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RESEARCH ARTICLE

Interpolation for Brill–Noether curves

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Abstract

In this paper, we determine the number of general points through which a Brill–Noether curve of fixed degree and genus in any projective space can be passed.

Contents

1. Introduction

The interpolation problem has occupied a central position in mathematics for several millennia. Roughly speaking, it asks:

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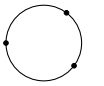
Question. When can a curve of a given type be drawn through a given collection of points?

The first results on the interpolation problem date to classical antiquity. Two such results appear in Euclid's *Elements* (circa 300 B.C.): A line can be drawn through any two distinct points in the plane (the first postulate), and a circle can be drawn through any three noncollinear points in the plane (Proposition 5 of Book IV).



The study of the interpolation problem in antiquity culminated in the work of Pappus (circa 340 A.D.), who showed in his *Collection* that a conic section can be drawn through any five points in the plane (Proposition 14 of Book VIII).

The introduction of algebraic techniques to geometry enabled a second wave of results in the 18th century, and cast the interpolation problem firmly in the then-emerging field of algebraic geometry. For example, Cramer generalized Pappus's result to plane curves of arbitrary degree n, which he showed can pass through $n(n+3)/2$ general points in 1750 [11]. Then Waring solved the interpolation problem for graphs of polynomial functions in 1779 [32]. (Lagrange independently rediscovered this result in 1795 [21] and thus it is often known as the 'Lagrange interpolation formula'.) Cauchy [6], Hermite and Borchardt [19] and Birkhoff [5], all subsequently generalized Waring's result in several different directions. These results are of interest far outside algebraic geometry and even outside of mathematics. For example, they play essential roles in the Newton–Cotes method for numerical integration, in Shamir's cryptographic secret sharing protocol [28], and in Reed–Solomon error-correcting codes [26] (which currently power most digital storage media).

The key prerequisite to the modern study of the interpolation problem was the development of Brill–Noether theory in the 20th century, which studies maps from general curves to projective space, and thus identifies the most natural class of curves for which to study the interpolation problem. Namely, let C be a general curve of genus g. From our perspective here, the two key facts are:

1. There exists a nondegenerate map $C \to \mathbb{P}^r$ of degree d if and only if the quantity
$$\rho(d,g,r) := (r+1)d - rg - r(r+1)$$
satisfies $\rho \geq 0$. [Proven in 1980 by Griffiths and Harris [17].]
2. In this case, the *universal* space of such maps has a unique component dominating $\overline{M}_g$. [Proven in the 1980s by Fulton and Lazarsfeld [15], Gieseker [16] and Eisenbud and Harris [12].]

We call stable maps $f\colon C \to \mathbb{P}^r$ corresponding to points in this unique component Brill–Noether curves (BN-curves). (The general such curve is an embedding of a smooth curve for $r \geq 3$.) This language then gives us a *precise* and *natural* formulation of the interpolation problem:

Question. Let d, g, r, n be nonnegative integers with $\rho(d,g,r) \geq 0$. When can we pass a BN-curve of degree d and genus g through n general points in $\mathbb{P}^r$?

Equivalently, writing $M^\circ_{g,n}(\mathbb{P}^r, d)$ for the component corresponding to BN-curves, this question is asking when the evaluation map $M^\circ_{g,n}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^n$ is dominant. It is evidently necessary for:

$$rn = \dim(\mathbb{P}^r)^n \leq \dim M^\circ_{g,n}(\mathbb{P}^r, d) = (r+1)d - (r-3)(g-1) + n,$$

or upon rearrangement,

$$(r-1)n \leq (r+1)d - (r-3)(g-1).$$

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Despite cases where this is not sufficient, it is a folklore conjecture that it *usually* suffices:

Conjecture 1.1. *Let d, g, r, n be nonnegative integers with $\rho(d, g, r) \geq 0$. Then there is a BN-curve of degree d and genus g through n general points in $\mathbb{P}^r$ if and only if*

$$(r-1)n \leq (r+1)d - (r-3)(g-1),$$

apart from finitely many exceptions.

This conjecture has been studied intensely in recent years. As in previous eras, this attention has been motivated by both intrinsic interest, as well as by striking applications to a wide range of other interesting geometric problems. Recent examples of such problems include smoothing curve singularities [30], constructing moving curves in $\overline{M}_g$ [3, 7], the first author's resolution of Severi's 1915 maximal rank conjecture [22], as well as various generalizations thereof [4].

