^{-1}(\eta_{i+1}(x)) \cap \mathbb{R}^k(2\delta_0)\\
&\quad = \left(\widetilde{W}^i_{x,2\delta_0}(\hat{y})\right) \cap \Phi_x^{-1}(\eta_{i+1}(x))\\
&\quad = \Phi_x^{-1}(\eta_i(y)).
\end{aligned}$$

□

Note from Lemma 11.2(b), for $1 \le i \le r$ we have

$$h_\mu(f,\eta_i) = H_\mu(f^{-1}\eta_{i+1} \mid \eta_i) \le H_\mu(f^{-1}\eta_{i+1} \mid \eta_{i+1}) = h_\mu(f,\eta_{i+1}).$$

In particular, the partitions η_i carry more of the entropy as i increases.

11.3. **Transverse metrics on η_i/η_{i-1} and transverse dimension.** Recall we fixed $x_0 \in S^0$ and $E \subset B(x_0, \frac{1}{2}\delta_0\ell_0^{-1})$ defined in the previous section. Also recall that for each $1 \le i \le r$ and almost every x we have that $\Phi_x^{-1}(\eta_i(x)) \subset W^i_{x,\delta_0}$ and $\Phi_x^{-1}(\eta_*(x)) \subset S^{cu}_{\delta_0,x}$.

Consider $x \in \Lambda_0$. For $1 \le i \le r+1$, let $V^i = \bigoplus_{j\ge i}\mathbb{R}^j$. Given $y, z \in \eta_*(x)$ note that $\widetilde{W}^{i-1}_{x,2\delta_0}\left(\Phi_x^{-1}(y)\right) \cap V^i$ and $\widetilde{W}^{i-1}_{x,2\delta_0}\left(\Phi_x^{-1}(z)\right) \cap V^i$ are singletons. Let

$$d^i_x(y,z) = \left\| \widetilde{W}^{i-1}_{x,2\delta_0}\left(\Phi_x^{-1}(y)\right) \cap V^i - \widetilde{W}^{i-1}_{x,2\delta_0}\left(\Phi_x^{-1}(z)\right) \cap V^i \right\|.$$

Then d^i_x defines a metric on η_{i-1}-equivalence classes in $\eta_*(x)$; below we will restrict our d^i_x to a metric on η_{i-1}-equivalence classes in $\eta_i(x)$, hence the notation in the superscript. As in [31, Lemma 2.3.2] and [32, Lemma 8.3.2], applying the dynamics in charts we have the following.

CLAIM 11.3. *For $z \in \eta_i(x)$,*

$$d^i_{f(x)}\left(f(x), f(z)\right) \le e^{\lambda_i + 3\varepsilon} d^i_x(x,z).$$

We construction an alternative metric that is independent of the choice of x. This is slightly different from the construction in [31, (4.2)] and [32, (8.4)]; in particular, we use the Lipschitzness of holonomies in the setting of $C^{1+\beta}$ diffeomorphisms established in [11]. For each $1 \le i \le r+1$, let

$$T_i \subset B\left(x_0, 1/2\delta_0\ell_0^{-1}\right)$$

be a $(\dim(\oplus_{j\ge i}E_j(x)))$-dimensional embedded disc that is uniformly transverse to each $V^{i-1}_{\mathrm{loc},y,\varepsilon}$ for $y \in E$ and such that for $1 \le i \le r$, $V^i_{\mathrm{loc},y,\varepsilon} \cap T_i$ is an embedded m_i-dimensional submanifold for each $y \in E$. For $x \in E$ and $y, z \in \eta_*(x)$ define $d^{T_i}(y,z)$ to be

$$d^{T_i}(y,z) := d^{T_i}\left(V^{i-1}_{\mathrm{loc},y,\varepsilon} \cap T_i, V^{i-1}_{\mathrm{loc},z,\varepsilon} \cap T_i\right),$$

where d^{T_i} is the metric on T_i obtained by a bi-Lipschitz identification of T_i with a subset of $\mathbb{R}^{\dim(\oplus_{j\ge i}E_j)}$. For $x \in E$, d^{T_i} defines a metric on η_{i-1}-equivalence classes in $\eta_*(x)$.

For $x \in E$, let $\widetilde{T}_i(x) = \Phi_x^{-1}(T_i)$. We have that $\widetilde{T}_i(x)$ and T_i are bi-Lipschitz equivalent with Lipschitz constant uniform over $x \in E$. Given $x \in E$, consider the holonomy map

$$V^i \cap \widetilde{W}^i_{x,2\delta_0} \to \widetilde{T}_i(x) \cap \widetilde{W}^i_{x,2\delta_0}$$

along $\widetilde{W}^{i-1}_{x,2\delta_0}$-leaves. The main result of [11] ensures this map is uniformly (over $x \in E$) bi-Lipschitz. Given $x \in E$ and $y, x \in \eta_*(x)$, define

$$\tilde{d}^{\widetilde{T}_i,i}_x(y,z) := \left\| \left(\widetilde{W}^{i-1}_{x,2\delta_0}(\Phi_x^{-1}(y)) \cap \widetilde{T}_i(x)\right) - \left(\widetilde{W}^{i-1}_{x,2\delta_0}(\Phi_x^{-1}(z)) \cap \widetilde{T}_i(x)\right) \right\|.$$

It follows that restricted to η_{i-1}-equivalence classes in $\eta_i(x) \subset \Phi_x\left(\widetilde{W}^i_{x,\delta_0}(0)\right)$, the metrics d^i_x and $\tilde{d}^{\widetilde{T}_i,i}_x$ are uniformly (over $x \in E$) bi-Lipschitz equivalent.

Using that the charts $\{\Phi_x : x \in E\}$ are uniformly Lipschitz embeddings, we obtain the following.

CLAIM 11.4. *There is a $N > 1$ (depending on ℓ_0) such that for all $x \in E$ and $y, z \in \eta_i(x)$,*

$$\frac{1}{N} d^{T_i}(y,z) \le d_x^i(y,z) \le N d^{T_i}(y,z).$$

We note the N in Claim 11.4 depends only on the choice of the set E (on which ℓ_0 is bounded from above) but not on the choice of $E' \subset E$ in the choice of $\mathscr{P}$ in Section 11.2. We now specify $E' \subset E$ to be any subset with

$$0 < \mu(E') < \frac{\varepsilon^2}{8 \log N}.$$

(Below, we will define a metric on η_{i-1}-equivalence classes in each η_i relative to which all hyperbolicity is concentrated on returns to E'. The choice of E' with small mass will be used to control bi-Lipschitz changes of metric that arise over subsequent returns to E'.)

Fix $x \in \Lambda_0$ and let

$$n(x) := \min\left\{n \ge 0 : f^{-n}(x) \in E'\right\}.$$

By the choice of $\mathscr{P}$, $n(x)$ is constant on elements of η_*. For $y, z \in \eta_*(x)$, define

$$d^{T_i}(y,z) := d^{T_i}\left(f^{-n(x)}(y), f^{-n(x)}(z)\right). \tag{32}$$

Observe that, as $\eta_{i-1}(f(x)) \subset f\eta_{i-1}(x)$, d^{T_i} defines a metric on η_{i-1}-equivalence classes in $\eta_i(x)$. In particular, we view the space of equivalence classes as a topological quotient $\eta_i(x)/\eta_{i-1}$ endowed with this metric.

For $1 \le i \le r+1$, let

$$B^{T_i}(x,\rho) := \left\{y \in \eta_i(x) : d^{T_i}(y,x) < \rho\right\}$$

denote the ball of η_{i-1}-equivalence classes in $\eta_i(x)$ of radius ρ centered at x; note, in particular, that $y \in B^{T_i}(x,\rho)$ implies $\eta_{i-1}(y) \subset B^{T_i}(x,\rho)$.

For $1 \le i \le r+1$, write

$$\tilde{\gamma}_i(x) = \liminf_{\rho \to 0} \frac{\log \mu_x^{\eta_i}\left(B^{T_i}(x,\rho)\right)}{\log \rho}. \tag{33}$$

Recall that $E^0(x)$ denotes the subspace corresponding to the zero Lyapunov exponent and $E^s(x)$ denotes the subspace corresponding to all negative Lyapunov exponents.

CLAIM 11.5. *For $1 \le i \le r$ and $x \in E$, $\tilde{\gamma}_i(x) \le m_i$ and $\tilde{\gamma}_{r+1}(x) \le \dim E^0(x) + \dim E^s(x)$.*

Proof. For $1 \le i \le r$, note that $\eta_i(x) \subset V^i_{\mathrm{loc},x,\varepsilon}$. For $x \in E$, we may isometrically identify $\eta_i(x)/\eta_{i-1}$ with a subset of $V^i_{\mathrm{loc},x,\varepsilon} \cap T_i$. Fixing bi-Lipschitz charts on $V^i_{\mathrm{loc},x,\varepsilon} \cap T_i$, it follows for almost every $x' \in \eta_i(x)$ (see [31, Lemma 4.1.4]) that $\tilde{\gamma}_i(x') \le m_i$. The result for $i = r+1$ holds identifying T^{r+1} with a subset of $\mathbb{R}^{\dim E^0 + \dim E^s}$. □

The following claim summarizes the discussion in [32, (11.3)–(11.4)] (using the Lipschitzness of holonomies established in [11] rather than the coordinates of [32, (8.4)]).

CLAIM 11.6. *For $1 \le i \le r$, suppose $c \ge 0$ is such that $\tilde{\gamma}_i(x) \ge c$ for almost every $x \in E$. Then*

$$c \le \underline{\dim}^i(\mu|\eta^+) - \underline{\dim}^{i-1}(\mu|\eta^+).$$

Proof. Note that for $x \in E$ we have $\xi^i(x) \subset \Phi_x(W^r_{x,1}) \subset \Phi_x(1)$. Applying the backwards dynamics, for some $m \ge 0$ we have $f^{-m}(x) \in E$ and $\Phi^{-1}_{f^{-m}(x)}\left(f^{-m}\left(\xi^i(x)\right)\right) \in \mathbb{R}^k(2\delta_0) \subset \mathbb{R}^k(\hat{\delta}/2)$. Then the globalized backwards dynamics (29) coincides with the local backwards dynamics on $\Phi^{-1}_x\left(f^{-m}(\xi^i(x))\right)$. Using the local Lipschitz property of holonomies established in [11] (for the globalized past dynamics restricted to $\tilde{f}^{-m}_x \Phi^{-1}_x(\xi^i(x))$ and pushing forward under $\tilde{f}^m$ to charts at x), for each $x \in E$ there is a bi-Lipschitz identification of $\xi^i(x)$ with a subset of $\mathbb{R}^{\dim \oplus_{j \le i} E^j}$ such that for $x' \in \xi^i(x)$, $\xi^{i-1}(x')$ is contained in a horizontal slice $\mathbb{R}^{\dim \oplus_{j<i} E^j} \times \{t'\}$ and if $x'' \notin \xi^{i-1}(x')$, then $\xi^{i-1}(x'')$ and $\xi^{i-1}(x')$ are contained in distinct horizontal slices $\mathbb{R}^{\dim \oplus_{j<i} E^j} \times \{t''\}$ and $\mathbb{R}^{\dim \oplus_{j<i} E^j} \times \{t'\}$, respectively.

Under this identification, we may push forward the measures $\mu_x^{\xi^i \vee \eta^+}$ and $\mu_y^{\eta_i}$ for all $y \in \xi^i(x)$. Note by construction that η_i refines $\xi^i \vee \eta^+(x)$. In particular,

$$\mu_x^{\xi^i \vee \eta^+} = \int \mu_y^{\eta_i} \, d\mu_x^{\xi^i \vee \eta^+}(y).$$

Note also that by definition, $\xi^{i-1}(y) \cap \eta_i(y) = \eta_{i-1}(y)$. As all identifications discussed are bi-Lipschitz, the claim follows from [32, Lemma 11.3.2] and [32, Lemma 11.3.1]. $\square$

Note that if $\hat{\eta}$ is a measurable partition of (M, μ) with $\eta^+ \prec \hat{\eta}^+$, then we may similarly define $\hat{\eta}_* = (\hat{\eta} \vee \mathscr{P})^+$ and $\hat{\eta}_i = \hat{\eta}_* \vee \xi^i$. We can then define for $1 \le i \le r+1$

$$\hat{\gamma}_i(x) = \liminf_{\rho \to 0} \frac{\log \mu_x^{\hat{\eta}_i}\left(B^{T_i}(x,\rho)\right)}{\log \rho}.$$

From [32, Lemma 11.3.2], it follows that

$$\operatorname{ess\,inf} \hat{\gamma}_i(x) \le \operatorname{ess\,inf} \tilde{\gamma}_i(x).$$

Claim 11.6 combined with Proposition 11.7 below and inequality (I) then imply Proposition 7.6.

11.4. **Key Proposition.**

The following proposition is the analogue of [31, Proposition 5.1] and [32, Proposition 11.2].

PROPOSITION 11.7. *For the family of partitions*

$$\eta_{r+1} \prec \eta_r \prec \cdots \prec \eta_2 \prec \eta_1$$

as in Section 11.2, for every $1 \le i \le r+1$ with $\lambda_i \ge 0$ and a.e. x we have

$$(\lambda_i + 3\varepsilon)\tilde{\gamma}_i(x) \ge (1-\varepsilon)\left(h_\mu(f,\eta_i) - h_\mu(f,\eta_{i-1}) - 2\varepsilon\right)$$

where $\tilde{\gamma}_i(x)$ is the transverse dimension (33) associated to $\eta_i(x)/\eta_{i-1}$.

The 3 inequalities (II), (III), and (IV) now follow from Proposition 11.7, Claims 11.6 and 11.5, and the arbitrariness of $\varepsilon > 0$. Indeed, using Corollary 8.6, for $1 \le i \le r$ the quantities

$$h_\mu(f, \eta_i) = h_\mu(f, \xi^i \vee \eta) = h_i(\eta)$$

are independent of the choice of the partition $\mathscr{P}$. Then (III) follows from Proposition 11.7 and Claim 11.5. Inequality (II) follows from Claim 11.6 with

$$c = \frac{(1-\varepsilon)\left(h_\mu(f, \eta_i) - h_\mu(f, \eta_{i-1}) - 2\varepsilon\right)}{(\lambda_i + 3\varepsilon)}$$

and Proposition 11.7 applied to $x \in E$.

For (IV), first note that

$$h_r(\eta) = h_\mu\left(f, \eta \vee \xi^r\right) = h_\mu\left(f, \eta \vee \xi^r \vee \mathscr{P}\right) \le h_\mu(f \mid \eta)$$

where the first equality holds by Proposition 9.1, the second equality holds by Corollary 8.6, and the inequality is from definition. Moreover, given any $M < h_\mu(f \mid \eta)$ we may assume $\mathscr{P}$ in Section 11.2 is chosen so that

$$h_\mu\left(f, \eta \vee \mathscr{P}\right) := h_\mu\left(f, \eta_{r+1}\right) > M. \tag{34}$$

If $\lambda_{r+1} < 0$, then $\eta_r = \eta_{r+1}$ and

$$M \le h_\mu(f, \eta_{r+1}) = h_\mu(f, \eta \vee \xi^r \vee \mathscr{P}) = h_\mu(f, \eta \vee \xi^u) = h_r(\eta) \le h_\mu(f \mid \eta)$$

and the result follows. If $\lambda_{r+1} = 0$, then Proposition 11.7 gives

$$3\varepsilon \dim\left(E^0 \oplus E^s\right) \ge (1-\varepsilon)(M - h_r(\eta) - 2\varepsilon).$$

As $\varepsilon < \varepsilon_0$ is arbitrary, we have $M \le h_r(\eta)$ and (IV) follows.

We remark in the degenerate situation $\lambda_1 < 0$ that η_* is the point partition and it follows that $h_\mu(f) = 0$. Similarly, if $\lambda_1 = 0$, then $\eta_* = \eta_1$ and η_0 is the point partition. Proposition 11.7 applies to this setting and shows again that $h_\mu(f) = 0$.

