Journal of Algebra 634 (2023) 554–562

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Idempotent completions of equivariant matrix factorization categories

Michael K. Brown [a,*], Mark E. Walker [b,1]

[a] *Department of Mathematics and Statistics, Auburn University, Auburn,AL 36849, United States of America*
[b] *Department of Mathematics, University of Nebraska-Lincoln, Lincoln,NE 68588, United States of America*

A R T I C L E   I N F O

Article history:
Received 2 January 2023
Available online 2 August 2023
Communicated by Claudia Polini

MSC:
13J10
13J15
18G35
18G80

Keywords:
Equivariant
Henselian
Idempotent completion
Matrix factorization

A B S T R A C T

We prove that equivariant matrix factorization categories associated to henselian local hypersurface rings are idempotent complete, generalizing a result of Dyckerhoff in the non-equivariant case.

* Corresponding author.
E-mail addresses: mkb0096@auburn.edu (M.K. Brown), mark.walker@unl.edu (M.E. Walker).
[1] The second author was partially supported by National Science Foundation grant DMS-2200732.

https://doi.org/10.1016/j.jalgebra.2023.07.023
0021-8693/

1. Introduction

It follows from a result of Dyckerhoff [3, Lemma 5.6] that matrix factorization categories associated to complete local hypersurface rings are idempotent complete. In this paper, we generalize this result to the equivariant case.

Throughout the paper, we let G be a finite group acting on a noetherian local ring $(Q, \mathfrak{m})$ in such a way that Q is module finite over the invariant subring Q^G, and we assume f is an element of $\mathfrak{m}$ that is fixed by G. (The assumption that $Q^G \hookrightarrow Q$ is module finite holds quite generally; see Remark 2.3.) We write $[\mathrm{mf}_G(Q, f)]$ for the (triangulated) homotopy category of G-equivariant matrix factorizations of f; see §3 for the definition. Our main result is:

Theorem 1.1. *In the setting above: if Q is henselian, then $[\mathrm{mf}_G(Q, f)]$ is idempotent complete.*

Suppose now that Q is regular, $(Q', \mathfrak{m}')$ is another regular local ring with G-action, and $\phi : Q \to Q'$ is a G-equivariant homomorphism of local rings. Setting $f' = \phi(f)$, we have an induced triangulated functor

$$\phi_* : [\mathrm{mf}_G(Q, f)] \to [\mathrm{mf}_G(Q', f')]$$

given by extension of scalars along ϕ. Building from the aforementioned result of Dyckerhoff [3, Lemma 5.6], we also prove:

Proposition 1.2. *Assume Q/f and Q'/f' have isolated singularities, $Q^G \subseteq Q$ and $(Q')^G \subseteq Q'$ are module finite, $|G|$ is a unit in Q, ϕ is flat, and the canonical map $Q' \otimes_Q Q/\mathfrak{m} \to Q'/\mathfrak{m}'$ is an isomorphism. The functor ϕ_* induces an equivalence of triangulated categories*

$$\phi_* : [\mathrm{mf}_G(Q, f)]^\vee \xrightarrow{\cong} [\mathrm{mf}_G(Q', f')]^\vee,$$

where $^\vee$ denotes idempotent completion.

From Theorem 1.1 and Proposition 1.2, we deduce:

Corollary 1.3. *If $|G|$ is a unit in Q, $Q^G \subseteq Q$ is module finite, and the local hypersurface Q/f has an isolated singularity, then the canonical functors*

$$[\mathrm{mf}_G(Q, f)]^\vee \to [\mathrm{mf}_G(Q^h, f)] \to [\mathrm{mf}_G(\widehat{Q}, f)]$$

are equivalences of triangulated categories, where Q^h and $\widehat{Q}$ are the henselization and $\mathfrak{m}$-adic completion of Q, respectively.

