DIFFUSING ORBITS ALONG CHAINS OF CYLINDERS

Marian Gidea[✉][*][1] and Jean-Pierre Marco[✉][2]

[1]Department of Mathematical Sciences, Yeshiva University,
215 Lexington Ave, New York, NY 10016, USA

[2]Sorbonne Université (Paris 6) and Institut de Mathématiques de Jussieu-Paris Rive Gauche,
4 Place Jussieu, 75005 Paris cedex 05, France

(Communicated by Amadeu Delshams)

Abstract. We develop a geometric mechanism to prove the existence of orbits that drift along a prescribed sequence of cylinders, under some general conditions on the dynamics. This mechanism can be used to prove the existence of Arnold diffusion for large families of perturbations of Tonelli Hamiltonians on $\mathbb{A}^3$. Our approach can also be applied to more general Hamiltonians that are not necessarily convex.

The main geometric objects in our framework are 3–dimensional invariant cylinders with boundary (not necessarily hyperbolic), which are assumed to admit center-stable and center-unstable manifolds. These enable us to define *chains of cylinders*, i.e., finite, ordered families of cylinders where each cylinder admits homoclinic connections, and any two consecutive cylinders in the chain admit heteroclinic connections.

Our main result is on the existence of diffusing orbits which drift along such chains of cylinders, under precise conditions on the dynamics on the cylinders – i.e., the existence of Poincaré sections with the return maps satisfying a *tilt condition* – and on the geometric properties of the intersections of the center-stable and center-unstable manifolds of the cylinders – i.e., certain compatibility conditions between the *tilt map* and the homoclinic maps associated to its essential invariant circles.

We give two proofs of our result, a very short and abstract one, and a more constructive one, aimed at possible applications to concrete systems.

1. **Introduction and main result.** The present work originates in a geometric proof of the Arnold conjecture in the Mather *a priori stable* setting[1] for 3-degree-of-freedom Hamiltonian systems (see [47, 48] which give the necessary geometric framework and the application to the proof). Our paper is however self-contained, and our results can also be applied to other contexts, as it will be detailed below.
1. Let us first informally recall the convex setting for Arnold diffusion. Given $n \geq 1$, we denote by $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ the cotangent bundle of the torus $\mathbb{T}^n$, equipped

---

[1]We pay homage to the fundamental role played by Mather in the development of the field, and in providing many ideas and inspiration [52, 53, 54].

with its natural angle-action coordinates (θ, r) and its exact-symplectic form $\Omega = \sum_{i=1}^{n} dr_i \wedge d\theta_i$. Consider a Hamiltonian of class C^κ on $\mathbb{A}^3$, of the form

$$H(\theta, r) = h(r) + f(\theta, r), \qquad (\theta, r) \in \mathbb{A}^3, \tag{1}$$

where κ is large enough, and h is strictly convex and with superlinear growth at infinity w.r.t. the actions[2]. We think of the Hamiltonian H as a small perturbation of the Hamiltonian h. This setting of the Arnold diffusion problem, where the dynamics of h is described in terms of angle-action variables only, is referred to as the *a priori stable* case, whereas the setting where the dynamics of h is the product of an angle-action factor with a hyperbolic one is referred to as the *a priori unstable* case.

The main problem is the *a priori* stable case. Namely, fix a regular level set $h^{-1}(\mathbf{e})$ of the unperturbed energy in the action space $\mathbb{R}^3$, consider an arbitrary family $(\widehat{O}_i)_{1 \le i \le m}$ of small open subsets in $\mathbb{R}^3$ which intersect $h^{-1}(\mathbf{e})$, and set $O_i = \mathbb{T}^3 \times \widehat{O}_i$. The diffusion problem is on proving that the system H admits orbits visiting each O_i, for a large class of perturbations f in $C^\kappa(\mathbb{A}^3)$. Such orbits will be referred to as *diffusing orbits*. The *a priori* unstable case is an intermediate but very interesting problem which serves as a guide for developing the methods.

Three independent approaches of the diffusion problem in the *a priori* stable setting were developed recently: see [14, 38, 48] and references therein. A common feature of the works [14, 38, 48] is the use of "normally hyperbolic cylinders" which form "chains" intersecting the open sets O_i. Once the existence of such chains of cylinders is shown (under appropriate nondegeneracy conditions on f), proving the existence of diffusing orbits amounts to proving the existence of orbits "drifting along the cylinders as well as from one cylinder to the next", possibly under additional nondegeneracy conditions on f.

The very definition of normally hyperbolic cylinders and chains is *not* the same in the three approaches, mainly regarding the invariance condition of the cylinders under the Hamiltonian flow. In [48] the cylinders are 3-dimensional, *genuinely invariant*, normally hyperbolic and compact (with boundary), which ensures the existence of well-defined asymptotic manifolds, namely the center-stable and center-unstable manifolds consisting of orbits that approach the cylinder in forward and backward time, respectively.

2. In this paper we consider a more general situation, that of *tame cylinders*, which possess asymptotic manifolds and admit a lambda-lemma type property, see below. Crossings between these asymptotic manifolds enables us to define homoclinic connections for each cylinder, as well as heteroclinic connections between consecutive cylinders, which allows us to formulate a natural geometric definition of a *chain of cylinders*. The main interest in this notion is that non-convex integrable systems yield invariant manifolds which are, as a rule, not normally hyperbolic but only tame.

Once a chain of tame cylinders is given, the central idea of our approach is to perform a systematic reduction to two dimensional dynamics, which allows us to go back to very simple arguments to detect the existence of diffusing orbits. Here again, we have non-convex situations in mind and we assume that the dynamics inside our cylinders admit a section whose return map is a *tilt map* (instead of the more usual assumption of being a *twist map*). Then, we will not need more than

---

[2]The strict convexity assumption is not present in Arnold's original formulation of the problem, while the superlinear growth is assumed only to get compact energy levels.

the usual Birkhoff theory of twist maps[3], properly generalized here following an initial idea by Moeckel (see [55]), to produce diffusing pseudo-orbits along the chain (see the definition below). Thanks to the lambda-lemma property and the Poincaré recurrence theorem, a suitable shadowing process then easily yields the existence of true orbits of the perturbed system following those pseudo-orbits, and therefore drifting along the chain.

3. We now briefly describe our set-up and the main results. We consider a C^κ Hamiltonian function H on $\mathbb{A}^3$, with $\kappa \geq 2$. Although it is not absolutely necessary, we find it convenient to restrict our study to regular levels $H^{-1}(\mathbf{e})$ on which the Hamiltonian vector field is complete. In this case we say that $\mathbf{e}$ (or the level $H^{-1}(\mathbf{e})$) is *completely regular*.

In this paper, a *cylinder* is a compact submanifold with boundary of $\mathbb{A}^3$, diffeomorphic to $\mathbb{T}^2 \times [0, 1]$. We consider cylinders invariant under the flow generated by a Hamiltonian function H on $\mathbb{A}^3$, and contained in a given, completely regular energy level $H^{-1}(\mathbf{e})$ of H. In this paper we consider more general conditions on the cylinders rather than the usual normal hyperbolicity, and call them *tame cylinders*; details will be given in Section 2.1. In particular, a tame cylinder $\mathscr{C}$ is 3-dimensional, admits 4-dimensional center-stable and center-unstable manifolds (with boundary) $W^\pm(\mathscr{C})$, which are themselves foliated by the 1-dimensional stable and unstable manifolds $W^\pm(x)$, respectively, associated to the points $x \in \mathscr{C}$. Moreover, a tame cylinder admits a lambda-lemma property[4].

A *chain of cylinders* for H is a finite ordered sequence $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ of tame cylinders contained in the same energy level $H^{-1}(\mathbf{e})$, such that each cylinder $\mathscr{C}_k$, for $1 \leq k \leq k_*$, admits homoclinic connections, that is

$$W^-(\mathscr{C}_k) \cap W^+(\mathscr{C}_k) \neq \emptyset,$$

and each consecutive pair of cylinders $\mathscr{C}_k$ and $\mathscr{C}_{k+1}$ in the chain, for $1 \leq k \leq k_* - 1$, admits heteroclinic connections, that is

$$W^-(\mathscr{C}_k) \cap W^+(\mathscr{C}_{k+1}) \neq \emptyset.$$

To ensure the existence of orbits drifting along a chain, we will require additional conditions on the dynamics on the cylinders, and their homoclinic and heteroclinic connections. In this paper, a cylinder (resp. a chain) satisfying those additional properties is called a *good cylinder* (resp. a *good chain*, see Figure 1); see Section 2.3 for the definitions.

Given a good cylinder $\mathscr{C}$, we define an *essential subtorus* of $\mathscr{C}$ as a 2–dimensional torus (not necessarily differentiable) contained in $\mathscr{C}$ and invariant under the Hamiltonian flow, which intersects a certain Poincaré section $\Sigma \sim \mathbb{T} \times [a, b]$ along an essential circle[5] (see Definition 2.4). Examples of essential subtori are the components of the boundary $\partial \mathscr{C}$; any essential subtorus is homotopic in $\mathscr{C}$ to each of these components. The family of essential subtori will serve us as a guide to building our drifting orbits.

Definition 1.1. Consider a good chain $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ contained in some completely regular energy level $H^{-1}(\mathbf{e})$. Given $\delta > 0$, we say that an orbit of the Hamiltonian flow is *δ-admissible for the chain* when it intersects the δ-neighborhood in $H^{-1}(\mathbf{e})$ of any essential subtorus of $\mathscr{C}_k$, $1 \leq k \leq k_*$.

---

[3] With its now well-known extensions to tilt maps.

[4] A normally hyperbolic invariant cylinder is a particular case of a tame cylinder.

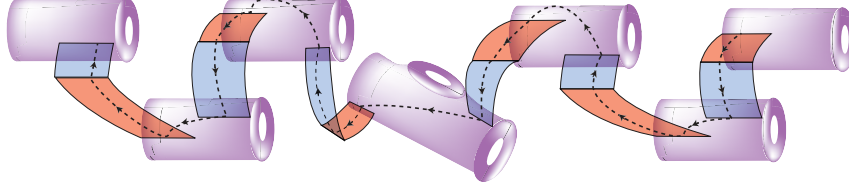[5] That is, homotopic to $\mathbb{T} \times \{a\}$ inside Σ.

FIGURE 1. An illustration of a cylinder chain, consisting of a sequence of tame cylinders with boundary, connected via homoclinic and heteroclinic orbits. This chain also contains a singular cylinder (shown in the center of the figure); see Section 3.3 for details.

The notion of δ-admissible orbits reveals itself to be useful in [48] to obtain the existence of diffusing orbits for systems (1) once a chain of cylinders that intersects each open set O_i is constructed[6]. A related notion is that of a δ-good chain, which involves a quantitative control on the homoclinic and heteroclinic connections of the cylinders. This will be introduced in Definitions 2.13 and 2.18.

The first main result of this paper is the following.

Theorem 1.2. *Let H be a C^κ Hamiltonian on $\mathbb{A}^3$, with $\kappa \geq 2$, and let $\mathbf{e}$ be a completely regular value of H. Fix $\delta > 0$. Then, for any δ-good chain of cylinders contained in $H^{-1}(\mathbf{e})$, there exists a δ-admissible orbit for the chain.*

Otherwise explicitly mentioned, the energies $\mathbf{e}$ and the energy levels we consider in the following will be assumed to be completely regular.

The proof of **Theorem 1.2** is divided into two parts. In the first part, we introduce a *polysystem* of maps and homoclinic or heteroclinic correspondences associated with the chain of cylinders, and prove the existence of "drifting pseudo-orbits" for this polysystem. This is the content of **Theorem 3.1**, which is stated and proved in Section 3 (with an algorithmic version in Section 5). In the second part we derive a shadowing process (similar to but simpler than that in [28]) to prove the existence of genuine orbits of the Hamiltonian system which intersect arbitrarily small neighborhoods of each point of the pseudo-orbits of **Theorem 1.2**; see also the related shadowing results in [9, 17, 21, 31, 26]. This is the content of **Theorem 4.1**, stated and proved in Section 4. We now describe informally both theorems.

4. We first describe the main objects involved in the construction of drifting orbits and pseudo-orbits. Here we call a polysystem on some space X a dynamical system formed by a finite family of maps $(f_i)_{i \in I}$, each defined on some subset of X (possibly the whole set X), which can be iterated in any order for which the composition of maps is well defined.

Consider a good chain $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ contained in a level $H^{-1}(\mathbf{e})$. Via the shadowing process developed in Section 4, we will reduce the dynamics in the neighborhood of the cylinders $\mathscr{C}_k$ and along their homoclinic and heteroclinic connections to that of a polysystem on 2-dimensional annuli $(\Sigma_k)_{1 \leq k \leq k_*}$, which are diffeomorphic to $\mathbb{T} \times [0,1]$. Let us describe this polysystem.

---

[6]Under suitable assumptions, each open set O_i contains an essential torus.

In the neighborhood of a single cylinder $\mathscr{C} := \mathscr{C}_k$ of the chain, we will consider a polysystem $(f_i)_{i \in I}$ defined on a global section Σ of the Hamiltonian flow restricted to $\mathscr{C}$. The polysystem consists of a pair (φ, ψ), where φ is a diffeomorphism on $\mathscr{C}$, and ψ is a homoclinic *correspondence*[7] on $\mathscr{C}$, whose existence relies on additional conditions imposed on good cylinders. By construction, the global section Σ is endowed with symplectic coordinates $(\theta, r) \in \mathbb{T} \times [a, b]$ for some $0 < a < b$. The diffeomorphism φ is the Poincaré return map associated to Σ, so φ is symplectic, We assume φ to tilt all vertical lines $\theta = \text{cst.}$ to the right (or to the left). Other mild conditions on φ will also be required (see Definition 2.2). The essential subtori of $\mathscr{C}$ are those which intersect Σ along essential invariant circles of φ.

The homoclinic correspondence ψ is determined by the homoclinic orbits to $\mathscr{C}$. A first natural homoclinic correspondence is defined on $\mathscr{C}$ rather than on Σ, and associates to each element $x \in \mathscr{C}$ the subset of all $y \in \mathscr{C}$ such that $W^-(x)$ intersects $W^+(y)$ (with additional transversality requirements)[8].

To derive the homoclinic correspondence ψ on Σ from the homoclinic correspondence on $\mathscr{C}$, we perform a "reduction" to the section Σ, which is achieved by a suitable transport by the Hamiltonian flow inside $\mathscr{C}$. The correspondence ψ turns out to be a collection of local diffeomorphisms of Σ, whose domains may intersect one another.

This way, each cylinder $\mathscr{C}_k$ of the chain is equipped with a polysystem (φ_k, ψ_k). Finally, the complete polysystem attached to the chain $(\mathscr{C}_k)_{1 \le k \le k_*}$ is formed by the collection of the polysystems attached to each cylinder, together with additional local diffeomorphisms corresponding to the heteroclinic maps or to the transitions maps (defined along the flow) between consecutive cylinders.

5. We now describe briefly **Theorem 3.1**, which is given in Section 3. This theorem asserts the existence, for every $\delta > 0$, of a δ-*admissible pseudo-orbit* for the polysystems (φ_k, ψ_k), that is, an orbit of the polysystem which passes δ-close to every essential invariant circle contained in each of the sections $\Sigma_k \subset \mathscr{C}_k$. In particular, such pseudo-orbits drift along the chain in a natural sense.

Let us restrict ourselves first to a single cylinder $\mathscr{C} := \mathscr{C}_k$ of the chain, equipped with a section $\Sigma \subset \mathscr{C}$ and a polysystem (φ, ψ). To prove the existence of δ-admissible pseudo-orbits, we will require additional compatibility conditions between φ and ψ, as explained below.

Our result generalizes the approach introduced by Moeckel in [55] and developed by Le Calvez in [45], see also [56] and references therein. Their main result is that "generically", any polysystem formed by an area-preserving tilt map[9] φ on $\mathbf{A} = \mathbb{T} \times [a, b]$ and a globally defined area-preserving diffeomorphism ψ of $\mathbf{A}$ admits a finite "connecting pseudo-orbit" whose first point is arbitrarily close to $\mathbb{T} \times \{a\}$ and whose last point is arbitrarily close to $\mathbb{T} \times \{b\}$. The single map φ would not in general admit a connecting orbit, due to the existence of essential invariant circles that separate the annular section. The role of the diffeomorphism ψ is to allow the pseudo-orbits of the polysystem to "jump" over the essential invariant circles of φ

---

[7] In Section 2.3 we will consider homoclinic correspondences consisting of families of local diffeomorphisms on $\mathscr{C}$, whose domains are not necessarily mutually disjoint.

[8] This type of correspondence was previously studied in [17, 18], where the authors use an appropriate restriction of the correspondence, which they refer to as the *scattering map*.

[9] We recall here that a tilt map is a map that tilts every vertical line always to the right (or always to the left); for instance any twist map, or any finite composition of twist maps (all being right-twists or left-twists), is a tilt map.

– under the crucial assumption that such an invariant circle is *not* invariant under ψ.

In our case, the main difficulty is that ψ is *not* everywhere defined (and, in general, is multivalued). The compatibility conditions that we require generalize in some natural sense the previous "no simultaneous invariant circle condition." We now require that the range of ψ intersects every essential invariant circle Γ of φ and that ψ "breaks" this circle, in the sense that $\psi^{-1}(\Gamma)$ admits a "topologically transverse" intersection[10] with Γ (see Definition 2.8). Under this mild condition, we prove that the polysystem (φ, ψ) admits connecting pseudo-orbits similar to those in [55] and [45]. A further restriction on the size of the domain of the correspondence ψ (the δ-bounded property, see Definition 2.13) guarantees that connecting pseudo-orbits are δ-admissible.

Once the existence of δ-admissible pseudo-orbits along a single cylinder is proved, the existence of δ-admissible pseudo-orbits for the full polysystem along the whole chain of cylinders follows immediately, once we require some additional quantitative conditions on the heteroclinic maps between consecutive sections.

The proof of **Theorem 3.1** is given in Section 3.2. It relies on **Proposition 3.2**. We will give two different proofs of **Proposition 3.2**.

The first proof of **Proposition 3.2**, given in Section 3.1, is non-constructive and extends quite directly the methods introduced in [55] to the case of polysystems of correspondences. It requires very mild nondegeneracy conditions, which facilitate the applications to occurrence of diffusion under "generic" conditions on the perturbation.

The second proof of **Proposition 3.2** relies on **Proposition 5.1**, whose statement and proof are given in Section 5. The assumptions on the polysystem in **Proposition 5.1** are slightly more stringent, but enable us to use an iterative "Birkhoff procedure." Due to the "algorithmic" nature of the underlying construction, we expect this approach to be applicable to specific examples.

6. We now briefly describe **Theorem 4.1** from Section 4, on the shadowing of pseudo-orbits of a polysystem.

We first consider the case of a single good cylinder. Given a pseudo-orbit $(x_n)_{1 \leq n \leq n_*}$ of the polysystem (φ, ψ) on Σ, where $\psi = (\psi_i)_{i \in I}$, with ψ_i being a local diffeomorphism of Σ for $i \in I$. In particular, ψ can consist of a single global diffeomorphism. By the definition of a pseudo-orbit, either $x_{n+1} = \varphi(x_n)$ or there exists $i_n \in I$ such that $x_{n+1} = \psi_{i_n}(x_n)$, for $1 \leq n \leq n_* - 1$.

In the first case, since φ is a flow-induced return map, there is a time $T_n \geq 0$ such that $x_{n+1} = \Phi_H^{T_n}(x_n)$ (where Φ_H^T stands for the time-T map of the Hamiltonian flow).

In the second case, using the definition of ψ as a reduced homoclinic correspondence, together with the Poincaré recurrence theorem applied to φ, it turns out that there exist a point $\xi \in H^{-1}(\mathbf{e})$ and a time $T_n > 0$ such that ξ is arbitrarily close to x_n and $\Phi_H^{T_n}(\xi)$ is arbitrarily close to x_{n+1}. The possibility of using the recurrence properties of φ follows directly from the symplectic nature of our setting and the compactness of Σ (hence, its finite measure), and therefore is intimately related to our definition of cylinders. The idea of using the recurrence property of the dynamics in such a context was used in [28] (see also [56]).

---

[10]Specifically, we require that $\psi^{-1}(\Gamma)$ contains certain arcs emanating from Γ which are below Γ; we can think of $\psi^{-1}(\Gamma)$ as a union of arcs, since ψ is a collection of local diffeomorphisms.

One can expect that slightly perturbing and "gluing together" the previous pieces of Hamiltonian orbits $\Phi_H\big([0, T_n] \times \{x_n\}\big)$ yield a genuine orbit of the Hamiltonian flow which contains points located arbitrarily close to each point x_n of the initial orbit. This is the main statement of **Theorem 4.1**. The proof uses the inclination property (a λ-lemma for arcs in the setting of tame cylinders) to construct a positively invariant sequence of balls B_n centered on the unstable manifolds of the points x_n (and arbitrarily close to them), such that $\Phi_H^{T_n}(B_n) \subset B_{n+1}$ for a suitable $T_n > 0$. This proves our claim; for similar approaches, see also [9, 17, 25, 28].

Finally, the shadowing process for the complete polysytem along the chain of cylinders is based on the same idea and involves similar considerations regarding the heteroclinic transitions.

With the constructive method used in the proof of **Proposition 5.1**, one can expect to obtain a quantitative control on the recurrence times of the Hamiltonian flow (in the specific zones under consideration), and avoid the use of the Poincaré recurrence theorem in specific models.

Theorem 1.2 immediately follows from **Theorem 3.1** and **Theorem 4.1**.

7. As mentioned above, the present paper considers the problem of generic drift in an abstract setting. Let us now describe its possible applications and extensions.

7.1. The conditions we impose to our chains are also satisfied by the usual examples of *a priori unstable* systems, limited to a single cylinder. In this respect, the present work is an abstract approach of the problems considered in [16, 8, 7, 5, 6, 9, 15, 17, 18, 19, 25, 26, 27, 29, 30, 31, 28, 55, 59, 23], amongst others. However, our assumptions are less stringent than the usual ones.

7.2. Our approach can be applied to Hamiltonian systems that do not satisfy the strict convexity assumption mentioned earlier, but instead it satisfies the weaker condition that the return map to the sections is a twist map (hence, a tilt map). Consider the following Hamiltonian on $\mathbb{A}^3$:

$$H(\theta, r) = \frac{r_1^4}{4} + r_2 + \frac{r_3^2}{2} + \varepsilon \cos(2\pi\theta_3) + \varepsilon\mu f(\theta), \tag{2}$$

with $\mu \ll 1$. When $\mu = 0$, the unperturbed Hamiltonian h has a normally hyperbolic annulus $\mathbf{A} = \{(\theta, r) \,|\, \theta_3 = 0, r_3 = 0\}$ diffeomorphic to $\mathbb{A}^2$. The intersection of $\mathbf{A}$ with a level $H^{-1}(\mathbf{e})$ is transverse and yields a cylinder $\mathscr{C}$ diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, with coordinates $(\theta_1, \theta_2, r_1)$. The return map to the section $\Sigma = \{\theta_2 = 0\}$ inside this cylinder has the twist property but the global system is not convex (nor quasi-convex) in the actions. These properties persist for $\mu > 0$ sufficiently small.

This system would not in principle be accessible to weak KAM theory or usual variational methods.

7.3. Our method can be applied to Hamiltonian systems that satisfy the even weaker condition that the return map to the section is a tilt map (rather than a twist map). Consider the system (2) when the unperturbed Hamiltonian h is changed to

$$h(\theta, r) = \frac{r_1^4}{4} + r_2 + \frac{r_3^2}{2} + a \cos(2\pi\theta_1) + b \cos(2\pi\theta_3).$$

The annulus $\mathbf{A}$ defined above is normally hyperbolic if $a \ll b$. Now consider the dynamics inside the cylinder $\mathscr{C}$ as above. The return map to the section $\Sigma = \{\theta_2 = 0\}$ inside $\mathscr{C}$ is a tilt map but not a twist map when a is large enough. Such systems occur for (large) perturbations of integrable Hamiltonians at the crossing of two independent simple resonances. Our method would still apply in this case, which is "far from convex and far from perturbative".

7.4. We mainly developed the present method in view of a proof of the Mather version of the Arnold conjecture (see [52, 54]) for a precise formulation of the problem). The main difficulty in this setting is to get ride of any quantitative estimate on the so-called "splitting of separatrices", since one cannot expect it to be bounded from below in generic situations. In particular, usual transversality arguments are not adapted to this setting and this is the main reason to shift to measure theoretic ones, as explained above. See [50] for more details and "simple" examples.

7.5. Non-hyperbolic cylinders naturally appear in quantitative questions related to diffusion. The main results in quantitative perturbation theory have been obtained in the case of perturbations of convex or quasi-convex systems (e.g., optimality of the Nekhoroshev stability times in various functional classes (C^k, $G^{\alpha,L}$, C^ω), optimality of the splitting of invariant tori). However, a much more natural geometric class for the unperturbed systems would be the steep one, introduced by Nekhoroshev in [57]. The optimality of Nekhoroshev stability times for steep unperturbed systems is not proved yet, and a general study clearly involves more general objects than normally hyperbolic ones. As for another example, regarding the problem of the optimality of the measure of the wandering domains in exact-symplectic perturbations of integrable diffeomorphisms on the annulus $\mathbb{A}^n$, it turns out that the introduction of non-hyperbolic objects is unavoidable (see [41]). We hope that our work will motivate a more thorough study of these objects, probably using blowing-up methods from symplectic geometry.