11.5. **Proof of Proposition 11.7.** The proof is identical to [31, Proposition 5.1] (and [32, Proposition 11.2]); we include it for completeness. For $1 \le i \le r+1$ with $\lambda_i \ge 0$, define measurable functions $g, g_\delta, g_* \colon M \to \mathbb{R}$ as follows:

$$\begin{aligned}
g(x) &:= \mu_x^{\eta_{i-1}}\left(f^{-1}\eta_i(x)\right) = \mu_x^{\eta_{i-1}}\left(f^{-1}\eta_{i-1}(x)\right)\\
g_\delta(x) &:= \frac{1}{\mu_x^{\eta_i}\left(B^{i,T}(x,\delta)\right)} \int_{B^{i,T}(x,\delta)} \mu_z^{\eta_{i-1}}\left(\left(f^{-1}\eta_i\right)(x)\right)\, d\mu_x^{\eta_i}(z)\\
&= \frac{\mu_x^{\eta_i}\left[\left(B^{i,T}(x,\delta)\right) \cap (f^{-1}\eta_i)(x)\right]}{\mu_x^{\eta_i}\left(B^{i,T}(x,\delta)\right)}\\
g_*(x) &:= \inf_{\delta > 0} g_\delta(x).
\end{aligned}$$

The equality in the definition of $g(x)$ follows from Lemma 11.2.

For $1 \le i \le r+1$ and almost every x, the metric d^i identifies the quotient $\left(\eta_i(x)/\eta_{i-1}\right)$ with a subset of $\mathbb{R}^{\dim(\oplus_{j\ge i} E_j)}$. This identification gives a measurable map $\eta_i(x) \to \mathbb{R}^{\dim(\oplus_{j\ge i} E_j)}$ that is constant on elements of η_{i-1}. We may apply [31, Lemma 4.1.3] to this setting (taking α in the statement of [31, Lemma 4.1.3] to be the partition $f^{-1}\eta_i$ of $\eta_i(x)$) and obtain

$$\int -\log g_*(z)\; d\mu_x^{\eta_i}(z) \le H_{\mu_x^{\eta_i}}(f^{-1}\eta_i) + \log c + 1$$

where c is a constant depending only on the dimension of $\mathbb{R}^{\dim(\oplus_{j\ge i}E_j)}$ coming from the Besicovitch Covering Lemma (see [31, (4.1)].) It follows that $g_\delta \to g$ μ-a.e. and that

$$\begin{aligned}\int -\log g_*(z)\; d\mu(z) &\le \int H_{\mu_x^{\eta_i}}(f^{-1}\eta_i)\; d\mu(x) + \log c + 1\\ &\le H(f^{-1}\eta_i \mid \eta_i) + \log c + 1\\ &< \infty.\end{aligned}$$

Recall our choice of $E' \subset E$ with $0 < \mu(E') < \frac{\varepsilon^2}{8\log N}$ relative to which our metric in (32) is defined. Recall that relative to the metric $d^{T_i}(\cdot,\cdot)$, all hyperbolicity is seen only on subsequent returns to E'.

Consider x whose forwards orbit returns to E' infinitely often and let $r_0 \le 0 < r_1 < \dots$ denote the distinct times when $f^{r_i}(x) \in E'$. For n sufficiently large and for $0 \le k < n$ with $r_j \le k < r_{j+1}$ write

$$a(x;n,k) := B^{T_i}\left(f^k(x), e^{-(\lambda_i+3\varepsilon)(n-r_j)}N^{2j}\right) \subset \eta_i(f^k(x)).$$

We check from definition the following.

CLAIM 11.8. $a(x;n,k) \cap \left(f^{-1}\eta_i\right)\left(f^k(x)\right) \subset f^{-1}(a(x;n,k+1))$.

Proof. First consider $r_j \le k < r_{j+1}-1$ for some $j \ge 0$. By definition of the metric $d^{T_i}(\cdot,\cdot)$,

$$f(a(x;n,k)) \cap \eta_i(f^{k+1}(x)) = B^{T_i}\left(f^{k+1}(x), e^{-(\lambda_i+3\varepsilon)(n-r_j)}N^{2j}\right) = a(x;n,k+1).$$

If $k = r_{j+1}-1$ for some $j \ge 0$, then, recursively applying the above,

$$\begin{aligned}&f(a(x;n,k)) \cap \eta_i(f^{k+1}(x))\\ &\quad = f^{r_{j+1}-r_j}\left(B^{T_i}\left(f^{r_j}(x), e^{-(\lambda_i+3\varepsilon)(n-r_j)}N^{2j}\right)\right) \cap \eta_i(f^{r_{j+1}}(x))\\ &\quad \subset B^{T_i}\left(f^{r_{j+1}}(x), e^{-(\lambda_i+3\varepsilon)(n-r_j)}e^{(\lambda_i+3\varepsilon)(r_{j+1}-r_j)}N^{2j}N^2\right)\\ &\quad = a(x,n,k+1)\end{aligned}$$

where the growth of $e^{(\lambda_i+3\varepsilon)(r_{j+1}-r_j)}$ along the orbit is bounded above by Claim 11.3 and the factor of N^2 comes by converting between the metrics d^{T_i} and $d^i_{f^{r_j}(x)}$ and $d^i_{f^{r_{j+1}}(x)}$ using Claim 11.4. □

Consider the quantity $\mu_x^{\eta_i}(a(x;n,0))$. Write $p = p(n) := \lfloor (1-\varepsilon)n \rfloor$. Then

$$\mu_x^{\eta_i}(a(x;n,0)) = \mu_{f^p(x)}^{\eta_i}(a(x;n,p)) \prod_{k=0}^{p-1} \frac{\mu_{f^k(x)}^{\eta_i}(a(x;n,k))}{\mu_{f^{k+1}(x)}^{\eta_i}(a(x;n,k+1))}$$
$$\leq \prod_{k=0}^{p-1} \frac{\mu_{f^k(x)}^{\eta_i}(a(x;n,k))}{\mu_{f^{k+1}(x)}^{\eta_i}(a(x;n,k+1))}.$$

Renormalizing and applying Claim 11.8,

$$\begin{aligned}
&\frac{\mu_{f^k(x)}^{\eta_i}(a(x;n,k))}{\mu_{f^{k+1}(x)}^{\eta_i}(a(x;n,k+1))} \\
&= \mu_{f^k(x)}^{\eta_i}(a(x;n,k)) \frac{\mu_{f^k(x)}^{\eta_i}((f^{-1}\eta_i)(f^k(x)))}{\mu_{f^k(x)}^{\eta_i}(f^{-1}(a(x;n,k+1)))} \\
&\leq \frac{\mu_{f^k(x)}^{\eta_i}(a(x;n,k))}{\mu_{f^k(x)}^{\eta_i}\left[(f^{-1}\eta_i)(f^k(x)) \cap (a(x;n,k))\right]} \mu_{f^k(x)}^{\eta_i}((f^{-1}\eta_i)(f^k(x))) \\
&= \left[g_{\delta(x;n,k)}(f^k(x))\right]^{-1} e^{-I(f^k(x))}
\end{aligned}$$

where

$$\delta(x;n,k) := e^{-(\lambda_i+3\varepsilon)\left(n-r_{j_k}\right)} N^{2j_k}$$

for $r_{j_k} \leq k < r_{j_k+1}$ and

$$I(y) := -\log \mu_y^{\eta_i}\left(\left(f^{-1}\eta_i\right)(y)\right).$$

Then

$$\begin{aligned}
(\lambda_i+3\varepsilon) \liminf_{r\to 0} &\frac{\log\left(\mu_x^{\eta_i} B^{T_i}(x,r)\right)}{\log r} \\
&= (\lambda_i+3\varepsilon) \liminf_{n\to\infty} \frac{\log\left(\mu_x^{\eta_i} B^{T_i}\left(x, e^{-(\lambda_i+3\varepsilon)n}\right)\right)}{\log e^{-(\lambda_i+3\varepsilon)n}} \\
&= (\lambda_i+3\varepsilon) \liminf_{n\to\infty} \frac{\log\left(\mu_x^{\eta_i} a(x;n,0)\right)}{\log e^{-(\lambda_i+3\varepsilon)n}} \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{p(n)} \log g_{\delta(x;n,k)}\left(f^k(x)\right) + \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{p(n)} I\left(f^k(x)\right).
\end{aligned} \tag{35}$$

For almost every x, $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{p(n)} I(f^k(x))$ exists and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{p(n)} I(f^k(x)) = (1-\varepsilon) H_\mu(\eta_i \mid f\eta_i) = (1-\varepsilon) h_\mu(f,\eta_i). \tag{36}$$

Now, let

$$A_{\delta'} := \left\{x : -\log g_\delta(x) \leq -\log g(x) + \varepsilon \text{ for all } \delta \leq \delta'\right\}.$$

Since $\int -\log g_* < \infty$, taking δ' sufficiently small we can ensure

$$\int_{M \smallsetminus A_{\delta'}} -\log g_* < \varepsilon.$$

For a.e. x, we may take n sufficiently large so that

$$J_n = \#\left\{0 \le i \le n : f^i(x) \in E'\right\} \le 2n\mu(E').$$

By the choice of $\mu(E')$, for large enough n we have

$$\delta(x;n,k) := e^{-(\lambda_i+3\varepsilon)(n-r_{j_k})} N^{2j_k} \le e^{-\varepsilon(n-p(n))+2\log N J_n} \le e^{(-\varepsilon^2+\varepsilon^2/2)n} \le \delta'$$

for all $0 \le k \le p(n) = \lfloor(1-\varepsilon)n\rfloor$. Then, for such x and n,

$$\begin{aligned}
&\sum_{k=0}^{p(n)} -\log g_{\delta(x;n,k)}\left(f^k(x)\right)\\
&\le \sum_{\{0\le k\le p(n): f^k(x)\in A_{\delta'}\}} -\log g_{\delta(x;n,k)}\left(f^k(x)\right) + \sum_{\{0\le k\le p(n): f^k(x)\notin A_{\delta'}\}} -\log g_{\delta(x;n,k)}\left(f^k(x)\right)\\
&\le \sum_{\{0\le k\le p(n): f^k(x)\in A_{\delta'}\}} \left(-\log g(f^k(x)) + \varepsilon\right) + \sum_{\{0\le k\le p(n): f^k(x)\notin A_{\delta'}\}} -\log g_*\left(f^k(x)\right)
\end{aligned}$$

whence (from the pointwise ergodic theorem)

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{p(n)} -\log g_{\delta(x;n,k)}\left(f^k(x)\right)\\
&\le (1-\varepsilon)\left[\int_{A'_\delta} \left(-\log g + \varepsilon\right)\, d\mu + \int_{M\smallsetminus A_{\delta'}} -\log g_*\, d\mu\right]\\
&\le (1-\varepsilon)\left[\int -\log g\, d\mu + 2\varepsilon\right]\\
&= (1-\varepsilon)\left[H_\mu\left(f^{-1}\eta_{i-1}|\eta_{i-1}\right) + 2\varepsilon\right]\\
&= (1-\varepsilon)\left[h_\mu(f,\eta_{i-1}) + 2\varepsilon\right].
\end{aligned}$$

Combining the above with (35) and (36), the proposition follows. □

12. PROOF OF THEOREM 7.2 AND PROPOSITION 7.3

Let $\mathscr{F}$ be an f-invariant, $C^{1+\beta}$-tame, measurable foliation. Let $\varepsilon < \varepsilon_0$ be sufficiently small, let $\{\Phi_x\}$ be a family of ε-Lyapunov charts, and let $\{\Psi_x\}$ be a family of charts adapted to $\mathscr{F}$ built from Φ_x as in Proposition 5.4.

Let S^0_ρ and $V^{\mathscr{F},u}(x,\rho)$ be as in Section 8.2.1 and let

$$S^{\mathscr{F},u}_\rho = \bigcup_{x\in S^0_\rho} V^{\mathscr{F},u}(x,\rho).$$

For $y \in S^{\mathscr{F},u}_\rho$, select $x \in S^0_\rho$ with $y \in V^{\mathscr{F},u}(x,\rho)$ and write $D^{\mathscr{F},u}(y) = V^{\mathscr{F},u}(x,\rho)$.

Let $\hat{\xi}$ be the partition

$$\hat{\xi}(y) = \begin{cases} D^{\mathscr{F},u}(y) & y \in S_\rho^{\mathscr{F},u} \\ U \setminus S_\rho^{\mathscr{F},u} & y \notin S_\rho^{\mathscr{F},u} \end{cases} \tag{37}$$

and let $\tilde{\xi}^{\mathscr{F}^u} := \hat{\xi}^+$. Note that $V^{\mathscr{F},u}(x,\rho) \subset V^r_{\mathrm{loc},x,\varepsilon}$ for $x \in S_\rho^{\mathscr{F},u}$ and thus distances along each $V^{\mathscr{F},u}(x,\rho)$ contract exponentially fast under backwards dynamics. Choosing from a full measure set of ρ, we similarly obtain an increasing generator $\tilde{\xi}^{\mathscr{F}^u}$ subordinate to $\mathscr{F}^u$ satisfying analogues of (1)–(4) of Section 8.2. Moreover, since $V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}$ is an embedded submanifold of each $V^u_{\mathrm{loc},x,\varepsilon}$, setting

$$\xi^{\mathscr{F}^u} := \tilde{\xi}^{\mathscr{F}^u} \vee \xi^u$$

then $\xi^{\mathscr{F}^u}$ is still an increasing generator subordinate to $\mathscr{F}^u$ satisfying analogues of (1)–(4) of Section 8.2.

Let η be a measurable partition of (M,μ). As in Lemma 8.7, if $\xi_1^{\mathscr{F}^u}$ and $\xi_2^{\mathscr{F}^u}$ are any two partitions satisfying the analogues of (1)–(4) of Section 8.2, then

$$h_\mu\left(f, \xi_1^{\mathscr{F}^u} \vee \eta\right) = h_\mu\left(f, \xi_2^{\mathscr{F}^u} \vee \eta\right). \tag{38}$$

We have the following which immediately implies Proposition 7.3.

CLAIM 12.1. *For any partition $\xi^{\mathscr{F}^u}$ as above we have*

$$h_\mu(f \mid \mathscr{F} \vee \eta) = h_\mu\left(f, \eta \vee \xi^{\mathscr{F}^u}\right) = H_\mu\left(\eta^+ \vee \xi^{\mathscr{F}^u} \mid f\left(\eta^+ \vee \xi^{\mathscr{F}^u}\right)\right).$$

Proof. Given $b > 0$, let ξ be a measurable partition subordinate to $\mathscr{F}$ such that

$$h_\mu(f \mid \mathscr{F} \vee \eta) - b \le h_\mu(f \mid \eta \vee \xi) = h_\mu(f, \eta \vee \xi \vee \xi^u)$$

where the equality follow from Proposition 8.3. We construct a partition $\xi^{\mathscr{F}^u}$ satisfying the analogues of (1)–(4) of Section 8.2 such that

$$h_\mu(f \mid \mathscr{F} \vee \eta) - b \le h_\mu(f, \xi^{\mathscr{F}^u} \vee \eta) \le h_\mu(f \mid \mathscr{F} \vee \eta). \tag{39}$$

The claim follows since by (38), the value of middle term of (39) is independent of the choice of such $\xi^{\mathscr{F}^u}$ and thus (39) holds for all $b > 0$.

From Remark 6.5, we may further assume $\xi(x)$ is precompact in $\mathscr{F}'(x)$ for a positive measure set of x. Fix ρ, S_ρ^0, $S_\rho^{\mathscr{F},u}$, and S_ρ^r used in the construction of partitions $\xi^{\mathscr{F}^u}$ and ξ^u. We may moreover assume S_ρ^0 was chosen so that in addition to the properties in Section 8.2.1, there exists $\hat{\rho}_0$ such that for $0 < \rho < \hat{\rho}_0$ and $x \in S_\rho^0$,

1. $V^{\mathscr{F}}(x,\rho) \subset \xi(x)$ (where $V^{\mathscr{F}}(x,\rho)$ is as in Section 8.2.1);
2. $\xi(x)$ is precompact in $\mathscr{F}'(x)$ and if $y \in \xi(x) \cap B(x_0,\rho)$, then $y \in V^{\mathscr{F}}(x,\rho)$.

Recall for $x \in S_\rho^0$ we also have

3. $V^{\mathscr{F}}(x,\rho) \cap V^u(x,\rho) = V^{\mathscr{F},u}(x,\rho)$.

Given $0 < \delta < \min\{e^{-2\lambda_0 - 4\varepsilon}, \hat{\rho}_0\}$, let $\mathscr{P}$ be a measurable partition of (M,μ) satisfying the following:

1. $H_\mu(\mathscr{P}) < \infty$;

2. $\mathscr{P}$ refines $\left\{S_\rho^{\mathscr{F},u}, M \setminus S_\rho^{\mathscr{F},u}\right\}$;
3. $\mathscr{P}$ is adapted to $\{\Phi, \delta\}$.

