Optimal Pursuit of Surveilling Agents near a High Value Target*

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Abstract. We introduce a tracking evasion game comprising a single mobile pursuer, two mobile trackers and one static high value target. The trackers rely on individual measurements of the location of the target using, for instance, their individual distance to the target and are assumed to be slower than the pursuer. The pursuer seeks to minimize the square of the instantaneous distance to one of the trackers, while the trackers aim to jointly maximize a weighted combination of the determinant of the Fisher Information Matrix and the square of the distance between the pursuer and the tracker being pursued. This formulation models the objective of the trackers which is to maximize the information gathered about the target, while delaying capture. We show that the optimization problem for the trackers can be transformed into a Quadratically Constrained Quadratic Program. We then establish that the game admits a Nash equilibrium in the space of pure strategies and provide several numerical insights into the trajectories and the payoff of the mobile agents. Finally, we outline how this work can be generalized to the case of multiple trackers and multiple targets.

Keywords: Pursuit Evasion · Game Theory · Target Tracking.

1 Introduction

The decreasing cost and increasing capabilities of Unmanned Aerial Vehicles (UAVs) have led to their widespread use in many applications such as environmental monitoring, surveillance and defense [3,19]. However, ease of access to UAV technology has found adversarial use [23]. A commonly reported adversarial application is deploying multiple adversarial UAVs (or intruders) to breach a perimeter [27,2]. In most of the works on perimeter defense, it is assumed that the location of the perimeter is known to the adversary, which may not be true in all applications. For instance, the location of a high value defense/research facility (target/perimeter) is not precisely known to the adversary. In such scenarios, prior to deploying intruders to breach the perimeter, the adversary will typically obtain the estimates of the location of the facility by deploying adversarial UAVs (or trackers) equipped with some low cost range sensor [5]. To

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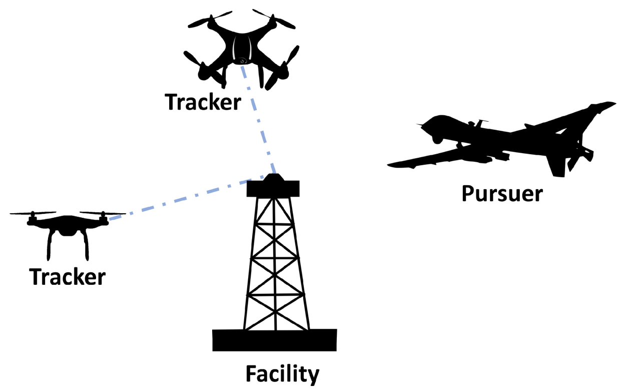


Fig. 1: Problem Description. Adversarial UAVs (trackers) move to maximize the information gathered using the distance measurements to the facility (target) while simultaneously evading from the pursuer.

counter these UAVs, the defense system can release mobile pursuers (cf. Fig. 1) that have the ability to intercept and disable the UAVs or to corrupt the information gathered by obstructing the line-of-sight. These scenarios raise an important question which has not yet been fully explored in the literature – *how does the motion strategy of adversarial trackers change in the presence of one or many pursuers?*

This work is a first step towards formulating an adversarial information gathering problem in presence of a mobile pursuer. Specifically, we introduce a *tracking pursuit* game in a planar environment comprising one single static high value target, a single mobile pursuer and two mobile trackers (adversarial UAVs). We assume that the trackers are slower than the pursuer and can only measure their individual distances from the target. The trackers jointly seek to maximize the tracking performance while simultaneously evading the pursuer at every time instant. On the other hand, the pursuer seeks to capture one of the trackers to hinder the tracking objective of the trackers. Although we considers a planar environment and range-only measurements, we also show how this work can be extended to other sensing models such as bearing measurements.

1.1 Related Works

General target tracking problems involve a static/moving target whose state (e.g., the position and velocity) needs to be estimated by trackers using measurements based on the distance or bearing or both [8,4,17,20]. A generic approach in these works is to optimize a measure (e.g., trace or determinant) of the estimation error covariance matrix obtained from an Extended Kalman Filter (EKF) used to estimate the state. We refer to [7] and the references therein for the application of state estimation to various target tracking scenarios. Since the relative geometry of the target and the trackers plays an important role in the tracking process, many works [16,4,30,31,24] have focused on identifying such geometries and motion strategies that optimize the tracking performance such as the determinant of the Fisher Information Matrix (FIM) or the trace of the estimation error covariance of the EKF. Tracking based on metrics of observability have also been considered [9,21,22].

All of the above mentioned works only focus on determining optimal trajectories for the trackers to optimize a certain tracking performance, but do not consider the presence of a pursuer. Authors in [14] design strategies for the pursuers to optimize the tracking performance while maintaining a desired formation. In [25], an adaptive sampling approach is considered to track mobile targets and maintain them in the field of view. Authors in [1] propose an algorithm based on rapidly exploring random trees for pursuers to detect and track a target.

Pursuit of mobile agents (or evaders) in the presence of a target has been extensively studied as a differential game known as Target-Attacker-Defender (TAD) games [29,12,10,11]. In these works, the attacker tries to capture a target while simultaneously evading a defender. The objective in these works is to determine optimal cooperative strategies for the the target and the defender to delay the time taken to capture or evade the attacker. This paper differs from the aforementioned TAD games as the trackers do not seek to capture the target. Instead, through the measurements obtained, the trackers aim to maximize the information gathered about the target while, evading the pursuer. Another variant of pursuit evasion games is pursuit tracking [26,18,32] where the objective of the pursuer is to track the evader by maintaining a fixed distance or Line of Sight to it. In contrast, in this work, the pursuer seeks to capture the trackers which are tracking a static target.

1.2 Preliminaries and Contributions

Recall that one of the objectives of the trackers is to maximize the information obtained from the set of range measurements to the target. This motivates the use of Fisher Information Matrix (FIM). The FIM is a symmetric, positive definite matrix that characterizes the amount of information provided by the measurements for the position of the target that is to be estimated. In other words, by moving to locations that provide the highest information, the trackers aim to improve the outcome of the estimation process. Maximizing the FIM can be achieved by maximizing a real-valued scalar function (or a metric) of the FIM. The most commonly used metrics are the trace, determinant and the eigenvalues of the FIM, also known as the A-optimality, D-optimality and E-optimality criteria, respectively [28]. Although the trace of the FIM is easy to compute, we consider the determinant as a metric to be maximized by the trackers. This is because the trace of FIM may be non zero even when the FIM is singular, implying that optimizing the trace of the FIM can result in singular configurations.