As another application, we combine Theorem 1.1 and a result of Spellmann-Young [11] to conclude that $[\mathrm{mf}_G(Q,f)]$ is equivalent to the category of G-equivariant objects in the triangulated category $[\mathrm{mf}(Q,f)]$; see Corollary 5.2 below for the precise (and more general) statement. This gives an analogue of a result of Elagin involving bounded derived categories of equivariant sheaves [5, Theorem 9.6]. This consequence of the statement of Theorem 1.1 was observed by Spellmann-Young [11, Remark 3.7] and provided a main source of motivation for this work.

Acknowledgments. We thank Jack Jeffries and Anurag Singh for suggesting the argument in the proof of Proposition 2.2, and we are grateful to the anonymous referee for many helpful suggestions.

2. Background

2.1. Twisted group rings

Let A be a commutative ring with action of a finite group G.

Definition 2.1. The *twisted group ring*, written $A\#G$, has underlying set given by formal sums $\sum_{g\in G} a_g g$, with $a_g \in A$ for all g, and multiplication determined by the rule

$$ag \cdot bh = ab^g gh$$

for $a, b \in A$ and $g, h \in G$, where b^g is the result of acting by g on b.

The map $A \to A\#G$ sending a to ae_G is a ring homomorphism, but beware that $A\#G$ is only an A-algebra when the action of G on A is trivial, in which case $A\#G$ coincides with the group ring $A[G]$. In general, letting A^G denote the ring of invariants $\{a \in A \mid a^g = a \text{ for all } g \in G\}$, the composition $A^G \hookrightarrow A \to A\#G$ exhibits $A\#G$ as an A^G-algebra.

A left module over $A\#G$ is the same thing as a set M that is equipped with a left A-module structure and a left G-action such that $g(am) = a^g(gm)$ for all $g \in G$, $a \in A$ and $m \in M$.

Suppose A is local with maximal ideal $\mathfrak{m}$. Since G is finite, the inclusion $A^G \hookrightarrow A$ is integral, and it is therefore a consequence of the Going Up Theorem that the invariant ring A^G is also local. We observe also that the henselization A^h and $\mathfrak{m}$-adic completion $\widehat{A}$ inherit canonical G-actions. In more detail, every morphism of local rings $\phi : A \to B$ induces unique morphisms on henselizations $\phi^h : A^h \to B^h$ and completions $\widehat{\phi} : \widehat{A} \to \widehat{B}$ that cause the evident squares to commute. Applying this when $B = A$ and ϕ ranges over the isomorphisms determined by the actions of the group elements of G gives the actions of G on A^h and $\widehat{A}$.

Proposition 2.2. *Assume a finite group G acts on a local ring A in such a way that the extension $A^G \subseteq A$ is module finite. Both of the extensions $(A^h)^G \subseteq A^h$ and $(\widehat{A})^G \subseteq \widehat{A}$ are module finite, $(A^h)^G$ is henselian, and $(\widehat{A})^G$ is complete.*

Remark 2.3. The assumption that the extension $A^G \subseteq A$ is module finite holds in many cases of interest. For instance, if A is a finite type F-algebra for a field F contained in A^G, then, since $A^G \subseteq A$ is integral and finite type, it is module finite. More generally, if A is any equivariant localization of an example of this kind, then $A^G \subseteq A$ is module finite.

Additionally, if A is a noetherian domain, and $|G|$ is invertible in A, then $A^G \subseteq A$ is module finite [9, Proposition 5.4]. More generally, if A is any equivariant quotient of an example of this kind, then $A^G \subseteq A$ is module finite.

Indeed, we know of no examples where $A^G \subseteq A$ fails to be module finite when $|G|$ is invertible in A, but there are examples of such failure when $A = F[[x, y]]$, F is a field of infinite transcendence degree over the field with p elements for a prime p, and G is cyclic of order p; see [7].

