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Failed power domination in grids, cylinders, and tori

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The power domination number of a graph is the minimum number of vertices required to monitor the graph. Here, the notion of monitoring is given by a set of rules for power system monitoring where vertices model phasor management units (PMU) in a power network. We consider the failed power domination number of a graph G, $\gamma_{\mathrm{fp}}(G)$, a recently introduced graph parameter. Any set of vertices of G whose cardinality is greater than $\gamma_{\mathrm{fp}}(G)$ will dominate the graph. The failed power domination number also allows one to consider PMU (or node) failure. Indeed, any set of $\gamma_{\mathrm{fp}}(G)+i+1$ vertices will monitor the network even in the presence of i node failures. We establish the failed power domination number for products of paths and cycles including square grids, tori, and hypercubes and provide bounds for the failed power domination number of square cylinders.

1. Introduction

The power domination number $\gamma_p(G)$ of a graph G is the smallest number of vertices that can monitor the graph. Introduced by Haynes, Hedetniemi, Hedetniemi, and Henning [Haynes et al. 2002], power domination is now a well-studied concept and one of many vertex domination problems. Here, vertices in a power dominating set correspond to phasor management units (PMUs) in a power network. The power domination number is motivated by applications where PMUs are expensive, so minimizing the number needed is important for cost savings. In general, placement of the PMUs is crucial, meaning that not every collection of $\gamma_p(G)$ vertices will be power dominating. Thus, clever placement of $\gamma_p(G)$ the PMUs may be required to monitor the network. While the setting described here is in terms of power networks, the concept applies more broadly to any system in which nodes are selected to monitor the network in such a way that their neighbors are monitored

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and a propagation rule allows for each of these nodes to monitor a neighbor precisely when it has exactly one presently unmonitored neighbor.

This notion is formalized as follows. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The closed neighborhood of a set of vertices $S \subseteq V$, $N[S]$, is the set S together with all vertices adjacent to vertices in S. Given $S \subseteq V(G)$, the set of monitored vertices at the i-th step of propagation, $P^i(S)$, is defined as follows for $i \geq 0$:

(1) $P^0(S) = N[S]$.

(2) $P^{i+1}(S) = P^i(S) \cup \{w : \{w\} = N[v] \setminus P^i(S) \text{ for some } v \in P^i(S)\}$.

Let $P^\infty(S) := \bigcup_{i=0}^{\infty} P^i(S)$. The set S is said to be a power dominating set if and only if $P^\infty(S) = V(G)$, and the power domination number of G is

$$\gamma_p(G) := \min\{|S| : S \text{ is a power dominating set of } G\}.$$

In this paper, we consider how large a set of vertices can be that still fails to monitor the network. The failed power domination number $\gamma_{\text{fp}}(G)$ of a graph G was introduced by Glasser, Jacob, Lederman, and Radziszowski to capture this quantity. In [Glasser et al. 2020], they prove that the computation of $\gamma_{\text{fp}}(G)$ is NP-hard, consider graphs which realize the extreme values of $\gamma_{\text{fp}}(G)$, and determine $\gamma_{\text{fp}}(G)$ for complete bipartite graphs $G = K_{m,n}$, ladder graphs $G = K_m \square P_2$, and products $G = K_m \square P_n$.

Understanding the failed power domination number of a family of graphs is important, because any set of vertices whose cardinality is greater than γ_{fp} will dominate the graph regardless of which vertices are selected. The failed power domination number also allows a shift of focus from minimizing cost to preventing network failure. It allows the guarantee of network monitoring even in situations where nodes may go offline, so to speak, and are no longer able to provide monitoring. The failed power domination number of a graph is relevant to PMU placement because this number determines the minimum number of PMUs that are needed to monitor the network regardless of placement, as well as the minimum number required to guarantee that the network is monitored even if some nodes fail to provide monitoring.

This paper is organized as follows. Section 2 provides the necessary background and notation to be used throughout this work. Section 3 contains results on the failed power domination number of grids, cylinders, and tori, and those for hypercubes are found in Section 4. The paper concludes with a summary and open problems in Section 5.

2. Preliminaries

This section includes background material on failed domination as well as notation that will be used in the paper.

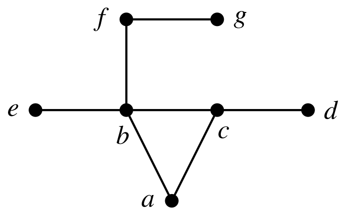

Figure 1. A graph G with $\gamma_{\rm fp}(G) = 3$ in which $\{a, c, d\}$ is a failed power dominating set, while $\{a, bc\}$ is not.

Definition 1. A set $S \subseteq V(G)$ is a failed power dominating set of a graph G if $P^\infty(S) \neq V(G)$. The failed power domination number of G, $\gamma_{\rm fp}(G)$, is the maximum cardinality achieved by a set of vertices which is not a power dominating set of G, meaning

$$\gamma_{\rm fp}(G) := \max\{|S| : S \text{ is not a power dominating set of } G\}.$$

We say that a vertex $v \in V(G)$ is power dominated by S if and only if $v \in P^\infty(S)$; if S is clear from the context, we simply say that v is power dominated. A set is said to be stalled if $P^0(S) = P^\infty(S)$. We use the terms monitor and power dominate interchangeably. These ideas are illustrated in the next example.