In this paper, we seek to understand the role of a pursuer in tracking problems. Equivalently, we aim to understand how the cost of evasion combined with the tracking cost affects the trajectories, and consequently, the payoff of the trackers. In particular, we consider an instantaneous two player zero sum game between the pursuer and the trackers wherein the pursuer seeks to minimize the (square of) distance to one of the trackers at every time instant whereas the trackers aim to jointly maximize a weighted combination of the determinant of the FIM at every time instant and the distance from the pursuer. Our main contributions are as follows:

1. **Tracking-Pursuit Game with a target:** We introduce a tracking-pursuit problem, modelled as a zero sum game, in a planar environment which consists of a single mobile pursuer, two mobile trackers and a single static target. For ease of presentation, we assume that the tracking agents can only measure the distance to the target and are assumed to be slower than the pursuer. At every time instant, the pursuer aims to minimize the square of the distance to a tracker, whereas the trackers aim to jointly maximize a weighted combination of the determinant of the FIM and the square of the distance to the pursuer from the nearest tracker. The game terminates when the pursuer captures a tracker.
2. **Computing Nash Equilibrium Strategies:** We first establish the optimal strategy for the pursuer. Although the payoff for the trackers is a non-convex function, we show that the optimization problem can be converted to a Quadratically Constrained Quadratic Program (QCQP). We further establish that the optimal strategies obtained for the pursuer and the trackers form a Nash equilibrium of this game.
3. **Numerical Insights:** We provide several numerical examples highlighting the trajectories of the mobile agents and the affect on the instantaneous payoff. In particular, we show that due to the presence of pursuer the determinant of the FIM achieves a lower value. We also show, through one of the examples, that the pursuer can capture a tracker even when the tracker is faster than the pursuer.
4. **Extension to multiple trackers and targets:** Finally, we thoroughly describe how this work extends to the scenarios when there are multiple trackers or multiple targets.

This paper is organized as follows. Section 2 comprises the formal problem definition. In section 3, we derive optimal strategies for the pursuer and the trackers, Section 4 provides several numerical insights into the problem and Section 5 describes the extension of this work to the case of multiple targets and trackers. Finally, Section 6 summarizes this work and outlines future directions for this work.

2 Problem Description

We consider a *tracking evasion* problem in a planar environment which consists of a single static target, a single mobile pursuer and two mobile trackers. We denote the two trackers as E_1 and E_2, respectively (cf. Fig. 2). Each mobile

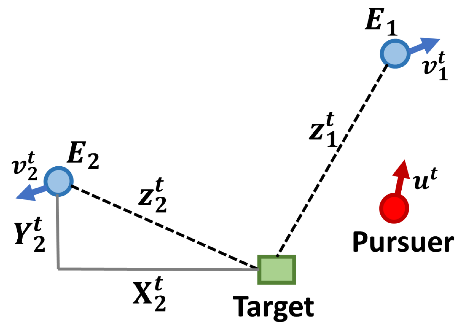


Fig. 2: Problem Description with $\alpha = 1$. The trackers and the pursuer is denoted by the blue and the red circles, respectively. The (static) high value target is denoted by a green square.

agent is modelled as a single order integrator with bounded maximum speed. The pursuer is assumed to be faster than the trackers and can move with a maximum speed normalized to unity. At every time instant t, each tracker i, where $i \in \{1, 2\}$, has access to the range measurements, z_i^t, to the target located at $s \triangleq [s_x \quad s_y]' \in \mathbb{R}^2$, where r' denotes the transpose of some vector r. Let $e_i^t \triangleq \begin{bmatrix} e_{x,i}^t & e_{y,i}^t \end{bmatrix}' \in \mathbb{R}^2$ (resp., $p^t \triangleq \begin{bmatrix} p_x^t & p_y^t \end{bmatrix}' \in \mathbb{R}^2$) denote the position of the i^{th} tracker (resp. the pursuer) and let v_i^t (resp. u^t) denote the i^{th} tracker's (pursuer's) control. Then, the motion model and the measurements are given by

$$\begin{aligned} e_i^{t+1} &= e_i^t + v_i^t + w_i^t, \qquad \|v_i^t\| \le \mu_i < 1, \ \forall i \in \{1, 2\}, \\ p^{t+1} &= p^t + u^t + w_p^t, \qquad \|u^t\| \le 1, \\ z_i^t &= \|s - e_i^t\| + \nu_i^t, \ \forall i \in \{1, 2\}, \end{aligned} \tag{1}$$

where, $\nu_i^t \sim \mathcal{N}(0, \sigma_\nu^2), \forall i \in \{1, 2\}$ denotes the measurement noise, assumed to be mutually independent, $w_i^t \sim \mathcal{N}(0, \sigma_{w_e})$ as well as $w_p^t \sim \mathcal{N}(0, \sigma_{w_p})$ denotes the process noise.

A tracker is said to be *captured* by the pursuer at time instant t, if its location is within a unit distance from the pursuer at time t. Note that since the pursuer is faster than both trackers, the pursuer can always capture both trackers successively. However, we will see that the game ends when the pursuer captures any one of the trackers. A strategy for the pursuer is defined as $u^t : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$. Similarly, a strategy for an i^{th} tracker is defined as $v_i^t : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \forall i \in \{1, 2\}$.

We now determine the expression for the determinant of the FIM. Let $h(s, e_1^t, e_2^t) \triangleq [\|s - e_1^t\| \ \|s - e_2^t\|]'$ denote the measurement vector at time instant t and $\nabla_s \triangleq [\frac{\partial}{\partial s_x} \ \frac{\partial}{\partial s_y}]'$. Then, for a given model (1) and a measurement vector

$h(s, e_1^t, e_2^t)$, the FIM at time instant $t+1$ is [16]

$$f(s, e_1^t, e_2^t, v_1^t, v_2^t) = \frac{1}{\sigma_\nu^2}(\nabla_s h(s, e_1^t, e_2^t))'(\nabla_s h(s, e_1^t, e_2^t)),$$
$$= \frac{1}{\sigma_\nu^2}\sum_{i=1}^{2}\frac{1}{S_i^2}\begin{bmatrix}(X_i^t - v_{x,i}^t)^2 & (X_i^t - v_{x,i}^t)(Y_i^t - v_{y,i}^t)\\ (X_i^t - v_{x,i}^t)(Y_i^t - v_{y,i}^t) & (Y_i^t - v_{y,i}^t)^2\end{bmatrix},$$

where $X_i^t = s_x - e_{x,i}^t$ (resp. $Y_i^t = s_y - e_{y,i}^t$) denotes the difference in the x (resp. y)-coordinate of the target and the i^{th} tracker (cf. Fig. 2) and $S_i = \sqrt{(X_i^t - v_{x,i}^t)^2 + (Y_i^t - v_{y,i}^t)^2}$. Since the trackers do not know the location of the target, we replace s by its estimate $\hat{s}$ which is obtained from a centralized EKF. Thus, we obtain the determinant of the FIM as

$$\begin{aligned}\det(f(\hat{s}, e_1^t, e_2^t, v_1^t, v_2^t)) &= \frac{1}{\sigma_\nu^2}\Big[\sum_{i=1}^{2}\frac{1}{\hat{S}_i^2}(\hat{X}_i^t - v_{x,i}^t)^2\sum_{i=1}^{2}\frac{1}{\hat{S}_i^2}(\hat{Y}_i^t - v_{y,i}^t)^2 - \\ &\qquad \Big(\sum_{i=1}^{2}\frac{1}{\hat{S}_i^2}(\hat{X}_i^t - v_{x,i}^t)(\hat{Y}_i^t - v_{y,i}^t)\Big)^2\Big] \\ &= \frac{1}{\sigma_\nu^2\hat{S}_1^2\hat{S}_2^2}((\hat{X}_1^t - v_{x,1}^t)(\hat{Y}_2^t - v_{y,2}^t) - (\hat{Y}_1^t - v_{y,1}^t)(\hat{X}_2^t - v_{x,2}^t))^2, \quad (2)\end{aligned}$$

where $\hat{X}_i^t = \hat{s}_x - e_{x,i}^t$ and $\hat{Y}_i^t = \hat{s}_y - e_{y,i}^t$ for all $i \in \{1, 2\}$. Note that the determinant of the FIM for a single tracker is equal to zero implying that the configuration is always singular in the case of one tracker.