By Corollary 8.6,

$$h_\mu(f \mid \mathscr{F} \vee \eta) - b \le h_\mu\left(f, \mathscr{P} \vee \eta \vee \xi \vee \xi^u\right). \tag{40}$$

Take $\zeta = (\eta \vee \mathscr{P} \vee \xi)^+ \vee \xi^u$.

Define $\hat{\xi}$ as in (37) and take $\tilde{\xi}^{\mathscr{F}^u}(y) = \hat{\xi}^+$ and $\xi^{\mathscr{F}^u}(y) = \tilde{\xi}^{\mathscr{F}^u}(y) \vee \xi^u$. Note that $S_\rho^{\mathscr{F},u} \subset S_\rho^r$ and

$$S_\rho^{\mathscr{F},u} \subset \bigcup_{x \in S_\rho^0} V^{\mathscr{F}}(x, \rho) \subset \bigcup_{x \in S_\rho^0} \xi(x).$$

For any y, any $z \in \zeta(y)$, and all $m \ge 0$ we have the following:

1. $f^{-m}(z) \in S_\rho^{\mathscr{F},u}$ if and only if $f^{-m}(y) \in S_\rho^{\mathscr{F},u}$.
2. For all $m \ge 0$ such that $f^{-m}(y) \in S_\rho^{\mathscr{F},u}$, there is $x \in S_\rho^0$ with

$$f^{-m}(z), f^{-m}(y) \in \xi^u(x) \subset V^u(x, \rho)$$

and

$$f^{-m}(y), f^{-m}(z) \in \xi(x) \cap B(x_0, \rho) \subset V^{\mathscr{F}}(x, \rho).$$

In particular, for all such m, we have $f^{-m}(z), f^{-m}(y) \in V^{\mathscr{F}}(x,\rho) \cap V^u(x,\rho) = V^{\mathscr{F},u}(x, \rho)$.

It follows that $z \in \tilde{\xi}^{\mathscr{F}^u}(y)$. Since we also have $z \in \xi^u(y)$, it follows that $z \in \xi^{\mathscr{F}^u}(y)$ whence $\xi^{\mathscr{F}^u} \prec \zeta$.

By definition we have

$$h_\mu(f \mid \mathscr{F} \vee \eta) \ge h_\mu(f \mid \mathscr{F}^u \vee \eta) \ge h_\mu\left(f, \xi^{\mathscr{F}^u} \vee \eta\right).$$

On the other hand, (using (40), that $\xi^{\mathscr{F}^u} \prec \zeta$, and Claim 8.5) we have that

$$h_\mu(f \mid \mathscr{F} \vee \eta) - b \le h_\mu(f, \zeta) = h_\mu(f, \zeta \vee \xi^{\mathscr{F}^u}) \le h_\mu(f, \xi^{\mathscr{F}^u} \vee \eta).$$

This completes the proof of the claim. □

We turn to proof of Theorem 7.2. Let $\xi^{\mathscr{F}^u}$ be an increasing generator subordinate to $\mathscr{F}^u$ as above. From Claim 12.1, we have that $h_\mu(f \mid \mathscr{F}) = h_\mu\left(f, \xi^{\mathscr{F}^u}\right)$. Let $\eta = \xi^{\mathscr{F}^u}$. We first claim for $1 \le i \le r$ that

$$\gamma^i(\mu \mid \eta) \le m_i(\mathscr{F}).$$

Indeed we may assume that the transversals T_i in Section 11.3 were taken to be in general position with $V^{\mathscr{F}}_{\mathrm{loc},x,\varepsilon}$ for every $x \in E \subset S_\rho^0$. Let $\mathscr{P}$ be as in Section 11.2 and let $\eta_* = \xi^{\mathscr{F}^u} \vee \mathscr{P}$. Then for $x \in E$, we have $\eta_i(x) \subset V^i_{\mathrm{loc},x,\varepsilon} \cap V^{\mathscr{F}}_{\mathrm{loc},x,\varepsilon} \cap T_i$ and the metric d^{T_i} identifies $\eta_i(x)/\eta_{i-1}$ with a subset of a $m_i(\mathscr{F})$-dimensional submanifold of $\mathbb{R}^{\dim(\oplus_{j \ge i} E_j)}$. It follows that $\tilde{\gamma}_i(x) \le m_i(\mathscr{F})$. Inequality (I), Proposition 11.7, and Claim 11.5 of Section 11 then show that $\gamma^i(\mu|\xi^{F^u}) \le m_i(\mathscr{F})$ whence the inequality in Theorem 7.2 then follows from Theorem 7.7.

To complete the proof of Theorem 7.2 we claim that if

$$h_\mu\left(f,\xi^{\mathscr{F}^u}\right)=\sum_{1\le i\le r}\lambda_i m_i(\mathscr{F}),$$

then, exactly as in [28, Theorem 3.4] or [31, (6.1)], Jensen's inequality implies that $\mu_x^{\xi^{\mathscr{F},u}}$ is absolutely continuous along $\mathscr{F}^u$ for almost every x. We sketch the details in the remainder.

Recall that for $x\in S^0_\rho$ we have $\xi^{\mathscr{F}^u}(x)\subset V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}$. Define

$$n(x)=\min\left\{n\ge 0: f^n(x)\in S^0_\rho\right\}$$

and

$$\bar{\xi}^{\mathscr{F}^u}(x)=f^{-n(x)}\left(\xi^{\mathscr{F}^u}\left(f^{n(x)}(x)\right)\right).$$

We then have

1. $\bar{\xi}^{\mathscr{F}^u}$ is a partition of (M,μ) subordinate to $\mathscr{F}^u$;
2. $\bar{\xi}^{\mathscr{F}^u}(x)\subset V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}$ for almost every x;
3. $\bar{\xi}^{\mathscr{F}^u}\prec f^{-1}\bar{\xi}^{\mathscr{F}^u}$;
4. $h_\mu(f\mid\mathscr{F})=h_\mu\left(f,\bar{\xi}^{\mathscr{F}^u}\right)$.

Replacing $\xi^{\mathscr{F}^u}$ with $\bar{\xi}^{\mathscr{F}^u}$ we may assume $\xi^{\mathscr{F}^u}$ satisfies (2) for the remainder. This choice ensures that D_zf is defined for all $z\in\xi^{\mathscr{F}^u}(x)$ and almost every x.

For each $C\in\xi^{\mathscr{F}^u}$ and $x\in C$ restrict the ambient Riemannian metric to $V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}$ and consider the induced Riemannian volume m_x on C. Note that $m_x=m_y$ if $y\in C$ and, since $\xi^{\mathscr{F}^u}$ is subordinate to $\mathscr{F}^u$, m_x is a positive measure for almost every x.

Define $\Delta_x:\xi^{\mathscr{F}^u}(x)\to[0,\infty)$ by

$$\Delta_x(y)=\lim_{n\to\infty}\frac{\prod_{i=1}^n J^{\mathscr{F},u}(f^{-i}(x))}{\prod_{i=1}^n J^{\mathscr{F},u}(f^{-i}(y))}$$

where for $z\in\xi^{\mathscr{F}^u}(x)$ we define

$$J^{\mathscr{F},u}(z)=\left|\det D_zf\restriction_{T_zV^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}}\right|$$

and the determinant is computed against the Riemannian metric on U_0. Note that as $\xi^{\mathscr{F},u}(x)\subset V^{\mathscr{F},u}_{\mathrm{loc},x,\varepsilon}$ for a.e. x, for all $y\in\xi^{\mathscr{F}^u}(x)$ we have $f^{-n}(y)\in U_0$ and $J^{\mathscr{F},u}(f^{-n}(y))$ is defined for all $n\ge 0$.

Arguing in ε-charts $\{\Phi_x\}$, we have that as in [31, (6.1)] the following.

CLAIM 12.2.

(a) *For almost every x, the limit defining Δ_x converges and Δ_x is uniformly β-Hölder on $\xi^{\mathscr{F}^u}(x)$.*

(b) *Defining $L(x):=\int_{\xi^{\mathscr{F}^u}(x)}\Delta_x(y)\,dm_x(y)$, for almost every x*

$$0<L(x)<\infty.$$

(c) $L(f(x))=J^{\mathscr{F}^u}(x)\int_{(f^{-1}\xi^{\mathscr{F}^u})(x)}\Delta_x(y)\,dm_x(y)\le J^{\mathscr{F}^u}(x)L(x).$

(d) *Defining* ν_x *on* $\xi^{\mathscr{F}^u}(x)$ *by*

$$d\nu_x(y) = \frac{\Delta_x(y)}{L(x)} dm_x(y)$$

we have that $\nu_x = \nu_y$ *if* $y \in \xi^{\mathscr{F}^u}(x)$, $\{\nu_x\}$ *is a measurable family of probability measures, and*

$$\int -\log \nu_x(f^{-1}\xi^{\mathscr{F}^u})(x)\ d\mu(x) = \int \log J^{\mathscr{F}^u}(x)\ d\mu(x).$$

Let ν be the probability measure defined on M by

$$\nu(A) = \int \nu_x(A)\ d\mu(x)$$

for any Borel set A. As the derivative of the charts in Section 3.2 is controlled by a slowly increasing function, we have (see for example [29, Proposition 2.1])

$$\int \log\left(J^{\mathscr{F}^u}(x)\right)\ d\mu(x) = \sum m_i(\mathscr{F})\lambda_i.$$

Exactly as in [31, Lemma 6.1.3] (see also [28, Theorem 3.4]), Jensen's inequality gives the following.

CLAIM 12.3. *If* $\int \log J^{\mathscr{F}^u}(x)\ d\mu(x) = H_\mu(f^{-1}(\xi^{\mathscr{F}^u}) \mid \xi^{\mathscr{F}^u})$, *then* ν *and* μ *coincide on the* σ*-algebra generated by* $f^{-1}(\xi^{\mathscr{F}^u})$.

Iterating, one has that ν and μ coincide on the σ-algebra generated by $f^{-n}(\xi^{\mathscr{F}^u})$ for all $n \geq 0$. As $\xi^{\mathscr{F}^u}$ generates, it follows that the measures ν and μ coincide. Theorem 7.2 then follows.

Part III. Product structure of entropy by Aaron Brown, Federico Rodriguez Hertz and Zhiren Wang

For a smooth action of $\mathbb{Z}^d$ preserving a Borel probability measure, we show that entropy satisfies a certain "product structure" along coarse unstable manifolds. Moreover, given two smooth $\mathbb{Z}^d$-actions—one of which is a measurable factor of the other—we show that all coarse Lyapunov exponents contributing to the entropy of the factor system are coarse Lyapunov exponents of the total system and derive an Abramov–Rohlin formula for entropy subordinated to coarse Lyapunov manifolds.

13. STATEMENT OF RESULTS

As in Part I, we take M to be a C^∞ manifold equipped with a Borel probability measure μ. Let $\alpha\colon \mathbb{Z}^d \times M \to M$ be an action by measure-preserving, measurable transformations. We moreover assume (M,μ) and α satisfy the standing hypotheses (either hypotheses I or II) of Section 3.2. We further assume for simplicity of statements and proofs that μ is ergodic.

13.1. **Product structure and subadditivity of entropy.** Our first main result is the following "product structure of entropy" formula. Write $\mathscr{L}$ for the Lyapunov exponents of α with respect to μ; at times we also write $\mathscr{L} = \mathscr{L}^{\alpha}(\mu)$ to emphasize that we consider Lyapunov exponents for the action of α and the invariant measure μ. Let $\widehat{\mathscr{L}}$ denote the coarse Lyapunov exponents of α with respect to μ. For $\chi \in \widehat{\mathscr{L}}$, let $\mathscr{W}^{\chi}$ denote the corresponding foliation by coarse Lyapunov manifolds.

THEOREM 13.1. *Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation and let η be an α-invariant measurable partition. Then for $n \in \mathbb{Z}^d$,*

$$h_{\mu}(\alpha(n) \mid \mathscr{F} \vee \eta) = \sum_{\{\chi \in \widehat{\mathscr{L}} : \chi(n) > 0\}} h_{\mu}\big(\alpha(n) \mid \mathscr{F} \vee \mathscr{W}^{\chi} \vee \eta\big).$$

In particular, we have the following.

COROLLARY 13.2 (Product structure of entropy)**.**

$$h_{\mu}(\alpha(n)) = \sum_{\{\chi \in \widehat{\mathscr{L}} : \chi(n) > 0\}} h_{\mu}\big(\alpha(n) \mid \mathscr{W}^{\chi}\big). \tag{41}$$

Note that if $f\colon M \to M$ and $g\colon N \to N$ are diffeomorphisms preserving μ_1 and μ_2, respectively, then there is a natural $\mathbb{Z}^2$-action on $M \times N$ preserving $\mu_1 \times \mu_2$. In this case, (41) follows immediately from the classical Ledrappier–Young entropy formula. Our result (41) suggests—at least at the level of entropy—that unstable conditional measures associated to an ergodic α-invariant measure behave like a product measure along coarse Lyapunov manifolds. It would be of interest to know if the unstable conditional measures are always products of conditional measures along coarse Lyapunov manifolds. In homogeneous settings such as those considered in [17, 15, 16], similar product structures of entropy are established by first establishing a product structure of the measure along total unstable manifolds.

As explained in Lemma 14.12 below, the expression

$$n \mapsto \sum_{\{\chi \in \widehat{\mathscr{L}} : \chi(n) > 0\}} h_{\mu}\big(\alpha(n) \mid \mathscr{F} \vee \mathscr{W}^{\chi} \vee \eta\big)$$

extends from the set $\{n \in \mathbb{Z}^d : \chi(n) > 0\}$ to a linear functional $\mathbb{R}^d \to \mathbb{R}$. In particular, combined with Theorem 13.1 above, we recover the subadditivity of entropy of $\mathbb{Z}^d$-actions first obtained by Hu in [21] for commuting C^2 diffeomorphisms of compact manifolds.

THEOREM 13.3 (c.f. [21, Theorem B])**.** *Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation and let η be an α-invariant measurable partition. Then for all $n, m \in \mathbb{Z}^d$*

1. *$h_{\mu}(\alpha(n+m) \mid \mathscr{F} \vee \eta) \le h_{\mu}(\alpha(n) \mid \mathscr{F} \vee \eta) + h_{\mu}(\alpha(m) \mid \mathscr{F} \vee \eta)$;*
2. *moreover, if $\mathscr{F} \vee \mathscr{W}^u_n = \mathscr{F} \vee \mathscr{W}^u_m$, then equality holds.*

Indeed, observe that if $\chi(n+m) > 0$, then either $\chi(n) > 0$ or $\chi(m) > 0$. Both conclusions then follow from Theorem 13.1 and the linearity in Lemma 14.12.

The main technical results we establish to prove Theorem 13.1 are Proposition 15.4 and Corollary 15.6, below. As a direct consequence of Corollary 15.6, we obtain a certain exact dimensionality formula for measures invariant under $\mathbb{Z}^k$-actions. Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation and let η be an α-invariant measurable partition. For a coarse Lyapunov exponent $\chi \in \widehat{\mathscr{L}}$, let $d^{\mathscr{F},\chi,\eta}(\mu)$ be the almost-surely constant value of the pointwise dimension of μ along $\mathscr{F} \vee \mathscr{W}^{\chi} \vee \eta$ (see Section 7.3). For $n \in \mathbb{Z}^d$, let $d_n^{\mathscr{F},u,\eta}(\mu)$ be the almost-surely constant value of the pointwise dimension of μ along $\mathscr{F} \vee \mathscr{W}_n^u \vee \eta$.

Corollary 15.6 (with $\mathscr{I} := \{\lambda_i \in \mathscr{L} : \lambda_i(n) > 0\}$) immediately implies the following.

COROLLARY 13.4 (Product structure of unstable dimension). *For any $n \in \mathbb{Z}^d$,*

$$d_n^{\mathscr{F},u,\eta}(\mu) = \sum_{\chi(n)>0} d^{\mathscr{F},\chi,\eta}(\mu).$$

13.2. **Measurable factors and coarse-Lyapunov Abramov–Rohlin formula.** Consider a second action $\overline{\alpha}$ of $\mathbb{Z}^d$ on (N,ν) satisfying the standing hypotheses II of Section 3.2. We say that $\overline{\alpha}$ is a *measurable factor of α* if there is a measurable map $\psi\colon M \to N$ with $\psi_*\mu = \nu$ and $\psi \circ \alpha(n) = \overline{\alpha}(n) \circ \psi$ for all $n \in \mathbb{Z}^d$. Let $\mathscr{A}^{\psi}$ denote the α-invariant partition of (M,μ) into level sets of ψ.