We emphasize that non-hyperbolic objects are in general difficult to control from the point of view of persistence theory, so that the aforementioned applications are (up to now) limited to the construction of particular examples for which the perturbations can be suitably chosen - as is the case for instance in the original Arnold example.

8. We summarize the distinguished features of our approach, and emphasize the differences from other approaches.

- it uses homoclinic and heteroclinic "maps" that are only *locally defined*, and requires only very mild topological transversality assumptions; between the stable and unstable manifolds of the essential tori contained in the cylinders; this makes the method well-adapted to passing to the singular limit when the "hyperbolicity" of the cylinders tends to 0;
- there is no requirement on the "lower bounds of the splitting of tori", nor that they form countable families;
- the aforementioned singular limit generates signficant difficulties due to the fact that the cylinders have "wild embeddings" in the *a priori* stable case when the perturbation parameter tends to 0, which makes the variational methods difficult to apply, while the symplectic geometric ones (closely linked to our approach in this paper) are easier;
- our method does not require the system restricted to the cylinder to be close to integrable, since we only assume the return map to have a tilt property, in particular it does not require any precise knowledge on the invariant objects for the dynamics restricted to the cylinders (apart from their homology);
- it does not require the cylinders to be normally hyperbolic, nor the inner dynamics to possess strong torsion property, which allows us to expect easy applications to non-convex settings (as well as to more quantitative questions, see [41])

- our construction of pseudo-orbits can be made "algorithmic" from a numerical point of view, which makes it accessible to computer-assisted methods (some numerical approaches showing the existence of diffusing orbits along normally hyperbolic cylinders, via computer assisted proofs, can be found in [11, 12]).

9. The structure of the paper is the following. Section 2 is devoted to the description of the general setting and the definition good cylinders and good chains of cylinders. In Section 3 we state and prove **Theorem 3.1**. **Theorem 4.1** and **Theorem 1.2** are proved in Section 4. Section 5 is devoted to the algorithmic method to prove **Theorem 3.1**. Finally, we recall in Appendix A some basic results on tilt maps, and in particular a strong form of the Birkhoff transition lemma through a Birkhoff zone.

2. **The setting: Good chains of cylinders.** In this section we make precise the dynamical features of good cylinders and good chains.

2.1. **Tame cylinders.** The main objects in [47, 48] are normally hyperbolic invariant cylinders with boundary[11], satisfying some additional conditions. In this paper we adopt a slightly more general setting, relaxing the normal hyperbolicity assumption but preserving all essential features of this situation (existence of center-stable and center-unstable manifolds foliated by stable and unstable manifolds of points, λ-lemma for arcs, and existence of an invariant measure). We call here *tame cylinders* the resulting objects, for which we give a formal definition gathering together the necessary properties.

Some examples of tame cylinders that are not normally hyperbolic, as well as applications to the problem of wandering domains, can be found in [41]. Other examples appear in concrete systems from Celestial Mechanics, see [33]. Also, an example in the case of a rotator-pendulum type system is presented at the end of this section.

1. For a manifold with boundary L, we denote by ∂L the boundary of L and by $\mathrm{int}\, L = L \setminus \partial L$ the interior of the manifold.

Let X be a complete vector field on a smooth manifold M. We say that $\mathscr{C} \subset M$ is an *invariant cylinder* for X if $\mathscr{C}$ is a C^1–submanifold of M, C^1–diffeomorphic to $\mathbb{T}^2 \times [0,1]$, such that X is everywhere tangent to $\mathscr{C}$ and is moreover tangent to $\partial \mathscr{C}$ at each point of $\partial \mathscr{C}$. Note that $\mathscr{C}$ is compact and invariant under the flow of X.

In the following we consider $M = \mathbb{A}^3$, a C^κ-Hamiltonian function $H : \mathbb{A}^3 \to \mathbb{R}$, with $\kappa \geq 2$. We denote the Hamiltonian vector field associated to H by X_H, and its Hamiltonian flow by Φ_H. Recall that we only consider energies $\mathbf{e}$ and levels $H^{-1}(\mathbf{e})$ which are completely regular. The following definition has "minimal" requirements on the regularity of the various objects of interest, however in the future constructions these objects can have higher regularity.

Definition 2.1. Let H be a C^κ Hamiltonian function on $\mathbb{A}^3$, with $\kappa \geq 2$, and fix an energy $\mathbf{e}$. We denote by d the usual distance on $\mathbb{A}^3$. We define a *tame cylinder at energy* $\mathbf{e}$ for H as a cylinder $\mathscr{C}$ contained in $H^{-1}(\mathbf{e})$ and invariant under Φ_H, which satisfies the following properties:

---

[11]Or normally hyperbolic *singular* cylinders, whose dynamical study will be reduced here to the non-singular case, see Section 3.3.

- There exists a 5-dimensional manifold with boundary $U \subset H^{-1}(\mathbf{e})$ containing $\mathscr{C}$, with $\operatorname{int} \mathscr{C} \subset \operatorname{int} U$ and $\partial \mathscr{C} \subset \partial U$, such that the subsets

$$W_U^\pm(\mathscr{C}) = \left\{ x \in U \mid \Phi_H^t(x) \in U, \ \forall t \in \mathbb{R}^\pm, \ \text{and} \ \lim_{t \to \pm\infty} d\big(\Phi_H(x), \mathscr{C}\big) = 0 \right\} \quad (3)$$

  are 4-dimensional C^1 embedded submanifolds of U, with boundaries contained in ∂U. We fix U once and for all and set

$$W^-(\mathscr{C}) = \bigcup_{t \geq 0} \Phi_H^t\big(W_U^-(\mathscr{C})\big), \qquad W^+(\mathscr{C}) = \bigcup_{t \leq 0} \Phi_H^t\big(W_U^+(\mathscr{C})\big), \quad (4)$$

  which are Φ_H-invariant C^1 immersed submanifolds of $H^{-1}(\mathbf{e})$.
- There exist C^1-diffeomorphisms

$$J^\pm : \mathscr{C} \times \,]-1, 1[\,\to W_U^\pm(\mathscr{C}) \quad (5)$$

  such that $J^\pm(x, 0) = x$ for $x \in \mathscr{C}$, and, setting

$$W_U^\pm(x) = J^\pm\big(\{x\} \times \,]-1, 1[\big), \qquad x \in \mathscr{C}, \quad (6)$$

  then $J^\pm(x, \cdot) : \,]-1, 1[\,\to W_U^\pm(x)$ is a C^1 diffeomorphism, and

$$\forall y \in W_U^\pm(x), \quad \lim_{t \to \pm\infty} d\big(\Phi_H^t(x), \Phi_H^t(y)\big) = 0. \quad (7)$$

- We set

$$W^-(x) = \bigcup_{t \geq 0} \Phi_H^t\big(W_U^-(\Phi_H^{-t}(x))\big), \qquad W^+(x) = \bigcup_{t \leq 0} \Phi_H^t\big(W_U^+(\Phi_H^{-t}(x))\big), \quad (8)$$

  which are C^1 immersed 1-dimensional submanifolds of $H^{-1}(\mathbf{e})$.
- The manifolds $W^\pm(x)$ satisfy the equivariance property

$$\Phi_H^t\big(W^\pm(x)\big) = W^\pm\big(\Phi_H^t(x)\big), \qquad \forall x \in \mathscr{C}, \ \forall t \in \mathbb{R}. \quad (9)$$

- There is a negatively invariant neighborhood $\mathscr{N}^- \subset W_U^-(\mathscr{C})$ of $\mathscr{C}$ in $W^-(\mathscr{C})$ such that for $x \in \mathscr{C}$ and $y \in \mathscr{N}^- \cap W^-(x)$,

$$d\big(\Phi_H^{-t}(x), \Phi_H^{-t}(y)\big) \leq d(x, y), \qquad \forall t \geq 0, \quad (10)$$

  and a positively invariant neighborhood $\mathscr{N}^+ \subset W_U^+(\mathscr{C})$ of $\mathscr{C}$ in $W^+(\mathscr{C})$ such that for $x \in \mathscr{C}$ and $y \in \mathscr{N}^+ \cap W^+(x)$,

$$d\big(\Phi_H^t(x), \Phi_H^t(y)\big) \leq d(x, y), \qquad \forall t \geq 0. \quad (11)$$

- **The λ-property.** For $x \in \mathscr{C}$, set $J_x^- = J^-(x, \cdot) : \,]-1, 1[\,\to W_U^-(x)$. Fix $x \in \mathscr{C}$. Then for any 1-dimensional C^1 submanifold Δ of $H^{-1}(\mathbf{e})$ which intersects $W^+(\mathscr{C})$ transversely in $H^{-1}(\mathbf{e})$ at $\xi \in W^+(x)$, there exist a family $(\Delta_t)_{t \geq t_0}$ of submanifolds of Δ containing ξ and C^1 parametrizations $\ell_t : \,]-1, 1[\,\to H^{-1}(\mathbf{e})$ of the images $\Phi_H^t(\Delta_t)$ such that:

$$\lim_{t \to +\infty} \|\ell_t - J_{\Phi_H^t(x)}^-\|_{C^0(]-1,1[)} = 0. \quad (12)$$

- There exists a Φ_H-invariant Borel measure $\boldsymbol{\mu}$ on $\mathscr{C}$ such that $\boldsymbol{\mu}(\mathscr{C}) < \infty$, and $\boldsymbol{\mu}(O) > 0$ for any nonempty open subset O of $\mathscr{C}$.
- The cylinder $\mathscr{C}$ is contained in the interior $\operatorname{int} \widehat{\mathscr{C}}$ of a cylinder $\widehat{\mathscr{C}}$ which satisfies the previous six conditions, with natural continuation conditions (that is, with obvious notation, $U \subset \widehat{U}$, $\widehat{J}_{|U}^\pm = J^\pm$, $\mathscr{N}^- \subset \widehat{\mathscr{N}^-}$). Any such cylinder $\widehat{\mathscr{C}}$ is said to be a continuation of $\mathscr{C}$.

Here we use the terminology of stable/unstable manifolds to refer to the leaves $W^\pm(x)$ associated to points $x \in \mathscr{C}$, and of center-stable/unstable manifolds[12] to refer to the invariant manifolds $W^\pm(\mathscr{C})$ associated to the cylinder $\mathscr{C}$.

In the setting of Definition 2.1, the global manifolds $W^\pm(\mathscr{C})$ could depend on the choice of U, but this will cause no trouble in what follows.

Normally hyperbolic invariant cylinders with boundary are particular cases of tame cylinders [13]. In such settings, a natural choice of the invariant measure $\boldsymbol{\mu}$ is the Liouville measure on $\mathscr{C}$ induced by the symplectic form on $\mathbb{A}^3$. However, for the general setting of Definition 2.1, there is no additional requirement on the Borel measure $\boldsymbol{\mu}$.

However, there are some notable differences between tame cylinders and normal hyperbolic ones. In the definition of a tame cylinder we explicitly require the existence of (center-)stable and (center-)unstable manifolds of certain regularity, while in the case of normally hyperbolic invariant manifolds the existence of the such objects and their regularity follows from the definition, from the regularity of the system, and from the spectral data. Also tame cylinder do not necessarily persist under small perturbations, as it is the case for normally hyperbolic invariant manifolds.

2. We set out now some remarks and conventions. Given a subset A of a tame cylinder $\mathscr{C}$, we set

$$W^\pm(A) = \bigcup_{x \in A} W^\pm(x). \tag{13}$$

Thus $W^\pm(A)$ are invariant when A is invariant. This will be the case in particular when $A = \operatorname{int} \mathscr{C}$.

Observe that since the parametrization J^+ is a C^1-diffeomorphisms, the (stable) characteristic projection

$$\Pi^+ : W^+(\mathscr{C}) \to \mathscr{C}, \tag{14}$$

which associates to a point $y \in W^+(\mathscr{C})$ the unique point x of $\mathscr{C}$ such that $y \in W^+(x)$, is also C^1. A similar property holds for the (unstable) characteristic projection $\Pi^- : W^-(\mathscr{C}) \to \mathscr{C}$.

Convention. *In the following, given a tame cylinder $\mathscr{C}$, we will choose once and for all one continuation of $\mathscr{C}$, which we always denote by $\widehat{\mathscr{C}}$.*

3. An example of a tame cylinder. Let us give an example, similar to that in [41], of a system that has a tame cylinder which is not normally hyperbolic. Consider the following variant of the Arnold example on $\mathbb{A}^3$:

$$H_\varepsilon(\theta, r) = \tfrac{1}{2}(r_0^2 + r_1^2 + r_2^2) - (\cos(2\pi\theta_2) - 1)^2 + \varepsilon f(\theta), \qquad (\theta, r) \in \mathbb{A}^3, \tag{15}$$

with $\varepsilon \geq 0$ and where $\operatorname{Supp} f \subset \{\theta_2 \in [1/4, 3/4]\}$. When $\varepsilon = 0$, the system is the uncoupled product of the integrable system generated by $\frac{1}{2}(r_0^2 + r_1^2)$ on $\mathbb{A}^2$ and the pendulum-like system $\frac{1}{2}r_2^2 - (\cos(2\pi\theta_2) - 1)^2$ on $\mathbb{A}$. This latter system admits the fixed point O of coordinates $(\theta_2, r_2) = (0, 0)$. Clearly O is degenerate since the expansion of the Hamiltonian in the neighborhood of O reads $\frac{1}{2}r_2^2 - 4\pi^4\theta_2^4 + \cdots$. Note that O admits well-defined stable and unstable manifolds, and the pendulum-like system satisfies the usual λ-lemma by easy 2-dimensional arguments.

---

[12]This terminology is used, for instance, in [2]; it is however more common to refer to the global manifolds as stable/unstable manifolds, rather than center-stable/unstable manifolds.

[13]In particular, the cylinders considered in [47] are readily seen to be tame cylinder, by usual normal hyperbolicity arguments.

The annulus $\mathscr{A} = \mathbb{A}^2 \times \{O\}$ is invariant under the flow, but not normally hyperbolic. This is also the case when $\varepsilon > 0$, thanks to the condition on the support of f.

For ε small enough, the level $H_\varepsilon^{-1}(0)$ is compact and regular, and its intersection with $\mathscr{A}$ is the 3-dimensional torus

$$\mathscr{T} = \left\{(\theta, r) \in \mathbb{A}^3 \mid (\theta_2, r_2) = O, \ \tfrac{1}{2}(r_0^2 + r_1^2) = 1\right\},$$

which is invariant in $H_\varepsilon^{-1}(1)$ but not normally hyperbolic. However $\mathscr{T}$ clearly admits global invariant manifolds, which are the product of $\mathscr{T}$ with the stable and unstable manifolds of O. In the same way, the stable and unstable manifolds of the points of $\mathscr{T}$ are the product of those points with the stable and unstable manifolds of O. They are uniquely defined.

Moreover, given a point $\mathbf{a} = (a_0, a_1)$ on the circle C of equation $\tfrac{1}{2}(r_0^2 + r_1^2) = 1$, the 2-dimensional torus

$$T_{\mathbf{a}} = \left\{(\theta, r) \in \mathbb{A}^3 \mid (\theta_2, r_2) = O, \ (r_0, r_1) = \mathbf{a}\right\} \subset \mathscr{T}$$

is invariant under the flow. Varying $\mathbf{a}$ yields a foliation of $\mathscr{T}$ by invariant tori. Given now any compact connected arc $C_{\mathbf{a}}^{\mathbf{a}'} \subset C$ with extremities $\mathbf{a}$ and $\mathbf{a}'$, the subset

$$\mathscr{C} = \left\{(\theta, r) \in \mathbb{A}^3 \mid (\theta_2, r_2) = O, \ (r_0, r_1) \in C_{\mathbf{a}}^{\mathbf{a}'}\right\}$$

immediately satisfies our definition of a tame cylinder at energy 1 for H_ε: the two components of its boundary are the invariant tori $T_{\mathbf{a}}$ and $T_{\mathbf{a}'}$. The existence of continuations follows from the choice of larger compact arcs $C_{\mathbf{b}}^{\mathbf{b}'} \supset C_{\mathbf{a}}^{\mathbf{a}'}$. The λ-property follows from easy two-dimensional considerations around O. The invariant measure $\boldsymbol{\mu}$ is the Liouville measure on $H^{-1}(0) \cap \mathscr{A}$ deduced from the induced symplectic form on $\mathscr{A}$.

The interest in such examples comes from the fact that the distance between two consecutive periodic invariant curves in the "pendulum" part of (15) is polynomially small with respect to the period, while it is exponentially small in the case of the usual simple pendulum. This is of crucial importance in constructing examples of perturbed initially stable systems with "large" wandering domains (see [41]).

2.2. Tilt sections, homoclinic correspondences and splitting arcs. We introduce in this section the main ingredients to reduce our problem to a two-dimensional one.

1. *Two-dimensional tilt sections inside a cylinder.* We first introduce the notion of a *tilt section*, which will be our main tool to reduce our problem to the Birkhoff theory in dimension 2. Given $a < b$ we set:

$$\mathbf{A}(a, b) = \mathbb{T} \times [a, b], \qquad \Gamma(a) = \mathbb{T} \times \{a\}, \qquad \Gamma(b) = \mathbb{T} \times \{b\}.$$

We often abbreviate $\mathbf{A}(a, b)$ to $\mathbf{A}$ when there is no risk of confusion.

We refer to Appendix A for the usual definition of a tilt map, which we always assume to tilt the vertical to the right (or always to the left). We denote by $\mathrm{Ess}(\varphi)$ the set of essential invariant circles of an area-preserving tilt map $\varphi : \mathbf{A} \to \mathbf{A}$. Elements $\Gamma \in \mathrm{Ess}(\varphi)$ are graphs of uniformly Lipschitz functions $\ell_\Gamma : \mathbb{T} \to [a, b]$, by the Birkhoff theorem [35, 51, 60]. We endow $\mathrm{Ess}(\varphi)$ with the Hausdorff topology or, equivalently, with the uniform C^0 topology on the corresponding functions ℓ_Γ.

Definition 2.2. We say that an area-preserving tilt map φ of $\mathbf{A}$ is *good* if

- φ does not admit any essential invariant circle with rational rotation number;
- The boundaries $\Gamma(a)$ and $\Gamma(b)$ are both dynamically minimal;

- Each boundary $\Gamma(a)$, $\Gamma(b)$ is accumulated by a sequence of dynamically minimal elements of $\mathrm{Ess}(\varphi)$.

The last two conditions are always satisfied in most cases of interest (twist maps on annuli with Diophantine boundaries, under the KAM theorem conditions).

Given $\Gamma \in \mathrm{Ess}(\varphi)$, Γ^- denotes the connected component of $\mathbf{A} \setminus \Gamma$ located below Γ in $\mathbf{A}$, and Γ^+ denotes the connected component of $\mathbf{A} \setminus \Gamma$ located above Γ in $\mathbf{A}$. In the sequel we will crucially use the following result, which is proved in Appendix A.

Lemma 2.3. *Let φ be a good area-preserving tilt map φ of $\mathbf{A}$. Then the following properties hold true:*

(i) Any two distinct elements of $\mathrm{Ess}(\varphi)$ are disjoint, so that the set $\mathrm{Ess}(\varphi)$ admits a natural order $\leq$ given by $\Gamma_1 \leq \Gamma_2$ if $\ell_{\Gamma_1}(\theta) \leq \ell_{\Gamma_2}(\theta)$ for all $\theta \in \mathbb{T}$, where Γ_j is the graph of ℓ_{Γ_j}, $j = 1, 2$;

(ii) Given a nonempty subset $E \subset \mathrm{Ess}(\varphi)$, the greatest lower bound $\mathrm{Inf}\, E$ and the least upper bound $\mathrm{Sup}\, E$ relative to $\leq$ exist;

(iii) Every invariant essential circle $\Gamma \subset \big(\mathbf{A} \setminus \Gamma(a)\big)$ (resp., $\Gamma \subset \big(\mathbf{A} \setminus \Gamma(b)\big)$) is either the upper (resp., lower) boundary of a Birkhoff zone of φ, or is accumulated by a sequence of elements of $\mathrm{Ess}(\varphi)$ located in Γ^- (resp. Γ^+).

We can now introduce our main notion. We consider a C^2 Hamiltonian function H on $\mathbb{A}^3$, we fix an energy $\mathbf{e}$ and a tame cylinder $\mathscr{C}$ at energy $\mathbf{e}$. Given a topological space X and two subsets $A \subset B$ of X with A connected, we denote by

$$\mathrm{cc}(B, A)$$

the connected component of B which contains A.

Definition 2.4. A *tilt section* for $\mathscr{C}$ is a quadruple $(\Sigma, \mathbf{A}, \chi, \varphi)$ such that:

- Σ is a global Poincaré section for the flow $(\Phi_H)_{|\mathscr{C}}$, with return map φ;
- χ is a C^1 embedding of $\mathbf{A}$ in $\mathscr{C}$ with image $\Sigma = \chi(\mathbf{A})$;
- $\chi^{-1} \circ \varphi \circ \chi$ is a good area-preserving tilt map of $\mathbf{A}$.

Given such a tilt section on $\mathscr{C}$, we set

$$\partial_\bullet \Sigma = \chi\big(\Gamma(a)\big), \qquad \partial^\bullet \Sigma = \chi\big(\Gamma(b)\big). \tag{16}$$

These two circles are contained in the two boundary components of $\partial \mathscr{C}$. We set

$$\partial_\bullet \mathscr{C} = \mathrm{cc}\big(\partial \mathscr{C}, \partial_\bullet \Sigma\big), \qquad \partial^\bullet \mathscr{C} = \mathrm{cc}\big(\partial \mathscr{C}, \partial^\bullet \Sigma\big). \tag{17}$$

We finally define a continuation of $(\Sigma, \mathbf{A}, \chi, \varphi)$ for the continuation $\widehat{\mathscr{C}}$ as a tilt section $(\widehat{\Sigma}, \widehat{\mathbf{A}}, \widehat{\chi}, \widehat{\varphi})$ for $\widehat{\mathscr{C}}$ which continues the previous one in the natural way.

2. *Homoclinic correspondences.* The homoclinic correspondences that we consider will always be families of locally defined diffeomorphisms (with possibly intersecting domains). The closely related *scattering maps* (see [20] and references therein) are local diffeomorphisms on cylinders, while our homoclinic diffeomorphisms are local diffeomorphisms on tilt sections (see also [21]). This is a crucial step in order to reduce our approach to two-dimensional dynamics.

- We first recall the usual context for scattering maps and examine the transversality properties of the center-stable and center-unstable manifolds of the cylinder.

Definition 2.5. We consider a C^2 Hamiltonian function H on $\mathbb{A}^3$, we fix an energy $\mathbf{e}$ and consider a tame cylinder $\mathscr{C}$ at energy $\mathbf{e}$. We define the *transverse homoclinic intersection* of the continuation $\widehat{\mathscr{C}}$ as the set

$$\mathbf{Homt}(\widehat{\mathscr{C}}) \subset W^+(\operatorname{int}\widehat{\mathscr{C}}) \cap W^-(\operatorname{int}\widehat{\mathscr{C}}) \tag{18}$$

formed by the points ξ such that

$$W^-\big(\Pi^-(\xi)\big) \pitchfork_\xi W^+(\operatorname{int}\widehat{\mathscr{C}}) \quad \text{and} \quad W^+\big(\Pi^+(\xi)\big) \pitchfork_\xi W^-(\operatorname{int}\widehat{\mathscr{C}}), \tag{19}$$

where $\Pi^\pm$ are the characteristic projections defined in (14) and where $\pitchfork_\xi$ stands for "intersects transversely at ξ relatively to $H^{-1}(\mathbf{e})$".

Note that $W^\pm(\operatorname{int}\widehat{\mathscr{C}}) = \operatorname{int}W^\pm(\widehat{\mathscr{C}})$, following our definitions. The following lemma is an immediate consequence of the implicit function theorem.

Lemma 2.6. *Fix $\xi \in \mathbf{Homt}(\widehat{\mathscr{C}})$ and write $x^\pm = \Pi^\pm(\xi) \in \widehat{\mathscr{C}}$. Then the following hold:*

  (i) $W^+(\widehat{\mathscr{C}})$ *and* $W^-(\widehat{\mathscr{C}})$ *intersect transversely at ξ in $H^{-1}(\mathbf{e})$;*
  (ii) *There exists a 3-dimensional open neighborhood $\mathscr{O}$ of ξ in $W^+(\operatorname{int}\widehat{\mathscr{C}}) \cap W^-$ $(\operatorname{int}\widehat{\mathscr{C}})$ such that the condition (19) is satisfied at all points $\eta \in \mathscr{O}$, and, moreover,*

$$\begin{aligned} W^-\big(\Pi^-(\xi)\big) \pitchfork_\eta \mathscr{O} \text{ relative to } W^-(\operatorname{int}\widehat{\mathscr{C}}), \text{ and}\\ W^+\big(\Pi^-(\xi)\big) \pitchfork_\eta \mathscr{O} \text{ relative to } W^+(\operatorname{int}\widehat{\mathscr{C}}). \end{aligned} \tag{20}$$

  (iii) *Provided that the open neighborhood $\mathscr{O}$ of ξ in $W^+(\operatorname{int}\widehat{\mathscr{C}}) \cap W^-(\operatorname{int}\widehat{\mathscr{C}})$ is chosen sufficiently small, there exist open neighborhoods $O^\pm$ of $x^\pm$ in $\operatorname{int}\widehat{\mathscr{C}}$ such that the restrictions $\Pi^\pm_{|\mathscr{O}}$ are C^1 diffeomorphisms from $\mathscr{O}$ onto $O^\pm$.*

In [20] the open set $\mathscr{O}$ is referred to as a *homoclinic channel*, and the map

$$S = \Pi^+ \circ (\Pi^-_{\mathscr{O}})^{-1} : O^- \to O^+$$

as the *scattering map* associated to $\mathscr{O}$.