At the first time instant ($t = 1$), the pursuer selects the tracker which is closest to the pursuer. This selection is characterized by $\alpha \in \{0, 1\}$. Specifically, if tracker E_1 is closest to the pursuer, then $\alpha = 1$. Otherwise, $\alpha = 0$. The pursuer, then selects its control, u^t, such that the square of the distance to the selected tracker is minimized at every time instant $t \geq 1$. On the other hand, the trackers jointly select their control at every time instant $t \geq 1$ to maximize a weighted combination of the determinant of the FIM and the square of the distance between the selected tracker and the pursuer. We assume that the trackers have information of the location of the pursuer and thus, the choice of α is known to the trackers. Since the two trackers jointly maximize the payoff, we model the interaction between the trackers and the pursuer as a two player zero sum game with the payoff, at time instant $t+1$, defined as

$$\begin{aligned}J(v_1^t, v_2^t, u^t) = \det(f(\hat{s}, e_1^t, e_2^t, v_1^t, v_2^t)) + \delta(\alpha\|e_1^t + v_1^t - p^t - u^t\|^2 + \\ (1-\alpha)\|e_2^t + v_2^t - p^t - u^t\|^2),\end{aligned} \quad (3)$$

where $\delta \in \mathbb{R}$ is a fixed weight associated with the evasion cost (distance between the pursuer and the selected tracker) and is assumed to be known by all agents. The game terminates when the pursuer captures the selected tracker since the determinant of the FIM is always zero for one tracker. We use t_f to denote the time instant when the game terminates.

We now provide two definitions that will be helpful in establishing our main result in Section 3.

Definition 1 (Best Response). *For a two player zero sum game with the payoff defined as $J(\gamma, \sigma)$, the strategy $\gamma^* \in \Gamma_1$ for player 1 (minimizer) is called the best response to player 2's (maximizer) strategy $\sigma \in \Gamma_2$ if the following holds*

$$J(\gamma^*, \sigma^*) \leq J(\gamma, \sigma^*),\ \forall \gamma \in \Gamma_1.$$

Note that the best response for the maximizer can be analogously defined.

Definition 2 (Nash Equilibrium). *Given a strategy $\gamma \in \Gamma_1$ for player 1 and a strategy $\sigma \in \Gamma_2$ for player 2 in a two player zero sum game with the payoff $J(\gamma, \sigma)$, the pair of strategies (γ^*, σ^*) is said to be a saddle-point equilibrium strategy if the following holds.*

$$J(\gamma^*, \sigma) \leq J(\gamma^*, \sigma^*) \leq J(\gamma, \sigma^*),\ \forall \gamma \in \Gamma_1,\ \sigma \in \Gamma_2. \tag{4}$$

Observe that equation (4) in Definition 2 can be rewritten as [13]

$$J(\gamma^*, \sigma^*) = \min_{\gamma \in \Gamma_1} J(\gamma, \sigma^*),\ \text{and}\ J(\gamma^*, \sigma^*) = \max_{\sigma \in \Gamma_2} J(\gamma^*, \sigma),$$

implying that the pair of strategies (γ^*, σ^*) form a Nash equilibrium if γ^* (resp. σ^*) is the best response to σ^* (resp. γ^*).

We now formally state our objective for the above model.

Problem 1. The aim of this work is to determine saddle-point strategies $u^{t^*} \in \mathbb{R}^2$ and $\{v_1^{t^*}, v_2^{t^*}\} \in \mathbb{R}^2 \times \mathbb{R}^2$ at every time instant $t < t_f$ such that

$$\max_{v_1^t, v_2^t} \min_{u^t}\ J(v_1^t, v_2^t, u^t) = \min_{u^t} \max_{v_1^t, v_2^t}\ J(v_1^t, v_2^t, u^t)$$

holds subject to the individual agents maximum speed constraints, i.e.,

$$\|u^t\| \leq 1,\ \|v_1^t\| \leq \mu_1 < 1,\ \|v_2^t\| \leq \mu_2 < 1.$$

For the problem to be well-posed, we make the following assumption.

Assumption 1 (EKF Convergence) *There exists a time instant $t_e < t_f$ at which the estimates obtained by the trackers are equal to the true location of the target, i.e., $||\hat{s} - s|| = 0, \forall t \geq t_e$.*

3 Optimal Strategies

In this section, we determine the optimal strategies for the pursuer and the trackers. We start with the optimal strategy of the pursuer followed by that of the trackers.

3.1 Optimal Strategy of the Pursuer

Without loss of generality, let $\alpha = 1$ at the first time instant and suppose that v_1^t was known to the pursuer. Then, the pursuer solves the following optimization problem.

$$\min_{u^t} \ \delta(\|e_1^t + v_1^t - p^t - u^t\|)^2, \quad \text{subject to } \|u^t\| \leq 1 \tag{5}$$

where we used the fact that the term $\det(f(\hat{s}, e_1^t, e_2^t, v_1^t, v_2^t))$ is not a function of the pursuer's control u^t. It follows directly that the solution to the optimization problem (5) is

$$u^{t^*}(v_1^t) = \frac{e_1^t + v_1^t - p^t}{\|e_1^t + v_1^t - p^t\|},$$

meaning that the pursuer moves directly towards the first tracker, E_1, with unit speed as long as the tracker's position at time instant $t+1$, i.e., $e_1^t + v_1^t$, is not within a unit distance from the current position of the pursuer. Otherwise, the optimal pursuer strategy is $e_1^t + v_1^t - p^t$, implying that the tracker is guaranteed to be captured (evasion cost is zero) at time instant $t + 1 = t_f$. This further implies that the trackers will move only to maximize the determinant of FIM at time instant $t = t_f - 1$ as tracker E_1 is guaranteed to be captured at time $t+1$.

