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Steady translating hollow vortex pair in weakly compressible flow

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ABSTRACT

The weakly compressible translating hollow vortex pair is examined using the Imai–Lamla formula and a direct conformal mapping approach applied to a Rayleigh–Jansen expansion in the Mach number, M, taken to be small. The incompressible limit has been studied by Crowdy et al. (Eur. J. Mech. B Fluids 37 (2013), 180–186) who found explicit formulas for the solutions using conformal mapping. We improve upon their results by finding an explicit formula for the conformal map which had been left as an integral in Crowdy et al. (2013). The weakly compressible problem requires the solution of two boundary value problems in an annulus. This results in a linear parameter problem for the perturbed propagation velocity and speed along the vortex boundary. We find that two additional constraints are required to solve for the perturbed parameters. We require that the perturbation in vortex area and the perturbation in centroid separation to vanish. Three possible centroid definitions are given and the results worked out for each case. It is found that the correction to the propagation velocity is always negative, so that the vortex always slows down at first order due to compressible effects. Numerical results are consistent in the limit of small vortex size with the previous result by Leppington (J. Fluid Mech. 559 (2006), 45–55) that the propagation parameter is unchanged to $O(M^2)$.

1. Introduction

The study of incompressible vortices has a long history going back to Helmholtz. In contrast, vortices in compressible flows have been much less studied despite their importance [1]. In recent years there has been a renewal of interest in theoretical studies of weakly compressible flows via perturbation methods using Rayleigh–Jansen expansions [2–4]. Many of these studies model a vortex as a hollow vortex, which is a bounded region of fluid at constant pressure with a non-zero circulation around it. Some studies have also considered point vortices in compressible flows, using methods of matched asymptotic expansions to deal with the supersonic flow region present in compressible flow near a point vortex [2,3,5,6].

Early studies of hollow vortices in compressible flow favored the hodograph method, which is based on taking the velocity field in polar coordinates as the independent variables. In this hodograph plane, the governing equations are linear. Baker et al. [7] studied a singly-periodic row of incompressible hollow vortices by combining this technique with a conformal mapping approach to deal with the unknown shape of the hollow vortex boundary. The corresponding compressible flow was studied by Ardalan et al. [8] using a Rayleigh–Jansen expansion in the small Mach number limit and employing numerical methods for larger Mach numbers. They found the existence of transonic shock-free solutions for a suitable choice of the parameters in the problem.

For point vortices in incompressible flow, solutions to the two-vortex problem are relative equilibria in which the inter-vortex distances are constant [9]. Two vortices with equal and opposite circulations translate at a constant speed. The incompressible hollow vortex pair was studied by Pocklington [10] who found solutions in terms of elliptic functions. This problem was revisited more recently by Crowdy et al. [11], who used a conformal mapping approach employing the Schottky–Klein prime function [12].

The translating vortex pair is a model for a vortex ring and the effect of compressibility on a thin vortex ring was studied by Moore [13]. Moore and Pullin [14] considered a hollow vortex pair in compressible flow and found that for small vortex sizes (i.e., the point vortex limit) the propagation parameter, which encodes the translation speed of the pair as well as a suitably defined vortex separation, changes at first-order in a perturbation expansion. Later, Leppington [2] modeled the vortices as point vortices, and showed that the first-order correction to the propagation parameter is in fact zero. This was done by a careful consideration of the force-free condition on point vortices in weakly compressible flows.

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This centrally important point was recently further clarified by Crowdy and Krishnamurthy [3] via an extension of the Blasius theorem to weakly compressible flows (see also Barsony-Nagy [15]). Their approach makes use of the Imai–Lamla formula [16] to derive a closed-form expression for the first-order correction to the classical formula for the translation speed of the incompressible Kármán vortex street. The Imai–Lamla formula combined with a direct conformal mapping approach (without transforming to a hodograph plane) was used by Crowdy and Krishnamurthy [4] to find solutions for weakly compressible translating hollow vortex streets. These solutions are perturbations of the incompressible hollow vortex street studied by Crowdy and Green [17].

In this paper, we study the weakly compressible translating hollow vortex pair using the Imai–Lamla formula and a direct conformal mapping approach. In Section 2 we review the basic equations and set up the problem in the complex plane using Rayleigh–Jansen expansions for the physical quantities combined with a perturbed conformal mapping. In Section 3 we review the incompressible hollow vortex pair as studied by Crowdy et al. [11], to which we add an explicit formula for the conformal map found in this paper. In Section 4 we formulate the weakly compressible problem. We set up two boundary value problems to be solved in an annulus in order to obtain the weakly compressible solutions. In Section 5 we discuss the results of solving the parameter problem arising in Section 4. We conclude with a summary of our findings in Section 6.

2. Governing equations for weakly compressible flow

We consider the steady, irrotational, isentropic two-dimensional flow of an inviscid barotropic fluid. Let (x, y) be the flow plane, (u, v) the two components of the fluid velocity field, $p(x, y)$ the pressure and $\nu(x, y)$ the density of the fluid. The basic equation of motion for such a flow is the Bernoulli equation

$$\frac{u^2+v^2}{2} + \int^{\nu} \frac{p'(\hat{\nu})}{\hat{\nu}}\, \mathrm{d}\hat{\nu} = \text{const.} \tag{1}$$

The pressure–density relationship for isentropic flows is

$$p \propto \nu^{\gamma}, \tag{2}$$

where γ is the ratio of specific heats. The speed of sound c is defined by $c^2 = p'(\nu)$. Since the flow is irrotational there exists a velocity potential $\phi(x, y)$ such that

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y}. \tag{3}$$

Further, since the flow is steady, the conservation of mass implies the existence of a stream function $\psi(x, y)$ such that

$$u = \frac{\nu_0}{\nu}\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\nu_0}{\nu}\frac{\partial \psi}{\partial x}. \tag{4}$$

Here ν_0 is the (constant) incompressible density. The Eqs. (1), (2), (3) and (4) provide a complete set of equations for the unknown functions u, v, ϕ, ψ, p and ν.

It is convenient to work in the complex plane through the formal change of variables $(x, y) \mapsto (z, \overline{z})$ where $z = x + \mathrm{i}y$. Throughout this paper, overbars denote complex conjugates. Even though the flow is compressible, we can define a complex potential $f(z, \overline{z})$ as

$$f(z, \overline{z}) = \phi(z, \overline{z}) + \mathrm{i}\psi(z, \overline{z}). \tag{5}$$

Note that the complex potential is not an analytic function of z, but instead also depends on the conjugate variable $\overline{z}$, due to the compressibility of the flow. We may combine the definitions (3) and (4) using

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) \tag{6}$$

to define derivatives with respect to z and $\overline{z}$. We get the two equivalent equations

$$\frac{\partial f}{\partial z} = \frac{1}{B(\nu)}\frac{\partial \overline{f}}{\partial z} \tag{7a}$$

$$\frac{\partial f}{\partial \overline{z}} = B(\nu)\frac{\partial \overline{f}}{\partial \overline{z}} \tag{7b}$$

where the real-valued function $B(\nu)$ is defined as

$$B(\nu) = \frac{1-\nu/\nu_0}{1+\nu/\nu_0}. \tag{8}$$

The two Eqs. (7a) and (7b) are equivalent in the sense that they are complex conjugates of each other. It is seen from (8) that $B(\nu) = 0$ when $\nu \equiv \nu_0$, (7b) then implies that $\partial f/\partial \overline{z} \equiv 0$ and we recover the incompressible case.

We can define the complex velocity field as

$$\xi(z, \overline{z}) = u(z, \overline{z}) - \mathrm{i}v(z, \overline{z}), \tag{9}$$

and obtain using (3), (5) and (7a)

$$\xi(z, \overline{z}) = \frac{\partial f}{\partial z} + \frac{\partial \overline{f}}{\partial z} = \frac{\partial f}{\partial z}(1 + B(\nu)). \tag{10}$$

Using the pressure–density relationship (2), the Bernoulli equation (1) can be re-written:

$$\left(\frac{\gamma-1}{c_s^2}\right)\frac{|\xi|^2}{2} = 1 - \left(\frac{\nu}{\nu_s}\right)^{\gamma-1}. \tag{11}$$

Here c_s is the speed of sound at the stagnation point and ν_s is the corresponding density . We set $\nu_s = \nu_0$ without loss of generality. The real-valued equation (11) and the complex-valued equation (7a) form a closed system of equations for the unknown complex potential and fluid density.

2.1. Imai–Lamla formula

The foregoing discussion is applicable to compressible flows at any Mach number; we now introduce Rayleigh–Jansen expansions for weakly compressible flows. Let $M = U_0/c_s$ be a Mach number for the fluid flow, where U_0 is some characteristic velocity scale in the fluid. We expand the complex potential $f(z,\overline{z})$ in a perturbative series, in powers of the Mach number M, about a known incompressible complex potential $f_0(z)$. We also expand the velocity field $\xi(z,\overline{z})$ about the corresponding incompressible velocity field $\xi_0(z)$, to obtain:

$$f(z,\overline{z}) = f_0(z) + M^2 f_1(z,\overline{z}) + O(M^4), \tag{12a}$$

$$\xi(z,\overline{z}) = \xi_0(z) + M^2 \xi_1(z,\overline{z}) + O(M^4). \tag{12b}$$

The first-order corrections $f_1(z,\overline{z})$ and $\xi_1(z,\overline{z})$ are unknowns to be found. Combining these Rayleigh–Jansen expansions with (7a) and the Bernoulli equation (11), we obtain the Imai–Lamla formula, which provides an expression for $f_1(z,\overline{z})$ in terms of $\xi_0(z) = f_0'(z)$ (primes denote derivatives with respect to the argument) and an arbitrary analytic function $G(z)$:

$$f_1(z,\overline{z}) = \frac{1}{4U_0^2}\,\xi_0(z)\overline{I(z)} + G(z), \tag{13a}$$

$$I(z) = \int^z (\xi_0(\hat{z}))^2 \,\mathrm{d}\hat{z}. \tag{13b}$$

We call $I(z)$ the Imai–Lamla integral. We can also derive an expression for $\xi_1(z,\overline{z})$ using (10) and (12a):

$$\xi_1(z,\overline{z}) = \frac{1}{4U_0^2}\left[(\xi_0(z))^2\overline{\xi_0(z)} + \xi_0'(z)\overline{I(z)}\right] + G'(z). \tag{14}$$

The problem is thus reduced to finding a single unknown analytic function $G(z)$, for a given incompressible solution. An important point to note is that the ratio of specific heats, γ, does not appear in the solutions at first order. For details on the derivation of the above formulas, see Crowdy and Krishnamurthy [3,4].

2.2. Solutions via a perturbed conformal mapping method

Since the problem we are interested in involves free boundaries, we adopt a conformal mapping approach to obtain solutions. Let $z(\zeta)$ be a conformal map from an auxiliary ζ-plane to the complex flow plane. We further consider that the conformal map $z(\zeta)$ can be expanded as a Rayleigh–Jansen expansion

$$z(\zeta) = z_0(\zeta) + M^2 z_1(\zeta) + O(M^4), \tag{15}$$

where $z_0(\zeta)$ is the known map in terms of which the incompressible solution is given, while $z_1(\zeta)$ is an unknown first-order correction to be computed. The conformal map is always an analytic function, and hence the corrections to the map are also analytic functions. The Rayleigh–Jansen expansions (12) in the ζ-plane are

$$f(\zeta,\overline{\zeta}) = f_0(\zeta) + M^2 f_1(\zeta,\overline{\zeta}) + O(M^4), \tag{16a}$$

$$\xi(\zeta,\overline{\zeta}) = \xi_0(\zeta) + M^2 \xi_1(\zeta,\overline{\zeta}) + O(M^4). \tag{16b}$$

Here the known incompressible solutions are taken to be given in terms of the conformal map, so that $f_0(\zeta) = f_0(z_0(\zeta))$ and $\xi_0(\zeta) = \xi_0(z_0(\zeta))$. Usage of the same function name in the z-plane as well as the ζ-plane should not cause any confusion since the function meant is clear by context. The first order corrections f_1 and ξ_1 in the ζ-plane are now the unknown functions to be found.