We assume μ and thus ν are ergodic. To distinguish data associated to each action, let $\widehat{\mathscr{L}^{\overline{\alpha}}}(\nu)$ and $\hat{\mathscr{L}}^{\alpha}(\mu)$ denote, respectively, the coarse Lyapunov exponents for the actions $\overline{\alpha}$ and α of $\mathbb{Z}^d$ on (N,ν) and (M,μ).

Consider a coarse Lyapunov exponent $\overline{\chi} \in \widehat{\mathscr{L}^{\overline{\alpha}}}(\nu)$ of $\overline{\alpha}$ and suppose that

$$h_\nu\left(\overline{\alpha}(n) \mid \mathscr{W}^{\overline{\chi}}\right) > 0 \tag{42}$$

for some $n \in \mathbb{Z}^d$ with $\overline{\chi}(n) > 0$. Let $E = \ker(\overline{\chi}) \subset \mathbb{R}^d$ be the *Lyapunov hyperplane* determined by $\overline{\chi}$. It follows from Corollary 13.2 and (42) that for an open cone $C \subset \mathbb{R}^d$ containing E, the function

$$n \mapsto h_\nu(\overline{\alpha}(n))$$

is not a linear function on $C \cap \mathbb{Z}^d$. By the classical Abramov-Rohlin formula (21), for every $n \in \mathbb{Z}^d$ we have that

$$h_\nu(\overline{\alpha}(n)) = h_\mu(\alpha(n)) - h_\mu\left(\alpha(n) \mid \mathscr{A}^{\psi}\right).$$

If no Lyapunov exponent of α were proportional to $\overline{\chi}$ then, taking any open cone $C' \subset C \subset \mathbb{R}^d$ containing E and disjoint from the kernels of all non-zero Lyapunov exponents in $\mathscr{L}^{\alpha}(\mu)$, it follows from Theorem 13.3 that both $h_\mu(\alpha(n))$ and $h_\mu(\alpha(n) \mid \mathscr{A}^{\psi})$ coincide with linear functions on $C' \cap \mathbb{Z}^d$, contradicting the properties of C above.

It thus follows that every coarse Lyapunov exponent $\overline{\chi} \in \widehat{\mathscr{L}^{\overline{\alpha}}}(\nu)$ that contributes entropy to $\overline{\alpha}$ is proportional to a Lyapunov exponent of α. We say $\chi \in \widehat{\mathscr{L}^{\overline{\alpha}}}(\nu)$ is *essential* if

$$h_\nu\left(\overline{\alpha}(n) \mid \mathscr{W}^{\chi}\right) > 0$$

for some (and hence all) $n \in \mathbb{Z}^d$ with $\chi(n) > 0$. Let $\widehat{\mathscr{L}}^{\overline{\alpha}}_{\mathrm{ess}}(\nu)$ denote the essential coarse Lyapunov exponents of the action $\overline{\alpha}$ of $\mathbb{Z}^d$ on (N, ν). As remarked above, all essential exponents $\overline{\chi}$ of $\overline{\alpha}$ are proportional to Lyapunov exponents of α. We show that they are, in fact, positively proportional.

THEOREM 13.5. *For every $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}_{\mathrm{ess}}(\nu)$ there exists $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ with χ positively proportional to $\overline{\chi}$.*

Analogous statements to Theorem 13.5 are established (for all coarse Lyapunov exponents) in [24, Section 6.2] and [22, Lemma 2.3] under the assumption that the factor map ψ is Hölder continuous using exponential contraction of the dynamics along stable manifolds. Our more general statement in Theorem 13.5 follows for measurable factors using only entropy considerations and Theorem 7.7.

REMARK 13.6. Consider two ergodic actions of $\mathbb{Z}^d$ (satisfying the standing hypotheses II of Section 3.2), one denoted α on (M, μ) and the other denoted $\overline{\alpha}$ on (N, ν). A necessary condition for these actions to be measurable isomorphic is that the entropy functions $\mathbb{Z}^d \to [0, \infty)$

$$n \mapsto h_\mu(\alpha(n)), \qquad n \mapsto h_\nu(\overline{\alpha}(n)) \tag{43}$$

coincide. By [21, Theorem B], both functions (43) extend to semi-norms on $\mathbb{R}^d$; moreover, the unit balls are convex (possibly non-complact) polytopes whose faces meet only along kernels of coarse Lyapunov exponents. By Theorem 13.1, the faces of the unit ball in each norm meets precisely along the kernels of those coarse Lyapunov exponents that contribute entropy to the system.

We note that coincidence of the entropy functions (43) is not sufficient to conclude the actions α and $\overline{\alpha}$ are measurably conjugate; furthermore, as noted by the referee, coincidence of the entropy functions also does not imply our Theorem 13.5. In particular, Theorem 13.5 provides perhaps a finer mechanisms to distinguish $\mathbb{Z}^d$ actions up to measurable isomorphism.

As an example, consider $\mathbb{Z}^2$ generated by the standard basis $\{e_1, e_2\}$ in $\mathbb{R}^2$. Consider the linear functionals on $\mathbb{R}^2$,

$$\lambda_1 = e_1^*, \quad \lambda_2 = -e_1^*, \quad \lambda_3 = e_2^*, \quad \lambda_4 = -e_2^*, \quad \lambda_5 = e_1^* + e_2^*, \quad \lambda_6 = -e_1^* + -e_2^*.$$

Suppose the exponents of the action α on (M, μ) are $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, and λ_6, each with conditional dimension exactly 1. Then the entropy function of α is

$$(n_1 e_1, n_2 e_2) \mapsto 2\max\{|n_1|, |n_2|, |n_1 + n_2|\}. \tag{44}$$

Suppose the exponents of the action $\overline{\alpha}$ on (N, ν) are λ_2, λ_4, and λ_5, each with conditional dimension exactly 2. Then, the entropy function of $\overline{\alpha}$ is again (44). In particular, while the entropy functions coincide, Theorem 13.5 implies the actions α and $\overline{\alpha}$ are not measurably isomorphic as λ_1 contributes entropy to α but $\overline{\alpha}$ has no exponent positively proportional to λ_1. Similarly, Theorem 13.5 implies the actions $n \mapsto \overline{\alpha}(n)$ and $n \mapsto \overline{\alpha}(-n)$ on (N, ν) are not measurably conjugate.

We return to the setup at the beginning of Section 13.2. Given $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$, let $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ denote the equivalence class of exponents positively proportional to those in χ if such a class exists; if no such class exists let $\overline{\chi}$ denote the 0 functional. If $\overline{\chi} = 0$, let $\mathscr{W}^{\overline{\chi}}$ denote the point partition on (N, ν). Note that in this case $h_\nu\left(\overline{\alpha}(n) \mid \mathscr{W}^{\hat{\chi}}\right) = 0$.

Recall the classical Abramov–Rohlin formula (21). We establish an analogous formula for entropy subordinate to coarse Lyapunov foliations under a measurable factor map ψ intertwining smooth $\mathbb{Z}^d$-actions α and $\overline{\alpha}$.

THEOREM 13.7 (Coarse-Lyapunov Abramov–Rohlin formula)**.** *Let $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$. Then for $n \in \mathbb{Z}^d$ with $\chi(n) > 0$ we have*

$$h_\mu\left(\alpha(n) \mid \mathscr{W}^{\chi}\right) = h_\nu\left(\overline{\alpha}(n) \mid \mathscr{W}^{\overline{\chi}}\right) + h_\mu\left(\alpha(n) \mid \mathscr{W}^{\chi} \vee \mathscr{A}^{\psi}\right). \tag{45}$$

We note that in Proposition 3.1 and Corollary 3.4 of [17], an analogous result is derived in the context of joinings of homogeneous actions in which case the factor maps are smooth.

14. PRELIMINARIES

14.1. Lyapunov hyperplanes, (sub)chambers, and complete classes of exponents.

In the rest of this article, the letter χ (possibly with superscripts and subscripts) always denotes a coarse Lyapunov exponent and the letter λ always denotes a Lyapunov exponent or more general linear functional.

Consider an abstract collection $\mathscr{L} = \{\lambda_1, \dots, \lambda_p\}$ of linear functionals $\lambda_i : \mathbb{R}^d \to \mathbb{R}$. As in Definition 4.9, declare that $\lambda_i, \lambda_j \in \mathscr{L}$ are equivalent if they are positively propositional. Let $\widehat{\mathscr{L}}$ denote the equivalence classes in $\mathscr{L}$. Previewing our main application where $\mathscr{L}$ are the Lyapunov exponents for an ergodic action of $\mathbb{Z}^d$, given a non-zero $\lambda_i \in \mathscr{L}$, the *Lyapunov hyperplane* associated to λ_i is the kernel of λ_i in $\mathbb{R}^d$. Note that if λ_i and λ_j are proportional, then λ_i and λ_j induce the same Lyapunov hyperplane.

An (open) *chamber* $W \subset \mathbb{R}^d$ (associated to $\mathscr{L}$) is a connected component of the complement of all Lyapunov hyperplanes in $\mathbb{R}^d$. An (open) *subchamber* $U \subset \mathbb{R}^d$ (associated to $\mathscr{L}$) is a connected component of the complement in W of all hyperplanes associated to all non-zero linear functionals of the form $\lambda_i - \lambda_j$ and λ_i for all non-zero $\lambda_i, \lambda_j \in \mathscr{L}$ with $\lambda_i \neq \lambda_j$. Every subchamber is an open subset of some chamber. Given a chamber $W \subset \mathbb{R}^d$, we say a non-zero linear functional $L : \mathbb{R}^d \to \mathbb{R}$ is *in the wall* of W if the intersection of the boundary of W with the kernel of L, $\partial W \cap \ker L$, is not contained in any proper subspace of $\ker L$; in this case, $\partial W \cap \ker L$ is a closed, convex subset with nonempty interior in $\ker L$. Similarly, a non-zero linear functional $L : \mathbb{R}^d \to \mathbb{R}$ is *in the wall* of a subchamber U if $\partial U \cap \ker L$ is not contained in any proper subspace of $\ker L$.

Elements $v \in \mathbb{R}^d$ of (open) subchambers are said to be *generic*; that is, v is generic if it is outside the kernels of all non-zero linear functionals of the form λ_i and $\lambda_i - \lambda_j$ for $\lambda_i, \lambda_j \in \mathscr{L}$.

For each $v \in \mathbb{R}^d$, fix a permutation $\sigma(v)\colon \{1,\dots,p\} \to \{1,\dots,p\}$ so that

$$\lambda_{\sigma(v)(1)}(v) \geq \lambda_{\sigma(v)(2)}(v) \geq \cdots \geq \lambda_{\sigma(v)(p)}(v).$$

Note that if v is generic, then the above inequalities are strict, the permutation $\sigma(v)$ is uniquely defined and is constant on the subchamber.

DEFINITION 14.1. We say two chambers W_1 and W_2 are *adjacent* if the intersection of their boundaries, $\partial W_1 \cap \partial W_2$, has nonempty interior in both ∂W_1 and ∂W_2. Equivalently, W_1 and W_2 are adjacent if there is a nonzero linear functional $L\colon \mathbb{R}^d \to \mathbb{R}$ (proportional to some functional $\lambda_i \in \mathscr{L}$) in the wall of W_1 and W_2 such that $\partial W_1 \cap \partial W_2$ is not contained in any proper subspace of $\ker L$. To emphasize the role of L, we sometimes say W_1 and W_2 are *adjacent through L*.

Similarly, two subchambers U_1, U_2 are *adjacent* if the intersection of their boundaries, $\partial U_1 \cap \partial U_2$, has nonempty interior in both ∂U_1 and ∂U_2 or, equivalently, if there is a nonzero linear functional $L\colon \mathbb{R}^d \to \mathbb{R}$ (proportional to a functional of the form λ_i or $\lambda_i - \lambda_j$) in the wall of U_1 and U_2 such that $\partial U_1 \cap \partial U_2$ is not contained in any proper subspace of $\ker L$.

LEMMA 14.2. *Let U_1 and U_2 be distinct subchambers adjacent through some $L\colon \mathbb{R}^d \to \mathbb{R}$. Then for all $v_1 \in U_1$ and $v_2 \in U_2$ the permutations $\sigma(v_1), \sigma(v_2)$ differ by disjoint transpositions. Moreover, if $\sigma(v_2)(k) \neq \sigma(v_1)(k)$ and if $\lambda_{\sigma(v_1)(k)}$ is not proportional to L, then $\lambda_{\sigma(v_1)(k)}$ and $\lambda_{\sigma(v_2)(k)}$ are linearly independent.*

More specifically, (up to permutation) the set of indices $\{1,\dots,p\}$ may be written as the disjoint union of intervals $I_1,\dots,I_r$, each of the form $I_j = \{\ell_j, \ell_j + 1,\dots,\ell_j + t_j\}$ for some $t_j \geq 0$, such that

1. *for every $1 \leq j \leq r-1$, every $k_1 \in I_j$ and $k_2 \in I_{j+1}$, and every $v \in U_1 \cup U_2$,*
$$\lambda_{\sigma(v)(k_1)}(v) > \lambda_{\sigma(v)(k_2)}(v);$$
2. *if $k \in I_j$ is of the form $k = \ell_j + t$ for $0 \leq t \leq t_j$, then*
$$\sigma(v_2)(k) = \sigma(v_2)(\ell_j + t) = \sigma(v_1)(\ell_j + t_j - t);$$
3. *in particular, if $t_j = 0$ so that $\#I_j = 1$, then for $k \in I_j$,*
$$\sigma(v_2)(k) = \sigma(v_1)(k);$$
4. *for each j with $\#I_j \geq 2$, either the linear functionals*
$$\left\{\lambda_{\sigma(v_1)(k)} : k \in I_j\right\} = \left\{\lambda_{\sigma(v_2)(k)} : k \in I_j\right\}$$
are pairwise linearly independent or are all proportional to L.

The lemma essentially describes the following phenomena: for elements v inside the wall $U_1 \cap U_2 \cap \ker L$ between U_1 and U_2, the values of different Lyapunov exponents may coincide; however, the pattern of coincidence does not depend on the choice of v. Each individual group of Lyapunov exponents that coincide along the wall can have their indices written as an interval I_j. The union $U_1 \cup U_2$ can be viewed as a neighborhood of $U_1 \cap U_2 \cap \ker L$ in which the ordering of exponents are perturbed, but such perturbation would respect the blocks I_j. Moreover, if λ, λ' are both in the group represented by I_j, then $\lambda - \lambda'$

is proportional to L. In other words, the Lyapunov exponents in the same group can be arranged in a line in $(\mathbb{R}^d)^*$ in the direction of L. Crossing from one side of the wall to the other changes the sign of the value of L, and hence reverses the ordering within this group of Lyapunov exponents.

Proof of Lemma 14.2. Fix a nonzero $L\colon \mathbb{R}^d \to \mathbb{R}$ such that U_1 and U_2 are adjacent through L. Let $\{L, \xi_2, \dots, \xi_d\}$ be a basis for the dual space $(\mathbb{R}^d)^*$. Let $H = \ker L$ and let $Z = \bigcap_{i=2}^d \ker(\xi_i)$. Let $\Pi_H\colon \mathbb{R}^d \to H$ and $\Pi_Z\colon \mathbb{R}^d \to Z$ denote the projections relative to the direct sum decomposition $\mathbb{R}^d = H \oplus Z$ and for $v \in \mathbb{R}^d$, write $v = v_H + v_Z$ where $v_H = \Pi_H(v) \in H$ and $v_Z = \Pi_Z(v) \in Z$. For every $\lambda \in \mathscr{L}$, the functionals $\lambda \circ \Pi_H$ and $\lambda \circ \Pi_Z$ are linearly independent when both are non-zero; in particular, for any $\lambda \in \mathscr{L}$, there exists a unique $a_\lambda \in \mathbb{R}$ such that

$$\lambda = \lambda \circ \Pi_H + a_\lambda L \circ \Pi_Z. \tag{46}$$

Let $v_0 \in H = \ker L$ be an element in the interior (in $\ker L$) of $\partial U_1 \cap \partial U_2$ that is generic for the restriction of $\mathscr{L}$ to the subspace H. Fix a nonzero $u_0 \in Z$. Let $q_1 < q_2 < \cdots < q_r$ be the distinct values of $\lambda_i(v_0)$ for $\lambda_i \in \mathscr{L}$. Partition the index set $\{1, \dots, p\}$ of $\mathscr{L}$ into r sets $Q_j := \{i : \lambda_i(v_0) = q_j\}$. Take I_j to be the preimage of the Q_j under the permutation $\sigma(v_1)$ for some (and hence all) $v_1 \in U_1$. Since v_0 is generic in H, given $i_1, i_2 \in Q_j$ we have $\lambda_{i_1}\restriction_H = \lambda_{i_2}\restriction_H$ whence

$$\lambda_{i_1} \circ \Pi_H = \lambda_{i_2} \circ \Pi_H.$$

If $i_1 \neq i_2$ are distinct elements of Q_j, it follows from the uniqueness of the decomposition (46) that the coefficients $a_{\lambda_{i_k}}$ are distinct over all indices $i_k \in Q_j$. Assuming $q_j \neq 0$, then $\lambda_i \circ \Pi_H \neq 0$ for all $i \in Q_j$. Conclusion (4) then follows from (46).