Thus, the optimal strategy for the pursuer at every time instant $t < t_f - 1$ is

$$u^{t^*}(v_1^t, v_2^t) = \begin{cases} \frac{e_1^t + v_1^t - p^t}{\|e_1^t + v_1^t - p^t\|}, \text{if } \alpha = 1, \\ \frac{e_2^t + v_2^t - p^t}{\|e_2^t + v_2^t - p^t\|}, \ \text{otherwise.} \end{cases} \tag{6}$$

Further, the optimal strategy for the pursuer at time instant $t = t_f - 1$ is

$$u^{t^*}(v_1^t, v_2^t) = \begin{cases} e_1^t + v_1^t - p^t, \text{if } \alpha = 1, \\ e_2^t + v_2^t - p^t, \ \text{otherwise} \end{cases} \tag{7}$$

if $\|e_1^t + v_1^t - p^t\| < 1$ (resp. $\|e_2^t + v_2^t - p^t\| < 1$) for $\alpha = 1$ (resp. $\alpha = 0$). Otherwise, the optimal strategy for the pursuer at time instant $t_f - 1$ is given by equation (6). Note that although we assumed that $v_i^t, \forall i \in \{1, 2\}$ was known to the pursuer, in reality, the pursuer does not have this information. This means that the optimal strategy defined in (6) is an *anticipatory* strategy of the pursuer based on the belief of the trackers' strategy. As will be clear from the next subsection, we use the optimal strategy, $u^{t^*}(v_1^t, v_2^t)$, of the pursuer to determine the optimal state-feedback strategies of the trackers $v_1^{t^*}$ and $v_2^{t^*}$. Substituting $v_1^{t^*}$ and $v_2^{t^*}$ into (6) implies that u^{t^*} is a state-feedback strategy.

In the next section, we determine the optimal strategies of the trackers. Since maximizing only the determinant of the FIM has been extensively studied [16], we only focus on the case that the position of the tracker being pursued at time instant $t_f - 1$ is more than a unit distance from the current position of the pursuer. In other words, the pursuer's optimal strategy is given by equation (6) at time $t_f - 1$. Note that the optimal strategy of the pursuer for any time instant $t < t_f - 1$ is given by equation (6). Further, for ease of presentation, we drop the dependency on time from the notations in the next subsection.

3.2 Optimal Strategies of the Trackers

For a given value of α, the trackers jointly solve the following optimization problem.

$$\begin{aligned}\max_{v_1, v_2}\ &\det(f(\hat{s}, e_1, e_2, v_1, v_2)) + \delta(\|e_1 + v_1 - p - u\|)^2 \\ &\qquad\qquad\text{subject to } \|v_1\| \leq \mu_1, \|v_2\| \leq \mu_2,\end{aligned}$$

where, without loss of generality, we assumed that $\alpha = 1$. Substituting $u^*(v_1, v_2)$ from equation 6 as well as the expression of the determinant yields

$$\begin{aligned}\max_{v_1, v_2}\ &\frac{1}{\sigma_\nu^2 \hat{S}_1^2 \hat{S}_2^2}\Big((\hat{X}_1 - v_{x,1})(\hat{Y}_2 - v_{y,2}) - (\hat{Y}_1 - v_{y,1})(\hat{X}_2 - v_{x,2})\Big)^2 + \\ &\qquad\qquad\qquad\qquad \delta(\|e_1 + v_1 - p\| - 1)^2, \\ &\text{subject to } \|v_1\| \leq \mu_1,\ \|v_2\| \leq \mu_2.\end{aligned} \tag{8}$$

Although the constraints are convex, the objective is a non-convex function of $v_i, \forall i \in \{1, 2\}$ and thus, computing a global maximizer is difficult. In what follows, we show that this optimization problem is equivalent to solving a quadratically constrained quadratic program (QCQP) [6].

For ease of presentation, we use the following notation in the next result.

Let $V = \begin{bmatrix} v_{x,1} & v_{y,1} & v_{x,2} & v_{y,2} & v_{x,1}v_{y,2} & v_{x,2}v_{y,1} \end{bmatrix}' \in \mathbb{R}^6$ and let

$$z^2 = \frac{1}{\hat{S}_1^2 \hat{S}_2^2}\big((\hat{X}_1 - v_{x,1})(\hat{Y}_2 - v_{y,2}) - (\hat{Y}_1 - v_{y,1})(\hat{X}_2 - v_{x,2})\big)^2.$$

Lemma 1. *Suppose $\alpha = 1$ and let $m = \|e_1 + v_1 - p\|$. Then, the optimization problem defined in* (8) *is equivalent to solving a QCQP given by*

$$\begin{aligned}&\max_{\tilde{V}} \tilde{V}' P \tilde{V} \\ &\textit{subject to} \\ &\qquad \tilde{V}' Q_j \tilde{V} \leq 0, \forall j \in \{1, 2\} \\ &\qquad \tilde{V}' F \tilde{V} = 0, \\ &\qquad \tilde{V}' M_1 \tilde{V} = 0, \\ &\qquad \tilde{V}' L_g \tilde{V} = 0, \forall g \in \{1, \ldots, 10\} \\ &\qquad \tilde{V}_8 = 1,\end{aligned} \tag{9}$$

where $\tilde{V} \triangleq \begin{bmatrix} V' & m & 1 & zV' & zv_{x,1}v_{x,2} & zv_{y,1}v_{y,2} & z \end{bmatrix}' \in \mathbb{R}^{17}$, $\tilde{V}_k$ *denotes the* k^{th} *entry of vector* $\tilde{V}$ *and the matrices* $P, M_1, Q_j, \forall j \in \{1,2\}$ *and* $L_g, \forall g \in \{1,\ldots,10\}$ *are as defined in the Appendix.*

Proof. By replacing $\|e_1 + v_1 - p\|$ by m, the optimization problem defined in (8) can be rewritten as

$$\max_{v_1,v_2,m,z} \ \frac{1}{\sigma_\nu^2} z^2 + \delta(m-1)^2$$
$$\text{subject to } \|v_1\| \le \mu_1, \ \|v_2\| \le \mu_2, \ m^2 = \|e_1 + v_1 - p\|^2,$$
$$\Big((\hat{X}_1 - v_{x,1})(\hat{Y}_2 - v_{y,2}) - (\hat{Y}_1 - v_{y,1})(\hat{X}_2 - v_{x,2})\Big)^2 - z^2 \hat{S}_1^2 \hat{S}_2^2 = 0.$$

Observe that the optimization problem is now a polynomial in the original optimization variables and the additional variables z and m. By adding some extra variables corresponding to the terms that are polynomial in the optimization variables $v_{x,i}$ and $v_{y,i}, \forall i \in \{1,2\}$, we now convert the aforementioned optimization problem into a QCQP.