We can write the Imai–Lamla formula (13a) and the Imai–Lamla integral (13b) in the ζ-plane as

$$f_1(\zeta,\overline{\zeta}) = \frac{1}{4U_0^2}\,\xi_0(\zeta)\overline{I(\zeta)} + G(\zeta), \tag{17a}$$

$$I(\zeta) = \int^\zeta (\xi_0(\hat{\zeta}))^2 z_0'(\hat{\zeta})\,\mathrm{d}\hat{\zeta}. \tag{17b}$$

The first-order correction to the velocity field (14) becomes

$$\xi_1(\zeta,\overline{\zeta}) = \frac{1}{4U_0^2}\left[(\xi_0(\zeta))^2\overline{\xi_0(\zeta)} + \frac{\xi_0'(\zeta)}{z_0'(\zeta)}\overline{I(\zeta)}\right] + \frac{G'(\zeta)}{z_0'(\zeta)} - \xi_0(\zeta)\frac{z_1'(\zeta)}{z_0'(\zeta)}. \tag{18}$$

The additional term, dependent on the correction to the conformal map, appears because the correct expression relating the incompressible velocity, complex potential and conformal map is $\xi_0(\zeta) = f_0'(\zeta)/z_0'(\zeta)$ whereas the derivatives in (10) are with respect to z. The solution here is thus determined by two unknown analytic functions $G(\zeta)$ and $z_1(\zeta)$.

3. Incompressible hollow vortex pair

The solution for an incompressible pair of hollow vortices was obtained by Pocklington [10]. A modern re-derivation of Pocklington's results using a conformal mapping method as well as a stability analysis is found in Crowdy et al. [11]. Consider a pair of hollow vortices with equal and opposite circulations $\pm\Gamma$ as shown in Fig. 1. A particular form of solutions for the hollow vortex pair is sought based on the behavior of a point vortex pair with equal and opposite circulations:

1. the hollow vortex pair must be uniformly translating with some speed U_0,
2. in the rest frame of the vortices there are exactly two stagnation points in the velocity field due to the vortex pair, and
3. additional conditions are imposed so that the two vortices are symmetric reflections of each other.

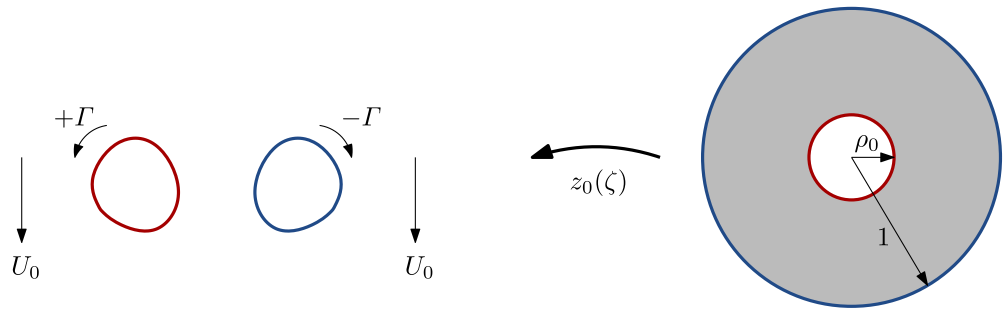

Fig. 1. An incompressible hollow vortex pair in the rest frame of the vortices. In this rest frame, the velocity at infinity is U_0 along the negative y-axis, where U_0 is the translation speed of the vortices in the lab frame. A conformal map $z_0(\zeta)$ from an annulus $\rho_0 \leq |\zeta| \leq 1$ in a ζ-plane to the exterior of the hollow vortices in the z-plane (flow plane) is used to obtain solutions in Crowdy et al. [11]. The inner boundary of the annulus is mapped to the vortex with circulation $+\Gamma$ while the outer boundary is mapped to the vortex with circulation $-\Gamma$.

The hollow vortex pair is taken to be uniformly translating along the positive y-axis. The solutions are described in the rest frame of the hollow vortex pair, in which there exists a uniform flow U_0 in the negative y-direction at infinity. The solutions for this free boundary problem are obtained using a conformal mapping approach, where an auxiliary ζ-plane is introduced and explicit formulas for the complex potential $f_0(\zeta)$ and the velocity field $\xi_0(\zeta)$ are obtained. The conformal mapping $z_0(\zeta)$ itself is then obtained using the simple but important formula (chain rule)

$$z_0'(\zeta) = \frac{f_0'(\zeta)}{\xi_0(\zeta)}. \tag{19}$$

The solutions are given in terms of a parameter ρ_0 which appears in the conformal mapping, which is the inner radius of the annulus $\rho_0 \leq |\zeta| \leq 1$; see Fig. 1. Symmetric solutions are sought, so that this single parameter ρ_0 determines the size, shape and separation of the hollow vortices. We note the following symmetries that we require of the solutions under the mapping $\zeta \mapsto \rho_0/\zeta$, i.e. reflection in the circle $|\zeta| = \sqrt{\rho_0}$:

$$f_0(\zeta) = -f_0(\rho_0/\zeta), \qquad \xi_0(\zeta) = \xi_0(\rho_0/\zeta), \qquad z_0(\zeta) = -z_0(\rho_0/\zeta). \tag{20}$$

The explicit solutions are given in terms of the Schottky–Klein prime function on the annulus. A brief introduction to this special function is given in Appendix A. We make extensive use of the properties listed there in the rest of this paper.

The conformal map has a simple pole at the pre-image of infinity. Let $\zeta = \beta_0$ map to infinity, then,

$$z_0(\zeta) = \frac{a_0}{\zeta - \beta_0} + \text{Taylor series} \quad \text{as } \zeta \to \beta_0, \tag{21}$$

where a_0 is some length scale which will be fixed later. Since we are in the rest frame of the vortex pair, we have the condition at infinity

$$\xi_0(\zeta) = \mathrm{i}U_0 \quad \text{at } \zeta = \beta_0. \tag{22}$$

The same boundary conditions apply on both the hollow vortex boundaries due to symmetry; these conditions are defined on the pre-images $|\zeta| = \rho_0, 1$ of the two boundaries. First, the no penetration condition leads to

$$\mathrm{Im}\left[f_0(\zeta)\right] = \begin{cases} C_0 & \text{on } |\zeta| = 1, \\ -C_0 & \text{on } |\zeta| = \rho_0, \end{cases} \tag{23}$$

where C_0 is some constant. Second, the constant speed q_0 on the vortex boundary is related to U_0 via

$$|\xi_0(\zeta)|^2 = \xi_0(\zeta)\,\overline{\xi_0(\zeta)} = q_0^2 = \begin{cases} U_0^2/\chi & \text{on } |\zeta| = 1, \\ U_0^2/\chi & \text{on } |\zeta| = \rho_0, \end{cases} \tag{24}$$

where χ is a constant defined further below (see (27)).

3.1. Exact solutions for incompressible flow

The formulation presented above is slightly different from that used by Crowdy et al. [11], where the symmetry conditions are not imposed explicitly in all the formulas. In terms of the present formulation, the solutions of Crowdy et al. [11] can be written

$$f_0(\zeta) = \frac{\mathrm{i}U_0 a_0}{\beta_0}\left[K(\zeta/\beta_0, \rho_0) + K(\zeta\beta_0, \rho_0) - \frac{1}{2}\right] - \frac{\mathrm{i}\Gamma}{2\pi}\log\frac{\zeta}{\sqrt{\rho_0}}, \tag{25a}$$

$$\xi_0(\zeta) = \frac{\mathrm{i}U_0\rho_0}{\sqrt{\chi}}\frac{P(\zeta/\alpha_0, \rho_0)P(\zeta/\overline{\alpha}_0, \rho_0)}{\zeta P(\zeta\alpha_0, \rho_0)P(\zeta\overline{\alpha}_0, \rho_0)}, \tag{25b}$$

$$z_0'(\zeta) = \hat{C}\left[\frac{P(\zeta\alpha_0, \rho_0)P(\zeta\overline{\alpha}_0, \rho_0)}{P(\zeta\beta_0, \rho_0)P(\zeta/\beta_0, \rho_0)}\right]^2. \tag{25c}$$

Here α_0, $\overline{\alpha}_0$ are the pre-images of the stagnation points. The constant $\hat{C}$ is given in terms of a_0 via the formula

$$a_0 = -\rho_0\hat{C}\left(\frac{P(\alpha_0\beta_0, \rho_0)P(\overline{\alpha_0}\beta_0, \rho_0)}{\hat{P}(1, \rho_0)P(\rho_0, \rho_0)}\right)^2, \tag{26}$$

but we will not use this formula in the following development. The constant χ is

$$\chi = \beta_0^2 \left[\frac{P(\alpha_0/\beta_0, \rho_0) P(\overline{\alpha}_0/\beta_0, \rho_0)}{P(\alpha_0\beta_0, \rho_0) P(\overline{\alpha}_0\beta_0, \rho_0)} \right]^2. \tag{27}$$

It can be shown using the properties discussed in Appendix A that the conditions at infinity (21) and (22) as well as the boundary conditions (23) and (24) are satisfied by the solutions (25). The symmetry requirements (20) can be satisfied by the solutions (25) if we choose

$$\beta_0 = \sqrt{\rho_0} \quad \text{and} \quad \alpha_0 = \sqrt{\rho_0}\, \mathrm{e}^{\mathrm{i}\theta_0}, \tag{28}$$

where $\theta_0 \in (0, 2\pi)$. We have thus identified five real parameters in the incompressible problem, namely $\Gamma, a_0, U_0, \rho_0, \theta_0$. Finally, we note that the constant C_0 can be calculated to be

$$C_0 = \frac{a_0 U_0}{2\beta_0} + \frac{\Gamma}{4\pi} \log \rho_0. \tag{29}$$

3.2. The conformal map and the parameter problem

The solution (25c) provides an expression for the derivative of the conformal map $z_0'(\zeta)$ but not the map $z_0(\zeta)$ itself. In Crowdy et al. [11] the map was found by numerical integration, but in Appendix B we show that the expression (25c) for $z_0'(\zeta)$ can be integrated to

$$z_0(\zeta) = \frac{a_0}{\beta_0} \left[K(\zeta/\beta_0, \rho_0) + \chi\, K(\zeta\beta_0, \rho_0) - \frac{1}{2} \right]. \tag{30}$$

The derivative of the map can be written as

$$\zeta z_0'(\zeta) = \frac{a_0}{\beta_0} \left[L(\zeta/\beta_0, \rho_0) + \chi L(\zeta\beta_0, \rho_0) \right], \tag{31}$$

which is equivalent to the expression (25c) for $z_0'(\zeta)$. We see from (25c) that the zeros of $z_0'(\zeta)$ are at $\zeta = 1/\alpha_0, 1/\overline{\alpha}_0$. We need to impose these as conditions on the parameters, since the expression (30) only fixes the poles of $z_0(\zeta)$:

$$L(\alpha_0\beta_0, \rho_0) + \chi L(\alpha_0/\beta_0, \rho_0) = 0. \tag{32}$$

The condition at $1/\overline{\alpha}_0$ is just the complex conjugate of (32) and is therefore equivalent to (32). The derivative $f_0'(\zeta)$ must also be zero at the points $\zeta = 1/\alpha_0, 1/\bar{\alpha}_0$, which leads to

$$L(\alpha_0/\beta_0, \rho_0) + L(\alpha_0\beta_0, \rho_0) = \frac{\Gamma\beta_0}{2\pi U_0 a_0}. \tag{33}$$

We use (33) to determine the translation speed U_0 in terms of the other parameters.

We solve the parameter problem as follows: first, we set the time and length scales by fixing the values of Γ and a_0. Then we choose the value of $0 < \rho_0 < 1$, and solve (32) for θ_0. Finally, U_0 is determined by (33). We thus obtain a one-parameter family of solutions, parameterised by ρ_0.

4. Weakly compressible hollow vortex pair

When the flow is weakly compressible, the complex potential $f(z, \overline{z})$ and velocity field $\xi(z, \overline{z})$ are no longer analytic functions, and the theory outlined in Section 2 applies. The velocity scale U_0 is taken to be the incompressible velocity of the vortex pair. The speed of the vortex pair and the speed on the vortex boundary are perturbed to

$$U = U_0 + M^2 U_1 + O(M^4), \tag{34a}$$

$$q = q_0 + M^2 q_1 + O(M^4), \tag{34b}$$

where U_1 and q_1 are the unknown first-order corrections due to compressibility. Since the incompressible solutions are described in terms of the conformal map $z_0(\zeta)$, the Rayleigh-Jansen expansions for the complex potential $f(\zeta, \overline{\zeta})$ and the velocity field $\xi(\zeta, \overline{\zeta})$ are given by (16). In addition, the conformal map itself has a Rayleigh-Jansen expansion (15) where $z_1(\zeta)$ is the unknown first-order correction to the conformal map $z_0(\zeta)$. The domain of the perturbed conformal map $z(\zeta)$ is $\rho \le |\zeta| \le 1$. The pre-image of infinity under $z(\zeta)$ is β, whereas the circle $|\zeta| = \rho$ maps to one of the hollow vortex boundaries. The circle $|\zeta| = 1$ maps to the other hollow vortex boundary. The Imai–Lamla formula (17a), together with (18), shows that there are two unknown analytic functions $G(\zeta)$ and $z_1(\zeta)$ in the theory.

