



APPROXIMATE SOLUTIONS TO SECOND-ORDER PARABOLIC EQUATIONS: EVOLUTION SYSTEMS AND DISCRETIZATION

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In loving memory of Rosa Maria (Rosella) Mininni

ABSTRACT. We study the discretization of a linear evolution partial differential equation when its Green's function is known or well approximated. We provide error estimates both for the spatial approximation and for the time stepping approximation. We show that, in fact, an approximation of the Green function is almost as good as the Green function itself. For suitable time-dependent parabolic equations, we explain how to obtain good, explicit approximations of the Green function using the Dyson-Taylor commutator method that we developed in *J. Math. Phys.* **51** (2010), n. 10, 103502 (reference [15]). This approximation for short time, when combined with a bootstrap argument, gives an approximate solution on any fixed time interval within any prescribed tolerance.

1. Introduction. We consider an initial value problem (IVP) of the form

$$\begin{cases} \partial_t u(t) - L(t)u(t) = f, & 0 \leq s \leq t, \\ u(s) = h, \end{cases} \quad (1)$$

where we require $u(t)$ and h to belong to certain Sobolev spaces on $\mathbb{R}^N$.

Let us assume $f = 0$. The *solution operator*, if it exists, is then $U^L(t, s)h = u(t)$. It defines what is called an *evolution system* [1, 43, 48] (we recall the definition of an evolution system in Definition 2.1). We have

$$[U^L(t, s)h](x) = \int_{\Omega} \mathcal{G}_{t,s}^L(x, y)h(y)dy \quad (2)$$

when such a distribution $\mathcal{G}_{t,s}^L(x, y)$ exists. We call this distribution $\mathcal{G}_{t,s}^L(x, y)$ the *Green function* of the evolution system U^L . The existence of $\mathcal{G}$ under mild conditions on $L(t)$ follows by the Schwartz Kernel Theorem (see e.g. [16]). (In the cases considered in this paper, it will be a *true function*. We shall also say that $\mathcal{G}_{t,s}^L(x, y)$ is the *Green function* of $\partial_t - L$. The terminology *fundamental solution* of $\partial_t - L$ is also used for $\mathcal{G}_{t,s}^L(x, y)$.)

In this paper we consider the following problems:

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- (1) Assuming that the Green function $\mathcal{G}_{t,s}^L(x,y)$ of the evolution system U^L is known, establish the properties of the approximations of $u(t)$ in suitable discretization spaces S .
- (2) Show that suitable good approximations of the Green function are (almost) as good as the Green function itself.
- (3) Provide a method to find good approximations of the Green function, including complete error estimates.

We address the problems above under some mild, customary assumptions. First, we assume that

$$L(t) := \sum_{i,j}^N a_{ij}(t,x) \partial_i \partial_j + \sum_i^N b_i(t,x) \partial_i + c(t,x), \quad (3)$$

with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $\partial_k := \frac{\partial}{\partial x_k}$, and $t \in I$, where, throughout this paper, $I \subset [0, \infty)$ is an interval containing 0. Most of the results pertaining to points (1) and (2) above extend to bounded domains $\Omega \subset \mathbb{R}^N$ of sufficient regularity under suitable boundary conditions. However, the Green function approximation in point (3) changes significantly. Therefore, we choose to work on $\mathbb{R}^N$ in this paper. The coefficients a_{ij} , b_i , and c all their derivatives are assumed smooth and bounded (i.e., they are assumed to be in $W^{\infty,\infty}(\mathbb{R}^+ \times \mathbb{R}^N) = \mathcal{C}_b^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$). For simplicity, we assume as well that these coefficients are real and that the resulting matrix is symmetric, namely $a_{ij} = a_{ji}$ for all i, j . We impose a *uniform strong ellipticity* condition on the operators $L(t)$, meaning that there exists a constant $\gamma > 0$ such that

$$\sum a_{ij}(t,x) \xi_i \xi_j \geq \gamma \|\xi\|^2, \quad \forall t \geq 0, x, \xi \in \mathbb{R}^N, \xi \neq 0. \quad (4)$$

We collectively denote by $\mathbb{L}_\gamma$ the class of operators $L = (L(t))_{t \in I}$ of the form (3) satisfying the ellipticity condition (4) and the coefficients of which, together with all their derivatives, are bounded (see Definition 4.1).

Let us discuss in more detail the three main contributions of our work to the problems (1)-(3)

(1) The contribution to the first problem (“Assuming that the Green function $\mathcal{G}_{t,s}^L(x,y)$ of the evolution system U^L is known, to establish the properties of the approximations of $u(t)$ in suitable discretization spaces S ”) addresses a very natural question. Even if, theoretically, the knowledge of the initial data h and of the Green functions $\mathcal{G}_{t,s}^L(x,y)$ determines the solution u via integration: $u(t,x) = \int_{\mathbb{R}^N} \mathcal{G}_{t,0}^L(x,y) h(y) dy$, applying this result in practice leads to at least two issues. The first one is that we can store only a *finite dimensional* space V of potential solutions and initial data computationally. We thus need to *discretize* our equation and to *approximate* both the initial data and the solution with elements of V . Our first result, Theorem 3.3 gives a “proof of concept” result on how such a discretization (in the space variable) works. The main point of the result is that the projection error has to decrease in time at the same order as the time itself (unlike in the time independent case, see Theorem 3.3, especially the Condition 9). In our setting, we know few error estimates of this kind, but in the general framework of Finite Difference or Finite Element methods for evolution equations, there are some similar results [23, 33, 37, 49, 54].

(2) Our contribution to the second problem (“To show that suitable good approximations of the Green function are (almost) as good as the Green function itself”) addresses another natural question, which is what kind of approximations of the

Green functions would be acceptable in case the Green function itself is not known. Thus, let us assume that an approximate Green function $\tilde{\mathcal{G}}_{t,s}^L(x, y)$ is given. We also assume that the discretization in space is to divide the time interval $[0, T]$ in n equal size intervals (in this paper, we will always use this very common discretization). If the error $\|\mathcal{G}_{t,s}^L - \tilde{\mathcal{G}}_{t,s}^L\|$ is of the order of $(t-s)^\alpha$, then we show that the order of the error due to time discretization (or bootstrap) is of the order $n^{1-\alpha}$. This shows that we need a good approximation of the Green function (*i.e.*, $\alpha > 1$). The bootstrap method is the one we developed in [13, 14]. It is a common method in Finite Difference and Finite Element methods [23, 33, 37, 49, 54]. For Green functions, a similar method was more recently suggested in [41].

A common issue in both space and time discretization (*i.e.*, in (1) and (2)) is that we need to find error estimates that are at least of the order of $(t-s)$ (in fact, even better for (2)). We know very few earlier results in the line of (1) and (2).

(3) Our contribution to the third problem (“*To provide a method to find good approximations of the Green function, including complete error estimates*”) fits into a very long sequence of results concerning heat kernel approximations and Dyson series expansions. The literature on the subject is truly vast, but we nevertheless mention the papers [9, 10, 17, 18, 21, 27, 30, 31], which are some of the papers preceding and most closely related to the articles [15, 13, 12, 14] (in chronological order), in which we have developed the Dyson-Taylor commutator method used in this paper. Let us mention also the more recent papers [20, 22, 25, 26, 28, 29, 42, 57], where the reader will be able to find further references. Some general related monographs include [19, 32, 39].

For the Green function approximation, we use the Dyson-Taylor commutator method developed in [15, 13, 12, 14], which we also expand and make more precise. A similar method was employed more recently in [41, 42]. The main result regarding this third questions is a sharp error estimate in weighted Sobolev spaces. This error estimate, when combined with the results of (2) and using the bootstrap argument we developed in [13] gives an approximate solution on any fixed time interval within any prescribed tolerance. Our method is such that also derivatives of the solution can be effectively approximated with verified bounds (with the price of increasing the order of approximation). Our error estimates are in *exponentially weighted Sobolev spaces* $W_a^{r,p}(\mathbb{R}^N) = e^{-a\langle x \rangle} W^{r,p}(\mathbb{R}^N)$, $r \geq 0$, $1 < p < \infty$, $a \in \mathbb{R}$, defined in Equation (17), where $\langle x \rangle := \sqrt{1 + |x|^2}$ is given in (16).

Our main result is the following. (The $\mathbb{L}_\gamma$ was introduced above, but see 4.1 for more details.)

Theorem 1.1. *Let L be an operator in the class $\mathbb{L}_\gamma$. Then L generates an evolution system U^L in the Sobolev space $W_a^{r,p}(\mathbb{R}^N)$, $r \geq 0$, $1 < p < \infty$, $a \in \mathbb{R}$. Given $\mu \in \mathbb{N}$, there exists an explicitly computable smooth function $\tilde{\mathcal{G}}_{t,s}^{[\mu]}(x, y)$, given in Definition 6.2, such that the distribution kernel $\mathcal{G}_{t,s}^L(x, y)$ of $U^L(t, s)$ (that is, the Green function of $\partial_t - L$) can be represented as*

$$\mathcal{G}_{t,s}^L(x, y) := \tilde{\mathcal{G}}_{t,s}^{[\mu]}(x, y) + (t-s)^{(\mu+1)/2} \tilde{E}_{t,s}^{[\mu]}(x, y),$$

where the remainder $\tilde{E}_{t,s}^{[\mu]}$, when regarded as an integral operator, satisfies

$$\|\tilde{E}_{t,s}^{[\mu]} g\|_{W_a^{r+k,p}} \leq C (t-s)^{-k/2} \|g\|_{W_a^{r,p}}, \quad 0 \leq s < t \leq T, \quad k \in \mathbb{N}$$

with a bound C depending on L, μ, a, k, r, p , and $0 < T < \infty$, but independent of g and $s, t \in [0, T]$, $s \leq t$.

Together with Theorem 3.6, this theorem yields an approximation of the solution u of our Initial Value problem (1).

The paper is organized as follows. In Section 2, we remind some standard facts about non-autonomous, second-order initial value problems $(\partial_t - L(t))u(t, x) = 0$ and the evolution system they generate. In Section 3, we establish space discretization and time discretization (bootstrap) error estimates in a general, abstract setting. The setting is that of an evolution system that satisfies some standard exponential bounds. These exponential bounds are satisfied both in the parabolic and hyperbolic settings, so they are realistic. (They are automatically satisfied if L is independent of time.) Beginning with Section 4, we specialize to the case of operators $L \in \mathbb{L}_\gamma$. In that section, we introduce weighted Sobolev spaces and we study the evolution system generated by $L \in \mathbb{L}_\gamma$. Using the theory of analytic semigroups, we establish explicit mapping properties that allow us to make sense of the integrals appearing in the iterative time-ordered expansions that we use (the resulting formulas are sometimes called Dyson-series and are well known and much used in the Physics literature). The time-ordered expansion is obtained, as usual, using Duhamel's principle iteratively. Section 5 contains a formal derivation of the asymptotic expansion of the solution operator for the Equation (1). This derivation allows us to use the method from [15] for computing the time-ordered integral appearing in the resulting Dyson series expansion using Hadamard's formula:

$$e^A B = \left(B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \right) e^A. \quad (5)$$

Here we use the crucial observation in [15] that, in the cases of interest for us, this series reduces to a finite, explicit sum. In Section 6, we introduce our approximate Green function, we prove Theorem 1.1, and we complete our error analysis. Technically, this section is one of the most demanding.

Throughout the paper, unless explicitly mentioned, C will denote a generic constant that may be different each time when it is used. We employ standard notation for function spaces throughout, in particular $W^{r,p}$, $1 \leq p \leq \infty$, $r \in \mathbb{R}$ for standard L^p -based Sobolev spaces on $\mathbb{R}^n$, and $H^s = W^{s,2}$. We also denote the space of continuous functions (which may take values in a Banach space) with $\mathcal{C}$, and by $W^{\infty,\infty}$ the Sobolev space of bounded functions with bounded derivatives of all orders. By the Sobolev Embedding Theorem, the elements of $W^{\infty,\infty}$ are smooth functions.

The results of this paper are based in great part and extend some results in [12] and an unpublished 2011 IMA preprint [14]. See [25, 42, 57] for some recent, related results to that preprint. However, Section 3 is essentially new. Also, we did not include the numerical test and the explicit calculations of the SABR model from [14] in order to keep this paper more focused (and to limit its size).

Convention: we use throughout the usual multi-index notation for derivatives with respect to the space variable x , that is, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^N$, and $|\alpha| = \sum_{i=1}^N \alpha_i$, $\partial_j = \frac{\partial}{\partial x_j}$, while $\partial_t = \frac{\partial}{\partial t}$.

2. Preliminaries on evolution systems . We refer the reader to [1, 43, 48] for further results and details on the functional analytic framework that we employ. Let $(X, \|\cdot\|)$ be a Banach space and let $A : \mathcal{D}(A) \rightarrow X$ be a (possibly unbounded) closed linear operator with domain $\mathcal{D}(A) \subset X$. We let $\rho(A)$ denote its *resolvent set*, that is, the set of $\lambda \in \mathbb{C}$ such that $A - \lambda : \mathcal{D}(A) \rightarrow X$ is a bijection. We let $R(\lambda, A) := (\lambda - A)^{-1} : X \rightarrow X$ be its *resolvent*, for $\lambda \in \rho(A)$.

Throughout, $\mathcal{L}(X_1, X_2)$ is the space of all bounded linear operators on $X_1 \rightarrow X_2$ for two normed spaces X_1 and X_2 . We let $\mathcal{L}(X) = \mathcal{L}(X, X)$. For ease of notation, we let $\|\cdot\|_{X_1, X_2}$ and $\|\cdot\|_X$ denote the corresponding norms. Let $\arg : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ be the imaginary part of the branch of log that satisfies $\log(1) = 0$.

2.1. Properties of evolution systems. Recall that, throughout this paper, $I \subset [0, \infty)$ is an interval containing 0. In this section, we show that $L = (L(t))_{t \in I}$ generates an evolution system on Sobolev spaces. We recall below the definition of an evolution system and some basic properties for the reader's convenience. (We refer to [43] for an in-depth discussion. See also [1, 48])

Definition 2.1. Let $I \subset [0, \infty)$ be an interval containing 0 (as always). A two parameter family of bounded linear operators $U(t, t')$ on X , $0 \leq t' \leq t \in I$, is called an *evolution system* if the following three conditions are satisfied

1. $U(t, t) = 1$, the identity operator, for all $t \in I$;
2. $U(t, t')U(t', t'') = U(t, t'')$ for $0 \leq t' \leq t'' \leq t \in I$;
3. $U(t, t')$ is strongly continuous in t and t' for all $0 \leq t' \leq t \in I$.