Let $V = \begin{bmatrix} v_{x,1} & v_{y,1} & v_{x,2} & v_{y,2} & v_{x,1}v_{y,2} & v_{x,2}v_{y,1} \end{bmatrix}' \in \mathbb{R}^6$. Then, we define a vector of optimization variables $\tilde{V} \in \mathbb{R}^{17}$ as

$$\tilde{V} = \begin{bmatrix} V' & m & 1 & zV' & zv_{x,1}v_{x,2} & zv_{y,1}v_{y,2} & z \end{bmatrix}'.$$

Taking the square on both sides of the norm constraints, the above optimization problem yields the QCQP form as defined in (9). Note that the constraint $\tilde{V}'M_1\tilde{V} \equiv \|e_1 + v_1 - p\|^2 - m^2$. Further, the set of constraints $\tilde{V}'L_g\tilde{V} = 0, \forall g \in \{1,\ldots,10\}$ characterize the relationship between the elements of $\tilde{V}$. As described in [15], the equality constraints in optimization problem (9) can be replaced with two inequality constraints ,thus, reducing the optimization problem in the standard QCQP form. This concludes the proof. □

Following similar steps, an analogous optimization problem when $\alpha = 0$ is

$$\begin{aligned} &\max_{\tilde{V}} \tilde{V}'P\tilde{V} \\ \text{subject to}\quad & \\ &\tilde{V}'Q_j\tilde{V} \le 0, \forall j \in \{1,2\} \\ &\tilde{V}'M_0\tilde{V} = 0, \\ &\tilde{V}'F\tilde{V} = 0, \\ &\tilde{V}'L_g\tilde{V} = 0, \forall g \in \{1,\ldots,10\} \\ &\tilde{V}_8 = 1, \end{aligned} \tag{10}$$

where matrix M_0 is as defined in the Appendix. Note that all of the matrices $Q_j, \forall j \in \{1,2\}$, P, M_0, M_1, F and $L_g, \forall g \in \{1,\ldots,10\}$ are sparse matrices.

We now establish the main result of this paper, i.e., that the pair of strategies $(u^{t^*}, \{v_1^{t^*}, v_2^{t^*}\})$ form a pair of Nash equilibrium. Note that if the optimal strategies of the trackers and the pursuer form a pair of Nash equilibrium, there is no incentive for the trackers to deviate from their optimal strategy (see Definition 2). This means that the pursuer has the correct belief of the trackers strategy and can determine its state-feedback strategy, u^{t^*} by first solving the QCQP to determine $v_1^{t^*}$ and $v_2^{t^*}$ and then substituting these into (6). However, to determine the strategy of the trackers, the pursuer needs the information of the estimates that the trackers have of the target's location. Since the pursuer does not have this information, we propose that the pursuer uses the true value of the target's location to solve the QCQP and consequently determine u^{t^*}.

Theorem 1. *At every instant $t_e \leq t < t_f$, the pair of strategies $(u^{t^*}, \{v_1^{t^*}, v_2^{t^*}\})$ defined in* (6) *and obtained by solving the optimization problem* (8)*, form a pair of Nash equilibrium strategies for the payoff function $J(\hat{s}, e_i^t, p^t, v_i^t, u^t)$ as defined in* (3)*.*

Proof. Observe that once the estimates about the location of the target converges to the true value, all of the mobile agents use the same value of the target's location to solve the QCQP. Further, at every time instant $t_e \leq t < t_f$, the optimal strategy of the pursuer defined in equation (6) is the best-response of the pursuer to the trackers strategy. Similarly, the optimal strategy of the trackers obtained by solving the optimization problem defined (9) (if $\alpha = 1$) and (10) (otherwise) is the best response of the trackers to an optimal pursuer strategy. The result then follows directly from Definition 2. This concludes the proof. □

We now briefly describe how the game is solved. At every time instant, depending on the value of α, the pursuer solves the optimization problem 9 or 10 using the true location of the target (s) to obtain $v_1^{t^*}$ and $v_2^{t^*}$. The pursuer then moves to the location by determining its control via 6. On the other hand, the trackers jointly solve the same optimization problem using the estimates of the target ($\hat{s}$) and move to the next location using $v_1^{t^*}$ and $v_2^{t^*}$.

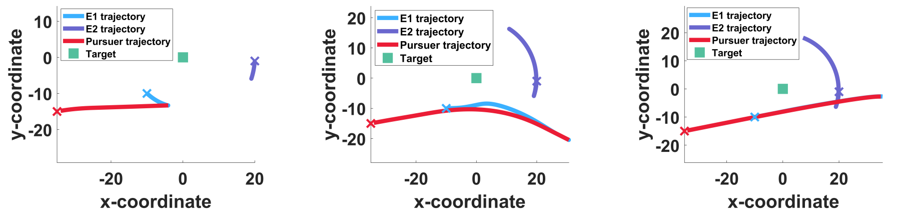
Remark 1 (Bearing Measurements). If the trackers use a sensor that measures the bearing (angle) of the target relative to their positions instead of range measurements, then the determinant of the FIM is given by [20]

$$f(\hat{s}, e_1^t, e_2^t, v_1^t, v_2^t) = \frac{1}{\sigma_\nu^2 \hat{S}_1^4 \hat{S}_2^4}\Big((\hat{X}_1^t - v_{x,1}^t)(\hat{Y}_2^t - v_{y,2}^t) - (\hat{Y}_1^t - v_{y,1}^t)(\hat{X}_2^t - v_{x,2}^t)\Big)^2.$$

As the pursuer's optimal strategy does not change, by following similar steps as in Section 3.2, the optimization problem for the trackers can similarly be expressed as a QCQP and thus, this work easily extends to scenarios when trackers have access to bearing measurements.

4 Numerical Observations

We now present numerical simulations of the optimal strategies defined in Section 3 and highlight the trajectories of the mobile agents. In all of our simulations, the

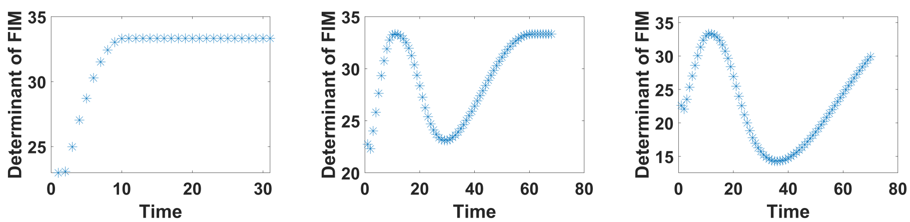


(a) Trajectories of the mobile agents for $\delta = 0$. $t_f = 44$.

(b) Trajectories of the mobile agents for $\delta = 0.2$. $t_f = 68$.

(c) Trajectories of the mobile agents for $\delta = 5$. $t_f = 71$.

Fig. 3: Trajectories of the pursuer and the trackers for different values of δ. The cross represents the starting locations of the mobile agents. The target is denoted by the green square.



(a) $\delta = 0$

(b) $\delta = 0.2$

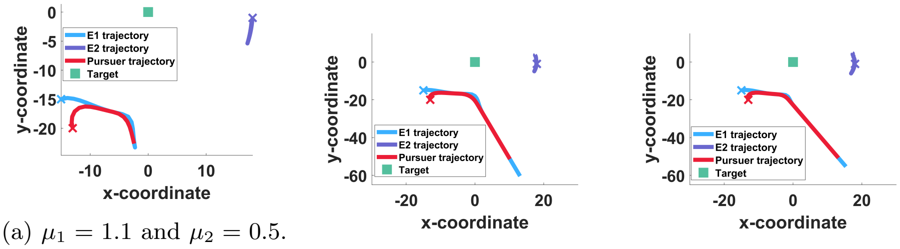
(c) $\delta = 5$

Fig. 4: Determinant of FIM vs Time plots for different values of δ.

parameter σ_ν^2 was kept fixed to 0.03 and the target's location was chosen to be $(0, 0)$. Due to the number and size of the sparse matrices in the proposed QCQP optimization problem (9), generating the trajectories was time consuming. Thus, we use fmincon function in MATLAB to determine the optimal strategies of the trackers which was verified to be consistent with the strategies obtained by solving optimization problem in (9).