Our main variation relative to the usual definition of a scattering map consists in a *two-dimensional reduction* by transport to the section Σ via the Hamiltonian flow. We also allow for the transversality condition (20) to hold only almost everywhere, which prove to be useful to be enough in our subsequent constructions of drifting orbits)[14].

Let us first informally describe our two-dimensional reduction. Consider an energy $\mathbf{e}$, and suppose that $\mathscr{C}$ is a tame cylinder with continuation $\widehat{\mathscr{C}}$ and continued tilt section $\widehat{\Sigma}$ in the corresponding level. Assume that we are given an (in general uncountable) family of scattering maps on $\widehat{\mathscr{C}}$

$$S = (S_i)_{i\in I}, \qquad S_i : \operatorname{Dom}S_i \to \operatorname{Im}S_i, \tag{21}$$

where $\operatorname{Dom}S_i \subset \operatorname{int}\widehat{\mathscr{C}}$ and $\operatorname{Im}S_i \subset \operatorname{int}\widehat{\mathscr{C}}$ are open subsets and where each S_i is a *measure preserving*[15] C^1-diffeomorphism from $\operatorname{Dom}S_i$ to $\operatorname{Im}S_i$. Moreover, assume

---

[14]In particular, we will be able to deal with points $x^- \in \mathscr{C}$ and $S(x^-) = x^+ \in \mathscr{C}$ with $\xi \in W^-(x^-) \cap W^+(x^+)$ such that $W^+(\widehat{\mathscr{C}})$, $W^-(\widehat{\mathscr{C}})$ are not transverse at ξ in $H^{-1}(\mathbf{e})$.

[15]In the normally hyperbolic situation, this comes from the definition of the Borel measure on $\mathscr{C}$ together with the symplectic properties of its stable and unstable manifolds.

that for every $i \in I$ there is an open subset $\mathrm{Domt}\, S_i \subset \mathrm{Dom}\, S_i$, with full measure in $\mathrm{Dom}\, S_i$, such that

$$\mathrm{Im}\, S_i \cap \mathrm{int}\, \widehat{\Sigma} \neq \emptyset. \tag{22}$$

and

$$\forall y \in \mathrm{Domt}S_i, \quad W^-(y) \cap W^+\big(S_i(y)\big) \cap \mathbf{Homt}(\widehat{\mathscr{C}}) \neq \emptyset. \tag{23}$$

Assume finally that there exists an open subset $D_i \subset \mathrm{int}\, \widehat{\Sigma}$ and a non-negative C^1 function $\tau_i : D_i \to \mathrm{Dom}\, S_i$ such that for all $x \in D_i$

$$\Phi_H^{\tau_i(x)}(x) \in \mathrm{Dom}\, S_i, \qquad S_i\big(\Phi_H^{\tau_i(x)}(x)\big) \in \mathrm{Im}\, S_i \cap \mathrm{int}\, \widehat{\Sigma} \tag{24}$$

(this is not restrictive : such a function always exists if $\mathrm{Dom}\, S_i$ is small enough, since $\widehat{\Sigma}$ is a global Poincaré section). Note that the function τ_i together with is open domain $\mathrm{Dom}\, \tau_i$ need not be uniquely determined. This description leads to the following precise definition.
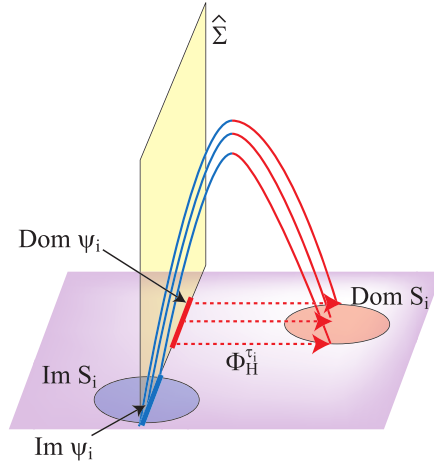

FIGURE 2. Homoclinic correspondence on the cylinder and the induced homoclinic correspondence on the Poincaré section.

Definition 2.7. We consider a C^2 Hamiltonian function H on $\mathbb{A}^3$, and we fix an energy $\mathbf{e}$ and a tame cylinder $\mathscr{C}$ at energy $\mathbf{e}$, with continuation $\widehat{\mathscr{C}}$ and continued tilt section $\widehat{\Sigma}$.

A *homoclinic correspondence* associated with these data is an (in general uncountable) family of C^1 local diffeomorphisms of $\mathrm{int}\, \widehat{\Sigma}$:

$$\psi = (\psi_i)_{i \in I}, \qquad \psi_i : \mathrm{Dom}\, \psi_i \to \mathrm{Im}\, \psi_i, \tag{25}$$

where $\mathrm{Dom}\, \psi_i \subset \mathrm{int}\, \widehat{\Sigma}$ and $\mathrm{Im}\, \psi_i \subset \mathrm{int}\, \widehat{\Sigma}$ are open subsets, for which there exists a family of measure preserving homoclinic maps $S = (S_i)_{i \in I}$, $S_i : \mathrm{Dom}\, S_i \to \mathrm{Im}\, S_i$ on $\widehat{\mathscr{C}}$ such that S_i and ψ_i satisfy (21), (23), (22), (24), where

$$\forall x \in \mathrm{Dom}\, \psi_i, \qquad \psi_i(x) := S_i\Big(\Phi_H^{\tau_i(x)}(x)\Big). \tag{26}$$

Homoclinic correspondences are not uniquely defined, and the domains $\operatorname{Dom} \psi_i$ (resp. $\operatorname{Dom} S_i$) are not necessarily pairwise disjoint. In the following we identify $\widehat{\Sigma}$ with $\widehat{\mathbf{A}} = \mathbb{T} \times [\widehat{a}, \widehat{b}]$ via the embedding $\widehat{\chi}$, and we indifferently consider our homoclinic correspondences as defined on $\widehat{\Sigma} = \widehat{\chi}(\widehat{\mathbf{A}})$ or on $\widehat{\mathbf{A}}$.

3. *Splitting arcs and associated domains.* An *arc* in $\widehat{\mathbf{A}}$ is a continuous injective map $\zeta : [0, 1] \to \widehat{\mathbf{A}}$, and we write $\widetilde{\zeta} = \zeta([0, 1]) \subset \widehat{\mathbf{A}}$ for its image. Denote by $\pi : \widehat{\mathbf{A}} \to \mathbb{T}$ the projection onto the θ-coordinate. Given two points θ, θ' of $\mathbb{T}$ at a distance less than $1/2$, we write $[\theta, \theta'] \subset \mathbb{T}$ for the unique segment bounded by θ and θ' according to the natural orientation of $\mathbb{T}$. In the following definition we implicitly assume that the pairs points in $\mathbb{T}$ that we consider are always close enough so that the previous convention applies.

Definition 2.8. We consider a tame cylinder $\mathscr{C}$ with section $(\Sigma, \mathbf{A}, \chi, \varphi)$ and continuations $\widehat{\mathscr{C}}$, $(\widehat{\Sigma}, \widehat{\mathbf{A}}, \widehat{\chi}, \widehat{\varphi})$, respectively. We let $\psi = (\psi_i)_{i \in I} : \widehat{\Sigma} \to \widehat{\Sigma}$ be a homoclinic correspondence. Fix $\Gamma \in \operatorname{Ess}(\widehat{\varphi})$ contained in $\widehat{\mathbf{A}} \setminus \Gamma(\widehat{a})$ and let α_0 be a point in Γ.

- A *splitting arc* based at α_0 is an arc ζ in $\widehat{\mathbf{A}}$ for which

$$\zeta(0) = \alpha_0, \quad \zeta(]0, 1]) \subset \Gamma^-; \quad \exists i \in I, \ \zeta(]0, 1]) \subset \operatorname{Dom} \psi_i, \quad \psi_i(\zeta(]0, 1])) \subset \Gamma.$$

- A *non-vertical splitting arc* is a splitting arc ζ for which there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in $]0, 1]$ with $\lim_{n \to \infty} s_n = 0$ such that

$$\pi\big(\zeta(s_n)\big) \neq \pi\big(\zeta(0)\big), \qquad \forall n \in \mathbb{N}.$$

- A *right (left) splitting arc* is a splitting arc ζ for which there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in $]0, 1]$ with $\lim_{n \to \infty} s_n = 0$ such that

$$\pi\big(\zeta(s_n)\big) > \pi\big(\zeta(0)\big), \qquad (\text{resp. } \pi\big(\zeta(s_n)\big) < \pi\big(\zeta(0)\big)), \qquad \forall n \in \mathbb{N}.$$

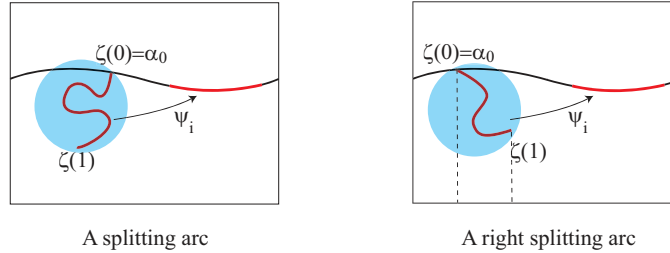
A splitting arc   A right splitting arc

FIGURE 3. Splitting arcs.

See Figure 3. We say that a splitting arc is based on Γ when it is based at some point of Γ.

Remark 2.9. Here we collect several remarks on splitting arcs.

- If ζ is a right (resp. left) splitting arc, then the restriction $\zeta_{|[0,s_*]}$ (up to reparametrization) is also a right (resp. left) splitting arc, for any $0 \leq s^* \leq 1$.
- A sufficient condition for a splitting arc ζ to be a right (left) splitting arc is that it admits a derivative $\zeta'(0) = (u, v)$ with $u > 0$ (resp. $u < 0$).
- A non-vertical splitting arc is either a right splitting arc, or a left splitting arc, or both.

- For a vertical splitting arc (that is, a splitting arc for which there exists $s_* > 0$ such that $\pi\big(\gamma(s)\big) = \pi\big(\gamma(0)\big)$ for all $0 \leq s \leq s_*$), its image under the tilt map $\widehat{\varphi}$ is a left arc (resp., its image under $\widehat{\varphi}^{-1}$ is a right arc); this is not necessarily a splitting arc, since for instance $\widehat{\varphi}(\widetilde{\zeta})$ (resp. $\widehat{\varphi}^{-1}(\widetilde{\zeta})$) may not be contained in any $\mathrm{Dom}\,\psi_i$, $i \in I$.

Given a point $x_0 = (\theta_0, r_0)$ in $\widehat{\mathbf{A}}$, we denote by

$$V(x_0) = \big\{(\theta_0, r) \mid r \in [\widehat{a}, \widehat{b}]\big\}$$

the vertical through x_0 in $\widehat{\mathbf{A}}$, by

$$V^-(x_0) = \big\{(\theta_0, r) \mid r \in [\widehat{a}, r_0]\big\}$$

the semi-vertical below x_0 in $\widehat{\mathbf{A}}$, and by

$$V^+(x_0) = \big\{(\theta_0, r) \mid r \in [r_0, \widehat{b}]\big\}$$

the semi-vertical above x_0 in $\widehat{\mathbf{A}}$.

Definition 2.10. Let $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ contained in $\widehat{\mathbf{A}} \setminus \Gamma(\widehat{a})$, be the graph of the continuous function $\gamma : \mathbb{T} \to [\widehat{a}, \widehat{b}]$ and $\alpha_0 \in \Gamma$.

Let ζ be a right splitting arc based at $\alpha_0 = \zeta(0)$, let α_* be a point in Γ such that

$$\pi(\alpha_0) < \pi(\alpha_*) < \max_{s \in [0,1]} \pi(\zeta(s)),$$

and let $\beta_* = \zeta(s_*)$ be the point in $V^-(\alpha_*) \cap \widetilde{\zeta}$ with the maximal r-coordinate. Let C be the Jordan curve formed by the concatenation of the arcs $\zeta([0, s_*])$, $[\beta_*, \alpha_*]$, and $[\alpha_*, \alpha_0] \subseteq \Gamma$. We denote by $D(\zeta_{|[0,s_*]})$ the bounded connected component of the complement of C in $\mathbb{T} \times \mathbb{R}$. We say that $D(\zeta_{|[0,s_*]})$ is a *triangular domain associated with* ζ. We define a triangular domain associated with a left splitting arc similarly.
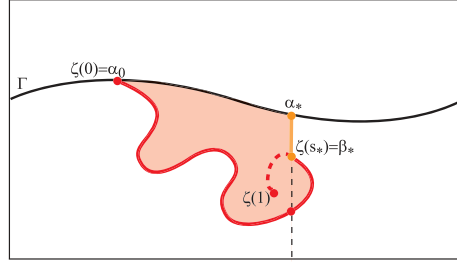

FIGURE 4. A triangular domain associated to a right splitting arc.

Let us now define the notion of *tilted arcs* (see Appendix A for more details). Given u, v two vectors in $\mathbb{R}^2$, let $\angle(u, v)$ be the angle from u to v in $[0, 2\pi[$, measured counterclockwise. Denote $\partial_r = (0, 1) \in \mathbb{R}^2$.

A C^1 arc emanating from Γ with $\eta'(s) \neq 0$ for $s \in [0, 1]$ is said to be *positively tilted* (resp. *negatively tilted*) when

- $\angle\big(\partial_r, \eta'(0)\big) \in]0, \pi[$ (resp. $\angle\big(\partial_r, \eta'(0)\big) \in]-\pi, 0[$), and
- the continuous lift $s \in [0, 1] \mapsto \angle\big(\partial_r, \eta'(s)\big) \in \mathbb{R}$ is positive (resp. negative) for all $s \in [0, 1]$.

The main property of the triangular domains defined above is the following (see Appendix A for a proof).

Lemma 2.11. *Consider an essential invariant circle $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ contained in $\widehat{\mathbf{A}} \setminus \Gamma(\widehat{a})$, and $\Gamma_\bullet \subset \widehat{\mathbf{A}}$ an essential circle in Γ^-. Assume that ζ is a right (resp. left) splitting arc based on Γ such that $\widetilde{\zeta}$ is contained in the region $\Gamma_\bullet^+$ above $\Gamma_\bullet$, and let $D(\zeta_{|[0,s_*]})$ be a triangular domain associated to ζ as in Definition 2.10. Assume that η be a negatively (resp. positively) tilted arc with $\eta(0) \in \Gamma_\bullet$, $\eta(]0,1]) \subset \Gamma_\bullet^+ \cap \Gamma^-$, and $\eta(1) \in D(\zeta_{|[0,s_*]})$.*
Then

$$\eta(]0,1[) \cap \zeta(]0,1]) \neq \emptyset.$$

In the previous lemma, the circle $\Gamma_\bullet$ is not assumed to be invariant by the return map.

2.3. **Good cylinders.** We now define the main first notion of this section.

Definition 2.12. Consider a tame cylinder $\mathscr{C}$ equipped with a tilt section (Σ, φ), with continuations $\widehat{\mathscr{C}}$, $(\widehat{\Sigma}, \widehat{\varphi})$, and homoclinic correspondence ψ. We say that $\mathscr{C}$ is a good cylinder when for every essential invariant circle $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ in $\mathrm{int}\,\widehat{\Sigma}$ there exists a splitting arc based on Γ.

Above, it is implicitly assumed that for each Γ there exists ψ_i such that Cl $(\mathrm{Dom}\,\psi_i) \cap \Gamma \neq \emptyset$ and $\mathrm{Im}\,\psi_i$ contains a segment of Γ.

An additional quantitative definition will be necessary to produce δ-admissible orbits. We denote by $\mathrm{Cl}\,(E)$ the closure of a subset $E \subset \widehat{\Sigma}$.

Definition 2.13. With the same assumptions as in the previous definition, fix $\delta > 0$. We say that $\psi = (\psi_i)_{i \in I}$ is δ-*bounded* when for each essential invariant circle $\Gamma \in \mathrm{Ess}(\varphi)$

$$\mathrm{Sup}\,\left\{\mathrm{Max}\,\left\{\mathrm{dist}\,(x,\Gamma), \mathrm{dist}\,(\psi(x),\Gamma)\right\} \mid x \in \Gamma^- \cap \psi^{-1}(\Gamma^+)\right\} < \delta, \qquad (27)$$

where dist is the point-set distance on $\widehat{\Sigma}$ induced by the canonical distance on $\mathbb{A}^3$ and where $\psi^{-1}(\Gamma^+)$ is the set of all points $x \in \widehat{\Sigma}$ such that there exists $i \in I$ with $x \in \mathrm{Dom}\,\psi_i$ and $\psi_i(x) \in \Gamma^+$.

We say that $\mathscr{C}$ is δ-good when ψ is δ-bounded.

The δ-bounded condition says that for every essential invariant circle Γ there is a homoclinic map ψ_i whose domain and range are both contained in a δ-neighborhood of Γ.

The following Lemma proves that assuming that a good cylinder admits a δ-bounded correspondence is not restrictive.

Lemma 2.14. *Consider a good cylinder $\mathscr{C}$ equipped with a tilt section (Σ, φ), with continuations $\widehat{\mathscr{C}}$, $(\widehat{\Sigma}, \widehat{\varphi})$, and homoclinic correspondence $\psi = (\psi_i)_{i \in I}$ on Σ. Then given $\delta > 0$, there is a homoclinic correspondence $\widetilde{\psi} = (\widetilde{\psi}_i)_{i \in J}$ such that $\mathscr{C}$ is a δ-good cylinder when endowed with $\widetilde{\psi}$.*

Proof. For each essential invariant circle Γ, select one homoclinic map $\psi_{i(\Gamma)}$ which produces a splitting arc $\widetilde{\zeta} \subset \mathrm{Dom}\,\psi_{i(\Gamma)}$ for Γ. Let α_0 be the base point of $\widetilde{\zeta}$. By the definition of a splitting arc, the closure of the domain of $\psi_{i(\Gamma)}$ intersects Γ. Consider the intersection D of the domain $\mathrm{Dom}\,\psi$ with a δ-ball centered at α_0 and with the inverse image under $\psi_{i(\Gamma)}^{-1}$ of a δ-ball centered at $\psi_{i(\Gamma)}(\alpha_0)$. As a consequence, the

diameter of $(\psi_{i(\Gamma)|D})^{-1}(\Gamma^+)$ and the diameter of $\psi_{i(\Gamma)}(D) \cap \Gamma^+$ are both less than δ. Let $\widetilde{\psi}_i = \psi_{i|D}$. Carrying out this process for each essential circle gives rise to a new homoclinic correspondence $(\widetilde{\psi}_j)_{j \in J}$, with a new set of indices $J \subset I$, such that $\mathscr{C}$ equipped with $(\widetilde{\psi}_j)_{j \in J}$ is still a good cylinder. $\qquad\square$

Convention. *In the following, given a good cylinder $\mathscr{C}$, we implicitly choose once and for all a section $(\Sigma, \mathbf{A}, \chi, \varphi)$, a continuation $\widehat{\mathscr{C}}$, a continuation $(\widehat{\Sigma}, \widehat{\mathbf{A}}, \widehat{\chi}, \widehat{\varphi})$ and a homoclinic correspondence ψ, which will always be denoted this way.*

2.4. **Good chains.** Recall we want to be able to prove the existence of drifting orbits along chains of cylinders, and not only along a single one. Again, this requires slightly more involved definitions. We consider a C^2 Hamiltonian function H on $\mathbb{A}^3$, and fix an energy $\mathbf{e}$.

Definition 2.15. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be good cylinders at energy $\mathbf{e}$ for H, with continuations $\widehat{\mathscr{C}}_i$ and characteristic projections $\Pi_i^\pm : W^\pm(\widehat{\mathscr{C}}_i) \to \widehat{\mathscr{C}}_i$. We define the *transverse heteroclinic intersection of $\widehat{\mathscr{C}}_1$ and $\widehat{\mathscr{C}}_2$* as the set

$$\mathbf{Hett}(\widehat{\mathscr{C}}_1, \widehat{\mathscr{C}}_2) \subset W^-(\mathrm{int}\,\widehat{\mathscr{C}}_1) \cap W^+(\mathrm{int}\,\widehat{\mathscr{C}}_2) \qquad (28)$$

formed by the points ξ satisfying

$$W^-\big(\Pi_1^-(\xi)\big) \pitchfork_\xi W^+(\mathrm{int}\,\mathscr{C}_2) \quad \text{and} \quad W^+\big(\Pi_2^+(\xi)\big) \pitchfork_\xi W^-(\mathrm{int}\,\mathscr{C}_1). \qquad (29)$$

As in Lemma 2.6, when the characteristic projections are C^1, for $\xi \in \mathbf{Hett}(\widehat{\mathscr{C}})$ we set $x^- = \Pi_1^-(\xi) \in \widehat{\mathscr{C}}_1$ and $x^+ = \Pi_2^+(\xi) \in \widehat{\mathscr{C}}_2$, so $W^+(\widehat{\mathscr{C}}_1)$ and $W^-(\widehat{\mathscr{C}}_2)$ intersect transversely at ξ in $H^{-1}(\mathbf{e})$. Then there exist a 3-dimensional open neighborhood $\mathscr{O}$ of ξ in $W^+(\mathrm{int}\,\widehat{\mathscr{C}}_1) \cap W^-(\mathrm{int}\,\widehat{\mathscr{C}}_2)$, and open neighborhoods O_1^- of x^- in $\mathrm{int}\,\widehat{\mathscr{C}}_1$ and O_2^+ of x^+ in $\mathrm{int}\,\widehat{\mathscr{C}}_2$, such that the restrictions $(\Pi_1^-)_{|\mathscr{O}}$ and $(\Pi_2^+)_{|\mathscr{O}}$ are C^1 *measure preserving* diffeomorphisms from $\mathscr{O}$ onto their images O_1^-, O_2^+, respectively. Hence one can define the local diffeomorphism

$$S_{1 \to 2} = \Pi_2^+ \circ \big((\Pi_1^-)_{|\mathscr{O}}\big)^{-1} : O_1^- \to O_2^+.$$

This motivates the following definition for the notion of heteroclinic correspondences[16].

Definition 2.16. A *heteroclinic map* associated with $(\widehat{\mathscr{C}}_1, \widehat{\mathscr{C}}_2)$ is a C^1 diffeomorphism

$$\psi_{1 \to 2} : \mathrm{Dom}\,\psi_{1 \to 2} \to \mathrm{Im}\,\psi_{1 \to 2} \qquad (30)$$

where $\mathrm{Dom}\,\psi_{1 \to 2}$ is open in $\mathrm{int}\,\widehat{\Sigma}_1$ and $\mathrm{Im}\,\psi_{1 \to 2}$ is open in $\mathrm{int}\,\widehat{\Sigma}_2$, for which there exists a C^1 measure preserving diffeomorphism

$$S_{1 \to 2} : \mathrm{Dom}\,S_{1 \to 2} \to \mathrm{Im}\,S_{1 \to 2} \qquad (31)$$

where $\mathrm{Dom}\,S_{1 \to 2}$ and $\mathrm{Im}\,S_{1 \to 2}$ are open in $\mathrm{int}\,\widehat{\mathscr{C}}_1$ and $\mathrm{int}\,\widehat{\mathscr{C}}_2$ respectively, which satisfies the following conditions:

- there is an open subset $\mathrm{Domt}S_{1 \to 2} \subset \mathrm{Dom}\,S_{1 \to 2}$, with full measure in $\mathrm{Dom}\,S_{1 \to 2}$, such that

$$\forall y \in \mathrm{Domt}\,S_{1 \to 2}, \qquad W^-(y) \cap W^+\big(S_{1 \to 2}(y)\big) \cap \mathbf{Hett}(\widehat{\mathscr{C}}_1, \widehat{\mathscr{C}}_2) \neq \emptyset; \qquad (32)$$

---

[16]It turns out that in the subsequent constructions of [48] the projections are locally C^1 in a neighborhood of the heteroclinic points we are interested in.

