BESSEL MODELS FOR REAL UNITARY GROUPS: THE TEMPERED CASE

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Abstract

We prove the local Gan–Gross–Prasad conjecture for tempered L-packets of real unitary groups. The proof is based on theta lifts and is very simple.

1. Introduction

The goal of this paper is to prove the local Gan–Gross–Prasad (GGP) conjecture, as stated in [3, Conjecture 17.3], for all tempered L-packets of *real* unitary groups $\mathrm{U}(n+2t+1)\times\mathrm{U}(n)$. In the simplest case $t=0$, the local GGP conjecture seeks to characterize the nonvanishing of the space $\mathrm{Hom}_{\mathrm{U}(n)}(\pi\,\widehat{\otimes}\,\sigma,\mathbb{C})$, where π and σ are irreducible representations of (not necessarily compact) $\mathrm{U}(n+1)$ and $\mathrm{U}(n)$, respectively. When the unitary groups in question are indeed compact, the characterization is given by Weyl's celebrated branching rule in [22, Section 18.2, p. 391]. In this case, irreducible representations of compact unitary groups are all finite-dimensional and are parameterized by their highest weights. If π and the dual of σ, respectively, have highest weights

$$a_1\geq\cdots\geq a_{n+1},\qquad b_1\geq\cdots\geq b_n,$$

then Weyl's branching rule states that $\mathrm{Hom}_{\mathrm{U}(W)}(\pi\otimes\sigma,\mathbb{C})$ is at most 1-dimensional, and it is 1-dimensional precisely when

$$a_1\geq b_1\geq a_2\geq b_2\geq\cdots\geq b_n\geq a_{n+1}.$$

In general, when the unitary groups are not necessarily compact, it is also known that this Hom space is at most 1-dimensional (see [19]). The local GGP conjecture roughly states that in each generic Vogan L-packet of $\mathrm{U}(n+1)\times\mathrm{U}(n)$, there is a unique pair (π,σ) such that $\mathrm{Hom}_{\mathrm{U}(W)}(\pi\,\widehat{\otimes}\,\sigma,\mathbb{C})\neq 0$, and this pair can be specified by the local root numbers. We prove this conjecture for all tempered L-packets. The precise definitions and the statement of the theorem will be given below in Section 2.

DUKE MATHEMATICAL JOURNAL
Vol. 172, No. 5, © 2023 DOI 10.1215/00127094-2022-0018
Received 6 May 2021. Revision received 26 January 2022.
First published online 24 March 2023.
2020 *Mathematics Subject Classification*. 11F70.

In a subsequent paper [23], we will prove this conjecture for real unitary groups in general by reducing it to the tempered case. Because the techniques used in each paper are so different in nature, we chose to separate the two.

The proof of the tempered case here is remarkably simple, both conceptually and technically. Previously, Beuzart-Plessis [2], following the strategy of Waldspurger, proved the "multiplicity one in a Vogan packet" part of the local GGP conjecture for all tempered packets of $\mathrm{U}(n+2t+1)\times\mathrm{U}(n)$ (and the full local GGP conjecture for tempered L-packets of p-adic unitary groups). The argument is based on local trace formulas and is long and difficult, but has the advantage of being uniform for both real and p-adic groups. Our proof follows a completely different approach and applies exclusively to real unitary groups. The proof is based on theta lifts. Via theta lifts, He [9] proved the local GGP conjecture for $\mathrm{U}(n+1)\times\mathrm{U}(n)$ for square-integrable representations, and Gan and Ichino [6] studied the relation between $\mathrm{U}(n+1)\times\mathrm{U}(n)$ and $\mathrm{U}(n)\times\mathrm{U}(n)$. Part of our argument is inspired by their work. One notable point is that, contrary to the usual expectations, our argument is not based on the results in [9] and it does not reduce the tempered case to the square-integrable case. An outline of the proof will be given after the precise statement of the theorem. The reason why the method applies to real unitary groups but not p-adic ones will be clear from the discussion there. Some ideas in the proof have other applications, for example, studying the local linear forms appearing in the Ichino–Ikeda conjecture (see [12]). We will pursue these ideas in subsequent work.

Notation and conventions. Throughout this paper, we keep the following notation. Let $\psi:\mathbb{R}\to\mathbb{C}^\times$ be the additive character given by $\psi(x)=e^{-2\pi ix}$ and $\psi^{\mathbb{C}}(z)=e^{2\pi(\overline{z}-z)}$. By a character of $\mathbb{C}^\times$ we mean a unitary character. If χ is a character of $\mathbb{C}^\times$, then we put $\chi^c(z)=\chi(\overline{z})$. The character χ is called *conjugate self-dual* of sign $+1$ (resp., -1) if $\chi|_{\mathbb{R}^\times}$ is trivial (resp., the sign character), or equivalently, $\chi(z)=(z/\sqrt{z\overline{z}})^m$ for an even (resp., odd) integer m.

We denote by 1_n the $n\times n$ identity matrix, and by $\mathrm{diag}(x_1,\dots,x_n)$ the $n\times n$ diagonal matrix with entries $x_1,\dots,x_n$.

If V is a Hermitian or skew-Hermitian space, then the Hermitian or skew-Hermitian form is denoted by h_V. If V is Hermitian of signature $(n-q,q)$, then we define $\mathrm{disc}\,V=(-1)^{\frac{n(n-1)}{2}-q}$. If V is skew-Hermitian, then we let $-iV$ be the Hermitian space with the underline vector space V and Hermitian form $-ih_V$. We denote by L_{+1} and L_{-1} the positive and negative Hermitian lines, respectively; that is, the underline vector spaces are $\mathbb{C}$ and the Hermitian forms are given by $h(x,y)=x\overline{y}$ and $h(x,y)=-x\overline{y}$, respectively.

Unless otherwise explicitly mentioned, by *representation* we always mean a unitary Casselman–Wallach representation of finite length (Frechet representation of moderate growth; see [21, Chapter XII]). The inner product on a representation is

denoted by $\langle -, - \rangle$. Let π be a representation. We denote by $\pi^\vee$ the space of continuous linear functionals on π endowed with the strong topology (uniform convergence on bounded subsets). The smooth dual of π, that is, the subspace of smooth vectors in $\pi^\vee$, is identified with $\overline{\pi}$. By a *tempered representation* of a reductive group G, we mean a (unitary) representation whose matrix coefficients are in $L^{2+\epsilon}(G/Z)$ for any $\epsilon > 0$, where Z stands for the center of G.

2. Restriction problems

2.1. Langlands–Vogan packets

As explained in [3, Section 8], a tempered L-parameter ϕ for a unitary group in n variables is an n-dimensional conjugate self-dual semisimple continuous representation of $\mathbb{C}^\times$ of sign $(-1)^{n-1}$. As $\mathbb{C}^\times$ is abelian, we can write

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_k\phi_k, \tag{2.1}$$

where $m_1, \ldots, m_k$ are integers, $m_1 + \cdots + m_k = n$, and $\phi_1, \ldots, \phi_k$ are distinct characters of $\mathbb{C}^\times$. The parameter ϕ being conjugate self-dual of sign $(-1)^{n-1}$ means the following. If ϕ_i is conjugate self-dual of sign $(-1)^{n-1}$, then m_i is arbitrary. If ϕ_i is conjugate self-dual of sign $(-1)^n$, then m_i is even. If ϕ_i is not conjugate self-dual, then there is an i' so that $\phi_i^{c,-1} = \phi_{i'}$ and $m_i = m_{i'}$. In particular, the number of conjugate self-dual characters of sign $(-1)^{n-1}$ (counting multiplicity) has the same parity with n.

A component group A_ϕ is defined for each ϕ (see [3, Section 4]). In our case it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where r is the number of distinct conjugate self-dual characters of sign $(-1)^{n-1}$ contained in ϕ. By relabeling, we may assume that ϕ_i is self-dual of sign $(-1)^{n-1}$ precisely when $1 \le i \le r$. We then label elements in A_ϕ as

$$\bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \tag{2.2}$$

where a_i is a symbol corresponding to ϕ_i. Without stating otherwise, we will follow this convention of labeling characters in the L-parameter.

Let V be a Hermitian space of dimension n. To each ϕ, the Langlands correspondence (see [3, Sections 9 and 10]) assigns a finite set Π_ϕ^V of irreducible tempered representations of $\mathrm{U}(V)$ called the *L-packet attached to ϕ*. The Vogan packet attached to ϕ, denoted by Π_ϕ, is the (disjoint) union of all Π_ϕ^V as V ranges over all (isomorphism classes of) Hermitian spaces of dimension n:

$$\Pi_\phi = \bigcup_{V:\dim V = n} \Pi_\phi^V.$$

Attached to each representation $\pi \in \Pi_\phi$ is a character $\eta : A_\phi \to \langle \pm 1 \rangle$ defining a bijection between Π_ϕ and all characters of A_ϕ. This bijection $\Pi_\phi \to \mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ depends on the choice of (an equivalence class of) Whittaker datum. When n is odd, we choose it to be the unique Whittaker datum (up to equivalence) of $\mathrm{U}(V)$, where V is of signature $(\frac{n+1}{2}, \frac{n-1}{2})$. When n is even as explained in [3, Section 10], this is equivalent to choosing a nontrivial additive character of $\mathbb{C}$ which is trivial on $\mathbb{R}$. Throughout this paper, we will take this additive character to be $\overline{\psi^{\mathbb{C}}}$. This choice agrees with the one used in [1], where the author picked the Whittaker datum $\mathfrak{w}_+$ in his notation (see the last sentence of the paper; see also [1, p. 66] for the definition of $\mathfrak{w}_+$ for odd unitary groups, and [1, p. 67] for that of even unitary groups—his notation is μ_+).

Let V' be the skew-Hermitian space, and let $V = -iV'$. The unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$ are physically the same. The use of $-i$ instead of i is compatible with the identifications in [1], [16], and [17]. In [16, Section 1.1] and [1, Section 3.1], the (skew-Hermitian) unitary group preserving skew-Hermitian form $i\,\mathrm{diag}(1_p, -1_q)$ is identified with a unitary group of signature (p, q). We have a similar Langlands correspondence for $\mathrm{U}(V')$ as in the case of $\mathrm{U}(V)$. The bijection between Π_ϕ and $\mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ again depends on the choice of a Whittaker datum. Following [3, Section 12], we choose the Whittaker datum so that under the natural identification of $\mathrm{U}(V)$ and $\mathrm{U}(V')$, a representation of $\mathrm{U}(V)$ and hence $\mathrm{U}(V')$ corresponds to the same character of A_ϕ.

Properties of the bijection $\Pi_\phi \leftrightarrow \mathrm{Hom}(A_\phi, \langle \pm 1 \rangle)$ have been summarized in [1, Theorem 2.1] and [3, Sections 9 and 10]. Those most frequently used are recalled in the following proposition.

PROPOSITION 2.1

We have the following assertions.

(1) *[1, Theorem 2.1(5)] Suppose that $V = X \oplus V_0 \oplus Y$, where X, Y are isotropic lines and $X \oplus Y$ is a 2-dimensional split Hermitian space perpendicular to V_0. Let $P = MN$ be the parabolic subgroup stabilizing X, and let $M \simeq \mathbb{C}^\times \times \mathrm{U}(V_0)$ be the Levi subgroup. Let π_0 be an irreducible tempered representation of $\mathrm{U}(V_0)$ whose Langlands–Vogan parameter is (ϕ_0, η_0), and let ξ be a character of $\mathbb{C}^\times$ that is not conjugate self-dual of sign $(-1)^{n-1}$. Then the parabolic induction*

$$\pi = \mathrm{Ind}_P^{\mathrm{U}(V)} \xi \otimes \pi_0$$

is an irreducible tempered representation of $\mathrm{U}(V)$ whose Langlands–Vogan parameter is (ϕ, η), where

$$\phi = \phi_0 \oplus \xi \oplus \xi^{c,-1}, \qquad A_\phi = A_{\phi_0}, \qquad \eta = \eta_0.$$

(2) *[1, Theorem 2.1(8)] Suppose that π is an irreducible tempered representation of* $\mathrm{U}(V)$ *with Langlands–Vogan parameter* (ϕ, η). *Then* $\pi \otimes \det^k$ *has parameter* $(\phi \otimes \chi^k, \eta)$, *where* $\chi_k(z) = (z/\overline{z})^k = (z/\sqrt{z\overline{z}})^{2k}$.

(3) *[1, Theorem 2.1(6)] If π is an irreducible tempered representation* $\mathrm{U}(V)$ *with Langlands–Vogan parameter* (ϕ, η), *then the smooth dual of* π, *which has been identified with* $\overline{\pi}$, *has Langlands–Vogan parameter* $(\phi^\vee, \eta^\vee)$, *where*

$$\phi^\vee = m_1\phi_1^{-1} \oplus \cdots \oplus m_k\phi_k^{-1}, \qquad A_{\phi^\vee} = A_\phi, \qquad \eta^\vee(a_i) = (-1)^{n-1}\eta(a_i).$$

(4) *[3, Section 10] Let π be an irreducible tempered representation of* $\mathrm{U}(V)$, *where V is of signature* (p, q) *and π has Langlands–Vogan parameter* (ϕ, η). *Then*

$$\eta(a_1 + \cdots + a_r) = (-1)^{\frac{n(n-1)}{2} - q} = \operatorname{disc} V.$$

Given any nontrivial additive character ψ' of $\mathbb{C}$, one can define the root number $\epsilon(\phi, \psi')$. We only record the following properties which will be used often and refer readers to [3, Section 5] for a more detailed discussion. We have

$$\epsilon(\phi, \overline{\psi'}) = \det\phi(-1)\epsilon(\phi, \psi'), \qquad \epsilon(\phi, \psi')\epsilon(\phi^\vee, \overline{\psi'}) = 1. \tag{2.3}$$

If ψ' is trivial on $\mathbb{R}$, and ξ is a character of $\mathbb{C}^\times$, then

$$\epsilon(\phi, \psi') = \epsilon(\phi \oplus \xi \oplus \xi^{c,-1}, \psi'). \tag{2.4}$$

We also note that the additive character $\psi^{\mathbb{C}}$ satisfies the condition of being trivial on $\mathbb{R}$.