Example 2. Let G be the graph in Figure 1. Consider $S = \{a\}$. Then $P^0(S) = \{a, b, c\}$, being the closed neighborhood of a. Then d is power dominated by $\{c\}$, and $P^1(S) = \{a, b, c, d\} = P^\infty(S) = V(G) \setminus \{e, f, g\}$. Thus, S is a failed power dominating set since $P^\infty(S) \neq V(G)$. On the other hand, $T = \{b\}$ is a power dominating set as $P^0(T) = \{a, b, c, e, f\}$ and $P^1(T) = V(G)$ as $N[c] \setminus P^0(T) = \{d\}$ and $N[f] \setminus P^0(T) = \{g\}$.

We claim that $\gamma_{\rm fp}(G) = 3$. Notice that $W := \{a, c, d\}$ is a failed power dominating set of G. Indeed, $P^0(W) = \{a, b, c, d\} = P^\infty(W) \neq V(G)$ as $|N[b] \setminus P^0(W)| = 2$. Thus, $\gamma_{\rm fp}(G) \geq 3$.

To verify that $\gamma_{\rm fp}(G) \leq 3$, consider $U \subseteq V(G)$ that is a failed power dominating set. Notice that $b \notin U$; otherwise, $\{b\}$ being a power dominating set would force U to also be power dominating. If $b \notin N[U]$, then $|U| \leq 2$. It remains to consider the situation in which $b \in N[U]$, meaning at least one of the vertices a, c, e, f is in U. If $a \in U$, then $e, f, g \notin U$, as otherwise U is a power dominating set. Thus, $U \subseteq \{a, c, d\}$. Suppose that $a \notin U$. If $c \in U$, then $g \notin U$ (otherwise U is a power dominating set); we may also rule out the case where $e, f \in U$, as then U would be a power dominating set. Hence, $U \subseteq \{c, d, e\}$ or $U \subseteq \{c, d, f\}$. Suppose now that $a, c \notin U$. If $e \in U$, then either $U \subseteq \{d, e\}$ or $U \subseteq \{e, f, g\}$, as it is not possible that $d \in U$ along with $f \in U$ or $g \in U$. In the remaining case, $U \subseteq \{d, f, g\}$. We conclude that $|U| \leq 3$, proving that $\gamma_{\rm fp}(G) = 3$.

Not every set of vertices with cardinality 3 is a failed power dominating set, as $\{a, b, c\}$ illustrates. However, every set of four vertices is power dominating.

Because the failed power domination number is the maximum number of vertices that fail to form a power dominating set, any set of $\gamma_{\text{fp}}(G)+1$ vertices is a power dominating set of G, meaning any set of $\gamma_{\text{fp}}(G)+1$ vertices can monitor G. Hence:

- Any set of $\gamma_{\text{fp}}(G)+1$ PMUs can monitor the network.
- Any set of $\gamma_{\text{fp}}(G)+2$ PMUs can monitor the network even in the event of a single PMU failure.
- Any set of $\gamma_{\text{fp}}(G)+i+1$ PMUs can monitor the network even in the event of i node failures.

This can be seen in Example 2. Any set of four vertices of G can monitor the network, and the largest failed power dominating set has size 3. Moreover, given any collection of five vertices of G, omitting any one of them leaves four vertices which are guaranteed to be a power dominating set as $\gamma_{\text{fp}}(G)=3$. This allows for a single vertex to fail to provide monitoring and yet the entire network is still monitored.

Notation. The set of nonnegative integers is denoted by $\mathbb{N}$, and the set of positive integers is written as $\mathbb{Z}^+$. Given $n \in \mathbb{Z}^+$, $[n] := \{1, 2, \dots, n\}$. All graphs considered in this paper are simple graphs. Given a graph G and vertices $u, v \in V(G)$, uv denotes the edge incident with u and v. For $n \in \mathbb{Z}^+$, the path on n vertices is denoted by P_n, and the cycle on n vertices is denoted by C_n. Given two graphs G and H, $G \square H$ denotes their Cartesian (or box) product, meaning $V(G\square H)=V(G)\times V(H)$ and $(u, v)(u', v') \in E(G\square H)$ if and only if $u=u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v=v'$.

3. Grids, cylinders, and tori

In this section, we consider the failed power domination numbers of grids, cylinders, and tori. These graphs have a common vertex set

$$V_{mn}=[m]\times[n],$$

where $m, n \in \mathbb{Z}^+$, and we will see that they share a key property with respect to failed power domination.