Proof of Proposition 2.2. For any module finite extension of local rings $B \subseteq A$, the induced map $B^h \subseteq A^h$ is also a module finite extension, and the canonical map $B^h \otimes_B A \xrightarrow{\cong} A^h$ is an isomorphism [1, Lemma 10.156.1]. Applying this when $B = A^G$, and using that $(A^G)^h \subseteq (A^h)^G \subseteq A^h$, we obtain the first result. Similarly, we have $\widehat{A} \cong \widehat{A^G} \otimes_{A^G} A$ is module finite over $\widehat{A^G}$, and $\widehat{A^G} \subseteq (\widehat{A})^G \subseteq \widehat{A}$, hence $(\widehat{A})^G \subseteq \widehat{A}$ is module finite.

Let us return to the general setting of a module finite extension of local rings $B \subseteq A$. If A is complete, then B is complete. To see this, note that $\widehat{B} \otimes_B A \cong \widehat{A} = A$, and hence $B \subseteq \widehat{B} \subseteq A$, so that $\widehat{B}$ is a module finite extension of B. Thus, $\widehat{B} \otimes_B (\widehat{B}/B) = 0$. Since $\widehat{B}$ is a faithfully flat B-module, it follows that $B = \widehat{B}$. Likewise, if A is henselian, so is B. Indeed, we have $B \subseteq B^h \subseteq A$, since $B^h \subseteq A^h$ and $A = A^h$. Thus, B^h is module finite over B. Taking completions, and using that $\widehat{(B^h)} \cong \widehat{B}$, we get $\widehat{B} \otimes_B B^h/B = 0$, and hence $B = B^h$. □

3. Proof of the main theorem

Let us first recall the notion of an idempotent complete additive category:

Definition 3.1. An additive category A is *idempotent complete* if, given an object X in A and an idempotent endomorphism e of X, there exists an object Y in A and morphisms $\pi: X \to Y$, $\iota: Y \to X$ such that $\pi\iota = \mathrm{id}_Y$ and $\iota\pi = e$. The *idempotent completion of* A is, roughly speaking, the smallest idempotent complete additive category containing A. More precisely: the idempotent completion $A^\vee$ of A is the category with objects given by all pairs (X, e), where X is an object in A and e is an idempotent endomorphism of X. A morphism $(X, e) \to (X', e')$ in $A^\vee$ is a morphism $f: X \to X'$ in A such that $e'f = fe = f$.

Recall that a not-necessarily-commutative ring E is called *nc local* if $E/J(E)$ is a division ring, where $J(E)$ denotes the Jacobson radical of E (i.e., the intersection of all the maximal left ideals of R). An additive category A is called *Krull-Schmidt* if every object is a finite direct sum of objects with nc local endomorphism rings. By a result of Krause [8, Corollary 4.4], every Krull-Schmidt additive category is idempotent complete. We observe that if A is a Krull-Schmidt additive category and B is a quotient of A, by which we mean B has the same objects of A and the hom groups of B are quotients of the hom groups of A, then B is also Krull-Schmidt. In particular, any quotient of a Krull-Schmidt additive category is idempotent complete.

Lemma 3.2. *Suppose A is an R-linear additive category, where R is a henselian (local noetherian) ring. If A is idempotent complete, and the endomorphism ring of every object of A is finitely generated as an R-module, then A is Krull-Schmidt.*

Proof. Our argument follows the proof of [9, 1.8]. The assumptions imply that the endomorphism ring of every object is noetherian, and it follows that every object of A is a finite direct sum of indecomposable objects. Given any indecomposable object X of A, set $E := \mathrm{End}_A(X)$. By assumption, E is a module finite R-algebra, and so, since R is henselian, every idempotent of $E/J(E)$ lifts to an idempotent of E [9, A.30]. Since X is indecomposable and A is idempotent complete, E has no nontrivial idempotents. We conclude that $E/J(E)$ has no nontrivial idempotents. Again using that E is module finite over R, by [9, 1.7] we have $\mathfrak{m}_R E \subset J(E)$ and thus $E/J(E)$ is a module finite algebra over the field $R/\mathfrak{m}_R$. This shows $E/J(E)$ is artinian and hence semi-simple. Since it has no nontrivial idempotents, it must be a division ring. □