If $U(t, t')$ depends only on $t - t'$, then U is called *autonomous*.

Informally, we shall say that the family of unbounded operators $L = (L(t))_{t \in I}$ generates the evolution system U if $\partial_t U(t, s)\xi = L(t)U(t, s)\xi$ for all $t_0 \leq s < t \in I$ and ξ in a suitable large subspace. We prefer not to give a formal definition for what “large” means in this setting, as for the families L that we will consider, this will happen *everywhere*.

Definition 2.2. Let $I \subset [0, \infty)$ be an open interval containing 0, as always in this paper. A family of operators $L = (L(t))_{t \in I}$, $L(t) : \mathcal{D}(L(t)) \subset X \rightarrow X$, $t \in I$, will be called *uniformly sectorial* if the following conditions are satisfied:

1. The domains $\mathcal{D}(L(t)) =: \mathcal{D}$ are independent of t and dense in X ;
2. $\mathcal{D}$ can be endowed with a Banach space norm such that the injection $\mathcal{D} \hookrightarrow X$ is continuous and $I \ni t \rightarrow L(t) \in \mathcal{L}(\mathcal{D}, X)$ is uniformly Hölder continuous with exponent $\alpha \in (0, 1]$.
3. There exist $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, and $M > 0$ such that, for any $t \in [0, T]$,

$$\begin{cases} \rho(L(t)) \supset S_{\theta, \omega} := \{\lambda \in \mathbb{C}, \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \|R(\lambda, L(t))\| \leq \frac{M}{|\lambda - \omega|}, \quad \forall \lambda \in S_{\theta, \omega}. \end{cases}$$

The following well known proposition (see again [43, page 43] for a proof) gives a sufficient condition that guarantees the sectoriality of an operator.

Proposition 2.3. Let $A : \mathcal{D}(A) \subset X \rightarrow X$ be a linear operator. Assume that there exist $\omega \in \mathbb{R}$ and $M > 0$ such that $\rho(A)$ contains the half plane $\{\lambda \in \mathbb{C}, \operatorname{Re} \lambda \geq \omega\}$ and

$$\|\lambda R(\lambda, A)\|_X \leq M, \quad \forall \operatorname{Re} \lambda \geq \omega.$$

Then A is sectorial.

We recall that uniform sectoriality implies generation of an evolution system [43, page 212]. (This is the “uniform parabolic case,” see also sections 5.6 and 5.7 in [48].) Specifically, we have the following result that applies to our setting, which is introduced in Section 4. (See, for example, [43, Corollary 6.1.8, page 219], for a proof.)

Theorem 2.4. Suppose $L = (L(t))_{t \in I}$ is uniformly sectorial with common domain $\mathcal{D}$, then there exists an evolution system $U(t, s)$, $0 \leq s \leq t \in I$, such that $\partial_t U(t, s)\xi = L(t)U(t, s)\xi$ for all $\xi \in \mathcal{D}$ and $s < t$. This evolution system is unique and the following hold:

1. The functions

$$\|U(t, s)\|_X, \quad (t - s)\|U(t, s)\|_{X, \mathcal{D}}, \quad \|L(t)U(t, s)\|_{\mathcal{D}, X}$$

are uniformly bounded for $0 \leq s < t \in I$ and

2. $\partial_s U(t, s) = -U(t, s)L(s)$, for $s < t$.

If L and U are as in the above theorem, then we shall say that L generates U and denote it U^L . We now return to the study of the IVP (1). We shall use the following notion of solution (see e.g. [43, pages 123-124]).

Definition 2.5. Let X be a Banach space, $h \in X$, and $f \in L^1((0, T), X)$.

1. By a *strong solution in X* of (1) on the interval $[0, T)$, we mean a function

$$u \in \mathcal{C}([0, T), X) \cap W^{1,1}((0, T), X) \quad (6)$$

such that $u(t) \in D(L)$ and $\partial_t u(t) = L(t)u(t) + f(t)$ in X for almost all $t \in (0, T)$, and $u(0) = h$.

2. By a *classical solution in X* of (1) on the interval $[0, T)$, we mean a function

$$u \in \mathcal{C}([0, T), X) \cap \mathcal{C}^1((0, T), X) \cap \mathcal{C}((0, T), \mathcal{D}(L(t))) \quad (7)$$

such that $\partial_t u(t) = L(t)u(t) + f(t)$ in X for $0 < t < T$, and $u(0) = h$.

Every classical solution is also a strong solution. Theorem 2.4 shows that, if $f = 0$ and $L(t)$ is uniformly sectorial, then the IVP (1) has a unique strong solution for all $h \in X$ and this solution is also a classical solution. It will be convenient to formalize our presentation using the following definitions.

Definition 2.6. Let J be an arbitrary index set. A family of norms $||| \cdot |||_t$, $t \in J$, on X will be called *uniformly equivalent* to the given norm $\| \cdot \|$ on X if there exists $C > 0$ with the property that, for all $x \in X$ and all $t \in J$, we have

$$C^{-1}\|x\| \leq |||x|||_t \leq C\|x\|.$$

The following concept will play an important role in what follows.

Definition 2.7. We shall say that an evolution system $U(t, s)$, $0 \leq s \leq t \in I$, has *exponential bounds* if there exist $\omega_U \in \mathbb{R}$ and $M_U > 0$ such that, for all $x \in X$ and all $0 \leq s \leq t \in I$, we have the estimate

$$\|U(t, s)x\| \leq M_U e^{\omega_U(t-s)} \|x\|.$$

Clearly, any autonomous evolution system has exponential bounds (a simple consequence of the Banach-Steinhaus uniform boundedness principle) [1, 43, 48]. We will need the following result (see again [1, 43, 48]).

Lemma 2.8. Assume that $U(t, s)$, $0 \leq s \leq t \in I$, is an evolution system that has exponential bounds (with bounds M_U and ω_U , as in Definition 2.7). Then, there exists a family of norms $||| \cdot |||_t$, $t \in I$, on X that are uniformly equivalent to the given norm of X such that, for all $0 \leq s \leq t \in I$ and all $x \in X$, they satisfy

$$|||U(t, s)x|||_t \leq e^{\omega_U(t-s)} |||x|||_s.$$

The point of this lemma is, of course, that we can assume $M_U = 1$ in Definition 2.7, with the price of admitting time-dependent norms. We include a proof for completeness.

Proof. Set $V(t, s) = e^{-\omega_U(t-s)}U(t, s)$, then it is clear that $V(t, s)$ is uniformly bounded by M_U . We define a new norm as

$$|||x|||_s := \sup_{s \leq t \in I} \|V(t, s)x\|.$$

From the first part, we then obtain $\|x\| \leq |||x|||_s \leq M_U\|x\|$, for all $s \in I$. Thus, the family $|||\cdot|||_s$, $0 \leq s \in I$, is *uniformly* equivalent to $\|\cdot\|$ on X . Note that by our definition, for all $0 \leq s \leq t \in I$,

$$\begin{aligned} |||V(t, s)x|||_t &= \sup_{t \leq r \in I} \|V(r, t)V(t, s)x\| = \sup_{t \leq r \in I} \|V(r, s)x\| \\ &\leq \sup_{s \leq r \in I} \|V(r, s)x\| =: |||x|||_s. \end{aligned}$$

Substituting $V(t, s) = e^{-\omega_U(t-s)}U(t, s)$, we obtain the desired estimate. $\square$

We now state the desired form of this result.

Corollary 2.9. *Assume that the family $L = (L(t))_{t \in I}$ of operators on a Banach space $(X, \|\cdot\|)$ is uniformly sectorial and let U^L be the evolution system it generates. Then U^L has exponential bounds. Let ω_U be as in Definition 2.7 (with U replaced with U^L). Consequently, there exists a uniformly equivalent family of time-dependent norms $|||\cdot|||_t$, $t \in I$, such that, for all $0 \leq s \leq t \in I$ and all $x \in X$,*

$$|||U^L(t, s)x|||_t \leq e^{\omega_U(t-s)}|||x|||_s.$$

Proof. The first part is well known [1, 43, 48]. The second part follows easily from Lemma 2.8 and is also known. $\square$

Again, the main point of this result is that there is no additional factor M_U in front of the factor $e^{\omega_U(t-s)}$. This will be crucial in the error estimates of the following section.

3. Discretization and bootstrap error estimates. In this section, we study the discretization error when we compress our evolution system $U = (U(t, s))_{0 \leq s \leq t \in I}$, $U(t, s) \in \mathcal{L}(X)$, to a subspace $S \subset X$ and the bootstrap error when we approximate U with some other two-parameter family of operators K . Recall that, throughout this paper, $I \subset [0, \infty)$ is an interval containing 0.

Throughout this section, let $U(t, s)$, $0 \leq s \leq t \in I$, be an evolution system acting on some Banach space $(X, \|\cdot\|)$. We assume that U has exponential bounds, see Definition 2.7. In particular, $\omega_U \in \mathbb{R}$ and $M_U > 0$ will be as in that definition (i.e. $\|U(t, s)x\| \leq M_U e^{\omega_U(t-s)}\|x\|$ for all $x \in X$ and all $0 \leq s \leq t \in I$). Recall then from Lemma 2.8 that there exists a uniformly equivalent family of time-dependent norms $|||\cdot|||_t$, $t \in I$, on X which makes the factor M_U unnecessary. That is, there exist $C_U > 0$ such that

$$\begin{aligned} C_U^{-1}\|x\| &\leq |||x|||_t \leq C_U\|x\| \quad \text{and} \\ |||U(t, s)x|||_t &\leq e^{\omega_U(t-s)}|||x|||_s, \end{aligned} \tag{8}$$

for all $x \in X$ and all $0 \leq s \leq t \in I$ (with ω_U as above). We stress that there is *no additional bound* M_U in front of the exponential in the last estimate, and this is indeed crucial in our error estimates below. The need for such estimates is one

feature that is specific to time dependent equations. Below, U , C_U , and ω_U will always be as in the above equation. We recall that evolution systems generated by uniform parabolic (the case in the following sections) or uniform hyperbolic generators will satisfy our assumptions [48].

Let $X_s = X$, but with the norm $|||\cdot|||_s$. If $T \in \mathcal{L}(X)$, we let $|||T|||_{s,t} := \|T\|_{X_s, X_t}$, the norm on $\mathcal{L}(X_s, X_t)$. We shall need the following simple lemmata.

Lemma 3.1. *We let C_U and the norms $|||\cdot|||_t$ on X be as in Equation (8). Then, for all $Q \in \mathcal{L}(X)$, we have $|||Q|||_{s,t} \leq C_U^2 \|Q\|_X$.*

The proof is immediate. We have stated this lemma only for the purpose of referencing it.

Lemma 3.2. *Let $V(t, s), G(t, s) \in \mathcal{L}(X)$, $0 \leq s \leq t \in I$, and $|||\cdot|||_{s,t} := \|\cdot\|_{X_s, X_t}$, as above. Let $S \subset X$ be a closed subspace. Suppose that there exist $\omega \in \mathbb{R}$, $\alpha \geq 1$, and $C_G > 0$ such that following conditions hold for all $0 \leq s \leq t \in I$:*

1. $|||V(t, s)|||_{s,t} \leq e^{\omega(t-s)}$.
2. $\|V(t, s) - G(t, s)\|_{S,X} \leq C_G(t-s)^\alpha$.

Then there exists $\omega' \in \mathbb{R}$ such that $|||G(t, s)\xi|||_t \leq e^{\omega'(t-s)} \|\xi\|_s$ for all $0 \leq s \leq t \in I$ and $\xi \in S$.

Proof. We first notice that, by Lemma 3.1, for all $\xi \in S$ and all $0 \leq s \leq t \in I$, we have $|||(V(t, s) - G(t, s))\xi|||_t \leq C_U^2 C_G(t-s)^\alpha \|\xi\|_s$. Then, we notice that, for large ω' fixed, we have

$$\sup_{0 \leq s, t \in \mathbb{R}} \frac{e^{\omega|t-s|} + C_U^2 C_G |t-s|^\alpha}{e^{\omega'|t-s|}} \leq 1,$$

since $\alpha \geq 1$. The result then follows from the triangle inequality. $\square$

We remark that, if $\alpha < 1$, then, in general, the lemma will not be true anymore. We are ready now to prove an error estimate for the spatial discretization. To simplify the notation, in the following, we shall let $U_k := U((k+1)\delta, k\delta)$.