Recall that the $m\times n$ grid $G_{m\times n}$, cylinder $C_{m\times n}$, and torus $T_{m\times n}$ have vertex set V_{mn} with edge sets as follows:

$$\begin{aligned}
E(G_{m\times n}) &= \{(i, j)(i, j+1) : i \in [m],\, j \in [n-1]\} \\
&\qquad\qquad \cup \{(i, j)(i+1, j) : i \in [m-1],\, j \in [n]\}, \\
E(C_{m\times n}) &= E(G_{m\times n}) \cup \{(1, j)(m, j) : j \in [n]\}, \\
E(T_{m\times n}) &= E(C_{m\times n}) \cup \{(i, 1)(i, n) : i \in [m]\}.
\end{aligned}$$

Alternatively, we may consider these graphs as products: $G_{m\times n} = P_m \square P_n$, $C_{m\times n} = C_m \square P_n$, and $T_{m\times n} = C_m \square C_n$. In such a graph G we set the notation

$$\mathrm{Row}_r(G) := \{r\} \times [n],$$
$$\mathrm{Col}_c(G) := [m] \times \{c\}$$

for $r \in [m]$ and $c \in [n]$ and refer to these sets of vertices as a row and column respectively. For convenience, we set $\mathrm{Row}_r(G_{m\times n}) = \mathrm{Col}_c(G_{m\times n}) = \mathrm{Col}_c(C_{m\times n}) = \varnothing$ for $r \notin [m]$ and $c \notin [n]$. Moreover, for $\mathrm{Row}_r(C_{m\times n})$ and $\mathrm{Row}_r(T_{m\times n})$, we consider the indices as $r \pmod m$; similarly, we take the index to be $c \pmod n$ in $\mathrm{Col}_c(C_{m\times n})$. In addition, we use the adjective square to refer to grids, cylinders, and tori where $m = n$.

Lemma 3. *Consider a graph G which is a grid, cylinder, or torus. If S is a failed power dominating set of G, then for all $i \in [m]$ and all $j \in [n]$ we have* $\mathrm{Row}_i(G), \mathrm{Col}_j(G) \nsubseteq S$. *Moreover, for all $t \in \mathbb{N}$ there exists $i \in [m-1]$ such that* $\mathrm{Row}_i(G), \mathrm{Row}_{i+1}(G) \nsubseteq P^t(S)$. *Similarly, for all $t \in \mathbb{N}$ there exists $j \in [n-1]$ such that* $\mathrm{Col}_j(G), \mathrm{Col}_{j+1}(G) \nsubseteq P^t(S)$.

Proof. Let G be any $m \times n$ grid, cylinder, or torus with vertex set $V_{mn} = [m] \times [n]$ and failed power dominating set S.

Suppose there exists $r \in [m]$ such that $\mathrm{Row}_r(G) \subseteq P^t(S)$. Consider the smallest such t. If $t = 0$, then $\mathrm{Row}_{r-1}(G) \cup \mathrm{Row}_{r+1}(G) \subseteq P^1(S)$. Then $\mathrm{Row}_{r+2}(G) \cup \mathrm{Row}_{r-2}(G) \subseteq P^2(S)$. This pattern continues until $P^\infty(S) = P^k(S) = V(G)$ for some $k \in \mathbb{Z}^+$. Hence, $t > 0$, demonstrating that $\mathrm{Row}_r(G) \subseteq P^0(S)$. A similar argument applies in the situation where $\mathrm{Col}_c(G) \subseteq P^t(S)$ for some $c \in [n]$ and $t \in \mathbb{N}$.

Suppose $\mathrm{Row}_r(G), \mathrm{Row}_{r+1}(G) \subseteq P^t(S)$ for some $t \in \mathbb{N}$ and $r \in [m-1]$. Then in $P^{t-1}(S)$, there is a vertex whose only neighbor outside of $P^{t-1}(S)$ is (r, j) for some $j \in [n]$. Moreover, every vertex in $\mathrm{Row}_r(G)$ has at most one neighbor outside of $P^t(S)$. Thus, $\mathrm{Row}_{r-1}(G) \cup \mathrm{Row}_{r+1}(G) \subseteq P^{t+1}(S)$. This will continue so that there exists $k \in \mathbb{N}$ such that $P^k(S) = P^\infty(S) = V(G)$. A similar argument applies in the situation where $\mathrm{Col}_c(G), \mathrm{Col}_{c+1}(G) \subseteq P^t(S)$ for some $c \in [n]$ and $t \in \mathbb{N}$. □

According to Lemma 3, a failed power dominating set of a grid, cylinder, or torus cannot contain a set that power dominates two consecutive rows or columns of vertices of G. This will be useful in determining the failed power domination number for these families of graphs.

Theorem 4. *The failed power domination number of the square grid $G_{n\times n}$ with $n \geq 2$ is*

$$\gamma_{\mathrm{fp}}(G_{n\times n}) = (n-1)(n-2).$$

Proof. We begin by showing that $\gamma_{\mathrm{fp}} \geq (n-1)(n-2)$. Let

$$S = \{(i, j) : i \geq j+2 \text{ or } j \geq i+2\},$$

so that S includes all vertices except those on the main diagonal and the two diagonals next to it. Then $|S| = n^2 - n - 2(n-1) = n^2 - 3n + 2 = (n-1)(n-2)$. We claim that S is not a power dominating set.