Since v_0 was chosen in the interior of $\partial U_1 \cap \partial U_2 \cap \ker L$ and since Z is transverse to H, replacing u_0 with $-u_0$ if necessary, for all $c > 0$ sufficiently small we have $v_0 + cu_0 \in U_1$ and $v_0 - cu_0 \in U_2$. Since the set $\{a_\lambda : \lambda \in \mathscr{L}\}$ is finite, given any $\varepsilon > 0$, for all $c > 0$ sufficiently small we have

$$|\lambda(v_0 \pm cu_0) - \lambda(v_0)| < \varepsilon$$

for all $\lambda \in \mathscr{L}$. Taking $\varepsilon > 0$ sufficiently small and using that the permutations are constant on subchambers, we verify for $v_i \in U_i$ that $I_j := \sigma(v_1)^{-1} Q_j = \sigma(v_2)^{-1} Q_j$ and that conclusion (1) holds.

Finally, note that $L(u_0) \neq 0$. Consider the case $L(u_0) > 0$. Take $v_1 = v_0 + cu_0$ and $v_2 = v_0 - cu_0$ for $c > 0$ sufficiently small. Then for each $1 \le j \le r$ and writing the interval I_j as $I_j = \{\ell_j, \ell_j + 1, \dots, \ell_j + t_j\}$, we have

$$a_{\lambda_{\sigma(v_1)(\ell_j)}} > a_{\lambda_{\sigma(v_1)(\ell_j+1)}} > \cdots > a_{\lambda_{\sigma(v_1)(\ell_j+t_j)}}.$$

From the decomposition (46),

$$\lambda_{\sigma(v_1)(\ell_j)}(v_1) > \cdots > \lambda_{\sigma(v_1)(\ell_j+t_j)}(v_1)$$

and

$$\lambda_{\sigma(v_1)(\ell_j)}(v_2) < \cdots < \lambda_{\sigma(v_1)(\ell_j+t_j)}(v_2).$$

A similar analysis holds if $L(u_0) < 0$. Conclusion (2) then follows. □

14.2. **Complete and strongly integrable collections of exponents.** We now return to the special case that $\mathscr{L} = \mathscr{L}^\alpha(\mu)$ are the Lyapunov exponent functionals for an ergodic action α of $\mathbb{Z}^d$ on (M,μ). We recall that for each Lyapunov exponent $\lambda_i \in \mathscr{L}^\alpha(\mu)$, the function $\lambda_i\colon \mathbb{Z}^d \to \mathbb{R}$ extends uniquely to a linear function $\lambda_i\colon \mathbb{R}^d \to \mathbb{R}$. Recall we take M to be a C^∞ manifold equipped with a Borel probability measure μ and take $\alpha\colon \mathbb{Z}^d \times M \to M$ to be an action satisfying the hypotheses (either hypotheses I or II) of Section 3.2 with μ ergodic.

DEFINITION 14.3. Given a subset $\Sigma \subset V$ where $V \cong \mathbb{R}^d$,

1. Define the *positive cone* of Σ as $C(\Sigma) := \{\xi \in V^* : \xi(v) > 0 \text{ for all } v \in \Sigma\}$;
2. If $C(\Sigma) \neq \emptyset$, define the *positive convex hull* of Σ as $H(\Sigma) := C(C(\Sigma)) = \{v \in V : \xi(v) > 0 \text{ for all } \xi \in C(\Sigma)\}$.

REMARK 14.4. The following are always true:

1. $C(\Sigma)$ is convex and closed under scalar multiplication by positive real numbers;
2. $C(\Sigma)$ is open if Σ is finite;
3. $H(\Sigma) \supset \Sigma$.

For our application, we will only study $C(\mathscr{I})$ and $H(\mathscr{I})$ where $\mathscr{I}$ is a subset of Lyapunov exponents of the action and hence finite.

The following lemma is easy to establish and we omit its proof.

LEMMA 14.5. *If $C(\Sigma) \neq \emptyset$, then $H(\Sigma)$ is the smallest positive convex cone (i.e., a convex subset that is closed under scalar multiplication by positive real numbers) that contains Σ.*

Given a subset $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$, write

$$E_x^{\mathscr{I}} = \bigoplus_{\lambda \in \mathscr{I}} D_0\phi_x E_\lambda(x)$$

for the subspace of T_xM spanned by Oseldec's subspaces associated to $\lambda \in \mathscr{I}$. (Here, ϕ_x are the dynamical charts introduced in Standing Hypotheses II of Section 3.2.)

Note in the case that $\mathscr{I} = \chi$ is a single coarse Lyapunov exponent, $C(\mathscr{I})$ is a open half-space. For $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$, $C(\mathscr{I})$ is a union of Weyl chambers.

With the above notation, we have the following definitions.

DEFINITION 14.6. Let $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$ be such that $C(\mathscr{I}) \neq \emptyset$.

1. We say $\mathscr{I}$ is *complete* if $\mathscr{I} = H(\mathscr{I}) \cap \mathscr{L}^\alpha(\mu)$.
2. The *completion of* $\mathscr{I}$ is $H(\mathscr{I}) \cap \mathscr{L}^\alpha(\mu)$.
3. We say a collection $\mathscr{I}'$ is *integrable* if there is an α-invariant, $C^{1+\beta}$-tame measurable foliation $\mathscr{F}$ with $T_x\mathscr{F} = E_x^{\mathscr{I}'}$ for almost every x.
4. Given a complete collection $\mathscr{I} \subset \mathscr{L}$, we say a collection $\mathscr{I}'$ is *strongly integrable with respect to* $\mathscr{I}$ if there exists generic $n_1, \dots, n_k \in C(\mathscr{I})$ and $\kappa_1, \dots, \kappa_k \in (0, \infty)$ such that
$$\mathscr{I}' = \{\lambda \in \mathscr{I} : \lambda(n_i) \geq \kappa_i \text{ for all } 1 \leq i \leq k\}.$$

REMARK 14.7. We make a number of observations about the above definitions

(1) If $\mathscr{I}$ is complete, then it is a union of coarse Lyapunov exponents.
(2) From Proposition 4.6, every complete or strongly integrable collection $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$ is integrable; we write $\mathscr{W}^{\mathscr{I}}$ for the associated α-invariant, $C^{1+\beta}$-tame, measurable foliation.
(3) $\mathscr{I}$ is strongly integrable w.r.t. itself. If $\mathscr{I}'$ is strongly integrable (w.r.t. $\mathscr{I}$), then given $n \in C(\mathscr{I})$ and $\kappa > 0$, the collection $\{\lambda_i \in \mathscr{I}' : \lambda_i(n) \geq \kappa\}$ is strongly integrable.

LEMMA 14.8. *Fix a complete collection $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$ with $C(\mathscr{I}) \neq \varnothing$. Suppose for some coarse Lyapunov exponent χ there are two Weyl chambers W_1, W_2 adjacent through* $\ker\chi$ *with $W_1 \subset C(\mathscr{I})$. Then $\mathscr{I} \smallsetminus \chi$ is a complete collection.*

We also note that if $\mathscr{I}$ is complete and $\chi \subset \mathscr{I}$, then the completeness of $\mathscr{I} \smallsetminus \chi$ necessarily implies $\ker\chi$ contains a face of $C(\mathscr{I})$. Note that in the assumption of the lemma, W_2 may or may not be in $C(\mathscr{I})$. However, if $\chi \subset \mathscr{I}$, then W_2 must be outside of $C(\mathscr{I})$.

Proof of Lemma 14.8. If $\chi \not\subset \mathscr{I}$, then by Remark 14.7(1), $\mathscr{I} \smallsetminus \chi = \mathscr{I}$ and thus the statement is trivial. So we assume $\chi \subset \mathscr{I}$. It is clear that $C(\mathscr{I} \smallsetminus \chi) \supset C(\mathscr{I})$ and thus $H(\mathscr{I} \smallsetminus \chi) \subset H(\mathscr{I}) = \mathscr{I}$ by completeness of $\mathscr{I}$. In addition $(\mathscr{I} \smallsetminus \chi) \subset H(\mathscr{I} \smallsetminus \chi)$ by construction. By Remark 14.7(1), it suffices to show that $\chi \not\subset H(\mathscr{I} \smallsetminus \chi)$.

Choose $n_1 \in W_1$, $n_2 \in W_2$. Then for $\lambda \in \mathscr{L}^\alpha(\mu)$, $\lambda(n_1)$ and $\lambda(n_2)$ have same signs, except when λ is proportional to χ. Moreover, $n_1 \in C(\mathscr{I})$. Because $C(\mathscr{I})$ is non-empty and $\chi \in \mathscr{I}$, we know $\mathscr{I} \cap (-\chi) = \varnothing$. Hence for all $\lambda \in \mathscr{I} \smallsetminus \chi$, $\lambda(n_1) > 0$ and $\lambda(n_2) > 0$. Thus $n_2 \in C(\mathscr{I} \smallsetminus \chi)$.

On the other hand, for $\lambda' \in \chi$, $\lambda'(n_1)$ and $\lambda'(n_2)$ have different signs. As $\chi \subset \mathscr{I}$, $\lambda'(n_1) > 0$ and $\lambda'(n_2) < 0$. Thus $\lambda' \notin C(C(\mathscr{I} \smallsetminus \chi)) = H(\mathscr{I} \smallsetminus \chi)$. This completes the proof. $\square$

DEFINITION 14.9. Fix a complete collection $\mathscr{I}$ with $C(\mathscr{I}) \neq \varnothing$. Let $U_1, \ldots, U_\ell$ be the subchambers contained in $C(\mathscr{I})$ and for each $1 \leq j \leq \ell$ select $n_j \in U_j \cap \mathbb{Z}^d$. Given $\lambda \in \mathscr{I}$ set

$$\mathscr{I}(\lambda) := \left\{\lambda' \in \mathscr{I} : \lambda'(n_j) \geq \lambda(n_j) \text{ for all } 1 \leq j \leq \ell\right\}.$$

Note the above definition is independent of the choice of $n_i \in U_i$ since the relative order of all Lyapunov exponents is constant on subchambers. This, in particular, implies $\lambda'(n) \geq \lambda(n)$ for all $\lambda' \in \mathscr{I}(\lambda)$ and $n \in C(\mathscr{I})$.

Given a complete collection $\mathscr{I}$ and $\lambda \in \mathscr{I}$, clearly $\mathscr{I}(\lambda)$ is strongly integrable with respect to $\mathscr{I}$. Similarly, we collect the following observation.

LEMMA 14.10. *Let $\mathscr{I}$ be a complete collection and let $\mathscr{I}' \subset \mathscr{I}$ be strongly integrable w.r.t. $\mathscr{I}$. Suppose there exists $\lambda' \in \mathscr{I}'$ such that*

$$\lambda'(n) \leq \lambda(n) \text{ for all } \lambda \in \mathscr{I}' \text{ and all } n \in C(\mathscr{I}). \tag{47}$$

Then $\mathscr{I}' = \mathscr{I}(\lambda')$.

Proof. From the hypothesis (47), it is clear that $\mathscr{I}' \subset \mathscr{I}(\lambda')$.

On the other hand, there are generic $n_1, \dots, n_k \in C(\mathscr{I})$ such that

$$\begin{aligned}\mathscr{I}(\lambda') &= \{\lambda \in \mathscr{I} : \lambda(n) \geq \lambda'(n) \text{ for all } n \in C(\mathscr{I})\\ &\subset \{\lambda \in \mathscr{I} : \lambda(n_i) \geq \lambda'(n_i) \text{ for all } 1 \leq i \leq k\}\\ &= \mathscr{I}'\end{aligned}$$

where the first equality is from the definition and the second equality follows from Definition 14.6 and hypothesis (47). □

14.3. **Sufficient subsets of $\mathbb{Z}^d$.** Fix a collection $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$ such that $C(\mathscr{I}) \neq \varnothing$. The union of all subchambers $U \subset C(\mathscr{I})$ in $C(\mathscr{I})$ is an open dense subset of $C(\mathscr{I})$. We say a subset $S \subset C(\mathscr{I}) \cap \mathbb{Z}^d$ is *sufficient* in $C(\mathscr{I})$ if

1. every $n \in S$ is generic, and
2. for every subchamber $U \subset C(\mathscr{I})$, the set $S \cap U$ is a spanning set of $\mathbb{R}^d$.

Since subchambers are open, if $C(\mathscr{I}) \neq \varnothing$, then there exists a finite subset $S \subset \mathbb{Z}^d$ that is sufficient in $C(\mathscr{I})$.

14.4. **Increasing partitions subordinate to expanding foliation.** Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation. Define the *positive cone* of $\mathscr{F}$ to be

$$C(\mathscr{F}) := \left\{ n \in \mathbb{Z}^d : \mathscr{F} \text{ is expanding for } \alpha(n) \right\}.$$

Following the discussion in Section 12, for each $n \in C(\mathscr{F})$, there is a measurable partition $\xi_n^{\mathscr{F}}$ of (M,μ) subordinate to $\mathscr{F}$ and increasing for $\alpha(n)$. By adapting the constructions in Sections 8.2 and 12, as in [21, Section 8] we obtain the following.

PROPOSITION 14.11. *Let $n_1, \dots, n_\ell \in C(\mathscr{F})$. Then there exists a measurable partition $\xi^{\mathscr{F}}$ of (M,μ) with*

1. *$\xi^{\mathscr{F}}$ subordinate to $\mathscr{F}$;*
2. *$\alpha(n_i)\xi^{\mathscr{F}} \prec \xi^{\mathscr{F}}$ for $i = 1, \dots, \ell$;*
3. *$\xi^{\mathscr{F}}$ generates for $\alpha(n_i)$; that is $\bigvee_{k=0}^{\infty} \alpha(-kn_i)\xi^{\mathscr{F}}$ is the point partition.*

Note that if $\alpha(n)\xi^{\mathscr{F}} \prec \xi^{\mathscr{F}}$ and $\alpha(m)\xi^{\mathscr{F}} \prec \xi^{\mathscr{F}}$, then

$$\alpha(n+m)\xi^{\mathscr{F}} \prec \xi^{\mathscr{F}}.$$

The construction of $\xi^{\mathscr{F}}$ is essentially the same as that in Section 12. The main adaptation is that, instead of letting $\tilde{\xi}^{\mathscr{F}^u} := \hat{\xi}^+ = \bigvee_{k=0}^{\infty} f^k \hat{\xi}$, here we define

$$\xi^{\mathscr{F}} := \bigvee_{k_1=0}^{\infty} \cdots \bigvee_{k_\ell=0}^{\infty} \alpha\left(\sum_{i=1}^{\ell} k_i n_i\right) \hat{\xi}.$$

The parameter ρ is again chosen from a full measure set, whose definition is modified accordingly to avoid boundary concentration for all partitions from the countable family $\left\{\alpha\left(\sum_{i=1}^{\ell} k_i n_i\right) \hat{\xi} : k_1, \cdots, k_\ell \geq 0\right\}$.

14.5. **Linearity of entropy on positive cones.** We have the following adaptation of [21, Proposition 9.1].

LEMMA 14.12. *Let $\mathscr{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation. Let $n_1, n_2 \in C(\mathscr{F})$ and let η be a measurable partition that is increasing for $\alpha(n_1)$ and $\alpha(n_2)$. Then*

$$h_\mu(\alpha(n_1+n_2) \mid \mathscr{F} \vee \eta) = h_\mu(\alpha(n_2) \mid \mathscr{F} \vee \eta) + h_\mu(\alpha(n_1) \mid \mathscr{F} \vee \eta).$$

In particular, if η is α-invariant, then

$$n \mapsto h_\mu(\alpha(n) \mid \mathscr{F} \vee \eta)$$

coincides on $C(\mathscr{F})$ with a linear function.