For a group G acting on a commutative ring Q and an element $f \in Q$ fixed by the action, we write $\mathrm{mf}_G(Q, f)$ for the additive category of equivariant matrix factorizations. Objects are pairs $P = (P, d)$ with P a $\mathbb{Z}/2$-graded module over the twisted group ring $Q\#G$ that is finitely generated and projective as a Q-module and d a $Q\#G$-linear endomorphism of P of odd degree that squares to $f \cdot \mathrm{id}_P$. (We do not assume $|G|$ is a unit in Q here; if it is, then such a P is finitely generated and projective as a module over $Q\#G$.) We write $[\mathrm{mf}_G(Q, f)]$ for the quotient of $\mathrm{mf}_G(Q, f)$ obtained by modding out by homotopy in the usual sense.

Our main result, Theorem 1.1, is an immediate consequence of the following slightly stronger statement:

Theorem 3.3. *Let G be a finite group acting on a commutative ring Q, and assume $f \in Q$ is fixed by G. If Q is (local noetherian) henselian, and the ring extension $Q^G \hookrightarrow Q$ is module finite, then $[\mathrm{mf}_G(Q, f)]$ is a Krull-Schmidt category and hence idempotent complete.*

Proof. Since Q is henselian, Q^G is also henselian by Proposition 2.2. Since Q^G belongs to the center of $Q\#G$, the additive category $\mathrm{mf}_G(Q, f)$ is Q^G-linear. The endomorphism ring of every object P of $\mathrm{mf}_G(Q,f)$ is contained in $\mathrm{End}_Q(P)$ and hence is module finite over Q^G. So, since $[\mathrm{mf}_G(Q,f)]$ is a quotient of $\mathrm{mf}_G(Q,f)$, by Lemma 3.2 it suffices to prove $\mathrm{mf}_G(Q,f)$ is idempotent complete.

Let e be an idempotent endomorphism of an object (P,d) in $\mathrm{mf}_G(Q,f)$. The category of all modules over $Q\#G$ is certainly idempotent complete, and so P decomposes as $P = \ker(e) \oplus \mathrm{im}(e)$ over this ring. Since P is Q-projective, so are both $\ker(e)$ and $\mathrm{im}(e)$. Since e commutes with d, we have $d(\ker(e)) \subseteq \ker(e)$ and $d(\mathrm{im}(e)) \subseteq \mathrm{im}(e)$. Thus, $(\ker(e), d|_{\ker(e)})$ and $(\mathrm{im}(e), d|_{\mathrm{im}(e)})$ are objects of $\mathrm{mf}_G(Q,f)$, and the canonical maps $p : (P,d) \twoheadrightarrow (\mathrm{im}(e), d|_{\mathrm{im}(e)})$ and $i : (\mathrm{im}(e), d|_{\mathrm{im}(e)}) \hookrightarrow (P,d)$ are morphisms in $\mathrm{mf}_G(Q,f)$. Since $e = i \circ p$, this proves $\mathrm{mf}_G(Q,f)$ is idempotent complete. □

4. Proofs of Proposition 1.2 and Corollary 1.3

Recall that, if $\mathcal{T}$ is a triangulated category, a subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *thick* if $\mathcal{S}$ is full, triangulated, and closed under summands. Given a collection $\mathcal{X}$ of objects of $\mathcal{T}$, the *thick closure* of $\mathcal{X}$ in $\mathcal{T}$, written $\mathrm{Thick}_{\mathcal{T}}(\mathcal{X})$, is the intersection of all thick subcategories of $\mathcal{T}$ that contain $\mathcal{X}$. Let us say that an object X of $\mathcal{T}$ *builds* $\mathcal{T}$ if $\mathrm{Thick}_{\mathcal{T}}(\{X\}) = \mathcal{T}$. Concretely, this means that every object of $\mathcal{T}$ is obtained from X by a finite process of taking mapping cones, suspensions, and summands.