The streamline condition leads to the following (symmetric) boundary condition on the complex potential

$$\mathrm{Im}[f(\zeta, \overline{\zeta})] = \begin{cases} C & \text{on} \quad |\zeta| = 1, \\ -C & \text{on} \quad |\zeta| = \rho. \end{cases} \tag{35}$$

The constant C is related to the incompressible constant C_0 in (23) via the Rayleigh–Jansen expansion

$$C = C_0 + M^2 C_1 + O(M^4). \tag{36}$$

The fluid speed on the hollow vortex boundaries remains a constant in the compressible flow. We express this as

$$|\xi(\zeta, \overline{\zeta})|^2 = \xi(\zeta, \overline{\zeta})\, \overline{\xi(\zeta, \overline{\zeta})} = q^2 \quad \text{on } |\zeta| = 1 \text{ and } |\zeta| = \rho. \tag{37}$$

This is analogous to (24) with q given by the Rayleigh–Jansen expansion Eq. (34b).

4.1. Properties of the perturbed conformal map

The parameters in $z(\zeta)$ are perturbatively related to the parameters in $z_0(\zeta)$. We have the Rayleigh–Jansen expansions

$$\rho = \rho_0 + M^2\rho_1 + O(M^4), \quad \beta = \beta_0 + M^2\beta_1 + O(M^4), \quad a = a_0 + M^2 a_1 + O(M^4). \tag{38}$$

We require the symmetry properties (20) to also hold for the weakly compressible flow so that we have $\beta = \sqrt{\rho}$. Using (38), this symmetry property implies

$$\rho_1/\rho_0 = 2\beta_1/\beta_0. \tag{39}$$

Of the five parameters Γ, a_0, U_0, ρ_0 and θ_0 in the incompressible solution, we leave the circulation Γ unchanged due to compressibility.

We require that $z(\zeta)$ have a simple pole at $\zeta = \beta$, like in (21):

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{Taylor series} \quad \text{as } \zeta \to \beta. \tag{40}$$

Using (38), (21), and (15) in (40) we obtain the required behavior of the correction $z_1(\zeta)$ near $\zeta = \beta_0$:

$$z_1(\zeta) = \frac{a_1}{\zeta - \beta_0} + \frac{a_0\beta_1}{(\zeta - \beta_0)^2} + \text{Taylor series}, \quad \text{as } \zeta \to \beta_0. \tag{41}$$

Let us take the conformal map $z(\zeta)$ to be of the following form, consistent with (40):

$$z(\zeta) = \frac{a}{\beta} K(\zeta/\beta, \rho_0) + \frac{a_0}{\beta_0}\chi K(\zeta\beta_0, \rho_0) - \frac{1}{2} + O(M^2). \tag{42}$$

Here the $O(M^2)$ terms are regular with a Taylor series everywhere in the domain. Using (38) for β and a, we see that

$$\frac{a}{\beta} K(\zeta/\beta, \rho_0) = \frac{a_0}{\beta_0} K(\zeta/\beta_0, \rho_0) + M^2\left[\frac{a_1}{\beta_0} K(\zeta/\beta_0, \rho_0) - \frac{a_0}{\beta_0}\frac{\beta_1}{\beta_0}\left[K(\zeta/\beta_0, \rho_0) + L(\zeta/\beta_0, \rho_0)\right]\right] + O(M^4), \tag{43}$$

which leads to

$$z(\zeta) = z_0(\zeta) + M^2\left[z_{1\mathrm{s}}(\zeta) + z_{1\mathrm{r}}(\zeta)\right] + O(M^4). \tag{44}$$

We have decomposed the correction as

$$z_1(\zeta) = z_{1\mathrm{s}}(\zeta) + z_{1\mathrm{r}}(\zeta) \tag{45}$$

with singular part $z_{1\mathrm{s}}(\zeta)$ and regular part $z_{1\mathrm{r}}(\zeta)$. The singular part can be taken to be

$$z_{1\mathrm{s}}(\zeta) = -\frac{a_0}{\beta_0}\frac{\beta_1}{\beta_0}\tilde{z}_{1s\beta}(\zeta) + \frac{a_0}{\beta_0}\frac{a_1}{a_0}\tilde{z}_{1sa}(\zeta), \tag{46}$$

where we have defined

$$\tilde{z}_{1s\beta}(\zeta) = K(\zeta/\beta_0, \rho_0) + L(\zeta/\beta_0, \rho_0) - \frac{1}{2} + \chi L(\zeta\beta_0, \rho_0), \tag{47a}$$

$$\tilde{z}_{1sa}(\zeta) = K(\zeta/\beta_0, \rho_0) - \frac{1}{2}. \tag{47b}$$

This expression for $z_{1\mathrm{s}}(\zeta)$ is consistent with the required behavior (41) near $\zeta = \beta_0$. Note that $z_{1s}(\zeta)$ has two terms, one proportional to β_1/β_0 and another proportional to a_1/a_0.

The conformal map $z(\zeta)$ has to obey the same symmetries as the incompressible map $z_0(\zeta)$, i.e.,

$$z(\rho/\zeta) = -z(\zeta) \quad \text{for} \quad \rho \le |\zeta| \le 1. \tag{48}$$

Using (15) and (38) we find

$$z(\rho/\zeta) = -z_0(\zeta) + M^2\left[z_1(\rho_0/\zeta) + \frac{\rho_1}{\rho_0}\zeta z_0'(\zeta)\right] + O(M^4) \tag{49a}$$

so that the symmetry condition for $z_1(\zeta)$ is

$$z_1(\rho_0/\zeta) + \frac{\rho_1}{\rho_0}\zeta z_0'(\zeta) = -z_1(\zeta). \tag{50}$$

This equation can be rewritten as

$$z_{1\mathrm{r}}(\zeta) + z_{1\mathrm{r}}(\rho_0/\zeta) = 0. \tag{51}$$

4.2. Imai-Lamla integral

The first step in solving the weakly compressible problem is to evaluate the Imai–Lamla integral (17b) in which $\xi_0(\zeta)$ is given by (25b) and $z_0'(\zeta)$ is given by (31). Taking derivatives of (17b) and multiplying by ζ on both sides, we get

$$\zeta\, I'(\zeta) = [\xi_0(\zeta)]^2\,\zeta\, z_0'(\zeta) = [\xi_0(\zeta)]^2\, Q_{z_0}(\zeta) = \xi_0(\zeta)\, Q_{f_0}(\zeta), \tag{52}$$

where we have defined the functions

$$Q_{z_0}(\zeta) = \zeta z_0'(\zeta), \qquad Q_{f_0}(\zeta) = \zeta f_0'(\zeta). \tag{53}$$

Differentiating (25a) and applying the property (106) leads to

$$Q_{f_0}(\zeta) = -\overline{Q_{f_0}(1/\overline{\zeta})}. \tag{54}$$

The Imai–Lamla integral can be evaluated using a combination of properties (54) and the analytic continuation of (24) inside the annulus $\rho_0 < |\zeta| < 1$. We have using (19)

$$Q_{f_0}(\zeta) = -\left[\frac{U_0^2/\chi}{\xi_0(\zeta)\,\overline{\xi_0(1/\overline{\zeta})}}\right]\overline{Q_{f_0}(1/\overline{\zeta})} = -\left[\frac{U_0^2/\chi}{\xi_0(\zeta)}\right]\overline{Q_{z_0}(1/\overline{\zeta})}, \tag{55}$$

implying

$$\xi_0(\zeta)Q_{f_0}(\zeta) = -(U_0^2/\chi)\,\overline{Q_{z_0}(1/\overline{\zeta})}. \tag{56}$$

Now it is clear from (31) that $\overline{Q_{z_0}(1/\overline{\zeta})} = (1/\zeta)\, z_0'(1/\zeta)$. Therefore we get

$$\zeta\, I'(\zeta) = -\left(\frac{U_0^2}{\chi}\right)\left(\frac{1}{\zeta}\right) z_0'(1/\zeta). \tag{57}$$

Integrating, we obtain

$$I(\zeta) = \left(\frac{U_0^2}{\chi}\right) z_0(1/\zeta) + \text{constant}, \tag{58}$$

where the constant is found to be

$$-\frac{aU_0^2}{2\beta_0}\left(1+\frac{1}{\chi}\right) \tag{59}$$

by imposing the symmetry property $I(\rho_0/\zeta) = -I(\zeta)$. The Imai-Lamla integral thus evaluates to

$$I(\zeta) = -\frac{aU_0^2}{\beta_0}\left[K(\zeta/\beta_0,\rho_0) + \frac{1}{\chi}\, K(\zeta\beta_0,\rho_0) - \frac{1}{2}\right]. \tag{60}$$

4.3. Streamline boundary condition

The two unknown analytic functions $G(\zeta)$ and $z_1(\zeta)$ are found by deriving two boundary value problems from the two boundary conditions on the hollow vortex boundaries. In this subsection we describe how to set up the boundary value problem for $G(\zeta)$.

We first need to ensure that the velocity correction $\xi_1(\zeta,\overline{\zeta})$ has the correct required behavior (34a) at infinity, remembering that $\zeta = \beta$ maps to infinity under the perturbed map. We decompose $G(\zeta)$ as

$$G(\zeta) = \frac{1}{4U_0^2}\xi_0(\zeta)I(\zeta) + G_{\rm s}(\zeta) + G_{\rm r}(\zeta), \tag{61}$$

where $G_{\rm s}(\zeta)$ is the singular part of $G(\zeta)$, and $G_{\rm r}(\zeta)$ is a regular function with a Taylor series throughout the domain. We find the following behavior as $\zeta \to \beta_0$:

$$z_0'(\zeta) = -\frac{a_0}{(\zeta-\beta_0)^2} + \text{Taylor series}, \tag{62a}$$

$$\frac{1}{z_0'(\zeta)} = -\frac{(\zeta-\beta_0)^2}{a_0} + O\left((\zeta-\beta_0)^4\right), \tag{62b}$$

$$I(\zeta) = -\frac{a_0U_0^2}{\zeta-\beta_0} + \text{Taylor series}, \tag{62c}$$

$$\frac{G'(\zeta)}{z_0'(\zeta)} = -\frac{\mathrm{i}U_0}{4} + \frac{G_{\rm s}'(\zeta)}{z_0'(\zeta)}. \tag{62d}$$

Substituting (61) and (62) into (18), we get

$$\left.\xi_1(\zeta,\overline{\zeta})\right|_{\zeta=\beta_0} = \frac{G_{\rm s}'(\zeta)}{z_0'(\zeta)} - \xi_0(\zeta)\frac{z_{1\rm s}'(\zeta)}{z_0'(\zeta)}. \tag{63}$$

We choose $G_{\rm s}(\zeta)$ such that the velocity field $\xi_1(\zeta,\overline{\zeta})$ is well defined at $\zeta = \beta$, with

$$\left.\xi(\zeta,\overline{\zeta})\right|_{\zeta=\beta} = \mathrm{i}U, \tag{64}$$

where U is given by (34a). With the choice

$$G_{\rm s}(\zeta) = \xi_0(\zeta)z_{1\rm s}(\zeta) + \frac{U_1}{U_0}\left[f_0(\zeta) + \frac{\mathrm{i}\Gamma}{2\pi}\log\zeta\right] \tag{65}$$

we get

$$\frac{G_{\rm s}'(\zeta)}{z_0'(\zeta)} - \xi_0(\zeta)\frac{z_{1\rm s}'(\zeta)}{z_0'(\zeta)} = \xi_0'(\zeta)\frac{z_{1\rm s}(\zeta)}{z_0'(\zeta)} + \frac{U_1}{U_0}\left[\frac{f_0'(\zeta)}{z_0'(\zeta)} + \frac{\mathrm{i}\Gamma}{2\pi\zeta}\frac{1}{z_0'(\zeta)}\right]. \tag{66}$$

Using (66), (41) for $z_{1s}(\zeta)$, and (62) we find

$$\xi_1(\zeta,\overline{\zeta})\Big|_{\zeta=\beta_0} = \mathrm{i}U_1 - \beta_1\xi_0'(\beta_0) + O(M^4). \tag{67}$$

But since

$$\xi(\zeta,\overline{\zeta})\Big|_{\zeta=\beta} = \xi_0(\beta_0) + M^2\beta_1\xi_0'(\beta_0) + M^2\xi_1(\zeta,\overline{\zeta})\Big|_{\zeta=\beta_0} + O(M^4), \tag{68}$$

we see that (64) is satisfied.