Proof. Take a measurable partition $\xi^{\mathscr{F}}$ of (M,μ) with $\xi^{\mathscr{F}}$ subordinate to $\mathscr{F}$ and increasing for $\alpha(n_1)$ and $\alpha(n_2)$ as in Proposition 14.11. It follows from Claim 12.1 that

$$h_\mu(\alpha(m) \mid \mathscr{F} \vee \eta) = H_\mu(\alpha(-m)(\xi^{\mathscr{F}} \vee \eta) \mid \xi^{\mathscr{F}} \vee \eta)$$

for $m = n_1, n_2$ and $m = n_1 + n_2$. Moreover,

$$\begin{aligned}
&h_\mu(\alpha(n_1+n_2) \mid \mathscr{F} \vee \eta)\\
&= H_\mu(\alpha(-n_1-n_2)(\xi^{\mathscr{F}} \vee \eta) \mid \xi^{\mathscr{F}} \vee \eta)\\
&= H_\mu(\alpha(-n_1)(\alpha(-n_2)(\xi^{\mathscr{F}} \vee \eta)) \vee (\alpha(-n_2)(\xi^{\mathscr{F}} \vee \eta)) \mid \xi^{\mathscr{F}} \vee \eta)\\
&= H_\mu(\alpha(-n_2)(\xi^{\mathscr{F}} \vee \eta) \mid \xi^{\mathscr{F}} \vee \eta) + H_\mu(\alpha(-n_1)(\alpha(-n_2)(\xi^{\mathscr{F}} \vee \eta)) \mid \alpha(-n_2)(\xi^{\mathscr{F}} \vee \eta))\\
&= h_\mu(\alpha(n_2) \mid \mathscr{F} \vee \eta) + h_\mu(\alpha(n_1) \mid \mathscr{F} \vee \eta).
\end{aligned}$$

□

15. KEY PROPOSITION AND PROOF OF THEOREM 13.1

15.1. **Conditional and transverse dimensions dependent on choice of filtration.** Consider an α-invariant, $C^{1+\beta}$-tame, measurable foliation $\mathscr{F}$. Fix an integrable collection $\mathscr{I}' \subset \mathscr{L}^\alpha(\mu)$ of Lyapunov exponents with $C(\mathscr{I}') \neq \varnothing$; let $\mathscr{W}^{\mathscr{I}'}$ denote the foliation tangent to the subspaces $E_x^{\mathscr{I}'}$. Write $\mathscr{F}^{\mathscr{I}'} = \mathscr{F} \vee \mathscr{W}^{\mathscr{I}'}$. According to Proposition 4.6, for any $n \in C(\mathscr{I}')$ we obtain a filtration by α-invariant, $C^{1+\beta}$-tame, measurable foliations

$$\{x\} \subset \mathscr{F}_n^{1,\mathscr{I}'}(x) \subset \mathscr{F}_n^{2,\mathscr{I}'}(x) \subset \cdots \subset \mathscr{F}_n^{u(n),\mathscr{I}'}(x) := \mathscr{F}^{\mathscr{I}'}(x) \tag{48}$$

where $\mathscr{F}_n^{j,\mathscr{I}'} := \mathscr{W}_n^{u,j} \vee \mathscr{F}^{\mathscr{I}'}$.

Note that even for generic n, when the cardinality $\#\mathscr{I}'$ is small (relative to $u(n)$) many of the foliations appearing in the filtration (48) coincide. Let $m = \#\mathscr{I}'$. For generic $n \in C(\mathscr{I}')$, let

$$\{j_1, \ldots, j_m\} = \sigma(n)^{-1}\left\{i : \lambda_i \in \mathscr{I}'\right\}$$

ordered so that $j_1 > \cdots > j_m$. Then we also have filtration

$$\{x\} \subsetneq \mathscr{F}_n^{j_1,\mathscr{I}'}(x) \subsetneq \mathscr{F}_n^{j_2,\mathscr{I}'}(x) \subsetneq \cdots \subsetneq \mathscr{F}_n^{j_m,\mathscr{I}'}(x) := \mathscr{F}^{\mathscr{I}'}(x). \tag{49}$$

Fix a generic $n \in C(\mathscr{I})$ and let η be a measurable partition such that $\alpha(n)\eta \prec \eta$. For each $1 \le j \le u(n)$, let $\xi(\mathscr{F}_n^{j,\mathscr{I}'})$ be an increasing measurable partition of (M,μ) subordinate to $\mathscr{F}_n^{j,\mathscr{I}'}$. We write

$$\dim_n^j\left(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\right) := \lim_{r\to 0} \frac{\log\left(\mu_x^{\xi\left(\mathscr{F}_n^{j,\mathscr{I}'}\right)\vee\eta}(B(x,r))\right)}{\log(r)}.$$

The limit exists a.e. by Proposition 7.4 as $\xi\left(\mathscr{F}_n^{j,\mathscr{I}'}\right) \vee \eta$ is increasing for $\alpha(n)$. Although μ need not be ergodic for the action of $\alpha(n)$, α-invariance of the filtration (48) and the measure μ implies that

$$x \mapsto \lim_{r\to 0} \frac{\log\left(\mu_x^{\xi\left(\mathscr{F}_n^{j,\mathscr{I}'}\right)\vee\eta}(B(x,r))\right)}{\log(r)}$$

is α-invariant and hence constant by α-ergodicity of μ.

With $\dim_n^0(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}') := 0$, for $n \in S \cap C(\mathscr{I}')$ and $\lambda_i \in \mathscr{L}^\alpha(\mu)$ such that $\lambda_i(n) > 0$ we write

$$\begin{aligned}\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}') := {} & \dim_n^{\sigma(n)^{-1}(i)}(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}') \\ & - \dim_n^{\sigma(n)^{-1}(i)-1}(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}').\end{aligned} \tag{50}$$

For $j_{k-1} \le j < j_k$ observe that $\mathscr{F}_n^{j,\mathscr{I}'}(x) = \mathscr{F}_n^{j_{k-1},\mathscr{I}'}(x)$ a.e. whence

$$\dim_n^j(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}') = \dim_n^{j_{k-1}}(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}').$$

In particular, we have the following.

CLAIM 15.1. *If $n \in C(\mathscr{I}') \cap S$ and if $\lambda_i(n) > 0$ for some $\lambda_i \notin \mathscr{I}'$, then*

$$\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}') = 0.$$

Moreover, for $\lambda_i \in \mathscr{I}'$, we may alternatively define $\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}')$ as follows: if $\sigma(n)(j_k) = i$, then

$$\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}') = \dim_n^{j_k}(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}') - \dim_n^{j_k-1}(\mu \mid \mathscr{F} \vee \eta \mid \mathscr{I}'). \tag{51}$$

From Proposition 7.6 we have the following.

CLAIM 15.2. *Let $\mathscr{I}'' \subset \mathscr{I}'$ be integrable collections and fix $n \in C(\mathscr{I}')$. If η and $\hat\eta$ are measurable partitions with $\eta \prec \hat\eta$, $\alpha(n)\hat\eta \prec \hat\eta$, and $\alpha(n)\eta \prec \eta$ then*

$$\gamma_n(\lambda_i \mid \mathscr{F} \vee \hat\eta \mid \mathscr{I}'') \le \gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}').$$

Proof. Take $\xi_n^{\mathscr{I}',\mathscr{F}}$ and $\xi_n^{\mathscr{I}'',\mathscr{F}}$ with $\xi_n^{\mathscr{I}',\mathscr{F}} \prec \xi_n^{\mathscr{I}'',\mathscr{F}}$ to be measurable partitions of (M,μ) subordinated to $\mathscr{W}^{\mathscr{I}'\vee\mathscr{F}}$ and $\mathscr{W}^{\mathscr{I}''\vee\mathscr{F}}$, respectively, with $\alpha(n)\xi_n^{\mathscr{I}',\mathscr{F}} \prec \xi_n^{\mathscr{I}',\mathscr{F}}$ and $\alpha(n)\xi^{\mathscr{I}'',\mathscr{F}} \prec \xi^{\mathscr{I}'',\mathscr{F}}$. Then

$$\eta \vee \xi^{\mathscr{I}',\mathscr{F}} \prec \hat\eta \vee \xi^{\mathscr{I}'',\mathscr{F}}$$

whence by Proposition 7.6 we have

$$\gamma_n(\lambda_i \mid \mathscr{F} \vee \hat\eta \mid \mathscr{I}'') \le \gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}'). \qquad \square$$

15.2. **Main proposition.** Now consider a complete collection $\mathscr{I} \subset \mathscr{L}^\alpha(\mu)$. Let $S \subset \mathbb{Z}^d$ be a finite subset of generic elements that is sufficient in $C(\mathscr{I})$. Let η be a measurable partition such that $\alpha(n)\eta \prec \eta$ for all $n \in S$.

Before stating our main proposition, we first observe the following.

CLAIM 15.3. *For $\lambda_i \in \mathscr{I}$, the quantity*

$$\gamma_n\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_i)\big)$$

is independent of the choice of $n \in C(\mathscr{I}) \cap S$.

Indeed, by definition of $\mathscr{I}(\lambda_i)$, the right-most and second-to-right-most elements of the filtration (49) coincide for all $n \in C(\mathscr{I})$; since the right-most foliation in the filtration (49) corresponds to λ_i for all $n \in C(\mathscr{I}) \cap S$, the conclusion follows from (51).

Given Claim 15.3, write $\gamma\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_i)\big)$ for the constant value of $\gamma_n\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_i)\big)$ over $n \in C(\mathscr{I}) \cap S$.

Our main technical proposition and its corollaries show that the inequality in Claim 15.2 is often an equality and remains constant as the choice of n or $\mathscr{I}$ vary.

PROPOSITION 15.4. *Let $\mathscr{I}$ be a complete set of exponents, let $S \subset \mathbb{Z}^d$ be a finite subset that is sufficient in $C(\mathscr{I})$, and let η be a measurable partition such that $\alpha(n)\eta \prec \eta$ for all $n \in S$.*

Then, for every $\lambda_i \in \mathscr{I}$ and $n \in S \cap C(\mathscr{I})$, we have

$$\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}) = \gamma\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_i)\big). \tag{52}$$

In particular, $\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I})$ is independent of the choice of $n \in S \cap C(\mathscr{I})$.

Before giving the proof of Proposition 15.4, we collect the main corollaries that we use in the sequel.

COROLLARY 15.5. *Let $\chi_0 \in \widehat{\mathscr{L}^\alpha}(\mu)$ be a non-zero coarse Lyapunov exponent. Let $S \subset \mathbb{Z}^d$ be a finite set that is sufficient in the half-space $C(\chi_0)$ and suppose η is a measurable partition such that $\alpha(n)\eta \prec \eta$ for all $n \in S$.*

Let $\mathscr{I}$ be a complete collection with $\chi_0 \subset \mathscr{I}$. Then for $\lambda_i \in \chi_0$,

$$\gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}) = \gamma_n\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \chi_0\big) \tag{53}$$

for every $n \in C(\mathscr{I}) \cap S$.

We postpone the proof of Corollary 15.5 but state the following special case when η is an α-invariant partition.

COROLLARY 15.6. *Let η be an α-invariant measurable partition. Given any complete collection $\mathscr{I}$ with $C(\mathscr{I}) \neq \varnothing$, any coarse Lyapunov exponent $\chi \subset \mathscr{I}$, and any $\lambda_i \in \chi$ we have*

$$\gamma_n\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}\big) = \gamma_n\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \chi\big)$$

for every generic $n \in C(\mathscr{I})$.

Proof. Take a sufficient set S in the half-space $C(\chi)$ containing n. Then apply Corollary 15.5. $\square$

With Corollary 15.6 at hand, we are now ready to prove Corollary 13.4.

Proof of Corollary 13.4. For a coarse Lyapunov exponent $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ with $\chi(n) > 0$, let $d^{\mathscr{F},\chi,\eta}(\mu)$ be the almost-surely constant value of the pointwise dimension of μ along $\mathscr{F} \vee \mathscr{W}^{\chi} \vee \eta$ as in Section 7.3). In the notation above, this means

$$\begin{aligned} d^{\mathscr{F},\chi,\eta}(\mu) &= \lim_{r\to 0} \frac{\log\left(\mu_x^{\xi^{\mathscr{F}\vee\mathscr{W}^{\chi}}\vee\eta}(B(x,r))\right)}{\log(r)} \\ &= \dim_n^{j_{m_\chi}}(\mu \mid \mathscr{F}\vee\eta \mid \chi) = \sum_{\lambda_i\in\chi} \gamma_n\left(\lambda_i \mid \mathscr{F}\vee\eta \mid \chi\right) \\ &= \sum_{\lambda_i\in\chi} \gamma\left(\lambda_i \mid \mathscr{F}\vee\eta \mid \chi\right) \\ &= \sum_{\lambda_i\in\chi} \gamma\left(\lambda_i \mid \mathscr{F}\vee\eta \mid \mathscr{W}_n^u\right), \end{aligned} \tag{54}$$

where m_χ denotes the length of the filtration (49) for $\mathscr{I}' = \chi$, and the two last equalities are respectively given by Proposition 15.4 and Corollary 15.6.

We also let $d_n^{\mathscr{F},u,\eta}(\mu)$ be the almost-surely constant value of the pointwise dimension of μ along $\mathscr{F} \vee \mathscr{W}_n^u \vee \eta$. One can similarly deduce that

$$d_n^{\mathscr{F},u,\eta}(\mu) = \sum_{\lambda_i\in\mathscr{L}^{\alpha}(\mu):\lambda_i(n)>0} \gamma\left(\lambda_i \mid \mathscr{F}\vee\eta \mid \mathscr{W}_n^u\right). \tag{55}$$

It suffices to recognize that (55) is the sum of (54) over all coarse Lyapunov exponents χ with $\chi(n) > 0$. □

15.3. **Proof of Proposition 15.4 and Corollary 15.5.** Fix a complete collection $\mathscr{I}$ as in Proposition 15.4. Let $S \subset \mathbb{Z}^d$ be a finite set of generic elements that is sufficient in $C(\mathscr{I})$ and let η be a measurable partition such that $\alpha(n)\eta \prec \eta$ for all $n \in S$.

Proposition 15.4 follows immediately from the following induction hypothesis.

LEMMA 15.7. *Let $1 \le m \le \#\mathscr{I} - 1$ be an integer such that the following holds: for every strongly integrable (w.r.t. $\mathscr{I}$) collection $\mathscr{I}' \subset \mathscr{I}$ with $\#\mathscr{I}' \le m$, every $\lambda \in \mathscr{I}'$, and every $n \in C(\mathscr{I}) \cap S$,*

$$\gamma_n(\lambda \mid \mathscr{F}\vee\eta \mid \mathscr{I}') = \gamma\big(\lambda \mid \mathscr{F}\vee\eta \mid \mathscr{I}(\lambda)\big). \tag{56}$$

Then the same holds for every strongly integrable (w.r.t. $\mathscr{I}$) collection $\mathscr{I}_0 \subset \mathscr{I}$ with $\#\mathscr{I}_0 = m+1$.

When $m = 1$, each strongly integrable collection $\mathscr{I}'$ with $\#\mathscr{I}' = 1$ has a single exponent $\mathscr{I}' = \{\lambda_i\}$. From Lemma 14.10, we have $\mathscr{I}' = \mathscr{I}(\lambda_i)$. Then, for any $n \in C(\mathscr{I}) \cap S$, $\gamma_n\left(\lambda_i \mid \mathscr{F}\vee\eta \mid \mathscr{I}'\right) = \gamma\big(\lambda_i \mid \mathscr{F}\vee\eta \mid \mathscr{I}(\lambda_i)\big)$.

Given any $n \in C(\mathscr{I}) \cap S$, we find strongly integrable collections $\mathscr{I}'_1 \subset \cdots \subset \mathscr{I}'_{\#\mathscr{I}} = \mathscr{I}$ (with respect to $\mathscr{I}$) given by the filtration (49) associated with the ordering on $\mathscr{I}$ associated with n. By induction on m, Proposition 15.4 follows directly Lemma 15.7.

Proof of Lemma 15.7. Fix a strongly integrable (w.r.t. $\mathscr{I}$) collection $\mathscr{I}_0 \subset \mathscr{I}$ with $\#\mathscr{I}_0 = m+1$. Fix $n_0 \in C(\mathscr{I}) \cap S$. We show (56) holds for $n = n_0$ and all $\lambda \in \mathscr{I}_0$. Fix $\lambda_k \in \mathscr{I}_0$ such that

$$\lambda_k(n_0) = \min\{\lambda_i(n_0) : \lambda_i \in \mathscr{I}_0\}.$$

Recall n_0 is generic and so $\lambda_k(n_0) < \lambda_i(n_0)$ for all $\lambda_i \in \mathscr{I}_0 \smallsetminus \{\lambda_k\}$.