2.2. *Bessel models*

Let t be a nonnegative integer. Let $W \subset V$ be a pair of Hermitian spaces of dimensions n and $n + 2t + 1$, respectively. We say that the pair (W, V) is relevant if $V = W \oplus^\perp Z$ and we can find a basis $z_0, z_{\pm i}, i = 1, \ldots, t$ of Z with

$$h_V(z_i, z_j) = (-1)^n\delta_{i,-j}, \quad i, j = 0, \pm 1, \ldots, \pm t.$$

Note that this definition is slightly more restrictive than the one in [3, Section 2], where it is only required that Z be split. To study the restriction problem and prove the local GGP conjecture, we however only need to consider the case at hand, because if $h_V(z_0, z_0) = (-1)^{n+1}$, then we may analyze the restriction problem for the embedding of groups $\mathrm{U}(-W) \subset \mathrm{U}(-V)$ instead.

Let P be the parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag of completely isotropic subspaces

$$\langle z_t\rangle \subset \langle z_t, z_{t-1}\rangle \subset \cdots \subset \langle z_t, \ldots, z_1\rangle.$$

Let N be the unipotent radical of P, and let $H = \mathrm{U}(W) \ltimes N$, which is a subgroup of $\mathrm{U}(V)$. We define a character of N as follows. Let $u \in N$. We define a character

$$\nu(u) = \psi\Big(\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} \sum_{i=0}^{t-1} h_V(z_{-i-1}, uz_i)\Big).$$

As ν is invariant under the conjugation action of $\mathrm{U}(W)$, it admits a unique extension to H which is trivial on $\mathrm{U}(W)$ and which we still denote by ν.

Let π and σ be representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$, respectively. We denote by $\mathrm{Hom}_H(\pi \mathbin{\widehat{\otimes}} \sigma, \nu)$ the space of continuous linear forms $\ell : \pi \mathbin{\widehat{\otimes}} \sigma \to \mathbb{C}$ with the property that

$$\ell\big(\pi(h)v \otimes \sigma(h)w\big) = \nu(h)\ell(v \otimes w), \quad \text{for all } h \in H,\ v \in \pi \text{ and } w \in \sigma.$$

We put $m(\pi, \sigma) = \dim \mathrm{Hom}_H(\pi \mathbin{\widehat{\otimes}} \sigma, \nu)$. By [13] and [19], if π and σ are irreducible, then $m(\pi, \sigma) \leq 1$.

If $t = 0$, then $H = \mathrm{U}(W)$ and ν is trivial. Then $\mathrm{Hom}_{\mathrm{U}(W)}(\pi \mathbin{\widehat{\otimes}} \sigma, \mathbb{C})$ is the space of $\mathrm{U}(W)$-invariant continuous linear forms on $\pi \mathbin{\widehat{\otimes}} \sigma$.

Assume that π and σ are irreducible and tempered. Let (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$ be the parameters of π and σ, respectively. Write

$$\phi_\pi = m_1\chi_1 \oplus \cdots \oplus m_k\chi_k, \qquad \phi_\sigma = n_1\mu_1 \oplus \cdots \oplus n_l\mu_l,$$

and

$$A_{\phi_\pi} = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \qquad A_{\phi_\sigma} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j.$$

The following theorem is the main result of this article. It confirms the local GGP conjecture for real unitary groups (see [3, Conjecture 17.3]).

THEOREM 2.2
Assume that π and σ are irreducible and tempered. Then $m(\pi, \sigma) = 1$ if and only if

$$\eta_\pi(a_i) = \epsilon(\chi_i \otimes \phi_\sigma, \psi^{\mathbb{C}}), \qquad \eta_\sigma(b_j) = \epsilon(\phi_\pi \otimes \mu_j, \psi^{\mathbb{C}}),$$

for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

2.3. Tempered intertwining

Let $W \subset V$ be a relevant pair of Hermitian spaces of dimensions n and $n + 2t + 1$, respectively. Let π and σ be tempered representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$, respectively. In [2, Section 7.2], an $H \times H$ bi-invariant continuous linear form

$$\mathcal{L}_{\pi,\sigma} : \pi \,\widehat{\otimes}\, \overline{\pi} \,\widehat{\otimes}\, \sigma \,\widehat{\otimes}\, \overline{\sigma} \to \mathbb{C}$$

is defined. We remark that it is not required in this definition that π and σ be irreducible. Assume now that $t = 0$. Then when restricted to $\pi \otimes \overline{\pi} \otimes \sigma \otimes \overline{\sigma}$, the linear form $\mathcal{L}_{\pi,\sigma}$ is given by the more familiar integration of matrix coefficients (see [12])

$$\mathcal{L}_{\pi,\sigma}(e, e', v, v') = \int_{\mathrm{U}(W)} \langle \pi(h)e, e' \rangle \langle \sigma(h)v, v' \rangle \, \mathrm{d}h. \tag{2.5}$$

The integral is absolutely convergent (see [12]). In general, if $t > 0$, then a regularization is needed in the definition (see [2, Section 7.1]), but we will not need the expression of $\mathcal{L}_{\pi,\sigma}$ in this case.

Let us recall two results from [2, Chapter 7] about $\mathcal{L}_{\pi,\sigma}$. In the local trace formula approach, these results need to be established before one can analyze the spectral side. It does not rely on any local trace formula argument.

In general (any t), it follows from the definition that if $\mathcal{L}_{\pi,\sigma} \neq 0$, then $m(\pi, \sigma) \neq 0$. The following result is [2, Theorem 7.2.1], which proves the converse.

PROPOSITION 2.3
If π and σ are irreducible and tempered, then $m(\pi, \sigma) = 1$ if and only if $\mathcal{L}_{\pi,\sigma} \neq 0$.

Let $W_1 = V \oplus \langle z_0' \rangle$ with $h_{W_1}(z_0', z_0') = (-1)^{n+1}$. Let X be the isotropic subspace of W_1 spanned by

$$\langle z_0 + z_0', z_1, \ldots, z_t \rangle,$$

and let Q be the parabolic subgroup of $\mathrm{U}(W_1)$ stabilizing X. The Levi subgroup M of Q is isomorphic to $\mathrm{GL}_{t+1}(\mathbb{C}) \times \mathrm{U}(W)$. Let τ be a tempered representation of $\mathrm{GL}_{t+1}(\mathbb{C})$, and consider the induced representation

$$\sigma_1 = \mathrm{Ind}_{Q}^{\mathrm{U}(W_1)} \tau \,\widehat{\otimes}\, \sigma.$$

We usually denote this induced representation by $\tau \rtimes \sigma$.

The following result is [2, Proposition 7.4.1]. Proposition 2.3 is deduced from it by an argument using the Plancherel formula.

PROPOSITION 2.4
For any tempered representation τ, we have $\mathcal{L}_{\pi,\sigma} \neq 0$ if and only if $\mathcal{L}_{\sigma_1,\pi} \neq 0$.

Propositions 2.3 and 2.4 reduce Theorem 2.2 to the case $t = 0$.

COROLLARY 2.5
Theorem 2.2 in the case $t = 0$ implies all other cases $t > 0$.

Proof

We keep the notation from Theorem 2.2. Let τ be an irreducible principal series representation of $\mathrm{GL}_{t+1}(\mathbb{C})$, induced from the character $\xi_1 \otimes \cdots \otimes \xi_{t+1}$ of $(\mathbb{C}^\times)^{t+1}$, where $\xi_1, \ldots, \xi_{t+1}$ are not conjugate self-dual. Then by Proposition 2.1(1), σ_1 is irreducible and the parameter of σ_1 is given by

$$\phi_{\sigma_1} = \phi_\sigma \oplus \xi_1 \oplus \xi_1^{c,-1} \oplus \cdots \oplus \xi_{t+1} \oplus \xi_{t+1}^{c,-1}, \qquad A_{\phi_{\sigma_1}} = A_{\phi_\sigma}, \qquad \eta_{\sigma_1} = \eta_\sigma.$$

Therefore, by (2.4), we have $\epsilon(\chi_i \otimes \phi_{\sigma_1}, \psi^{\mathbb{C}}) = \epsilon(\chi_i \otimes \phi_\sigma, \psi^{\mathbb{C}})$. By Propositions 2.3 and 2.4, Theorem 2.2 for (π, σ) follows from that for (σ_1, π). □

2.4. Outline of the proof

We now outline the proof of Theorem 2.2 when $t = 0$. Assume that ϕ_π contains a conjugate self-dual characters of sign $(-1)^n$ and that ϕ_σ contains b conjugate self-dual characters of sign $(-1)^{n-1}$ (counting multiplicity). Note that a has the same parity as $n+1$ and b has the same parity as n. Thus $a+b$ is odd. The trick is to make an induction on $a+b$, not on the size of the unitary group.

The base case is $a+b=1$. In this case, both π and σ are irreducible principal series representations. We have $m(\pi,\sigma)=1$, and both η_π and η_σ are trivial characters. One in fact checks directly in this case that $\mathcal{L}_{\pi,\sigma}$ is not identically zero. Theorem 2.2 thus holds when $a+b=1$.

Now we assume that $a+b \geq 3$ and proceed with induction. We should have either $a \geq 2$ or $b \geq 2$.

Assume first that $a \geq 2$. The key observation is that, up to twists by characters of the form $\det^k$ for some integer k, π can be constructed by a composition of two theta lifts. More precisely we can find the following data:

- a Hermitian space V_0 of dimension $n-1$ and a skew-Hermitian space V' of dimension n;
- an irreducible tempered representation of π_0 of $\mathrm{U}(V_0)$ and an irreducible tempered representation π' of $\mathrm{U}(V')$;

with the property that $\pi' \otimes \det^{k'}$ is the theta lift of π_0 and $\pi \otimes \det^k$ is the theta lift of π' (the integers k and k' depend on π and π'). In this case, via a familiar seesaw argument, Theorem 2.2 for (π,σ) is reduced to that for (σ_0,π_0), where σ_0 (up to twist) is an irreducible tempered representation in the same Vogan packet as σ. The L-parameter of σ_0 contains b conjugate self-dual characters of sign $(-1)^{n-1}$ and the L-parameter of π_0 contains $a-2$ conjugate self-dual characters of sign $(-1)^n$ (see the discussion of the relation between parameters and theta lifts in Section 3). Thus we are done by induction.

Finally, assume that $b \geq 2$. Let W_1 be the (orthogonal) direct sum of W and a hyperplane, and let P be the parabolic subgroup stabilizing an isotropic line in

this hyperplane. Then the Levi subgroup is isomorphic to $\mathbb{C}^\times \times \mathrm{U}(W)$. We choose a character χ that is not conjugate self-dual. Consider the induced representation $\sigma_1 = \chi \rtimes \sigma$. Then σ_1 is irreducible and the L-parameter contains b conjugate self-dual characters of sign $(-1)^{n-1}$. We then apply the result in the case $a \geq 2$ to (σ_1, π) and prove Theorem 2.2 for (σ_1, π). Using tempered intertwining, we see that Theorem 2.2 for (σ_1, π) is equivalent to Theorem 2.2 for (π, σ). Then we are done.

The reason why the argument applies exclusively to real unitary groups is now clear. An L-parameter of a real unitary group is a sum of characters and most of the representations (up to twists) can be constructed via theta lifts from smaller unitary groups. Of course, this is not true for p-adic unitary groups. For similar reasons, the technique does not seem to work without modification for the local GGP conjecture for orthogonal groups.

3. Theta lifts

3.1. Weil representations and theta lifts

Let V and V' be a Hermitian space and a skew-Hermitian space of dimensions n and n', respectively. Then $\mathrm{U}(V) \times \mathrm{U}(V')$ is a dual pair in $\mathrm{Sp}(2nn', \mathbb{R})$ in the sense of Howe. Let ω_ψ be the Weil representation of $\mathrm{Mp}(2nn', \mathbb{R})$ associated to the additive character ψ. The choice of ψ agrees with the choices made in [16] and [17] by [11, Lemma 7.10]. It also agrees with the choice made in [1, p. 32]. In what follows, the character ψ is always fixed and will be suppressed from the notation when there is no confusion.

By choosing a conjugate self-dual character χ_V of sign $(-1)^{n'}$ (resp., χ'_V of sign $(-1)^n$) of $\mathbb{C}^\times$, we get a splitting $\mathrm{U}(V) \to \mathrm{Mp}(2nn', \mathbb{R})$ (resp., $\mathrm{U}(V') \to \mathrm{Mp}(2nn', \mathbb{R})$) (see [1, Section 3.1]). We denote again by ω the restriction of the Weil representation to $\mathrm{U}(V) \times \mathrm{U}(V')$. We remark that it is not a representation by our convention as it is not of finite length. It is a continuous action of $\mathrm{U}(V) \times \mathrm{U}(V')$ on some Schwartz space, and it is unitary. In particular, the underline space is a nuclear Frechet space. We will describe realizations of the Weil representation on mixed models below. If $V = L_{+1}$, and we use a conjugate self-dual character μ of sign (-1) to split the metaplectic cover over $\mathrm{U}(V')$, then we denote the Weil representation of $\mathrm{U}(V')$ by $\omega_{\psi,\mu}$. This is the Weil representation that will appear in the Fourier–Jacobi model below.

Let π be an irreducible representation of $\mathrm{U}(V)$. Then $\mathrm{U}(V)$-coinvariance $(\omega \mathbin{\widehat{\otimes}} \overline{\pi})_{\mathrm{U}(V)}$ is denoted by $\Theta(\pi)$. It is a (not necessarily unitary) Casselman–Wallach representation of finite length of $\mathrm{U}(V')$ and its maximal semisimple quotient $\theta(\pi)$ of $\Theta(\pi)$ is irreducible (see [10, Theorem 1A]). If $\dim V - \dim V' = 0, \pm 1$ and π is tempered, then it is expected that $\Theta(\pi)$ is itself irreducible and hence unitary, and $\Theta(\pi) = \theta(\pi)$.