We are now in a position to formulate a boundary value problem for the regular function $G_\mathrm{r}(\zeta)$. To do this, we have to find the boundary conditions on $f_1(\zeta,\overline{\zeta})$ first. Expanding the left hand side of (35) using (12a), and substituting (36) on the right hand side, (35) becomes

$$\mathrm{Im}[f_0(\zeta)] + M^2\,\mathrm{Im}[f_1(\zeta,\overline{\zeta})] + O(M^4) = \begin{cases} C_0 + M^2C_1 + O(M^4) & \text{on } |\zeta| = 1, \\ -C_0 - M^2C_1 + O(M^4) & \text{on } |\zeta| = \rho. \end{cases} \tag{69}$$

Using (23) and the expansion

$$\mathrm{Im}[f_0(\zeta)]\Big|_{|\zeta|=\rho} = \mathrm{Im}[f_0(\zeta)]\Big|_{|\zeta|=\rho_0} + M^2\frac{\rho_1}{\rho_0}\,\mathrm{Im}[\zeta f_0'(\zeta)]\Big|_{|\zeta|=\rho_0} + O(M^4), \tag{70}$$

we find the boundary condition for $f_1(\zeta,\overline{\zeta})$ from (69):

$$\mathrm{Im}[f_1(\zeta,\overline{\zeta})] = \begin{cases} C_1 & \text{on} \quad |\zeta| = 1, \\ -C_1 - \dfrac{\rho_1}{\rho_0}\,\mathrm{Im}[\zeta f_0'(\zeta)] & \text{on} \quad |\zeta| = \rho_0. \end{cases} \tag{71}$$

Since the correction ρ_1 is an unknown at this stage, we have rewritten the boundary condition in terms of ρ_0 by noting that the point of evaluation of an $O(M^2)$ term can be safely switched from ρ to ρ_0. We will use this observation throughout what follows. Substituting (61) and (65) into (17a) and using this in (71), we get the boundary conditions for $G_\mathrm{r}(\zeta)$:

$$\mathrm{Re}[\mathrm{i}G_\mathrm{r}(\zeta)] = \begin{cases} \widetilde{C}_1 + S(\zeta,\overline{\zeta}) + \frac{\beta_1}{\beta_0}S_\beta(\zeta,\overline{\zeta}) + \frac{a_1}{a_0}S_a(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = 1, \\ \widetilde{C}_2 + S(\zeta,\overline{\zeta}) + \frac{\beta_1}{\beta_0}S_\beta(\zeta,\overline{\zeta}) + \frac{a_1}{a_0}S_a(\zeta,\overline{\zeta}) + \frac{\rho_1}{\rho_0}S_\rho(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = \rho_0. \end{cases} \tag{72}$$

Here we define the functions

$$S(\zeta,\overline{\zeta}) = \frac{1}{2U_0^2}\mathrm{Im}[\xi_0(\zeta)]\mathrm{Re}[I(\zeta)], \tag{73a}$$

$$S_\beta(\zeta,\overline{\zeta}) = \mathrm{Im}[-\frac{a_0}{\beta_0}\xi_0(\zeta)\tilde{z}_{1s\beta}(\zeta)], \tag{73b}$$

$$S_a(\zeta,\overline{\zeta}) = \mathrm{Im}[\frac{a_0}{\beta_0}\tilde{z}_{1sa}(\zeta)], \tag{73c}$$

$$S_\rho(\zeta,\overline{\zeta}) = \mathrm{Im}[\zeta f_0'(\zeta)], \tag{73d}$$

and note that these are known on both boundaries. The constants $\widetilde{C}_1$ and $\widetilde{C}_2$ are

$$\widetilde{C}_1 = -C_1 + \frac{U_1}{U_0}C_0 \quad \text{and} \quad \widetilde{C}_2 = C_1 - \frac{U_1}{U_0}C_0 + \frac{U_1}{U_0}\frac{\Gamma}{2\pi}\log\rho_0. \tag{74}$$

Given the real part of a regular analytic function on the boundaries of an annulus, we can find the function within the annulus by solving a "Villat problem". The appendix in Crowdy and Krishnamurthy [4] provides details of this solution procedure. We need to satisfy the Villat consistency condition, which states that the constant terms in the given data on the two boundaries must match, in order to obtain well-defined solutions in the annulus. The boundary condition (72) is not yet a boundary value problem for $G_\mathrm{r}(\zeta)$ because of the unknown parameters β_1 and ρ_1. To remedy this, we now split (72) into four parts:

$$\mathrm{Re}[\mathrm{i}G_{\mathrm{r}S}(\zeta)] = \begin{cases} S(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = 1, \\ S(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = \rho_0. \end{cases} \tag{75a}$$

$$\mathrm{Re}[\mathrm{i}G_{\mathrm{r}\beta}(\zeta)] = \begin{cases} S_\beta(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = 1, \\ S_\beta(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = \rho_0. \end{cases} \tag{75b}$$

$$\mathrm{Re}[\mathrm{i}G_{\mathrm{r}a}(\zeta)] = \begin{cases} S_a(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = 1, \\ S_a(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = \rho_0. \end{cases} \tag{75c}$$

$$\mathrm{Re}[\mathrm{i}G_{\mathrm{r}\rho}(\zeta)] = \begin{cases} 0 & \text{on} \quad |\zeta| = 1, \\ S_\rho(\zeta,\overline{\zeta}) & \text{on} \quad |\zeta| = \rho_0. \end{cases} \tag{75d}$$

The four Villat problems (75) can be solved for the four functions $G_{\mathrm{r}S}(\zeta)$, $G_{\mathrm{r}\beta}(\zeta)$, $G_{\mathrm{r}a}(\zeta)$ and $G_{\mathrm{r}\rho}(\zeta)$. The final solution is then given by

$$G_\mathrm{r}(\zeta) = G_{\mathrm{r}S}(\zeta) + \frac{\beta_1}{\beta_0}G_{\mathrm{r}\beta}(\zeta) + \frac{a_1}{a_0}G_{\mathrm{r}a}(\zeta) + \frac{\rho_1}{\rho_0}G_{\mathrm{r}\rho}(\zeta), \tag{76}$$

which is known once the parameters β_1, a_1, and ρ_1 are determined. Note that $G_{\mathrm{r}S}(\zeta)$, $G_{\mathrm{r}\beta}(\zeta)$, $G_{\mathrm{r}a}(\zeta)$ and $G_{\mathrm{r}\rho}(\zeta)$ do not need to satisfy the Villat consistency condition individually, since we only need the final solution $G_\mathrm{r}(\zeta)$ to do so. From (72) and (74) it is clear that once β_1, a_1, and ρ_1

are known we can equate the constants in the data on the two boundaries to obtain a single equation for the single unknown C_1, and the Villat consistency condition can be satisfied this way. Since C_1 is a constant in the complex potential, its value does not alter the physical solution and we make no further mention of it.

4.4. Bernoulli condition

We now turn to formulating a boundary value problem for $z_{1\mathrm{r}}(\zeta)$. Since we have already solved the boundary value problem in Section 4.3, the function $G_{\mathrm{r}}(\zeta)$ is known at this stage, although the parameters are still unknown. The condition (37) becomes

$$\begin{aligned}|\xi(\zeta)|^2 &= \xi_0(\zeta)\,\overline{\xi_0(\zeta)} + 2M^2\mathrm{Re}[\xi_1(\zeta)\,\overline{\xi_0(\zeta)}] + O(M^4)\\ &= q_0^2 + 2M^2 q_0 q_1 + O(M^4),\end{aligned} \tag{77}$$

on $|\zeta| = 1$ and $|\zeta| = \rho$, where we have used (24) and (34b).

We first have to take into account the change from ρ_0 to ρ due to compressibility. The expansion

$$\xi_0(\zeta)\Big|_{|\zeta|=\rho} = \xi_0(\zeta)\Big|_{|\zeta|=\rho_0} + M^2\frac{\rho_1}{\rho_0}\left[\zeta\xi_0'(\zeta)\right]\Big|_{|\zeta|=\rho_0} + O(M^4), \tag{78}$$

together with (24), leads to

$$\frac{|\xi_0(\zeta)|^2}{q_0^2} = \frac{|\xi_0(\zeta)|^2}{q_0^2}\begin{cases}1 & \text{on } |\zeta| = 1,\\ 1 + M^2\frac{2\rho_1}{\rho_0}\mathrm{Re}\left[\frac{\zeta\xi_0'(\zeta)}{\xi_0(\zeta)}\right] + O(M^4) & \text{on } |\zeta| = \rho_0.\end{cases} \tag{79}$$

The boundary condition (77) along with (79) and (12b) gives a condition on $\overline{\xi_0(\zeta)}\,\xi_1(\zeta,\overline{\zeta})$ on the boundaries:

$$\mathrm{Re}\left[\frac{\overline{\xi_0(\zeta)}\,\xi_1(\zeta,\overline{\zeta})}{q_0^2}\right] = \begin{cases}\frac{q_1}{q_0} & \text{on } |\zeta| = 1,\\ \frac{q_1}{q_0} - \frac{\rho_1}{\rho_0}\mathrm{Re}\left[\frac{\zeta\xi_0'(\zeta)}{\xi_0(\zeta)}\right] & \text{on } |\zeta| = \rho_0.\end{cases} \tag{80}$$

Since the right hand side of (80) is known and the only unknown on the left hand side is $z_{1\mathrm{r}}(\zeta)$ in $\xi_1(\zeta,\overline{\zeta})$, we can rewrite (80) as a boundary value problem for $g(\zeta) = z_{1\mathrm{r}}'(\zeta)/z_0'(\zeta)$. Substituting (45) and (61) into (18), then using (60), (46) and (66), gives, after some algebra,

$$\mathrm{Re}[g(\zeta)] = \begin{cases}T(\zeta,\overline{\zeta}) + \dfrac{\beta_1}{\beta_0}T_\beta(\zeta,\overline{\zeta}) + \dfrac{a_1}{a_0}T_a(\zeta,\overline{\zeta}) + \dfrac{U_1}{U_0}T_U(\zeta,\overline{\zeta}) + \dfrac{\rho_1}{\rho_0}T_\rho(\zeta,\overline{\zeta}) - \dfrac{q_1}{q_0} & \text{on } |\zeta| = 1,\\ T(\zeta,\overline{\zeta}) + \dfrac{\beta_1}{\beta_0}T_\beta(\zeta,\overline{\zeta}) + \dfrac{a_1}{a_0}T_a(\zeta,\overline{\zeta}) + \dfrac{U_1}{U_0}T_U(\zeta,\overline{\zeta}) & \\ \qquad + \dfrac{\rho_1}{\rho_0}\left(T_\rho(\zeta,\overline{\zeta}) + \mathrm{Re}\left[\dfrac{\zeta\xi_0'(\zeta)}{\xi_0(\zeta)}\right]\right) - \dfrac{q_1}{q_0} & \text{on } |\zeta| = \rho_0.\end{cases} \tag{81}$$

The various functions here are defined below:

$$T(\zeta,\overline{\zeta}) = \mathrm{Re}\left[\frac{1}{4\chi} + \frac{1}{4U_0^2}\left(\frac{\xi_0'(\zeta)}{f_0'(\zeta)}(I(\zeta) + \overline{I(\zeta)}) + (\xi_0(\zeta))^2\right) + \frac{G_{\mathrm{r}S}'(\zeta)}{f_0'(\zeta)}\right], \tag{82a}$$

$$T_\beta(\zeta,\overline{\zeta}) = \mathrm{Re}\left[-\frac{a_0}{\beta_0}\frac{\xi_0'(\zeta)}{f_0'(\zeta)}\tilde{z}_{1s\beta}(\zeta) + \frac{G_{\mathrm{r}\beta}'(\zeta)}{f_0'(\zeta)}\right], \tag{82b}$$

$$T_a(\zeta,\overline{\zeta}) = \mathrm{Re}\left[\frac{a_0}{\beta_0}\frac{\xi_0'(\zeta)}{f_0'(\zeta)}\tilde{z}_{1sa(\zeta)} + \frac{G_{\mathrm{r}a}'(\zeta)}{f_0'(\zeta)}\right], \tag{82c}$$

$$T_\rho(\zeta,\overline{\zeta}) = \mathrm{Re}\left[\frac{G_{\mathrm{r}\rho}'(\zeta)}{f_0'(\zeta)}\right], \tag{82d}$$

$$T_U(\zeta,\overline{\zeta}) = 1 + \mathrm{Re}\left[\frac{\mathrm{i}\Gamma}{2\pi\zeta f_0'(\zeta)}\right]. \tag{82e}$$