We consider two cases.

Case 1. Consider first the case that

$$\lambda_k(n) = \min\{\lambda_i(n) : \lambda_i \in \mathscr{I}_0\}$$

for all $n \in C(\mathscr{I}) \cap S$. Let $\mathscr{I}' = \mathscr{I}_0 \smallsetminus \{\lambda_k\}$; observe that $\mathscr{I}'$ is strongly integrable.

By Lemma 14.10 we have $\mathscr{I}_0 = \mathscr{I}(\lambda_k)$. Then clearly

$$\gamma_n\big(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\big) = \gamma\big(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_k)\big)$$

holds for any choice of $n \in C(\mathscr{I}) \cap S$. As in Claim 15.3, for any $n \in C(\mathscr{I}_0) \cap S$, the element of the filtration (49) associated to λ_k is the right-most. Thus, for any $\lambda_j \in \mathscr{I}_0 \smallsetminus \{\lambda_k\}$ and $n \in C(\mathscr{I}) \cap S$ we have

$$\gamma_n\big(\lambda_j \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\big) = \gamma_n\big(\lambda_j \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\big).$$

Since $\mathscr{I}'$ is strongly integrable and $\#\mathscr{I}' = m$, by the inductive hypothesis we have

$$\gamma_n\big(\lambda_j \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\big) = \gamma\big(\lambda_j \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_j)\big)$$

The conclusion then holds.

Case 2. Now suppose there is $n_* \in C(\mathscr{I}) \cap S$ and $\lambda_* \in \mathscr{I}_0$ such that

$$\lambda_k(n_*) > \lambda_*(n_*).$$

Then there exists a sequence of subchambers $U_0, U_1, \dots, U_\ell \subset C(\mathscr{I})$ with $n_0 \in U_0$ and $\lambda' \in \mathscr{I}_0$ with $\lambda' \neq \lambda_k$ such that for any choice of $n_j \in U_j \cap S$ for $1 \le j \le \ell$ the following hold:

1. U_j is adjacent to U_{j-1} for every $1 \le j \le \ell$;
2. $\lambda_k(n_j) = \min\big\{\lambda_i(n_j) : \lambda_i \in \mathscr{I}_0\big\}$ for every $0 \le j \le \ell - 1$;
3. $\lambda'(n_\ell) = \min\{\lambda_i(n_\ell) : \lambda_i \in \mathscr{I}_0\}$.

Note, in particular, that $\lambda'(n_\ell) < \lambda_k(n_\ell)$.

Let $\mathscr{I}' = \mathscr{I}_0 \smallsetminus \{\lambda_k\}$, $\mathscr{I}'' = \mathscr{I}_0 \smallsetminus \{\lambda'\}$ and $\mathscr{I}''' = \mathscr{I}_0 \smallsetminus \{\lambda_k, \lambda'\}$; observe $\mathscr{I}'$, $\mathscr{I}'$ and $\mathscr{I}'''$ are strongly integrable with $\#\mathscr{I}' = \#\mathscr{I}'' = m$ and $\#\mathscr{I}''' = m-1$. By the inductive hypothesis and examining the filtration (49) on the subchambers $U_{\ell-1}$ and U_ℓ, for all $\lambda_i \in \mathscr{I}'''$ we have

$$\begin{aligned}\gamma_{n_{\ell-1}}\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\big) &= \gamma_{n_{\ell-1}}\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\big)\\ &= \gamma\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_i)\big)\\ &= \gamma_{n_\ell}\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}''\big)\\ &= \gamma_{n_\ell}\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\big).\end{aligned} \tag{57}$$

Moreover, by the inductive hypothesis,

$$\gamma_{n_{\ell-1}}\big(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\big) = \gamma_{n_{\ell-1}}\big(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\big) = \gamma\big(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda')\big)$$

and

$$\gamma_{n_\ell}\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\right) = \gamma_{n_\ell}\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}''\right) = \gamma\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_k)\right).$$

By Lemma 14.12, the maps

1. $n \mapsto h_\mu\left(\alpha(n) \mid \mathscr{F} \vee \eta \vee \mathscr{W}^{\mathscr{I}'}\right)$
2. $n \mapsto h_\mu\left(\alpha(n) \mid \mathscr{F} \vee \eta \vee \mathscr{W}^{\mathscr{I}''}\right)$
3. $n \mapsto h_\mu\left(\alpha(n) \mid \mathscr{F} \vee \eta \vee \mathscr{W}^{\mathscr{I}'''}\right)$
4. $n \mapsto h_\mu\left(\alpha(n) \mid \mathscr{F} \vee \eta \vee \mathscr{W}^{\mathscr{I}_0}\right)$

extend from $C(\mathscr{I}) \cap S$, respectively, to linear functionals $L_1, L_2, L_3, L_0 \colon \mathbb{R}^d \to \mathbb{R}$.

From Theorem 7.7 and the structure of the filtration (49), for $n \in U_{\ell-1} \cap S$ we have

$$L_0(n) = L_1(n) + \gamma_{n_{\ell-1}}\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\right)\lambda_k(n)$$

and

$$L_1(n) = L_3(n) + \gamma_{n_{\ell-1}}\left(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\right)\lambda'(n).$$

Similarly, for $n \in U_\ell \cap S$ we have

$$L_0(n) = L_2(n) + \gamma_{n_\ell}\left(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\right)\lambda'(n)$$

and

$$L_2(n) = L_3(n) + \gamma_{n_\ell}\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}''\right)\lambda_k(n).$$

Combining the equalities above, after applying (57) and canceling common linear terms, we get that for all $n \in S \cap C(\mathscr{I})$,

$$\begin{aligned}&\left[\gamma\big(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_k)\big) - \gamma_{n_{\ell-1}}(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0)\right]\lambda_k(n)\\ &\quad - \left[\gamma\big(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda')\big) - \gamma_{n_\ell}(\lambda' \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0)\right]\lambda'(n) = 0.\end{aligned}$$

Here we used the fact that both $U_{\ell-1} \cap S$ and $U_\ell \cap S$ are spanning as S is a sufficient set.

Since $U_{\ell-1}$ and U_ℓ are adjacent and λ_k and λ' are positive on $U_{\ell-1}$ and U_ℓ and swap relative order when crossing the subchamber wall from U_1 and U_2, by Lemma 14.2, λ_k and λ' are linearly independent. It follows that

$$\gamma_{n_0}\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\right) = \gamma_{n_{\ell-1}}\left(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\right) = \gamma\big(\lambda_k \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_k)\big)$$

Moreover, for $\lambda_i \in \mathscr{I}_0 \setminus \{\lambda_k\}$ we have by the inductive hypothesis that

$$\gamma_{n_0}\left(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}_0\right) = \gamma_{n_0}\left(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}'\right) = \gamma\big(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}(\lambda_i)\big).$$

The conclusion follows. □

We conclude with the proof of Corollary 15.5.

Proof of Corollary 15.5. Fix $n_0 \in C(\mathscr{I})$.

If $\mathscr{I} = \chi_0$, then the conclusion follows from definition. We reduce to this case by backwards induction on the number of coarse exponents in $\mathscr{I}$.

Suppose $\mathscr{I}$ contains at least two distinct coarse exponents. Since $C(\mathscr{I}) \neq \varnothing$, there is a coarse exponent χ_1 that is not proportional to χ_0; in particular, the kernel of χ_1 has nonempty intersection with $C(\chi_0)$. It follows that $C(\mathscr{I})$ is a proper subset of $C(\chi_0)$ and there exist adjacent chambers $W_1, W_2 \subset C(\chi_0)$ with $W_1 \subset C(\mathscr{I})$ and $W_2 \subset C(\chi_0) \smallsetminus C(\mathscr{I})$.

Let $\chi \in \mathscr{I}$ be such that W_1 and W_2 are adjacent through χ. Let $U \subset W_1$ be a subchamber such that χ is in the wall of U; by Lemma 14.2,

$$\lambda(n_1) < \lambda'(n_1)$$

for all $n_1 \in U, \chi' \in \mathscr{I} \smallsetminus \{\chi\}$, $\lambda \in \chi$, and $\lambda' \in \chi'$. Let $\mathscr{I}' = \mathscr{I} \smallsetminus \chi$. Then $\mathscr{I}'$ is complete by Lemma 14.8 and for all $\lambda \in \mathscr{I}'$ and $n_1 \in U \cap S$,

$$\gamma_{n_1}(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}) = \gamma_{n_1}(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}').$$

By Proposition 15.4 and Claim 15.3, the above remain are constant as n_1 varies across $C(\mathscr{I}) \cap S$ and $C(\mathscr{I}') \cap S$, respectively. It follows that

$$\gamma_{n_0}(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}) = \gamma_{n_0}(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{I}').$$

By backwards induction on $\#\{\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu) : \chi \in \mathscr{I}\}$, we reduce to the case that $\mathscr{I} = \{\chi_0\}$. □

15.4. **Proof of Theorem 13.1.** Theorem 13.1 follows immediately from the Corollary 15.6.

Proof of Theorem 13.1. First consider a generic $n \in \mathbb{Z}^d$. Let

$$\mathscr{U}(n) := \left\{\lambda_i \in \mathscr{L}^{\alpha}(\mu) : \lambda_i(n) > 0\right\}$$

be the collection of positive Lyapunov exponents for $\alpha(n)$; then $\mathscr{U}(n)$ is complete. From Theorem 7.7, with $f = \alpha(n)$ we have that

$$h_\mu(\alpha(n) \mid \mathscr{F} \vee \eta) = \sum_{\lambda_i(n)>0} \gamma_n(\lambda_i \mid \mathscr{F} \vee \eta \mid \mathscr{U}(n))\lambda_i(n).$$

On the other hand, for any coarse Lyapunov exponent $\chi \in \hat{\mathscr{L}}^{\alpha}(\mu)$ with $\chi(n) > 0$, applying Theorem 7.7 again we have that

$$h_\mu\left(\alpha(n) \mid \mathscr{F} \vee \mathscr{W}^{\chi} \vee \eta\right) = \sum_{\lambda_i \in \chi} \gamma_n\left(\lambda_i \mid \mathscr{F} \vee \eta \mid \chi\right) \lambda_i(n).$$

The conclusion then follows for all generic $n \in \mathbb{Z}^d$ by Corollary 15.6.

Now, consider a non-generic $m \in \mathbb{Z}^d$. From Claim 12.1 we have

$$h_\mu(\alpha(m) \mid \mathscr{F} \vee \eta) = h_\mu\left(\alpha(m) \mid \mathscr{F} \vee \mathscr{W}^u_m \vee \eta\right).$$

Take $\widehat{\mathscr{F}} = \mathscr{F} \vee \mathscr{W}^u_m$. Then $\hat{\mathscr{F}}$ is expanding for $\alpha(m)$.

Since $C(\widehat{\mathscr{F}}) \neq \varnothing$, it follows that $C(\widehat{\mathscr{F}})$ contains a spanning set of generic points. From Lemma 14.12, the functions

$$n \mapsto h_\mu\left(\alpha(n) \mid \widehat{\mathscr{F}} \vee \eta\right) \quad \text{and} \quad n \mapsto \sum_{\{\chi \in \widehat{\mathscr{L}} : \chi(n) > 0\}} h_\mu\left(\alpha(n) \mid \widehat{\mathscr{F}} \vee \mathscr{W}^{\chi} \vee \eta\right)$$

extend from $C(\widehat{\mathscr{F}})$ to linear functions on $\mathbb{R}^d$; moreover the two functions coincide on a spanning set of $n \in \mathbb{Z}^d$. It follows that they agree on $C(\widehat{\mathscr{F}})$. As $\widehat{\mathscr{F}} \vee \mathscr{W}^{\chi} = \mathscr{F} \vee \mathscr{W}^{\chi}$ for all $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ with $\chi(m) > 0$, the conclusion extends by linearity to non-generic $m \in \mathbb{Z}^d$. □

16. PROOF OF THEOREMS 13.5 AND 13.7

We retain all notation from Section 13.2. In particular, $\overline{\alpha}$ is an action of $\mathbb{Z}^d$ on (N,ν) which is a measurable factor of α induced by $\psi\colon M \to N$; $\mathscr{A}^\psi$ denotes the α-invariant measurable partition on (M,μ) induced by the factor map ψ.

16.1. **Key Lemma.** Consider any non-zero Lyapunov exponent $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$. Note that we have not yet shown the equivalence class $\overline{\chi}$ is positively propositional to an equivalence class in $\widehat{\mathscr{L}}^{\alpha}(\mu)$. Take $H = C(\overline{\chi})$ to be the Lyapunov half-space associated with $\overline{\chi}$.

Considering all non-zero $\overline{\lambda} \in \mathscr{L}^{\overline{\alpha}}(\nu)$ and $\lambda \in \mathscr{L}^{\alpha}(\mu)$ as linear functionals on $\mathbb{R}^d$, define *joint chambers* and *joint subchambers* relative to the collection of linear functionals $\mathscr{L} = \mathscr{L}^{\overline{\alpha}}(\nu) \cup \mathscr{L}^{\alpha}(\mu)$. Then H is saturated by joint subchambers and we may take a finite set $S \subset H \cap \mathbb{Z}^d$ that is sufficient (for the collection of joint subchambers) in H.

Take $\overline{\eta}$ to be a measurable partition of (N,ν) as in Proposition 14.11 that is subordinate to $\mathscr{W}^{\overline{\chi}}$ with $\overline{\alpha}(n)(\overline{\eta}) \prec \overline{\eta}$ for all $n \in S$. Let $\eta = \psi^{-1}(\overline{\eta})$.

The key observation in the proof of Theorems 13.5 and 13.7 is that for $n \in S \subset H$, every coarse Lyapunov exponent $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ with $\chi \neq \overline{\chi}$ contributes only fiber-entropy to the quantity $h_\mu(\alpha(n) \mid \eta)$. Write

$$\mathscr{U}^{\alpha}_{\mu}(n) := \left\{\lambda' \in \mathscr{L}^{\alpha}(\mu) : \lambda'(n) > 0\right\}.$$

LEMMA 16.1. *Let $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ be as fixed above. For $n \in S \subset H$ and any $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ such that χ is not proportional to $\overline{\chi}$ and $\chi(n) > 0$ we have*

$$\gamma_n\left(\lambda_j \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) = \gamma_n\left(\lambda_j \mid \mathscr{A}^\psi \mid \chi\right)$$

for all $\lambda_j \in \chi$.

In Lemma 16.1 and in what follows, the dimension $\gamma_n(\cdot)$ are always relative to the action α.

Proof. First note that for any $n \in S$ with $\chi(n) > 0$, from Claim 15.2 we have

$$\gamma_n\left(\lambda_j \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) \geq \gamma_n\left(\lambda_j \mid \eta \mid \chi\right) \geq \gamma_n\left(\lambda_j \mid \mathscr{A}^\psi \mid \chi\right) \tag{58}$$

for all $\lambda_j \in \chi$.

To prove the reverse inequality note that, as χ is not proportional to $\overline{\chi}$, the Lyapunov hyperplane associated to χ intersects the interior of H. In particular, there are joint chambers $W_1 \subset H \cap C(\chi)$ and $W_2 \subset H \smallsetminus C(\chi)$ that are adjacent through χ. Let U_1 and U_2 be subchambers, respectively, of W_1 and W_2 that are adjacent through χ and fix $n_i \in U_i \cap S \subset H$.

Let $W_0 \subset C(\chi) \cap H$ be the joint chamber containing the n in the statement of the lemma. As S is sufficient in $C(\chi) \cap H$ there is a sequence of subsequently adjacent joint chambers $W_0 = W^1, W^2, \dots, W^{\ell+1} = W_1$ in $C(\chi) \cap H$ and a sequence of coarse Lyapunov exponents $\chi'_1, \chi'_2, \dots, \chi'_\ell$ in $\widehat{\mathscr{L}}^{\alpha}(\mu) \cup \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ with $\chi'_j \neq \chi$ for

$1 \le j \le \ell$ such that each pair W^j, W^{j+1} is adjacent through χ'_j. Note also that since each $W^j \subset H$, we have $\chi'_j \neq \overline{\chi}$ for $1 \le j \le \ell$. Let

$$\mathscr{I} = \mathscr{U}^{\alpha}_{\mu}(n) \smallsetminus \left(\chi'_1 \cup \chi'_2 \cup \cdots \cup \chi'_\ell\right).$$

Then also

$$\mathscr{I} = \mathscr{U}^{\alpha}_{\mu}(n_1) \smallsetminus \left(\chi'_1 \cup \chi'_2 \cup \cdots \cup \chi'_\ell\right).$$

Repeatedly using Lemma 14.8 and Proposition 15.4 when crossing chamber walls, we know $\mathscr{I}$ is complete, and for $\lambda_j \in \chi$ the following hold:

1. $\gamma_n(\lambda_j \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)) = \gamma_{n_1}(\lambda_j \mid \eta \mid \mathscr{I})$;
2. $\gamma_{n_1}(\lambda_j \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n_1)) = \gamma_n(\lambda_j \mid \eta \mid \mathscr{I})$;
3. $\gamma_n(\lambda_j \mid \eta \mid \mathscr{I}) = \gamma_{n_1}(\lambda_j \mid \eta \mid \mathscr{I})$;
4. $\gamma_n(\lambda_j \mid \mathscr{A}^{\psi} \mid \chi) = \gamma_{n_1}(\lambda_j \mid \mathscr{A}^{\psi} \mid \chi)$.