The analogous result for p-adic unitary groups holds by the work of Gan and Ichino [5, Appendix C]. A rigorous proof for real unitary groups is unfortunately missing in the literature. If we had this, then we would be able to prove Theorem 2.2 and Proposition 2.3 simultaneously by combining the seesaw argument in Section 3.4 below and the abstract seesaw identity (see [1, Proposition 3.11]), without appealing to the results of [2, Chapter 7].

We are going to deal with several theta lifts at the same time, so we add various subscripts to $\theta(\pi)$ to distinguish them. When we need to specify them, for instance, we denote the above $\theta(\pi)$ by $\theta_{V,V',\chi_V,\chi_{V'},\psi}(\pi)$.

We now describe the realization of ω on a mixed model following [6, Section 7.4], and use it to deduce an estimate of matrix coefficients of ω. Though [6] considers only non-Archimedean local fields, formulas for the Weil representation presented there are valid for all local fields of characteristic zero. Let s be the Witt index of V, and let V_0 be the anisotropic kernel of V, $\dim V_0 = n_0 = n - 2s$. Choose a basis $\{v_i, v_i^* \mid i = 1, \dots, s\}$ of $V_0^\perp$ so that for all $i, j = 1, \dots, s$,

$$\langle v_i, v_j \rangle = \langle v_i^*, v_j^* \rangle = 0, \qquad \langle v_i, v_j^* \rangle = \delta_{ij}.$$

Let $P = MN$ be a minimal parabolic subgroup of $\mathrm{U}(V)$ stabilizing the flag

$$\mathbb{C}\{v_1\} \subset \mathbb{C}\{v_1, v_2\} \subset \cdots \subset \mathbb{C}\{v_1, \dots, v_s\}.$$

Then $M \simeq (\mathbb{C}^\times)^s \times \mathrm{U}(V_0)$, and we let $A \simeq (\mathbb{R}_{>0})^s$ be the identity component of the maximal split torus in M. Let Δ_P be the roots of A in N, and let

$$A^+ = \big\{ b \in A \;\big|\; |\alpha(b)| \le 1, \forall \alpha \in \Delta_P \big\} = \big\{ (b_1, \dots, b_s) \;\big|\; 0 < b_1 \le \cdots \le b_s \le 1 \big\}.$$

Similarly, let r be the Witt index of V', and let V_0' be the anisotropic kernel of V', $\dim V_0' = n_0' = n' - 2r$. Then we have a minimal parabolic subgroup P' and the identity component of a maximal split torus A', and

$$A'^+ = \big\{ a = (a_1, \dots, a_r) \;\big|\; 0 < a_1 \le \cdots \le a_r \le 1 \big\}.$$

To compare the notation with [6], the spaces V and W there are our V and V', respectively, the dimensions m and n are our n and n', respectively, and the Witt indices r and s are our s and r, respectively.

The Weil representation ω is realized on the mixed model. Let us first fix a realization $\mathcal{S}_{00}$ of the Weil representation ω_{V_0', V_0} of $\mathrm{U}(V_0') \times \mathrm{U}(V_0)$. The Weil representation $\omega_{V_0', V}$ of $\mathrm{U}(V_0') \times \mathrm{U}(V)$ is then realized on $\mathcal{S}_0 = \mathcal{S}(V_0'^s) \mathbin{\widehat{\otimes}} \mathcal{S}_{00}$. Finally, the Weil representation $\omega_{V',V}$ of $\mathrm{U}(V') \times \mathrm{U}(V)$ is realized on $\mathcal{S} = \mathcal{S}(V^r) \mathbin{\widehat{\otimes}} \mathcal{S}_0$. We view elements in $\mathcal{S}$ as Schwartz functions on $V^r \times V_0'^s$ valued in $\mathcal{S}_{00}$. We do not need the fully detailed description of the action as in [6, Section 7.4], but only the following. Let $a = (a_1, \dots, a_r) \in A'^+$, let $b = (b_1, \dots, b_s) \in A^+$, and let $\phi \in \mathcal{S}$. Then

$$\omega_{V',V}(a,b)\phi(z,w)$$
$$= \chi_V(a_1\cdots a_r)\chi_{V'}(b_1\cdots b_s)(a_1\cdots a_r)^n(b_1\cdots b_s)^{n'-2r}\phi(b^{-1}za, wb), \quad (3.1)$$

where $z \in V^r$ (resp., $w \in V_0'^s$) is viewed as a row vector with entries in V (resp., V_0'), and when multiplied from the right, a and b are viewed as diagonal matrices with entries $a_1,\dots,a_r$ and $b_1,\dots,b_s$, respectively.

We fix an inner product on $\mathcal{S}_{00}$. Then an inner product on $\mathcal{S}$ is given by

$$\langle\phi,\phi'\rangle = \int_{V^r\times V_0'^s} \langle\phi(z,w),\phi'(z,w)\rangle \,\mathrm{d}z\,\mathrm{d}w.$$

This makes sense since ϕ, ϕ' are Schwartz. Define a function on $\mathbb{R}_{>0}$ by

$$\Upsilon(x) = \begin{cases} 1 & x \le 1, \\ x^{-1} & x > 1. \end{cases}$$

LEMMA 3.1
There is a continuous seminorm ν on $\mathcal{S}$ such that

$$\left|\langle\omega_{V,V'}(b,a)\phi,\phi'\rangle\right| \le \prod_{i=1}^{r} a_i^n \prod_{j=1}^{s} b_j^{n'-2r} \prod_{i=1}^{r}\prod_{j=1}^{s} \Upsilon(a_i b_j^{-1})^2 \nu(\phi)\nu(\phi'), \quad (3.2)$$

where $a = (a_1,\dots,a_r) \in A'^+$, $b = (b_1,\dots,b_s) \in A^+$.

Proof
This is a slightly refined estimate of the one given in the proof of [5, Lemma D.1]. From the explicit formula (3.1) we see that we only need to prove the following more general result. Let m be an integer, take $\phi,\phi' \in \mathcal{S}(\mathbb{R}^m)\,\widehat{\otimes}\,\mathcal{S}_{00}$, viewed as Schwartz functions valued in $\mathcal{S}_{00}$, and let $\lambda = (\lambda_1,\dots,\lambda_m) \in (\mathbb{R}_{>0})^m$. Then there is a seminorm ν on $\mathcal{S}(\mathbb{R}^m)\,\widehat{\otimes}\,\mathcal{S}_{00}$ such that

$$\left|\int_{\mathbb{R}^m} \langle\phi(\lambda_1 x_1,\dots,\lambda_m x_m),\phi(x_1,\dots,x_m)\rangle \,\mathrm{d}x_1\cdots\mathrm{d}x_m\right|$$
$$\le \prod_{i=1}^{m} \Upsilon(\lambda_i^{-1})\nu(\phi)\nu(\phi'). \quad (3.3)$$

By relabeling, we may assume that

$$\lambda_1 \ge \cdots \ge \lambda_l \ge 1 > \lambda_{l+1} \ge \cdots \ge \lambda_m.$$

Make changes of variables $x_i \mapsto \lambda_i^{-1}x_i$, $i = 1,\dots,l$. The left-hand side of (3.3) then becomes $(\lambda_1\cdots\lambda_l)^{-1}$ times

$$\Big|\int_{\mathbb{R}^m}\big\langle\phi(x_1,\dots,x_l,\lambda_{l+1}x_{l+1},\dots,\lambda_m x_m),\\ \phi'(\lambda_1^{-1}x_1,\dots,\lambda_l^{-1}x_l,x_{l+1},\dots,x_m)\big\rangle\,\mathrm{d}x_1\cdots\mathrm{d}x_m\Big|. \tag{3.4}$$

Choose positive polynomial functions $p(x_1,\dots,x_l)$ and $p'(x_{l+1},\dots,x_m)$ such that

$$\int_{\mathbb{R}^m} p(x_1,\dots,x_l)^{-1}\,p'(x_{l+1},\dots,x_m)^{-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_m$$

is convergent. Then we have that (3.4) is bounded by the product of this integral and

$$\sup_{x_1,\dots,x_m}\big|\big\langle p(x_1,\dots,x_l)\phi(x_1,\dots,x_l,\lambda_{l+1}x_{l+1},\dots,\lambda_m x_m),\\ p'(x_{l+1},\dots,x_m)\phi'(\lambda_1^{-1}x_1,\dots,\lambda_l^{-1}x_l,x_{l+1},\dots,x_m)\big\rangle\big|. \tag{3.5}$$

The Cauchy–Schwarz inequality gives that

$$(3.5)\le \sup_{x_1,\dots,x_m}\big\|\big\langle p(x_1,\dots,x_l)\phi(x_1,\dots,x_l,\lambda_{l+1}x_{l+1},\dots,\lambda_m x_m)\big\|\\ \times\sup_{x_1,\dots,x_m}\big\|p'(x_{l+1},\dots,x_m)\phi'(\lambda_1^{-1}x_1,\dots,\lambda_l^{-1}x_l,x_{l+1},\dots,x_m)\big\rangle\big\|,$$

where $\|\cdot\|$ is the norm induced by the inner product on $\mathcal{S}_{00}$. Note that p does not contain the variables $x_{l+1},\dots,x_m$, while p' does not contain the variables $x_1,\dots,x_l$. Thus both sup terms are independent of the λ_i's. Both sup terms are continuous seminorms on $\mathcal{S}(\mathbb{R}^m)\,\widehat{\otimes}\,\mathcal{S}_{00}$ by the definition of the Frechet space structure of $\mathcal{S}(\mathbb{R}^m)\,\widehat{\otimes}\,\mathcal{S}_{00}$, and we can find another continuous seminorm ν that dominates them. The desired estimate (3.3) follows, since

$$\prod_{i=1}^{m}\Upsilon(\lambda_i)=(\lambda_1\cdots\lambda_l)^{-1}$$

by our assumption. □

We now deduce from Lemma 3.1 several convergence lemmas which we need later. First we fix some notation and recall some standard estimates from [2, Section 1.5] and [12, Section 4]. Let G be a real reductive group, and let Ξ^G be the Harish-Chandra Xi function on G (see [2, Section 1.5]). We fix a logarithmic norm ς_G on G (see [2, Section 1.2]). Fix a maximal connected split torus A of G and a minimal parabolic subgroup P_0 of G. Let Δ_{P_0} be the set of roots of A in P_0, and let

$$A^+=\big\{a\in A\;\big|\;|\alpha(a)|\le 1,\forall\alpha\in\Delta_{P_0}\big\}.$$

Fix a maximal compact subgroup K of G such that we have the Cartan decomposition $G=KA^+K$. For any $f\in L^1(G)$, we have the following integration formula:

$$\int_G f(g)\,\mathrm{d}g = \int_{A^+} \varphi(a) \iint_{K\times K} f(k_1 a k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}a, \tag{3.6}$$

where $\varphi(a)$ is a nonnegative function on A^+ satisfying

$$\varphi(a) \le C\delta_{P_0}^{-1}(a), \tag{3.7}$$

where C is a positive constant independent of a and δ_{P_0} stands for the modulus character of P_0. Let π be an irreducible tempered representation of G, and let α be a matrix coefficient of π. Then there is a constant A such that $|\alpha(g)| \le A\Xi^G(g)$ for all $g \in G$. The result of [18] is more precise. There is a seminorm ν on π such that

$$\left|\left\langle \pi(g)v, v'\right\rangle\right| \le \Xi^G(g)\nu(v)\nu(v'), \quad v, v' \in \pi. \tag{3.8}$$

Moreover, by [2, Proposition 1.5.1], there are constants $B, C' > 0$ such that if $a \in A^+$, then we have

$$\Xi^G(a) \le C'\delta_{P_0}^{\frac{1}{2}}(a)\varsigma_G(a)^B.$$

We now come back to our setup of unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$. We keep the notation prior to Lemma 3.1.

LEMMA 3.2

Assume that $n \ge n'$. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$. Take $v, v' \in \pi'$ and $\phi, \phi' \in \mathcal{S}$. Then there are continuous seminorms $\nu_{\pi'}$ on π' and $\nu_{\mathcal{S}}$ on $\mathcal{S}$ such that

$$\left|\int_{\mathrm{U}(V')} \overline{\left\langle \pi'(g)v, v'\right\rangle}\left\langle \omega(g,1)\phi, \phi'\right\rangle \mathrm{d}g\right| \le \nu_{\pi'}(v)\nu_{\pi'}(v')\nu_{\mathcal{S}}(\phi)\nu_{\mathcal{S}}(\phi'). \tag{3.9}$$

Proof

To shorten notation, we write $\omega(g)$, $g \in \mathrm{U}(V')$, instead of $\omega(g,1)$. We write $a = (a_1, \dots, a_r) \in A^+$, and we have

$$\delta_P(a) = \prod_{i=1}^{r} a_i^{2n'-4i+2}.$$

Let us fix a maximal compact subgroup K of $\mathrm{U}(V')$ such that we have the Cartan decomposition $\mathrm{U}(V') = KA'^+K$. By the integration formula (3.6), estimates (3.7) and (3.8), and Lemma 3.1, the left-hand side of (3.9) is bounded by

$$\int_{0<a_1\le\cdots\le a_r\le 1} \prod_{i=1}^{r} a_i^{-\frac{1}{2}(2n'-4i+2)} \Big(1 - \sum_{i=1}^{r} \log a_i\Big)^B \prod_{i=1}^{r} a_i^n\,\mathrm{d}^\times a_1 \cdots \mathrm{d}^\times a_r$$
$$\times \int_K \int_K \widetilde{\nu}_\pi\big(\pi(k)v\big)\widetilde{\nu}_\pi\big(\pi(k'^{-1})v'\big)\widetilde{\nu}_{\mathcal{S}}\big(\omega(k)\phi\big)\widetilde{\nu}_{\mathcal{S}}\big(\omega(k'^{-1})\phi'\big)\,\mathrm{d}k\,\mathrm{d}k',$$

where B is a positive constant, $\mathrm{d}^\times a_i = a_i^{-1}\,\mathrm{d}a_i$ is a multiplicative measure on $\mathbb{R}_{>0}$, and $\widetilde{\nu}_\pi$ (resp., $\widetilde{\nu}_{\mathcal{S}}$) is a continuous seminorm on π (resp., $\mathcal{S}$). The first term simplifies to

$$\int_{0<a_1\le\cdots\le a_r\le 1}\prod_{i=1}^{r} a_i^{n-n'+2i-1}\Big(1-\sum_{i=1}^{r}\log a_i\Big)^B \mathrm{d}^\times a_1\cdots\mathrm{d}^\times a_r.$$