The easiest cases of this conjecture are when d is large relative to g and r, and such cases have therefore been the focus of significant work. For example, Sacchiero proved Conjecture 1.1 for rational curves in 1980 [27]; Ran later gave an independent proof in this case in 2007 [25]. Subsequently, the first author, in joint work with Atanasov and Yang, proved Conjecture 1.1 when $d \geq g + r$ in characteristic zero [2]. Another case of interest is the proof of Conjecture 1.1 for canonical curves in characteristic zero in a pair of papers by Stevens from 1989 and 1996 [29, 30]. Many authors have also considered this conjecture in low dimensions. For example, Ellingsrud and Hirschowitz in 1984 [13], Perrin in 1987 [24] and later Atanasov in 2014 [3], all made significant progress on Conjecture 1.1 for space curves, but their analysis left infinitely many cases unsolved. This effort culminated in the proof of Conjecture 1.1 for space curves in characteristic zero by the second author in 2018 [31], and for curves in $\mathbb{P}^4$ in characteristic zero by both authors in 2021 [23].

Nevertheless, despite this significant interest, fundamental limitations of previous techniques have prevented the resolution of Conjecture 1.1 in general and limited even partial results largely to characteristic zero. Our main result gives the first comprehensive answer to the interpolation problem.

Theorem 1.2. *Conjecture 1.1 holds in full generality and in any characteristic. More precisely: Let d, g, r, n be nonnegative integers with $\rho(d, g, r) \geq 0$. There is a BN-curve of degree d and genus g through n general points in $\mathbb{P}^r$ if and only if*

$$(r-1)n \leq (r+1)d - (r-3)(g-1), \tag{1.1}$$

except in the following four exceptional cases:

$$(d, g, r) \in \{(5, 2, 3), (6, 4, 3), (7, 2, 5), (10, 6, 5)\}.$$

Since the normal bundle N_C controls the deformation theory of C, Conjecture 1.1 is closely related to a certain property, also known as interpolation, for N_C.

Definition 1.3. A vector bundle E on a curve C satisfies interpolation if $H^1(E) = 0$, and for every $n > 0$, there exists an effective divisor D of degree n such that

$$H^0(E(-D)) = 0 \quad \text{or} \quad H^1(E(-D)) = 0. \tag{1.2}$$

For C irreducible, $\mathrm{Sym}^n C$ is also irreducible, so interpolation is equivalent to (1.2) for D general.

Given $p_1, \ldots, p_n \in C \subset \mathbb{P}^r$, a standard argument in deformation theory (see [24, Theorem 1.5]) implies that the evaluation map $M^\circ_{g,n}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^n$ is smooth at the point $(C, p_1, \ldots, p_n)$, and hence dominant, if $H^1(N_C(-p_1 - \cdots - p_n)) = 0$. Since $\chi(N_C(-p_1 - \cdots - p_n)) = (r+1)d - (r-3)(g-1) - (r-1)n$, we have $\chi(N_C(-p_1 - \cdots - p_n)) \geq 0$ precisely when equation (1.1) is satisfied. Therefore, interpolation for N_C implies that $H^1(N_C(-p_1 - \cdots - p_n)) = 0$ when equation (1.1) is satisfied, which implies

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Conjecture 1.1. In fact, interpolation for N_C is a slightly stronger condition, the essential differences being:

1. It implies an analog of Conjecture 1.1 where the general points are replaced by general linear spaces: There is a BN-curve of degree d and genus g incident to general linear spaces Λ_i of dimension λ_i if and only if
$$\sum(r-1-\lambda_i) \leq (r+1)d-(r-3)(g-1). \tag{1.3}$$
(This implication can be deduced from [3, Theorem 8.1], cf. the introduction to loc. cit.)
2. It implies that $M^\circ_{g,n}(\mathbb{P}^r,d)\to(\mathbb{P}^r)^n$ is generically smooth, rather than merely dominant. (This is of course equivalent in characteristic zero but is a stronger statement in positive characteristic.)

Theorem 1.2 is a consequence of our main theorem, which asserts:

Theorem 1.4. *Let d,g,r be nonnegative integers with $\rho(d,g,r)\geq 0$, and $C\subset\mathbb{P}^r$ be a general BN-curve of degree d and genus g. Then N_C satisfies interpolation if and only if neither of the following hold:*

1. *The tuple (d,g,r) is one of the following five exceptions:*
$$(d,g,r)\in\{(5,2,3),(6,4,3),(6,2,4),(7,2,5),(10,6,5)\}. \tag{1.4}$$
2. *The characteristic is 2, and $g=0$, and $d\not\equiv 1$ mod $r-1$.*

There are several exceptions in Theorem 1.4 that are not exceptions for Theorem 1.2:

1. The case $(d,g,r)=(6,2,4)$: Such curves have the expected behavior for passing through points, but *not* for incidence to linear spaces of arbitrary dimension. More precisely, a naive dimension count suggests that they can pass through nine general points and meet a general line, but this is not true.
2. The cases in characteristic 2: In these cases, the evaluation map $M^\circ_{g,n}(\mathbb{P}^r,d)\to(\mathbb{P}^r)^n$ is dominant but *not* generically smooth.