The parameters β_1, ρ_1, a_1, U_1 and q_1 are unknowns at this stage and as in Section 4.3 we split (81) into five separate Villat problems. These five problems can be solved individually since their boundary data is completely known.

$$\mathrm{Re}[g_T(\zeta)] = \begin{cases}T(\zeta,\overline{\zeta}) & \text{on } |\zeta| = 1,\\ T(\zeta,\overline{\zeta}) & \text{on } |\zeta| = \rho_0,\end{cases} \tag{83a}$$

$$\mathrm{Re}[g_\beta(\zeta)] = \begin{cases}T_\beta(\zeta,\overline{\zeta}) & \text{on } |\zeta| = 1,\\ T_\beta(\zeta,\overline{\zeta}) & \text{on } |\zeta| = \rho_0,\end{cases} \tag{83b}$$

$$\mathrm{Re}[g_a(\zeta)] = \begin{cases}T_a(\zeta,\overline{\zeta}) & \text{on } |\zeta| = 1,\\ T_a(\zeta,\overline{\zeta}) & \text{on } |\zeta| = \rho_0,\end{cases} \tag{83c}$$

$$\mathrm{Re}[g_\rho(\zeta)] = \begin{cases}T_\rho(\zeta,\overline{\zeta}) & \text{on } |\zeta| = 1,\\ T_\rho(\zeta,\overline{\zeta}) + \mathrm{Re}\left[\frac{\zeta\xi_0'(\zeta)}{\xi_0(\zeta)}\right] & \text{on } |\zeta| = \rho_0,\end{cases} \tag{83d}$$

$$\mathrm{Re}[g_U(\zeta)] = \begin{cases} T_U(\zeta,\overline{\zeta}) & \text{on } |\zeta| = 1, \\ T_U(\zeta,\overline{\zeta}) & \text{on } |\zeta| = \rho_0. \end{cases} \tag{83e}$$

The solution to a Villat problem is unique up to an imaginary constant. We will set this constant by requiring that Im $\oint g_T(\zeta)|\mathrm{d}\zeta| = 0$ on the boundaries (and similarly for $g_\beta(\zeta)$, $g_a(\zeta)$, $g_\rho(\zeta)$ and $g_U(\zeta)$). Therefore we set the constants C's to be real. The final solution is then given by $z'_{1\mathrm{r}}(\zeta) = g(\zeta)z'_0(\zeta)$, where $g(\zeta)$ is related to the solutions of (83) via

$$g(\zeta) = \left(g_T(\zeta) + \frac{\beta_1}{\beta_0} g_\beta(\zeta) + \frac{a_1}{a_0} g_a(\zeta) + \frac{\rho_1}{\rho_0} g_\rho(\zeta) + \frac{U_1}{U_0} g_U(\zeta) - \frac{q_1}{q_0} \right). \tag{84}$$

Once the Laurent series for $g(\zeta)$ has been found, the Laurent series for $z_{1r}(\zeta)$ can be found by integrating $z'_{1r}(\zeta)$.

4.5. Finding the parameters

Completing the solutions to the weakly compressible problem requires us to find the five unknown parameters, namely β_1/β_0, a_1/a_0, ρ_1/ρ_0, U_1/U_0, and q_1/q_0. We impose five conditions on the solutions in order to find the parameters. These five parameters appear as linear coefficients in all the conditions, which means that we solve a linear algebra problem to obtain them at the end. First, the solution to (81) has to satisfy the Villat consistency condition, although the individual boundary value problems in (83) do not need to do so. Let the constants in the boundary data (82) on $|\zeta| = 1$ be $C_T^{(1)}$, $C_\beta^{(1)}$, $C_a^{(1)}$, $C_\rho^{(1)}$, $C_U^{(1)}$, and the constants on $|\zeta| = \rho_0$ be $C_T^{(\rho_0)}$, $C_\beta^{(\rho_0)}$, $C_a^{(\rho_0)}$, $C_\rho^{(\rho_0)}$, $C_U^{(\rho_0)}$, respectively. The Villat consistency condition for (81) is then

$$C_T^{(1)} + \frac{\beta_1}{\beta_0} C_\beta^{(1)} + \frac{a_1}{a_0} C_a^{(1)} + \frac{\rho_1}{\rho_0} C_\rho^{(1)} + \frac{U_1}{U_0} C_U^{(1)} = C_T^{(\rho_0)} + \frac{\beta_1}{\beta_0} C_\beta^{(\rho_0)} + \frac{a_1}{a_0} C_a^{(\rho_0)} + \frac{\rho_1}{\rho_0} C_\rho^{(\rho_0)} + \frac{U_1}{U_0} C_U^{(\rho_0)}. \tag{85}$$

The second condition is that the correction to the conformal map is finite at $\zeta = \beta_0$, so that $g(\beta_0) = 0$, since $z'_0(\zeta)$ has a singularity at $\zeta = \beta_0$. We next require that the boundaries of the vortices, which are the images of the circles $|\zeta| = 1$ and $|\zeta| = \rho_0$, be closed curves. This means that the Laurent series of $z'_{1\mathrm{r}}(\zeta) = g(\zeta)z'_0(\zeta)$ cannot contain any ζ^{-1} terms. This is a separate condition because we have so far only ensured that $z'_{1r}(\zeta)/z'_0(\zeta)$ as well as $z'_{1r}(\zeta)$ are analytic in the annulus. By symmetry, this condition is the same on both boundaries. Note that the conditions so far do not involve q_1/q_0.

We impose two further conditions on the solutions. The circulation is automatically conserved at $O(M^2)$, as shown in Appendix C, so this is not an additional constraint. The first condition is to keep the area $\mathcal{A}$ of the vortices fixed at its incompressible value $\mathcal{A}_0$. The second condition is to keep the vortex centroid C fixed at its incompressible value C_0. We explain these choices in detail below.

The area of the two vortices can be computed from

$$\mathcal{A} = \pm\frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} \overline{z(\zeta)} z'(\zeta)\,\mathrm{d}\zeta = \mp\frac{1}{2\mathrm{i}} \oint_{|\zeta|=\rho} \overline{z(\zeta)} z'(\zeta)\,\mathrm{d}\zeta \tag{86}$$

The choice of sign depends on the direction of integration along the contour, and the areas are equal. We will work along the outer contour, in which case the minus sign is appropriate. Then

$$\mathcal{A}_0 = \frac{\mathrm{i}}{2} \oint_{|\zeta|=1} \overline{z_0(\zeta)} z'_0(\zeta)\,\mathrm{d}\zeta, \qquad \mathcal{A}_1 = \frac{\mathrm{i}}{2} \oint_{|\zeta|=1} [\overline{z_1(\zeta)} z'_0(\zeta) + \overline{z_0(\zeta)} z'_1(\zeta)]\,\mathrm{d}\zeta, \tag{87}$$

where $\mathcal{A}_0$ and $\mathcal{A}_1$ are the areas in the Rayleigh–Jansen expansion $\mathcal{A} = \mathcal{A}_0 + M^2\mathcal{A}_1 + O(M^4)$. Expanding $z_1(\zeta)$ as in (45), using (84), inserting into (87), and setting $\mathcal{A}_1 = 0$ gives a fourth linear relation between the unknown parameters.

The first vortex centroid we define is the geometric centroid

$$C_\zeta^{(1)} = \frac{\oint_{|\zeta|=1} z(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}}{\oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}} = \frac{1}{2\pi} \oint_{|\zeta|=1} z(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}. \tag{88}$$

The superscript in $C_\zeta^{(1)}$ refers to the vortex boundary corresponding to $|\zeta| = 1$. This centroid definition can be interpreted either as the ratio of the z-weighted perimeter of the vortex in the ζ-plane (since $|\mathrm{d}\zeta| = \mathrm{d}\zeta/\mathrm{i}\zeta$) to the perimeter in the ζ-plane (which is 2π) or as the ratio of the z-weighted area in the ζ-plane to the area in the ζ-plane, since the latter would be the integral of $\overline{\zeta} = \zeta^{-1}$. So it can be viewed as either the area- or arclength-weighted centroid in the ζ-plane. One can show from (30) and (102) that in the incompressible case

$$C_\zeta^{(1)} = \frac{1}{2\pi} \oint_{|\zeta|=1} z_0(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{89a}$$

$$= \frac{a_0}{\beta_0} \left[\frac{1}{2\pi} \oint_{|\zeta|=1} \left[\mathrm{d}\log P(\zeta/\beta_0) + \chi\,\mathrm{d}\log P(\zeta\beta_0)\right] - \frac{1}{2} \right] \tag{89b}$$

$$= \frac{a_0}{\beta_0} \left[1 + 0 - \frac{1}{2} \right] = \frac{a_0}{2\beta_0}; \tag{89c}$$

similarly $C_\zeta^{(\rho_0)} = -a_0/2\beta_0$. Let us set the length and time scale by fixing $\Gamma = 1$ and $a_0 = \beta_0$. Setting $a_0 = \beta_0$ fixes the distance between the geometric centroids of the vortices to be 1.

We consider two other definitions of the vortex centroid. The arclength-weighted centroid in the z-plane is defined as

$$C_z^{(1)} = \frac{\oint_{|\zeta|=1} z(\zeta)|z'(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}}{\oint_{|\zeta|=1} |z'(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}}. \tag{90}$$

This is the definition used in Crowdy and Krishnamurthy [4], and corresponds to weighting by the vorticity, i.e., it defines the center of vorticity. A final possibility is the area-weighted centroid in the z-plane defined by

$$C_{\mathcal{A}}^{(1)} = \frac{\oint_{|\zeta|=1} z(\zeta)\overline{z(\zeta)} z'(\zeta)\,\mathrm{d}\zeta}{\oint_{|\zeta|=1} \overline{z(\zeta)} z'(\zeta)\,\mathrm{d}\zeta}. \tag{91}$$

Table 1
The minima of the compressible corrections to the speeds U_1/U_0 and q_1/q_0 as functions of ρ_0 and $\mathcal{A}_0$, for the three choices of vortex centroid.

	$\min U_1/U_0$	q_1/q_0	ρ_0	$\mathcal{A}_0$	$\min q_1/q_0$	U_1/U_0	ρ_0	$\mathcal{A}_0$
ζ-centroid	−0.4420	−0.3252	0.2781	1.6854	−0.5516	−0.3254	0.2935	1.8568
area-centroid	−0.5332	−0.2808	0.2370	1.2850	−0.2809	−0.5331	0.2310	1.2328
z-centroid	−0.5491	−0.2745	0.2268	1.1974	−0.2745	−0.5491	0.2268	1.1974

Following Crowdy and Krishnamurthy [4] we expand the expressions for the centroids by substituting (15) and (45), obtaining

$$C_0 = \frac{N_0}{N_2}, \qquad C_1 = \frac{N_1}{N_2} - \frac{N_0 N_3}{N_2^2} \tag{92}$$

as the general expression. We denote the centroid expansions as $C_\zeta^{(1)} = C_{\zeta,0}^{(1)} + M^2 C_{\zeta,1}^{(1)} + O(M^4)$, and so on. The quantities N_1 and N_3 can be decomposed into terms proportional to the unknowns, and the result will be a linear equation when setting $C_1 = 0$. For the ζ-centroid we get,

$$N_0 = \oint_{|\zeta|=1} z_0(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} = \frac{1}{2}, \qquad N_1 = \oint_{|\zeta|=1} z_1(\zeta)\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}, \qquad N_2 = 2\pi, \qquad N_3 = 0, \tag{93}$$

for the z-centroid

$$N_0 = \oint_{|\zeta|=1} z_0(\zeta)|z_0'(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}, \qquad N_1 = \oint_{|\zeta|=1} \left\{ z_1(\zeta) + z_0(\zeta)\mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right]\right\} |z_0'(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}, \tag{94a}$$

$$N_2 = \oint_{|\zeta|=1} |z_0'(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}, \qquad N_3 = \oint_{|\zeta|=1} \mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] |z_0'(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}, \tag{94b}$$

while for the $\mathcal{A}$-centroid

$$N_0 = \oint_{|\zeta|=1} z_0(\zeta)\overline{z_0(\zeta)}z_0'(\zeta)\,\mathrm{d}\zeta, \qquad N_2 = -2\mathrm{i}\mathcal{A}_0 \tag{95a}$$

$$N_1 = \oint_{|\zeta|=1} [z_1(\zeta)\overline{z_0(\zeta)}z_0'(\zeta) + z_0(\zeta)\overline{z_1(\zeta)}z_0'(\zeta) + z_0(\zeta)\overline{z_0(\zeta)}z_1'(\zeta)]\,\mathrm{d}\zeta, \qquad N_3 = -2\mathrm{i}\mathcal{A}_1. \tag{95b}$$

In order to set the $O(M^2)$ correction in any of the centroid positions to zero, we set the corresponding $C_1 = 0$ in (92). Substituting for N_0, N_1, N_2 and N_3 from the corresponding equations above, we get a linear relation between the five unknown parameters, providing us with the required fifth condition. Note that all the centroids and their corrections are real-valued due to the symmetries in the problem.