4.1 Example 1 ($\alpha = 1$)

Our first numerical simulation (cf. Fig. 3) focuses on the trajectories of the mobile agents when the pursuer moves to capture the first tracker, E_1. Specifically, we select the initial locations such that $\alpha = 1$. To highlight the role of evasion by the trackers, we provide a numerical plot with $\delta = 0$ in Fig. 3, i.e., the evaders move to maximize only the determinant of the FIM. Note that the time taken by the pursuer to capture E_1 is mentioned in the description of each sub-figure in Fig. 3. The initial locations for all of the simulations presented in Fig. 3 were kept the same and selected to be $(-10, -10)$, $(20, -1)$ and $(-35, -15)$ for E_1, E_2 and the pursuer, respectively. Further, the parameters μ_1 and μ_2 were set to be 0.65 and 0.5, respectively.



(a) $\mu_1 = 1.1$ and $\mu_2 = 0.5$. $t_f = 17$. (b) $\mu_1 = 1.2$ and $\mu_2 = 0.5$. (c) $\mu_1 = 1.1$ and $\mu_2 = 0.65$.

Fig. 5: Trajectories of the pursuer and the trackers for $\delta = 0.14$ and different values of $\mu_i, \forall i \in \{1, 2\}$.

Observe that in Fig. 3a, the trackers move to position themselves such that the angle subtended at the target by the position of the trackers is $\frac{\pi}{2}$. This is consistent with trajectories that maximize only the FIM as reported in [4]. Upon reaching that position, the trackers remain at that position until tracker E_1 is captured by the pursuer. Based on the value of δ in Fig. 3b as well as in Fig. 3c, observe that the tracker E_1 first moves away from the pursuer and then it moves away from the target, maximizing both the time to capture as well as the determinant of the FIM. Fig. 3b shows the cooperative behaviour of E_2. In particular, although the pursuer does not move towards E_2, tracker E_2 first moves downwards and then changes its direction in order maximize the determinant of the FIM by moving to a location such that the position of the trackers subtend an angle of $\frac{\pi}{2}$ at the target. Once the angle between the position of the trackers is $\frac{\pi}{2}$, tracker E_2 remains stationary at its location while E_1 evades

Finally, in Fig. 4, observe that the determinant of the FIM monotonically increases in Fig. 4a and then converges to 33.33. Although in Fig. 4b the determinant of FIM reaches the value 33.3, the value then decreases as the trackers cannot stay at that position due to the evasion cost. Note that at time $t = 60$, the cost converges to 33.3 highlighting the fact that the angle subtended by the position of the trackers to the target is now at $\frac{\pi}{2}$, and thus, tracker E_2 remains at its position whereas tracker E_1 moves in a straight line maintaining the same angle. Similar trend is observed in Fig. 4c. However, tracker E_1 is captured before the angle subtended by the trackers to the target is $\frac{\pi}{2}$. This is due to the higher value of δ as compared to that in Fig. 4b because of which tracker E_1 moves directly away from the pursuer. Thus, the trackers require more time to reach the positions from which the angle subtended to the target is $\frac{\pi}{2}$.

4.2 Example 2 (Faster trackers)

This numerical simulation considers a scenario that at one tracker is faster than the pursuer. The initial locations of the trackers and the pursuer was set to $(18, -1)$, $(-15, -15)$ and $(-13, -20)$, respectively. Finally the parameter δ was kept fixed to 0.14 and from the initial locations, $\alpha = 1$.

In Fig. 5a, although tracker E_1 is faster ($\mu_1 = 1.1$ and $\mu_2 = 0.5$) than the pursuer, the pursuer is able to capture tracker E_1. However, for the same initial locations of all of the mobile agents, the pursuer is unable to capture tracker E_1 when $\mu_1 = 1.2$ and $\mu_2 = 0.5$ (cf. Fig. 5b), implying that for faster trackers, there may exist *winning regions* for the pursuer as well as the trackers. Specifically, it may be possible to partition the environment into a winning region (Ω_P) for the pursuer, i.e., the pursuer can always capture a tracker if the initial locations of all of the mobile agents lie inside Ω_P. Similarly, it may be possible to characterize the winning region (Ω_T) for the trackers, i.e., the trackers can always evade the pursuer if the initial locations of all of the mobile agents lie inside Ω_T. Finally, observe that for the same initial locations and $\mu_1 = 1.1$ (cf. Fig. 5c), the pursuer cannot capture tracker E_1 if the speed of the tracker E_2 is increased from 0.5 (Fig. 5a) to 0.65.

We now describe how this work extends to two different scenarios. We start with a scenario with multiple targets followed by a scenario with multiple trackers.

5 Extensions

In this section, we describe how our analysis extends to the case of multiple targets and multiple trackers. We also show that in both scenarios the pursuer's optimal strategy remains the same as established in Section 3. We further establish that the optimization problem for the trackers can be converted to a QCQP.

5.1 Multiple targets

In this scenario, we consider that there are $N > 1$ targets, two mobile trackers and a single mobile pursuer. Each tracker has access only to range measurements from each of the N targets. Thus, in this case, the measurement vector is $h(s_1, \ldots, s_N, e_1^t, e_2^t) = \left[\|s_1 - e_1^t\| \; \|s_1 - e_2^t\| \ldots \|s_N - e_1^t\| \; \|s_N - e_2^t\|\right]'$, where $s_1, \ldots, s_N$ denote the fixed locations of the N targets. By taking the partial derivatives with respect to the locations of the targets and replacing $s_j \forall j \in \{1, \ldots, N\}$ with its estimate $\hat{s}_j$, the FIM at time instant $t+1$ becomes a block diagonal matrix given by

$$F(\hat{s}_1, \ldots, \hat{s}_N, e_i^t, v_i^t) = \begin{bmatrix} f(\hat{s}_1, e_i^t, v_i^t) & \mathbf{0}_{2\times 2} & \ldots & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & f(\hat{s}_2, e_i^t, v_i^t) & \ldots & \mathbf{0}_{2\times 2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{2\times 2} & \ldots & \ldots & f(\hat{s}_N, e_i^t, v_i^t) \end{bmatrix}$$

where $f(\hat{s}_j, e_i^t, v_i^t), \forall 1 \le j \le N$ is the FIM defined analogously as $f(\hat{s}_1, e_i^t, v_i^t)$ (see Section 2). Using the fact that determinant of a block diagonal matrix is the product of the determinant of its blocks yields

$$\det(F(\hat{s}_1,\dots,\hat{s}_N,e_i^t,v_i^t)) = \prod_{j=1}^{N} \det(f(\hat{s}_j,e_i^t,v_i^t)).$$