To prove the lemma it thus suffices to show for $\lambda_i \in \chi$ that

$$\gamma_{n_1}(\lambda_j \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n_1)) = \gamma_{n_1}\left(\lambda_j \mid \mathscr{A}^{\psi} \mid \chi\right). \tag{59}$$

From Theorem 7.7, for any $n \in S$ we have

$$h_\mu(\alpha(n) \mid \eta) = \sum_{\lambda_i \in \mathscr{U}^{\alpha}_{\mu}(n)} \lambda_i(n)\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right).$$

Moreover, from the conditional Abramov–Rohlin formula (20), Theorem 7.7, and Theorem 13.1 (applied to the trivial foliation and α-invariant partition $\mathscr{A}^{\psi}$) we have for any $n \in S$ that

$$\begin{aligned}
h_\mu(\alpha(n) \mid \eta) &= h_\nu(\overline{\alpha}(n) \mid \overline{\eta}) + h_\mu\left(\alpha(n) \mid \mathscr{A}^{\psi}\right)\\
&= h_\nu(\overline{\alpha}(n) \mid \overline{\eta}) + \sum_{\chi' \subset \mathscr{U}^{\alpha}_{\mu}(n)} h_\mu\left(\alpha(n) \mid \mathscr{W}^{\chi'} \vee \mathscr{A}^{\psi}\right).\\
&= h_\nu(\overline{\alpha}(n) \mid \overline{\eta}) + \sum_{\chi' \subset \mathscr{U}^{\alpha}_{\mu}(n)} \sum_{\lambda_i \in \chi'} \lambda_i(n)\gamma_n\left(\lambda_i \mid \mathscr{A}^{\psi} \mid \chi'\right),
\end{aligned}$$

whence

$$h_\nu(\overline{\alpha}(n) \mid \hat{\eta}) = \sum_{\lambda_i \in \mathscr{U}^{\alpha}_{\mu}(n)} \lambda_i(n)\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) - \sum_{\chi' \subset \mathscr{U}^{\alpha}_{\mu}(n)} \sum_{\lambda_i \in \chi'} \lambda_i(n)\gamma_n\left(\lambda_i \mid \mathscr{A}^{\psi} \mid \chi'\right).$$

Recall the partition $\overline{\eta}$ of N is subordinate to $\mathscr{W}^{\overline{\chi}}$. From Lemma 14.12, it follows that the restriction of the map

$$n \mapsto h_\nu(\overline{\alpha}(n) \mid \hat{\eta})$$

to $S \subset H$ coincides with the restriction of a linear function $L\colon \mathbb{R}^d \to \mathbb{R}$. From the above analysis, for all $n \in W_1 \cap S$ and $m \in W_2 \cap S$ we have

$$L(n) = \sum_{\lambda_i \in \mathscr{U}^{\alpha}_{\mu}(n_1)} \lambda_i(n)\gamma_{n_1}\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n_1)\right) - \sum_{\chi' \subset \mathscr{U}^{\alpha}_{\mu}(n_1)} h_\mu\left(\alpha(n) \mid \mathscr{W}^{\chi'} \vee \mathscr{A}^{\psi}\right) \tag{60}$$

$$L(m) = \sum_{\lambda_i \in \mathscr{I}^*} \lambda_i(m)\gamma_{n_2}\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n_2)\right) - \sum_{\chi' \subset \mathscr{I}*} h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi'} \vee \mathscr{A}^{\psi}\right) \tag{61}$$

where $\mathscr{I}^* = \left(\mathscr{U}_\mu^\alpha(n_1) \setminus \chi\right) \cup -\chi$ or $\mathscr{I}^* = \mathscr{U}_\mu^\alpha(n_1) \setminus \chi$ depending, respectively, on whether or not $-\chi$ is a coarse Lyapunov exponent in $\widehat{\mathscr{L}}^\alpha(\mu)$.

Consider $\chi' \in \widehat{\mathscr{L}}^\alpha(\mu)$ with $\chi' \subset \mathscr{U}_\mu^\alpha(n_1) \setminus \{\pm\chi\}$ and $\lambda_i \in \chi'$. Since W_1 and W_2 are adjacent through χ, from Proposition 15.4 we have

$$\begin{aligned}\gamma_{n_1}\left(\lambda_i \mid \eta \mid \mathscr{U}_\mu^\alpha(n_1)\right) &= \gamma_{n_1}\left(\lambda_i \mid \eta \mid \mathscr{I}^*\right)\\ &= \gamma_{n_2}\left(\lambda_i \mid \eta \mid \mathscr{I}^*\right)\\ &= \gamma_{n_2}\left(\lambda_i \mid \eta \mid \mathscr{U}_\mu^\alpha(n_2)\right)\end{aligned}$$

and

$$\gamma_{n_1}\left(\lambda_i \mid \mathscr{A}^\psi \mid \chi'\right) = \gamma_{n_2}\left(\lambda_i \mid \mathscr{A}^\psi \mid \chi'\right).$$

Let L_1 be the function

$$L_1(n) = \sum_{\lambda_i \in \chi} \lambda_i(n)\gamma_{n_1}\left(\lambda_i \mid \eta \mid \mathscr{U}_\mu^\alpha(n_1)\right) - \sum_{\lambda_i \in \chi} \lambda_i(n)\gamma_{n_1}\left(\lambda_i \mid \mathscr{A}^\psi \mid \chi\right)$$

and (if $-\chi$ is a coarse Lyapunov exponent) let

$$L_2(n) = \sum_{\lambda_i \in -\chi} \lambda_i(n)\gamma_{n_2}\left(\lambda_i \mid \eta \mid \mathscr{U}_\mu^\alpha(n_2)\right) - \sum_{\lambda_i \in -\chi} \lambda_i(n)\gamma_{n_2}\left(\lambda_i \mid \mathscr{A}^\psi \mid -\chi\right).$$

Comparing righthand sides of (60) and (61) and canceling common linear terms it follows that either

$$L_1 = 0 \qquad \text{or} \qquad L_1 = L_2.$$

From (58) we have $L_1(n) \geq 0$ and $L_2(n) \leq 0$ for $n \in W_1$ whence either case above implies

$$\sum_{\lambda_i \in \chi} \lambda_i(n_1)\left(\gamma_{n_1}\left(\lambda_i \mid \eta \mid \mathscr{U}_\mu^\alpha(n_1)\right) - \gamma_{n_1}\left(\lambda_i \mid \mathscr{A}^\psi \mid \chi\right)\right) = 0.$$

Conclusion (59) then follows from (58). □

16.2. **Proof of Theorem 13.5.** Theorem 13.5 follows directly from Lemma 16.1.

Proof of Theorem 13.5. We retain all notations from above. In particular, we take $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ with

$$h_\nu\left(\overline{\alpha}(n) \mid \overline{\chi}\right) > 0$$

for some n with $\overline{\chi}(n) > 0$. Take a measurable partition $\overline{\eta}$ as in Proposition 14.11 that is subordinate to $\mathscr{W}^{\overline{\chi}}$ with $\overline{\alpha}(n)(\overline{\eta}) \prec \overline{\eta}$ for all $n \in S$. Suppose that no coarse Lyapunov exponent $\chi \in \widehat{\mathscr{L}}^\alpha(\mu)$ is positively proportional to $\overline{\chi}$. Then, by the conditional Abramov-Rohlin formula (20), Theorem 7.7, and Lemma 16.1 we obtain a contradiction, as for any $n \in S \subset H$ we have

$$\begin{aligned}&h_\nu\left(\overline{\alpha}(n) \mid \mathscr{W}^{\overline{\chi}}\right)\\ &\quad= h_\nu(\overline{\alpha}(n) \mid \overline{\eta}) = h_\mu(\alpha(n) \mid \eta) - h_\mu\left(\alpha(n) \mid \mathscr{A}^\psi\right)\\ &\quad= \sum_{\lambda_i \in \mathscr{U}_\mu^\alpha(n)} \lambda_i(n)\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}_\mu^\alpha(n)\right) - \sum_{\chi \in \widehat{\mathscr{L}}^\alpha(\mu):\chi(n)>0} \sum_{\lambda_i \in \chi} \lambda_i(n)\gamma_n\left(\lambda_i \mid \mathscr{A}^\psi \mid \chi\right)\\ &\quad= 0.\end{aligned}$$

□

Having established Theorem 13.5, given any coarse Lyapunov exponent $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ contributing entropy to the factor system $\overline{\alpha}$, it follows there is $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ positively proportional to $\overline{\chi}$.

Let $\overline{\chi} \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$, $\overline{\eta}$, η, and $S \subset H = C(\overline{\chi})$ be as above.

COROLLARY 16.2. *Suppose $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ is positively proportional to $\overline{\chi}$. For $n \in S$ we have*

$$h_\mu\left(\alpha(n) \mid \mathscr{W}^{\chi} \vee \eta\right) = h_\nu(\overline{\alpha}(n), \overline{\eta}) + h_\mu\left(\alpha(n) \mid \mathscr{A}^{\psi} \vee \mathscr{W}^{\chi}\right).$$

Proof. From Corollary 15.6, for $n \in S$, $\chi' \subset \mathscr{U}^{\alpha}_{\mu}(n)$, and $\lambda_i \in \chi'$ we have the equality

$$\gamma_n\left(\lambda_i \mid \mathscr{A}^{\psi} \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) = \gamma_n\left(\lambda_i \mid \mathscr{A}^{\psi} \mid \chi'\right)$$

which is well-defined and independent of the choice of $n \in C(\chi')$. Let $\gamma\left(\lambda_i \mid \mathscr{A}^{\psi}\right)$ denote this constant.

From Lemma 16.1 and Theorem 7.7 we have

$$\begin{aligned}
&h_\mu(\alpha(n) \mid \eta)\\
&\quad = \sum_{\lambda_i \in \mathscr{U}^{\alpha}_{\mu}(n)} \lambda_i(n)\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right)\\
&\quad = \sum_{\lambda_i \in \mathscr{U}^{\alpha}_{\mu}(n)} \lambda_i(n)\gamma\left(\lambda_i \mid \mathscr{A}^{\psi}\right) + \sum_{\lambda_i \in \chi} \lambda_i(n)\left(\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) - \gamma\left(\lambda_i \mid \mathscr{A}^{\psi}\right)\right)\\
&\quad = h_\mu(\alpha(n) \mid \mathscr{A}^{\psi}) + \sum_{\lambda_i \in \chi} \lambda_i(n)\left(\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) - \gamma\left(\lambda_i \mid \mathscr{A}^{\psi}\right)\right)
\end{aligned}$$

whence

$$\begin{aligned}
h_\nu(\overline{\alpha}(n), \overline{\eta}) &= h_\mu(\alpha(n) \mid \eta) - h_\mu\left(\alpha(n) \mid \mathscr{A}^{\psi}\right)\\
&= \sum_{\lambda_i \in \chi} \lambda_i(n)\left(\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) - \gamma\left(\lambda_i \mid \mathscr{A}^{\psi}\right)\right).
\end{aligned}$$

From Corollary 15.6, we have for $\lambda_i \in \chi$ that

$$\gamma_n\left(\lambda_i \mid \eta \mid \mathscr{U}^{\alpha}_{\mu}(n)\right) = \gamma_n(\lambda_i \mid \eta \mid \chi).$$

Thus

$$\begin{aligned}
h_\nu(\overline{\alpha}(n), \overline{\eta}) &= \sum_{\lambda_i \in \chi} \lambda_i(n)\left(\gamma_n(\lambda_i \mid \eta \mid \chi) - \gamma\left(\lambda_i \mid \mathscr{A}^{\psi}\right)\right)\\
&= h_\mu\left(\alpha(n) \mid \mathscr{W}^{\overline{\chi}} \vee \eta\right) - h_\mu\left(\alpha(n) \mid \mathscr{A}^{\psi} \vee \mathscr{W}^{\chi}\right).
\end{aligned}$$

□

16.3. **Proof of Theorem 13.7.** We obtain Theorem 13.7 from Corollary 16.2.

Proof of Theorem 13.7. Consider a fixed, non-zero $\chi \in \widehat{\mathscr{L}}^{\alpha}(\mu)$. Fix $n \in \mathbb{Z}^d$ with $\chi(n) > 0$ and let W be the chamber of $\widehat{\mathscr{L}}^{\alpha}(\mu)$ containing n. Fix a finite set S that is sufficient in W.

Choose an enumeration of all coarse exponents $\chi_j \subset \mathscr{U}^{\alpha}_{\mu}(n)$ with $\chi = \chi_0$. For each $\chi_j \in \widehat{\mathscr{L}}^{\alpha}(\mu)$ let $\overline{\chi}_j \in \widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ be the coarse Lyapunov exponent positively proportional to χ_j if such a coarse exponent exists; in this case let $\overline{\eta}_j$ be a measurable partition of (N,ν) as in Proposition 14.11 that is subordinate to $\mathscr{W}^{\overline{\chi}_j}$ with $\overline{\alpha}(m)(\overline{\eta}_j) \prec \overline{\eta}_j$ for all $m \in W \cap S$. For $\chi_j \in \widehat{\mathscr{L}}^\alpha(\mu)$ such that no coarse exponent in $\widehat{\mathscr{L}}^{\overline{\alpha}}(\nu)$ is positively proportional to χ_j, take $\overline{\chi}_j$ to be the 0 functional and take $\overline{\eta}_j$ to be the point partition. For all j, let $\eta_j = \psi^{-1}(\overline{\eta}_j)$.

From Theorems 13.1 and 13.5 we have for $m \in W \cap S$

1. $h_\nu(\overline{\alpha}(m)) = \sum_{\overline{\chi} \in \mathscr{L}^{\overline{\alpha}}_{\mathrm{ess}} \cap \mathscr{U}^{\overline{\alpha}}_\nu(m)} h_\nu\left(\overline{\alpha}(m) \mid \mathscr{W}^{\overline{\chi}}\right) = \sum_{\chi_j \in \mathscr{U}^\alpha_\mu(m)} h_\nu(\overline{\alpha}(m), \overline{\eta}_j)$;
2. $h_\mu(\alpha(m) \mid \mathscr{A}^\psi) = \sum_{\chi_j \in \mathscr{U}^\alpha_\mu(m)} h_\mu\left(\alpha(m) \mid \mathscr{A}^\psi \vee \mathscr{W}^{\chi_j}\right)$;
3. $h_\mu(\alpha(m)) = \sum_{\chi_j \in \mathscr{U}^\alpha_\mu(m)} h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j}\right)$.

Combined with Corollary 16.2, it follows that for $m \in W \cap S$,

$$\begin{aligned} \sum_{\chi_j \in \mathscr{U}^\alpha_\mu(m)} h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j} \vee \eta_j\right) &= h_\nu(\overline{\alpha}(m)) + h_\mu(\alpha(m) \mid \mathscr{A}^\psi) \\ &= h_\mu(\alpha(m)) \\ &= \sum_{\chi_j \in \mathscr{U}^\alpha_\mu(m)} h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j}\right). \end{aligned} \tag{62}$$

From Claim 8.5, for each j we have

$$h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j} \vee \eta_j\right) \le h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j}\right)$$

for every $m \in W \cap S$. Combined with (62) it follows for each j and $m \in W \cap S$ that

$$h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j} \vee \eta_j\right) = h_\mu\left(\alpha(m) \mid \mathscr{W}^{\chi_j}\right).$$

The conclusion of Theorem 13.7 then follows for $m \in W \cap S$ from Corollary 16.2.

Since the restrictions of the three terms in (45) to $C(\chi)$ extend to linear functions, the conclusion of Theorem 13.7 extends to all $n \in C(\chi)$ by linearity. □

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