We discuss these two cases in more depth in Sections 2 and 9.

Our approach to Theorem 1.4 will be via degeneration to reducible curves $X\cup Y$. In general, although the restrictions $N_{X\cup Y}|_X$ and $N_{X\cup Y}|_Y$ admit nice descriptions, fitting these together to describe $N_{X\cup Y}$ is extremely challenging outside a handful of special cases. This fundamental obstacle has limited previous attempts to study the interpolation problem. For example, the key innovation of [2] was an essentially complete description of $N_{X\cup Y}$ in the special case that Y was a one- or two-secant line. Considering only such degenerations leads to two severe limitations:

1. Only nonspecial curves can be obtained by successively adding one- and two-secant lines.
2. Since the set of degenerations used is so limited, only a few types of elementary modifications to the normal bundle appear. Because there are only a few types of modifications, it is difficult to produce desired modifications by combining them, in a way reminiscent of the Frobenius coin problem. Circumventing this difficulty requires additional tools that work only in characteristic zero.

Previous attempts to overcome these difficulties were limited to ad-hoc constructions in low-dimensional projective spaces. The present paper introduces two key innovations:

1. We consider a third degeneration, where Y is an $(r+1)$-secant rational curve of degree $r-1$ contained in a hyperplane H, which allows us to obtain any BN-curve. Describing how $N_{X\cup Y}|_X$ and $N_{X\cup Y}|_Y$ fit together to give $N_{X\cup Y}$ is intractable even in a degeneration of this complexity. Nevertheless, thanks to our detailed study of this setup in Sections 5.3 and 13.1, we are able to reduce interpolation for $N_{X\cup Y}$ to interpolation for a certain modification of N_X.
2. Although Y does not have many interesting degenerations in H, we show in Section 7 that as H becomes tangent to C, a plethora of such degenerations appear. As in the Frobenius coin problem, this plethora of additional degenerations makes it possible to produce the desired modifications by combining them. This eliminates the restriction to characteristic zero that plagued previous work.

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The techniques we develop here hold promise of application to other problems about the geometry of normal bundles. Indeed, in [10], they have already been applied to prove semistability of normal bundles of canonical curves, thereby establishing a (slightly weaker version of a) conjecture of Aprodu, Farkas and Orgeta [1]. (A vector bundle of integral slope which satisfies interpolation is necessarily semistable, c.f. [31, Remark 1.6], but in these cases the slope is not integral. Nevertheless, our techniques may be applied.)

Structure of the paper

We begin, in Section 2, by discussing the various exceptional cases appearing in Theorem 1.4. Then in Section 3, we introduce the notation we shall use for the remainder of the paper and discuss a few other preliminary points.

In Section 4, we explain the basic strategy of proof for the hard direction of Theorem 1.4, that is, that there are no other exceptional cases besides those mentioned in the statement of Theorem 1.4 above. After explaining the basic strategy, we give a roadmap to the proof, which occupies the remainder of the paper.

2. Counterexamples

2.1. Counterexamples in all characteristics

We start with the five counterexamples to Theorem 1.4 that occur in all characteristics:

$$(d, g, r) \in \{(5, 2, 3), (6, 4, 3), (6, 2, 4), (7, 2, 5), (10, 6, 5)\}.$$

In each of these cases, we will construct a certain surface S containing C, and see that S prevents Theorem 1.2 (or the generalization (1.3) thereof) from holding. Indeed, if S cannot be made to pass through the requisite number of points (or be made incident to the requisite linear spaces), then C cannot either. Since Theorem 1.4 implies Theorem 1.2 (and the generalization (1.3)), this implies that these five cases must also be counterexamples to Theorem 1.4.

Remark 2.1. An alternative approach, the details of which we leave to the interested reader, would be to see directly that the geometry of S obstructs Theorem 1.4. The basic idea is that, for any effective divisor D on C, we have $h^0(N_C(-D)) \geq h^0(N_{C/S}(-D))$; in the five exceptional cases, this inequality will prevent N_C from satisfying interpolation.