Thus, the expression for the payoff is given by

$$\begin{aligned} J(\hat{s}_1,\dots,\hat{s}_N,e_i^t,p^t,v_i^t,u^t) = {} & \det(F(\hat{s}_1,\dots,\hat{s}_N,e_i^t,v_i^t)) + \\ & \delta(\alpha\|e_1^t+v_1^t-p^t-u^t\|^2 + (1-\alpha)\|e_2^t+v_2^t-p^t-u^t\|^2). \end{aligned}$$

Since the determinant of the FIM is not a function of the pursuer's control, it follows that the pursuer's strategy remains the same as defined in (6). Observe that $\det(F(\hat{s}_1,\dots,\hat{s}_N,e_i^t,v_i^t))$ is a polynomial function of $v_{x,i}^t$ and $v_{y,i}^t$ for all $i \in \{1,2\}$. Therefore, following similar steps as in Section 3 and from the fact that any polynomial can be expressed into the standard QCQP form [15], it follows that the optimization problem obtained for the trackers after substituting $u^{t^*}(v_1^t,v_2^t)$ can also be converted into a QCQP of the same form as defined in Lemma 1. Finally, given that the pair of strategies $(u^{t^*},\{v_1^{t^*},v_2^{t^*}\})$ are best responses to each other, it follows that the pair of strategies forms a Nash equilibrium.

5.2 Multiple trackers

We now consider the scenario with a single target, $M > 2$ trackers and a single mobile pursuer.

Let at time instant $t < t_f$, $\alpha \triangleq \begin{bmatrix}\alpha_1 \dots \alpha_M\end{bmatrix}' \in \mathbb{R}^M$ such that $\sum_{j=1}^{M}\alpha_j = 1$ and $\alpha_j \in \{0,1\}, \forall j \in \{1,\dots,M\}$. Let $\mathbf{D} \in \mathbb{R}^M$ denote a vector consisting of the distance between the pursuer and the trackers, i.e., $\begin{bmatrix}\|p^t - e_1^t\| \dots \|p - e_M^t\|\end{bmatrix}'$. Then, the payoff is given by

$$J(\hat{s},e_1^t,\dots,e_M^t,v_1^t,\dots,v_M^t,p^t) = \det(f(\hat{s},e_1^t,\dots,e_M^t,v_1^t,\dots,v_M^t)) + \delta\alpha_t'\mathbf{D},$$

where

$$\begin{aligned} \det(f(\hat{s},e_1^t,\dots,e_M^t,v_1^t,\dots,v_M^t)) = \frac{1}{\sigma_\nu^2}\sum_{j=1}^{M}\sum_{l=j+1}^{M}\frac{1}{\hat{S}_j^2\hat{S}_l^2}\Big(& (\hat{X}_j^t - v_{x,j}^t)(\hat{Y}_l^t - v_{y,l}^t) - \\ & (\hat{Y}_j^t - v_{y,j}^t)(\hat{X}_l^t - v_{x,l}^t)\Big)^2. \end{aligned}$$

For a given vector α at the first time instant, the strategy of the pursuer is the same as defined in Section 3 and thus, following similar steps, the payoff for the trackers can be expressed as a polynomial function in the optimization variables $v_{x,i}^t$ and $v_{y,i}^t$ for all $i \in \{1,\dots,M\}$. Hence, following similar steps as in Section 3 and given the fact that any polynomial can be expressed into the standard QCQP form [15], it follows that the optimization problem obtained for the trackers after substituting $u^{t^*}(v_1^t, v_2^t)$ can also be converted into a QCQP of the same form as defined in Lemma 1.

Finally, given that the pair of strategies $(u^{t^*}, \{v_1^{t^*}, \ldots, v_M^{t^*}\})$ are best responses to each other, it follows that the pair of strategies forms a Nash equilibrium.

6 Conclusion and Future Directions

This paper introduced a tracking-evasion game consisting of a single pursuer, two trackers and a single target. The pursuer seeks to deter the tracking performance of the trackers by minimizing the square of the distance to the closest tracker, whereas, the trackers aim to jointly maximize a weighted combination of the determinant of the Fisher Information Matrix and the square of the distance between the pursuer to the tracker being pursued. We determined optimal strategies of the pursuer and and showed that the optimal strategies of the trackers can be obtained by solving a Quadratically Constrained Quadratic Program. We then established that the pair of strategies form a Nash equilibrium and provided several numerical observations highlighting the trajectories and the payoff. Finally, we discussed the extension of this work to multiple trackers and multiple targets.

Apart from leveraging the sparse-structure of the matrices for the optimization problem, a key future direction includes a generalized setup with multiple pursuers and trackers with motion and energy constraints. Further, we conjecture that by relaxing Assumption 1, an $\bar{\epsilon}$-Nash Equilibrium may exist. This conjecture will also be explored in the subsequent works.

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7 Appendix

In this section, we provide the expression for the matrices P, Q_j, M and L, respectively. For ease of notation, denote $a_i = \hat{X}_i^t$, $b_i = \hat{Y}_i^t$. Further, let $\mathbf{I}_{n\times p}$ (resp. $\mathbf{0}_{n\times p}$) denote the identity (resp. zero) matrix of dimension $n \times p$. Then,

$$P = \frac{1}{\sigma_\nu^2} \times \begin{bmatrix} \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times 1} & \mathbf{0}_{6\times 1} & \mathbf{0}_{6\times 8} & \mathbf{0}_{6\times 1} \\ \mathbf{0}_{1\times 6} & \delta\sigma_\nu^2 & -\delta\sigma_\nu^2 & \mathbf{0}_{1\times 8} & 0 \\ \mathbf{0}_{1\times 6} & -\delta\sigma_\nu^2 & \delta\sigma_\nu^2 & \mathbf{0}_{1\times 8} & 0 \\ \mathbf{0}_{8\times 6} & \mathbf{0}_{8\times 1} & \mathbf{0}_{8\times 1} & \mathbf{0}_{8\times 8} & \mathbf{0}_{8\times 1} \\ \mathbf{0}_{1\times 6} & 0 & 0 & \mathbf{0}_{1\times 8} & 1 \end{bmatrix}, F = \begin{bmatrix} F_1 & \mathbf{0}_{8\times 9} \\ \mathbf{0}_{9\times 8} & F_2 \end{bmatrix},$$

where $F_1 =$

$$\begin{bmatrix} b_2^2 & -a_2b_2 & -b_1b_2 & 2a_1b_2 - a_2b_1 & -b_2 & b_2 & 0 & a_2b_1b_2 - a_1b_2^2 \\ -a_2b_2 & a_2^2 & 2a_2b_1 - a_1b_2 & -a_1a_2 & a_2 & -a_2 & 0 & a_1a_2b_2 - b_1a_2^2 \\ -b_1b_2 & 2a_2b_1 - a_1b_2 & b_1^2 & -a_1b_1 & b_1 & -b_1 & 0 & a_1b_1b_2 - a_2b_1^2 \\ 2a_1b_2 - a_2b_1 & -a_1a_2 & -a_1b_1 & a_1^2 & -a_1 & a_1 & 0 & a_1a_2b_1 - b_2a_1^2 \\ -b_2 & a_2 & b_1 & -a_1 & 1 & -1 & 0 & 0 \\ b_2 & -a_2 & -b_1 & a_1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2b_1b_2 - a_1b_2^2 & a_1a_2b_2 - b_1a_2^2 & a_1b_1b_2 - a_2b_1^2 & a_1a_2b_1 - b_2a_1^2 & 0 & 0 & 0 & (a_1b_2 - a_2b_1)^2 \end{bmatrix}$$