Since $n \ge n'$, it is absolutely convergent. Since K is compact, the second term is bounded by

$$(\mathrm{vol}\,K)^2 \sup_{k\in K}\widetilde{\nu}_\pi\big(\pi(k)v\big)\sup_{k\in K}\widetilde{\nu}_{\mathcal{S}}\big(\omega(k)\phi\big)\sup_{k\in K}\widetilde{\nu}_\pi\big(\pi(k^{-1})v'\big)\sup_{k\in K}\widetilde{\nu}_{\mathcal{S}}\big(\omega(k^{-1})\phi'\big).$$

Each sup term defines a continuous seminorm on the corresponding space by the uniform boundedness principle (see [20, Theorem 33.1]). The desired estimate (3.9) then follows. □

LEMMA 3.3

Assume that $V = L_{+1}$ is a positive Hermitian line. Let π' and σ' be irreducible tempered representations of $\mathrm{U}(V')$, and let $v, v' \in \pi'$, $e, e' \in \sigma'$. Let $\phi, \phi' \in \mathcal{S}$. Then there are seminorms $\nu_{\pi'}$, $\nu_{\sigma'}$, and $\nu_{\mathcal{S}}$ on π', σ', and $\mathcal{S}$, respectively, such that

$$\int_{\mathrm{U}(V')}\big|\langle\pi'(h)v, v'\rangle\langle\sigma'(h)e, e'\rangle\overline{\langle\omega_{\psi,\mu}(h)\phi, \phi\rangle}\big|\,\mathrm{d}h \le \nu_{\pi'}(v)\nu_{\pi'}(v')\nu_{\sigma'}(e)\nu_{\sigma'}(e')\nu_{\mathcal{S}}(\phi)\nu_{\mathcal{S}}(\phi').$$

Proof

The argument is very similar to that of the previous lemma, so we will be brief. Using the integration formula (3.6), and applying estimates (3.7), (3.8), and Lemma 3.1, the left-hand side of the desired inequality is bounded by the product of

$$\int_{0<a_1\le\cdots\le a_r\le 1} a_1\cdots a_r\Big(1-\sum_{i=1}^{r}\log a_i\Big)^B \mathrm{d}^\times a_1\cdots\mathrm{d}^\times a_r$$

and

$$\sup_{k\in K}\widetilde{\nu}_{\pi'}\big(\pi'(k)v\big)\sup_{k\in K}\widetilde{\nu}_{\pi'}\big(\pi'(k)v'\big)\sup_{k\in K}\widetilde{\nu}_{\sigma'}\big(\sigma'(k)e\big) \times\sup_{k\in K}\widetilde{\nu}_{\sigma'}\big(\sigma'(k)e'\big)\sup_{k\in K}\widetilde{\nu}_{\mathcal{S}}\big(\omega'(k)\phi\big)\sup_{k\in K}\widetilde{\nu}_{\mathcal{S}}\big(\omega'(k)\phi'\big),$$

where B is a positive constant and the $\widetilde{\nu}$.'s are various continuous seminorms. The convergence of the first integral is clear. The sup terms are continuous seminorms again by the uniform boundedness principle. □

In the following proposition, if σ is a representation, then, by a dense subspace of matrix coefficients of σ, we mean a space of functions generated by $\langle \sigma(h)e, e' \rangle$, where $e \in U$ and $e' \in U'$ and U, U' are dense subspaces of σ.

PROPOSITION 3.4
Assume that either $n = n'$ or $n = n' + 1$. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$. Take $v, v' \in \pi'$ and $\phi, \phi' \in \mathcal{S}$.

(1) *The linear form*

$$(v, v', \phi, \phi') \mapsto \int_{\mathrm{U}(V')} \overline{\langle \pi'(g)v, v' \rangle} \langle \omega(g, 1)\phi, \phi' \rangle \, \mathrm{d}g$$

continuously extends to a linear form on $\overline{\pi'} \mathbin{\widehat{\otimes}} \pi' \mathbin{\widehat{\otimes}} \omega \mathbin{\widehat{\otimes}} \overline{\omega}$. It is not identically zero if and only if $\theta_{V',V}(\pi') \neq 0$.

(2) *Assume that $\theta_{V',V}(\pi') \neq 0$. The function*

$$h \mapsto \int_{\mathrm{U}(V')} \overline{\langle \pi'(g)v, v' \rangle} \langle \omega(g, h)\phi, \phi' \rangle \, \mathrm{d}g$$

defines a (possibly zero) matrix coefficient of $\theta_{V',V}(\pi')$. When v, v' range over a dense subspace of π', and ϕ, ϕ' range over a dense subspace of $\mathcal{S}$, functions of this form generate a dense subspace of matrix coefficients of $\theta_{V',V}(\pi')$.

Proof
The absolute convergence and continuity follow from Lemma 3.2. The nonvanishing is quite subtle. If $n' = n$, then it is proved in [14, Proposition 5.2]; if $n' = n + 1$, then it is proved in [7, Proposition 11.5]. This proves the first assertion.

For the second assertion, the integral indeed defines some Hermitian form on $\Theta_{V',V}(\pi')$. The semipositivity of this Hermitian form follows from [8, Theorem 1.1]. It then defines an inner product on $\Theta_{V',V}(\pi')/K$, where K is the kernel of it. Therefore $\Theta_{V',V}(\pi')/K$ must be semisimple, and thus coincides with the $\theta_{V',V}(\pi')$. The positivity of this pairing on $\Theta_{V',V}(\pi')/K$ needs some explanation. In fact, it follows from the following general fact. If U is a vector space and q is a nonzero semipositive definite Hermitian form, and L the kernel of it, then q descends to an inner product on U/L. To see this, if there is an $x \notin L$ such that $q(x, x) = 0$, then take some $y \in U$ with $q(x, y) \neq 0$ and consider $q(ax + y, ax + y)$ for some $a \in \mathbb{C}$, which equals $q(y, y) + 2\,\mathrm{Re}\, a q(x, y)$. Since a is arbitrary and $q(x, y) \neq 0$, it cannot be nonnegative for all a.

Finally, we fix a surjective homomorphism $p : \overline{\pi'} \mathbin{\widehat{\otimes}} \omega \to \theta_{V',V}(\pi')$. The integral in the lemma takes the form $\langle \theta_{V',V}(\pi')p(v, \phi), p(v', \phi') \rangle$, where this $\langle -, - \rangle$ is an inner product on $\theta_{V',V}(\pi')$. The last assertion of the lemma then follows from the surjectivity of p and the density of $\overline{\pi'} \otimes \omega$ in $\overline{\pi'} \mathbin{\widehat{\otimes}} \omega$. □

3.2. Parameters of theta lifts

Let π be an irreducible tempered representation of $\mathrm{U}(V)$ with the parameter (ϕ, η). In the case $n' - n = 0, 1$, the parameters of theta lifts can be described. We fix χ_V and $\chi_{V'}$ to split the metaplectic cover over $\mathrm{U}(V)$ and $\mathrm{U}(V')$, respectively.

Let us write the parameter of π as

$$\phi = m_1\phi_1 \oplus \cdots \oplus m_k\phi_k, \tag{3.10}$$

where $\phi_1, \ldots, \phi_r$ are self-dual characters of sign $(-1)^{n-1}$ while $\phi_{r+1}, \ldots, \phi_k$ are not. Then $A_\phi = \bigoplus_{i=1}^r (\mathbb{Z}/2\mathbb{Z})a_i$ as in (2.2).

We first consider the equal rank case $n = n'$. There exists a unique skew-Hermitian space V' with $\dim V' = n$ so that $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \neq 0$. Put

$$\theta(\phi) = \phi \otimes \chi_V^{-1}\chi_{V'}. \tag{3.11}$$

Then $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \in \Pi_{\theta(\phi)}$, which corresponds to the character $\theta(\eta) : A_\phi \to \langle \pm 1 \rangle$ given by

$$\theta(\eta)(a_i) = \epsilon(\phi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}})\eta(a_i). \tag{3.12}$$

The discriminants of V and V' are related by

$$\operatorname{disc} V \operatorname{disc}(-iV') = \epsilon(\phi \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}}).$$

These results were proved by Paul and were stated in [16, Theorem 6.1] in a different language. We will make a translation after Lemma 3.5 for the reader's convenience.

We now consider the almost equal rank situation. Put

$$\theta(\phi) = (\phi \otimes \chi_V^{-1}\chi_{V'}) \oplus \chi_{V'}. \tag{3.13}$$

There are two cases. If ϕ does not contain χ_V, then there are exactly two skew-Hermitian spaces V' with $\dim V' = n' = n + 1$ such that $\theta_{V,V',\chi_V,\chi_{V'}}(\pi)$ is nonzero. We have

$$\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \in \Pi^{V'}_{\theta(\phi)}, \qquad A_{\theta(\phi)} = A_\phi \oplus (\mathbb{Z}/2\mathbb{Z})a_{V'},$$

where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ comes from $\chi_{V'}$. The representation $\theta_{V,V',\chi_V,\chi_{V'}}(\pi)$ corresponds to the character $\theta(\eta) : A_{\theta(\phi)} \to \langle \pm 1 \rangle$ such that

$$\theta(\eta)|_{A_\phi} = \eta, \qquad \theta(\eta)(a_{V'}) = \eta(a_1)\cdots\eta(a_r)(-1)^{\frac{n(n+1)}{2}-q}, \tag{3.14}$$

where $-iV'$ is of signature (p, q). In this case, the theta lift defines a bijection

$$\Pi_\phi \leftrightarrow \bigcup_{\operatorname{disc}(-iV')=\epsilon} \Pi^{V'}_{\theta(\phi)},$$

where $\epsilon = \pm 1$ is a fixed sign and V' ranges over all (isomorphism classes of) skew-Hermitian spaces with $\mathrm{disc}(-iV') = \epsilon$. If ϕ contains χ_V, then there is a unique V' such that $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) \neq 0$, and it corresponds to $(\theta(\phi), \theta(\eta))$ where

$$A_{\theta(\phi)} = A_\phi, \qquad \theta(\eta) = \eta. \tag{3.15}$$

In this case, the theta lift defines a bijection

$$\Pi_\phi \leftrightarrow \Pi_{\theta(\phi)}.$$

These results are due to Paul [17]. Again, it is not stated in this language in [17]. We will make the translation after Lemma 3.5.

One direct consequence of this description of the parameters is the following lemma. It is easy but plays a crucial role in our argument.

LEMMA 3.5
Assume that $n' = n + 1$. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$ with Langlands–Vogan parameter $(\phi_{\pi'}, \eta_{\pi'})$. Assume that $\phi_{\pi'}$ contains at least one conjugate self-dual character of sign $(-1)^{n'-1}$. Then there is an integer k, a Hermitian space V with $\dim V = n$, and an irreducible tempered representation π of $\mathrm{U}(V)$ so that

$$\theta_{V,V',\chi_V,\chi_{V'}}(\pi) = \pi' \otimes \det^k .$$

Proof
By assumption, we have $\phi_{\pi'} = \chi' \oplus \phi'_0$, where $\chi'(z) = (z/\sqrt{z\overline{z}})^m$ and m has the same parity with $n(= n' - 1)$. Suppose that $\chi_{V'} = (z/\sqrt{z\overline{z}})^\nu$. Then ν has the same parity with m (and with n). Let χ_k be the character $\chi_k(z) = (z/\overline{z})^k$. Then there is an integer k such that $\chi' \otimes \chi_k = \chi_V$, and hence by Proposition 2.1(2) the L-parameter of $\pi' \otimes \det^k$ contains the character $\chi_{V'}$. Put $\phi = \phi'_0 \otimes \chi_k \otimes \chi_V \chi_{V'}^{-1}$. Then ϕ is an L-parameter of unitary groups in n variables, and A_ϕ is a subgroup of index at most two of $A_{\phi_{\pi'}}$. Let $\eta = \eta_{\pi'}|_{A_\phi}$. Then the representation π with the Langlands–Vogan parameter (ϕ, η) is an irreducible tempered representation of $\mathrm{U}(V)$ with $\dim V = n$, and $\theta_{V,V',\chi_V,\chi_{V'}}(\pi) = \pi' \otimes \det^k$ by (3.13) and (3.14). □

So far we have considered only the theta lifts from $\mathrm{U}(V)$ to $\mathrm{U}(V')$. We can also consider the theta lifts from $\mathrm{U}(V')$ to $\mathrm{U}(V)$. As groups, we have identifications $\mathrm{U}(V') = \mathrm{U}(-iV')$ and $\mathrm{U}(V) = \mathrm{U}(iV)$. Moreover, $V' \otimes V = (-iV') \otimes (iV)$ as skew-Hermitian spaces so the Weil representation does not change. Furthermore, we picked the Whittaker datum so that under the Langlands–Vogan parameterization a representation of $\mathrm{U}(-iV')$ (resp., $\mathrm{U}(iV)$) and hence $\mathrm{U}(V')$ (resp., $\mathrm{U}(V)$) corresponds to

the same Langlands–Vogan parameter. Therefore, the results described above do not change.

The rest of the subsection is devoted to the translation of Paul's results into the language of Langlands–Vogan parameterization. This material will not be used in the proof of the main theorem. Tempered representations are irreducible parabolic inductions from limits of discrete series representations. By the Langlands–Vogan parameters of parabolic inductions (see Proposition 2.1(1)) and the induction principle of theta lifts (see [16, Theorem 4.6.4]), the crux of the matter is the case when ϕ is an L-parameter of a limit of discrete series representation. Assume from now on that this is the case. Then the ϕ_i's are all conjugate self-dual characters of sign $(-1)^{n-1}$, in the expression (3.10).