Theorem 3.3. *Let $U(t, s)$, $0 \leq s \leq t \in I$ be an evolution system on a Banach space X as in Equation (8). Let $P : X \rightarrow S \subset X$ be a continuous linear projection and let $C_P > 0$ be such that*

$$\|(1-P)U(t, s)P\|_X \leq C_P(t-s) \tag{9}$$

for all $0 \leq s \leq t \in I$. Then there exists $\omega' \geq 0$ with the following property. Let $n \in \mathbb{N}$, $T_0 \in I$, $\delta := T_0/n$, $x_0 \in X$, $y_0 \in S$, $U_k := U((k+1)\delta, k\delta)$, and $x_{k+1} = U_k x_k$ and $y_{k+1} = P U_k y_k$, for $k = 0, \dots, (n-1)$. Then

$$\|x_n - y_n\| \leq C_U^2 e^{\omega' T_0} (\|x_0 - y_0\| + C_U^2 C_P T_0 \|y_0\|).$$

Proof. We let ω_U, C_U , and the norms $|||\cdot|||_t$ be as in Equation (8). The families of operators $V := U$ and $G := PU$ satisfy the assumptions of Lemma 3.2 with $\omega = \omega_U$, $\alpha = 1$, and the given subspace S since, for all $0 \leq s \leq t \in I$, we have

$$\begin{cases} |||V(t, s)|||_{s,t} = |||U(t, s)|||_{s,t} \leq e^{\omega_U(t-s)} & \text{by (8)} \\ \|V(t, s) - G(t, s)\|_{S,X} \leq \|U(t, s)P - PU(t, s)P\|_X \leq C_P(t-s) & \text{by (9)}. \end{cases}$$

That lemma then shows that there exists $\omega' \in \mathbb{R}$ such that, for all $0 \leq s \leq t \in I$ and $\xi \in S$,

$$|||PU(t, s)\xi|||_t \leq e^{\omega'(t-s)} |||\xi|||_s. \tag{10}$$

Then, from this equation, by induction on k since all $y_k \in S$, we obtain that

$$|||y_k|||_{k\delta} \leq e^{k\omega'\delta} |||y_0|||_0. \quad (11)$$

Since we can always increase ω' , we may assume that $\omega' \geq \max\{\omega_U, 0\}$ without loss of generality. Let us then prove by induction the estimate

$$|||x_k - y_k|||_{k\delta} \leq e^{k\omega'\delta} (|||x_0 - y_0|||_0 + C_U^2 C_P k\delta |||y_0|||_0), \quad (12)$$

for all $0 \leq k \leq n$. Indeed, it is true for $k = 0$ (we even have equality in that case). Assume it next to be true for k , and let us prove it for $(k+1)$. Let $U_k := U((k+1)\delta, k\delta)$, as before. We then have.

$$\begin{aligned} |||x_{k+1} - y_{k+1}|||_{(k+1)\delta} &= |||U_k x_k - P U_k y_k|||_{(k+1)\delta} \\ &\leq |||U_k(x_k - y_k)|||_{(k+1)\delta} + |||(1-P)U_k y_k|||_{(k+1)\delta} \\ &\leq |||U_k|||_{(k+1)\delta, k\delta} |||x_k - y_k|||_{k\delta} + |||(1-P)U_k P|||_{(k+1)\delta, k\delta} |||y_k|||_{k\delta} \\ &\leq e^{\omega'\delta} [|||x_k - y_k|||_{k\delta} + C_U^2 C_P \delta |||y_k|||_{k\delta}] \\ &\leq e^{\omega'\delta} \left[e^{k\omega'\delta} (|||x_0 - y_0|||_0 + C_U^2 C_P k\delta |||y_0|||_0) + C_U^2 C_P \delta e^{k\omega'\delta} |||y_0|||_0 \right] \\ &= e^{(k+1)\omega'\delta} (|||x_0 - y_0|||_0 + C_U^2 C_P (k+1)\delta |||y_0|||_0), \end{aligned}$$

where the last two inequalities are obtained, in order, from Lemma 3.1, from the assumption (9), from the induction hypothesis (12), and from the estimate (11). (The other inequalities are obvious.) This proves (12) for all $k \leq n$. The result follows from this relation for $k = n$, using also Lemma 3.1, since $n\delta = T_0$. $\square$

In applications, the following remark may be useful

Remark 3.4. The bound ω' depends on C_P and it is a non-decreasing function of C_P . Thus, if a sequence of projections P_k is given such that C_{P_k} is bounded, then we can choose ω' independent of k .

Remark 3.5. We stress that the appearance of the factor $(t-s)$ in Equation (9) is crucial and is a typical feature of the conditions needed for the error estimates in our bootstrap method. Let $L = (L(t))$ be the generator of U . This condition can be achieved if the commutator $[P, L(t)] := PL(t) - L(t)P$ is bounded on X . In turn, if $L = \Delta$, for instance and $X = L^2(\mathbb{R}^N)$, then we can construct a subspace S with these properties using a periodic partition of unity and GFEM discretization spaces. (See [6, 7, 8, 24, 45] for some general references to GFEM. See [50] and [24] for papers specifically devoted to evolution equations.) The constant C_P , on the other hand, can account for the spatial discretization error.

The last theorem is relevant if we know $U(t, s)$ explicitly. This is however rarely the case. Instead (and this is one of the reasons why we are writing this paper), we can usually approximate $U(t, s)$. A general example of how to do that will be given in Section 5. We keep the settings of Lemma 3.2.

Theorem 3.6. Let $V(t, s), G(t, s) \in \mathcal{L}(X)$, $0 \leq s \leq t \in I$, $\alpha \geq 1$, and $C_G > 0$ be as in Lemma 3.2, but $S = X$ (and hence $\|V(t, s) - G(t, s)\|_X \leq C_G(t-s)^\alpha$ for all $0 \leq s \leq t \in I$). Then there exists $\omega' \geq 0$ such that

$$1. \quad |||G(t, s)|||_{s,t} \leq e^{\omega'(t-s)} \text{ for all } 0 \leq s \leq t \in I.$$

2. Let $n \in \mathbb{N}$, $T_0 \in I$, $\delta := T_0/n$, $x_0, y_0 \in X$ satisfy $x_{k+1} = V((k+1)\delta, k\delta)x_k$ and $y_{k+1} = G((k+1)\delta, k\delta)y_k$. Then

$$\|x_n - y_n\| \leq C_U^2 e^{\omega' T_0} \left(\|x_0 - y_0\| + C_U^2 C_G \frac{T_0^\alpha}{n^{\alpha-1}} \|y_0\| \right).$$

Proof. Let $C_U > 0$ and $\omega_U \in \mathbb{R}$ be as in Equation (8). Lemma 3.1 then gives that there exists $\omega' \in \mathbb{R}$ that satisfies (1). Hence, if $G_k := G((k+1)\delta, k\delta)$ satisfies $\|G_k\|_{(k+1)\delta, k\delta} \leq e^{\omega' \delta}$.

We proceed as in the proof of Theorem 3.3. First, we similarly obtain, by induction, that

$$\|y_k\|_{k\delta} \leq e^{k\omega' \delta} \|y_0\|_0. \quad (13)$$

By increasing ω' , if necessary, we can assume that $\omega' \geq \omega_U$ in what follows (and hence that $\|V(t, s)\|_{s,t} \leq e^{\omega'(t-s)}$ for all $0 \leq s \leq t \in I$). Let $V_k := V((k+1)\delta, k\delta)$. Let $C_2 := C_U^2 C_G$. The result will then follow from the estimate

$$\|x_k - y_k\|_{k\delta} \leq e^{k\omega' \delta} \left(\|x_0 - y_0\|_0 + C_2 k \delta^\alpha \|y_0\|_0 \right), \quad (14)$$

valid for all $0 \leq k \leq n$, which we prove again by induction on k . Indeed, the estimate is true for $k = 0$ (we even have equality in that case). Assume it next to be true for k , and let us prove it for $(k+1)$. Lemma 3.1 gives that $\|V(t, s) - G(t, s)\|_{s,t} \leq C_2(t-s)^\alpha$ for all $0 \leq s \leq t \in I$, with $C_2 := C_U^2 C_G$. We have

$$\begin{aligned} \|x_{k+1} - y_{k+1}\|_{(k+1)\delta} &= \|V_k x_k - G_k y_k\|_{(k+1)\delta} \\ &\leq \|V_k(x_k - y_k)\|_{(k+1)\delta} + \|[V_k - G_k]y_k\|_{(k+1)\delta} \\ &\leq \|V_k\|_{(k+1)\delta, k\delta} \|x_k - y_k\|_{k\delta} + \|V_k - G_k\|_{(k+1)\delta, k\delta} \|y_k\|_{k\delta} \\ &\leq e^{\omega' \delta} (\|x_k - y_k\|_{k\delta} + C_2 \delta^\alpha \|y_k\|_{k\delta}) \\ &\leq e^{\omega' \delta} \left[e^{k\omega' \delta} (\|x_0 - y_0\|_0 + C_2 k \delta^\alpha \|y_0\|_0) + e^{k\omega' \delta} C_2 \delta^\alpha \|y_0\|_0 \right] \\ &= e^{(k+1)\omega' \delta} \left(\|x_0 - y_0\|_0 + C_2 (k+1) \delta^\alpha \|y_0\|_0 \right), \end{aligned}$$

where the last two inequalities are obtained, in order, from (1), from the estimates $\|V(t, s) - G(t, s)\|_{s,t} \leq C_2(t-s)^\alpha$, from Equation (14) (the induction hypothesis for k), from the estimate (13). This proves (14) for all k . The result follows from this relation for $k = n$, using also Lemma 3.1. $\square$

Since the first condition of Lemma 3.2 is automatically satisfied by an evolution system that satisfies exponential bounds. By taking $x_0 = y_0$ in the previous theorem, we obtain the following result.

Corollary 3.7. *Let $U(t, s)$ be an evolution system on X that satisfies exponential bounds and $G(t, s) \in \mathcal{L}(X)$, $0 \leq s \leq t \in I$. Assume that there exist $\alpha \geq 1$ and $C_G > 0$ such that $\|U(t, s) - G(t, s)\|_X \leq C_G(t-s)^\alpha$ for all $0 \leq s \leq t \in I$. Then there are $\omega' \geq 0$ and $C_{U,G} > 0$ with the following property. Let $n \in \mathbb{N}$, $T_0 \in I$, $\delta := T_0/n$, $y_0 \in X$, and $y_{k+1} = G((k+1)\delta, k\delta)y_k$, for $0 \leq k \leq n$. Then*

$$\|y_n - U(T_0, 0)y_0\| \leq \frac{C_{U,G} e^{\omega' T_0} T_0^\alpha}{n^{\alpha-1}} \|y_0\|.$$

Here, of course, $C_{U,G}$ is independent of n , y_0 , and T_0 . In particular,

Corollary 3.8. *Using the notation of Corollary 3.7, we have that, for any $n \in \mathbb{N}$ and any $T_0 \in I$,*

$$\left\| U(T_0, 0) - \prod_{k=0}^{n-1} G\left(\frac{(k+1)T_0}{n}, \frac{kT_0}{n}\right) \right\| \leq \frac{C_{U,G} e^{\omega' T_0} T_0^\alpha}{n^{\alpha-1}}.$$

See [23, 33, 37, 40, 49, 54] for some more general results on evolution equations that put our results into perspective.

4. Analytic semigroups and Duhamel's formula. In this section, we introduce the class of uniformly strongly elliptic operators that we study and we particularize to them the theory recalled in Section 2. These operators are particularly well suited to be studied via perturbative expansions. In particular, in this section, using the theory of analytic semigroups, we carefully check that all the integrals appearing in Duhamel's formula and in perturbative series expansions are well defined.

4.1. Properties of the class $\mathbb{L}_\gamma$. Since the dimension N is fixed throughout the paper, we will usually write $W^{r,p}$ for $W^{r,p}(\mathbb{R}^N)$. Similarly, we shall often write L^p instead of $L^p(\mathbb{R}^N)$. When $1 < p < \infty$, the dual of $W^{r,p}$ is the Sobolev space $W^{-r,p'}$ with $1/p + 1/p' = 1$.

Definition 4.1. Let $I \subset [0, \infty)$ be an interval containing 0. Let $\mathbb{L}$ be the set of second-order differential operators $L = (L(t))_{t \in I}$ of the form

$$L(t) = \sum_{i,j=1}^N a_{ij}(t, x) \partial_i \partial_j + \sum_{k=1}^N b_k(t, x) \partial_k + c(t, x), \quad (15)$$

where the matrix $[a_{ij}]$ is symmetric and $a_{ij}, b_k, c \in W^{\infty, \infty}(I \times \mathbb{R}^N)$ are *real valued*. Let $\mathbb{L}_\gamma$ be the subset of operators $L \in \mathbb{L}$ satisfying the uniformly strong ellipticity condition (4) with given ellipticity constant γ .

We shall use the symbol calculus for pseudo-differential operators (Ψ DOs for short) to establish several results. We begin by recalling some basic facts about Ψ DOs. (See [34, 38, 51, 52, 56] for the definition and basic properties of pseudodifferential operators.)

We deal only with classical symbols in Hörmander's class $S_{1,0}^m$, $m \in \mathbb{R}$, and denote the symbol of a pseudo-differential operator P by $\sigma(P)$ with $\sigma_0(P)$ its principal symbol. Conversely, given a symbol in $S_{1,0}^m$, we denote the associated pseudo-differential operator with $P = \sigma(x, D)$, $D = \frac{1}{i} \partial$. We recall that any operator with symbol in $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S_{1,0}^m$ is a smoothing operator. We let $\Psi_{1,0}^m$ denote the space of operators with symbols in $S_{1,0}^m$. Every Ψ DO has distributional kernel $\sigma(x, D)(x, y)$ by the Schwartz Kernel Theorem (see e.g. [53]). We will need to deal only with integral operators with smooth kernels.

Notation. If an operator T has smooth kernel, we will denote it by $T(x, y)$.

If $P = \sigma(x, D)$ is smoothing, then there is a one-to-one correspondence between the symbol and the kernel:

$$\sigma(x, D)(x, y) = (\mathcal{F}_2^{-1} \sigma)(x, x - y),$$

where $\mathcal{F}_2$ the Fourier transform in the second variable of a function of two variables. We will also use the standard fact that multiplication with a smoothing symbol is continuous on any symbol class.

We recall that elliptic Ψ DOs in $\Psi_{1,0}^m$, $m \in \mathbb{Z}$, generate equivalent norms in Sobolev spaces [44]. This is a general fact that holds in the greater generality of manifolds with bounded geometry [5, 3, 11, 44]. We shall use this fact for $\mathbb{L}_\gamma \subset \Psi_{1,0}^2$. In particular, we have the following result.

Lemma 4.2. *Suppose $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$, $1 < p < \infty$, and $m \in \mathbb{Z}_+$. Then the following two norms are equivalent*

$$\|u\|_{W^{2m,p}} \sim \|u\|_{L^p} + \|L^m(t)u\|_{L^p},$$

with bounds that are uniform in $t \in I$.

Next we show that if $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$, then $L(t)$ is Hölder continuous in t , and *uniformly* sectorial for $t \in I$ between the Sobolev spaces $W^{2k+2,p}$ and $W^{2k,p}$, $1 < p < \infty$, for each $k \in \mathbb{Z}_+$. These properties in turn give the needed mapping bounds for the evolution system discussed in Subsection 2.1. (See [1, 43, 48] for instance.)

Proposition 4.3. *Recall that $W^{k,p} = W^{k,p}(\mathbb{R}^N)$. Let $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$, $k \in \mathbb{Z}_+$, and $1 < p < \infty$.*

1. *The function $I \ni t \rightarrow L(t) \in \mathcal{L}(W^{k+2,p}, W^{k,p})$ is uniformly Lipschitz continuous.*
2. *For each $t \in I$ and k , the operator $L(t) : W^{2k+2,p} \rightarrow W^{2k,p}$ is sectorial.*
3. *The family $(L(t))_{t \in I}$ is uniformly sectorial.*

Proof. The first part follows from the fact that coefficients of the operators $L(t)$ depend smoothly on time, with bounded derivatives as functions from I to $W^{k,\infty}$, for each k , by the definition of the space $\mathbb{L} \supset \mathbb{L}_\gamma$. We first note that, by definition, $L(t)$ defines a continuous map $W^{2k+2,p} \rightarrow W^{2k,p}$, and that, by Gårding's inequality and the isomorphism $(1 + L(t))^k : W^{s+2k,p} \rightarrow W^{s,p}$, the resolvent set $\rho(L(t))$ of $L(t)$ contains a half plane $\{\lambda \in \mathbb{C}, \operatorname{Re} \lambda \geq \omega\}$. For $p = 2$, this result is standard (see e.g. [1, 43, 48]). For $p \neq 2$, we use here that the spectrum, and hence the resolvent, of a uniformly elliptic operator on L^p is independent of $p \in (1, \infty)$ (see e.g. [36, 46] and references therein).