- the range of $S_{1\to2}$ intersects the interior of $\widehat{\Sigma}_2$, i.e.,

$$\operatorname{Im} S_{1\to2} \cap \operatorname{int} \widehat{\Sigma}_2 \neq \emptyset; \tag{33}$$

- there exists a non-negative C^1 function $\tau : \operatorname{Dom} \psi_{1\to2} \to \mathbb{R}$ such that

$$\forall x \in \operatorname{Dom} \psi_{1\to2}, \qquad \Phi_H^{\tau(x)}(x) \in \operatorname{Dom} S_{1\to2} \quad and \quad \psi_{1\to2}(x) = S_{1\to2}\Big(\Phi_H^{\tau(x)}(x)\Big). \tag{34}$$

For purely technical reasons, we finally have to consider the case where the cylinders $\mathscr{C}_i$ are subsets of the same cylinder and they share a common boundary. The motivation, which appears in [47], is to avoid the construction of sophisticated normal forms along "long cylinders" and make much easier the use of symplectic methods to detect homoclinic intersections (see also [50]).

In this case we introduce a *transition map* between the sections, defined as the natural flow-induced map.

Definition 2.17. Assume that the good cylinders $\mathscr{C}_1$ and $\mathscr{C}_2$ are contained in a cylinder $\mathscr{C}$ and satisfy $\partial^{\bullet}\mathscr{C}_1 = \partial_{\bullet}\mathscr{C}_2$, in which case we say they are *adjacent*. A *transition map* from $\widehat{\mathscr{C}}_1$ to $\widehat{\mathscr{C}}_2$ is a homoclinic map (relative to $\mathscr{C}$)

$$\psi_{1\to2} : \operatorname{Dom} \psi_{1\to2} \to \operatorname{Im} \psi_{1\to2} \tag{35}$$

where $\operatorname{Dom} \psi_{1\to2}$ is an open neighborhood of the circle $\Gamma = \partial^{\bullet}\Sigma_1 = \partial_{\bullet}\Sigma_2$ in $\widehat{\Sigma}_1$, and $\operatorname{Im} \psi_{1\to2}$ is an open neighborhood of Γ in $\widehat{\Sigma}_2$. More precisely, there exists a non-negative C^1 function $\tau : \operatorname{Dom} \psi_{1\to2} \to \mathbb{R}$ such that

$$\psi_{1\to2}(x) = \Phi_H^{\tau(x)}(x), \qquad \forall x \in \operatorname{Dom} \psi_{1\to2}.$$

In the following our heteroclinic maps and transition maps will play the same role in our constructions and we will not make any distinction between them (including terminology). This yields our final definition.

Definition 2.18. Fix $\delta > 0$. A δ-*good chain of cylinders* at energy $\mathbf{e}$ is a finite ordered family $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ of δ-good cylinders at energy $\mathbf{e}$, such there exists a heteroclinic map $\psi_{k\to k+1}$ from $\widehat{\mathscr{C}}_k$ to $\widehat{\mathscr{C}}_{k+1}$ which satisfies the following condition:

- for any open neighborhood O of $\partial^{\bullet}\Sigma_k$ in $\widehat{\Sigma}_k$, the image $\psi_{k\to k+1}(O)$ intersects a dynamically minimal essential invariant circle in $\operatorname{Ess}(\varphi_{k+1})$ located in a δ-neighborhood of $\partial_{\bullet}\Sigma_{k+1}$ in $\widehat{\Sigma}_{k+1}$.

Again, the δ-neighborhoods are defined relatively to the induced distance on the section $\widehat{\Sigma}_{k+1}$.

2.5. **Polysystems.** Let us first make the definition of an orbit of a polysystem more precise (see [55, 46] for a formal definition). Let A be some set and consider a set $f = \{f_i \mid i \in I\}$ of locally defined maps $f_i : \operatorname{Dom} f_i \subseteq A \to A$. We say that a finite sequence $(x_n)_{0 \leq n \leq n_*}$, $n_* \geq 1$, of points of A is a *finite orbit of f, of length $n_* + 1$*, provided that there exists $\omega = (i_n)_{0 \leq n \leq n_* - 1} \in I^{n_*}$ such that for $0 \leq n \leq n_* - 1$:

$$x_{n+1} = f_{i_n}(x_n), \text{ with } x_n \in \operatorname{Dom} f_{i_n} \text{ and } x_{n+1} \in \operatorname{Dom} f_{i_{n+1}}.$$

We use the following short-hand notation

$$x_{n_*} = f^{\omega}(x_0).$$

We consider the point x_0 as being the length-1 orbit of x_0.

Given a subset $B \subset A$, we set

$$f^\omega(B) = \bigcup_{x \in B_\omega} f^\omega(x)$$

where B_ω is the subset of B formed by the points x such that $f^\omega(x)$ is well-defined.

The *full orbit of $B \subset A$* under f is the subset of A formed by the union of all $f^\omega(B)$ for all sequences (of any length) ω. In particular B is contained in its full orbit under f.

Conventions. *Given locally defined maps $f_i : \mathrm{Dom}\, f_i \to \mathrm{Im}\, f_i$ on A, we write $f_i \circ f_j$ for the map defined by composition on the subset $\mathrm{Dom}\, f_i \cap \mathrm{Im}\, f_j$ and for a subset $B \subset A$, we write $f_i(B)$ for $f_i\big(\mathrm{Dom}\, f_i \cap B\big)$.*

Given a finite set of polysystems $f, g, \ldots$ on A, we write $\{f, g, \ldots\}$ or (f, g) for the polysystem formed by their union.

Given a polysystem $f = \{f_i \mid i \in I\}$ on a set A, and a subset A_* of A, we define the restriction $f_{|A_*}$ of f to A_* as the polysystem formed by the maps

$$f_i^* : \mathrm{Dom}\, f_i \cap A_* \cap f_i^{-1}(A_*) \to A_*.$$

Note that f_i^* is not necessarily open, even when f_i is.

Remark 2.19. By Lemma 2.14, any pseudo-orbit of $(\varphi, \widetilde{\psi})$ is a pseudo-orbit of (φ, ψ). As a consequence, given $\delta > 0$, one can always assume the homoclinic correspondence of a good cylinder to be δ-bounded.

Definition 2.20. Let H be a C^2 Hamiltonian on $\mathbb{A}^3$ and fix an energy $\mathbf{e}$. Let $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ be a good chain of cylinders in $H^{-1}(\mathbf{e})$. Its *associated polysystem* is the following set of (locally) defined diffeomorphisms:

$$\mathscr{F} = \big\{ \widehat{\varphi}_k \mid 1 \leq k \leq k_* \big\} \cup \big\{ \psi_{k,i} \mid 1 \leq k \leq k_*, \ i \in I_k \big\} \cup \big\{ \psi_{k \to k+1} \mid 1 \leq k \leq k_* - 1 \big\}, \tag{36}$$

where $\widehat{\varphi}_k$ is the continuation of φ_k, $\psi_k = (\psi_{k,i})_{i \in I_k}$ is the homoclinic correspondence associated to $\mathscr{C}_k$, and $\psi_{k \to k+1}$ is either a heteroclinic map or a transition map from $\widehat{\mathscr{C}}_k$ to $\widehat{\mathscr{C}}_{k+1}$.

This polysystem will be indifferently considered to be defined on $\bigcup_{1 \leq k \leq k_*} \widehat{\mathbf{A}}_k$, or on $\bigcup_{1 \leq k \leq k_*} \widehat{\Sigma}_k$.

The finite orbits of the polysystem will be called *pseudo-orbits along the chain*.

The pseudo-orbits above are therefore generated by successive applications of the maps $\widehat{\varphi}_k, \psi_{k,i}, \psi_{k \to k+1}$ in any possible order.

3. **Pseudo orbits along a good chain of cylinders.** Our aim in this section is to prove the following result.

Theorem 3.1. *Let H be a C^2 Hamiltonian on $\mathbb{A}^3$ and fix an energy $\mathbf{e}$. Let $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ be a δ-good chain of cylinders at energy $\mathbf{e}$ for some $\delta > 0$.*

Then there exists a pseudo-orbit $(x_n)_{0 \leq n \leq n_*}$ of the polysystem $\mathscr{F}$ along the chain such that for any essential invariant circle Γ in $\bigcup_{1 \leq k \leq k_*} \mathrm{Ess}(\varphi_k)$, there is a $\nu \in \{0, \ldots, n_*\}$ with $d(x_\nu, \Gamma) < \delta$.

We first prove in Section 3.1 the existence of pseudo-orbits in the case where the chain is reduced to a single cylinder. Then **Theorem 3.1** is easily deduced from this preliminary result in Section 3.2. The main remark in our proof comes from

Lemma C.1 of Appendix C: it is enough to prove the existence of pseudo-orbits for the "symmetrized" polysystem

$$
\begin{aligned}
\mathscr{G} = &\big\{\widehat{\varphi}_k,\ \widehat{\varphi}_k^{-1} \mid 1 \leq k \leq k_*\big\} \\
&\cup \big\{\psi_{k,i} \mid 1 \leq k \leq k_*,\ i \in I_k\big\} \cup \big\{\psi_{k \to k+1} \mid 1 \leq k \leq k_* - 1\big\}
\end{aligned}
\tag{37}
$$

and obtain the conclusion by the density of the pseudo-orbits of $\mathscr{F}$ relative to the set of the pseudo-orbits of $\mathscr{G}$.

3.1. **Pseudo-orbits in the case of a single cylinder.** Given $\nu > 0$, we define a ν-*ball* of $\mathbb{T} \times \mathbb{R}$ as a subset $B = B_\theta \times B_r$ where B_θ and B_r are intervals of $\mathbb{T}$ and $\mathbb{R}$ respectively, such that

$$
\text{length } B_r > \nu \text{ length } B_\theta. \tag{38}
$$

The *center of B* is (a_θ, a_r), where a_θ, a_r are the mid-points of B_θ and B_r, respectively. We keep the notation and conventions of the last section for the good cylinders and their continuations.

Proposition 3.2. *Let H be a C^2 Hamiltonian on $\mathbb{A}^3$ and fix an energy $\mathbf{e}$. Let $\mathscr{C}$ be a good cylinder at energy $\mathbf{e}$ for H. Let $g = (\widehat{\varphi}, \widehat{\varphi}^{-1}, \psi)$ be the associated symmetrized) polysystem on $\widehat{\mathbf{A}}$, with $\psi = (\psi_i)_{i \in I}$. Fix $\Gamma_\bullet \in \mathrm{Ess}(\widehat{\varphi})$ with $\Gamma_\bullet \subset \mathbf{A} \setminus \Gamma(b)$. Fix a neighborhood V of $\Gamma_\bullet$ in $\mathbf{A}$ (see Figure 5).*

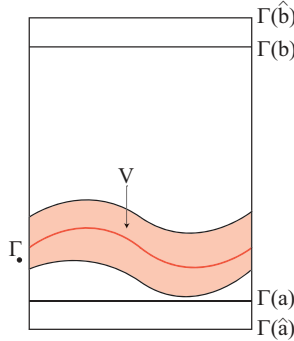*Then the full orbit of V under g contains $\Gamma(b)$.*

FIGURE 5. The setting of Proposition 3.2

The assumption $\Gamma_\bullet \subset \mathbf{A} \setminus \Gamma(b)$ is a matter of simplification and can easily be relaxed in practical cases.

Proof of Proposition 3.2. We assume for example that $\widehat{\varphi}$ tilts the vertical to the right, the other case being exactly similar. Let $\mathbf{A}_*$ be the $\widehat{\varphi}$–invariant subannulus of $\widehat{\mathbf{A}}$ limited by the (disjoint) circles $\Gamma_\bullet$ and $\Gamma(\widehat{b})$.

Without loss of generality, one can assume that for every circle $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ contained in $\mathbf{A}_* \setminus \{\Gamma_\bullet\}$, for every map ψ_i such that $\mathrm{Im}\,\psi_i \cap \Gamma \neq \emptyset$, we have that $\mathrm{Dom}\,\psi_i \subset \mathbf{A}_*$ and $\mathrm{Im}\,\psi_i \subset \mathbf{A}_*$, therefore the good cylinder condition is still satisfied on $\mathbf{A}_*$.

1. We consider the restricted polysystem $g_{|\mathbf{A}_*}$ instead of g and we consider a neighborhood V of $\Gamma_\bullet$ *in* $\mathbf{A}_*$. For simplicity, we still denote by $\widehat{\varphi}$ and $\psi = (\psi_i)_{i \in I}$ the restrictions of $\widehat{\varphi}$ and ψ to $\mathbf{A}_*$. Thanks to our previous remark, the maps ψ_i are *open*.

We also assume without loss of generality that the set V is open in $\mathbf{A}_*$ and connected. Let U be the full orbit of V under the polysystem $g = \big(\widehat{\varphi}, \widehat{\varphi}^{-1}, \psi = (\psi_i)_{i \in I}\big)$ on $\mathbf{A}_*$. Note that $\widehat{\varphi}(U) = U$ and $\psi_i(U) \subset U$. Set $U_c = \mathrm{cc}(U, \Gamma_\bullet)$. Then U_c is open and contains V, so $\widehat{\varphi}(U_c) = U_c$. It is therefore enough to prove that U_c intersects $\Gamma(b)$: since $\Gamma(b)$ is dynamically minimal, this yields $\Gamma(b) \subset U_c \subset U$.

Let us assume by contradiction that $U_c \cap \Gamma(b) = \emptyset$, so that U_c is contained in the lower connected component of $\mathbf{A}_* \setminus \Gamma(b)$. We want to apply the Birkhoff graph theorem to describe the frontier of U_c (see Appendix A), however U_c is not homeomorphic to an annulus. The following paragraph describes a standard way to "fill the holes" of U_c without altering its frontier, making it possible to apply the Birkhoff theory.

2. Set $O = \mathbf{A}_* \setminus \overline{U_c}$, so that O is open, contains $\Gamma(\widehat{b})$, and $O \cap V = \emptyset$. Moreover, since $\widehat{\varphi}(\overline{U_c}) = \overline{U_c}$,

$$\widehat{\varphi}(O) = \mathbf{A}_* \setminus \widehat{\varphi}(\overline{U_c}) = \mathbf{A}_* \setminus \overline{U_c} = O.$$

Then $\widehat{\varphi}\big(\mathrm{cc}(O, \Gamma(\widehat{b}))\big) = \mathrm{cc}(O, \Gamma(\widehat{b}))$ and so $\widehat{\varphi}(\overline{\mathrm{cc}(O, \Gamma(\widehat{b}))}) = \overline{\mathrm{cc}(O, \Gamma(\widehat{b}))}$. Let

$$\widehat{U} = \mathbf{A}_* \setminus \overline{\mathrm{cc}(O, \Gamma(\widehat{b}))},$$

so that $\widehat{U}$ is open and $\widehat{\varphi}(\widehat{U}) = \widehat{U}$, and set finally

$$\mathcal{U} = \mathrm{cc}(\widehat{U}, \Gamma_\bullet), \tag{39}$$

hence $\mathcal{U}$ is open, connected and $\widehat{\varphi}(\mathcal{U}) = \mathcal{U}$. Moreover clearly

$$\overline{\mathcal{U}} \subset \mathbf{A}_* \setminus \mathrm{cc}(O, \Gamma(\widehat{b})), \tag{40}$$

and

$$U_c \subset \mathcal{U}, \tag{41}$$

since $\overline{O} = \mathbf{A}_* \setminus \mathrm{Int}\,(\overline{U_c}) \subset \mathbf{A}_* \setminus U_c$, so $\overline{\mathrm{cc}(O, \Gamma(\widehat{b}))} \subset \mathbf{A}_* \setminus U_c$ and $U_c \subset \mathbf{A}_* \setminus \overline{\mathrm{cc}(O, \Gamma(\widehat{b}))} = \widehat{U}$, which proves (41) since $\Gamma_\bullet \subset U_c$.

The conclusion of this step is obtaining the set $\mathcal{U}$ by "filling the holes" of U_c.

3. We denote by $\mathrm{Fr}\,A$ the frontier $\overline{A} \setminus \mathrm{Int}\,A$ of a subset in a topological space. Let $\Gamma := \mathrm{Fr}\,\mathcal{U} \cap \Gamma_\bullet^+$. We shall prove that Γ is a Lipschitz graph over $\mathbb{T}$, invariant under $\widehat{\varphi}$, by the Birkhoff theorem (see Appendix A). By local connectedness of $\mathbf{A}_*$, one readily proves that $\mathrm{Int}\,\overline{\mathcal{U}} = \mathcal{U}$, since $\mathcal{U}$ is a connected component of the complement of the closure of an open set. Moreover $\widehat{\varphi}(\mathcal{U}) = \mathcal{U}$. Let now $\mathbf{S}$ be the quotient of $\mathbf{A}_*$ by the identification of each boundary circle to one point, so that $\mathbf{S}$ is homeomorphic to S^2. Up to this quotient, $\mathcal{U}$ is a connected component of the complement in $\mathbf{S}$ of a compact connected subset, so is homeomorphic to a disk. Going back to the initial space $\mathbf{A}_*$ proves that $\mathcal{U}$ is homeomorphic to $\mathbb{T} \times [0, 1[$. So by the Birkhoff theorem, $\Gamma = \partial \mathcal{U}$ is a Lipschitz graph over $\mathbb{T}$, invariant under $\widehat{\varphi}$ (see [55] for more details).

4. Let us now prove that $\Gamma \subset \overline{U_c}$, and so $\Gamma \subset \mathrm{Fr}\,(U_c) = \mathrm{Cl}\,(U_c) \setminus U_c$. Assume that $x \in \Gamma$ is not in $\overline{U_c}$, so that there exists a small ball $B(x, \varepsilon)$ with $B(x, \varepsilon) \cap \overline{U_c} = \emptyset$. Let z be some point on the vertical through x, located under Γ and inside $B(x, \varepsilon)$. Let us show that the semi-vertical $\sigma = V^+(z)$ over z in $\mathbf{A}_*$ is disjoint from $\overline{U_c}$. First $\Gamma \cap \sigma = \{x\}$, since Γ is a graph, so that $\sigma = [z, x] \cup [x, \xi]$, with $\xi \in \Gamma(\widehat{b})$. Clearly $[z, x] \subset B(x, \varepsilon)$ so $[z, x] \cap \overline{U_c} = \emptyset$, and $]x, \xi] \cap \overline{\mathcal{U}} = \emptyset$ since $\Gamma = \partial \mathcal{U}$ is a graph. Since $U_c \subset \mathcal{U}$, this proves that $\sigma \cap \overline{U}_c = \emptyset$.

As a consequence $\sigma \cup \Gamma(\widehat{b})$ is a connected set which satisfies $(\sigma \cup \Gamma(\widehat{b})) \cap \overline{U_c} = \varnothing$. Therefore $(\sigma \cup \Gamma(\widehat{b})) \subset \mathrm{cc}(O, \Gamma(\widehat{b}))$ and thus $(\sigma \cup \Gamma(\widehat{b})) \cap \overline{\mathcal{U}} = \varnothing$ by (40). This is a contradiction since $x \in \Gamma \subset \overline{\mathcal{U}}$. Therefore $\Gamma \subset \overline{U_c}$.

5. Since Γ is an invariant essential circle for the good tilt map $\widehat{\varphi}$, there are only two possibilities:

– either Γ is the upper boundary of a Birkhoff zone,

– or Γ is accumulated from below by essential invariant circles in the Hausdorff topology.

We will prove that both possibilities yield a contradiction with the initial assumption that $U_c \cap \Gamma(b) = \emptyset$.

We call attention that the argument from this point on is different from [55], who considers a polysystem consisting of two globally defined mappings on the annulus. In our case the polysystem contains not only the maps $\widehat{\varphi}, \widehat{\varphi}^{-1}$ which are globally defined, but also the maps ψ_i, which, by the definition of a good cylinder, are only locally defined around points on essential invariant circles.

6. Assume first that Γ is the upper boundary of a Birkhoff zone $\mathscr{Z}$ and let Γ_* be the lower boundary of $\mathscr{Z}$. Let ν be the Lipschitz constant of Γ_*. Since Γ_* is a graph and U_c is open, connected, contains V and satisfies $\overline{U_c} \cap \Gamma \neq \varnothing$, then $U_c \cap \Gamma_* \neq \varnothing$. So there exists a ν-ball $\mathbf{B} \subset U_c$ centered on Γ_*.

Since $\mathscr{C}$ is a good cylinder, there exists a splitting arc ζ based at some point α_0 of Γ (see Section 2). Either ζ is a non-vertical arc, in which case it is a right splitting arc or a left splitting arc, or it is a vertical arc.

If ζ is a right (resp. left) splitting arc, there exists a triangular domain $D := D(\zeta_{|[0,s_*]})$ associated to ζ such that $D \subset \mathscr{Z}$.

First, assume that $\widetilde{\zeta}$ is a right splitting arc. By Proposition A.6, there exist $z_0 \in \mathbf{B}$ and $n \in \mathbb{N}$ such that $z_n := \widehat{\varphi}^n(z_0) \in D$. By Lemma A.2 there exists a *negatively tilted arc* η with image in $\mathbf{B}$ emanating from Γ_* and ending at z_0. Therefore, by Lemma A.4, $\eta_n := \widehat{\varphi}^n \circ \eta$ is a *negatively* tilted arc emanating from Γ_* and ending at $z_n \in D$. By Lemma 2.11, the image $\widetilde{\eta}_n$ intersects $\widetilde{\zeta}$.

Second, assume that $\widetilde{\zeta}$ is a left splitting arc. We use $\widehat{\varphi}^{-1}$ instead of $\widehat{\varphi}$. This yields a point $z_0 \in \mathbf{B}$ such that $z_{-n} := \widehat{\varphi}^{-n}(z_0) \in D$, and a *positively tilted arc* η emanating from Γ_* and ending at z_0. Then $\eta_n := \widehat{\varphi}^{-n} \circ \eta$ is a *positively tilted arc* emanating from Γ_* and ending at $z_{-n} \in D$, whose image $\widetilde{\eta}_n$ intersects $\widetilde{\zeta}$.

Third, assume that $\widetilde{\zeta}$ is a vertical splitting arc. Then $\widehat{\varphi}^{-1}(\widetilde{\zeta})$ is a right arc (not necessarily splitting), see Remark 2.9. Let D be a triangular domain associated to $\widehat{\varphi}^{-1}(\widetilde{\zeta})$. As above, there exists $z_0 \in \mathbf{B}$, a *negatively tilted arc* η with image in $\mathbf{B}$, and $m \in \mathbb{N}$ such that the negatively tilted arc $\widehat{\varphi}^m(\widetilde{\eta})$, which emanates from Γ_* and ends at z_m, intersects $\widehat{\varphi}^{-1}(\widetilde{\zeta})$. Since $\widehat{\varphi}^m(\widetilde{\eta}) \cap \widehat{\varphi}^{-1}(\widetilde{\zeta}) \neq \varnothing$, we obtain $\widehat{\varphi}^{m+1}(\widetilde{\eta}) \cap \widetilde{\zeta} \neq \varnothing$. Setting $n = m + 1$ and $\widetilde{\eta}_n = \widehat{\varphi}^n(\widetilde{\eta})$ we obtain $\widetilde{\eta}_n \cap \widetilde{\zeta} \neq \varnothing$.

As a consequence, in either case we have that $U_c \cap \widetilde{\zeta} \neq \varnothing$ since $\widetilde{\eta}_n \subset U_c$, and therefore there is a small open ball $B \subset U_c$ centered on $\zeta(]0,1])$ and an index $i \in I$ such that $B \subset \mathrm{Dom}\,\psi_i$. Thus $\psi_i(B)$ is an open set which intersects Γ, and therefore also U_c since $\Gamma \subset \overline{U_c}$. This proves that $\psi_i(B) \subset U_c$ by connectedness, so that U_c contains points strictly above the circle Γ. This is a contradiction with the construction of $\Gamma = \mathrm{Fr}\,\mathcal{U}$ and the inclusion $U_c \subset \mathcal{U}$, which ensures that all points of U_c are located below Γ.

7. Assume now that Γ is accumulated from below by an increasing sequence $(\Gamma_m)_{m \geq 1}$ of essential invariant circles for $\widehat{\varphi}$. Let ζ be a splitting arc based on Γ. Let S_m be the closed strip limited by Γ_m and Γ_{m+1}. For m large enough, $S_m \cap \widetilde{\zeta}$ contains a C^0 curve ℓ which intersects both Γ_m and Γ_{m+1}. Now $\Gamma \subset \overline{U_c}$, so that $U_c \cap S_m$ contains a C^0 curve ℓ' which also intersects both Γ_m and Γ_{m+1}. Therefore, by Lemma A.3, there exists an integer n such that $\widehat{\varphi}^n(U_c) \cap \ell \neq \varnothing$, and so by invariance of U_c under $\widehat{\varphi}$, $U_c \cap \ell \neq \varnothing$. Since $\ell \subset \widetilde{\zeta} \subset \mathrm{Dom}\,\psi_i$ for some $i \in I$, there exists a ball $B \subset U_c$ centered on $\ell \subset \widetilde{\zeta}$ and contained in $\mathrm{Dom}\,\psi_i$. This yields the same contradiction as in the previous paragraph.

8. We have therefore proved that the two possibilities of our alternative yield a contradiction, which proves that our initial assumption $U_c \cap \Gamma(b) = \emptyset$ is false. This concludes the proof of the proposition. $\qquad\square$

As a consequence of Lemma C.1, under the assumptions of Proposition 3.2, for any $\delta > 0$, the polysystem $f = (\widehat{\varphi}, \psi)$ admits orbits which intersect the δ-neighborhoods of $\Gamma_\bullet$ and $\Gamma(b)$. We now take into account the additional assumption that ψ is δ-bounded.