In [16] and [17], theta lifts are viewed as correspondences between genuine representations of covers over unitary groups. Let $\widetilde{\mathrm{U}(V)}$ and $\widetilde{\mathrm{U}(V')}$ be the inverse image of $\mathrm{U}(V)$ and $\mathrm{U}(V')$ in $\mathrm{Mp}(2nn', \mathbb{R})$. Then by [16, Section 1.2], we have

$$\widetilde{\mathrm{U}(V)} = \left\{(g, z) \in \mathrm{U}(V) \times \mathbb{C}^\times \;\middle|\; (\det g)^{\nu'} = z^2\right\},$$

where ν' is an integer with the same parity as n'. Different choices of ν' give isomorphic groups, and do not affect the result of theta lifts on the covers. However, the choice of ν' does affect how the identifications of representations of $\mathrm{U}(V)$ and $\widetilde{\mathrm{U}(V)}$ are made. There is a genuine character of $\widetilde{\mathrm{U}(V)}$ given by

$$\det^{\frac{\nu'}{2}} : \widetilde{\mathrm{U}(V)} \to \mathbb{C}^\times, \qquad (g, z) \mapsto z.$$

Thus there is a natural identification of irreducible representations of $\mathrm{U}(V)$ and those of $\widetilde{\mathrm{U}(V)}$ given by $\pi \leftrightarrow \pi \otimes \det^{-\frac{\nu'}{2}}$. This choice is made so that the relation (3.16) below holds. Similarly, we have

$$\widetilde{\mathrm{U}(V')} = \left\{(g, z) \in \mathrm{U}(V') \times \mathbb{C}^\times \;\middle|\; (\det g)^{\nu} = z^2\right\},$$

where ν is an integer with the same parity as n. Thus there is a natural identification of irreducible representations of $\mathrm{U}(V')$ and those of $\widetilde{\mathrm{U}(V')}$ given by $\pi' \leftrightarrow \pi' \otimes \det^{-\frac{\nu}{2}}$. The result of [10] in the setting of covers of unitary groups is the following. If $\widetilde{\pi}$ is an irreducible genuine representation of $\widetilde{\mathrm{U}(V)}$, then the maximal semisimple quotient the $\widetilde{\mathrm{U}(V)}$-coinvariant $(\omega \mathbin{\widehat{\otimes}} \widetilde{\pi})_{\widetilde{\mathrm{U}(V)}}$ is a genuine irreducible representation of $\widetilde{\mathrm{U}(V')}$. We denote it by $\widetilde{\theta_{V,V'}}(\widetilde{\pi})$.

Suppose that the splitting characters χ_V and $\chi_{V'}$ are, respectively, of the form

$$\chi_V(z) = \left(\frac{z}{\sqrt{z\overline{z}}}\right)^{\nu'}, \qquad \chi_{V'}(z) = \left(\frac{z}{\sqrt{z\overline{z}}}\right)^{\nu}.$$

Let π be an irreducible representation of $\mathrm{U}(V)$. Then by [1, Proposition 3.5], we have

$$\theta_{V,V',\chi_V,\chi_{V'}}(\pi)\otimes\det^{-\frac{\nu}{2}}=\widetilde{\theta_{V,V'}}(\pi\otimes\det^{-\frac{\nu'}{2}}). \tag{3.16}$$

Strictly speaking, (3.16) is not explicitly stated in [1], but the calculation in the proof gives it.

Limits of discrete series representations of $\mathrm{U}(V)$ are parameterized by the pair (λ,Ψ) in [16] and [17]. Here

$$\lambda=(\underbrace{\lambda_1,\dots,\lambda_1}_{p_1},\dots\underbrace{\lambda_k,\dots,\lambda_k}_{p_k};\underbrace{\lambda_1,\dots,\lambda_1}_{q_1},\dots\underbrace{\lambda_k,\dots,\lambda_k}_{q_k})$$

with

- $\lambda_i\in\mathbb{Z}+\frac{n-1}{2}$, and $\lambda_1>\lambda_2>\cdots>\lambda_k$,
- $p_i,q_i\geq 0$, $(p_i,q_i)\neq(0,0)$, and $|p_i-q_i|\leq 1$,
- (p,q) is the signature of V, where $p=p_1+\cdots+p_k$, $q=q_1+\cdots+q_k$,

and Ψ is a set of positive roots (of the diagonal Cartan subalgebra) in $\mathfrak{gl}_n(\mathbb{C})=\mathrm{Lie}(\mathrm{U}(V))\otimes\mathbb{C}$ satisfying certain properties. The set Ψ can be specified by a real diagonal matrix in $\mathfrak{gl}_n(\mathbb{C})$. The conditions that Ψ needs to satisfy are stated explicitly in [11, Section 3].

Genuine limits of discrete series representations of $\widetilde{\mathrm{U}(V)}$ are parameterized in the same way, except that the λ_i's are required to be in $\mathbb{Z}+\frac{1}{2}$ (resp., in $\mathbb{Z}$) if $n'-n$ is even (resp., odd). If π is a limit of discrete series representations of $\mathrm{U}(V)$ with the parameter (λ,Ψ), then the representation $\pi\otimes\det^{-\frac{\nu'}{2}}$ of $\widetilde{\mathrm{U}(V)}$ has the parameter $(\lambda-\frac{\nu'}{2},\Psi)$, where $\lambda-\frac{\nu'}{2}$ means subtracting $\frac{\nu'}{2}$ from each entry of λ.

For a limit of discrete series representation, the Langlands–Vogan parameterization (ϕ,η) and the parametrization (λ,Ψ) are linked as follows. (This follows from [15, Théorème 1.1] and has been worked out in concrete terms in [11, Section 5.3.1].) Write the L-parameter ϕ as $\chi_1\oplus\chi_2\oplus\cdots\oplus\chi_n$, where $\chi_i(z)=(z/\sqrt{z\bar{z}})^{\kappa_i+\nu'}$. We may order these characters so that $\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_n$. Define an elementary abelian 2-group

$$A_\phi^+=(\mathbb{Z}/2\mathbb{Z})e_1\oplus(\mathbb{Z}/2\mathbb{Z})e_2\oplus\cdots\oplus(\mathbb{Z}/2\mathbb{Z})e_n,$$

where e_i corresponds to κ_i. Then $\mathrm{Hom}(A_\phi,\langle\pm1\rangle)$ is identified with a subgroup of $\mathrm{Hom}(A_\phi^+,\langle\pm1\rangle)$ consisting of characters η with the property that $\eta(e_i)=\eta(e_j)$ if $\kappa_i=\kappa_j$. Let η be a character of A_ϕ, and put

$$(p_i,q_i)=\begin{cases}(1,0) & \text{if } \eta(e_i)=(-1)^{i-1},\\ (0,1) & \text{if } \eta(e_i)=(-1)^{i}.\end{cases}$$

If π is a representation of $\mathrm{U}(V)$ with Langlands–Vogan parameter (ϕ,η), then its parameter in terms of (λ,Ψ) is given by

$$\lambda = \frac{1}{2}\big((\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n};\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n}) + (\underbrace{\nu',\dots,\nu'}_{n})\big),$$

and the set Ψ is given by the diagonal matrix

$$\mathrm{diag}(\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_n,\dots,x_n}_{p_n};\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_n,\dots,x_n}_{q_n}),$$

with $x_1 > x_2 > \cdots > x_n$.

Finally, we can now make the translation of Paul's results. Consider the equal rank case $n = n'$ first. In this case, the κ_i's are odd integers. Suppose that we have $\kappa_1 \geq \cdots \geq \kappa_l > 0 > \kappa_{l+1} \geq \cdots \geq \kappa_n$. The parameter (λ, Ψ) of π is as described above. The parameter of the representation $\pi \otimes \det^{-\frac{\nu'}{2}}$ of $\widetilde{\mathrm{U}(V')}$ is then $(\widetilde{\lambda}, \Psi)$, where

$$\frac{1}{2}(\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n};$$
$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n}).$$

By [16, Theorem 6.1], the theta lift $\widetilde{\theta_{V,V'}}(\pi \otimes \det^{-\frac{\nu'}{2}})$ is a limit of discrete series representation of $\widetilde{\mathrm{U}(V')}$ and the parameter is given by $(\Gamma\widetilde{\lambda}, \Gamma\Psi)$, where

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$
$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}),$$

and $\Gamma\Psi$ is given by the diagonal matrix

$$\mathrm{diag}(\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_l,\dots,x_l}_{p_l},\underbrace{x_{l+1},\dots,x_{l+1}}_{q_{l+1}},\dots,\underbrace{x_n,\dots,x_n}_{q_n};$$
$$\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_l,\dots,x_l}_{q_l},\underbrace{x_{l+1},\dots,x_{l+1}}_{p_{l+1}},\dots,\underbrace{x_n,\dots,x_n}_{p_n}).$$

The definition of $\Gamma\Psi$ in [16, Section 5.2, (5.2.4b)] is given explicitly in terms of the roots, which is equivalent to our description here. Then by the relation (3.16), we conclude that $\theta_{V,V',\chi_V,\chi'_V}(\pi)$ is $(\Gamma\lambda, \Gamma\Psi)$, where

$$\Gamma\lambda = \frac{1}{2}\big((\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}) + (\underbrace{\nu,\dots,\nu}_{n})\big).$$

Now consider the representation with the Langlands–Vogan parameter $(\theta(\phi),\theta(\eta))$. The L-parameter is of the form $\chi_1' \oplus \chi_2' \oplus \cdots \oplus \chi_n'$, where $\chi_i'(z) = (z/\sqrt{z\overline{z}})^{\kappa_i+\nu}$. By the relation (3.12), we have

$$\theta(\eta)(e_i) = \epsilon(\chi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}})\eta(e_i).$$

By definition, $\chi_i \otimes \chi_V^{-1}$ is the character $z \mapsto (z/\sqrt{z\overline{z}})^{\kappa_i}$. Then by [4, Proposition 2.1] and (2.3), we have

$$\epsilon(\chi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}}) = \begin{cases} +1 & \kappa_i > 0, \text{ i.e., } 1 \le i \le l, \\ -1 & \kappa_i < 0, \text{ i.e., } l+1 \le i \le n. \end{cases}$$

By the definition of (p_i, q_i), the effect of multiplying $\epsilon(\chi_i \otimes \chi_V^{-1}, \overline{\psi^{\mathbb{C}}})$ to $\eta(e_i)$ is equivalent to keeping (p_i, q_i) intact when $1 \le i \le l$ and swapping (p_i, q_i) when $l+1 \le i \le n$. Comparing this with $(\Gamma\lambda, \Gamma\Psi)$, we conclude that the Langlands–Vogan parameter of $\theta_{V,V',\chi_V,\chi_V'}(\pi)$ is precisely $(\theta(\phi),\theta(\eta))$, which is what we are after.

Let us move to the almost equal rank case $n' = n+1$. In this case, the κ_i's are all even integers. We assume that u, v are integers with $|u-v| \le 1$ and $\kappa_{l+1} = \cdots = \kappa_{l_{u+v}} = 0$ and hence

$$\kappa_1 \ge \cdots \ge \kappa_l \ge 0 = \cdots = 0 \ge \kappa_{l+u+v+1} \ge \cdots \ge \kappa_n.$$

As in the equal rank case, the parameter $(\widetilde{\lambda}, \Psi)$ of the representation $\pi \otimes \det^{-\frac{v'}{2}}$ is

$$\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{0,\dots,0}_{u},\underbrace{\kappa_{l+u+v+1},\dots,\kappa_{l+u+v+1}}_{p_{l+u+v+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{0,\dots,0}_{v},\underbrace{\kappa_{l+u+v+1},\dots,\kappa_{l+u+v+1}}_{q_{l+u+v+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n}),$$

and the set Ψ is again given by the diagonal matrix

$$\operatorname{diag}(\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_n,\dots,x_n}_{p_n};\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_n,\dots,x_n}_{q_n}),$$

with $x_1 > x_2 > \cdots > x_n$. We distinguish two cases.

First, assume that $u = v = 0$; that is, ϕ does not contain χ_V. By [17, Theorem 3.4(c)], there are two Hermitian spaces V' such that $\widetilde{\theta_{V,V'}}(\pi \otimes \det^{-\frac{\nu'}{2}}) \neq 0$, and the parameters are given, respectively, by

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},0,\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}),$$

$$\Gamma\Psi = \operatorname{diag}(\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_l,\dots,x_l}_{p_l},x',\underbrace{x_{l+1},\dots,x_{l+1}}_{q_{l+1}}\cdots,\underbrace{x_n,\dots,x_n}_{q_n};$$

$$\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_l,\dots,x_l}_{q_l},\underbrace{x_{l+1},\dots,x_{l+1}}_{p_{l+1}}\underbrace{x_n,\dots,x_n}_{p_n}),$$

and

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},0,\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}),$$

$$\Gamma\Psi = \operatorname{diag}(\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_l,\dots,x_l}_{p_l},\underbrace{x_{l+1},\dots,x_{l+1}}_{q_{l+1}}\cdots,\underbrace{x_n,\dots,x_n}_{q_n};$$

$$\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_l,\dots,x_l}_{q_l},x',\underbrace{x_{l+1},\dots,x_{l+1}}_{p_{l+1}}\underbrace{x_n,\dots,x_n}_{p_n}).$$

Here in both cases, we have $x_1 > x_2 > \cdots x_l > x' > x_{l+1} > \cdots > x_n$. Again, the description of $\Gamma\Psi$ is explicitly in terms of roots in [17, Definition 3.1], which is equivalent to our description. The representation $\widetilde{\theta}_{V,V'}(\pi \otimes \det^{-\frac{\nu'}{2}}) \otimes \det^{\frac{\nu}{2}}$ of $\mathrm{U}(V')$ then has parameter

$$\Gamma\lambda = \frac{1}{2}\big((\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},0,\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}) + \underbrace{(\nu,\dots,\nu)}_{n+1}\big),$$

and

$$\Gamma\lambda = \frac{1}{2}\Big((\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{q_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},0,\underbrace{\kappa_{l+1},\dots,\kappa_{l+1}}_{p_{l+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}) + \underbrace{(\nu,\dots,\nu)}_{n+1}\Big),$$

respectively, and $\Gamma\Psi = \Gamma\widetilde{\Psi}$. Now consider the representation of $\mathrm{U}(V')$ whose Langlands–Vogan parameterization is given by $(\theta(\phi),\theta(\eta))$. We have $\theta(\phi) = \chi'_1 \oplus \cdots \oplus \chi'_{n'}$, where

$$\chi'_i = \begin{cases} (z/\sqrt{z\overline{z}})^{\kappa_i+\nu} & 1 \le i \le l, \\ (z/\sqrt{z\overline{z}})^{\nu} & i = l+1, \\ (z/\sqrt{z\overline{z}})^{\kappa_{i-1}+\nu} & l+2 \le i \le n'. \end{cases}$$

Therefore, the fact that $\theta(\eta)|_{A_{\theta(\phi)}} = \eta$ is equivalent to the fact that we are swapping p_i and q_i when $i \ge l+1$ in the parameters. The fact that $\theta(\eta)(a_{V'}) = \eta(a_1)\cdots\eta(a_r)(-1)^{\frac{n(n+1)}{2}-q}$ is equivalent to $\theta(\eta)(a_1+\cdots+a_r+a_{V'}) = \operatorname{disc} V'$ (see Proposition 2.1(4); recall that V' is of signature (p,q) and $\dim V' = n+1$).