Next, we fix $t = t_0$ and simply write $L_0 = L(t_0)$. For any $u \in W^{2k,p}$ and $\lambda \in \rho(L_0)$, we have $R(\lambda, L_0)u \in W^{2k,p}$, by the definition of the resolvent set $\rho(L_0)$. Then, using the norm equivalence of Lemma (4.2) twice, the fact that $L(t)$ is sectorial on L^p , and standard properties of the resolvent, we obtain

$$\begin{aligned} \|\lambda R(\lambda, L_0)u\|_{W^{2k,p}} &\leq C(\|\lambda R(\lambda, L_0)u\|_{L^p} + \|\lambda L_0^k R(\lambda, L_0)u\|_{L^p}) \\ &\leq C(\|u\|_{L^p} + \|L_0^k u\|_{L^p}) \leq C\|u\|_{W^{2k,p}}, \end{aligned}$$

with C independent of λ . Proposition 2.3 then imply that $L_0 : W^{2k+2,p} \rightarrow W^{2k,p}$ is sectorial. Lastly, all constants appearing in the proof of sectoriality depend only on bounds of the coefficients in $W^{\infty,\infty}$, which are uniform by hypothesis, hence the operator is uniformly sectorial. $\square$

Recall that, by Theorem 2.4, if $f \equiv 0$ and $L(t)$ is uniformly sectorial, then the IVP (1) has a unique strong and classical solution for all $h \in X$. In particular, if $L \in \mathbb{L}_\gamma$, we have well-posedness in $W^{k,p}$, $k \geq 0$, $1 < p < \infty$ for our IVP, Equation (1). By duality and interpolation, we can obtain mapping properties between fractional Sobolev spaces $W^{s,p}$.

Corollary 4.4. *Suppose $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$. Then L generates an evolution system U in $W^{s,p}$, $s \geq 0$, $1 < p < \infty$, such that, for any fixed $T_0 > 0$, the functions*

$$\|U(t, t')\|_{W^{s,p}, W^{s,p}}, \quad \|L(t)U(t, t')\|_{W^{s+2,p}, W^{s,p}}, \quad (t - t')\|U(t, t')\|_{W^{s,p}, W^{s+2,p}}$$

are uniformly bounded for $0 \leq t' \leq t \in I$, $t \leq t' + T_0$.

From Corollary 4.4, the fact that L is Lipschitz and U is bounded uniformly in time on I as elements of $\mathcal{L}(W^{s+2,p}, W^{s,p})$ implies the following.

Corollary 4.5. *Given $s \geq 0$, $1 < p < \infty$, there exists $C > 0$ such that, for any $0 \leq t' \leq t \in I$, $t \leq t' + 1$,*

$$\|U(t, t') - U(t', t')\|_{W^{s+2,p}, W^{s,p}} \leq C |t - t'|.$$

In particular,

$$[t', \infty) \cap I \ni t \rightarrow U(t, t') \in \mathcal{L}(W^{s+2,p}, W^{s,p})$$

defines a Lipschitz continuous map.

For the applications we have in mind, the initial data h may not be integrable. An example is provided by the payoff function of a European call option. To include such cases, we therefore introduce *exponentially weighted Sobolev spaces*. Given a fixed point $w \in \mathbb{R}^N$, we set

$$\langle x \rangle_w := \langle x - w \rangle = (1 + |x - w|^2)^{1/2}, \quad (16)$$

with $\langle, \rangle$ the Japanese bracket. For notational ease, we denote $\rho_a(x) = e^{a\langle x \rangle_w}$, with w implicit. Then, for $k \in \mathbb{Z}_+$, $a \in \mathbb{R}$, $1 < p < \infty$,

$$W_{a,w}^{k,p}(\mathbb{R}^N) := \{u : \mathbb{R}^N \rightarrow \mathbb{R}, \partial^\alpha(\rho_a u) \in L^p(\mathbb{R}^N) \mid |\alpha| \leq k\}, \quad (17)$$

with norm

$$\|u\|_{W_{a,w}^{k,p}}^p := \|\rho_a u\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \|\partial^\alpha(\rho_a u)\|_{L^p}^p.$$

Weighted fractional spaces $W_{a,w}^{s,p}$, $s \geq 0$, can then be defined by interpolation, and negative spaces by duality $W_{a,w}^{-s,p} = (W_{a,w}^{s,p'})'$, with p' the conjugate exponent to p . The parameter w will be called *the weight center*. Different choices of w give equivalent norms and we also write $W_{a,w}^{s,p} = W_a^{s,p}$, since this vector space does not depend on w .

Recall that $\rho_a(x) := e^{a\langle x \rangle_w}$. We study the operator $L(t)$ on the weighted spaces by conjugation. To this end, we define the operator $L_a(t) := \rho_a L(t) \rho_a^{-1}$ and observe that $L : W_{a,w}^{s,p} \rightarrow W_{a,w}^{s,p}$ if, and only if, $L_a : W^{s,p} \rightarrow W^{s,p}$.

Lemma 4.6. *If $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$ and $a \in \mathbb{R}$, then $\rho_a L \rho_a^{-1} = (L_a(t)) \in \mathbb{L}_\gamma$.*

Proof. We compute $L_a(t) - L(t)$:

$$[L_a(t) - L(t)]u = \rho_a^{-1} \left[\sum 2a_{ij} \partial_i \rho_a \partial_j + \left(\sum_{i,j} \partial_i \partial_j \rho_a + \sum b_i \partial_i \rho_a \right) \right] u,$$

for u regular enough. Since $\langle x \rangle_w$ has bounded derivatives, $L_a(t) - L(t)$ is a first order differential operator the coefficients of which are smooth with all their derivatives uniformly bounded. Hence $L_a(t)$ satisfies the same assumptions as $L(t)$. $\square$

Remark 4.7. By Lemma 4.6, we can then reduce to study the case $a = 0$. Therefore, for instance, $L_a(t) : W_{a,z}^{s+2,p} \rightarrow W_{a,z}^{s,p}$ is well defined and continuous for any a , since this is true for $a = 0$. More generally, the results of Corollary 4.5 and 4.9 apply with $W^{k,p}$ replaced by $W_{a,w}^{s,p}$ for any w and a .

See also [2, 3, 4, 44] for further, related results.

4.2. Analytic semigroups. In the construction of the asymptotic expansion for $U(t, 0)$ in Section 4 below, we will need smoothing properties for the semigroup generated by a certain time-independent operator L_0 related to L . To this end, we recall needed basic facts about analytic semigroups. (We refer again to [1, 43, 48] for a more complete treatment.)

If A is sectorial (and hence, in particular, densely defined), then it generates an analytic semigroup. One of the most important properties of analytic semigroups is the following smoothing properties, which we state only for time-independent operators L_0 in the class $\mathbb{L}_\gamma$ acting on the Sobolev space $W_{a,z}^{k,p}$.

Proposition 4.8. *Let $L_0 \in \mathbb{L}_\gamma$ be time independent. Then, e^{tL_0} is a C_0 -semigroup, and for $0 < t \leq T_0$,*

$$\|e^{tL_0} f\|_{W_{a,z}^{r,p}} \leq C(r, s, T_0) t^{(s-r)/2} \|f\|_{W_{a,z}^{s,p}}, \quad r \geq s,$$

with $C(r, s, T_0)$ independent of t .

A proof for generators of abstract analytic semigroups can be found in [48] [Theorem 6.13, p. 74] for instance. We use it here together with the fact that, as operators on L^p , $\mathcal{D}(L^\alpha) = W^{\alpha,p}$. When applied to the operator sL_0^z , the constant $C(r, s)$ can be chosen uniform in z at least if z belong to a bounded subset of $\mathbb{R}^N$.

An immediate consequence of the above result is the following corollary.

Corollary 4.9. *Let $s, r \in \mathbb{R}$ be arbitrary and $L_0 \in \mathbb{L}_\gamma$ be time independent. Then, the map*

$$(0, \infty) \ni t \rightarrow e^{tL_0} \in \mathcal{L}(W_{a,z}^{s,p}, W_{a,z}^{r,p})$$

is infinitely many times differentiable.

4.3. Duhamel's formula. We assume next that we are given a *time independent* operator $L_0 \in \mathbb{L}_\gamma$ for a fixed $\gamma > 0$ and let $L \in \mathbb{L}_\gamma$. We write

$$L(t) = L_0 + V(t), \tag{18}$$

and study the classical question of relating the evolution system $U(t, s)$ generated by L to the semigroup e^{tL_0} generated by L_0 [1, 43]. Typically, L_0 will be obtained from L by freezing its coefficients at $(0, z)$.

We write the general IVP for L_0 as

$$\begin{cases} \partial_t u(t, x) - L_0 u(t, x) = f(t, x), & \text{in } (0, \infty) \times \mathbb{R}^N \\ u(0, x) = h(x), & \text{on } \{0\} \times \mathbb{R}^N, \end{cases} \tag{19}$$

where h belongs to a suitable function space to be specified each time in what follows depending on the type of solution we seek.

Lemma 4.10. *Let $h \in L^p$, $1 < p < \infty$, and let $0 < T \leq \infty$. If $f \in L^1((0, T), L^p) \cap \mathcal{C}((0, T], L^p)$ and u is the unique strong solution to (19) on $[0, T]$, then u is given by*

$$u(t, x) = e^{tL_0} h + \int_0^t e^{(t-\tau)L_0} f(\tau) d\tau, \quad 0 \leq t \leq T.$$

If f satisfies in addition $f \in C^\alpha((0, T); L^p)$ for some $0 < \alpha$, then (19) has a unique strong solution u .

Proof. This proof is standard (see e.g. [48, Theorem 2.9, p. 107, Corollary 3.3, p. 123]), using the fact that L_0 generates an analytic semigroup. $\square$

We obtain the following consequence.

Corollary 4.11. *Let $u(t)$ be the unique classical solution of the IVP (1) with $f = 0$. Then u solves the Volterra-type equation*

$$u(t) = U(t, 0)h = e^{tL_0}h + \int_0^t e^{(t-\tau)L_0}V(\tau)u(\tau)d\tau$$

where V is given in (18).

Proof. By density, we first assume that $h \in W^{2,p}$, and observe that, formally, the solution the IVP (1) satisfied (19) with the forcing term f replaced by

$$Vu(t, x) = (L(t) - L_0)u(t, x) = u_t(t, x) - L_0U(t, 0)h.$$

Since the solution operator $U(t, 0)$ of the IVP (1) satisfies $U(t, 0) : W^{2,p} \rightarrow W^{2,p}$ as a bounded operator that is strongly continuous for $t \geq 0$ and continuously differentiable for $t > 0$, $L_0U(t, 0)h \in L^p$ has this regularity. But $u_t \in L^p$ share the same regularity, given that u is a classical solution. Therefore, by Lemma 4.10 and the uniqueness of classical solutions, u must agree with the solution of Corollary 4.11. Next, given $h \in L^p$, there exists $h_n \in W^{2,p}$, $h_n \rightarrow h$ in L^p . Let u_n be the strong solution with $u_n(0) = h_n$. Then u_n satisfies

$$u_n(t) = U(t, 0)h_n = e^{tL_0}h_n + \int_0^t e^{(t-\tau)L_0}V(\tau)u_n(\tau)d\tau.$$

We would like to pass to the limit $n \rightarrow \infty$ on the right-hand side of the expression above. In order to do so, we will use the mapping properties of the semigroup e^{tL_0} (Proposition 4.8) and of the evolution system $U(t, 0)$ (Corollary 4.4) to show that the integral is the action of a continuous operator on L^p . Indeed,

$$\begin{aligned} & \left\| \int_0^t e^{(t-\tau)L_0}V(\tau)U(\tau, 0)d\tau \right\|_{L^p} \\ & \leq \int_0^t \|e^{(t-\tau)L_0}\|_{W^{-1,p}, L^p} \|V(\tau)\|_{W^{1,p}, W^{-1,p}} \|U(\tau, 0)\|_{L^p, W^{1,p}} d\tau \\ & \leq \int_0^t \frac{1}{\sqrt{t-\tau}} \frac{1}{\sqrt{\tau}} d\tau < \infty \end{aligned} \tag{20}$$

The proof is complete. $\square$

Remark 4.12. Solutions to the Volterra-type equation of Corollary 4.11 are called *mild* solutions. Under the assumptions of the Lemma, classical and strong solutions of (19) are mild solutions, which are in particular unique. In fact, if f is locally Hölder's continuous in time, mild solutions are classical solutions (19) [48, Theorem 3.2, page 111].

Using this lemma, we can generalize the bounds contained in Corollary 4.4.