Corollary 3.3. *With the same assumptions and notation as in* Proposition 3.2, *if* ψ *is* δ-*bounded, there exists an orbit of* $f = (\widehat{\varphi}, \psi)$ *which intersects the* δ–*neighborhoods of* $\Gamma_\bullet$ *and* $\Gamma(b)$ *also intersects the* δ-*neighborhood of every element of* $\mathrm{Ess}(\varphi)$ *contained in* $\mathbf{A}$.

Proof. Consider an orbit $(x_n)_{0 \leq n \leq n_*}$ of f with x_0 in the δ-neighborhood of $\Gamma_\bullet$ and x_{n_*} in the δ-neighborhood of $\Gamma(b)$. Let $\Gamma \in \mathrm{Ess}(\varphi)$ be contained in the complement of these neighborhoods. Then there exists $\nu \in \{0, \ldots, n_*\}$ and ψ_i such that $x_\nu \in \mathrm{Dom}\,\psi_i \cap \Gamma^-$ and $\psi_i(x_\nu) \in \Gamma^+$. (Otherwise, an immediate induction would imply that $x_n \in \Gamma^-$ for $0 \leq n \leq n_*$). Using again that ψ is δ-bounded, we have that both the domain and range of ψ_i are contained in a δ-neighborhood of Γ, which proves that $\mathrm{dist}\,(x_\nu, \Gamma) < \delta$. $\qquad\square$

3.2. **End of proof of Theorem 3.1 and a remark on pseudo-orbits.** Let $\mathbf{A}_k = \mathbb{T} \times [a_k, b_k]$ and $\widehat{\mathbf{A}}_k = \mathbb{T} \times [\widehat{a}_k, \widehat{b}_k]$. Fix $\delta > 0$. Let $V_1 = \mathbb{T} \times [a_1, a_1 + \delta] \subset \mathbf{A}_1$. We proceed by induction. Given $k \in \{1, \ldots, k_*\}$, we set the following condition:

- (H_k): the full orbit $O := O(V_1)$ of V_1 under the polysystem $\mathscr{G}$ contains $\Gamma(b_k) \subset \mathbf{A}_k$,

where $\mathscr{G}$ was defined in (37).

By Proposition 3.2, the connected component of O containing $\Gamma(a_1)$ also contains $\Gamma(b_1)$, so that (H_1) is satisfied.

Assume that (H_k) is satisfied for some $k \in \{1, \ldots, k_* - 1\}$. Then by our definition of a δ-good chain, since O is a neighborhood of $\Gamma(b_k)$, $\psi_{k \to k+1}(O)$ intersects some dynamically minimal circle $\Gamma_{k+1} \in \mathrm{Ess}(\varphi_{k+1})$ located in the δ-neighborhood of $\Gamma(a_{k+1})$. As a consequence, the invariance of O under $\widehat{\varphi}_{k+1}$ proves that O contains a neighborhood V_{k+1} of Γ_{k+1}, and Proposition 3.2 applied to $(\widehat{\varphi}_{k+1}, \widehat{\varphi}_{k+1}^{-1}, \psi_{k+1})$ proves that O contains a neighborhood of $\Gamma(b_{k+1})$, so that (H_{k+1}) is satisfied.

By finite induction, (H_k) is satisfied for $k \in \{1, \ldots, k_*\}$.

The previous statement holds without any boundedness assumption on the homoclinic or heteroclinic maps. Assuming now that the full assumptions for good chains are satisfied, then Corollary 3.3 (and the remark before) proves the existence of an orbit of the initial polysystem $\mathscr{F}$ defined in (36) which intersects the

δ-neighborhood of each element of $\mathrm{Ess}(\varphi_k)$, $1 \leq k \leq k_*$, which concludes the proof of **Theorem 3.1**. □

We can finally take advantage of the Poincaré recurrence theorem and prescribe one possible form for the pseudo-orbits. This is the content of the next remark.

Remark 3.4. With the same assumptions as in **Theorem 3.1**, one can choose the pseudo-orbit $(x_n)_{1 \leq n \leq n_*}$ so that it is (2δ)-admissible and is a concatenation of segments of the form

$$\psi \circ \varphi^m(x)$$

with $m > 0$, where

$$\varphi \in \big\{\varphi_k \mid 1 \leq k \leq k_*\big\}, \text{ and}$$

$$\psi \in \big\{\psi_{k,i} \mid 1 \leq k \leq k_*, \ i \in I_k\big\} \cup \big\{\psi_{k \to k+1} \mid 1 \leq k \leq k_* - 1\big\}.$$

In other words, in the pseudo-orbit (x_n) each application of a homoclinic/heteroclinic maps is always interspersed with a positive number of applications of the flow-induced map on the cylinder.

The proof follows the same lines as that of Lemma C.1. Assume that a pseudo-orbit involves the consecutive use of two homoclinic/heteroclinic maps ψ and ψ' (we get rid of the indices), so that say $x_{n+2} = \psi' \circ \psi(x_n)$. Using the Poincaré recurrence theorem, one can slightly perturb the point x_n into x'_n in such a way that $\psi(x'_n)$ is positively recurrent with a sufficiently large return time, and produce a pseudo-orbit starting arbitrarily close to x_n and ending arbitrarily close to x_{n+2}, of the following form

$$\begin{aligned}
&x'_n, \ x'_{n+1} = \psi(x'_n), \\
&x'_{n+2} = \varphi(x'_{n+1}), \ldots, x'_{n+m+1} = \varphi^m(x'_{n+1}) \\
&x'_{n+m+2} = \psi'(x'_{n+m+1}).
\end{aligned}$$

Indeed, it suffices to choose m is such a way that x'_{n+m+1} is close enough to the initial image $\psi(x'_n)$ so as to be able to control the whole orbit, using the continuity of the maps involved in the polysystem.

3.3. **An addendum for singular cylinders.** We now state an additional result which will enable us in [48] to deal with "singular cylinders" exactly in the same way as if they were good cylinders. More precisely, singular cylinders are 3-dimensional manifolds diffeomorphic to the product of $\mathbb{T}^1$ with a 2-sphere with three open disks with disjoint closures cut off[17]. Our way to deal with singular cylinders is to "fill-in" one of the cut-off disks, and extend the dynamics to a tame 3-dimensional cylinder. Since the extended dynamics on the filled disk does not correspond to "true" orbits of the system, one is interested in obtaining pseudo-orbits that avoid the "filled" disk. See [48] for details.

Corollary 3.5. *Assume the conditions from Proposition 3.2. Assume that $K \subset \mathbb{T} \times \,]a + \delta, b - \delta[$ is a compact subset of $\mathbf{A}$, invariant under φ and contained in the interior of some Birkhoff zone of φ. Then $f = (\varphi, \psi)$ admits a pseudo-orbit which does not intersect K. If moreover ψ is δ-bounded, then there exists a δ-admissible pseudo-orbit.*

---

[17]Such singular cylinders appear near double-resonances in near-integrable systems.

Proof. Since the interior of the Birkhoff zone contains no element of $\mathrm{Ess}(\widehat{\varphi})$, one can reduce the domains of the heteroclinic correspondence in such a way that $\mathrm{Dom}\,\psi_i \cap K = \emptyset$ for $i \in I$, still preserving the good cylinder condition. Therefore there exists a pseudo-orbit $(x_n)_{0 \leq n \leq n_*}$ for (φ, ψ) such that x_0 is δ-close to $\Gamma(a)$ and x_n is δ-close to $\Gamma(b)$. This orbit cannot intersect K, for if $x_n \in K$, then by induction $x_m \in K$ for $m \geq n$. The existence of a δ-admissible pseudo-orbit is then immediate. $\square$

4. **Shadowing of pseudo-orbits and proof of Theorem 1.2.** The aim of this section is to prove that given an arbitrary pseudo-orbit along a good chain (as introduced in Definition 2.20), there exists an orbit of the Hamiltonian system (1), which passes arbitrarily close to each point of the pseudo-orbit. **Theorem 1.2** is then an immediate consequence of the existence of pseudo-orbits intersecting arbitrarily small neighborhoods of any essential invariant circle in the tilt sections of a good chain, as proved in **Theorem 3.1**. The main result of this section is the following.

Theorem 4.1. *Let H be a Hamiltonian of class C^2 on $\mathbb{A}^3$ and fix an energy* **e**. *Let $(\mathscr{C}_k)_{1 \leq k \leq k_*}$ be a good chain of cylinders at energy* **e**.

Then given a pseudo-orbit $(x_n)_{0 \leq n \leq n_*}$ along this chain and any $\delta > 0$, there exist a solution of the Hamiltonian vector field X_H, with initial condition a_0, and a finite sequence of times $(\tau_n)_{0 \leq n \leq n_*}$, with $\tau_0 = 0$, such that

$$\mathrm{dist}\left(\Phi_H(\tau_n, a_0), x_n\right) < \delta \ for \ 0 \leq n \leq n_*.$$

Proof. The proof generalizes those of [9, 17]. We keep the notation from Section 3 for the polysystem $\{\widehat{\varphi}_k, \psi_k, \psi_{k \to k+1}\}$, with $\psi_k = (\psi_{k,i})_{i \in I_k}$, which we consider to be defined on the union

$$\bigcup_{1 \leq k \leq k_*} \widehat{\Sigma}_k.$$

1. Let $(x_n)_{0 \leq n \leq n_*}$ be a pseudo-orbit along the chain.

First, we shall prove the existence of a sequence $(y_j)_{0 \leq j \leq j_*}$ of points of $\widehat{\mathbf{C}} = \bigcup_{1 \leq k \leq k_*} \widehat{\mathscr{C}}_k$, which satisfies the following two properties.

- (P_1) For each $n \in \{0, \ldots, n_*\}$, there exists $j_n \in \{0, \ldots, j_*\}$ such that $x_n = y_{j_n}$.
- (P_2) Fix $j \in \{0, \ldots, j_* - 1\}$ and let k be such that $y_j \in \widehat{\mathscr{C}}_k$. Then one of the following three conditions is satisfied:
    - (i) there exists $\tau_j \geq 0$ such that $y_{j+1} = \Phi_H^{\tau_j}(y_j)$,
    - (ii) $y_{j+1} = S_{k,i}(y_j)$,
    - (iii) $y_{j+1} = S_{k \to k+1}(y_j)$.

Here $S_{k,i}$ with $i \in I_k$ stands for the family of homoclinic maps associated to $\mathscr{C}_k$ and $S_{k \to k+1}$ stands for the family of heteroclinic maps from $\mathscr{C}_k$ to $\mathscr{C}_{k+1}$.

The existence of the sequence (y_j) is proved by induction, starting with $y_0 = x_0$. Assume that $y_{j_n} = x_n$. By definition

$$x_{n+1} = \varphi_k(x_n) \quad \text{or} \quad x_{n+1} = \psi_{k,i}(x_n) \quad \text{or} \quad x_{n+1} = \psi_{k \to k+1}(x_n).$$

In the first case, set $y_{j_n+1} = x_{n+1}$.

In the second case, by definition of $\psi_{k,i}$, there exists $\tau \geq 0$ such that $S_{k,i}\left(\Phi_H^\tau(x_n)\right) = x_{n+1}$. Set $y_{j_n+1} = \Phi_H^\tau(x_n)$ and $y_{j_n+2} = x_{n+1}$.

The last case is similar to the latter one when $\psi_{k \to k+1}$ is a heteroclinic map: there exists $\tau \in \mathbb{R}$ such that $S_{k \to k+1}\left(\Phi_H^\tau(x_n)\right) = x_{n+1}$, so set $y_{j_n+1} = \Phi_H^\tau(x_n)$ and $y_{j_n+2} = x_{n+1}$. When $\psi_{k \to k+1}$ is a transition map (that is, $\mathscr{C}_k$ and $\mathscr{C}_{k+1}$ are

adjacent cylinders, see Definition 2.17), by definition there exists $\tau \geq 0$ such that $x_{k+1} = \Phi_H^\tau(x_n)$, so we set $y_{j_n+1} = x_{n+1}$.

The induction stops after a finite number of steps and yields a sequence $(y_j)_{0 \leq j \leq j_*}$ with $j^* \leq 2n$, which satisfies (P_1) and (P_2).

2. This provides us with a sequence $\mathbf{g} := (g_0, \ldots, g_{j_*-1})$ of elements from the set

$$\left\{ \Phi_H^\tau \mid \tau \in [0, +\infty[\right\} \cup \{S_{k,i} \mid 1 \leq k \leq k_*,\, i \in I_k\} \cup \{S_{k \to k+1} \mid 1 \leq k \leq k_* - 1\},$$

such that

$$y_{j+1} = g_j(y_j), \qquad 0 \leq j \leq j_* - 1. \tag{42}$$

For a point z of $\widehat{\mathbf{C}}$ that is *close enough to* y_0, we call the $\mathbf{g}$-orbit of z the ordered sequence of points

$$\big(z, g_0(z), g_1 \circ g_0(z), \ldots, g_{j_*-1} \circ \cdots \circ g_0(z)\big). \tag{43}$$

We similarly define the $\mathbf{g}$-orbit of a subset of $\widehat{\mathbf{C}}$ that is close enough to y_0. These $\mathbf{g}$ orbits are well-defined, thanks to the continuity of the maps involved and the openness of their domains.

3. For each k, we fix a negatively invariant neighborhood $\mathscr{N}_k \subset W^-(\widehat{\mathscr{C}}_k)$ of $\widehat{\mathscr{C}}_k$ in $W^-(\widehat{\mathscr{C}}_k)$ such that, given $z \in \widehat{\mathscr{C}}_k$ and $w \in \mathscr{N}_k \cap W^-(z)$, then

$$d\big(\Phi_H^{-\tau}(z), \Phi_H^{-\tau}(w)\big) \leq d(z, w), \qquad \forall \tau \geq 0. \tag{44}$$

(see Definition 2.1, (11)). In the case of adjacent cylinders $\widehat{\mathscr{C}}_k$ and $\widehat{\mathscr{C}}_{k+1}$, on can moreover assume

$$\mathscr{N}_{k+1} \cap W^-(\widehat{\mathscr{C}}_k) \subset \mathscr{N}_k. \tag{45}$$

To see this, note that the union $\widehat{\mathscr{C}}_k \cup \widehat{\mathscr{C}}_{k+1}$ is itself a tame cylinder C, and fix a negatively invariant neighborhood $N \subset W^-(C)$ which satisfies (44). Then set $\mathscr{N}_k = N \cap W^-(\widehat{\mathscr{C}}_k)$ and $\mathscr{N}_{k+1} = N \cap W^-(\widehat{\mathscr{C}}_{k+1})$.

4. We say that a point $z \in \widehat{\mathscr{C}}_k$ is *recurrent* when it is positively and negatively recurrent under the restriction of Φ_H to $\widehat{\mathscr{C}}_k$. Recall that $\widehat{\mathscr{C}}_k$ admits a Borel measure which is positive on the open sets and invariant under Φ_H. The set of recurrent points has full measure in $\widehat{\mathscr{C}}_k$, by the Poincaré recurrence theorem.

5. We write $B(u, \rho)$ for the ball in $H^{-1}(\mathbf{e})$ centered at $u \in H^{-1}(\mathbf{e})$ with radius ρ. Fix $\delta > 0$. Let $\delta_0 > 0$ be so small that the $\mathbf{g}$-orbit $(D_j)_{0 \leq j \leq j_*}$ of $D_0 := B(y_0, \delta_0) \cap \widehat{\mathbf{C}}$ is well-defined (and so $D_{j+1} = g_j(D_j)$ for $0 \leq j \leq j_* - 1$) and satisfies

$$D_j \subset B(y_j, \delta/2), \qquad 0 \leq j \leq j_*. \tag{46}$$

We will prove the existence of a sequence $(z_j)_{1 \leq j \leq j_*}$, where the point z_j is in D_j and is recurrent, together with a sequence of balls $(B_j)_{0 \leq j \leq j_*}$, such that, for $0 \leq j \leq j_*$:

- B_j is centered at a point in $W^-(z_j) \cap \mathscr{N}$ and $B_j \subset B(z_j, \delta/2)$; (C_j)

- there exists a time $\sigma_j \geq 0$ with $\Phi_H^{\sigma_j}(B_j) \subset B_{j+1}$ for $0 \leq j \leq j_* - 1$. (T_j)

We will construct these sequences backwards, by finite induction. More precisely, given some recurrent point $z_{j+1} \in D_{j+1}$ together with a ball B_{j+1} satisfying (C_{j+1}), we will find a recurrent point $z_j \in D_j$, a ball B_j satisfying (C_j) and a time $\sigma_j \geq 0$ which satisfies (T_j).

6. Assume first that $g_j = \Phi_H^{\tau_j}$, in which case the sets D_j and D_{j+1} are either contained in the same cylinder, say $\mathscr{C}_k$, or consecutive adjacent cylinders $\mathscr{C}_k$, $\mathscr{C}_{k+1}$.

In the first case, let $z_j := \Phi_H^{-\tau_j}(z_{j+1})$, so $z_j \in D_j$ and z_j is a recurrent point of $\widehat{\mathscr{C}}_k$. Let w_{j+1} be the center of B_{j+1}, and set $w_j = \Phi_H^{-\tau_j}(w_{j+1})$. Then $w_j \in W^-(z_j) \cap \mathscr{N}_k$

by equivariance of the unstable foliation, and satisfies $d(w_j, z_j) \leq d(w_{j+1}, z_{j+1}) < \delta/2$, so that there exists a ball B_j centered at w_j and contained in $B(z_j, \delta/2)$, which satisfies

$$\Phi_H^{\tau_j}(B_j) \subset B_{j+1}$$

Hence Conditions (C_j) and (T_j) with $\sigma_j = \tau_j$ are satisfied.

In the case of adjacent cylinders, $z_j := \Phi_H^{-\tau_j}(z_{j+1})$ and z_{j+1} both belong to the intersection $\widehat{\mathscr{C}}_k \cap \widehat{\mathscr{C}}_{k+1}$ and z_j is again recurrent in $\widehat{\mathscr{C}}_k$ by invariance of $\widehat{\mathscr{C}}_k$ and the definition of adjacent cylinders. Let again w_{j+1} be the center of B_{j+1}, so that $w_{j+1} \in \mathscr{N}_{k+1} \cap W^-(\widehat{\mathscr{C}}_k)$. By (45), this proves that $w_j = \Phi_H^{-\tau_j}(w_{j+1})$ is in $\mathscr{N}_k$, since this latter neighborhood is negatively invariant. As above, $d(w_j, z_j) \leq d(w_{j+1}, z_{j+1}) < \delta/2$ and the conclusion follows.

7. Assume now that $g_j = S_{\ell,i}$, so that $z_{j+1} \in \widehat{\mathscr{C}}_\ell$. Let R_j and R_{j+1} be the full-measure subsets of D_j and D_{j+1} formed by the recurrent points. Since $S_{\ell,i}$ is C^1 and hence preserves the full-measure property:

$$R_{j+1} \cap S_{\ell,i}\Big(R_j \cap \mathrm{Domt}\, S_{\ell,i}\Big)$$

is a full measure subset of D_{j+1}. Therefore, there exists a point

$$\overline{z}_j \in R_j \cap \mathrm{Domt}\, S_{\ell,i}$$

such that $\overline{z}_{j+1} := S_{\ell,i}(\overline{z}_j)$ is recurrent and so close to z_{j+1} that (by continuity of the unstable foliation) $W^-(\overline{z}_{j+1})$ intersects the ball B_{j+1}.

By definition of $\mathrm{Domt}\, S_{\ell,i}$, the submanifold $W^-(\overline{z}_j)$ intersects $W^+(\mathrm{int}\, \widehat{\mathscr{C}}_\ell)$ transversely at some point $\xi \in W^+(\overline{z}_{j+1})$. Apply the λ-property (Definition 2.1) to $W^-(\overline{z}_j)$ in the neighborhood of ξ, together with the positive recurrence property of $\overline{z}_{j+1}$: there exists a (large) time τ such that $\Phi_H^\tau\big(W^-(\overline{z}_j)\big) = W^-\big(\Phi_H^\tau(\overline{z}_j)\big)$ intersects B_{j+1}. Fix

$$\zeta \in W^-\big(\Phi_H^\tau(\overline{z}_j)\big) \cap B_{j+1}$$

and note that $\Phi_H^\tau(\overline{z}_j)$ is negatively recurrent since $\overline{z}_j$ is. Hence there exist arbitrarily large times τ' such that $z_j := \Phi_H^{-\tau'+\tau}(\overline{z}_j) \in D_j$. In particular, one can choose τ' such that this latter condition is satisfied and

$$\Phi_H^{-\tau'}(\zeta) \in W^-(z_j) \cap B(z_j, \delta/2) \cap \mathscr{N}.$$

Therefore, there exists a ball B_j centered at $\Phi_H^{-\tau'}(\zeta)$ and contained in $B(z_j, \delta/2)$ such that $\Phi_H^{\tau'}(B_j) \subset B_{j+1}$. This proves (C_j) and (T_j) with $\sigma_j = \tau'$.

8. The case where $g_j = S_{\ell-1}^\ell$ is completely similar to the previous one.

9. This yields our sequences $(z_j)_{0 \leq j \leq j_*}$ and $(B_j)_{0 \leq j \leq j_*}$. To end the proof, observe that for any point $a_0 \subset B_0$, the point $a_j = \Phi_H^{\sigma_{j-1}+\cdots+\sigma_0}(a_0)$ is in B_j, $1 \leq j \leq j_*$. Therefore

$$\mathrm{dist}\,(a_j, y_j) \leq \mathrm{dist}\,(a_j, z_j) + \mathrm{dist}\,(z_j, y_j) \leq \delta, \qquad 0 \leq j \leq j_*.$$

In particular, by (P_1)

$$\mathrm{dist}\,(a_{j_n}, x_n) \leq \delta, \qquad 0 \leq n \leq n_*.$$

Therefore the solution $\Phi_H(\cdot, a_0)$ satisfies our requirement, with $\tau_n = \sigma_{j_n-1}+\cdots+\sigma_0$ for $1 \leq n \leq n_*$. $\qquad\square$

5. **An algorithmic construction of pseudo-orbits.** In this section we provide a second proof, of algorithmic nature, of **Proposition 3.2**.

Let $\mathscr{C}$ be a good cylinder with continuations $\widehat{\mathscr{C}}$, let (Σ, φ) and $(\widehat{\Sigma}, \widehat{\varphi})$ be the corresponding tilt sections, respectively, and ψ be a homoclinic correspondence. As before, we identify $\Sigma \sim \mathbb{T} \times [a, b]$, and $\widehat{\Sigma} \sim \mathbb{T} \times [\widehat{a}, \widehat{b}]$. The good cylinder condition assumes that for essential invariant circle $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ in $\mathrm{int}\,\widehat{\Sigma}$ there exists a splitting arc based at Γ.

We will first prove a version of **Proposition 3.2**, under some compatibility condition between the homoclinic correspondence and the tilt map, formulated below:

● if $\widehat{\varphi}$ tilts the verticals to the right (resp. left), then for each element $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$ that is the upper boundary of a Birkhoff zone in $\mathbb{T} \times]\widehat{a}, \widehat{b}[$, there exists a right (resp. left) splitting arc based on Γ.   (VG)

The compatibility between the orientation of the splitting arcs and the direction of the tilt of the map $\widehat{\varphi}$ enables us, via the following result, to prove the existence of diffusing pseudo-orbits without the use of the Poincaré recurrence theorem.

Proposition 5.1. *Let $\mathscr{C}$ be a δ-good cylinder. Assume the setting of Proposition 3.2, and the compatibility condition (VG).*

Fix $\Gamma_\bullet \subset \mathbf{A} \setminus \Gamma(b)$ in $\mathrm{Ess}(\varphi)$ and a neighborhood $U_\bullet$ of $\Gamma_\bullet$ in $\mathbf{A}$. Then, there exist a point $z_0 \in U_\bullet$, an integer $N \geq 1$, and, for $n \in \{0, \dots, N-1\}$, an index $i_n \in I$ and an integer $m_n > 0$ such that the orbit given by

$$z_{n+1} = \psi_{i_n} \circ \widehat{\varphi}^{m_n}(z_n), \quad n = 0, \dots, N-1,$$

has the following properties:

   (i) $z_N \in \Gamma(b)^+$,
   (ii) for every essential invariant circle Γ between $\Gamma_\bullet$ and $\Gamma(b)$, there exists a point z_n that is δ-close to Γ.

Proof of Proposition 5.1. In the sequel we will assume that $\widehat{\varphi}$ tilts the verticals to the right (the case when $\widehat{\varphi}$ tilts the verticals to the left can be dealt with similarly). For the sake of notational simplicity, we assume that $\Gamma_\bullet = \Gamma(a)$ and we denote $\mathbf{A}_* = \mathbb{T} \times [a, \widehat{b}]$. We still write $\widehat{\varphi}, \psi$ for the restrictions of the corresponding maps to $\mathbf{A}_*$. The proof immediately extends to the case where $\Gamma_\bullet$ is any element of $\mathrm{Ess}(\widehat{\varphi})$ in the same way as in the proof of **Proposition 3.2**. By Definition 2.13 we have that $\mathrm{diam}\,\mathrm{Dom}\,\psi_i < \delta$ and $\mathrm{diam}\,\mathrm{Im}\,\psi_i < \delta$ for $i \in I$.