2.1.1. The family $(d, g, r) = (r + 2, 2, r)$

Let C be a curve of genus 2 and L be a line bundle of degree $r + 2$ on C. Write $f\colon C \to \mathbb{P}^1$ for the hyperelliptic map. Then $E := f_*L$ is a vector bundle of rank 2 on $\mathbb{P}^1$ with

$$\chi(E) = \chi(L) = r + 1.$$

By Riemann–Roch, $c_1(E) = r - 1$. The inclusion $L \to f^*f_*L$ embeds C in the projective bundle $\mathbb{P}E^\vee$ so that $O_{\mathbb{P}E^\vee}(1)|_C \simeq L$. Therefore, the image of C in $\mathbb{P}^r$ under the complete linear series for L lies on the image S of $\mathbb{P}E^\vee$ under the complete linear series for $\mathcal{O}_{\mathbb{P}E^\vee}(1)$. The surface S is a scroll of degree equal to

$$[\mathcal{O}_{\mathbb{P}E^\vee}(1)]^2 = -c_1(E^\vee) \cdot \mathcal{O}_{\mathbb{P}E^\vee}(1) = r - 1.$$

By [8, Lemma 2.6], the dimension of the space of such scrolls is $r^2 + 2r - 6$.

If $(d, g, r) = (5, 2, 3)$ Then $r^2 + 2r - 6 = 9$. Since it is 1 condition for a surface in $\mathbb{P}^3$ to pass through a point, S cannot pass through more than nine general points. This contradicts (1.1), which predicts that C should be able to pass through 10 general points.

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If $(d, g, r) = (6, 2, 4)$ Then $r^2 + 2r - 6 = 18$. Since it is two conditions for a surface in $\mathbb{P}^4$ to pass through a point and one condition to meet a line, S cannot pass through nine general points while meeting a general line. This contradicts equation (1.3), which predicts that C should be able to pass through nine general points points while meeting a general line.

If $(d, g, r) = (7, 2, 5)$ Then $r^2 + 2r - 6 = 29$. Since it is three conditions for a surface in $\mathbb{P}^5$ to pass through a point, S cannot pass through more than nine general points. This contradicts equation (1.1), which predicts that C should be able to pass through 10 general points.

2.1.2. The case $(d, g, r) = (6, 4, 3)$

A general canonical curve of genus 4 is a cubic section of a quadric surface S. There is a nine-dimensional family of quadric surfaces, and it is one condition for a surface in $\mathbb{P}^3$ to pass through a point, so S cannot pass through more than two points. This contradicts equation (1.1), which predicts that C should be able to pass through 12 general points.

2.1.3. The case $(d, g, r) = (10, 6, 5)$

A general canonical curve of genus 6 is a quadric section of a quintic del Pezzo surface S. There is a 35-dimensional family of quintic del Pezzo surfaces, and it is three conditions for a surface in $\mathbb{P}^5$ to pass through a point, so S cannot pass through more than 11 points. This contradicts (1.1), which predicts that C should be able to pass through 12 general points.

2.2. Rational curves in characteristic 2

In this section, we explain the final infinite family of counterexamples to Theorem 1.4 that occurs only in characteristic 2. This phenomenon was already observed for space curves in [9] in relation to semistability. We begin by describing which vector bundles on a rational curve satisfy interpolation in terms of the Birkhoff–Grothendieck classification.

Lemma 2.2. *The bundle $E = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(e_i)$ satisfies interpolation if and only if for all i, j,*

$$|e_i - e_j| \leq 1 \quad \text{and} \quad e_i \geq -1.$$

Proof. For any $n \geq 0$,

$$\begin{aligned} h^0(E(-n)) = 0 \quad &\Leftrightarrow \quad n \geq 1 + \max(e_i) \\ h^1(E(-n)) = 0 \quad &\Leftrightarrow \quad n \leq 1 + \min(e_i). \end{aligned}$$

One of these two conditions holds for all $n \geq 0$ if and only if $|e_i - e_j| \leq 1$ for all i and j, and the second of these holds for $n = 0$ if and only if $e_i \geq -1$ for all i. □

As a consequence of the Euler sequence, the conormal bundle of C sits in the exact sequence

$$0 \to N_C^\vee(1) \to \mathcal{O}_C^{\oplus r+1} \to \mathscr{P}^1(\mathcal{O}_C(1)) \to 0,$$

where $\mathscr{P}^1(\mathcal{O}_C(1))$ is the bundle of first principle parts of $\mathcal{O}_C(1)$. If the characteristic is 2, and we write $\pi\colon C \to C^{(2)}$ for the (relative) Frobenius morphism, then we have

$$\mathscr{P}^1(\mathcal{O}_C(1)) \simeq \pi^*\pi_*\mathcal{O}_C(1).$$

Therefore, $N_C^\vee(1)$ is isomorphic to the pullback of a bundle under the Frobenius morphism, so every entry of its splitting type is even.