Now assume that $u+v \ge 1$; that is, ϕ contains χ_V. We make use of [17, Theorem 3.4(b)]. In this case, there is a unique V' such that $\widetilde{\theta_{V,V'}}(\pi \otimes \det^{-\frac{\nu'}{2}}) \ne 0$. If $u > v$ or if $u = v$ and $(p_{l+1}, q_{l+1}) = (0,1)$, then the parameter of $\widetilde{\theta_{V,V'}}(\pi \otimes \det^{-\frac{\nu'}{2}})$ is

$$\Gamma\widetilde{\lambda} = \frac{1}{2}(\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{0,\dots,0}_{u},$$

$$\underbrace{\kappa_{l+u+v+1},\dots,\kappa_{l+u+v+1}}_{q_{l+u+v+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};$$

$$\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{0,\dots,0}_{v+1},$$

$$\underbrace{\kappa_{l+u+v+1},\dots,\kappa_{l+u+v+1}}_{p_{l+u+v+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}),$$

$$\Gamma\Psi = \operatorname{diag}(\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_{l+u+v},\dots,x_{l+u+v}}_{p_{l+u+v}},$$

$$\underbrace{x_{l+u+v+1},\dots,x_{l+u+v+1}}_{q_{l+u+v+1}}\cdots,\underbrace{x_n,\dots,x_n}_{q_n};$$

$$\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_{l+u+v},\dots,x_{l+u+v}}_{q_{l+u+v}},x',$$

$$\underbrace{x_{l+u+v+1},\dots,x_{l+u+v+1}}_{p_{l+u+v+1}}\cdots,\underbrace{x_n,\dots,x_n}_{p_n}).$$

If $u<v$ or if $u=v$ and $(p_{l+1},q_{l+1})=(1,0)$, then the parameter of $\widetilde{\theta_{V,V'}}(\pi\otimes\det^{-\frac{v'}{2}})$ is

$$\begin{aligned}\Gamma\widetilde{\lambda}=\frac{1}{2}(&\underbrace{\kappa_1,\dots,\kappa_1}_{p_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{p_l},\underbrace{0,\dots,0}_{u+1},\\&\underbrace{\kappa_{l+u+v+1},\dots,\kappa_{l+u+v+1}}_{q_{l+u+v+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{q_n};\\&\underbrace{\kappa_1,\dots,\kappa_1}_{q_1},\dots,\underbrace{\kappa_l,\dots,\kappa_l}_{q_l},\underbrace{0,\dots,0}_{v},\\&\underbrace{\kappa_{l+u+v+1},\dots,\kappa_{l+u+v+1}}_{p_{l+u+v+1}},\dots,\underbrace{\kappa_n,\dots,\kappa_n}_{p_n}),\\\Gamma\Psi=\operatorname{diag}(&\underbrace{x_1,\dots,x_1}_{p_1},\dots,\underbrace{x_{l+u+v},\dots,x_{l+u+v}}_{p_{l+u+v}},x',\\&\underbrace{x_{l+u+v+1},\dots,x_{l+u+v+1}}_{q_{l+u+v+1}}\cdots,\underbrace{x_n,\dots,x_n}_{q_n};\\&\underbrace{x_1,\dots,x_1}_{q_1},\dots,\underbrace{x_{l+u+v},\dots,x_{l+u+v}}_{q_{l+u+v}},\\&\underbrace{x_{l+u+v+1},\dots,x_{l+u+v+1}}_{p_{l+u+v+1}},\dots,\underbrace{x_n,\dots,x_n}_{p_n}).\end{aligned}$$

In both cases, we have $x_1>x_2>\cdots>x_{l+u+v}>x'>x_{l+u+v+1}>\cdots>x_n$. As in the previous case, translating these results from $\widetilde{\mathrm{U}(V')}$ to $\mathrm{U}(V')$ gives the desired Langlands–Vogan parameter.

3.3. Fourier–Jacobi models

We will need an auxiliary model in our proof of Theorem 2.2. Let V' be a skew-Hermitian space of dimension n. Let π' and σ' be representations of $\mathrm{U}(V')$. Let μ be a conjugate self-dual character of $\mathbb{C}^\times$ of sign (-1); for example, we may take $\mu(z)=z(z\overline{z})^{-\frac{1}{2}}$. Let $\omega_{\psi,\mu}$ be the Weil representation of $\mathrm{U}(V')$ (see Section 3.1). We denote by $\mathrm{Hom}_{\mathrm{U}(V')}(\pi'\otimes\sigma'\otimes\overline{\omega_{\psi,\mu}},\mathbb{C})$ the space of continuous $\mathrm{U}(V')$-invariant linear forms on $\pi'\mathbin{\widehat{\otimes}}\sigma'\mathbin{\widehat{\otimes}}\overline{\omega_{\psi,\mu}}$ and by $m(\pi',\sigma')$ its dimension. If π' and σ' are irreducible, then $m(\pi',\sigma')\le 1$ by [19].

Define a linear form

$$\begin{aligned}&\mathcal{L}_{\pi',\sigma',\psi,\mu} : \pi' \mathbin{\widehat{\otimes}} \overline{\pi'} \mathbin{\widehat{\otimes}} \sigma' \mathbin{\widehat{\otimes}} \overline{\sigma'} \mathbin{\widehat{\otimes}} \overline{\omega_{\psi,\mu}} \mathbin{\widehat{\otimes}} \omega_{\psi,\mu} \to \mathbb{C},\\ &(v, v', e, e', \phi, \phi') \mapsto \int_{\mathrm{U}(V')} \langle \pi'(h)v, v' \rangle \langle \sigma'(h)e, e' \rangle \overline{\langle \omega_{\psi,\mu}(h)\phi, \phi \rangle}\, \mathrm{d}h.\end{aligned} \tag{3.17}$$

By Lemma 3.3 the integral is absolutely convergent and the linear form indeed extends continuously to the completed tensor product.

Assume that π' and σ' are both irreducible and tempered. Let $(\phi_{\pi'}, \eta_{\pi'})$ and $(\phi_{\sigma'}, \eta_{\sigma'})$ be the parameters of π' and σ', respectively. Let us write

$$\phi_{\pi'} = m_1\chi_1 \oplus \cdots \oplus m_k\chi_k, \qquad \phi_{\sigma'} = n_1\mu_1 \oplus \cdots \oplus n_l\mu_l,$$

and

$$A_{\phi'_\pi} = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \qquad A_{\phi'_\sigma} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j,$$

following the convention in (2.2).

PROPOSITION 3.6
Assume that π' and σ' are irreducible and tempered. Then $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$ if and only if

$$\eta_{\pi'}(a_i) = \epsilon(\chi_i \otimes \phi_{\sigma'} \otimes \mu^{-1}, \overline{\psi^{\mathbb{C}}}), \qquad \eta_{\sigma'}(b_j) = \epsilon(\phi_{\pi'} \otimes \mu_j \otimes \mu^{-1}, \overline{\psi^{\mathbb{C}}}).$$

This proposition will be proved along with Theorem 2.2 later. The local GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ (see [3, Conjecture 17.3]) is a similar statement with $\mathcal{L}_{\pi',\sigma'} \neq 0$ replaced by $m(\pi', \sigma') = 1$. Since $\mathcal{L}_{\pi',\sigma'} \neq 0$ clearly implies that $m(\pi', \sigma') \neq 0$, Proposition 3.6 proves half of this conjecture. There are two possible ways to prove $m(\pi', \sigma') = 1$ if and only if $\mathcal{L}_{\pi',\sigma'} \neq 0$. One is to mimic the argument of [2, Chapter 7]. The other is to use theta lift. If we had $\Theta(\pi) = \theta(\pi)$ for irreducible tempered representations in the almost equal rank situation, we would have an explicit relation between $m(\pi', \sigma')$ and $m(\pi, \sigma)$ when π' (resp., σ') and π (resp., σ) are connected via theta lifts (see [1, Proposition 3.11]). In this way, we can prove $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$ if and only if $m(\pi', \sigma') = 1$ along with Theorem 2.2 and Proposition 3.6.

3.4. Seesaw

We now explain the relation between the tempered intertwining maps (2.5) and (3.17) via theta lifts. We consider a Hermitian space V and a skew-Hermitian space V' and the Weil representation of $\mathrm{U}(V) \times \mathrm{U}(V')$. We fix the following choice of characters.

- We fix the additive character ψ in the definition of theta lifts. We let $\mu(z) = z(z\overline{z})^{-\frac{1}{2}}$ be a conjugate self-dual character of $\mathbb{C}^\times$ of sign (-1).
- If $\dim V = \dim V' = n$, then we use

$$\chi_V = \mu^n, \qquad \chi_{V'} = \mu^n.$$

- If $\dim V = \dim V' \pm 1 = n \pm 1$, then we use

$$\chi_V = \mu^n, \qquad \chi_{V'} = \mu^{n+(-1)^n}.$$

- If $V = L_{(-1)^n}$, $\dim V' = n$, then we use

$$\chi_V = \mu^n, \qquad \chi_{V'} = \mu^{(-1)^n}.$$

With these choices of the characters, the Weil representation of $\mathrm{U}(V) \times \mathrm{U}(V')$ is denoted by $\omega_{V,V'}$. Let π' be an irreducible representation of $\mathrm{U}(V')$. We denote by $\theta_{V',V}(\pi')$ the theta lift of π' to $\mathrm{U}(V)$. Recall also that we have the Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}(V')$.

Let us consider the following setup. Let $W \subset V$ be a relevant pair of Hermitian spaces of dimensions n and $n+1$, respectively. Let $L = W^\perp = L_{(-1)^n}$. Let V' be a skew-Hermitian space of dimension n. Let π' be an irreducible tempered representation of $\mathrm{U}(V')$, and let $\pi = \theta_{V',V}(\pi')$. Let σ be an irreducible tempered representation of $\mathrm{U}(W)$. The next two lemmas exploit the following seesaw diagram:

$$\begin{array}{ccc} \mathrm{U}(V') \times \mathrm{U}(V') & & \mathrm{U}(V) \\ | & \times & | \\ \mathrm{U}(V') & & \mathrm{U}(W) \times \mathrm{U}(L) \end{array}$$

LEMMA 3.7

Let α' be a matrix coefficient of π', let β be a matrix coefficient of σ, and let $\phi, \phi' \in \mathcal{S}$. Then the integral

$$\int_{\mathrm{U}(W)} \int_{\mathrm{U}(V)} \left|\alpha'(g)\beta(h)\langle\omega(g,h)\phi, \phi'\rangle\right| \mathrm{d}h \, \mathrm{d}g$$

is absolutely convergent.

Proof

Let us first record a simple calculus fact from which the lemma will eventually be deduced.

Fact. Let $r_1, \dots, r_N$ and B be real numbers, and let $r_1 + \cdots + r_i > 0$ for any $1 \le i \le N$. The integral

$$\int_{0<x_1\le\cdots\le x_N\le 1} x_1^{r_1}\cdots x_N^{r_N}\Big(1-\sum_{i=1}^{N}\log x_i\Big)^B \mathrm{d}^\times x_1\cdots\mathrm{d}^\times x_N$$

is convergent.

Now back to the proof of the lemma. Fix minimal parabolic subgroups $P \subset Q$ of $\mathrm{U}(W)$ and $\mathrm{U}(V)$, respectively. Let s be the split rank of $\mathrm{U}(W)$, and we fix maximal connected split tori $A_W \simeq (\mathbb{R}_{>0})^s \subset A_V$ of $\mathrm{U}(W)$ and $\mathrm{U}(V)$ contained in P and Q, respectively. As the notation before Lemma 3.1, we have

$$A_W^+ = \big\{(b_1,\dots,b_s) \in A_W \;\big|\; 0 < b_1 \le \cdots \le b_s \le 1\big\}.$$

Fix a maximal compact subgroup K of $\mathrm{U}(W)$ such that we have the Cartan decomposition $\mathrm{U}(W) = KA_W^+K$. Similarly, assume that the split rank of $\mathrm{U}(V')$ is r, and that we have a minimal parabolic subgroup P' of $\mathrm{U}(V')$, a maximal connected split torus $A' \simeq (\mathbb{R}_{>0})^r$ of $\mathrm{U}(V')$ contained in P', and

$$A'^+ = \big\{(a_1,\dots,a_r) \in A' \;\big|\; 0 < a_1 \le \cdots \le a_r \le 1\big\}.$$

Fix a maximal compact subgroup K' of $\mathrm{U}(V')$ such that we have the Cartan decomposition $\mathrm{U}(V') = K'A'^+K'$.