Lemma 4.13 (Mapping properties of $U(t, t')$). *Let $U(t, t')$, $0 \leq t' \leq t \in I$, be the evolution system generated by the operator $L \in \mathbb{L}_\gamma$ on $[0, T]$. For any $0 \leq k \leq r$, $a \in \mathbb{R}$, $1 < p < \infty$, $U(t, t') : W_{a,z}^{k,p} \rightarrow W_{a,z}^{r,p}$ if $0 \leq t' < t \leq t' + 1$, there exists $C > 0$ independent of t_1, t_2 such that*

$$\|U(t, t')\|_{W_{a,z}^{k,p}, W_{a,z}^{r,p}} \leq C(t - t')^{(k-r)/2}.$$

Proof. We set $a = 0$ by Lemma 4.6 and, as p is fixed, write $W^k = W^{k,p}$. We temporarily assume that $k \leq r < k + 2$. Using the properties of evolution systems in Definition 2.1, given $h \in W^k$, $v(t) = U(t, t_1)h$ solves:

$$\begin{cases} \partial_t v - L(t)v = 0, & t > t_1, \\ v(t_1) = h. \end{cases}$$

Hence from Corollary 4.11, for any $0 \leq t_2 \leq t_1 \leq 1$ and any $h \in W^k$,

$$U(t_1, t_2)h = e^{(t_1-t_2)L_0} h + \int_0^{t_1-t_2} e^{(t_1-t_2-\tau)L_0} V(\tau) U(t_2 + \tau, t_2)h d\tau.$$

From the triangle inequality, using the mapping properties for U and L_0 in Corollary 4.4 and Proposition 4.8, it follows that

$$\begin{aligned} \|U(t_1, t_2)\|_{W^k, W^r} &\leq \|e^{(t_1-t_2)L_0}\|_{W^k, W^r} \\ &+ \int_0^{\frac{t_1-t_2}{2}} \|e^{(t_1-t_2-\tau)L_0}\|_{W^{k-2}, W^r} \|V(\tau)\|_{W^k, W^{k-2}} \|U(\tau + t_2, t_2)\|_{W^k, W^k} d\tau \\ &+ \int_{\frac{t_1-t_2}{2}}^{t_1-t_2} \|e^{(t_1-t_2-\tau)L_0}\|_{W^k, W^r} \|V(\tau)\|_{W^{k+2}, W^k} \|U(\tau + t_2, t_2)\|_{W^k, W^{k+2}} d\tau \\ &\leq C \left((t_1 - t_2)^{\frac{k-r}{2}} + \int_0^{\frac{t_1-t_2}{2}} (t_1 - t_2 - \tau)^{\frac{k-2-r}{2}} d\tau \right. \\ &\quad \left. + \int_{\frac{t_1-t_2}{2}}^{t_1-t_2} (t_1 - t_2 - \tau)^{\frac{k-r}{2}} \tau^{-1} d\tau \right) \leq C(t_1 - t_2)^{\frac{k-r}{2}}, \end{aligned}$$

exploiting also that $0 < (r - k)/2 < 1$, by hypothesis. This proves the result for $k \leq r < k + 2$. Next, let $r \geq k$, otherwise arbitrary, and choose $m \in \mathbb{Z}_+$ such that $m > \frac{r-k}{2}$. Set $\delta = \frac{r-k}{m}$ and note that $0 \leq \delta < 2$. Then for $j = 1, \dots, m$, we can apply the estimate already obtained by replacing k with $k + (j-1)\delta$ and r with $k + j\delta$ and we apply it on the time interval $(t_1 - (j-1)\frac{t_1-t_2}{m}, t - j\frac{t_1-t_2}{m})$, obtaining

$$\left\| U\left(t_1 - (j-1)\frac{t_1-t_2}{m}, t - j\frac{t_1-t_2}{m}\right) \right\|_{W^{k+(j-1)\delta} \rightarrow W^{k+j\delta}} \leq C \left(\frac{t_1-t_2}{m} \right)^{\frac{k-r}{2m}},$$

for $j = 1, 2, \dots, m$. Therefore,

$$\|U(t_1, t_2)\|_{W^k \rightarrow W^r} \leq C \left(\frac{t_1-t_2}{m} \right)^{m \frac{k-r}{2m}} = C(t_1 - t_2)^{(k-r)/2},$$

where C depends on k, r, p but not on t_1, t_2 . $\square$

In particular, the solution operator $U(t, 0)$ of (1) is smoothing of infinite order on any Sobolev space $W_{a,z}^{k,p}$ with $k \geq 0$ (in fact, by duality, on any Sobolev space) if $t > 0$, as it is the case for e^{tL_0} .

Corollary 4.14. *If $L(t) \in \mathbb{L}_\gamma$ and $U(t, t') = U^L(t, t')$, $0 \leq t' \leq t \in I$ is the associated evolution system, then*

$$(t', +\infty) \ni t \rightarrow U(t, t') \in \mathcal{L}(W_{a,z}^{s,p}, W_{a,z}^{m,p})$$

is infinitely many times differentiable for any $s, m, a \in \mathbb{R}$, $1 < p < \infty$, and $z \in \mathbb{R}^N$.

We omit the proof as it is very similar to that of Corollary 4.9. Another consequence of Lemma 4.13 is that the distributional kernel of the operator U , the *Green's function* or *fundamental solution* of (1), $\mathcal{G}_{t,s}^L \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. In fact, $\mathcal{G}_{t,s}^L$ is given by

$$\mathcal{G}_{t,s}^L(x, y) = \langle \delta_x, U(t, s) \delta_y \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{C}^\infty(\mathbb{R}^N)$ and compactly supported distributions, and where δ_z is the Dirac delta centered at z . One of the goals of this work is to obtain *explicit* approximations of $\mathcal{G}_t^L(x, y)$ with good error bounds.

Remark 4.15. For each $k \in \mathbb{Z}_+$, we let

$$\begin{aligned} \Sigma_k &:= \{ \tau = (\tau_0, \tau_1, \dots, \tau_k) \in \mathbb{R}^{k+1}, \tau_j \geq 0, \sum \tau_j = 1 \} \\ &\simeq \{ \sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{R}^k, 1 \geq \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{k-1} \geq \sigma_k \geq 0 \}, \end{aligned}$$

the *standard unit simplex* of dimension k . The bijection above is given by $\sigma_j = \tau_j + \tau_{j+1} + \dots + \tau_k$. Using this bijection and the notation $d\sigma := d\sigma_k \dots d\sigma_1$, for any continuous, Banach space valued function F on $\mathbb{R}^N$, we define

$$\int_{\Sigma_k} F(\tau) dS_k(\tau) := \int_0^1 \int_0^{\sigma_1} \dots \int_0^{\sigma_{k-1}} F(1 - \sigma_1, \sigma_1 - \sigma_2, \dots, \sigma_{k-1} - \sigma_k, \sigma_k) d\sigma,$$

where dS_k is the infinitesimal measure induced by the projection along a coordinate axis. It is multiple of the measure induced by the Euclidean metric. For instance, $dS_1(\tau) = d\tau$, but in higher dimension the factor is no longer 1.

We begin with a preliminary technical lemma.

Lemma 4.16. *Let $k \in \mathbb{Z}_+$ and $L_j \in \mathbb{L}_\gamma$ and $V_j \in e^{-b_j \langle x \rangle z} \mathbb{L}$, $j = 1, \dots, k$, for some $b = (b_1, \dots, b_k) \in \mathbb{R}_+^k$. Assume that L_j is time independent for $j < k$ and that $E(\tau_k) = e^{\tau_k L_k(0)}$ or $E(\tau_k) = U^{L_k}(\tau_k, 0)$. Then, for all $a, r, s \in \mathbb{R}$ and $1 < p < \infty$,*

$$\Phi(\tau) = e^{\tau_0 L_0} V_1 e^{\tau_1 L_1} \dots e^{\tau_{k-1} L_{k-1}} V_k E(\tau_k), \quad \tau \in \Sigma_k,$$

defines a continuous function $\Phi : \Sigma_k \rightarrow \mathcal{L}(W_{a,z}^{s,p}, W_{a-|b|,z}^{r,p})$.

The assumption that L_j be time independent for $j < k$ is, of course, not necessary, but is the setting in which we will use our result. Also, recall that $W_{a,z}^{s,p} = W_{a,z}^{s,p}(\mathbb{R}^N)$.

Proof. It suffices to prove that Φ is continuous on each of the sets $\mathcal{V}_j := \{\tau_j > 1/(k+2)\}$, $j = 0, \dots, k$, since they cover Σ_k . It also suffices to consider the case $r \geq s$.

Let us prove the continuity on $\mathcal{V}_0$. We define recursively numbers $c_j = c_{j+1} - b_{j+1}$, $c_k = a$, $r_j = r_{j+1} - 4$, $r_k = s$ for $j = 1, \dots, k-1$. By the assumption on the V_j 's and thanks to Proposition 4.8 and Corollary 4.5, each of the functions

$$\begin{aligned} [0, \infty) \ni \tau_j &\rightarrow V_j e^{\tau_j L_j} \in \mathcal{L}(W_{c_j}^{r_j+4,p}, W_{c_j-b_j}^{r_j,p}), \quad 1 \leq j < k, \\ [0, \infty) \ni \tau_k &\rightarrow V_k E(\tau_k) \in \mathcal{L}(W_{c_k}^{r_k+4,p}, W_{c_k-b_k}^{r_k,p}), \end{aligned}$$

is continuous, and hence their composition is continuous as a bounded map $W_a^{s,p} \rightarrow W_{a-|b|}^{s-4k,p}$. Since $e^{\tau_0 L_0}$ is continuous as a bounded operator $W_{a-|b|}^{s-4k,p} \rightarrow W_{a-|b|}^{r,p}$ if $\tau_0 > \frac{1}{k+2}$ thanks to Corollary 4.5, we conclude that the map

$$\mathcal{V}_0 \ni \tau \rightarrow \Psi(\tau) = e^{\tau_0 L_0} V_1 e^{\tau_1 L_1} \dots V_k e^{\tau_k L_k} \in \mathcal{L}(W_a^{s,p}, W_{a-|b|}^{r,p})$$

is continuous. The continuity of the sets $\mathcal{V}_j$ with $j < k$ is completely similar.

For $\tau \in \mathcal{V}_k$, we use instead Proposition 4.8 to show continuity of $e^{\tau_0 L_0}$ in $W_{a-|b|}^{r,p}$ for $\tau_0 \in [0, +\infty)$, and Corollary 4.14 to show continuity of the map

$$(0, \infty) \ni \tau_k \rightarrow E(\tau_k) \in \mathcal{L}(W_{a-|b|}^{s-4k,p}, W_{a-|b|}^{r,p}).$$

This proves the continuity of Φ on $\mathcal{V}_k$ and completes our proof. $\square$

Remark 4.17. In particular, Corollary 4.11 gives the following equality

$$U(t, 0) = e^{tL_0} + \int_0^t e^{(t-\tau)L_0} V(\tau) U(\tau, 0) d\tau,$$

as operators in $\mathcal{L}(W_{a,z}^{s,p}, W_{a,z}^{m,p})$, for all $a, s, m \in \mathbb{R}$ and $1 < p < \infty$.

We can now state the well-known result giving an iterative time-order expansion of the operator $U(1, 0)$. Let $L = L_0 + V$ as in Equation (18) (that is, $L, L_0 \in \mathbb{L}_\gamma$ with L_0 time independent).

Proposition 4.18. *Recall that dS_k is the measure on the simplex Σ_k induced by projection (a multiple of the measure induced by the Euclidean metric). Let $V(t) = L(t) - L_0$ be as in (18) and U^L the evolution system generated by L . For any $d \in \mathbb{Z}_+$, we have the following perturbative expansion*

$$\begin{aligned} U^L(1, 1) &= e^{L_0} + \int_{\Sigma_1} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} dS_1(\tau) + \dots \\ &\quad + \int_{\Sigma_{d-1}} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} \dots e^{\tau_{d-1} L_0} V(\tau_d) e^{\tau_d L_0} dS_{d-1}(\tau) \\ &\quad + \int_{\Sigma_d} e^{\tau_0 L_0} V(\tau_1) e^{\tau_1 L_0} \dots e^{\tau_d L_0} V(\tau_{d+1}) U(\tau_{d+1}, 0) dS_d(\tau), \end{aligned} \quad (21)$$

where each integral is a well-defined Banach-valued Riemann-Stieltjes integral with values in $\mathcal{L}(W_{a,z}^{s,p}, W_{a,z}^{r,p})$.

The positive integer d will be called the *iteration level* of the approximation. Later on, V will be replaced by a Taylor approximation of L , so that V will have polynomial coefficients in x and t .

Proof. We proceed inductively on d . First, we note that each term in the perturbative expansion of $U = U^L$ is well defined by Lemma 4.16.

The perturbative expansion for $d = 1$ is just Volterra's formula of Corollary 4.11 written in terms of operators (see also Remark 4.17). Suppose now that the formula holds for $d - 1$, that is, that

$$\begin{aligned} U(1, 0) &= e^{L_0} + \int_0^1 e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma_1 \\ &\quad + \int_0^1 \int_0^{\sigma_1} e^{(1-\sigma_1)L_0} V(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} V(\sigma_2) e^{\sigma_2 L_0} d\sigma_2 d\sigma_1 + \dots \\ &\quad + \int_0^1 \int_0^{\sigma_1} \dots \int_0^{\sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \dots e^{(\sigma_{d-2}-\sigma_{d-1})L_0} V(\sigma_{d-1}) U(\sigma_{d-1}, 0) \prod_{j=1}^{d-1} d\sigma_{d-j}. \end{aligned}$$

Applying the formula for $d = 1$ to $U(\sigma_{d-1}, 0)$ then gives:

$$U(1, 0) = e^{L_0} + \int_0^1 e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma_1 + \dots$$

$$\begin{aligned}
& + \int_0^1 \int_0^{\sigma_1} \dots \int_0^{\sigma_{d-1}} e^{(1-\sigma_1)L_0} V(\sigma_1) \dots e^{(\sigma_{d-2}-\sigma_{d-1})L_0} V(\sigma_{d-1}) U(\sigma_{d-1}, 0) \prod_{j=1}^{d-1} d\sigma_{d-j} \\
& = e^{L_0} + \int_0^1 e^{(1-\sigma_1)L_0} V(\sigma_1) e^{\sigma_1 L_0} d\sigma_1 + \dots + \int_0^1 \int_0^{\sigma_1} \dots \int_0^{\sigma_{d-1}} \int_0^{\sigma_d} \\
& \quad e^{(1-\sigma_1)L_0} V(\sigma_1) \dots V(\sigma_{d-1}) e^{(\sigma_{d-1}-\sigma_d)L_0} V(\sigma_d) U(\sigma_d, 0) d\sigma_d \prod_{j=1}^{d-1} d\sigma_{d-j}.
\end{aligned}$$

which is the desired perturbative expansion for d . $\square$

By sending $d \rightarrow +\infty$, we formally represent the evolution system as a series of iterated, time-ordered integrals. Such series appear in different contexts and are known as *Dyson series* in the Physics literature.

5. Dilations and Taylor expansion. In this section we employ suitable space-time dilations to reduce the computation of the Green's function $\mathcal{G}_{t,t'}^{L_s}$ to that of a related operator L^s at a fixed time (say 1) where $s = \sqrt{t}$. For given, fixed $s > 0$, we then obtain an expression of the Green's function associated to L^s by Taylor expanding its coefficients as functions of s up to order n and combining such expansion with the time-ordered expansion of Proposition 4.18 up to level d . In particular, the Taylor expansion will provide a natural choice for the operator L_0 and $V(t)$ to which the splitting (18) of L^s applies. We follow here closely [15], which treats the case of time-independent operators. In particular, we use the crucial observation from that paper that, for any second order differential operator with constant coefficients L_0 and any differential operator with polynomial coefficients L_m , we have $e^{L_0} L_m = \tilde{L}_m e^{L_0}$ for some other differential operator with polynomial coefficients $\tilde{L}_m$. (We also extend this result to higher order operators L_0 .) Similar methods, including the time dependent case, were employed in [12, 13, 14, 41, 42, 47].