Suppose that $(i, j) \in S$. Then the neighbors of (i, j) are $\{(i+1, j), (i-1, j), (i, j+1), (i, j-1)\}$, and either $i \geq j+2$ or $j \geq i+2$. In the case $i \geq j+2$, we have $(i+1, j) \in S$ and $(i, i-1) \in S$. Additionally, $(i, j+1), (i-1, j) \in S$ if and only if $i \neq j+2$. In the case $j \geq i+2$, we have $(i-1, j) \in S$ and $(i, j+1) \in S$. Additionally, $(i, j-1), (i+1, j) \in S$ if and only if $j \neq i+2$. This indicates that there exist neighbors of (i, j) that are not in S only if $i = j+2$ or $j = i+2$. This leads to the conclusion that

$$P^0(S) = S \cup \{(i, j) : i = j+1 \text{ or } j = i+1\}.$$

For an arbitrary $(i, j) \in P^0(S) \backslash S$, either $i = j+1$ or $j = i+1$. It follows that if $(i, j) \in P^0(S) \backslash S$, then (i, j) has exactly two neighbors in $V \backslash P^0(S)$, namely (i, i) and (j, j). This implies that no additional vertices in $V(G_n)$ can be power dominated, so $P^\infty(S) = P^0(S)$, which means that this is a stalled set. Note that $(1, 1) \in P^\infty(S)$, which confirms that S is a failed power dominating set. Hence, $\gamma_{\text{fp}}(G_n) \geq (n-1)(n-2)$.

Next, we will show that $\gamma_{\text{fp}} < (n-1)(n-2)+1$. Consider $S \subseteq V(G_n)$ such that $|S| \geq n^2 - 3n + 3$. Note that $|V(G_n) \setminus S| \leq 3n - 3$, so there are at least three rows and columns with more than $n-3$ vertices in S. According to Lemma 3, if $|\text{Row}_r(G) \cap S| \leq n-1$ or $|\text{Col}_c(G) \cap S| \leq n-1$ for some $r \in [m]$ or $c \in [n]$, then S is a power dominating set.

Suppose each row and column of G has no more than $n-2$ vertices in S. Then there must be three rows and three columns with $n-2$ vertices in S while the rest have at least $n-3$ vertices in S. If $|\text{Row}_r(G) \cap S| = n-2$, then $\text{Row}_r(G) \cap (V(G) \setminus S) = \{(r, 1), (r, 2)\}$ or $\text{Row}_r(G) \cap (V(G) \setminus S) = \{(r, n-1), (r, n)\}$, as otherwise $\text{Row}_r(G) \subseteq P^0(S)$ as each vertex in $\text{Row}_r(G) \cap S$ would have a neighbor on that same row which is an element of S. Therefore, the vertices $(1, 2)$, $(2, 1)$, $(2, 2)$ are in $P^0(S)$, which means that $\text{Col}_2(G) \subseteq P^0(S)$. It follows that the vertex $(1, 1)$ is in $P^1(S)$. Consequently, $\text{Col}_1(G) \subseteq P^1(S)$ and $\text{Col}_2(G) \subseteq P^1(S)$. By Lemma 3, S is a power dominating set. Thus, $\gamma_{\text{fp}}(G) < n^2 - 3n + 3$. It follows that $\gamma_{\text{fp}}(G) = (n-1)(n-2)$. □

Example 5. Consider the 10×10 grid $G_{10\times 10}$ as in Figure 2. Elements of $S = \{(i, j) \in [10] \times [10] : i \geq j+2 \text{ or } j \geq i+2\}$ are the gray vertices, and the black vertices are the neighbors of S. Hence, the set of the gray and black vertices is precisely $P^0(S)$. Notice that, for all $v \in P^0(S)$, we have $|N[v] \setminus P^0(S)| \in \{0, 2\}$. As a consequence, $P^0(S) = P^\infty(S) = [10] \times [10] \setminus \{(i, i) : i \in [10]\}$ and the white vertices are never power dominated.

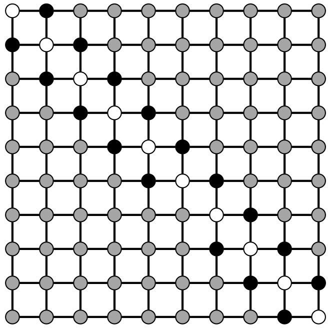

Figure 2. The 10×10 grid $G_{10\times 10}$ along with a maximal failed power dominating set $S = \{(i, j) \in [10]\times[10] : i \geq j+2 \text{ or } j \geq i+2\}$.

One might wish to compare the power domination and failed power domination number for square grids. According to [Dorfling and Henning 2006],

$$\gamma_p(G_{n\times n}) = \begin{cases} \lceil (n+1)/4 \rceil & \text{if } n \equiv 4 \pmod 8, \\ \lceil n/4 \rceil & \text{otherwise.} \end{cases}$$

While smart choice of $\lceil (n+1)/4 \rceil$ vertices yields a power dominating set of $G_{n\times n}$, there are subsets of vertices of cardinalities up to $(n-1)(n-2)$ which fail to be power dominating sets. Taking $n = 10$ as in Example 2 reveals $\gamma_p(G_{10\times n}) = 3$, whereas $\gamma_{\text{fp}}(G_{10\times n}) = 72$, indicating that a clever choice of three vertices provides monitoring of the network, but choosing vertices at random, up to 72 of them, may result in a set of vertices which fail to monitor the network.