Using the integration formula (3.6), estimates (3.7) and (3.8), and Lemma 3.1, we are reduced to prove that for any $B > 0$, the integral

$$\iint_{\substack{0<a_1\le\cdots\le a_r\le 1\\ 0<b_1\le\cdots\le b_s\le 1}} \prod_{i=1}^{r} a_i^{-(n+1-2i)}\Big(1-\sum_{i=1}^{r}\log a_i\Big)^B \prod_{j=1}^{s} b_j^{-(n+1-2j)}\Big(1-\sum_{j=1}^{s}\log b_j\Big)^B$$
$$\prod_{i=1}^{r} a_i^{n+1}\cdot\prod_{j=1}^{s} b_j^{n-2r}\cdot\prod_{i=1}^{r}\prod_{j=1}^{s}\Upsilon(a_ib_j^{-1})^2\,\mathrm{d}^\times a_1\cdots\mathrm{d}^\times a_r\,\mathrm{d}^\times b_1\cdots\mathrm{d}^\times b_s$$

is convergent. This simplifies to

$$\iint_{\substack{0<a_1\le\cdots\le a_r\le 1\\ 0<b_1\le\cdots\le b_s\le 1}} \prod_{i=1}^{r} a_i^{2i}\Big(1-\sum_{i=1}^{r}\log a_i\Big)^B \prod_{j=1}^{s} b_j^{2j-1-2r}\Big(1-\sum_{j=1}^{s}\log b_j\Big)^B$$
$$\prod_{i=1}^{r}\prod_{j=1}^{s}\Upsilon(a_ib_j^{-1})^2\,\mathrm{d}^\times a_1\cdots\mathrm{d}^\times a_r\,\mathrm{d}^\times b_1\cdots\mathrm{d}^\times b_s. \tag{3.18}$$

Let $(p_1,\dots,p_{s+1})$ be an $(s+1)$-tuple of nonnegative integers with $p_1 + \cdots + p_{s+1} = r$. Define $S_{p_1,\dots,p_{s+1}}$ to be the region in $A'^+ \times A_W^+$ defined by the inequalities

$$0 < a_1 \leq \cdots \leq a_{p_1} \leq b_1 \leq a_{p_1+1} \leq \cdots \leq a_{p_1+p_2} \leq b_2$$
$$\leq \cdots \leq a_{p_1+\cdots+p_s} \leq b_s \leq a_{p_1+\cdots+p_s+1} \leq \cdots \leq a_{p_1+\cdots+p_{s+1}} \leq 1.$$

The domain of integration is a finite union of the regions of the form $S_{p_1,\ldots,p_{s+1}}$. It is thus enough to show that the integral (3.18), with the domain of integration replaced by $S_{p_1,\ldots,p_{s+1}}$, is convergent. Elementary computation gives that when the a_i's and b_j's are in this region, we have

$$\prod_{\substack{i=1,\ldots,r\\ j=1,\ldots,s}} \Upsilon(a_i b_j^{-1}) = (a_{p_1+1}\cdots a_{p_1+p_2})^{-1} \cdot (a_{p_1+p_2+1}\cdots a_{p_1+p_2+p_3})^{-2}$$
$$\cdots (a_{p_1+\cdots+p_s+1}\cdots a_r)^{-s} \cdot b_1^{r-p_1} \cdot b_2^{r-p_1-p_2} \cdots b_s^{r-p_1-\cdots-p_s}.$$

We now check that the integration over $S_{p_1,\ldots,p_{s+1}}$ satisfies the condition of the fact that we noted at the beginning of the proof. The sum of the exponents of the terms up to $a_{p_1+\cdots+p_l+t}$ $(0 \leq t \leq p_{l+1})$ is

$$2+4+\cdots+2(p_1+\cdots+p_l+t)-2p_2-4p_3-\cdots-2(l-1)p_l-2lt$$
$$+l^2-2rl+2(r-p_1)+\cdots+2(r-p_1-\cdots-p_l).$$

Here the first line is the sum of the exponents of a_i and the second is the sum of the exponents of b_j. It simplifies to

$$(p_1+\cdots+p_l+t-l)^2+(p_1+\cdots+p_l+t)>0.$$

Similarly, the sum of the exponents of the terms up to b_l $(l \geq 1)$ is

$$2+4+\cdots+2(p_1+\cdots+p_l)-2p_2-4p_3-\cdots-2(l-1)p_l$$
$$+l^2-2rl+2(r-p_1)+\cdots+2(r-p_1-\cdots-p_l).$$

Here again the first line is the sum of the exponents of a_i and the second is the sum of the exponents of b_j. It simplifies to

$$(p_1+\cdots+p_l-l)^2+(p_1+\cdots+p_l)>0.$$

The convergence of the integral (3.18) is thus proved. □

LEMMA 3.8
We have $\mathcal{L}_{\pi,\sigma} \neq 0$ if and only if $\sigma' = \theta_{W,V'}(\overline{\sigma}) \neq 0$ and $\mathcal{L}_{\pi',\overline{\sigma'},\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$.

Proof
Consider the Weil representation $\omega_{V,V'}$ of $\mathrm{U}(V) \times \mathrm{U}(V')$. By [16, Lemma 2.8], we have

$$\omega_{V,V'}|_{\mathrm{U}(W)\times\mathrm{U}(V')} \simeq \omega_{W,V'} \mathbin{\widehat{\otimes}} \omega_{L,V'}.$$

Let $\phi, \phi' \in \omega_{W,V'}$ and $\varphi, \varphi' \in \omega_{L,V'}$. Then

$$\langle \omega_{V,V'}(g,h)\phi \otimes \varphi, \phi' \otimes \varphi' \rangle = \langle \omega_{W,V'}(g,h)\phi, \phi' \rangle \langle \omega_{L,V'}(h)\varphi, \varphi' \rangle$$

if $g \in \mathrm{U}(W)$ and $h \in \mathrm{U}(V')$.

Let $\alpha'(h)$ be a matrix coefficient of π'. Then by Proposition 3.4, the function

$$g \mapsto \int_{\mathrm{U}(V')} \overline{\alpha'(h)} \langle \omega_{V,V'}(g,h)\phi \otimes \varphi, \phi' \otimes \varphi' \rangle \, \mathrm{d}h, \quad g \in \mathrm{U}(V),$$

is a matrix coefficient of π, and when α' varies over all matrix coefficients of π' and ϕ, ϕ', φ, φ' vary over all elements in $\omega_{W,V'}$ and $\omega_{L,V'}$, respectively, the matrix coefficients of this form generate a dense subspace of all matrix coefficients of π. It follows the continuity of $\mathcal{L}_{\pi,\sigma}$ that if $\mathcal{L}_{\pi,\sigma} \neq 0$, then we can choose α', ϕ, ϕ', φ, φ', and a matrix coefficient β of σ such that

$$\int_{\mathrm{U}(W)} \beta(g) \Big(\int_{\mathrm{U}(V')} \overline{\alpha'(h)} \langle \omega_{W,V'}(g,h)\phi, \phi' \rangle \langle \omega_{L,V'}(h)\varphi, \varphi' \rangle \, \mathrm{d}h \Big) \, \mathrm{d}g \neq 0.$$

As π' and σ are both tempered, the double is absolutely convergent by Lemma 3.7.

We can then switch the order of integration and integrate over $\mathrm{U}(W)$ first. Then by Proposition 3.4 again, the integral

$$h \mapsto \int_{\mathrm{U}(W)} \beta(g) \langle \omega_{W,V'}(g,h)\phi, \phi' \rangle \, \mathrm{d}g$$

defines a nonzero matrix coefficient of $\sigma' = \theta(\overline{\sigma})$. In particular, $\sigma' \neq 0$. Denote this matrix coefficient by $\gamma(h)$. We conclude that

$$\int_{U(V')} \overline{\alpha'}(h)\gamma(h) \langle \omega_{L,V'}(h)\varphi, \varphi' \rangle \, \mathrm{d}h \neq 0.$$

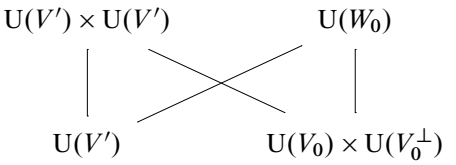
Note that as a representation of $\mathrm{U}(V')$, $\omega_{L,V'}$ is isomorphic to $\omega_{\psi,\mu}$ if n is even and to $\overline{\omega_{\psi,\mu}} = \omega_{\psi^{-1},\mu^{-1}}$ if n is odd (note that the splitting characters are different in these two cases). Therefore, $\mathcal{L}_{\pi',\overline{\sigma'},\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$.

Reversing the argument gives the converse implication. □

Now let V' be a skew-Hermitian space of dimension n, and let $V_0 \subset W_0$ be a relevant pair of Hermitian spaces of dimensions $n-1$ and n, respectively. Let π' and σ' be irreducible tempered representations of $\mathrm{U}(V')$. Assume that there is an irreducible tempered representation π_0 of $\mathrm{U}(V_0)$ so that $\pi' = \theta_{V_0,V'}(\pi_0)$. Let $\sigma_0 = \theta_{V',W_0}(\overline{\sigma'})$.

LEMMA 3.9
We have $\mathcal{L}_{\pi',\sigma',\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$ *if and only if* $\sigma_0 \neq 0$ *and* $\mathcal{L}_{\overline{\sigma_0},\pi_0} \neq 0$.

We omit the proof of this lemma, which exploits the following seesaw diagram and is exactly the same as that of Lemma 3.8:



4. Proof of the main theorem

4.1. Setup
We prove Theorem 2.2 in the case $t = 0$ and Proposition 3.6 in this section. Let us recall the setup.

$\mathrm{U}(n+1) \times \mathrm{U}(n)$.

- We have a relevant pair of Hermitian spaces $W \subset V$ of dimensions n and $n+1$, respectively.
- Let π and σ be irreducible tempered representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$, respectively. The parameters of π and σ are (ϕ_π, η_π) and $(\phi_\sigma, \eta_\sigma)$, respectively.

$\mathrm{U}(n) \times \mathrm{U}(n)$.

- We have a skew-Hermitian space V' of dimension n.
- We fix the character $\mu(z) = z(z\overline{z})^{-\frac{1}{2}}$ of $\mathbb{C}^\times$. We have a Weil representation $\omega_{\psi,\mu}$ of $\mathrm{U}(V')$.
- Let π' and σ' be irreducible tempered representations of $\mathrm{U}(V')$. The parameters of π' and σ' are $(\phi_{\pi'}, \eta_{\pi'})$ and $(\phi_{\sigma'}, \eta_{\sigma'})$, respectively.

Assume that ϕ_π contains a conjugate self-dual characters of sign $(-1)^n$ and that ϕ_σ contains b conjugate self-dual characters of sign $(-1)^{n-1}$ (both counting multiplicity). We prove by induction on $a + b = N$. Note that a (resp., b) has the same parity as $n+1$ (resp., n), and thus N is odd.

The base case is $a + b = 1$. In this case, π and σ are both irreducible (tempered) principal series representations. More precisely, assume that $W \subset V$ are split relevant Hermitian spaces, that is, their signatures are either (p, p), $(p+1, p)$ or $(p, p-1)$, (p, p), respectively. Let $B_2 = T_2U_2$ be a Borel subgroup of $\mathrm{U}(V)$ with U_2 the unipotent radical and T_2 a maximal torus. Let $B_1 = T_1U_1$ be a Borel subgroup of $\mathrm{U}(W)$ with U_1 the unipotent radical and T_1 a maximal torus. We fix the following data.

- Assume that $\dim V = 2p+1$. Let $\chi_1, \dots, \chi_p$ be characters of $\mathbb{C}^\times$, let χ_0 be a character of $\mathbb{C}^1$, and let $\underline{\chi}$ be a character of $T_2 \simeq (\mathbb{C}^\times)^p \times \mathbb{C}^1$ given by $\chi_1 \otimes \cdots \otimes \chi_p \otimes \chi_0$. Let $\mu_1, \dots, \mu_p$ be characters of $\mathbb{C}^\times$, and let $\underline{\mu}$ be a character of $T_1 \simeq (\mathbb{C}^\times)^p$ given by $\mu_1 \otimes \cdots \otimes \mu_p$.
- Assume that $\dim V = 2p$. Let $\chi_1, \dots, \chi_p$ be characters of $\mathbb{C}^\times$, and let $\underline{\chi}$ be a character of $T_2 \simeq (\mathbb{C}^\times)^p$ given by $\chi_1 \otimes \cdots \otimes \chi_p$. Let $\mu_1, \dots, \mu_{p-1}$ be characters of $\mathbb{C}^\times$, let μ_0 be a character of $\mathbb{C}^1$, and let $\underline{\mu}$ be a character of T_1 given by $\mu_1 \otimes \cdots \otimes \mu_{p-1} \otimes \mu_0$.

In both cases, let $\pi = \mathrm{Ind}_{B_2}^{\mathrm{U}(V)} \underline{\chi}$ and $\sigma = \mathrm{Ind}_{B_1}^{\mathrm{U}(W)} \underline{\mu}$ be principal series representations of $\mathrm{U}(V)$ and $\mathrm{U}(W)$, respectively.

LEMMA 4.1
We have $\mathcal{L}_{\pi,\sigma} \neq 0$. In particular, if π and σ are both irreducible, then Theorem 2.2 holds for (π, σ).

Proof
We prove $\mathcal{L}_{\pi,\sigma} \neq 0$ by induction on $\dim V = n+1$. The base case $n = 0$, $\dim V = 1$ is trivial. Assume now that $\dim V \geq 2$. Let $V_0 \subset W$ be relevant Hermitian spaces, and let $\dim V_0 = n-1$. Then $\mathrm{U}(V_0)$ is quasisplit. Let π_0 be the principal series representation of $\mathrm{U}(V_0)$ defined as follows. By conjugating the Borel subgroup B_2 suitably, we may assume that $B_0 = B_2 \cap \mathrm{U}(V_0)$ is a Borel subgroup of $\mathrm{U}(V_0)$ and that $T_0 = T \cap \mathrm{U}(V_0)$ is a maximal torus. Let $\underline{\chi_0}$ be the character of T_0 given by $\chi_1 \otimes \cdots \otimes \chi_{p-1} \otimes \chi_0$ if $\dim V = 2p+1$ and by $\chi_1 \otimes \cdots \otimes \chi_{p-1}$ if $\dim V = 2p$. Let $\pi_0 = \mathrm{Ind}_{B_0}^{\mathrm{U}(V_0)} \underline{\chi_0}$ be a principal series representation of $\mathrm{U}(V_0)$. Then $\pi = \chi_p \rtimes \pi_0$. By the induction hypothesis, we have $\mathcal{L}_{\sigma,\pi_0} \neq 0$. Then Proposition 2.4 implies that $\mathcal{L}_{\pi,\sigma} \neq 0$.