Throughout this section, we fix an arbitrary dilation center $z \in \mathbb{R}^N$.

5.1. Parabolic rescaling. For any sufficiently regular functions $v(t, x)$ and $f(x)$, we set

$$v^s(t, x) := v(s^2 t, z + s(x - z)), \quad (22a)$$

$$f^s(x) := f(z + s(x - z)). \quad (22b)$$

We therefore interpret s as the dilation factor and z as the dilation center. For any given operator $L(t) \in \mathbb{L}_\gamma$, we similarly define

$$\begin{aligned}
L^s(t) &:= \sum_{i,j=1}^N a_{ij}^s(s^2 t, z + s(x - z)) \partial_i \partial_j + s \sum_{i=1}^N b_i^s(s^2 t, z + s(x - z)) \partial_i \\
&\quad + s^2 c^s(s^2 t, z + s(x - z)).
\end{aligned} \quad (23)$$

It is not difficult to show that, if $u(t, x)$ is a solution of Equation (1), then $u^s(t, x)$ given by (22) is a solution of the following IVP:

$$\begin{cases} \partial_t u^s(t, x) - L^s u^s(t, x) = 0 & \text{in } (0, \infty) \times \mathbb{R}^N \\ u^s(0, x) = g^s(x), & \text{on } \{0\} \times \mathbb{R}^N. \end{cases} \quad (24)$$

Clearly, if $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$, then L^s is an operator in the same class, but with a possibly different I . Since our estimates will be uniform up to a finite

time, we shall assume from now on that $I = [0, T]$, for some fixed $T > 0$, and we shall consider $L^s(t)$ only for $s \in [0, 1]$ and $t \in [0, T] = I$. Based on our earlier discussion $L^s = (L^s(t))_{0 \leq t \leq T}$ generates an evolution system, which we denote by U^{L^s} . The Green's function of the IVP (24) will be denoted instead with $\mathcal{G}_t^{L^s}(x, y)$. The Green's functions for the original and for the dilated problems are simply related via a change of variables.

Lemma 5.1. *Given any $z \in \mathbb{R}^N$ and $s > 0$, we have*

$$\mathcal{G}_{t,t'}^L(x, y) = s^{-N} \mathcal{G}_{s^{-2}t, s^{-2}t'}^{L^s} \left(z + \frac{x-z}{s}, z + \frac{y-z}{s} \right).$$

In particular, when $s = \sqrt{t}$,

$$\mathcal{G}_{t,0}^L(x, y) = t^{-N/2} \mathcal{G}_{1,0}^{L^{\sqrt{t}}} \left(z + \frac{x-z}{\sqrt{t}}, z + \frac{y-z}{\sqrt{t}} \right). \quad (25)$$

By this lemma, it suffices to approximate $\mathcal{G}_{1,0}^{L^s}(x, y)$ and then set $s = \sqrt{t}$.

5.2. Taylor expansion of the operator L^s . We next Taylor expand the coefficients of the operator L^s , given by (23), up to order $n \in \mathbb{Z}_+$, as functions of $s > 0$. The purpose of this Taylor expansion is to replace the operator V in (18) with operators having polynomial coefficients for which the time-ordered integrals appearing in Proposition 4.18 can be explicitly computed as in [15].

We obtain the representation

$$L^s = L_0 + \sum_{m=1}^n s^m L_m + s^{n+1} L_{n+1}^{s,z}, \quad (26)$$

where

$$L_m = \frac{1}{m!} \left(\frac{d^m}{ds^m} L^s \right) \Big|_{s=0}, \quad 0 \leq m \leq n, \quad (27)$$

and $L_{n+1}^{s,z}$ comes from the remainder of the Taylor expansion. (Recall that z is the dilation center.) For m , $L_m = (L_m(t))_{0 \leq t \leq T}$ is a family of differential operators with coefficients that are polynomials in $(x-z)$ and t , but are independent of s . That is, for $m \leq n$,

$$L_m(t) = \sum_{ijk\alpha} a_{ijk\alpha}^{[\mu]} (x-z)^{\alpha} t^k \partial_i \partial_j + \sum_{ik\alpha} b_{ik\alpha}^{[\mu]} (x-z)^{\alpha} t^k \partial_i + \sum_{k\alpha} c_{k\alpha}^{[\mu]} (x-z)^{\alpha} t^k, \quad (28)$$

where $i, j = 1, \dots, d$, $0 \leq |\alpha| + 2k \leq m$, with the coefficients $a^{[\mu]} = a^{[\mu]}(z)$, $b^{[\mu]} = b^{[\mu]}(z)$, $c^{[\mu]} = c^{[\mu]}(z) \in \mathbb{R}$ obtained from the partial derivatives of the coefficients of L at $(t, x) = (0, z)$. However, $L_{n+1}^{s,z}$ does depend on s as well. Generally, we shall drop the dependence on z from the notation from now on.

We will obtain a perturbative expansion of the form appearing in Proposition 4.18 for $U^{L^s}(1, 0)$ with each $V(\tau_j)$ replaced by the operator L_j introduced above. The following easy corollary will justify that each term of this perturbative expansion is well defined. We record it for further use. We notice that $L_0(t)$ is independent of t , so we shall write simply L_0 . As in Remark 4.15, given $\tau = (\tau_1, \dots, \tau_k) \in \Sigma_k$, we let $\sigma_j := \tau_j + \tau_{j+1} + \dots + \tau_k$ for $j = 1, \dots, k$.

Corollary 5.2. *Let $L = (L(t))_{t \in I} \in \mathbb{L}_\gamma$, let $k \in \mathbb{Z}_+$, and let L_m , $0 \leq m \leq n+1$, be from the Taylor expansion of L , Equation (26). For $\tau \in \Sigma_k$, let us set*

$$\Phi(\tau) := e^{\tau_0 L_0} L_{j_1}(\sigma_1) e^{\tau_1 L_0} L_{j_2}(\sigma_2) \dots L_{j_{k-1}}(\sigma_{k-1}) e^{\tau_{k-1} L_0} L_{j_k}(\sigma_k) E(\tau_k),$$

with $0 \leq j_i \leq n+1$ and either $E(\tau_k) = e^{\tau_k L_0}$ or $E(\tau_k) = U^{L^s}(\tau_k, 0)$. Then, for any $\epsilon > 0$, $a, r, s \in \mathbb{R}$, and $1 < p < \infty$, $\Phi : \Sigma_k \rightarrow \mathcal{L}(W_{a,z}^{s,p}(\mathbb{R}^N), W_{a-\epsilon}^{r,p}(\mathbb{R}^N))$ is continuous

Proof. This follows from Lemma 4.16 with $b = (b_1, \dots, b_k) \in \mathbb{R}_+^k$, $b_j = \epsilon/k$. $\square$

5.3. Asymptotic expansion of the evolution system. In this section, we define an approximation $\tilde{\mathcal{G}}_{t,s}^{[\mu]}$ of the evolution system $U(t, s)$ satisfying the conditions of Theorem 3.6.

Definition 5.3 (Spaces of Differentials). Given non-negative integers a, b , we denote by $\mathcal{D}(a, b)$ the vector space of all differential operators of order at most b with coefficients that are polynomials in x and t of degree at most a . We extend this definition to negative indices by defining $\mathcal{D}(a, b) = \{0\}$ if either a or b is negative. By the degree of an operator $A \in \mathcal{D}(a, b)$, we mean the highest power of the polynomials appearing as coefficients of A .

Definition 5.4 (Adjoint Representation). For any two operators $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$ we define $\text{ad}_{A_1}(A_2)$ by

$$\text{ad}_{A_1}(A_2) := [A_1, A_2] = A_1 A_2 - A_2 A_1 = -[A_2, A_1],$$

and, for any integer $j \geq 1$, we define $\text{ad}_{A_1}^j(A_2)$ recursively by

$$\text{ad}_{A_1}^j(A_2) := \text{ad}_{A_1}(\text{ad}_{A_1}^{j-1}(A_2)), \quad \text{ad}_{A_1}^0(A_2) := A_2.$$

Above, the iterated commutators are well defined if we take the space $C_c^\infty(\mathbb{R}^N)$ as common domain $\mathcal{D}$ of A_1 and A_2 , for instance.

Lemma 5.5. Suppose $A_1 \in \mathcal{D}(a_1, b_1)$ and $A_2 \in \mathcal{D}(a_2, b_2)$. Then for any integer $k \geq 1$,

$$\text{ad}_{A_1}^k(A_2) \in \mathcal{D}(k(a_1 - 1) + a_2, k(b_1 - 1) + b_2).$$

Proof. A direct computation using the properties of the class $\mathcal{D}(a, b)$ and the definition of the commutator gives $\text{ad}_{A_1}(A_2) \in \mathcal{D}(a_1 - 1 + a_2, b_1 - 1 + b_2)$. The result then follows by iterating k times this relation. $\square$

As in [13, 15], we obtain the following consequence of this lemma.

Proposition 5.6. Let $Q \in \mathcal{D}(0, n)$ and $Q_m \in \mathcal{D}(m, m')$. We have the following:

1. $\text{ad}_Q^{m+1}(Q_m) = 0$;
2. the following sum is finite

$$\exp(\text{ad}_Q)(Q_m) := \sum_{j \geq 0} (j!)^{-1} \text{ad}_Q^j(Q_m);$$

3. $\exp(\text{ad}_Q)(P_1 P_2) = \exp(\text{ad}_Q)(P_1) \exp(\text{ad}_Q)(P_2)$ for all P_1, P_2 in the algebra $\mathcal{D} := \cup_{n,n'} \mathcal{D}(n, n')$;
4. assume that Q generates a C_0 -semigroup e^{tQ} on $L^2(\mathbb{R})$, $t \geq 0$, then

$$e^Q Q_m = \exp(\text{ad}_Q)(Q_m) e^Q.$$

Proof. The first relation follows from $\text{ad}_Q^k(Q_m) \in \mathcal{D}(m - k, m' + k(n - 1))$ and the fact that the later space is 0 when $k > m$. This then gives immediately that $\exp(\text{ad}_Q)$ is defined. The third relation follows from the fact that ad_Q is a derivation of $\mathcal{D}$ and the exponential of a derivation (when defined) is an algebra isomorphism (see e.g. [35]). Finally, to prove the last relation, let us consider the function

$F(t) := e^{tQ}Q_m - \exp(\text{ad}_Q)(Q_m)e^{tQ}$. It is a continuous function with values in $\mathcal{L}(\rho_w^{-a}L^2(\mathbb{R}^N), \rho_w^aL^2(\mathbb{R}^N))$ for a large ($a \geq m' + (m+1)(n-1)$). Then $F(0) = 0$ and $F'(t) = \text{ad}_Q(F(t))$. Hence $F(t) = 0$ for all $t > 0$. $\square$

A consequence of our discussions is that we obtain an automorphism $\phi_\theta : \mathcal{D} \rightarrow \mathcal{D}$ of the algebra $\mathcal{D} := \cup_{n,n'} \mathcal{D}(n, n')$, given by the formula $\phi_\theta(Q)e^{\theta L_0} = e^{\theta L_0}Q$. See also [12, 14, 41, 42, 47].

Lemma 5.7. *Let m be a fixed positive integer and L_m , $0 \leq m \leq n$, be defined as in (27), then for any $\theta \in \mathbb{R}$,*

$$e^{(1-\theta)L_0}L_m(\theta) = P_m(\theta, x-z, \partial)e^{(1-\theta)L_0}, \quad (29)$$

where $P_m(\theta, x-z, \partial) := \phi_{1-\theta}(L_m(\theta))$ is a differential operator with coefficients polynomials in θ and $(x-z)$. (There is no t , since we specialized at $t = \theta$ in the formula for L_m .)

Next, we rewrite the perturbative expansion of Proposition 4.18 in a more computable and explicit form. We recall that d is the level of the iteration in the Dyson series and n is the order of the Taylor expansion of L^s . In principle, d and n are unrelated, but we will find it convenient later on to choose $d = n$.

For ease of notation, we shall sometimes write $L_{n+1}^{s,z} = L_{n+1}$, even though this operator does depend on s and z . Inserting (26) into the perturbative expansion of Proposition 4.18 and collecting iterated integrals in the same number of variables, we have:

$$\begin{aligned} U^{L^s}(1) &= e^{L_0} + \sum_{k=1}^d \sum_{\substack{i=1,\dots,k \\ 1 \leq \alpha_i \leq n+1}} s^{\alpha_1+\dots+\alpha_k} \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \\ &\quad \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma + \sum_{\substack{i=1,\dots,d+1 \\ 1 \leq \alpha_i \leq n+1}} s^{\alpha_1+\dots+\alpha_{d+1}} \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} \\ &\quad \cdot L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U(\sigma_{d+1}, 0) d\sigma, \end{aligned} \quad (30)$$

where, for notational ease, we have set $d\sigma = d\sigma_\ell \dots d\sigma_1$ and where, in each integral term above, ℓ varies from 1 to $d+1$.

To simplify the above expression, we now introduce some helpful combinatorial notation to keep track of the indexes

Definition 5.8. For any integers $1 \leq k \leq d+1$ and $1 \leq \ell \leq (n+1)(d+1)$, we denote by $\mathfrak{A}_{k,\ell}$ the set of multi-indexes $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \{0, 1, \dots, n+1\}^k$, such that $|\alpha| := \sum \alpha_j = \ell$.

Clearly, since $\alpha_i \geq 1$, the set $\mathfrak{A}_{k,\ell}$ is empty if $\ell < k$. If $\alpha \in \mathfrak{A}_{k,\ell}$, then ℓ represents the order in powers of s of the corresponding term in (30), while k represents the level of iteration in the time-ordered expansion. For each $\alpha \in \mathfrak{A}_{k,\ell}$, we then set

$$\Lambda_\alpha = \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma, \quad (31a)$$

if $k < d+1$, and

$$\Lambda_\alpha = \int_{\Sigma_{d+1}} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \dots e^{(\sigma_d-\sigma_{d+1})L_0} L_{\alpha_{d+1}}(\sigma_{d+1}) U^{L^s}(\sigma_{d+1}, 0) d\sigma, \quad (31b)$$

if $k = d + 1$, respectively, using the notation $d\sigma$ of Equation (30). (We recall that we suppress the explicit dependence of L_k on s and z , if $k = d + 1$.) Also, since we keep the dilation center z fixed for the time being, we also suppress the explicit dependence on z .

A simple but useful result about Λ_α is the following lemma, which we record for later use.