As in the proof of **Proposition 3.2**, we want to prove that given any connected open neighborhood $U_\bullet$ of $\Gamma(a)$ in $\mathbf{A}_*$, the orbit U of $U_\bullet$ under the restricted polysystem $(\widehat{\varphi}, \psi)_{|\mathbf{A}_*}$ satisfies

$$\Gamma(b) \subset \mathrm{cc}(U, \Gamma(a)). \tag{47}$$

The following constructions and lemmas are part of the proof of **Proposition 5.1**.

1. *The Birkhoff procedure.* Consider *any* connected positively $\widehat{\varphi}$-invariant open subset $\mathbf{U} \subset \mathbf{A}_*$ containing $\Gamma(a)$ and such that $\overline{\mathbf{U}} \cap \Gamma(\widehat{b}) = \emptyset$. Set $O = \mathbf{A}_* \setminus \overline{\mathbf{U}}$.

We define the *filled subset* $\mathcal{U}$ associated with $\mathbf{U}$ as the open set

$$\mathcal{U} = \mathrm{cc}\big(\mathbf{A}_* \setminus \overline{\mathrm{cc}(O, \Gamma(\widehat{b}))}, \Gamma(a)\big). \tag{48}$$

One proves as in **Proposition 3.2** that $\mathrm{Int}\,\mathrm{Cl}\,(\mathcal{U}) = \mathcal{U}$, and that $\mathcal{U}$ is homeomorphic to $\mathbb{T} \times [0, 1[$ and satisfies $\widehat{\varphi}(\mathcal{U}) = \mathcal{U}$. As a consequence, by the Birkhoff theorem,

the frontier $\mathrm{Fr}\,\mathcal{U}$ is in $\mathrm{Ess}(\widehat{\varphi})$, and moreover one easily deduces from the proof of **Proposition 3.2** that
$$\mathrm{Fr}\,\mathcal{U} \subset \mathrm{Fr}\,\mathbf{U}.$$
We can now introduce our procedure. Let ν be a uniform Lipschitz constant for the circles of $\mathrm{Ess}(\widehat{\varphi})$. We define ν-balls of $\widehat{\mathbf{A}}$ as the intersections with $\widehat{\mathbf{A}}$ of open rectangles satisfying (38). Given $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$, we call a neighborhood V of Γ in $\widehat{\mathbf{A}}$ a ν-neighborhood if V is a union of ν-balls.

Recall that for two elements $\Gamma_1, \Gamma_2 \in \mathrm{Ess}(\widehat{\varphi})$, we write $\Gamma_1 > \Gamma_2$ (resp. $\Gamma_1 \geq \Gamma_2$) if $\gamma_1(\theta) > \gamma_2(\theta)$ (resp. $\gamma_1(\theta) \geq \gamma_2(\theta)$) for all $\theta \in \mathbb{T}$, where γ_i is the Lipschitz function whose graph is Γ_i, for $i = 1, 2$.

Denote by $\mathscr{P}$ the set of pairs (Γ, V), where $\Gamma \subset \mathbb{T} \times [a, \widehat{b}[$ is in $\mathrm{Ess}(\widehat{\varphi})$, and V is either a ν-neighborhood of Γ in $\mathbf{A}_*$, or a ν-ball centered on Γ.

We define the *Birkhoff procedure* as the map
$$\mathbb{B} : \mathscr{P} \longrightarrow \mathrm{Ess}(\widehat{\varphi}), \qquad \mathbb{B}(\Gamma, V) = \mathrm{Fr}\,\mathcal{U}, \tag{49}$$
where $\mathcal{U}$ is the filled subset associated with the connected $\widehat{\varphi}$-invariant open set
$$U = \bigcup_{n \geq 0} \widehat{\varphi}^n(\Gamma^- \cup V) = \Gamma^- \cup \Big(\bigcup_{n \geq 0} \widehat{\varphi}^n(V) \Big).$$

Recall that Γ^- stands for the connected component of $\mathbf{A}_* \setminus \Gamma$ located below Γ. Observe that since $\widehat{\varphi}$ is a good tilt map, $\mathbb{B}(\Gamma, V) > \Gamma$. Note that the region between Γ and $\mathbb{B}(\Gamma, V)$ is not necessarily a Birkhoff zone. However, if Γ is the lower boundary of a Birkhoff zone and if $V \cap \Gamma^+$ is contained in the Birkhoff zone, then $\mathbb{B}(\Gamma, V)$ is the upper boundary of that Birkhoff zone. One main property of $\mathbb{B}$ that we will use is the following easy transition lemma.

Lemma 5.2. *Fix $(\Gamma, V) \in \mathscr{P}$ and let $\Gamma' = \mathbb{B}(\Gamma, V)$. Then, for every open set V' intersecting Γ', there exists $n \geq 0$ such that $\widehat{\varphi}^n(V) \cap V' \cap (\Gamma')^- \neq \emptyset$.*

Proof. By definition, $\mathbb{B}(\Gamma, V) = \mathrm{Fr}\,\mathcal{U}$ so $\mathbb{B}(\Gamma, V) \subset \mathrm{Fr}\,U \subset \mathrm{Cl}\,U$. Hence $V' \cap U \neq \emptyset$, so that there exists $n \geq 0$ with $\widehat{\varphi}^n(V) \cap V' \neq \emptyset$. Moreover $V \subset (\Gamma')^-$ by construction, so $\widehat{\varphi}^n(V) \subset (\Gamma')^-$. $\qquad\square$

The following result is a crucial step in our subsequent construction of diffusing orbits. Recall that we write $\psi = (\psi_i)_{i \in I}$ for the homoclinic correspondence.

Lemma 5.3. *Consider a pair $(\Gamma, V) \in \mathscr{P}$ and let $\Gamma' = \mathbb{B}(\Gamma, V)$. Then there exist $n \geq 0$, $i \in I$ and a ν-ball V' centered on Γ' such that*
$$V' \subset \psi_i\big(\widehat{\varphi}^n(V)\big).$$

Proof. By the assumption that $\widehat{\varphi}$ is a good tilt map, either Γ' is the upper boundary of a Birkhoff zone, or is accumulated from below by a sequence of essential circles.

First, assume that Γ' is the upper boundary of a Birkhoff zone. By condition (VG) there exists a right splitting arc ζ based at a point $\alpha_0 \in \Gamma'$ such that $\widetilde{\zeta} \setminus \{\alpha_0\} = \zeta(]0, 1]) \subset \Gamma^+ \cap (\Gamma')^-$, and a homoclinic map ψ_i such that $\widetilde{\zeta} \setminus \{\alpha_0\} \subset \mathrm{Dom}\,\psi_i$.

Let D be a triangular domain associated with $\widetilde{\zeta}$. By Lemma 5.2, there exist $z_0 \in V$ and $n \geq 0$ such that $z_n = \widehat{\varphi}^n(z_0) \in D$. Since V is a ν-ball or a union of ν-balls, by Lemma A.2, one can find a negatively tilted arc η contained in V, emanating from some point of Γ and ending at z_0. Therefore, by Lemma A.4, $\eta_n = \widehat{\varphi}^n \circ \eta$ is a negatively tilted arc emanating from Γ and ending at $z_n \in D$. By Lemma 2.11, $\widetilde{\eta}_n \cap \big(\widetilde{\zeta} \setminus \{\alpha_0\}\big) \neq \emptyset$.

As a consequence, $\widehat{\varphi}^n(V) \cap (\widetilde{\zeta} \setminus \{\alpha_0\}) \neq \emptyset$, and therefore there exists a ball B centered on $\widetilde{\zeta}$ satisfying

$$B \subset (\mathrm{Dom}\,\psi_i \cap \widehat{\varphi}^n(V)).$$

So, by definition of $\widetilde{\zeta}$, $\psi_i(B) \subset \psi_i(\widehat{\varphi}^n(V))$ contains a ν-ball V' centered on Γ', which proves our claim.

Second, assume that Γ' is accumulated from below by a sequence of essential circles Γ_n, with $\Gamma_n < \Gamma_{n+1}$ for all n. Let ζ be a splitting arc based at a point in Γ'. There exists $\Gamma_{n_*} < \Gamma$ such that $\widetilde{\zeta} \subset (\Gamma')^-$ intersects both Γ_{n_*} and Γ_{n_*+1}. There exists a segment $\zeta[\sigma, \sigma']$ of $\widetilde{\zeta}$ such that $\zeta(\sigma) \in \Gamma_{n_*+1}$, $\zeta(\sigma') \in \Gamma_{n_*}$, and $\zeta]\sigma, \sigma'[$ is contained in the region between Γ_{n_*} and Γ_{n_*+1}. Lemma 5.2 applied to $\Gamma_{n_*+1}^+$, which is a neighborhood of Γ', implies that there exists $m \geq 0$ such that $\widehat{\varphi}^m(V)$ intersects both Γ_{n_*} and Γ_{n_*+1}. Hence, there exists an arc $\widetilde{\xi} = \xi([a, a']) \subset \widehat{\varphi}^m(V)$ that is contained in the region between Γ_{n_*} and Γ_{n_*+1}, with $\xi(a) \in \Gamma_{n_*+1}$ and $\xi(a') \in \Gamma_{n_*}$. By Lemma A.3, for some $m' \geq m$ sufficiently large we have $\widehat{\varphi}^{m'-m}(\widetilde{\xi}) \cap \zeta(]\sigma, \sigma'[) \neq \emptyset$. As $\widehat{\varphi}^{m'-m}(\widetilde{\xi}) \subset \widehat{\varphi}^{m'}(V)$, it follows that $\widehat{\varphi}^{m'}(V) \cap \zeta(]\sigma, \sigma'[) \neq \emptyset$. Since ζ is a splitting arc based at a point in Γ', there exists a homoclinic map ψ_i such that $\psi_i(\widehat{\varphi}^{m'}(V))$ contains a ν-ball centered on Γ'. This ends the proof of the lemma. $\qquad\square$

2. *Coherent sequences of circles and the existence of pseudo-orbits.*

Recall that $\mathrm{Ess}(\widehat{\varphi})$ is endowed with an order relation given by the natural order on graphs.

We say that an increasing sequence $(\Gamma_n)_{0 \leq n \leq n_*}$ of elements of $\mathrm{Ess}(\widehat{\varphi})$ is *coherent* if there exists a sequence $(V_n)_{0 \leq n \leq n_*}$, where each V_n is either ν-ball centered on Γ_n or a ν-neighborhood of Γ_n, and for every $n = 0, \ldots, n_* - 1$ we have

$$V_{n+1} \subset \psi_{i_n}(\widehat{\varphi}^{m_n}(V_n)), \tag{50}$$

for some $i_n \in I$ and $m_n \geq 0$.

We say that $(V_n)_{0 \leq n \leq n_*}$ is the system of *neighborhoods associated with* $(\Gamma_n)_{0 \leq n \leq n_*}$ and that Γ_{n_*} is the *upper* circle of the coherent sequence.

Lemma 5.4. *Fix a neighborhood $U_\bullet$ of $\Gamma(a)$ in $\mathbf{A}_*$. Choose an essential invariant circle Γ_* such that $\Gamma(a) < \Gamma_* \leq \Gamma(\widehat{b})$.*

Then there exists a coherent sequence $(\Gamma_n)_{0 \leq n \leq n_*}$, with associated neighborhoods $(V_n)_{0 \leq n \leq n_*}$, such that

$$\Gamma_0 = \Gamma(a), \qquad V_0 \subseteq U_\bullet, \qquad \Gamma_{n_*} = \Gamma_*.$$

Moreover, we can choose $(\Gamma_n)_{0 \leq n \leq n_*}$ with the additional property that any two consecutive circles $\Gamma_n < \Gamma_{n+1}$ in the sequence are either δ-close, i.e., $d_H(\Gamma_n, \Gamma_{n+1}) < \delta$, or they are at the boundary of a Birkhoff zone.

Proof. We recall that, since $\widehat{\varphi}$ is good, for any $\Gamma \leq \Gamma_*$, either $\Gamma < \Gamma_*$ or $\Gamma = \Gamma_*$. Let Coh be the set of all coherent sequences of circles originating at Γ_0, with $V_0 = V$, and such that $\Gamma_n \leq \Gamma_*$ for all $n = 0, \ldots, n_*$. This set is nonempty by Lemma 5.3 (while reduced to the a single coherent sequence when Γ_0 and Γ_* bound a Birkhoff zone). Let $\mathrm{Coh}^\bullet$ be the set of upper circles of all elements of Coh. Let

$$\Gamma^\bullet = \mathrm{Sup}\,\mathrm{Coh}^\bullet,$$

(see Lemma 2.3) so $\Gamma^\bullet > \Gamma(a)$. Assume by contradiction that $\Gamma^\bullet < \Gamma_*$.

Since $\widehat{\varphi}$ is good, one proves as in Lemma 2.3 that either $\Gamma^\bullet \in \mathrm{Coh}^\bullet$, or $\Gamma^\bullet \notin \mathrm{Coh}^\bullet$ and is accumulated from below in the Hausdorff topology by elements of $\mathrm{Coh}^\bullet$.

In the former case, there exists a coherent sequence $(\Gamma_n)_{1 \leq n \leq n_*}$ with $\Gamma_{n_*} = \Gamma^\bullet$, and with associated neighborhoods $(V_n)_{0 \leq n \leq n_*}$. Let $\Gamma := \mathbb{B}(\Gamma_{n_*}, V_{n_*})$. By Lemma 5.3 there exist $m \geq 0$, $i \in I$ and a ν-ball V centered on $\Gamma := \mathbb{B}(\Gamma_{n_*}, V_{n_*})$ such that

$$V \subset \psi_i\big(\widehat{\varphi}^m(V_{n_*})\big).$$

The neighborhood V_{n_*} can be chosen arbitrarily small, so that we can assume $V_{n_*} \cap \Gamma_* = \emptyset$. Hence, since $\Gamma_{n_*} = \Gamma^\bullet < \Gamma_*$, we also have $\Gamma = \mathbb{B}(\Gamma_{n_*}, V_{n_*}) \leq \Gamma_*$. Therefore one can extend the sequence $(\Gamma_n)_{0 \leq n \leq n_*}$ to a longer coherent sequence, by adjoining to the previous sequence $\Gamma_{n_*+1} = \Gamma$ and $V_{n_*+1} = V$. The resulting coherent sequence is in Coh, and has its upper circle located in $(\Gamma^\bullet)^+$, which is a contradiction.

Assume now that $\Gamma^\bullet \notin \mathrm{Coh}^\bullet$ and is accumulated from below by elements of $\mathrm{Coh}^\bullet$. In particular, $\Gamma^\bullet$ is not the upper boundary of a Birkhoff zone. By the good cylinder condition, there exists a splitting arc ζ based on $\Gamma^\bullet$, with $\psi_i\big(\zeta(]0,1])\big) \subset \Gamma^\bullet$ for some $i \in I$. So, by accumulation, there exists a coherent sequence $(\Gamma_n)_{1 \leq n \leq n_*}$ such that

$$\zeta(1) \in (\Gamma_{n_*})^-.$$

Consider the circle $\Gamma = \mathbb{B}(\Gamma_{n_*}, V_{n_*})$, which satisfies $\Gamma_{n_*} < \Gamma < \Gamma^\bullet$ (otherwise $\Gamma = \Gamma^\bullet$, which contradicts our initial assumption $\Gamma^\bullet \notin \mathrm{Coh}^\bullet$). The set $K = \zeta^{-1}(\Gamma)$ is compact and contained in $]0,1[$, set $\sigma = \mathrm{Max}\, K$. So $\zeta(\sigma) \in \Gamma$ and $\zeta(]\sigma,1]) \subset \Gamma^-$ by continuity. Moreover, there exists $\sigma' > \sigma$ such that $\zeta(\sigma') \in \Gamma_{n_*}$; let σ' be the smallest one with such property. Thus $\zeta([\sigma, \sigma'])$ is an arc contained in the closed annulus bounded by Γ_{n_*} and Γ, and has one endpoint on Γ_{n_*} and the other one on Γ.

By Lemma 5.5 below, there exists a point $z \in V_{n_*}$ with the property that for the vertical segment $V^-(z)$ below z there exists $m \geq 0$ such that $\varphi^m(V^-(z))$ intersects $\zeta(]\sigma, \sigma'[)$. Since V_{n_*} is a ν-ball, the vertical segment $V^-(z)$ is contained in V_{n_*}. It follows that $\varphi^m(V_{n_*}) \cap \zeta(]\sigma, \sigma'[) \neq \emptyset$.

Therefore, since $\psi_i(\widetilde{\zeta}) \subset \Gamma^\bullet$, the image $\psi_i\big(\widehat{\varphi}^m(V_{n_*})\big)$ contains a ν-ball $V^\bullet$ centered on $\Gamma^\bullet$. We can choose the neighborhood $V^\bullet$ small enough so that $V^\bullet \cap \Gamma_* = \emptyset$, hence $\mathbb{B}(\Gamma^\bullet, V^\bullet) \leq \Gamma_*$. Thus one can construct a coherent sequence in Coh given by

$$\Gamma_0, \ldots, \Gamma_{n_*}, \Gamma^\bullet, \mathbb{B}(\Gamma^\bullet, V^\bullet)$$

whose upper circle is located in $(\Gamma^\bullet)^+ \cap (\Gamma_*)^-$, which is a contradiction. This proves the first claim of the lemma[18].

Now we show the second claim of the lemma. We will restrict ourselves to coherent sequences with the property that the neighborhood V of $\Gamma(a)$ is contained in a δ-neighborhood of $\Gamma(a)$, and that each neighborhood V_n associated to the coherent sequence is contained in a δ-neighborhood of the corresponding Γ_n. Restricting to coherent sequences satisfying this condition is possible due to (50).

We first construct recursively a finite sequence of essential circles Γ^j, $j = 0, 1, \ldots, N$, with the following properties:

(i) $\Gamma(a) = \Gamma^0 < \Gamma^1 < \cdots < \Gamma^{N-1} < \Gamma^N = \Gamma_*$;
(ii) For each $j \in \{0, \ldots, N-1\}$, either
   (ii.a) the Hausdorff distance between Γ^j and Γ^{j+1} is less than δ, or
   (ii.b) the region between Γ^j and Γ^{j+1} is a Birkhoff zone.

---

[18]In which, as indicated at the beginning of the proof, one can eventually choose any $\Gamma_0 \in \mathrm{Ess}(\widehat{\varphi})$, $\Gamma_0 < \Gamma(\widehat{b})$, instead of $\Gamma(a)$ as first element of the coherent sequence.

We start by setting $\Gamma^0 = \Gamma(a)$. Let $0 < \delta' < \delta$. If Γ^j has been constructed, we define the set

$$\mathscr{T}_{\Gamma^j} = \{\Gamma \in \mathrm{Ess}(\varphi) \,|\, \Gamma^j < \Gamma < \Gamma_*, \text{ and } d_H(\Gamma^j, \Gamma) < \delta'\}, \qquad (51)$$

where d_H stands for the Hausdorff distance.

If $\mathscr{T}_{\Gamma^j} \neq \emptyset$ we define

$$\Gamma^{j+1} = \mathrm{Sup}\, \mathscr{T}_{\Gamma^j}. \qquad (52)$$

By construction, $d_H(\Gamma^j, \Gamma^{j+1}) \leq \delta' < \delta$.

If $\mathscr{T}_{\Gamma^j} = \emptyset$, it follows from Lemma 2.3 that Γ^j is the lower boundary of some Birkhoff zone. In this case we define Γ^{j+1} as the upper boundary of that Birkhoff zone. In this case we must have $d_H(\Gamma^j, \Gamma^{j+1}) \geq \delta'$.

This recursive construction ends after finitely many steps at Γ_*. Otherwise, we would obtain an infinite sequence Γ^j, $j = 0, 1, \ldots$, satisfying (ii), with $\Gamma^j < \Gamma_*$ for all j. Take $\Gamma^* = \mathrm{Sup}\,\Gamma^j$. Since the sequence Γ^j is infinite, it accumulates on Γ^* from below. Take a δ'-neighborhood of Γ^* in the Hausdorff topology. There exists $\Gamma^{j_*} < \Gamma^*$ that is contained in that neighborhood. It follows that

$$\Gamma^{j_*+1} = \mathrm{Sup}\ \mathscr{T}_{\Gamma^{j_*}}$$

is also contained in the same neighborhood, with moreover $\Gamma^{j_*} < \Gamma^{j_*+1} < \Gamma^*$. Thus, both Γ^{j_*+1} and Γ^* are δ'-close to Γ^{j_*}. This is in contradiction with the definition of Γ^{j_*+1}. Hence the recursive construction ends after finitely many steps, therefore it must end at Γ_*.

At this point we have constructed a finite sequence of essential circles Γ^j, $j \in \{0, \ldots, N\}$ satisfying properties (i) and (ii). The first statement of Lemma 5.4, applied successively to the pairs Γ^0 and $\Gamma_* = \Gamma^1$, then to $\Gamma_0 = \Gamma^1$ and $\Gamma_* = \Gamma^2$,..., and finally $\Gamma_0 = \Gamma^{N-1}$ and $\Gamma_* = \Gamma^N$, yields coherent sequences between each consecutive pair of essential circles. Concatenating these coherent sequences yields a coherent sequence satisfying the second claim of Lemma 5.4, which concludes its proof. $\qquad \square$

The following result (which can be viewed as a loose extension of Lemma A.3) has been used in the above argument.

Lemma 5.5. *Let $f : \mathbf{A} \to \mathbf{A}$ be an area-preserving good tilt map. Let $\Gamma_\bullet < \Gamma^\bullet$ be two nonintersecting essential invariant circles contained in $\mathbf{A}$. Let $\gamma : [0, 1] \to \mathbf{A}$ be a continuous function such that $\gamma(0) \in \Gamma_\bullet$ and $\gamma(1) \in \Gamma^\bullet$. Let $U_\bullet$ be a neighborhood of some point in $\Gamma_\bullet$. Then there exists a vertical segment υ originating at $\Gamma_\bullet$ and contained in $U_\bullet$, such that the positive orbit of υ under φ intersects $\widetilde{\gamma} = \gamma([0, 1])$.*

Proof of Lemma 5.5. Without loss of generality, we can assume that $U_\bullet$ is a ν-ball, that $U_\bullet \cap \widetilde{\gamma} = \emptyset$, and that $\widetilde{\gamma}$ is contained in the strip bounded by $\Gamma_\bullet$ and $\Gamma^\bullet$. Lifts to $\mathbb{R} \times [a, b]$ of all objects at hand will be denoted by boldface letters. We fix a lift $\boldsymbol{\varphi}$ of φ and a lift $\boldsymbol{\gamma}$ of γ. Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ in $\mathbb{R}$ be such that the image $\widetilde{\boldsymbol{\gamma}}$ is contained in the strip $[\widehat{\theta}_1, \widehat{\theta}_2] \times [a, b]$. Set $\delta = 1 + \widehat{\theta}_2 - \widehat{\theta}_1$.

Let $\rho_\bullet$ the rotation number of $\Gamma_\bullet$. Apply the Birkhoff procedure to $\Gamma_\bullet$ and $U_\bullet$, obtaining $\Gamma' = \mathbb{B}(\Gamma_\bullet, U_\bullet) \leq \Gamma^\bullet$. Since φ is a good tilt map, $\Gamma' \cap \Gamma_\bullet = \emptyset$, so the rotation number ρ' of Γ' satisfies $\rho' > \rho_\bullet$. Therefore, there exists $k > 0$ such that, for every $\boldsymbol{\xi} \in \boldsymbol{\Gamma}_\bullet$ and $\boldsymbol{\xi}' \in \boldsymbol{\Gamma}'$ such that $\pi(\boldsymbol{\xi}) - \pi(\boldsymbol{\xi}') \geq -1$:

$$\pi[\boldsymbol{\varphi}^k(\boldsymbol{\xi}')] - \pi[\boldsymbol{\varphi}^k(\boldsymbol{\xi})] > \delta + 1, \qquad (53)$$

where π stands for the first projection.

Given $u = (\theta, r) \in \mathbf{A}$, let $B(u, \varepsilon)$ be the ν-ball $]\theta - \varepsilon, \theta + \varepsilon[\times]r - \nu\varepsilon, r + \nu\varepsilon[\cap \mathbf{A}$. Choose a point $x' \in \Gamma'$, fix $\varepsilon \in]0, 1/4[$ such that $B(x', \varepsilon) \in \Gamma_{\bullet}^{+}$, and set

$$V' = \bigcap_{i=0}^{k} \varphi^{-i}\Big(B\big(\varphi^{i}(x'), \varepsilon\big)\Big), \tag{54}$$

so that $V' \subset \Gamma_{\bullet}^{+}$ is a open neighborhood of x'.