Next, we consider failed power domination for cylinders.

Proposition 6. *The failed power domination number of the square cylinder $C_{n\times n}$ with $n \geq 3$ satisfies*

$$\gamma_{\text{fp}}(C_{n\times n}) \geq n^2 - 6n + 10.$$

Proof. We consider the two cases depending on the parity of n and provide a failed power dominating set in each case.

First, consider the case where $n \geq 4$ is an even integer. Let

$$S' = \left\{ (i, j) : \begin{array}{ll} \frac{n}{2} - i + 3 \leq j \leq \frac{n}{2} + i - 1 & \text{if } 2 \leq i \leq \frac{n}{2}, \\ i - \frac{n}{2} + 1 \leq j \leq \frac{3n}{2} - i - 1 & \text{if } \frac{n}{2} + 1 \leq i \leq n-1 \end{array} \right\},$$

$$C = \left\{ (i, j) : \begin{array}{ll} j \leq \frac{n-2-2i}{2} \text{ or } j \geq \frac{n+6+2i}{2} & \text{if } 1 \leq i \leq \frac{n-4}{2}, \\ j \leq \frac{-n-6+2i}{2} \text{ or } j \geq \frac{3n+6-2i}{2} & \text{if } \frac{n+6}{2} \leq i \leq n \end{array} \right\}.$$

For $n = 4$, let $S = S'$; when $n \geq 6$, let $S = S' \cup C$. Clearly, $S \subseteq V(C_{n\times n})$. Notice that $|S| = n^2 - 6n + 10$.

We claim that S is a failed power dominating set. To see this, note that the neighbors of C that are not elements of S are elements of

$$C_1 = \left\{ (i,j) : \begin{array}{ll} j = \frac{n-2i}{2} \text{ or } j = \frac{n+4+2i}{2} & \text{if } 1 \le i \le \frac{n-4}{2}, \\ j = \frac{-n-4+2i}{2} \text{ or } j = \frac{3n+4-2i}{2} & \text{if } \frac{n+6}{2} \le i \le n \end{array} \right\},$$

and the neighbors of S' that are not elements of S are elements of

$$S'_1 = \left\{ (i,j) : \begin{array}{ll} j = \frac{n-2i+4}{2} \text{ or } j = \frac{n+2i}{2} & \text{if } 1 \le i \le \frac{n-4}{2}, \\ j = \frac{-n+2i}{2} \text{ or } j = \frac{3n-2i}{2} & \text{if } \frac{n+6}{2} \le i \le n \end{array} \right\}.$$

Hence, $P^0(S) = S \cup C_1 \cup S'_1$. It follows that for $(i,j) \in P^0(S) \backslash S$, either $(i,j) \in C_1$ or $(i,j) \in S'_1$. In either case, every such vertex will have two neighbors that are not elements of $P^0(S)$. Therefore, $P^\infty(S) = P^0(S)$.

One may see that $(1, n/2) \in V(C_{n\times n}) \setminus P^\infty(S)$, as its neighbors are $(1, n/2-1)$, $(1, n/2+1)$, $(2, n/2)$, none of which are elements of S. Indeed, the elements of S on $\text{Row}_1(C_{n\times n})$ are $(1, j)$ with $j \le n/2-2$ or $j \ge n/2+4$, and those on $\text{Row}_1(C_{n\times n})$ are $(2, j)$ with $j \le n/2-3$ or $j \ge n/2+5$. This confirms that S is a failed power dominating set, so $\gamma_{\text{fp}}(C_{n\times n}) \ge n^2 - 6n + 10$ for even $n \ge 4$.

Next, consider the case where $n \ge 3$ is an odd integer. Let

$$S' = \left\{ (i,j) : \text{for } i = \frac{n+1}{2} \pm k,\ 2+k \le j \le n-1-k \text{ for } k \in \{0\} \cup \left[\left\lceil \frac{n-3}{2} \right\rceil \right] \right\},$$

$$C = \left\{ (i,j) : \begin{array}{ll} j \le \frac{n-3-2i}{2} \text{ or } j \ge \frac{n+5+2i}{2} & \text{if } 1 \le i \le \frac{n-5}{2}, \\ j \le \frac{-n-5+2i}{2} \text{ or } j \ge \frac{3n+7-2i}{2} & \text{if } \frac{n+7}{2} \le i \le n \end{array} \right\}.$$