and $F_2 =$

$$\begin{bmatrix} -(a_2^2+b_2^2) & 0 & -2a_1a_2 & -2a_1b_2 & b_2 & 0 & a_2 & 0 & a_1(a_2^2+b_2^2) \\ 0 & -(a_2^2+b_2^2) & -2a_2b_1 & -2b_1b_2 & 0 & a_2 & 0 & b_2 & b_1(a_2^2+b_2^2) \\ -2a_1a_2 & -2a_2b_1 & -(a_1^2+b_1^2) & 0 & 0 & b_1 & a_1 & 0 & a_2(a_1^2+b_1^2) \\ -2a_1b_2 & -2b_1b_2 & 0 & -(a_1^2+b_1^2) & a_1 & 0 & 0 & b_1 & b_2(a_1^2+b_1^2) \\ b_2 & 0 & 0 & a_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_2 & 0 & a_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & b_2 & 0 & b_1 & 0 & 0 & 0 & 1 & 0 \\ a_1(a_2^2+b_2^2) & b_1(a_2^2+b_2^2) & a_2(a_1^2+b_1^2) & b_2(a_1^2+b_1^2) & 0 & 0 & 0 & 0 & -(a_1^2+b_1^2)(a_2^2+b_2^2) \end{bmatrix}.$$

Moreover,

$$Q_1 = \begin{bmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 6} & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{5\times 4} & \mathbf{0}_{5\times 4} & \mathbf{0}_{5\times 9} \\ \mathbf{0}_{1\times 7} & -\mu_1^2 & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{9\times 4} & \mathbf{0}_{9\times 4} & \mathbf{0}_{9\times 9} \end{bmatrix}, Q_2 = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 4} & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{2\times 2} & \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 4} & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 2} & \mathbf{0}_{3\times 4} & \mathbf{0}_{3\times 9} \\ \mathbf{0}_{1\times 2} & \mathbf{0}_{1\times 5} & -\mu_2^2 & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 5} & \mathbf{0}_{9\times 1} & \mathbf{0}_{9\times 9} \end{bmatrix},$$

$$M_1 = \begin{bmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 5} & [e_1^t - p^t] & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{4\times 2} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 2} & \mathbf{0}_{4\times 9} \\ \mathbf{0}_{1\times 6} & -1 & 0 & \mathbf{0}_{1\times 9} \\ [e_1^t - p^t]' & \mathbf{0}_{1\times 5} & \|e_1^t - p^t\|^2 & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 11} \end{bmatrix},$$

$$M_0 = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{2\times 2} & \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 3} & [e_2^t - p^t] & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 9} \\ \mathbf{0}_{1\times 2} & \mathbf{0}_{1\times 4} & -1 & 0 & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{1\times 2} & [e_2^t - p^t]' & \mathbf{0}_{1\times 3} & \|e_2^t - p^t\|^2 & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 2} & \mathbf{0}_{9\times 9} \end{bmatrix}.$$

We now define the matrices $L_g \in \mathbb{R}^{17\times 17}, \forall g \in \{1,\ldots,10\}$. Let $L_g(k,l)$ denote an element at the k^{th} row and the l^{th} column of the matrix $L_g, g \in \{1,\ldots,10\}$. Then,

$$L_1(k,l) = \begin{cases} 0.5, \text{ if } k=1, l=4, \\ 0.5, \text{ if } k=4, l=1, \\ -0.5, \text{ if } k=5, l=8, \\ -0.5, \text{ if } k=8, l=5, \\ 0 \text{ otherwise} \end{cases}, L_2(k,l) = \begin{cases} 0.5, \text{ if } k=2, l=3, \\ 0.5, \text{ if } k=3, l=2, \\ -0.5, \text{ if } k=6, l=8, \\ -0.5, \text{ if } k=8, l=6, \\ 0 \text{ otherwise} \end{cases},$$

$$L_3(k,l) = \begin{cases} 0.5, \text{ if } k=1, l=17, \\ 0.5, \text{ if } k=17, l=1, \\ -0.5, \text{ if } k=9, l=8, \\ -0.5, \text{ if } k=8, l=9, \\ 0 \text{ otherwise} \end{cases}, L_4(k,l) = \begin{cases} 0.5, \text{ if } k=2, l=17, \\ 0.5, \text{ if } k=17, l=2, \\ -0.5, \text{ if } k=10, l=8, \\ -0.5, \text{ if } k=8, l=10, \\ 0 \text{ otherwise} \end{cases},$$

$$L_5(k,l) = \begin{cases} 0.5, \text{ if } k = 3, l = 17, \\ 0.5, \text{ if } k = 17, l = 3, \\ -0.5, \text{ if } k = 11, l = 8, \\ -0.5, \text{ if } k = 8, l = 11, \\ 0 \text{ otherwise} \end{cases}, L_6(k,l) = \begin{cases} 0.5, \text{ if } k = 4, l = 17, \\ 0.5, \text{ if } k = 17, l = 4, \\ -0.5, \text{ if } k = 12, l = 8, \\ -0.5, \text{ if } k = 8, l = 12, \\ 0 \text{ otherwise} \end{cases},$$

$$L_7(k,l) = \begin{cases} 0.5, \text{ if } k = 5, l = 17, \\ 0.5, \text{ if } k = 17, l = 5, \\ -0.5, \text{ if } k = 13, l = 8, \\ -0.5, \text{ if } k = 8, l = 13, \\ 0 \text{ otherwise} \end{cases}, L_8(k,l) = \begin{cases} 0.5, \text{ if } k = 6, l = 17, \\ 0.5, \text{ if } k = 17, l = 6, \\ -0.5, \text{ if } k = 14, l = 8, \\ -0.5, \text{ if } k = 8, l = 14, \\ 0 \text{ otherwise} \end{cases},$$

$$L_9(k,l) = \begin{cases} 0.5, \text{ if } k = 3, l = 9, \\ 0.5, \text{ if } k = 9, l = 3, \\ -0.5, \text{ if } k = 15, l = 8, \\ -0.5, \text{ if } k = 8, l = 15, \\ 0 \text{ otherwise} \end{cases}, L_{10}(k,l) = \begin{cases} 0.5, \text{ if } k = 4, l = 10, \\ 0.5, \text{ if } k = 10, l = 4, \\ -0.5, \text{ if } k = 16, l = 8, \\ -0.5, \text{ if } k = 8, l = 16, \\ 0 \text{ otherwise} \end{cases},$$