By Lemma A.6, there exists $z \in U_{\bullet}$ and $n \geq 0$ such that $\varphi^{n}(z) \in V'$. Let x be the intersection point between the vertical line through z and $\Gamma_{\bullet}$. Since $U_{\bullet}$ is a ν-ball, the vertical segment $v = [x, z]$ is contained in $U_{\bullet}$. The image $\varphi^{n}(v)$ is a positively tilted arc emanating from $\varphi^{n}(x) \in \Gamma_{\bullet}$ and ending at $\varphi^{n}(z) \in V'$. Fix a lift $\boldsymbol{\sigma} = [\mathbf{x}, \mathbf{z}]$ of v and set $\boldsymbol{\xi}_{n} = \boldsymbol{\varphi}^{n}(\mathbf{x})$ and $\mathbf{z}_{n} = \boldsymbol{\varphi}^{n}(\mathbf{z})$. Since $\varphi^{n}(v)$ is positively tilted,

$$\pi(\mathbf{z}_{n}) \geq \pi(\boldsymbol{\xi}_{n}).$$

By (54), there is a unique lift $\boldsymbol{\xi}_{n}'$ of x' such that $\mathbf{z} \in B(\boldsymbol{\xi}_{n}', \varepsilon)$, and moreover

$$\boldsymbol{\varphi}^{k}(\mathbf{z}_{n}) \in B\big(\boldsymbol{\varphi}^{k}(\boldsymbol{\xi}_{n}'), \varepsilon\big).$$

In particular, $\pi(\boldsymbol{\xi}_{n}') - \pi(\boldsymbol{\xi}_{n}) \geq -1$ and by (53):

$$\pi[\boldsymbol{\varphi}^{k}(\boldsymbol{\xi}_{n}')] - \pi[\boldsymbol{\varphi}^{k}(\boldsymbol{\xi}_{n})] > \delta + 1.$$

Therefore

$$\pi[\boldsymbol{\varphi}^{k}(\mathbf{z}_{n})] - \pi[\boldsymbol{\varphi}^{k}(\boldsymbol{\xi}_{n})] > \delta.$$

As a consequence, there is an integer m such that $\boldsymbol{\varphi}^{k}\big(\boldsymbol{\varphi}^{n}(\boldsymbol{\sigma})\big)$ intersects $m + \widetilde{\gamma}$, and so $\varphi^{k+n}(v)$ intersects $\widetilde{\gamma}$, which proves our claim. $\quad\square$

3. *End of proof of Proposition 5.1.* By Lemma 5.4, applied to $\Gamma_{*} = \Gamma(b)$, there exists a coherent sequence of circles $(\Gamma_{n})_{0 \leq n \leq n_{*}}$, with associated neighborhoods $(V_{n})_{0 \leq n \leq n_{*}}$, such that

$$V_{0} \subseteq U_{\bullet}, \qquad \Gamma_{n_{*}} = \Gamma(b).$$

Then, by continuity, there exists a sequence $(W_{n})_{0 \leq n \leq n_{*}}$ of open sets with $W_{n_{*}} \subset V_{n_{*}}$, such, for $0 \leq n \leq n_{*} - 1$:

$$W_{n} \subset V_{n}, \qquad \psi_{i_{n}} \circ \widehat{\varphi}^{m_{n}}(W_{n}) = W_{n+1}$$

for suitable $i_{n} \in I$ and $m_{n} \geq 0$. As a consequence, since $W_{n_{*}} \cap \Gamma(b)^{+} \neq \emptyset$, there exists a point $z_{0} \in W_{0}$ such that $z_{n+1} = \psi_{i_{n}}(\widehat{\varphi}^{m_{n}}(z_{n})) \in W_{n+1}$, for $n = 0, \ldots, N-1$. In particular $z_{N} \in \Gamma(b)^{+}$.

At the beginning of the proof we assumed that the maps ψ_{i} are chosen so that the domain of each map has diameter less than δ. The coherent sequence obtained at the end of Lemma 5.4 has been constructed so that, for any two consecutive circles $\Gamma_{n} < \Gamma_{n+1}$, either $d_{H}(\Gamma_{n}, \Gamma_{n+1}) < \delta$, or the region between Γ_{n} and Γ_{n+1} is a Birkhoff zone. Each essential invariant circle is between a pair Γ_{n} and Γ_{n+1} as above, or is at the boundary of a Birkhoff zone between Γ_{n} and Γ_{n+1}, for some n. Hence the pseudo-orbit $(z_{n})_{0 \leq n \leq N}$ obtained above gets δ-close to every essential invariant circle.

This concludes the proof of Proposition 5.1. $\quad\square$

The same type of argument is used for the following corollary dedicated to the "singular cylinders", for which a constructive proof can also be deduced from the previous one.

Corollary 5.6. *We assume the same condition as in Proposition 5.1. Assume that $K \subset \mathbb{T} \times]a + \delta, b - \delta[$ is a compact subset of $\mathbf{A}$, invariant under φ and contained in the interior of some Birkhoff zone of φ.*

Then $f = (\varphi, \psi)$ admits a pseudo-orbit which does not intersect K. If moreover ψ is δ-bounded, then there exists a δ-admissible pseudo-orbit.

The proof is exactly the same as that of Corollary 3.5.

Second proof of Proposition 3.2. If the compatibility condition (VG) is satisfied, then the result follows immediately from Proposition 5.1.

Assume that the compatibility condition (VG) is not satisfied for some $\Gamma \in \mathrm{Ess}(\widehat{\varphi})$, for instance $\widehat{\varphi}$ tilts the vertical to the right, Γ is the upper boundary of a Birkhoff zone in $\mathbb{T} \times]\widehat{a}, \widehat{b}[$, and there exists a left splitting arc based on Γ.

The key observation is that $\widehat{\varphi}^{-1}$ tilts the verticals to the left, and the compatibility condition (VG) is locally satisfied for $\widehat{\varphi}^{-1}$ and ψ.

Then, as in the proof of Proposition 5.1 we obtain an orbit (z_n) such that, for each n, either

$$z_{n+1} = \psi_{i_n} \circ \widehat{\varphi}^{m_n}(z_n)$$

or

$$z_{n+1} = \psi_{i_n} \circ \widehat{\varphi}^{-m_n}(z_n).$$

Using the symmetrization result on polysystems, Lemma C.1 in Appendix C, yields the result in Proposition 3.2.

We note here that the proof of Proposition 5.1 does not require to use the Poincaré recurrence theorem, while this proof Proposition 3.2 requires to use the Poincaré recurrence theorem when the compatibility condition (VG) is not satisfied. $\square$

4. Remarks. The compatibility condition (VG) between the orientation of the splitting arcs and the direction of the tilt may not be generic, even in specific classes of examples as [40].

We describe an alternative way to eliminate the condition (VG) and obtain an algorithmic proof of **Proposition 3.2**, without using Poincaré recurrence. Instead we use the equivariance properties of the homoclinic correspondences (see Appendix B).

Assume, for instance, that φ tilts the verticals to the right, and that the homoclinic correspondence ψ is equivariant.

For the key part of the proof of **Proposition 5.1**, let $(\Gamma, V) \in \mathscr{P}$ and $\Gamma' = \mathbb{B}(\Gamma, V)$. Assume that there is a *left-splitting arc* $\widetilde{\zeta}$ for Γ' based at a point $\alpha_0 \in \Gamma'$, such that $\widetilde{\zeta} \setminus \{\alpha_0\} \subset \mathrm{Dom}(\psi_i)$ for some homoclinic map ψ_i, and $\psi_i(\widetilde{\zeta} \setminus \{\alpha_0\}) \subset \Gamma$. Since the inverse map φ^{-1} *tilts the verticals to the left*, the argument in the proof of Lemma 5.3 implies that $\varphi^{-n}(V) \cap (\widetilde{\zeta} \setminus \{\alpha_0\}) \neq \emptyset$ for some $n > 0$. Choose $m > n$. Then $\varphi^{m-n}(V) \cap \varphi^m(\widetilde{\zeta} \setminus \{\alpha_0\}) \neq \emptyset$. By the definition of an equivariant homoclinic correspondence, there exists a homoclinic map $\psi_{i(m)}$ satisfying

$$\mathrm{Dom}\, \psi_{i(m)} = \widehat{\varphi}^m\big(\mathrm{Dom}\, \psi_i\big), \qquad \mathrm{Im}\, \psi_{i(m)} = \widehat{\varphi}^m\big(\mathrm{Im}\, \psi_i\big),$$

and

$$\psi_{i(m)} \circ \widehat{\varphi}^m = \widehat{\varphi}^m \circ \psi_i.$$

Therefore

$$\psi_{i(m)}(\varphi^m(\widetilde{\zeta} \setminus \{\alpha_0\})) = \varphi^m(\psi_i(\widetilde{\zeta} \setminus \{\alpha_0\})) \subset \varphi^m(\Gamma) \subset \Gamma,$$

since Γ is φ-invariant. The rest of the proof goes as before.

Appendix A. **A reminder on twist and tilt maps.** Here we review some basic concepts on twist and tilt maps, following [10, 34, 35, 37, 51, 60, 42, 39], and prove some auxiliary results that are used in the paper.

Let $a < b$ be fixed. We set

$$\mathbf{A} = \mathbb{T} \times [a, b], \qquad \Gamma(a) = \mathbb{T} \times \{a\}, \qquad \Gamma(b) = \mathbb{T} \times \{b\}.$$

The closure of a subset $E \subset \mathbf{A}$ will denoted by $\mathrm{Cl}\, E$, and its interior will be denoted by $\mathrm{Int}\, E$. The set $\mathrm{Fr}\, E = \mathrm{Cl}\, E \setminus \mathrm{Int}\, E$ is the frontier of E. A disk is an open, connected and simply connected subset of $\mathbf{A}$.

An essential circle in $\mathbf{A}$ is a C^0 curve which is homotopic to $\Gamma(a)$. We denote by $\mathrm{Ess}(f)$ the set of all essential circles in $\mathbf{A}$.

Here we say that $f : \mathbf{A} \to \mathbf{A}$ is a *twist map* when it is a C^1 diffeomorphism, preserves $\Gamma(a)$ and $\Gamma(b)$ and twists the verticals *to the right* (resp., *to the left*), that is, for $f(\theta, r) = (\Theta, R)$ we have

$$\partial_r \Theta(\theta, r) > 0 \quad (\text{resp.} \quad \partial_r \Theta(\theta, r) < 0), \qquad \forall (\theta, r) \in \mathbf{A}.$$

We now recall the definition of tilted arcs.

Fix a circle $\Gamma \in \mathrm{Ess}(f)$. An *arc emanating from* Γ is a C^0 injective function $\eta : [0, 1] \to \mathbf{A}$ such that $\eta(0) \in \Gamma$ and $\eta(]0, 1]) \in \Gamma^+$. We say that such an arc *starts at* $\eta(0)$ and *ends at* $\eta(1)$.

Given u, v two vectors in $\mathbb{R}^2$, let $\angle(u, v)$ be the angle from u to v in $[0, 2\pi[$, measured counterclockwise. Denote $\partial_r = (0, 1) \in \mathbb{R}^2$.

A C^1 arc emanating from Γ with $\eta'(s) \neq 0$ for $s \in [0, 1]$ is said to be *positively tilted* (resp. *negatively tilted*) when

- $\angle\big(\partial_r, \eta'(0)\big) \in]0, \pi[$ (resp. $\angle\big(\partial_r, \eta'(0)\big) \in]-\pi, 0[$), and
- the continuous lift $s \in [0, 1] \mapsto \angle\big(\partial_r, \eta'(s)\big) \in \mathbb{R}$ is positive (resp. negative) for all $s \in [0, 1]$.
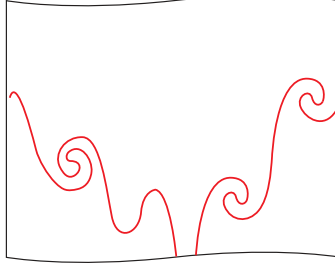

FIGURE 6. Positively and negatively tilted arcs

We now define tilt maps.

The map $f : \mathbf{A} \to \mathbf{A}$ is a *tilt map* when it is a C^1 diffeomorphism, preserves $\Gamma(a)$ and $\Gamma(b)$ and maps each vertical into a *positively tilted* (resp., *negatively tilted*) arc, that is, for each map $\ell : [0, 1] \to \mathbf{A}$ of the form $\ell(s) = (\theta_0, a + s(b - a))$, for some $\theta_0 \in \mathbb{T}$, we have

$$\angle(\partial_r, (f \circ \ell)'(s)) < 0 \quad (\text{resp.} \quad \angle(\partial_r, (f \circ \ell)'(s)) > 0), \qquad \forall s \in [0, 1].$$

We also say that the map tilts the verticals to right (resp. left) when it maps verticals onto *negatively tilted* (resp. *positively tilted*) arcs.

A right (resp. left) twist map is a map satisfying $-\pi < \angle(\partial_r, (f \circ \ell)'(s)) < 0$ (resp. $0 < \angle(\partial_r, (f \circ \ell)'(s)) < \pi$) for every vertical ℓ. That is, any twist map is a tilt map (but not every tilt map is a twist map).

Any composition of twist maps of the same type is a tilt map (but not necessarily a twist map). Any composition of tilt maps of the same type is a tilt map of the same type.

A continuous map $f : \mathbf{A} \to \mathbf{A}$ is said to be *area-preserving* when it leaves invariant a Radon measure which is positive on the open subsets of $\mathbf{A}$.

In the sequel we will always assume that f is a twist map or a tilt map that tilts the verticals to the right.

Below we list several results from the Birkhoff theory, which are valid for both twist and tilt maps.

Theorem (Birkhoff). *Let $f : \mathbf{A} \to \mathbf{A}$ be an area-preserving twist map (resp. tilt map).*

Then there exists $\nu > 0$ such that every essential circle invariant under f is the graph of some ν–Lipschitz function $\gamma : \mathbb{T} \to [a, b]$.

The second result from Birkhoff's theory that we need is the following.

Theorem (Birkhoff). *Let $f : \mathbf{A} \to \mathbf{A}$ be an area-preserving twist map (resp. tilt map). Assume that U is an open subset of $\mathbf{A}$ homeomorphic to $\mathbb{T} \times [0, 1[$, with $\Gamma(a) \subset U$, such that $f(U) \subset U$ and such that U is the interior of its closure. Then the frontier $\mathrm{Fr}\, U$ is an invariant essential circle.*

One easily deduces from the first Birkhoff theorem that the set $\mathrm{Ess}(f)$ of essential invariant circles of f, endowed with the Hausdorff topology, is compact. Given $\Gamma \in \mathrm{Ess}(f)$ with $\Gamma = \mathrm{Graph}\,(\gamma)$, we set

$$\Gamma^+ = \big\{ (\theta, r) \in \mathbf{A} \mid r > \gamma(\theta) \big\}, \quad \Gamma^- = \big\{ (\theta, r) \in \mathbf{A} \mid r < \gamma(\theta) \big\}. \qquad (55)$$

By the Poincaré theory, every $\Gamma \in \mathrm{Ess}(f)$ admits a rotation number in $\mathbb{T}$ for $f_{|\Gamma}$. One can choose a common lift to $\mathbb{R}$ for the rotation number of all circles, which yields a function $\rho : \mathrm{Ess}(f) \to \mathbb{R}$. This function is continuous and increasing, in the sense that if $\Gamma_i = \mathrm{Graph}\,\gamma_i$, $i = 1, 2$ are invariant with $\gamma_1 \leq \gamma_2$, then $\rho(\gamma_1) \leq \rho(\gamma_2)$. Moreover, $\rho(\gamma_1) < \rho(\gamma_2)$ when $\gamma_1 < \gamma_2$.

Definition A.1. Let $f : \mathbf{A} \to \mathbf{A}$ be an area-preserving twist map (resp. tilt map) of the annulus $\mathbf{A}$. Let $\gamma_\bullet, \gamma^\bullet : \mathbb{T} \to]a, b[$ be two functions with $\gamma_\bullet < \gamma^\bullet$, whose graphs $\Gamma_\bullet$ and $\Gamma^\bullet$ are in $\mathrm{Ess}(f)$.

The set

$$\mathscr{B} = \{ (\theta, r) \mid \theta \in \mathbb{T}, \ \gamma_\bullet(\theta) \leq r \leq \gamma^\bullet(\theta) \}$$

is said to be a *Birkhoff zone* when that there is no element $\Gamma = \mathrm{Graph}\,\gamma \in \mathrm{Ess}(f)$ such that $\gamma_\bullet \leq \gamma \leq \gamma^\bullet$ and $\gamma \neq \gamma_\bullet$, $\gamma \neq \gamma^\bullet$.

We now prove Lemma 2.3, for which we refer to Definition 2.2.

Proof of Lemma 2.3. The main property of a good twist (resp. tilt) map f, coming from the fact that no element of $\mathrm{Ess}(f)$ has rational rotation number, is that two distinct elements of $\mathrm{Ess}(f)$ are disjoint (see [39], Section 13.2). As a consequence, the rotation number $\rho : \mathrm{Ess}(f) \to \mathbb{R}$ is a homeomorphism onto its image $\mathcal{R} = \rho\big(\mathrm{Ess}(f)\big)$, by the compactness of $\mathrm{Ess}(f)$, where $\mathrm{Ess}(f)$ is endowed with the uniform topology. The boundaries of the Birkhoff zones are mapped by ρ onto the boundaries of the maximal intervals in the complement $\mathrm{Rot} \setminus \rho\big(\mathrm{Ess}(f)\big)$, where

Rot $= \big[\rho(\Gamma(a)), \rho(\Gamma(b))\big]$ is the rotation interval of f. The other claims easily follow. $\qquad\square$

The following easy result on negatively tilted arcs is used several times in our constructions.

Lemma A.2. *Let Γ be an essential circle of $\mathbf{A}$ which is the graph of a ν-Lipschitz function $\gamma : \mathbb{T} \to [0,1]$, and let B be a ν-ball centered on Γ. Then for every $z \in \Gamma^+ \cap B$, there exists a negatively tilted arc emanating from Γ and ending at z, whose image is contained in B.*

Proof. Set $B = B_\theta \times B_r$. Then since B is a ν-ball the graph of $\gamma_{|B_\theta}$ is contained in B. Given $z = (\theta, r) \in \Gamma^+ \cap B$ and $\theta_0 \in B_\theta$ with $\theta_0 < \theta$ close enough to θ so that

$$\frac{r - \gamma(\theta_0)}{\theta - \theta_0} > \nu,$$

and setting $z_0 = \big(\theta_0, \gamma(\theta_0)\big)$, the segment $[z_0, z]$ satisfies our requirements. $\qquad\square$

The proof of the following lemma is immediate.

Lemma A.3. *Let $f : \mathbf{A} \to \mathbf{A}$ be an area-reserving twist map (resp. tilt map). Let $\Gamma^\pm$ be two nonintersecting essential invariant circles contained in $\mathbf{A}$.*

Then for any continuous curves C and C' which intersect both circles $\Gamma^\pm$, the positive orbit of C under f intersects C'.

We refer to [43] and [44] for the proofs of the following two results from Birkhoff's theory.

Lemma A.4. *Let $f : \mathbf{A} \to \mathbf{A}$ be an area-preserving twist map (resp. tilt map), and let Γ be an essential invariant circle for f.*

Then, the inverse image $f^{-1} \circ \eta$ of a positively tilted arc η emanating from Γ is a positively tilted arc emanating from Γ. Also, the direct image $f \circ \eta$ of a negatively tilted arc η emanating from Γ is a negatively tilted arc emanating from Γ.

Given a point $x \in \mathbf{A}$, we define the lower vertical $V^-(x)$ as the vertical segment joining x to the corresponding point of $\Gamma(a)$.

Lemma A.5. *Let $f : \mathbf{A} \to \mathbf{A}$ be an area-preserving twist map (resp. tilt map). Let Γ be an essential invariant circle for f. Let X be a connected closed subset of $\mathbf{A}$ which disconnects the annulus $\mathbf{A}$ and such that $X \subset \Gamma^+$. Let $x \in \mathbf{A}$ be such that there exists a positively tilted arc γ and a negatively tilted arc η, both emanating from Γ and ending at x, such that the images of γ and η do not intersect X.*

Then the vertical $V^-(x)$ does not intersect X.

We can now prove our second lemma on good tilt maps and triangular domains associated with right or left splitting arcs, stated in Section 2.

Proof of Lemma 2.11. We provide the proof only for the case when ζ is a right splitting arc and η is a negatively tilted arc; the other case can be proved similarly. Let $D = D_{\zeta|[0,s_*]}$ as in Definition 2.11. With the notation from the definition D is bounded by the arcs $\zeta([0, s_*])$, $[\beta_*, \alpha_*]$, and $[\alpha_*, \alpha_0]$. We claim that in fact $\eta(]0, 1[) \cap \zeta([0, s_*]) \neq \emptyset$. Assume by contradiction that the arc η is such that $\eta(1) \in D$ but $\tilde{\eta} \cap \zeta([0, s_*]) = \emptyset$. Note that $\zeta([0, s_*])$ is the image of a Lipschitz arc (a piece of an essential circle) by a C^1 local diffeomorphism ψ_i^{-1}. There exists $0 < t < 1$ such that $\eta(t) \in]\beta_*, \alpha_*[$; let t_x be the largest t with this property, and

let $x = \eta(t_x)$. Note that if $t' > t_x$ then $\eta(t') \in D$. Let $x' = \eta(t_{x'}) = (\theta', r')$, with $t'_x > t_x$ sufficiently close to t_x so that the vertical segment from $x' = (\theta', r')$ to $\alpha' = (\theta', \gamma(\theta')) \in \Gamma$ is contained in D. Note that the vertical arc $V^-(x')$ below x' intersects $\zeta([0, s_*])$, since this curve bounds D from below.

The arc $\eta_{|[0,t_{x'}]}$ is a negatively tilted arc from $\eta(0) \in \Gamma_\bullet$ to $x' \in D$, by assumption.

Let $X = \Gamma \cup \zeta([0, s_*])$. There exists a positively tilted curve from $\Gamma_\bullet$ to x' that does not intersect X, as we will shown now. Choose $\theta'' > \max_{s \in [0, s_*]} \pi(\zeta(s))$, where π denotes the projection onto the θ-coordinate. Let $y' \in \Gamma$ be the intersection point of the vertical through x' with Γ, and $y'' \in \Gamma$ be given by $y'' = (\theta'', \gamma(\theta''))$. Let $[y'', y']_\Gamma$ be the segment of Γ between y' and y'', traversed from y'' to y'. Choose a point $z \in \Gamma_\bullet$ and a positively tilted arc from z to y'', which we denote by $[z, y'']$. We define a C^0 arc emanating from $z \in \Gamma_\bullet$ and ending at x' as the concatenation of the arcs $[z, y'']$, $[y'', y']_\Gamma$ and $[y', x']$. Approximating the segment $[y'', y']_\Gamma$ from below and "rounding the corners" yields a positively tilted C^1 arc ξ from z to x'.

Observe that $\eta[0, t_{x'}]$ is a negatively tilted curve from $\eta(0) \in \Gamma_\bullet$ to x', which does not intersect X by assumption, and that ξ is a positively tilted arc from $z \in \Gamma_\bullet$ to x' that does not intersect X. Therefore, Lemma A.5 implies $V^-(x')$ does not intersect X. This is a contradiction, since $V^-(x')$ must intersect $\zeta([0, s_*])$ as previously noted. □

The following strong connecting lemma appeared with a different proof in [31].

Proposition A.6. *Let $f : \mathbf{A} \to \mathbf{A}$ be a area-preserving twist map (resp. tilt map)[19]. Let $\Gamma_\bullet$ and $\Gamma^\bullet$ be the boundary components of some Birkhoff zone of instability for f. Fix a pair of open sets $V_\bullet, V^\bullet$ which intersect $\Gamma_\bullet$ and $\Gamma^\bullet$ respectively, with moreover $V_\bullet \subset (\Gamma^\bullet)^-$. Then there exist a point $z \in V_\bullet$ and an integer $n \geq 0$ such that $f^n(z) \in V^\bullet$. Moreover the integer n can be chosen arbitrarily large.*

Proof. Set

$$U = \bigcup_{n \geq 0} f^n(\Gamma_\bullet^- \cup V_\bullet) = \Gamma_\bullet^- \cup \Big(\bigcup_{n \geq 0} f^n(V_\bullet)\Big),$$

so that U is a connected and f-invariant neighborhood of $\Gamma(a)$, which satisfies

$$U \subset (\Gamma^\bullet)^-.$$

Hence the frontier $\Gamma := \mathrm{Fr}\,\mathcal{U}$ of its associated filled subset (see Section 5) is in $\mathrm{Ess}(f)$ and satisfies $\Gamma_\bullet \leq \Gamma \leq \Gamma^\bullet$. Therefore $\Gamma = \Gamma_\bullet$ or $\Gamma = \Gamma^\bullet$. The former equality is impossible by construction, so $\Gamma = \Gamma^\bullet$.