For $n \le 5$, let $S = S'$; when $n \ge 7$, let $S = S' \cup C$. Notice that $|S| = n^2 - 6n + 10$. The neighbors of C that are not elements of S are elements of

$$C_1 = \left\{ (i,j) : \begin{array}{ll} j = \frac{n-1-2i}{2} \text{ or } j = \frac{n+7+2i}{2} & \text{if } 1 \le i \le \frac{n-5}{2}, \\ j = \frac{-n-7+2i}{2} \text{ or } j = \frac{3n+7-2i}{2} & \text{if } \frac{n+5}{2} \le i \le n \end{array} \right\},$$

and the neighbors of S' that are not elements of S are elements of

$$S'_1 = \left\{ (i,j) : \begin{array}{ll} j = \frac{n+1 \pm 2i}{2} & \text{if } 1 \le i \le \frac{n+1}{2}, \\ j = \frac{(n+1) \pm (2n-2i)}{2} & \text{if } \frac{n+1}{2} \le i \le n \end{array} \right\}.$$

Hence, $P^0(S) = S \cup C_1 \cup S'_1$. It follows that for $(i,j) \in P^0(S) \backslash S$, either $(i,j) \in C_1$ or $(i,j) \in S'_1$. In either case, every such vertex will have two neighbors that are not in $P^0(S)$. Therefore $P^\infty(S) = P^0(S)$.

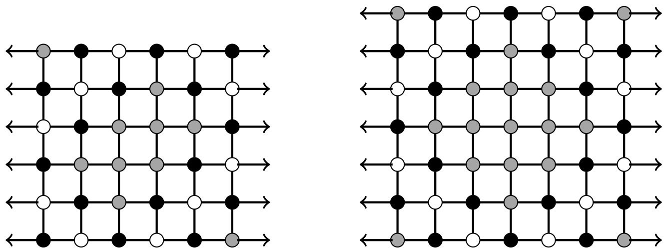

Figure 3. The 6×6 cylinder $C_{6\times6}$ on the left and the 7×7 cylinder $C_{7\times7}$ on the right with maximal failed power dominating sets highlighted in gray.

One may see that $(1, (n-1)/2) \in V(C_{n\times n}) \setminus P^{\infty}(S)$, as its neighbors are $(1, (n-1)/2-1)$, $(1, (n-1)/2+1)$, and $(2, (n-1)/2)$, none of which are elements of S. It follows that S is a failed power dominating set, so $\gamma_{\text{fp}}(C_n) \geq n^2 - 6n + 10$ for odd $n \geq 3$. □

Example 7. Consider the 6×6 cylinder $C_{6\times6}$ as in the left-hand side of Figure 3. Elements of $S = S' \cup C$ are the gray vertices, and the black vertices are the neighbors of elements of S. Hence, the set of gray and black vertices is precisely $P^0(S)$. It is easy to see that $P^{\infty}(S) = P^0(S)$. Indeed, for $v \in P^0(S)$, we have $|N[v] \setminus P^0(S)| \neq 1$. We can also confirm that $(1, 3) \notin P^{\infty}(S)$. Similarly, the odd case is illustrated by the 7×7 cylinder $C_{7\times7}$ in the right-hand side of Figure 3.

While one can verify that $\gamma_{\text{fp}}(C_n) \leq n(n-3)$, it remains an open question to determine a tight upper bound for the failed domination number of a cylinder.

It is interesting to compare the power domination number and the given bound on the failed power domination number for cylinders. According to [Koh and Soh 2019] (see also [Barrera and Ferrero 2011]), the power domination number of the $n \times n$ cylinder $C_{n\times n}$ with $n \geq 3$ is

$$\gamma_p(C_{n\times n}) = \begin{cases} \lceil (n+1)/4 \rceil & \text{if } n \equiv 4 \pmod 8, \\ \lceil n/2 \rceil & \text{otherwise,} \end{cases}$$

whereas Proposition 6 shows $\gamma_{\text{fp}}(C_{n\times n}) \geq n^2 - 6n + 10$. Taking $n = 6$ (resp. $n = 7$) as in Example 7 yields $\gamma_p(C_{6\times6}) = 3$ (resp. $\gamma_p(C_{7\times7}) = 4$), while $\gamma_{\text{fp}}(C_{6\times6}) \geq 10$ (resp. $\gamma_{\text{fp}}(C_{7\times7}) \geq 17$).

Next, we consider failed power domination for tori.

Theorem 8. *The failed power dominating number of the square torus $T_{n\times n}$ with $n \geq 4$ is*

$$\gamma_{\text{fp}}(T_{n\times n}) = n(n-3).$$

Proof. We begin by showing that $\gamma_{\text{fp}}(T_{n\times n}) \geq n(n-3)$. Let

$S = \{(i, j) \in [n] \times [n] : j \neq i-1 \pmod n,\ j \neq i-2 \pmod n,\ j \neq i-3 \pmod n\}$.

Note that in each row of $T_{n\times n}$, there are three consecutive vertices, meaning they form a P_3, which are not elements of S, and $|S| = n(n-3)$.

One can verify that

$$P^0(S)\backslash S = \{(i, j) : j = i-3 \pmod n \text{ or } j = i-1 \pmod n\}$$

and $P^0(S) = \{(i, j) : j \neq i-2 \pmod n\}$; therefore,

$$V(T_{n\times n})\backslash P^0(S) = \{(i, j) : j = i-2 \pmod n\}.$$

Let $(i, j) \in P^0(S)\backslash S$. Notice that (i, j) has exactly two neighbors in the set $V(T_{n\times n})\backslash P^0(S)$, namely $(i, (j+1) \pmod n)$ and $((i-1) \pmod n, j)$. This implies that $P^\infty(S) = P^0(S)$, which means that this is a stalled set. Note that $(1, n-1) \in V(T_{n\times n}) \setminus P^\infty(S)$. Thus, S is a failed power dominating set, and $\gamma_{\text{fp}}(T_{n\times n}) \geq n(n-3)$.