Given a dg-category $\mathcal{C}$, we write $[\mathcal{C}]$ for its homotopy category, which has the same objects as $\mathcal{C}$ and morphisms $\mathrm{Hom}_{[C]}(X,Y) \coloneqq H^0\mathrm{Hom}_C(X,Y)$. We say $\mathcal{C}$ is *pre-triangulated* if the image of the dg-Yoneda embedding $[\mathcal{C}] \hookrightarrow [\mathrm{Mod}_{\mathrm{d}g}(\mathcal{C})]$ is a triangulated subcategory of $[\mathrm{Mod}_{\mathrm{d}g}(C)]$. See, e.g., [10, Section 2.3] for more details; roughly this means that $\mathcal{C}$ has notions of suspension and mapping cone making $[\mathcal{C}]$ into a triangulated category. For example, the dg-category $\mathrm{mf}(Q,f)$ is pre-triangulated.

We use the following well-known fact:

Lemma 4.1. *Suppose $\phi : \mathcal{C} \to \mathcal{D}$ is a dg-functor between two pre-triangulated dg-categories. Assume there exists an object $X \in \mathcal{C}$ such that*

- *X builds $[\mathcal{C}]$,*
- *$\phi(X)$ builds $[\mathcal{D}]$, and*
- *the map $\phi : \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(\phi(X))$ of dga's is a quasi-isomorphism.*

The dg-functor ϕ induces an equivalence $[\mathcal{C}]^\vee \xrightarrow{\cong} [\mathcal{D}]^\vee$ on idempotent completions of the associated homotopy categories.

Proof. This essentially follows from [10, Proposition 2.7]. In more detail: given a pre-triangulated dg-category $\mathcal{A}$, let $\mathrm{Perf}(\mathcal{A})$ denote the (triangulated) homotopy category of the dg-category of perfect right $\mathcal{A}$-modules; see e.g. [10, Definition 2.3] and the surrounding discussion for additional background. As stated in e.g. [10, §2.3], the Yoneda embedding $[\mathcal{A}] \hookrightarrow \mathrm{Perf}(\mathcal{A})$ exhibits $\mathrm{Perf}(\mathcal{A})$ as the idempotent completion of $[\mathcal{A}]$.

We have the following commutative diagram of triangulated categories:

$$\begin{array}{ccccc} \mathrm{Perf}(\mathrm{End}_{\mathcal{C}}(X)) & \xrightarrow{\cong} & \mathrm{Perf}(\mathcal{C}) & \xleftarrow{\cong} & [\mathcal{C}]^\vee \\ \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow \\ \mathrm{Perf}(\mathrm{End}_{\mathcal{D}}(\phi(X))) & \xrightarrow{\cong} & \mathrm{Perf}(\mathcal{D}) & \xleftarrow{\cong} & [\mathcal{D}]^\vee; \end{array}$$

the vertical maps are induced by ϕ, the leftmost horizontal maps are induced by inclusions, and the rightmost maps are induced by the Yoneda embeddings. The rightmost horizontal maps are equivalences by the above discussion; since we assume X builds $\mathcal{C}$ and $\phi(X)$ builds $\mathcal{D}$, [10, Proposition 2.7] implies that the leftmost horizontal functors are equivalences as well. The leftmost vertical map is an equivalence since we assume $\mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{D}}(\phi(X))$ is a quasi-isomorphism. Thus, the rightmost vertical map is an equivalence. $\square$

Let $F : \mathrm{mf}_G(Q, f) \to \mathrm{mf}(Q, f)$ be the evident dg-functor that forgets the group action. Since $Q\#G$ is free of finite rank as a Q-module, given $(P, d) \in \mathrm{mf}(Q, f)$, the pair $((Q\#G) \otimes_Q P, \mathrm{id} \otimes d)$ is an object of $\mathrm{mf}_G(Q, f)$. We extend this to a rule on morphisms in the evident way to obtain a dg-functor $E : \mathrm{mf}(Q, f) \to \mathrm{mf}_G(Q, f)$.