Assume that π and σ are irreducible, that is, none of the χ_i's, $i \neq 0$, are conjugate self-dual of sign $(-1)^{\dim V - 1}$ and none of the μ_j's, $j \neq 0$ are conjugate self-dual of sign $(-1)^{\dim W - 1}$. If $\dim V = 2p+1$, then $A_{\phi_\pi} = \mathbb{Z}/2\mathbb{Z}$ and $A_{\phi_\sigma} = \{1\}$. If $\dim V = 2p$, then $A_{\phi_\pi} = \{1\}$ and $A_{\phi_\sigma} = \mathbb{Z}/2\mathbb{Z}$. In both cases, η_π and η_σ are trivial characters. We thus have $m(\pi, \sigma) = 1$ and η_π and η_σ are both trivial characters. This follows from our choice of the Whittaker datum in the Langlands–Vogan parameterization, as we picked the Whittaker data for $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$ if n is odd. Therefore, Theorem 2.2 holds. □

From now on let us assume that $a + b = N \geq 3$ and that Theorem 2.2 has been proved for all (π, σ) with $a + b < N$. This induction hypothesis is in effect throughout this section.

4.2. The case $a \geq 2$

We first look at the Fourier–Jacobi model. Assume that $\phi_{\pi'}$ (resp., $\phi_{\sigma'}$) contains a' (resp., b') conjugate self-dual characters of sign $(-1)^{n-1}$ (again both counting multiplicity).

LEMMA 4.2

Assume that $a' + b' < N$ and $a' \geq 1$. Then Proposition 3.6 holds for (π', σ').

Proof

We will be dealing with various theta lifts. We fix the choice of characters to split the metaplectic cover as in Section 3.4.

Note first that $\mathcal{L}_{\pi',\sigma',\psi,\mu}$ and the local root numbers in Proposition 3.6 will not change if π' and σ' are replaced by $\pi' \otimes \det^k$ and $\sigma' \otimes \det^{-k}$, respectively, for all integers k. It follows that Proposition 3.6 holds for (π', σ') if and only if it holds for $(\pi' \otimes \det^k, \sigma' \otimes \det^{-k})$. Because of the assumption that $a' \geq 1$, by Lemma 3.5, we may assume that there is a Hermitian space V_0 of dimension $n-1$ and an irreducible tempered representation π_0 of $\mathrm{U}(V_0)$ so that $\pi' = \theta_{V_0,V'}(\pi_0)$. The cases n being even or odd differ slightly in notation. We will treat the case n being even and leave the case n being odd to the interested reader.

Assume that n is even. Put $W_0 = V_0 \oplus L_{-1}$ and $\sigma_0 = \theta_{V',W_0}(\overline{\sigma'})$. By Lemma 3.9, we have $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$ if and only if $\sigma_0 \neq 0$ and $\mathcal{L}_{\overline{\sigma_0},\pi_0} \neq 0$.

Let us now compute the parameters. Since $\pi' = \theta_{V_0,V'}(\pi_0)$, by (3.13) and our choice of the splitting characters in Section 3.4, the parameter $\phi_{\pi'}$ contains μ^{n+1}. We write

$$\phi_{\pi'} = m_1\chi_1 \oplus \cdots \oplus m_k\chi_k, \qquad \phi_{\sigma'} = n_1\mu_1 \oplus \cdots \oplus n_l\mu_l,$$

where

- $\chi_i = \mu^{\alpha_i}$, $i = 1, \ldots, r$, with α_i distinct odd integers, $\alpha_r = n+1$, and χ_i, $i > r$, are not conjugate self-dual of sign $(-1)^{n-1}$;
- $\mu_j = \mu^{\beta_j}$, $i = 1, \ldots, s$, with β_j distinct odd integers, and μ_j, $j > s$, are not conjugate self-dual of sign $(-1)^{n-1}$.

Then

$$A_{\phi_{\pi'}} = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})a_i, \qquad A_{\phi_{\sigma'}} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j.$$

By (3.13), we have

$$\phi_{\pi_0} = m_1\mu^{\alpha_1 - 1} \oplus \cdots \oplus m_{r-1}\mu^{\alpha_{r-1}-1} \oplus (m_r - 1)\mu^{\alpha_r - 1} \oplus *$$

and

$$\phi_{\sigma_0} = n_1\mu^{-\beta_1} \oplus \cdots \oplus n_s\mu^{-\beta_s} \oplus *,$$

where $*$ consists of characters that are not conjugate self-dual of sign $(-1)^n$ in ϕ_{π_0} and of sign $(-1)^{n-1}$ in ϕ_{σ_0}, respectively.

Assume that $\mathcal{L}_{\pi',\sigma',\psi,\mu} \neq 0$. Then $\sigma_0 \neq 0$ and $\mathcal{L}_{\overline{\sigma_0},\pi_0} \neq 0$ and hence $\mathcal{L}_{\sigma_0,\overline{\pi_0}} \neq 0$. We distinguish two cases: $m_r > 1$ and $m_r = 1$.

Assume first that $m_r > 1$. Then $A_{\phi_{\pi'}} = A_{\phi_{\pi_0}}$ and $A_{\phi_{\sigma'}} = A_{\phi_{\sigma_0}}$. By the induction hypothesis, we have

$$\eta_{\pi_0}(a_i) = \eta_{\overline{\pi_0}}(a_i) = \epsilon(\phi_{\sigma_0} \otimes \mu^{-\alpha_i+1}, \psi^{\mathbb{C}}), \qquad \eta_{\sigma_0}(b_j) = \epsilon(\phi_{\overline{\pi_0}} \otimes \mu^{-\beta_j}, \psi^{\mathbb{C}}).$$

Here the first equality $\eta_{\pi_0} = \eta_{\overline{\pi_0}}$ follows from Proposition 2.1(3). Therefore again by (3.15), we have

$$\eta_{\pi'}(a_i) = \eta_{\pi_0}(a_i) = \epsilon(\phi_{\sigma_0} \otimes \mu^{-\alpha_i+1}, \psi^{\mathbb{C}}) = \epsilon(\phi_{\sigma'} \otimes \mu^{\alpha_i-1}, \overline{\psi^{\mathbb{C}}}), \quad i = 1, \ldots, r,$$

and

$$\begin{aligned}\eta_{\sigma'}(b_j) &= -\eta_{\sigma_0}(b_j)\epsilon(\mu^{-\beta_j-n}, \overline{\psi^{\mathbb{C}}}) = -\epsilon(\phi_{\pi_0} \otimes \mu^{\beta_j}, \overline{\psi^{\mathbb{C}}})\epsilon(\mu^{-\beta_j-n}, \overline{\psi^{\mathbb{C}}}) \\ &= \epsilon(\phi_{\pi'} \otimes \mu^{\beta_j-1}, \overline{\psi^{\mathbb{C}}}).\end{aligned}$$

Here the minus sign is a result of $\eta_{\overline{\sigma_0}}(b_j) = -\eta_{\sigma_0}(b_j)$, which again follows from Proposition 2.1(3). The last equality follows from the fact that $\phi_{\pi'} = (\phi_{\pi_0} \otimes \mu) \oplus \mu^{n+1}$ by (3.13) and the identity

$$\epsilon(\mu^{\beta_j+n}, \overline{\psi^{\mathbb{C}}})\epsilon(\mu^{-\beta_j-n}, \overline{\psi^{\mathbb{C}}}) = \mu^{\beta_j+n}(-1) = -1,$$

where the first equality follows from (2.3) and the second follows from the fact that β_j is odd and n is even. This proves that the parameters of π' and σ' are as described in Proposition 3.6 when $m_r > 1$.

Assume now that $m_r = 1$. Then

$$A_{\phi_{\pi_0}} = \bigoplus_{i=1}^{r-1} (\mathbb{Z}/2\mathbb{Z})a_i, \qquad A_{\phi_{\sigma_0}} = A_{\phi_{\sigma'}} = \bigoplus_{j=1}^{s} (\mathbb{Z}/2\mathbb{Z})b_j.$$

The values $\eta_{\pi'}(a_i)$, $i = 1, \ldots, r-1$ and $\eta_{\sigma'}(b_j)$, $j = 1, \ldots, s$ can be computed in the same way as in the case $m_r > 1$. It remains to compute $\eta_{\pi'}(a_r)$. Let us temporarily assume that the signature of W_0 is (p, q) and the signature of $-iV'$ is (p', q'). Then $\operatorname{disc}(-iV') = (-1)^{\frac{n(n-1)}{2}-q'}$ and $\operatorname{disc} W_0 = (-1)^{\frac{n(n-1)}{2}-q}$. By (3.14) and Proposition 2.1(4), we have

$$\epsilon(\phi_{\overline{\sigma'}} \otimes \mu^{-n}, \psi^{\mathbb{C}}) = \operatorname{disc} W_0 \operatorname{disc}(-iV') = (-1)^{-q'-q}.$$

We also have

$$\eta_{\pi'}(a_r) = \eta_{\pi'}(a_1 + \cdots + a_{r-1} + a_r)\eta_{\pi_0}(a_1 + \cdots + a_{r-1})$$
$$= (-1)^{\frac{n}{2}-q'}(-1)^{(\frac{n}{2}-1)-(q-1)} = (-1)^{-q'-q}.$$

It follows that

$$\eta_{\pi'}(a_r) = \epsilon(\phi_{\overline{\sigma'}} \otimes \mu^{-n}, \psi^{\mathbb{C}}) = \epsilon(\phi_{\sigma'} \otimes \mu^{n}, \overline{\psi^{\mathbb{C}}}).$$

This proves that the parameters of π' and σ' are as described in Proposition 3.6 if $m_r = 1$.

The converse implication can be proved by reversing this argument. □

LEMMA 4.3
Theorem 2.2 holds for (π, σ) *if* $a \geq 2$.

Proof
We again note first that Theorem 2.2 holds for (π, σ) if and only if it holds for $(\pi \otimes \det^k, \sigma \otimes \det^{-k})$ for any k. Thus we may assume that there is a skew-Hermitian space V' of dimension n and an irreducible tempered representation π' such that $\pi = \theta_{V',V}(\pi')$. By Proposition 2.3, $m(\pi, \sigma) = 1$ is equivalent to $\mathcal{L}_{\pi,\sigma} \neq 0$. Thus we are reduced to proving Theorem 2.2 with the condition $m(\pi, \sigma) = 1$ replaced by $\mathcal{L}_{\pi,\sigma} \neq 0$. Let $\sigma' = \theta_{W,V'}(\overline{\sigma})$. By Lemma 3.8, $\mathcal{L}_{\pi,\sigma} \neq 0$ if and only if $\sigma' = \theta_{W,V'}(\overline{\sigma}) \neq 0$ and $\mathcal{L}_{\pi',\overline{\sigma'},\psi^{(-1)^n},\mu^{(-1)^n}} \neq 0$. Assume that $\phi_{\pi'}$ and $\phi_{\sigma'}$ contain a' and b' conjugate self-dual characters of sign $(-1)^{n-1}$, respectively. Then by (3.10) and (3.13), we have $a' = a - 1 \geq 1$ and $b' = b$, $a' + b' = N - 1 < N$. Therefore, Proposition 3.6 holds for (π', σ') by Lemma 4.2. The rest of the argument is to deduce Theorem 2.2 for (π, σ) from Proposition 3.6 for (π', σ'). This does not differ much from Lemma 4.2 and we omit the details. □

4.3. The case $b \geq 2$

LEMMA 4.4
Theorem 2.2 holds for (π, σ) *if* $b \geq 2$.

Proof
Let χ be any character of $\mathbb{C}^\times$ that is *not* conjugate self-dual of sign $(-1)^{n-1}$. Let $W_1 = W \oplus^\perp \langle z_1, z_{-1} \rangle$, where $z_{\pm 1}$ are isotropic vectors with $h_{W_1}(z_1, z_{-1}) = 1$. Let $\sigma_1 = \chi \rtimes \sigma$ be the induced representation of $\mathrm{U}(W_1)$. Then σ_1 is irreducible and tempered, and we have $\phi_{\sigma_1} = \phi_\sigma \oplus \chi \oplus \chi^{c,-1}$, $A_{\phi_{\sigma_1}} = A_{\phi_\sigma}$, and $\eta_{\sigma_1} = \eta_\sigma$. Note that ϕ_{σ_1} contains b conjugate self-dual characters of sign $(-1)^{n+1}$, and ϕ_π contains a conjugate self-dual characters of sign $(-1)^n$.

Assume that $m(\pi,\sigma)=1$. Then $m(\sigma_1,\pi)=1$ by Proposition 2.4. Now apply Lemma 4.3 to (σ_1,π). We conclude that the parameters of σ_1 and π are as specified in Theorem 2.2. It then follows that the parameters of π and σ are as specified in Theorem 2.2.

Let us now assume that the parameters of π and σ are given as in Theorem 2.2. Then by our construction the parameters of σ_1 and π are given as in Theorem 2.2. It follows from Lemma 4.3 that $m(\sigma_1,\pi)=1$. Then $m(\pi,\sigma)=1$ by Propositions 2.3 and 2.4. □

4.4. End of the proof

We now prove Theorem 2.2 and Proposition 3.6. As $a+b\geq 3$, we have either $a\geq 2$ or $b\geq 2$. So we conclude by either Lemma 4.3 or 4.4. This completes the induction and proves Theorem 2.2. Proposition 3.6 then follows from Theorem 2.2 by an argument similar to that in Lemma 4.2.

Acknowledgments. I thank Wee Teck Gan, Atsushi Ichino, and Yifeng Liu for helpful discussions, and Ye Tian and Shou-Wu Zhang for their constant support. I also thank the anonymous referees for their many suggestions which helped improve the paper's readability. Special thanks go to my family and particularly my wife for their support throughout the difficult and stressful time of the coronavirus pandemic during which this manuscript was prepared. By taking on most of the child-caring duties during this period, my wife was instrumental in allowing me the time to "escape" and do math.

The author's work was partially supported by National Science Foundation grant DMS-1901862.

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