Lemma 5.9. *Recall the differential operators P_k of Lemma 5.7. For any given multi-index $\alpha \in \mathfrak{A}_{k,\ell}$ with $k \leq d$ and $1 \leq \alpha_i \leq n$, $i = 1, \dots, k$,*

$$\Lambda_\alpha = \mathcal{P}_\alpha(x - z, \partial)e^{L_0}$$

where

$$\mathcal{P}_\alpha(y, \partial) = \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, y, \partial) P_{\alpha_2}(\sigma_2, y, \partial) \cdots P_{\alpha_k}(\sigma_k, y, \partial) d\sigma$$

is a differential operator with coefficients polynomials in y (in particular, it is independent of t or s).

Proof. Applying Lemma 5.7 repeatedly gives

$$\begin{aligned} \Lambda_\alpha &= \int_{\Sigma_k} e^{(1-\sigma_1)L_0} L_{\alpha_1}(\sigma_1) e^{(\sigma_1-\sigma_2)L_0} \cdots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\ &= \int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x - z, \partial) e^{(1-\sigma_2)L_0} \cdots e^{(\sigma_{k-1}-\sigma_k)L_0} L_{\alpha_k}(\sigma_k) e^{\sigma_k L_0} d\sigma \\ &\quad \vdots \\ &= \left(\int_{\Sigma_k} P_{\alpha_1}(\sigma_1, x - z, \partial) P_{\alpha_2}(\sigma_2, x - z, \partial) \cdots P_{\alpha_k}(\sigma_k, x - z, \partial) d\sigma \right) e^{L_0}. \end{aligned}$$

This completes the proof. $\square$

To further simplify some of the formulas, we define

$$\Lambda^\ell := \sum_{k=1}^{\min(\ell, d+1)} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \Lambda_\alpha, \quad \ell \geq 1. \quad (32)$$

For convenience, we let $\Lambda^0 = e^{L_0}$.

We combine the results obtained so far in this section in the following representation theorem. We will perform an error analysis in the Sobolev spaces $W_{a,z}^{k,p}$ in Section 6.

Lemma 5.10 (Definition of the local approximation). *Let d be the iteration level in the time-ordered expansion the perturbative expansion of Proposition 4.18, let n be the order of the Taylor expansion (26) of L^s , as before, and let $m \in \mathbb{Z}_+$. Let*

$$E_{m,d,n}^s = \sum_{\ell=m+1}^{\infty} s^{\ell-m-1} \Lambda^\ell.$$

(The sum is actually finite.) Then

$$U^{L^s}(1, 0) = e^{L_0} + \sum_{\ell=1}^m s^\ell \Lambda^\ell + s^{m+1} E_{m,d,n}^s.$$

Assume that $\ell \leq \min\{d, n\}$. Then Λ^ℓ does not depend on d , n , or s , and, consequently, $E_{m,d,n}^s$ also does not depend on d and n , if $m \leq \min\{d, n\}$ as well.

Proof. This follows from the fact that, if $\alpha \in \mathfrak{A}_{k,\ell}$, then $k \leq \ell := \alpha_1 + \alpha_2 + \dots + \alpha_k$, since all $\alpha_i \geq 1$. $\square$

Consequently, when $m \leq \min\{d, n\}$, we shall write $E_m^s = E_{m,d,n}^{s,z}$, since $E_{m,d,n}^{s,z}$ does not depend on d and n and z is fixed.

Remark 5.11. The idea pursued here (following [15]) relies on the following three analysis points

- $U^{L^s}(t, t')$ depends smoothly on $s \in [0, 1]$;
- we can explicitly identify $U^{L^0}(t, t') = e^{(t-t')L_0}$;
- the sum $e^{L_0} + \sum_{\ell=1}^m s^\ell \Lambda^\ell$ is the Taylor polynomial of (the distribution kernel of) $U^{L^s}(1, 0)$ at $s = 0$.

Note that L_0 is obtained from the operator L by freezing its coefficients at $(0, z)$ ($t = 0$ in time and z in space). We can thus try to approximate $U^{L^s}(1, 0)$ with its Taylor polynomial. In turn, after rescaling back, this approximation will yield an approximation of $U^L(s^2, 0)$, that is, for short time. Note that $U^{L^s}(1, 0)$ does not exhibit any singularities at $s = 0$, but rescaling back introduces a strong singularity at $s = 0$ in $U^L(s^2, 0)$, however, repeating ourselves, that singularity is entirely due to the rescaling. The next section will make this construction explicit to define the approximate Green's function of $U^L(t, s)$ for $t - s > 0$ small.

6. The approximate Green's function and error analysis. In this section we introduce our approximate Green's function, we prove Theorem 1.1, and we complete our error analysis. Our error estimates are using the norm of linear maps between weighted Sobolev spaces. A different kind of estimate (pointwise in (x, y)) was obtained in [47].

6.1. Definition of the approximate Green's function. We are now ready to introduce our approximation of the Green's function

$$\mathcal{G}_{t,s}^L(x, y) := U^L(t, s)(x, y)$$

of the operator $U^L(t, s)$ following the idea outlined in Remark 5.11. Since the problem is translation invariant, we may assume $s = 0$. Soon, we will replace z (which was fixed in the previous section) with a function of x and y . We first introduce the conditions that such a function must satisfy.

Definition 6.1. A smooth function $z : \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$ will be called *admissible* if $z(x, x) = x$, for all $x \in \mathbb{R}^N$ and all partial derivatives (of positive orders) of z are bounded.

A typical example is $z(x, y) = \lambda x + (1 - \lambda)y$, for some fixed parameter λ . A simple application of the mean value theorem gives that $\langle z - x \rangle \leq C \langle y - x \rangle$ for some $C > 0$. From the point of view of application, $z(x, y) = x$ will give us the simplest formula to approximate the Green's function. However, as discussed in [13], other more suitable choices are possible, for instance, $z(x, y) = (x + y)/2$ seems to be often better. In what follows, we fix an admissible $z = z(x, y)$. We now fix for the rest of the paper an admissible function $z : \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$. It will be the dilation center used to approximate the Green's functions at (x, y) .

Assume we want an approximation of order m (that is, up to $s^m = t^{m/2}$). We shall use the formulas and the results of Lemma 5.10. We shall choose then in that Lemma $n, d \geq m$, so that the terms Λ^ℓ are independent of s (and t) and $E_{m,d,n}^s$

is independent of d and n , so we can write $E_{m,d,n}^s = E_m^s$ for the “error term.” Motivated by Lemmata 5.1 and 5.10, we now introduce the following.

Definition 6.2. We assume $\mu \leq \min\{d, n\}$ and let the *order μ approximation* $\tilde{\mathcal{G}}_{t,0}^{[\mu]}(x, y)$ of the Green’s function $\mathcal{G}_{t,0}^L(x, y)$ of $U^L(t, 0)$ be

$$\tilde{\mathcal{G}}_{t,0}^{[\mu]}(x, y) := \sum_{\ell=0}^{\mu} t^{(\ell-N)/2} \Lambda^{\ell} \left(z + \frac{x-z}{\sqrt{t}}, z + \frac{y-z}{\sqrt{t}} \right),$$

where z is some admissible function. By replacing L with a translation of size t' in time, we define similarly the approximation $\tilde{\mathcal{G}}_{t,t'}^{[\mu]}(x, y)$ using the μ -approximate kernel at $(t - t', 0)$ for this translated operator.

For this definition, it suffices to choose $n = d = m$, but for the proof of our error estimates, the freedom to choose much larger n and d will be useful. This will be especially the case when dealing with the error term:

$$\tilde{E}_{t,0}^{[\mu]}(x, y) := t^{-(\mu+1)/2} \left[\mathcal{G}_{t,0}^L(x, y) - \tilde{\mathcal{G}}_{t,0}^{[\mu]}(x, y) \right] \quad (33)$$

of Theorem 1.1.

6.2. Convergence analysis. In this section, we show that our approximate Green’s function $\tilde{\mathcal{G}}_{t,t'}^{[\mu]}(x, y)$ satisfies the assumptions of Theorem 3.6. We shall primarily use pseudo-differential techniques. For all relevant properties of pseudo-differential operators, we refer to [51, 55]. For the moment, we continue to keep z , the dilation center, fixed.

We start by analyzing in more detail the properties of the operators L_m in expansion (27). We recall that $\langle x \rangle_z = \langle x - z \rangle$. We also recall that L_m , $0 \leq m \leq n+1$, are second-order differential operator with polynomial coefficients, independent of the dilation factor s . Moreover, L_m has coefficients of degree at most m in $x - z$. An immediate consequence of this fact is recorded in the following lemma.

Lemma 6.3. *Let L_j , $0 \leq j \leq n+1$ be the operators of Equation (26), with $L_{n+1} = L_{n+1}^{s,z}$ depending also on s (all these operators depend on z). The family*

$$\{\langle x \rangle_z^{-j} L_j, \langle x \rangle_z^{-n-1} L_{n+1}^{s,z}; s \in (0, 1], z \in \mathbb{R}^N, j = 0, \dots, n+1\}$$

defines a bounded subset of $\mathbb{L}_{\gamma}$.

We recall that, for convenience, we have dropped the dependence on z in the notation of the polynomials L_m . The next Lemma allows to change the center of the dilation z . This change is needed when z is replaced by a function $z = z(x, y)$. It also allows to reduce to the case $a = 0$ to establish bounds in $W_{a,z}^{k,s}$, as long as a belongs to a bounded set.

Lemma 6.4. *For each given $\epsilon > 0$, the family*

$$\{e^{-\epsilon \langle z \rangle w} e^{-\epsilon \langle x \rangle w} L_j, s \in (0, 1], z, w \in \mathbb{R}^N, j = 0, \dots, n+1\}$$

is a bounded subset of $\mathbb{L}_{\gamma}$.

Proof. The triangle inequality shows that

$$\langle x - z \rangle - \langle x - w \rangle \leq \langle w - z \rangle.$$

Therefore $e^{\epsilon(\langle x-z \rangle - \langle x-w \rangle - \langle w-z \rangle)} \leq 1$, and hence the family

$$e^{\epsilon(\langle x-z \rangle - \langle x-w \rangle - \langle w-z \rangle)} e^{-\epsilon \langle x \rangle z} L_j^z = e^{-\epsilon \langle z-w \rangle} e^{-\epsilon \langle x \rangle w} L_j^z$$

is bounded in $W^{\infty, \infty}$ for $s \in (0, 1]$ and $j = 0, 1, 2, \dots, n+1$ as claimed, by Lemma 6.3. $\square$

Lemma 4.16 and Lemma 6.4 yield the following result.

Corollary 6.5. *For any $\alpha \in \mathfrak{A}_{k, \ell}$, $z, w \in \mathbb{R}^N$, $r, s \in \mathbb{R}$, $1 < p < \infty$, and $\epsilon > 0$,*

$$\Lambda_\alpha := \int_{\Sigma_k} e^{\tau_0 L_0} L_{\alpha_1}(\tau_1) e^{\tau_1 L_0} \dots e^{\tau_{k-1} L_0} L_{\alpha_k}(\tau_k) e^{\tau_k L_0} dS_k(\tau), \quad k \leq d$$

and

$$\Lambda_\alpha := \int_{\Sigma_{d+1}} e^{\tau_0 L_0} L_{\alpha_1}(\tau_1) e^{\tau_1 L_0} \dots e^{\tau_d L_0} L_{\alpha_{d+1}}(\tau_{d+1}) U(\tau_{d+1}, 0) dS_{d+1}(\tau)$$

are bounded linear operators from $W_{a, w}^{s, p}$ to $W_{a - \epsilon, w}^{r, p}$. Moreover, we have that

$$\|\Lambda_\alpha\|_{W_{a, w}^{s, p}, W_{a - \epsilon, w}^{r, p}} \leq C_{s, r, p, a, \epsilon} e^{k\epsilon \langle z - w \rangle},$$

for a bound $C_{s, r, p, a, \epsilon}$ that does not depend on z or w . In particular, each Λ_α is an operator with smooth kernel $\Lambda_\alpha(x, y)$.

In order to treat the resulting kernels and the resulting remainder term, Corollary 6.5 is not sufficient and we need refined estimates. We address first the terms comprising $\tilde{\mathcal{G}}_{t, 0}^{[\mu]}$ of the expansion introduced in Definition 6.2 via pseudo-differential calculus and treat the terms in the remainder next via direct kernel estimates.

6.3. Bounds on $\tilde{\mathcal{G}}_{t, 0}^{[\mu]}$. We bound each operator Λ_α appearing in Definition 6.2 separately, where Λ_α is defined in (31). To this end, we define the operator

$$\mathcal{L}_{s, \alpha} f(x) = s^{-N} \int_{\mathbb{R}^N} \Lambda_\alpha(z + s^{-1}(x - z), z + s^{-1}(y - z)) f(y) dy, \quad (34)$$

We show below that, for an admissible function z , and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathfrak{A}_{k, \ell}$, $k \leq n$, $\alpha_i \leq n$, the operator $\mathcal{L}_{s, \alpha}$ is a pseudo-differential operator with a good symbol. We shall then use symbol calculus to derive the desired operator estimates. By Lemma 4.6, it is enough to assume $a = 0$ in $W_{a, w}^{s, p}$ (in which case, w becomes unnecessary).

Since we keep z fixed, the operator L_0 is constant coefficient and its Green's function G can be computed explicitly as:

$$G(x, y) = \frac{e^{c^0 t}}{\sqrt{(4\pi t)^n \det(A^0)}} e^{\frac{(x + b^0 t - y)^t (A^0)^{-1} (x + b^0 t - y)}{4t}},$$

where A^0 is the matrix with entries $a_{ij}(z)$. A direct computation gives the following lemma, using the explicit form of the kernel G of e^{L_0} .

Lemma 6.6. *Fix $z \in \mathbb{R}^N$. Consider the operator $T = (x - z)^\beta \partial_x^\gamma e^{L_0}$, where β and γ are multi-indices. Then the distributional kernel of T is given by*

$$T(x, y) = (x - z)^\beta (\partial_x^\gamma G)(x - y).$$

The next theorem characterizes the symbol of $\mathcal{L}_{s, \alpha}$ belonging to the principal term of the expansion.