5. Results

We first show that some of the differences in the Villat consistency condition (85) vanish. These quantities are real, so that

$$C_U^{(1)} = \oint_{|\zeta|=1} g_U\,\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} = \oint_{|\zeta|=1}\left(1 + \mathrm{Re}\left[\frac{\mathrm{i}\Gamma}{2\pi\zeta f_0'(\zeta)}\right]\right)\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \tag{96a}$$

$$= \oint_{|\zeta|=1}\left(1 + \mathrm{Re}\left[\frac{\mathrm{i}\Gamma}{2\pi(\rho_0/\zeta)f_0'(\rho_0/\zeta)}\right]\right)\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \tag{96b}$$

$$= -\oint_{|\eta|=\rho_0}\left(1 + \mathrm{Re}\left[\frac{\mathrm{i}\Gamma}{2\pi\eta f_0'(\eta)}\right]\right)\frac{\mathrm{d}\eta}{(-2\pi\mathrm{i}\eta)} = C_U^{(\rho_0)}, \tag{96c}$$

using the derivative of the symmetry property in (20) in the form $f_0'(\zeta) = (\rho_0/\zeta^2)f_0'(\rho_0/\zeta)$. The underlying symmetry property is shared by $T(\zeta,\overline{\zeta})$. Hence $C_T^{(1)} - C_T^{(\rho_0)} = C_U^{(1)} - C_U^{(\rho_0)} = 0$. The consistency condition (85) simplifies to

$$\frac{\beta_1}{\beta_0}[C_\beta^{(1)} - C_\beta^{(\rho_0)}] + \frac{\rho_1}{\rho_0}[C_\rho^{(1)} - C_\rho^{(\rho_0)}] = 0, \tag{97}$$

which can be shown numerically to be the same as the symmetry relation (39). Together with the finiteness condition and the single-valuedness condition this gives three equations.

Fig. 2 shows a plot of the incompressible area $\mathcal{A}_0$ and the incompressible perimeter $\mathcal{P}_0$ as functions of the conformal mapping parameter ρ_0. These plots show that both $\mathcal{A}_0$ and $\mathcal{P}_0$ are monotonic functions of ρ_0. For the area constraint, the area is the same as the incompressible area (and this could be enforced for all orders in M^2), so we can use $\mathcal{A} = \mathcal{A}_0$ as the abscissa.

Fig. 3 shows the behavior of the first-order corrections to the speed of the hollow vortex pair U_1/U_0 and the speed on the boundaries of the vortices q_1/q_0 as functions of the conformal mapping parameter ρ_0 for the three different choices of the vortex centroid. In all cases, we see that both U_1/U_0 and q_1/q_0 are always strictly negative for $\rho_0 > 0$, showing that, at first order, the compressible hollow vortex pair is always slower than its incompressible counterpart. The behavior is however not monotonic, and both speeds attain minima for certain values of ρ_0. The minimum values of U_1/U_0 and q_1/q_0 and the corresponding values of q_1/q_0 and U_1/U_0 along with the values of ρ_0 and the area $\mathcal{A}_0$ at which these minima are attained are tabulated in Table 1. We find that when the correction to the ζ-centroid is set to zero, the minima occur for quite different values of ρ_0 and $\mathcal{A}_0$. In the case of the area-centroid, the values of ρ_0 and $\mathcal{A}_0$ are quite close together, but not exactly the same. Whereas for the z-centroid, the minima of both U_1/U_0 and q_1/q_0 occur at the same value of ρ_0, to machine precision.

Fig. 4 shows the behavior of U_1/U_0 and q_1/q_0 as functions of the vortex area $\mathcal{A}_0$. It is clear from both Figs. 3 and 4 that the compressible corrections go to zero in the limit of zero vortex area, i.e. point vortices. Leppington [2] showed that the propagation parameter $S = 4\pi U h/\Gamma$ has

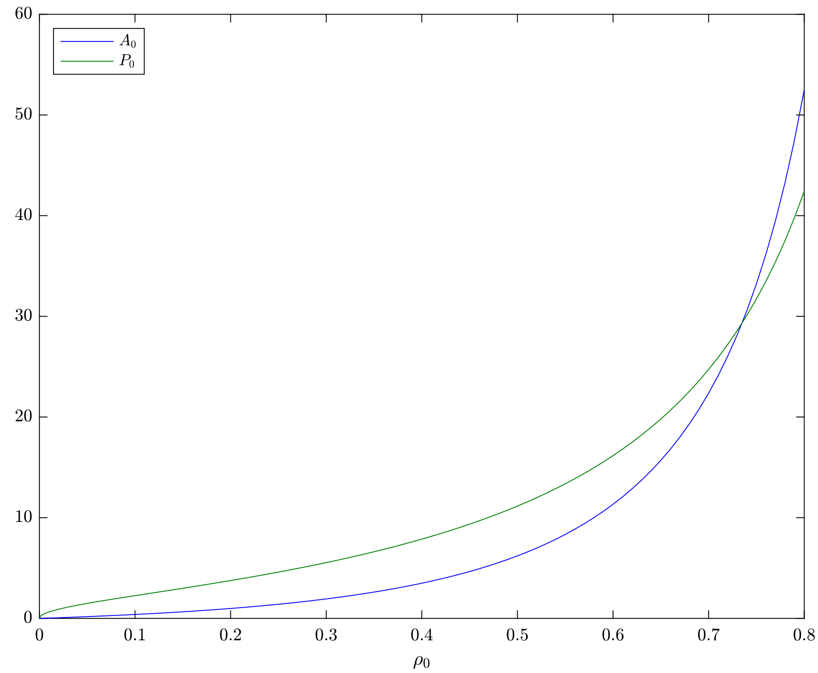

Fig. 2. Incompressible area $\mathcal{A}_0$ and perimeter $\mathcal{P}_0$ as functions of ρ_0.

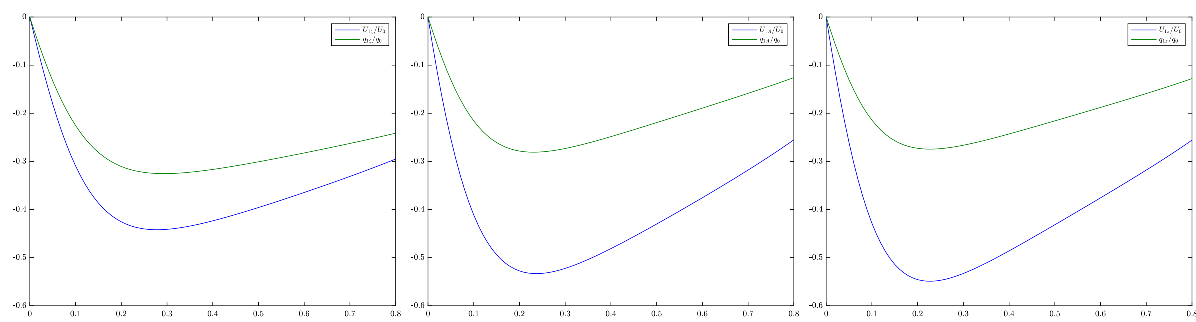

Fig. 3. Relative corrections to propagation velocity, U_1/U_0, and speed on boundary, q_1/q_0, as functions of ρ_0. The subscripts ζ, $\mathcal{A}$ and z denote the centroid positions calculated according to (88), (90) and (91).

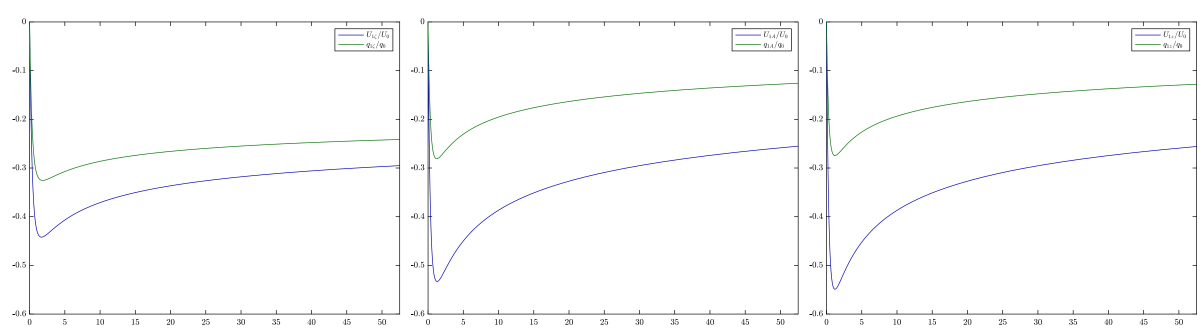

Fig. 4. Relative corrections to propagation velocity, U_1/U_0, and speed on boundary, q_1/q_0, as functions of $\mathcal{A}_0$. The subscripts ζ, $\mathcal{A}$ and z denote the centroid positions calculated according to (88), (90) and (91).

the behavior $S = 1 + o(M^2)$ in the point vortex limit so that it is unchanged at $O(M^2)$. Here $2h$ is the separation between the vortices and we take $2h = |C^{(1)} - C^{(\rho_0)}|$. The precise definition is irrelevant in the point vortex limit as $\rho_0 \to 0$. It is seen from Fig. 5 that the incompressible term tends to 1 in the point vortex limit.

On substituting (34a) and $h = h_0 + M^2 h_1 + O(M^4)$ into the propagation parameter, and setting $S = S_0 + M^2 S_1 + O(M^4)$ where $S_0 = 4\pi U_0 h_0/\Gamma$, we get

$$\frac{S_1}{S_0} = \frac{U_1}{U_0} + \frac{h_1}{h_0} = \frac{U_1}{U_0} + \frac{|C_1^{(1)} - C_1^{(\rho_0)}|}{|C_0^{(1)} - C_0^{(\rho_0)}|} = \frac{U_1}{U_0} + \frac{|C_1^{(1)}|}{|C_0^{(1)}|} = \frac{U_1}{U_0}. \tag{98}$$

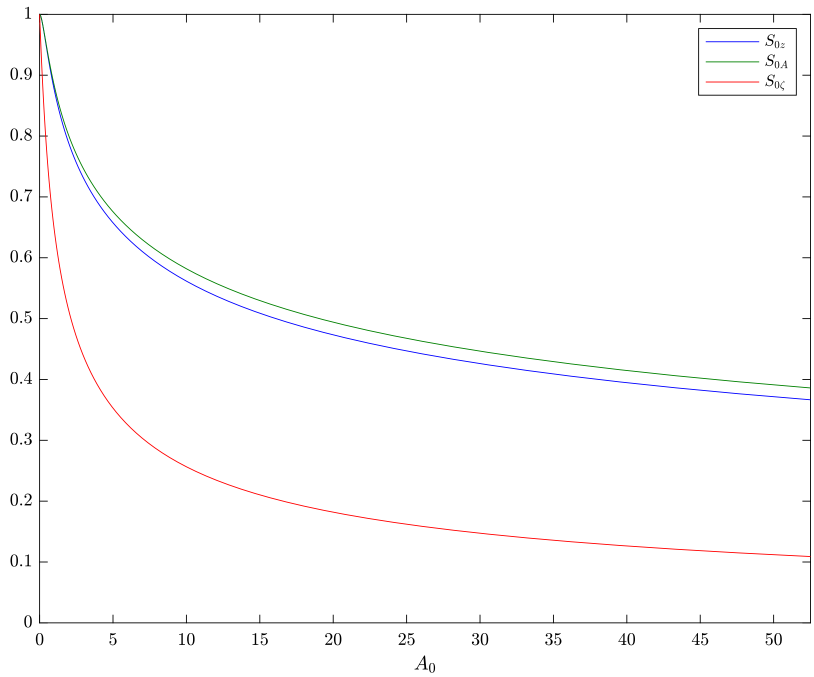

Fig. 5. The propagation parameter, $4\pi U_0 h_0/\Gamma$, using the 3 different centroid definitions.