Next, we will show that $\gamma_{\text{fp}} < n(n-3)+1$. Consider $S \subseteq V(T_{n\times n})$ such that $|S| = n^2 - 3n + 1 = n(n-3)+1$. Suppose that S is not a power dominating set. According to Lemma 3, for all $i \in [n]$ we have $\text{Row}_i(T_{n\times n}) \nsubseteq S$ and for all $t \in \mathbb{N}$ and $i \in [n-1]$ we have $\text{Row}_i(T_{n\times n}), \text{Row}_{i+1}(T_{n\times n}) \nsubseteq P^t(S)$. Thus, there exists at least one row $\text{Row}_i\ T_{n\times n}$ such that S contains at least $n-2$ of its vertices. Since every vertex in the row is adjacent to two other vertices in that row, it follows that every vertex in the row is adjacent to a vertex in S; that is, $\text{Row}_i\ T_{n\times n} \subseteq P^0(S)$. In addition, for all $r \in [n]\setminus\{i\}$, we have $|\text{Row}_r(T_{n\times n}) \cap P^0(S)| \leq 1$. Suppose $(i-1, j) \notin P^0(S)$ for some $j \in [n]$. Then $\text{Row}_{i-1}\ T_{n\times n} \cap S = \{i-1\} \times [n] \setminus \{j-1, j, j+1\}$, since $|\text{Row}_{i-1}\ T_{n\times n} \cap S| = n-3$. Thus, $(i-1, j-1), (i-1, j+1) \in P^0(S)$. If $(i-1, j) \in P^0(S)$, then S is a power dominating set by Lemma 3. Similarly, $(i+1, j) \in P^0(S)$. Note that $(i\pm2, n-1) \notin P^0(S)$. Continuing in this fashion, we see that $V(T_{n\times n}) \setminus P^0(S) \subseteq \text{Col}_{j-1}(T_{n\times n}) \cup \text{Col}_j(T_{n\times n})$. An application of Lemma 3 yields $P^\infty(S) = V(T_{n\times n})$, which is a contradiction. As a result, $\gamma_{\text{fp}}(T_{n\times n}) < n(n-3)+1$. We conclude that $\gamma_{\text{fp}}(T_{n\times n}) = n(n-3)$. □

Example 9. Figure 4 shows a 6×6 torus in which the gray vertices are precisely the elements of

$$S = \{(i, j) : j \neq i-1 \pmod n,\ j \neq i-2 \pmod n, \text{ and } j \neq i-3 \pmod n\},$$

and the set of gray and black vertices is precisely $P^0(S)$. The white vertices are never power dominated. Here, we see that up to half of the vertices of $T_{6\times 6}$ may fail to power dominate the network. However, any choice of 19 vertices will provide monitoring of the network.

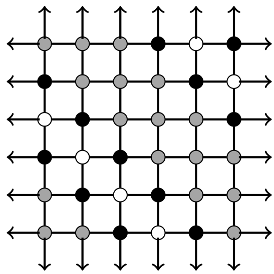

Figure 4. The 6×6 torus $T_{6\times 6}$ along with the maximal failed dominating set $S = \{(i, j) : j \neq i - 1 \pmod 6,\ j \neq i - 2 \pmod 6,$ and $j \neq i - 3 \pmod 6\}$.

It is interesting to compare the power domination and failed power domination numbers for tori. According to [Koh and Soh 2019] (see also [Barrera and Ferrero 2011]), the power domination number of the $n \times n$ cylinder $T_{n\times n}$ with $n \geq 3$ is

$$\gamma_p(T_{n\times n}) = \begin{cases} \lceil (n+1)/2 \rceil & \text{if } n \equiv 2 \pmod 4, \\ \lceil n/2 \rceil & \text{otherwise,} \end{cases}$$

whereas Theorem 8 shows $\gamma_{\text{fp}}(T_{n\times n}) = n(n-3)$. Considering the 6×6 torus as in Example 9, we see that $\gamma_p(T_{n\times n}) = 4$, whereas $\gamma_{\text{fp}}(T_{n\times n}) = 18$. This indicates that a smart choice of vertices allows the network to be monitored with only four vertices, yet there are configurations of up to 18 vertices that fail to do so.