Lemma 4.2. *Assume $|G|$ is a unit in Q. Given $P \in \mathrm{mf}(Q, f)$, if P builds $[\mathrm{mf}(Q, f)]$, then $E(P)$ builds $[\mathrm{mf}_G(Q, f)]$.*

Proof. Given objects X and Y in a triangulated category $\mathcal{T}$, we use the notation $X \models_{\mathcal{T}} Y$ as a shorthand for "X builds Y (in $\mathcal{T}$)". The goal is to prove $E(P) \models_{[\mathrm{mf}_G(Q,f)]} Y$ for all $Y \in [\mathrm{mf}_G(Q, f)]$. For any such Y, by assumption we have $P \models_{[\mathrm{mf}(Q,f)]} F(Y)$. Since E induces a triangulated functor on homotopy categories, it follows that $E(P) \models_{[\mathrm{mf}_G(Q,f)]} E(F(Y))$. It therefore suffices to prove $E(F(Y)) \models_{[\mathrm{mf}_G(Q,f)]} Y$; in fact, we show Y is a summand of $E(F(Y))$ in $\mathrm{mf}_G(Q, f)$.

The object $E(F(Y))$ has underlying module $(Q\#G) \otimes_Q Y$, with G-action through the left tensor factor (and the G action on Y ignored) and differential $\mathrm{id} \otimes d_Y$. There is an evident surjection $p : E(F(Y)) \twoheadrightarrow Y$ in $\mathrm{mf}_G(Q, f)$ given by multiplication. Define $j : Y \hookrightarrow E(F(Y))$ by $j(y) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \otimes y^g$. One readily verifies that (a) j is $Q\#G$ linear, (b) j commutes with the differentials, and (c) $p \circ j = \mathrm{id}_Y$, so that j is a splitting of p in $\mathrm{mf}_G(Q, f)$. $\square$

Proof of Proposition 1.2. Let $k = Q/\mathfrak{m}$ be the residue field of Q. For a sufficiently high Q/f-syzygy M of k, we have that M is a maximal Cohen-Macaulay (MCM) Q/f-module. By a Theorem of Eisenbud [4], the MCM module M determines an object in $\mathrm{mf}(Q, f)$; let k^{stab} be such a matrix factorization. (We note that the object k^{stab} depends on M only up to a shift in $[\mathrm{mf}(Q, f)]$.) Since Q/f has an isolated singularity, it follows from [3, Corollary 4.12] that k^{stab} builds $[\mathrm{mf}(Q, f)]$.

Let us write $(k')^{\text{stab}}$ for the image of k^{stab} in $[\text{mf}(Q', f')]$ under ϕ_*; that is, $(k')^{\text{stab}} = Q' \otimes_Q k^{\text{stab}}$. Since ϕ is flat, $(k')^{\text{stab}}$ is the matrix factorization associated to the MCM Q'/f'-module $Q' \otimes_Q M$, which is a high syzygy of $Q' \otimes_Q Q/\mathfrak{m} \cong Q'/\mathfrak{m}'$. Since Q'/f' is an isolated singularity, we see that $(k')^{\text{stab}}$ builds $[\text{mf}(Q', f')]$.