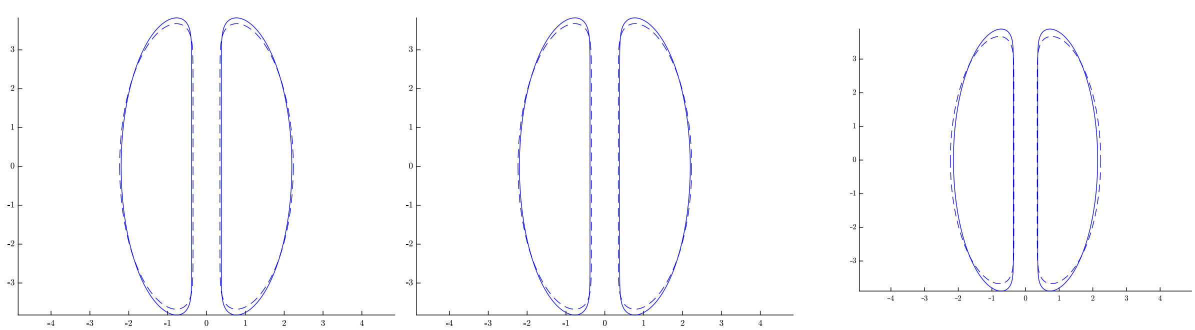

Fig. 6. Shapes of incompressible (dashed) and compressible (solid) pairs. From left to right, centroid positions calculated according to (88), (90) and (91).

The third equality above follows from the symmetry properties of C. We have already seen in Fig. 3 that $U_1/U_0 \to 0$ as $\rho_0 \to 0$. If one were to use a different centroid definition for C_0 and C_1, the last expression would not be valid. However, in the point vortex limit all the centroid definitions converge and we find from Figs. 3 and 4 that the propagation parameter goes to zero in all cases. It is found numerically that the condition continues to be satisfied.

Fig. 6 shows the shape of the incompressible (dashed curves) and compressible (solid curves) vortex pair for $\rho_0 = 0.6$ and $M = 0.4$. The correction to the area is zero so the two shapes have the same area. The compressible pair is slightly more elongated.

6. Summary

We have used a perturbed conformal mapping approach to study the steadily translating weakly compressible hollow vortex pair. The first-order complex potential and conformal map are computed by solving two Villat problems in an annulus. There are five free parameters in the problem which are obtained by imposing various conditions on the solutions. We find that three conditions are necessary for physically meaningful solutions, and impose two further conditions on the change in vortex area and change in the location of the vortex centroid. There are three distinct choices for vortex centroid for any non-zero vortex area and this leads to three sets of solutions for the parameters. The incompressible vortex area can be chosen as an independent variable on which all the other parameters depend.

For small vortex areas, we find that the $O(M^2)$ correction to the speed of the hollow vortex pair tends to zero. This is consistent with the result of Leppington [2] that the correction to the speed of a point vortex pair is zero at $O(M^2)$. For finite vortex areas, we find that the correction is always negative; thus the compressible hollow vortex pair is always slower compared to the incompressible pair. The $O(M^2)$ correction is not monotonic as a function of the area. It displays a minimum whose value depends on the choice of the vortex centroid correction set to zero, although it does display a minimum for each choice of vortex centroid.

Crowdy and Krishnamurthy [4] found that the speed of a hollow vortex street can increase or decrease due to compressibility effects relative to the incompressible speed, depending on the vortex area. The $O(M^2)$ correction to the speed is a monotonic function of the area and increases for areas smaller than a critical value and decreases for larger areas; further the value of the critical area depends on the aspect ratio of the vortices in the street. On the other hand, Crowdy and Krishnamurthy [3] found that the $O(M^2)$ speed of a weakly compressible von Kármán street of point vortices is not monotonic and displays a minimum, but as a function of the aspect ratio. Juxtaposition of the current results with these other recent results shows that the effects of compressibility on any arrangement of vortices may not be predicted a priori, but is a delicate function of the vortex geometry.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Stefan G. Llewellyn Smith reports financial support was provided by University of California, United States San Diego. Co-author serves as guest editor for this special issue.

Data availability

Data will be made available on request.

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Appendix A. Schottky–Klein prime function

In this appendix we present some properties of the Schottky-Klein prime function [12] that have been used extensively. We are concerned with the Schottky-Klein prime function defined on the annulus $\rho_0 \le |\zeta| \le 1$ in a complex ζ-plane and denoted by $P(\zeta,\rho_0)$. The prime function has the infinite product representation

$$P(\zeta,\rho_0) = (1-\zeta)\prod_{k=1}^{\infty}\left(1-\rho_0^{2k}\zeta\right)\left(1-\rho_0^{2k}/\zeta\right). \tag{99}$$

The following properties of the prime function can be verified directly from (99):

$$P(\rho_0^2\zeta,\rho_0) = -\frac{1}{\zeta}P(\zeta,\rho_0), \quad P(1/\zeta,\rho_0) = -\frac{1}{\zeta}P(\zeta,\rho_0), \quad P(\rho_0^2\zeta,\rho_0) = P(1/\zeta,\rho_0). \tag{100}$$

The definition

$$\hat{P}(\zeta,\rho_0) = \prod_{k=1}^{\infty}(1-\rho_0^{2k}\zeta)(1-\rho_0^{2k}/\zeta). \tag{101}$$

is useful in some calculations. Two other functions that are used throughout this paper are the K-function and the L-function, which are given in terms of the derivatives of the prime function:

$$K(\zeta,\rho_0) = \zeta\frac{P'(\zeta,\rho_0)}{P(\zeta,\rho_0)}, \tag{102}$$

$$L(\zeta,\rho_0) = \zeta K'(\zeta,\rho_0). \tag{103}$$

The following properties of the K-function can be derived from (100) and (102):

$$\begin{gathered} K(\rho_0^2\zeta,\rho_0) = K(\zeta,\rho_0)-1, \quad K(1/\zeta,\rho_0) = 1-K(\zeta,\rho_0), \\ K(\rho_0^2\zeta,\rho_0) = -K(1/\zeta,\rho_0). \end{gathered} \tag{104}$$

Using (104) we can check that

$$K(\rho_0,\rho_0) = 0, \quad \mathrm{Re}[K(\zeta,\rho_0)]\Big|_{|\zeta|=\rho_0} = 0, \quad \text{and} \quad K(-1,\rho_0) = +1/2. \tag{105}$$

Using (103) and (104) the L-function can be shown to satisfy

$$L(\rho_0^2\zeta,\rho_0) = L(\zeta,\rho_0), \quad L(1/\zeta,\rho_0) = L(\zeta,\rho_0). \tag{106}$$

Substituting (99) into (102) a series representation can be obtained for the K-function

$$K(\zeta,\rho_0) = -\frac{\zeta}{1-\zeta} - \sum_{k=1}^{\infty}\frac{-\rho_0^{2k}\zeta}{1-\rho_0^{2k}\zeta} + \sum_{k=1}^{\infty}\frac{\rho_0^{2k}/\zeta}{1-\rho_0^{2k}/\zeta}, \tag{107}$$

and substituting (107) into (103) a series representation

$$L(\zeta,\rho_0) = -\frac{\zeta}{(1-\zeta)^2} - \sum_{k=1}^{\infty}\frac{\rho_0^{2k}\zeta}{\left(1-\rho_0^{2k}\zeta\right)^2} - \sum_{k=1}^{\infty}\frac{\rho_0^{2k}/\zeta}{\left(1-\rho_0^{2k}/\zeta\right)^2} \tag{108}$$

can be obtained for the L-function. We note that in the annulus $\rho_0 \le |\zeta| \le 1$, the P-function has a simple zero at $\zeta = 1$, the K-function has a simple pole at $\zeta = 1$ and the L-function has a second order pole at $\zeta = 1$. The properties (104) and (106) can also be obtained directly from the series representations (107) and (108).

Appendix B. Analytical expression for the conformal map $z_0(\zeta)$

In this appendix we show how to integrate (25c) and obtain an analytical expression for the conformal map $z_0(\zeta)$. The function $Q_{z_0}(\zeta) = \zeta z_0'(\zeta)$ defined using (25c),

$$Q_{z_0}(\zeta) = C\zeta \left[\frac{P(\zeta\alpha_0,\rho_0)P(\zeta\overline{\alpha}_0,\rho_0)}{P(\zeta/\beta_0,\rho_0)P(\zeta\beta_0,\rho_0)}\right]^2, \tag{109}$$

is a loxodromic function satisfying the loxodromic property $Q_{z_0}(\rho_0^2\zeta) = Q_{z_0}(\zeta)$. In the annulus $\rho_0 \le |\zeta| \le 1/\rho_0$, $Q_{z_0}(\zeta)$ has second order poles at $\zeta = \beta_0$ and $\zeta = 1/\beta_0$. Hence there exists a "partial-fraction" expansion for Q_{z_0} of the form

$$Q_{z_0}(\zeta) = \hat{C}_1 K(\zeta/\beta_0,\rho_0) + \hat{C}_2 K(\zeta\beta_0,\rho_0) + \hat{C}_3 L(\zeta/\beta_0,\rho_0) + \hat{C}_4 L(\zeta\beta_0,\rho_0) + \hat{C}_5, \tag{110}$$

where the K- and L- functions are defined in (102) and (103). These functions have simple and second-order poles respectively (see (107) and (108)), at the two points $\zeta = \beta_0, 1/\beta_0$ in the annulus $\rho_0 \le |\zeta| \le 1/\rho_0$. Further, $\hat{C}_1,\dots,\hat{C}_5$ in (110) are constants that need to be determined to obtain a series representation for $Q_{z_0}(\zeta)$ that is equivalent to (109). It is a simple step to integrate (110) using the definitions (102) and (103):

$$\begin{aligned} z_0(\zeta) &= \hat{C}_1 \log P(\zeta/\beta_0,\rho_0) + \hat{C}_2 \log P(\zeta\beta_0,\rho_0) + \hat{C}_3 K(\zeta/\beta_0,\rho_0) \\ &\quad + \hat{C}_4 K(\zeta\beta_0,\rho_0) + \hat{C}_5 \log\zeta + \hat{C}_6, \end{aligned} \tag{111}$$

where $\hat{C}_6$ is an integration constant to be determined.

The constants $\hat{C}_1,\dots,\hat{C}_4$ are the coefficients of the poles in (110) and can be determined by matching local expansions of (109) and (110) near these poles. Near $\zeta = \beta_0$ (109) can be expressed as

$$Q_{z_0}(\zeta) = \frac{\hat{Q}(\zeta)}{P^2(\zeta/\beta_0,\rho_0)} \quad \text{with} \quad \hat{Q}(\zeta) = C\zeta\left[\frac{P(\zeta\alpha_0,\rho_0)P(\zeta\overline{\alpha}_0,\rho_0)}{P(\zeta\beta_0,\rho_0)}\right]^2. \tag{112}$$

First let us evaluate the Taylor series of $\hat{Q}(\zeta)$ near $\zeta = \beta_0$. Taking logarithmic derivatives of (112), evaluating at $\zeta = \beta_0$ and using (28) we get

$$\zeta\frac{\hat{Q}'(\zeta)}{\hat{Q}(\zeta)}\bigg|_{\zeta=\beta_0} = 1 + 2(K(\rho_0 e^{i\Theta},\rho_0) + K(\rho_0 e^{-i\Theta},\rho_0) - 2K(\rho_0,\rho_0)).$$

Using the properties (105) in this equation we obtain $\hat{Q}'(\beta_0) = \hat{Q}(\beta_0)/\beta_0$. The required Taylor series is then

$$\hat{Q}(\zeta) = \hat{Q}(\beta_0) + \frac{\hat{Q}(\beta_0)}{\beta_0}(\zeta-\beta_0) + O(\zeta-\beta_0)^2. \tag{113}$$

Using the definition (99), it can be shown that

$$\frac{1}{P^2(\zeta,\rho_0)} = \frac{1/\hat{P}^2(1,\rho_0)}{(1-\zeta)^2} + \text{Taylor series}, \quad \text{near } \zeta = 1. \tag{114}$$

Combining (113) and (114) we get

$$Q_{z_0}(\zeta) = \frac{\beta_0^2\hat{Q}(\beta_0)/\hat{P}^2(1,\rho_0)}{(\zeta-\beta_0)^2} + \frac{\beta_0\hat{Q}(\beta_0)/\hat{P}^2(1,\rho_0)}{(\zeta-\beta_0)} + \text{Taylor series}, \quad \text{near } \zeta = \beta_0. \tag{115}$$

At $\zeta = \beta_0$ (110) has two simple poles and a second-order pole,

$$Q_{z_0}(\zeta) = \frac{\hat{C}_1\beta_0}{\zeta-\beta_0} - \frac{\hat{C}_3\beta_0}{\zeta-\beta_0} - \frac{\hat{C}_3\beta_0^2}{(\zeta-\beta_0)^2} + \text{Taylor series} \quad \text{near } \zeta = \beta_0. \tag{116}$$

Comparing (115) and (116) we get

$$\hat{C}_3 = -\frac{\hat{Q}(\beta_0)}{\hat{P}^2(1,\rho_0)} \quad \text{and} \quad \hat{C}_1 = 0. \tag{117}$$

A similar analysis near $\zeta = 1/\beta_0$ yields the constants

$$\hat{C}_4 = -\frac{\tilde{Q}(1/\beta_0)}{\hat{P}^2(1,\rho_0)} \quad \text{and} \quad \hat{C}_2 = 0, \tag{118}$$

where $\tilde{Q}(\zeta) = Q_{z_0}(\zeta)P^2(\zeta\beta_0,\rho_0)$ is defined analogously to $\hat{Q}(\zeta)$. We can now check that $\chi = \hat{C}_4/\hat{C}_3$, where χ is given in (27). Using the behavior of $z_0'(\zeta)$ at $\zeta = \beta_0$, which can be deduced from (21), we get $\hat{C}_3 = a/\beta_0$. We set $\hat{C}_5 = 0$ since we do not seek a solution with a logarithmic branch point at $\zeta = 0$. The constant $\hat{C}_6$ is determined from the condition $z_0(-\beta_0) = 0$. Substituting for the constants in (111), we obtain (30).