4. Hypercubes

In this section, we turn our attention to failed power domination for hypercubes. Fix $n \in \mathbb{Z}^+$. The n-dimensional hypercube is

$$Q_n := P_2^n = \underbrace{P_2 \square \cdots \square P_2}_{n}.$$

It is convenient to set up some notation to describe Q_n in terms of its vertex set and edge set. Let $\mathbb{F}_2 := \{0, 1\}$. Consider $\mathbb{F}_2^n := \{(u_1, \ldots, u_n) : u_i \in \mathbb{F}_2 \text{ for all } i \in [n]\}$. Given $u, v \in \mathbb{F}_2^n$, the distance between them is

$$d(u, v) := |\{i \in [n] : u_i \neq v_i\}|,$$

and the weight of u is

$$\text{wt}(u) := |\{i \in [n] : u_i \neq 0\}|,$$

meaning the number of nonzero coordinates of u. The vertices of weight 1 are represented by the standard basis vector expressions $e_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, which have a single nonzero entry in the i-th component. Then Q_n has vertex set

$$V(Q_n) = \mathbb{F}_2^n$$

and edge set

$$E(Q_n) = \{uv : d(u, v) = 1\},$$

meaning $u, v \in V(Q_n)$ are adjacent if and only if there exists $= i \in [n]$ such that $u_i \neq v_i$ and $u_j = v_j$ for all $j \in [n] \setminus \{i\}$.

Proposition 10. *For $n \geq 3$, we have $\gamma_{\mathrm{fp}}(Q_n) = 2^n - \binom{n}{2} - n - 1$.*

Proof. First, we will show that $\gamma_{\mathrm{fp}}(Q_n) \geq 2^n - \binom{n}{2} - n - 1$. Let

$$S := \{v \in \mathbb{F}_2^n : \mathrm{wt}(v) \geq 3\}.$$

We claim that S is not a power dominating set for Q_n. To see this, note that

$$P^0(S) = \{v \in \mathbb{F}_2 : \mathrm{wt}(v) \geq 2\}.$$

For any $v \in P^0(S)$, we have $v = e_i + e_j$ for some distinct $i, j \in [n]$. Then $N[v] \setminus P^1(S) = \{e_i, e_j\}$, which implies that $P^2(S) = P^1(S)$. Thus,

$$P^\infty(S) = \{v \in \mathbb{F}_2^n : \mathrm{wt}(v) \geq 2\} \subsetneq V(Q_n),$$

and S is not a power dominating set for Q_n. Consequently,

$$\gamma_{\mathrm{fp}}(Q_n) \geq |S| = 2^n - \binom{n}{2} - n - 1.$$

Next, we will prove that $\gamma_{\mathrm{fp}}(Q_n) < 2^n - \binom{n}{2} - n$. To this end, consider $S \subseteq \mathbb{F}_2^n$ such that $|S| \geq 2^n - \binom{n}{2} - n$. Suppose that S is a failed power dominating set. Without loss of generality, we may assume that 0 is not power dominated. Then none of its neighbors are elements of S; that is, $e_i \notin S$ for all $i \in [n]$. The weight-0 and weight-1 vertices of Q_n account for $n + 1$ vertices of $Q_n \setminus S$. As a result, there are $\binom{n}{2} - 1$ vertices of weight at least 2 which are elements of $Q_n \setminus S$. Let $v \in S$ such that $\mathrm{wt}(v) = 2$. Note that v will power dominate two vertices of weight 1, say e_i and e_j, each of which must have a weight-2 neighbor which is not power dominated. Let e'_i and e'_j denote such a neighbor. Then $e'_i, e'_j \notin S$ and $e'_i \neq e'_j$. Note that the weight-3 neighbors of e'_i are not elements of S, and the same is true for e'_j, and there are $n - 2$ such vertices. Because there can be at most one vertex in common, there will be a total of either $2n - 3$ or $2n - 4$ vertices of weight 3 that are not in S. However, $2n - 3 \geq 2n - 4 > 1$, because $n \geq 3$. Therefore, it must be that there is not a single element of S of weight 2. Notice that the maximum number of vertices of weight 2 that are elements of S is $\lfloor n/2 \rfloor$, because each weight-1 vertex has a neighbor that is not power dominated and each weight-2 vertex only has 2 weight-1 neighbors that it can "protect". Therefore, if S contains k weight-2 vertices, we know that $1 < k \leq \lfloor n/2 \rfloor$. We also know that there are $k - 1$ vertices that are of weight at least 3 that are not elements of S. However, we also know that the number of weight-3 vertices that are not elements S must be at least $2n - 3$. This means that

$2n-3 \le k-1$, which implies that $2n-2 \le k$. This implies that $2n-2 \le k \le \lfloor n/2 \rfloor$, which leads to a contradiction because $2n-2 > \lfloor n/2 \rfloor$ for all $n \ge 3$. □

In [Dean et al. 2011], it is shown that $2^{n-1}/n \le \gamma_p(Q_n) \le 2^{n-\lfloor \log_2 n \rfloor -1}$ and if $n = 2^k$ then $\gamma_p(Q_n) = 2^{n-k-1}$. Determining the exact value of the power domination number of the n-dimensional hypercube when n is not a power of 2 remains open.

5. Conclusion

In this paper, we considered the failed power domination number for certain families of graphs. A closed form expression is provided for the failed power domination number of $n \times n$ grids, tori, and hypercubes, as well as bounds for $n \times n$ cylinders. Determining this exact value remains an open question. It also may be interesting to consider failed power domination for grids, cylinders, and tori which are not square, as well as that of other graph products.

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