Set $X = E(k^{\text{stab}})$ and $X' = E((k')^{\text{stab}})$, where E is the extension of scalars functor introduced above. By Lemma 4.2, we have that X and X' build $[\text{mf}_G(Q, f)]$ and $[\text{mf}_G(Q', f')]$, respectively. Moreover, X maps to X' under the functor $[\text{mf}_G(Q, f)] \to [\text{mf}_G(Q', f')]$. Since ϕ is flat, we have an isomorphism

$$Q' \otimes_Q H_*(\text{End}_{\text{mf}_G(Q,f)}(X)) \cong H_*(\text{End}_{\text{mf}_G(Q',f')}(X'))$$

of Q'-modules. Since the singularities are isolated, $H_*(\text{End}_{\text{mf}_G(Q,f)}(X))$ and $H_*(\text{End}_{\text{mf}_G(Q',f')}(X'))$ are finite length Q-modules, and so, since the natural map $Q' \otimes_Q Q/\mathfrak{m} \to Q'/\mathfrak{m}'$ is an isomorphism, the natural map

$$H_*(\text{End}_{\text{mf}_G(Q,f)}(X)) \xrightarrow{\cong} Q' \otimes_Q H_*(\text{End}_{\text{mf}_G(Q,f)}(X))$$

is an isomorphism as well. It follows that the map of dga's

$$\text{End}_{\text{mf}_G(Q,f)}(X) \to \text{End}_{\text{mf}_G(Q',f')}(X')$$

is a quasi-isomorphism, so that Lemma 4.1 yields an equivalence $[\text{mf}_G(Q, f)]^\vee \xrightarrow{\cong} [\text{mf}_G(Q', f')]^\vee$. $\square$

Proof of Corollary 1.3. By Proposition 2.2, each of $(Q^h)^G \subseteq Q^h$ and $\widehat{Q}^G \subseteq \widehat{Q}$ are module finite extensions. Proposition 1.2 therefore gives equivalences

$$[\text{mf}_G(Q, f)]^\vee \xrightarrow{\cong} [\text{mf}_G(Q^h, f)]^\vee \xrightarrow{\cong} [\text{mf}_G(\widehat{Q}, f)]^\vee;$$

applying Theorem 1.1 to both $[\text{mf}_G(Q^h, f)]$ and $[\text{mf}_G(\widehat{Q}, f)]$ finishes the proof. $\square$

5. Equivariant objects in the homotopy category of matrix factorizations

Finally, we address a remark of Spellmann-Young in [11]. Let $[\text{mf}(Q, f)]^G$ be the category of equivariant objects in $[\text{mf}(Q, f)]$, as defined, for instance, by Carqueville-Runkel in [2, §7.1]. There is a canonical functor

$$[\text{mf}_G(Q, f)] \to [\text{mf}(Q, f)]^G, \tag{5.1}$$

and it is proven in [11, Proposition 3.6] that, under certain circumstances, (5.1) exhibits the target as the idempotent completion of the source (in fact, this result applies more generally to Spellmann-Young's notion of Real equivariant matrix factorizations as well).

Spellmann-Young note in [11, Remark 3.7] that, if the map (5.1) (or its Real generalization) were an equivalence, some of their arguments could be shortened; we now apply Theorem 1.1 to prove this.

Corollary 5.2. *Suppose we are in the setting of Theorem 1.1, and assume further that $|G|$ is a unit in Q. The functor* (5.1) *is an equivalence.*

Proof. The proof of [11, Proposition 3.6] extends to our setting. In detail: by Theorem 1.1, the triangulated category $[\mathrm{mf}_G(Q, f)]$ is idempotent complete. As discussed in the proof of Lemma 4.1, the Yoneda embedding $[\mathrm{mf}_G(Q, f)] \hookrightarrow \mathrm{Perf}(\mathrm{mf}_G(Q, f))$ is an idempotent completion and hence an equivalence in our case. Since $|G|$ is a unit in Q, [6, Theorem 8.7] implies that there is an equivalence $\mathrm{Perf}(\mathrm{mf}_G(Q, f)) \to [\mathrm{mf}(Q, f)]^G$. Composing these two equivalences gives (5.1). □

Remark 5.3. While it is assumed in [11, Proposition 3.6] that the local hypersurface ring under consideration has an isolated singularity, this assumption is not necessary for Corollary 5.2.

Data availability

No data was used for the research described in the article.

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