Theorem 6.7. *Let $\alpha \in \mathfrak{A}_{k, \ell}$, $k \leq n$, $\alpha_i \leq n$. Let $z : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be an admissible function. Then there exists a uniformly bounded family $\{\varrho_s\}_{s \in (0, 1]}$ in $S^{-\infty}(\mathbb{R}^N \times \mathbb{R}^N)$ such that*

$$\mathcal{L}_{s, \alpha} = \sigma_s(x, D) := \varrho_s(x, sD), \quad \sigma_s(x, \xi) = \varrho_s(x, s\xi).$$

Proof. By Lemma 5.9, Λ_α is a finite sum of terms of the form $a(z)(x-z)^\beta \partial_x^\gamma e^{L_0^z}$ with $a \in W^{\infty, \infty}$. We recall that a is smooth with bounded derivatives of all orders. Let $k_z(x, y)$ be the distribution kernel of $a(z)(x-z)^\beta \partial_x^\gamma e^{L_0^z}$ and set

$$K_s(x, y) := s^{-N} k_z(z + s^{-1}(x - z), z + s^{-1}(y - z)), \quad z = z(x, y).$$

By abuse of notation, we shall denote also by K_s the integral operator with kernel K_s . It is enough to show that there exists a uniformly bounded family $\{\varrho_s\}_{s \in (0, 1]}$ in $S^{-\infty}$ such that

$$K_s = \varrho_s(x, sD).$$

A direct calculation shows that

$$K_s(x, y) = a(z) s^{-|\beta| - N} (x - z)^\beta \zeta(z, s^{-1}(x - y)), \quad z = z(x, y),$$

with $\zeta(z, x)$ the kernel of $\partial_x^\gamma e^{L_0^z}$. Then the symbol of K_s , $\sigma_s(x, \xi)$ is given by

$$\sigma_s(x, \xi) = \int_{\mathbb{R}^N} e^{-iy \cdot \xi} a(z) s^{-|\beta| - N} (x - z)^\beta \zeta(z, s^{-1}y) dy, \quad z = z(x, x - y).$$

If we denote

$$\varrho_s(x, \xi) = \int_{\mathbb{R}^N} e^{-iy \cdot \xi} a(z) s^{-|\beta|} (x - z)^\beta \zeta(z, y) dy, \quad z = z(x, x - sy),$$

we have $\sigma_s(x, \xi) = \varrho_s(x, s\xi)$. We show next that ϱ_s is a bounded family in $S^{-\infty}$. This follows from the continuity of multiplication with smoothing symbols, given that $a(z) \in S_{(1,0)}^0$ and $s^{-1}(x_j - z_j(x, x - sy)) \in S_{(1,0)}^0$ and they form bounded families for $s \in [0, 1]$. $\square$

A simple change of variables and the definition of the symbol class $S_{1,0}^m$ gives the lemma below.

Lemma 6.8. *Let $\varrho(x, \xi)$ be a symbol in $S^{-\infty}$, then $s^k \varrho(x, s\xi)$ is a symbol in $S_{1,0}^{-k}$ uniformly bounded in $(0, 1]$ with respect to s .*

The symbol calculus gives mapping properties on Sobolev spaces by standard results.

Theorem 6.9. *In the hypotheses of Theorem 6.7, for any $1 < p < \infty$, any $r \in \mathbb{R}$,*

$$s^k \|\mathcal{L}_{s,\alpha}\|_{W^{r,p}, W^{r+k,p}} \leq C_{k,r,p}, \quad (35)$$

for $C_{k,r,p}$ independent of s . The same estimate is valid for the integral operators with kernels $\tilde{\mathcal{G}}_{s^2,0}^{[\mu]}(x, y)$ (see Definition 6.2).

Proof. The first part follows from the definition of $\mathcal{L}_{s,\alpha}$, Theorem 6.7, and Lemma 6.8. The last part follows from the definition of $\tilde{\mathcal{G}}^{[\mu]}$ in terms of the operators $\mathcal{L}_{s,\alpha}$, Equation (32) and Definition 6.2. $\square$

By Definition 6.2, the above theorem translates into a corresponding bound on the principal part $\tilde{\mathcal{G}}_{t,0}^{[\mu]}$ of the asymptotic expansion for the Green's function.

Corollary 6.10. *Let $\tilde{\mathcal{G}}_{t,0}^{[\mu]} f(x) := \int_{\mathbb{R}^N} \tilde{\mathcal{G}}_{t,0}^{[\mu]}(x, y) f(y) dy$ (that is, the operator with kernel $\tilde{\mathcal{G}}_{t,0}^{[\mu]}(x, y)$). Let $T > 0$, $1 < p < \infty$, and $r \in \mathbb{R}$, then the family of operators $\tilde{\mathcal{G}}_{t,0}^{[\mu]}$ is uniformly bounded in $W^{r,p}$ for $t \in (0, T]$.*

6.4. Bounds on $\tilde{E}_{t,0}^{[\mu]}$. In this subsection, we study the error term $\tilde{E}_{t,0}^{[m]}$ in (33), which is a particular case of the error term $\tilde{E}_{t,t'}^{[m]}$ in Theorem 1.1. The case $t' \neq 0$ is obtained from the case $t' = 0$ by a translation in time. For our error estimate, we recall that, if d and n are large enough, both $\tilde{\mathcal{G}}_{t,0}^{[\mu]}(x, y)$ and $\tilde{E}_{t,0}^{[\mu]}$ are independent of d and n . Next, we replace μ with $M \geq \mu + r - 1$ in Definition 6.2, with $r > 0$ to be chosen. Then we increase d and n accordingly to satisfy $d, n \geq M$, remembering that $\tilde{E}_{t,0}^{[\mu]}(x, y)$ does not depend on d and n as long as $d, n \geq \mu$. We can decompose $\tilde{E}_{t,0}^{[\mu]}(x, y)$ as follows:

$$\begin{aligned} \tilde{E}_{t,0}^{[\mu]}(x, y) &= \sum_{\ell=\mu+1}^M t^{(\ell-N-\mu-1)/2} \Lambda^\ell(z + t^{-1/2}(x-z), z + t^{-1/2}(y-z)) \\ &\quad + t^{(M-\mu-N)/2} \tilde{E}_{t,0}^{[M]}(x, y). \end{aligned} \quad (36)$$

The first $M - \mu - 1$ terms in this expressions are pseudo-differential operators of the type discussed in Subsection 6.3. The last term contains operators Λ_α with either $\alpha \in \mathfrak{A}_{n+1,M}$ or for some $\alpha_i = n + 1$. In this range, we generally do not know whether Λ_α is a pseudo-differential operator or not. Instead of symbol calculus, it will be enough to apply a well-known result, sometimes referred to as Riesz's Lemma, which we recall for the reader's sake (see for example [53, Proposition 5.1, page 573]).

Lemma 6.11. *Assume K is an integral operator with kernel $k(x, y)$ on a measure space (X, μ) . If for all y and for all x , respectively,*

$$\int_X |k(x, y)| d\mu(x) \leq C_1, \quad \int_X |k(x, y)| d\mu(y) \leq C_2 \quad (37)$$

then K is a bounded operator on $L^p(X, \mu)$, $p \in [1, \infty]$. Moreover,

$$\|K\| \leq C_1^{1/p} C_2^{1/q}, \quad 1/p + 1/q = 1.$$

By Lemma 4.6, in the proofs following results, we need only consider the case $a = 0$ in $W_{a,w}^{s,p} = W_{a,w}^{s,p}(\mathbb{R}^N)$. In that case, the norms on our spaces become independent of w as well. The following result is similar to Theorem 6.9, but it has no constraints on the indices.

Lemma 6.12. *Let $z : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be admissible and let $1 < p < \infty$. Then, for any α and any $k \geq 0$, there exists $C_{k,p,\alpha} > 0$ such that*

$$s^k \|\mathcal{L}_{s,\alpha}\|_{W^{0,p}, W^{k,p}} \leq C_{k,p,\alpha}. \quad (38)$$

Proof. First, we assume that k is a positive integer. By Riesz's Lemma it suffices to show that, for any multi-index γ with $|\gamma| \leq k$,

$$\int_{\mathbb{R}^N} s^{|\gamma|} |\partial_x^\gamma \mathcal{L}_{s,\alpha}(x, y)| dy \leq C_1, \quad \int_{\mathbb{R}^N} s^{|\gamma|} |\partial_x^\gamma \mathcal{L}_{s,\alpha}(x, y)| dx \leq C_2, \quad (39)$$

where C_1 and C_2 are independent of x and y respectively. We observe that $\partial_x^\gamma \mathcal{L}_{s,\alpha}(x, y)$ is the sum of terms of the form

$$s^{-N-j} \partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_\alpha(z + s^{-1}(x-z), z + s^{-1}(y-z)) \cdot \xi(z), \quad (40)$$

where $j \leq |\gamma|$ and $\xi(z)$ is the product of derivatives of z with respect to x , which is bounded as z is admissible. This expression follows from (34) and the fact that Λ_α is a finite sum of terms of the form $(x-z)^\beta \partial_x^\gamma e^{L\tilde{z}}$ by Lemma 5.9. Keeping x, y

fixed, we bound each of these terms, using the Schwartz Kernel Theorem, since Λ_α is a smoothing operator:

$$\begin{aligned} |\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_\alpha(x, y)| &= |\langle \partial^\beta \delta_x, \partial_z^{\beta'} \Lambda_\alpha \partial_y^{\beta''} \delta_y \rangle| \\ &\leq C \|\partial^\beta \delta_x\|_{H_{-\epsilon}^{-q}} \|\partial_z^{\beta'} \Lambda_\alpha\|_{H^{-q} \rightarrow H_{-\epsilon}^q} \|\partial_y^{\beta''} \delta_y\|_{H^{-q}}, \end{aligned} \quad (41)$$

where $\langle \cdot, \cdot \rangle$ denotes again the pairing between smooth functions and compactly supported distributions. Above, we employed Corollary 6.5 with $p = 2$, $a = 0$, and $w = z$ (and $H_a^k := W_{a,w}^{k,2}$). Next we estimate the three norms at the right hand side of the above inequality. Choosing $q > N + |\beta|$ gives for all $\epsilon > 0$,

$$\|\partial^\beta \delta_x\|_{H_{-\epsilon}^{-q}} := \|e^{-\epsilon \langle x-z(x,y) \rangle} \partial^\beta \delta_x\|_{H^{-q}} \leq C e^{-\epsilon \langle x-z(x,y) \rangle}$$

and similarly for $\partial_y^{\beta''} \delta_y$. Since all the coefficients and their derivatives of $L(t)$ are bounded, $\partial_z^{\beta'} \Lambda_\alpha$ satisfies the same mapping properties as Λ_α . Thus by Corollary 6.5, $\|\partial_z^{\beta'} \Lambda_\alpha\|_{H^{-q} \rightarrow H_{-\epsilon}^q} \leq C e^{\epsilon \langle z-x \rangle}$. Consequently,

$$|\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_\alpha(x, y)| \leq C e^{\epsilon \langle z-x \rangle - \epsilon \langle x-z \rangle} \leq C,$$

and we obtain

$$|s^{-N-j} \partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_\alpha(z + s^{-1}(x-z), z + s^{-1}(y-z)) \cdot \xi(z)| \leq C s^{-N-|\gamma|}.$$

Finally, the change of variable $\lambda = \frac{y-x}{s}$ allows us to verify that (39) holds. The case of non-integer exponent k follows by interpolation. $\square$

Lemma 6.12 implies immediately

Corollary 6.13. *Let z be admissible, and let $k \in \mathbb{Z}_+$, $1 < p < \infty$. Then, for any $r \geq 0$ and α , there exists $C_{k,r,p,\alpha} > 0$ such that*

$$s^{k+r} \|\mathcal{L}_{s,\alpha}\|_{W^{r,p}, W^{r+k,p}} \leq C_{k,r,p,\alpha}. \quad (42)$$

Let $\tilde{E}_{t,0}^{[\mu]}$ denote also the “error” integral operator with kernel $\tilde{E}_{t,0}^{[\mu]}(x, y)$, as in Theorem 1.1.

Theorem 6.14. *Under the hypotheses of Theorem 6.9, the error operator $\tilde{E}_{t,0}^{[\mu]}$ of Theorem 1.1 satisfies*

$$\|\tilde{E}_{t,0}^{[\mu]}\|_{W^{r,p}, W^{r+k,p}} \leq C_{r,k,p,m} s^{-k}. \quad (43)$$

Proof. Recall the splitting (36). Then, applying Theorem 6.9 and Corollary 6.13 gives

$$\begin{aligned} \|\mathcal{E}_t^{[m]}\|_{W^{r,p}, W^{r+k,p}} &\leq \sum_{\ell=m+1}^M s^{\ell-m-1} \sum_{k=m+1}^{\ell} \sum_{\alpha \in \mathfrak{A}_{k,\ell}} \|\mathcal{L}_\alpha\|_{W^{r,p}, W^{r+k,p}} \\ &\quad + s^{M+1-m} \|\mathcal{E}_t^{[M]}\|_{W^{r,p}, W^{r+k,p}} \leq C s^{-k} (1 + s^{M+1-m} s^{-r}) \leq C s^{-k}. \end{aligned}$$

This completes the proof. $\square$

Our main result, Theorem 1.1, now follows from Definition 6.2, the expansion of the operator Λ_ℓ , and the error analysis of this section.

Remark 6.15. It is not difficult to show that the approximation introduced in Theorem 1.1 is invariant under affine transformations, a useful fact in applications. We refer to [12] for more details.

Combining our previous results we obtain the following.

Theorem 6.16. *Let $L \in \mathbb{L}_\gamma$ for $\gamma > 0$, and let $U = U^L$ be the evolution system generated by L on $W_{a,w}^{k,p}$, $k \in \mathbb{Z}_+$, $w \in \mathbb{R}^N$, $1 < p < \infty$, and $a \in \mathbb{R}$. Let $\tilde{\mathcal{G}}_{t,t'}^{[\mu]}$ be the μ^{th} -order approximation of the Green function for $\partial_t - L(t)$, $\mu \geq 1$ (Definition 6.2). Then there exist $\omega, M > 0$ such that, for all $t \geq 0$, we have*

$$\|U(t, 0) - \prod_{k=0}^{n-1} \left(\tilde{\mathcal{G}}_{(k+1)t/n, kt/n}^{[\mu]} \right) \|_{W_{a,w}^{k,p}} \leq M \frac{t^{(m+1)/2}}{n^{(m-1)/2}} e^{\omega t}.$$

Proof. Theorems 1.1 shows that the assumptions of Theorem 3.6 are satisfied with $\alpha = (\mu + 1)/2$. Substituting yields the claimed result. $\square$

In particular, we have the following convergence.

Corollary 6.17. *In the hypotheses of Theorem 6.16, if $\mu \geq 2$, then, for $t > 0$, we have*

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \left(\tilde{\mathcal{G}}_{(k+1)t/n, kt/n}^{[\mu]} \right) = U(t, 0),$$

in norm in $\mathcal{L}(W_{a,w}^{k,p})$.

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