As a consequence, $\Gamma^\bullet \subset \mathrm{Fr}\,\mathcal{U} \subset \mathrm{Fr}\,U$, so there exists an integer $n \geq 0$ such that

$$f^n(V_\bullet) \cap V^\bullet \neq \emptyset,$$

which proves our claim. Finally, observe that by choosing arbitrarily small open subsets $W_\bullet \subset V_\bullet$, $W^\bullet \subset V^\bullet$ and applying the previous result to the pair $W_\bullet$, $W^\bullet$, one can ensure that the integer n can be chosen arbitrarily large. □

Appendix B. **Equivariance properties for homoclinic correspondences.**
We describe here some equivariance properties of the homoclinic correspondences with respect to the Hamiltonian flow, see [20] for related equivariance results. These properties reveal themselves to be useful in the constructive framework of Section 5.

---

[19]Here the map is not necessarily assumed to be good.

Definition B.1. Consider a C^2 Hamiltonian function H on $\mathbb{A}^3$, an energy $\mathbf{e}$ and a tame cylinder $\mathscr{C}$ at energy $\mathbf{e}$, with continuation $\widehat{\mathscr{C}}$ and continued tilt section $\widehat{\Sigma}$ with return map $\widehat{\varphi}$. We say that a homoclinic correspondence $\psi = (\psi_i)_{i \in I}$ is *equivariant* when for each $i \in I$ and for each $m \in \mathbb{Z}$ there exists an index $i(m)$ such that

$$\mathrm{Dom}\,\psi_{i(m)} = \widehat{\varphi}^m\big(\mathrm{Dom}\,\psi_i\big), \quad \mathrm{Im}\,\psi_{i(m)} = \widehat{\varphi}^m\big(\mathrm{Im}\,\psi_i\big) \tag{56}$$

and

$$\psi_{i(m)} \circ \widehat{\varphi}^m = \widehat{\varphi}^m \circ \psi_i. \tag{57}$$

The main result of this section is the following.

Lemma B.2. *With the same assumptions as in the previous definition, any homoclinic correspondence $(\psi_i)_{i \in I}$ can be extended to an equivariant homoclinic correspondence, in the sense that there exists an equivariant homoclinic correspondence $(\boldsymbol{\psi}_i)_{i \in \mathbf{I}}$ with $\mathbf{I} \supset I$ and $\boldsymbol{\psi}_i = \psi_i$ for $i \in I$.*

Proof. Given a subset $A \subset \widehat{\mathscr{C}}$ and a function $T : A \to \mathbb{R}$, we denote by $\boldsymbol{\Phi}^T : A \to \widehat{\mathscr{C}}$ the map defined by

$$\boldsymbol{\Phi}^T(x) = \Phi_H\big(T(x), x\big), \quad x \in A.$$

1. Fix an element $\psi = \psi_i : \mathrm{Dom}\,\psi \to \mathrm{Im}\,\psi$ of the initial homoclinic correspondence. By definition, there exist a C^1 map $S : \mathrm{Dom}\,S \to \mathrm{Im}\,S$ and a C^1 function $\tau : \mathrm{Dom}\,\psi \to \mathbb{R}^+$ such that the following diagram

$$
\begin{array}{ccc}
\mathrm{Dom}\,\psi & \xrightarrow{\ \psi\ } & \mathrm{Im}\,\psi \\
\downarrow{\scriptstyle \boldsymbol{\Phi}^\tau} & & \downarrow{\scriptstyle j} \\
\mathrm{Dom}\,S & \xrightarrow{\ S\ } & \mathrm{Im}\,S
\end{array}
$$

commutes (where we denoted by j the canonical inclusion). The map S moreover satisfies condition (23) of Definition 2.7.

2. Since $\mathrm{Dom}\,S$ is an open subset of $\mathrm{int}\,\widehat{\mathscr{C}}$ which contains $\boldsymbol{\Phi}^\tau(\mathrm{Dom}\,\psi)$, one can find C^1 functions $\alpha, \beta : \mathrm{Dom}\,\psi \to \mathbb{R}$ such that $\alpha < \tau < \beta$ and

$$D(S) := \bigcup_{x \in \mathrm{Dom}\,\psi} \Phi_H\big(]\alpha(x), \beta(x)[\times \{x\}\big) \subset \mathrm{Dom}\,S.$$

By equivariance of S and commutativity of the previous diagram

$$I(S) := \bigcup_{x \in \mathrm{Im}\,\psi} \Phi_H\big(]\alpha(x) - \tau(x), \beta(x) - \tau(x)[\times \{x\}\big)$$

is an open subset of $\mathrm{Im}\,S$. We still denote by S the induced map from $D(S)$ to $I(S)$. Note that the previous diagram still commutes if $S : \mathrm{Dom}\,S \to \mathrm{Im}\,S$ is replaced with $S : D(S) \to I(S)$. Note moreover that if $D_t(S) := D(S) \cap \mathbf{Homt}\,\widehat{\mathscr{C}}$, then D_t is an open subset of full measure of $D(S)$ such that

$$\forall x \in D_t(S), \quad W^-(x) \cap W^+\big(S(x)\big) \cap \mathbf{Homt}\,\widehat{\mathscr{C}} \neq \emptyset. \tag{58}$$

3. For $m \in \mathbb{Z}$, we denote by $T^{(m)} : \widehat{\Sigma} \to \mathbb{R}^+$ the mth return time map associated with the Hamiltonian flow, so that $\widehat{\varphi}^m = \boldsymbol{\Phi}^{T^{(m)}} : \widehat{\Sigma} \to \widehat{\Sigma}$. We still denote by $T^{(m)}$ the "fiberwise continuation" of $T^{(m)}$ to the domains $D(S)$ and $I(S)$, that is:

$$\forall x \in \mathrm{Dom}\,\psi, \quad \forall z \in \Phi_H\big(]\alpha(x), \beta(x)[\times \{x\}\big), \qquad T^{(m)}(z) = T^{(m)}(x),$$

$$\forall y \in \mathrm{Im}\,\psi, \quad \forall z \in \Phi_H\big(]\alpha(y) - \tau(y), \beta(y) - \tau(y)[\times \{y\}\big), \qquad T^{(m)}(z) = T^{(m)}(y).$$

Then, observe that the map $S^{(m)} : \Phi^{T^{(m)}}\big(D(S)\big) \to \Phi^{T^{(m)}}\big(I(S)\big)$ defined by

$$S^{(m)} = \Phi^{T^{(m)}} \circ S \circ \Phi^{-T^{(m)}}$$

satisfies

$$\forall z \in \Phi^{T^{(m)}}\big(D_t(S)\big), \quad W^-(z) \cap W^+(S^{(m)}(z)) \cap \mathbf{Homt}\widehat{\mathscr{C}} \neq \emptyset,$$

by equivariance of the characteristic foliations and preservation of the transversality. Moreover, $\Phi^{T^{(m)}}\big(D_t(S)\big)$ is an open subset with full measure of $\Phi^{T^{(m)}}\big(D(S)\big)$.

4. For $m \in \mathbb{Z}$, let $\psi^{(m)}$ be the unique map such that the following diagram

$$
\begin{array}{ccc}
\widehat{\psi}^{(m)}(\mathrm{Dom}\,\psi) & \xrightarrow{\ \psi^{(m)}\ } & \widehat{\psi}^{(m)}(\mathrm{Im}\,\psi) \\
\Big\downarrow{\scriptstyle \Phi^\tau} & & \Big\downarrow{\scriptstyle j} \\
\mathbf{\Phi}^{T^{(m)}}\big(D(S)\big) & \xrightarrow{\ S^{(m)}\ } & \mathbf{\Phi}^{T^{(m)}}\big(I(S)\big)
\end{array}
$$

commutes. Then clearly $\mathrm{Dom}\,\psi^{(m)} = \widehat{\varphi}^m(\mathrm{Dom}\,\psi)$, $\mathrm{Im}\,\psi^{(m)} = \widehat{\varphi}^m(\mathrm{Im}\,\psi)$ and

$$\forall x \in \mathrm{Dom}\,\psi, \qquad \psi^{(m)}(x) \circ \widehat{\varphi}^m = \widehat{\varphi}^m \circ \psi. \tag{59}$$

5. Let us consider the new index set $\mathbf{I} = I \times \mathbb{Z}$ and set

$$\psi_{(i,m)} = \psi_i^{(m)}.$$

Then the previous construction proves that $(\psi_{(i,m)})_{(i,m)\in\mathbf{I}}$ is a homoclinic correspondence, with associated family $(S_i^{(m)})_{(i,m)\in\mathbf{I}}$.

6. It suffices now to show that it is equivariant. Fix $\mathbf{i} = (i,k) \in \mathbf{I}$, set $\boldsymbol{\psi} := \psi_{\mathbf{i}} = \psi_i^{(k)}$, fix $m \in \mathbb{Z}$ and set $\boldsymbol{\psi}^{(m)} = \psi_i^{(m+k)}$. Then

$$\mathrm{Dom}\,\boldsymbol{\psi}^{(m)} = \widehat{\varphi}^{m+k}(\mathrm{Dom}\,\psi_i) = \widehat{\varphi}^m(\mathrm{Dom}\,\boldsymbol{\psi}),$$

$$\mathrm{Im}\,\boldsymbol{\psi}^{(m)} = \widehat{\varphi}^{m+k}(\mathrm{Im}\,\psi_i) = \widehat{\varphi}^m(\mathrm{Im}\,\boldsymbol{\psi}),$$

and, by (59):

$$\boldsymbol{\psi}^{(m)} \circ \widehat{\varphi}^{m+k} = \widehat{\varphi}^{m+k} \circ \psi_i = \widehat{\varphi}^m \circ \widehat{\varphi}^k \circ \psi_i = \widehat{\varphi}^m \circ \boldsymbol{\psi} \circ \widehat{\varphi}^k$$

so that

$$\boldsymbol{\psi}^{(m)} \circ \widehat{\varphi}^m = \widehat{\varphi}^m \circ \boldsymbol{\psi}.$$

This proves the equivariance condition for $\psi_{\mathbf{i}}$, with $\mathbf{i}(m) = (i, m+k)$. $\qquad\square$

Observe that while homoclinic correspondences giving rise to good cylinders can always be modified in order to obtain δ-bounded correspondences, this is no longer the case for equivariant correspondences, due to the extension process.

Appendix C. **Symmetrization of polysystems.** The following general lemma has been used several times is specific contexts.

Lemma C.1. *Let A be a metric space endowed with a finite Borel measure, positive on the nonempty open subsets of A. Let φ be a measure-preserving homeomorphism of A and let $(\psi_i)_{i\in I}$ be a polysystem on A, where $\mathrm{Dom}\,\psi_i$ is open and the map $\psi_i : \mathrm{Dom}\,\psi_i \to \mathrm{Im}\,\psi_i$ is a homeomorphism, for all $i \in I$. Fix a nonempty open subset $V \subset A$. Let U_f and U_g be the full orbit of V under the polysystems $f = \big(\varphi, \psi = (\psi_i)_{i\in I}\big)$ and $g = \big(\varphi, \varphi^{-1}, \psi = (\psi_i)_{i\in I}\big)$ respectively. Then U_f is contained and dense in U_g.*

Proof. Since φ, φ^{-1} and ψ_i are open maps, the full orbits U_f and U_g are open in A, and clearly $U_f \subset U_g$. We assume that the index set I does not contain $\{-1, 1\}$ and we write $\mathbf{I} = \{-1, 1\} \cup I$. Set $\tau_{-1} = \varphi^{-1}$, $\tau_1 = \varphi$, $\tau_i = \psi_i$ for $i \in I$, so that

$$f = (\tau_1, (\tau_i)_{i \in I}), \qquad g = (\tau_{-1}, \tau_1, (\tau_i)_{i \in I}).$$

Fix a nonempty open subset $\mathbf{W} \subset U_g$. Then (by continuity of the maps) there exist a nonempty open subset W of V and a sequence $\omega = (\omega_n)_{0 \le n \le n_* - 1} \in \mathbf{I}^{n_*}$ such that $g^\omega(W) \subset \mathbf{W}$. We will iteratively modify the set W and the sequence ω so that we get a nonempty open set $\widetilde{W} \subset W$ and a sequence $\widetilde{\omega} \in (1 \cup I)^{m_*}$ such that

$$f^{\widetilde{\omega}}(\widetilde{W}) = g^\omega(\widetilde{W}) \subset \mathbf{W}.$$

At each step of the process, the number of occurrences of the index -1 in the sequence ω will decrease by 1, and so the process stops after a finite number of steps. Let us describe the first one.

We set, for $0 \le n \le n_* - 1$:

$$W_n = g^{(\omega_0, \dots, \omega_{n-1})}(W)$$

so that $W_{n_*} = g^\omega(W) \subset \mathbf{W}$. Let $\overline{n} \ge 0$ be the smallest index such that $\omega_{\overline{n}} = -1$. Since $\tau_1 = \varphi$ is measure-preserving, there exists a nonempty open subset $O \subset W_{\overline{n}}$ and an integer $\nu \ge 1$ such that

$$\tau_1^{\nu+1}(O) \subset W_{\overline{n}},$$

by the Poincaré recurrence theorem. Consider the new sequence $\omega' = (\omega'_0, \dots, \omega'_{n'_* + \nu - 2})$ with

$$\begin{aligned}
\omega'_n &= \omega_n, & 0 \le n \le \overline{n} - 1, \\
\omega'_n &= 1, & \overline{n} \le n \le \overline{n} + \nu - 1, \\
\omega'_n &= \omega_{n-\nu+1}, & \overline{n} + \nu \le n \le n_* + \nu - 2,
\end{aligned}$$

where the first line has to be omitted when $\overline{n} = 0$. Set

$$W' = \tau_{\omega_0}^{-1} \circ \cdots \circ \tau_{\omega_{\overline{n}-1}}^{-1}(O) \subset W,$$

and note that $W' \ne \emptyset$. Set

$$W'_0 = W', \qquad W'_n = g^{(\omega'_0, \dots, \omega'_{n-1})}(W'), \quad 1 \le n \le n_* + \nu - 1.$$

Therefore, by construction:

$$W'_n \subset W_n, \quad 0 \le n \le \overline{n} - 1, \quad \text{and} \quad W'_{\overline{n}} = O \subset W_{\overline{n}}.$$

Hence

$$W'_{\overline{n}+\nu} = \tau_1^\nu(W'_{\overline{n}}) = \tau_1^{-1}\big(\tau_1^{\nu+1}(W'_{\overline{n}})\big) = \tau_1^{-1}\big(\tau_1^{\nu+1}(O)\big) \subset \tau_{-1}(W_{\overline{n}}) = W_{\overline{n}+1}.$$

As a consequence, by definition of the sequence ω':

$$W'_n \subset W_{n-\nu+1}, \quad \overline{n} + \nu + 1 \le n \le n_* + \nu - 1.$$

In particular

$$W'_{n_*+\nu-1} \subset W_{n_*}$$

and if ℓ is the number of occurrences of -1 in ω, then -1 occurs $\ell - 1$ times in ω'. Iterating this process therefore yields a nonempty open set $\widetilde{W} \subset W$ and a sequence $\widetilde{\omega} \in \{0, 1\}^{m_*}$ such that $f^{\widetilde{\omega}}(\widetilde{W}) \subset W_{n_*} \subset \mathbf{W}$, so that the full orbit U_f of V under f intersects $\mathbf{W}$. Since $\mathbf{W}$ is arbitrary in U_g, this proves that U_f is dense in U_g. $\square$

REFERENCES

[1] V. I. Arnold, Instability of dynamical systems with several degrees of freedom, *Collected Works, Volume 1*, Springer-Verlag Berlin Heidelberg, 2009, 423-427.

[2] P. W. Bates, K. Lu and C. Zeng, Invariant foliations near normally hyperbolic invariant manifolds for semiflows, *Transactions of the American Mathematical Society*, **352** (2000), 4641-4676.

[3] P. Berger, Persistence of laminations, *Bull. Braz. Math. Soc.*, **41** (2010), 259-319.

[4] P. Berger and A. Bounemoura, A geometrical proof of the persistence of normally hyperbolic submanifolds, *Dynamical Systems: An International Journal*, **28** (2013), 567-581.

[5] P. Bernard, The dynamics of pseudographs in convex Hamiltonian systems, *J. Amer. Math. Soc.*, **21** (2008), 615-669.

[6] P. Bernard, V. Kaloshin and K. Zhang, Arnold diffusion in arbitrary degrees of freedom and normally hyperbolic invariant cylinders, *Acta Mathematica*, **217** (2016), 1-79.

[7] M. Berti, L. Biasco and P. Bolle, Drift in phase space: A new variational mechanism with optimal diffusion time, *J. Math. Pures Appl.*, **82** (2003), 613-664.

[8] U. Bessi, An approach to Arnold's diffusion through the calculus of variations, *Nonlinear Anal.*, **26** (1996), 1115-1135.

[9] S. Bolotin and D. Treschev, Unbounded growth of energy in nonautonomous Hamiltonian systems, *Nonlinearity*, **12** (1999), 365-387.

[10] P. Boyland, Rotation sets and Morse decompositions in twist maps, *Ergod. Th. & Dynam. Sys.*, **8** (1988), 33-61.

[11] M. J. Capiński and M. Gidea, Arnold diffusion, quantitative estimates and stochastic behavior in the three-body problem, *Communications on Pure and Applied Mathematics*.

[12] M. J. Capiński, J. Gonzalez, J.-P. Marco and J. D. Mireles James, Computer assisted proof of drift orbits along normally hyperbolic manifolds, *Communications in Nonlinear Science and Numerical Simulations*, **106** (2022), 105970.

[13] M. Chaperon, The Lipschitz core of some invariant manifold theorems, *Ergod. Th. & Dynam. Sys.*, **28** (2008), 1419-1441.

[14] C.-Q. Cheng, The genericity of Arnold diffusion in nearly integerable Hamiltonian systems, *Asian J. Math.*, **23** (2019), 401-438.

[15] C.-Q. Cheng and J. Yang, Arnold diffusion in Hamiltonian systems: Nearly integrable case, *J. Diff. Geom.*, **82** (2009), 229-277.

[16] L. Chierchia and G. Gallavotti, Drift and diffusion in phase space, *Ann. Inst. H. Poincaré Phys. Théor.*, **60** (1994), 1-144.

[17] A. Delshams, R. de la Llave and T. M. Seara, A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of T^2, *Comm. Math. Phys.*, **209** (2000), 353-392.

[18] A. Delshams, R. de la Llave and T. M. Seara, *A Geometric Mechanism for Diffusion in Hamiltonian Systems Overcoming the Large Gap Problem*, Memoirs AMS, 179, American Mathematical Soc., 2006.

[19] A. Delshams, R. de la Llave and T. M. Seara, Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows, *Adv. Math.*, **202** (2006), 64-188.

[20] A. Delshams, R. de la Llave and T. M. Seara, Geometric properties of the scattering map of a normally hyperbolic invariant manifold, *Adv. Math.*, **217** (2008), 1096-1153.

[21] A. Delshams, M. Gidea and P. Roldan, Transition map and shadowing lemma for normally hyperbolic invariant manifolds, *Discrete Contin. Dyn. Syst.*, **33** (2013), 1089-1112.

[22] A. Delshams and P. Gutiérrez, Splitting Potential and the Poincaré-Melnikov Method for Whiskered Tori in Hamiltonian Systems, *J. Nonlinear Sci.*, **10** (2000), 433-476.

[23] A. Delshams and R. G. Schaefer, Arnold diffusion for a complete family of perturbations, *Regular and Chaotic Dynamics*, **22** (2017), 78-108.

[24] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.*, **21** (1972), 193-226.

[25] E. Fontich and P. Martín, Hamiltonian systems with orbits covering densely submanifolds of small codimension, *Nonlinear Analysis*, **52** (2003), 315-327.

[26] V. Gelfreich and D. Turaev, Arnold diffusion in a priori chaotic symplectic maps, *Communications in Mathematical Physics*, **353** (2017), 507-547.

[27] M. Gidea and R. de la Llave, Topological methods in the instability problem of Hamiltonian systems, *Discrete Contin. Dyn. Syst.*, **14** (2006), 295-328.

[28] M. Gidea, R. de la Llave and T. M-Seara, A general mechanism of diffusion in Hamiltonian systems: Qualitative results, *Communications on Pure and Applied Mathematics*, **73** (2020), 150-209.

[29] M. Gidea and C. Robinson, Shadowing orbits for transition chains of invariant tori alternating with Birkhoff zones of instability, *Nonlinearity*, **20** (2007), 1115-1143.

[30] M. Gidea and C. Robinson, Obstruction argument for transition chains of tori interspersed with gaps, *Discrete Contin. Dyn. Syst. Ser. S*, **2** (2009), 393-416.

[31] M. Gidea and C. Robinson, Diffusion along transition chains of invariant tori and Aubry-Mather sets, *Ergodic Theory & Dynamical Systems*, **33** (2013), 1401-1449.

[32] N. Gourmelon, Adapted metrics for dominated splitting, *Erg. Th. and . Dyn. Syst.*, **27** (2007), 1839-1849.

[33] M. Guardia, T. M. Seara P. Martín and L. Sabbagh, Oscillatory orbits in the restricted elliptic planar three body problem, *Discrete and Continuous Dynamical Systems*, **37** (2017), 229-256

[34] G. R. Hall, *Birkhoff Orbits for Non-Monotone Twist Maps*, Ph.D thesis, Wisconsin Univ-Madison Mathematics Research Center, 1984.

[35] M.-R. Herman, Sur les courbes invariantes par les difféomorphismes de l'anneau, *Astérisque*, **103-104** 1983.

[36] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics 583, Springer-Verlag, 1977.

[37] S. Hu, A variational principle associated to positive tilt maps, *Commun. Math. Phys.*, **191** (1998), 627-639.

[38] V. Kaloshin and K. Zhang, *Arnold Diffusion for Smooth Systems of Two and a Half Degrees of Freedom*, (AMS-208), Princeton University Press, 2020.

[39] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press 54, 1997.

[40] L. Lazzarini and J.-P. Marco, From the Birkhoff theory of twist maps to Arnold diffusion in the a priori unstable case, preprint, 2016.

[41] L. Lazzarini, J.-P. Marco and D. Sauzin, *Measure and Capacity of Wandering Domains in Gevrey Near-Integrable Exact Symplectic Systems*, Mem. Amer. Math. Soc. 1235, 2019.

[42] P. Le Calvez, Propriétés dynamiques des difféomorphismes de l'anneau et du tore, *Astérisque*, **204** (1991), 131 pp.

[43] P. Le Calvez, Existence d'orbites quasi-périodiques dans les attracteurs de Birkhoff, *Comm. Math. Phys.*, **106** (1986), 383-394.

[44] P. Le Calvez, Propriétés dynamiques des régions d'instabilité, *Ann. Scient. Éc. Norm. Sup.*, **20** (1987), 443-464.

[45] P. Le Calvez, Drift orbits for families of twist maps of the annulus, *Ergodic Theory and Dynamical Systems*, **27** (2007), 869-879.

[46] J.-P. Marco, Modèles pour les applications fibrées et les polysystèmes, *C. R. Math. Acad. Sci. Paris*, **346** (2008), 203-208.

[47] J.-P. Marco, Chains of compact cylinders for cusp-residual nearly integrable systems on $\mathbb{A}^3$, preprint, 2016, `arXiv:1602.02399`.

[48] J.-P. Marco, Arnold diffusion in cusp-residual nearly integrable systems on $\mathbb{A}^3$, preprint, 2016, `arXiv:1602.02403`.

[49] J.-P. Marco, Twist maps and Arnold diffusion for diffeomorphisms, in *New Trends in Calculus of Variations', Series: Radon Series on Computational and Applied Mathematics*, Academic Press, 18, de Gruyter, 2017, 473-495.

[50] J.-P. Marco, A symplectic approach to Arnold diffusion problems, to appear, *Proccedings of HST*.

[51] J. N. Mather, Non-existence of invariant circles, *Ergodic Theory and Dynamical Ssystems*, **4** (1984), 301-309.

[52] J. N. Mather, Arnold diffusion, I: Announcement of Results, *J. Math. Sci.*, **24** (2004), 5275-5289.

[53] J. N. Mather, Order structure on action minimizing orbits, in *Symplectic Topology and Measure Preserving Dynamical Systems*, Contemp. Math., volume 512, Amer. Math. Soc., Providence, RI, (2010), 41-125.

[54] J. N. Mather, Arnold diffusion by variational methods, in *Essays in Mathematics and its Applications*, Springer, Heidelberg, 2012, 271-285.

[55] R. Moeckel, Generic drift on Cantor sets of annuli, in *Celestial Mechanics (eds. A. Chenciner R. Cushman, C. Robinson, Z.J. Xia)*, Contemp. Math., 292, Amer. Math. Soc., Providence, RI, 2002, 163-171

[56] M. Nassiri and E. R. Pujals, Robust transitivity in Hamiltonian dynamics, *Annales Scientifiques de l'École Normale Supérieure*, **45** (2012), 191-239.

[57] A.-N. Nekhoroshev, An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems, *Russian Mathematical Surveys*, **32** (1977), 5-66.

[58] L. Sabbagh, An inclination lemmas for normally hyperbolic invariant manifolds with applications to diffusion, *Ergodic Theory and Dynamical Systems*, **35** (2015), 2269-2291.

[59] D. Treschev, Evolution of slow variables in a priori unstable Hamiltonian systems, *Nonlinearity*, **17** (2004), 1803-1841.

[60] J-C. Yoccoz, Travaux de Herman sur les tores invariants, *Astérisque*, **206** (1992).