Appendix C. Conservation of circulation at $O(M^2)$

In this appendix we show that the circulation condition $\Gamma = q\mathcal{P}$ is identically satisfied at $O(M^2)$. Here the perimeter is given by

$$\mathcal{P} = \oint_{|\zeta|=1,\rho} |z'(\zeta)|\,|d\zeta|, \tag{119}$$

with the contour integral taken along either vortex, and the circulation Γ is fixed at its incompressible value. At leading order we have $\Gamma = q_0\mathcal{P}_0$ where

$$\mathcal{P}_0 = \oint_{|\zeta|=1} |z_0'(\zeta)|\,\frac{d\zeta}{i\zeta} = \oint_{|\zeta|=\rho_0} |z_0'(\zeta)|\rho_0\,\frac{d\zeta}{i\zeta} \tag{120}$$

Writing $\mathcal{P} = \mathcal{P}_0 + M^2\mathcal{P}_1 + O(M^4)$ and substituting in $\Gamma = q\mathcal{P}$ together with (34b), we find at $O(M^2)$ that

$$\mathcal{P}_1 + \frac{q_1}{q_0}\mathcal{P}_0 = 0. \tag{121}$$

Along the vortex corresponding to $|\zeta| = 1$, we find the $O(M^2)$ term

$$\mathcal{P}_1^{(1)} = \oint_{|\zeta|=1} \mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] |z_0'(\zeta)| \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} = \mathcal{P}_{1rT}^{(1)} + \frac{\beta_1}{\beta_0}[\mathcal{P}_{1s\beta}^{(1)} + \mathcal{P}_{1r\beta}^{(1)}] + \frac{a_1}{a_0}[\mathcal{P}_{1sa}^{(1)} + \mathcal{P}_{1ra}^{(1)}] + \frac{\rho_1}{\rho_0}\mathcal{P}_{1r\rho}^{(1)} + \frac{U_1}{U_0}P_{1rU}^{(1)} - \frac{q_1}{q_0}\mathcal{P}_0, \tag{122}$$

where

$$\mathcal{P}_{1rT}^{(1)} = \oint_{|\zeta|=1} T(\zeta,\bar{\zeta})|z_0'(\zeta)| \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}. \tag{123}$$

and so on, while

$$\mathcal{P}_{1s\beta}^{(1)} = \oint_{|\zeta|=1} \mathrm{Re}\left[-\frac{a_0}{\beta_0}\frac{\tilde{z}_{1s}'(\zeta)}{z_0'(\zeta)}\right] |z_0'(\zeta)| \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}. \tag{124}$$

We see that the q_1/q_0 terms in (122) cancels with the last term in (121), so that q_1/q_0 does not enter the circulation condition. Along the other vortex boundary, we find

$$\begin{aligned}\mathcal{P}_1^{(\rho_0)} =& \mathcal{P}_{1rT}^{(\rho_0)} + \frac{\beta_1}{\beta_0}[P_{1r\beta}^{(\rho_0)} + P_{1s\beta}^{(\rho_0)}] + \frac{a_1}{a_0}[\mathcal{P}_{1sa}^{(\rho_0)} + \mathcal{P}_{1ra}^{(\rho_0)}] + \frac{\rho_1}{\rho_0}\mathcal{P}_{1r\rho}^{(\rho_0)} + \frac{U_1}{U_0}P_{1rU}^{(\rho_0)} - \frac{q_1}{q_0}P_0\\ &+ \frac{\rho_1}{\rho_0}\oint_{|\zeta|=\rho_0}\left(1 + \mathrm{Re}\left[\frac{\zeta z_0''(\zeta)}{z_0'(\zeta)} + \frac{\zeta\xi_0'(\zeta)}{\xi_0(\zeta)}\right]\right)|z_0'(\zeta)|\rho_0 \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta},\end{aligned} \tag{125}$$

taking into account the fact that the integral is computed along $|\zeta| = \rho_0$. Once again the q_1/q_0 term in (125) cancels with the corresponding term in (121).

On the boundary the streamline condition (23) implies that $\zeta f_0'(\zeta)$ is pure imaginary along the boundaries in the ζ-plane, so that $\zeta f_0'(\zeta) = -\mathrm{i}|f_0'(\zeta)|$ on $|\zeta| = 1$. The conditions (19) and (24) give $|f_0'(\zeta)| = q_o|z_0'(\zeta)|$. The perimeter integrals all have real integrands, so the real part operator in front of the various g functions can be taken out of the integrand. As a result

$$\mathcal{P}_{1r\rho}^{(1)} = \mathrm{Re}\oint_{|\zeta|=1} \frac{G_{r\rho}'(\zeta)}{f_0'(\zeta)}|z_0'(\zeta)| \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} = \mathrm{Re}\oint_{|\zeta|=1} G_{r\rho}'(\zeta)\frac{\mathrm{d}\zeta}{q_0} = 0, \tag{126}$$

since the function $G_{r\rho}'(\zeta)$ is analytic inside the circle $|\zeta| = 1$. Similarly $\mathcal{P}_{1r\rho}^{(\rho_0)} = 0$. Along the inner vortex boundary, we need to compute the integral

$$\begin{aligned}\mathrm{Re}\oint_{|\zeta|=\rho_0}\left(1 + \left[\frac{\zeta z_0''(\zeta)}{z_0'(\zeta)} + \frac{\zeta\xi_0'(\zeta)}{\xi_0(\zeta)}\right]\right)|z_0'(\zeta)|\rho_0\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} &= \mathcal{P}_0 + \mathrm{Re}\oint_{|\zeta|=\rho_0}\frac{\zeta f_0''(\zeta)}{f_0'(\zeta)}|z_0'(\zeta)|\rho_0\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} && (127\text{a})\\ &= \mathcal{P}_0 + \mathrm{Re}\oint_{|\zeta|=\rho_0}\zeta f_0''(\zeta)\frac{\mathrm{d}\zeta}{q_0} && (127\text{b})\\ &= \mathcal{P}_0 + \frac{1}{q_0}\mathrm{Re}[\zeta f_0' - f_0]_{|\zeta|=\rho_0} && (127\text{c})\\ &= \mathcal{P}_0 - \frac{\Gamma}{q_0} = 0, && (127\text{d})\end{aligned}$$

by evaluating the logarithmic derivative of (19) and using the leading-order perimeter condition. Note that on the inner boundary $|\zeta| = \rho_0$, $\zeta f_0'(\zeta) = -\mathrm{i}\rho_0|f_0'(\zeta)|$.

We next find

$$\mathcal{P}_{1rU}^{(1)} = \oint_{|\zeta|=1}\left(1 + \mathrm{Re}\left[\frac{\mathrm{i}\Gamma}{2\pi\zeta f_0'(\zeta)}\right]\right)|z_0'(\zeta)|\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} = \mathcal{P}_0 - \frac{\Gamma}{2\pi q_0}\mathrm{Re}\oint_{|\zeta|=1}\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} = \mathcal{P}_0 - \frac{\Gamma}{q_0} = 0, \tag{128}$$

again using the leading-order circulation condition. Similarly $\mathcal{P}_{1rU}^{(\rho_0)} = 0$.

Using the same approach, we obtain

$$\mathcal{P}_{1rT}^{(1)} = \frac{\mathcal{P}_0}{4\chi} + \frac{1}{4U_0^2 q_0}\mathrm{Re}\oint_{|\zeta|=1}\xi_0'(\zeta)\overline{I(\zeta)}\,\mathrm{d}\zeta. \tag{129}$$

From (58) we have

$$\overline{I(\zeta)} = \left(\frac{U_0^2}{\chi}\right)z_0(\zeta) - \frac{aU_0^2}{2\beta_0}\left(1 + \frac{1}{\chi}\right). \tag{130}$$

Since $\xi_0'(\zeta)$ is analytic inside the contour, (129) becomes

$$\begin{aligned}\mathcal{P}_{1rT}^{(1)} &= \frac{\mathcal{P}_0}{4\chi} + \frac{1}{4\chi q_0}\mathrm{Re}\oint_{|\zeta|=1}\xi_0'(\zeta)z_0(\zeta)\,\mathrm{d}\zeta && (131\text{a})\\ &= \frac{\mathcal{P}_0}{4\chi} - \frac{1}{4\chi q_0}\mathrm{Re}\oint_{|\zeta|=1}\xi_0(\zeta)z_0'(\zeta)\,\mathrm{d}\zeta && (131\text{b})\\ &= \frac{\mathcal{P}_0}{4\chi} - \frac{1}{4\chi q_0}\mathrm{Re}\oint_{|\zeta|=1}f_0'(\zeta)\,\mathrm{d}\zeta && (131\text{c})\end{aligned}$$

$$= \frac{\mathcal{P}_0}{4\chi} - \frac{1}{4\chi q_0}\mathrm{Re}[f_0(\zeta)]_{|\zeta|=1} \tag{131d}$$

$$= \frac{\mathcal{P}_0}{4\chi} - \frac{\Gamma}{4\chi q_0} = 0, \tag{131e}$$

using (25) and the leading order circulation condition. Similarly $\mathcal{P}^{(\rho_0)}_{1rT} = 0$.

Next

$$\mathcal{P}^{(1)}_{1s\beta} + \mathcal{P}^{(1)}_{1r\beta} = \oint_{|\zeta|=1} \mathrm{Re}\left[-\frac{a_0}{\beta_0}\frac{\tilde{z}'_{1s\beta}(\zeta)}{z'_0(\zeta)} - \frac{a_0}{\beta_0}\frac{\xi'_0(\zeta)}{f'_0(\zeta)}\tilde{z}_{1s\beta}(\zeta) + \frac{G'_{r\beta}(\zeta)}{f'_0(\zeta)}\xi'_0(\zeta)z_0(\zeta)\right]|z'_0(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{132a}$$

$$= -\frac{a_0}{\beta_0}\mathrm{Re}\oint_{|\zeta|=1}[\xi_0(\zeta)\tilde{z}'_{1s\beta}(\zeta) + \xi'_0(\zeta)\tilde{z}_{1s\beta}(\zeta)]\,\frac{\mathrm{d}\zeta}{q_0} = 0. \tag{132b}$$

Similarly $\mathcal{P}^{(\rho_0)}_{1s\beta} + \mathcal{P}^{(\rho_0)}_{1r\beta} = 0$. Finally

$$\mathcal{P}^{(1)}_{1ra} + \mathcal{P}^{(1)}_{1ra} = \oint_{|\zeta|=1} \mathrm{Re}\left[\frac{a_0}{\beta_0}\frac{\tilde{z}'_{1sa}(\zeta)}{z'_0(\zeta)} + \frac{a_0}{\beta_0}\frac{\xi'_0(\zeta)}{f'_0(\zeta)}\tilde{z}_{1sa}(\zeta) + \frac{G'_{ra}(\zeta)}{f'_0(\zeta)}\xi'_0(\zeta)z_0(\zeta)\right]|z'_0(\zeta)|\,\frac{\mathrm{d}\zeta}{\mathrm{i}\zeta} \tag{133a}$$

$$= -\frac{a}{\beta_0}\mathrm{Re}\oint_{|\zeta|=1}[\xi_0(\zeta)\tilde{z}'_{1sa}(\zeta) + \xi'_0(\zeta)\tilde{z}_{1sa}(\zeta)]\,\frac{\mathrm{d}\zeta}{q_0} = 0. \tag{133b}$$

Similarly $\mathcal{P}^{(\rho_0)}_{1sa} + \mathcal{P}^{(\rho_0)}_{1ra} = 0$. This means that the condition (121) is identically satisfied.

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