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Lemma 2.3. *Assume that the characteristic of the ground field is* 2. *Let* $C \subset \mathbb{P}^r$ *be a rational curve of degree* d. *Then* N_C *satisfies interpolation only if*

$$d \equiv 1 (\mathrm{mod}\ r-1).$$

Proof. Suppose that $N_C \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(e_i)$. Since $N_C^\vee(1) \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(d-e_i)$, and every entry of its splitting type is even, every e_i satisfies $e_i \equiv d \bmod 2$. Applying Lemma 2.2, we conclude that N_C can only satisfy interpolation if all e_i are equal. This implies

$$(r+1)d - 2 = c_1(N_C) \equiv (r-1)d (\mathrm{mod}\ 2(r-1)),$$

and therefore $d \equiv 1 \bmod r-1$ as desired. □

3. Preliminaries

3.1. Elementary modifications of vector bundles

In this section, we give a brief overview of the key properties of elementary modifications of vector bundles. Our presentation will roughly follow the more detailed exposition given in Sections 2–4 of [2].

Definition 3.1. Let E be a vector bundle on a scheme X, and $D \subset X$ be a Cartier divisor and $F \subset E|_D$ be a subbundle of the restriction of E to D. We define the negative elementary modification of E along D towards F by

$$E[D \xrightarrow{-} F] := \ker\left(E \to E|_D/F\right).$$

We then define the (positive) elementary modification of E along D towards F as

$$E[D \xrightarrow{+} F] := E[D \xrightarrow{-} F](D).$$

Remark 3.2. This notation differs slightly from [2], in which negative modifications were denoted by $E[D \to F]$ (and no separate notation was given for positive modifications).

By construction, a modification of E along D is naturally isomorphic to E when restricted to the complement of D. If D_1 and D_2 are disjoint, then we may easily make sense of multiple modifications such as $E[D_1 \xrightarrow{+} F_1][D_2 \xrightarrow{+} F_2]$ by working locally. However, if D_1 and D_2 meet, then we do not have enough data to even define multiple modifications: For example, if $D_1 = D_2 = D$ and $F_1 = F_2 = F$, then we should have $E[D_1 \xrightarrow{+} F_1][D_2 \xrightarrow{+} F_2] \simeq E[2D \xrightarrow{+} F]$, so we must know how F extends over $2D$. To sidestep these issues, we suppose when defining multiple modifications – at least along divisors that meet – that we are given not just a subbundle F_i of $E|_{D_i}$, but a subbundle F_i of $E|_{U_i}$ where $U_i \subset X$ is an open neighborhood of D_i.

We first construct the modification $E[D_1 \xrightarrow{+} F_1]$, which is naturally isomorphic to E on $X \backslash D_1$, and so in particular on $U_2 \backslash D_1$. If the data of a subbundle $F_2 \subset E|_{U_2 \backslash D_1}$ extend to U_2, then it does so uniquely, and we may modify along D_2 towards this extension. However, this subbundle may not extend to U_2. The following situation where it does will include all situations we shall need in this paper.

Definition 3.3. Let $M = \{(D_i, U_i, F_i)\}_{i \in I}$ be a collection of modification data. For each point $x \in X$, define $I_x \subseteq I$ to be the set of indices for which $x \in D_i$. We say that M is tree-like if for all $x \in X$, and all subsets $I' \subset I_x$, the following condition holds: Whenever the fibers $\{F_i|_x\}_{i \in I'}$ are dependent, there exist indices $i, j \in I'$ and an open $U \subseteq U_i \cap U_j$ containing x such that $F_i|_U \subseteq F_j|_U$.

By [2, Proposition 2.17], we can transfer modification data as above when it is treelike. That is, given modification data M for E such that $\{(D, U, F)\} \cup M$ is treelike, we obtain modification data M' for $E[D \xrightarrow{+} F]$. In this way, we inductively define the multiple modification $E[M]$ for treelike modification

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data M. This is independent of the order in which the modifications from M are applied [2, Proposition 2.20].

Example 3.4. A simplifying special case is when F is a direct summand of E. Writing $E \simeq F \oplus E'$,

$$E[D \xrightarrow{+} F] = \ker(F \oplus E' \to E'|_D)(D) \simeq (F \oplus E'(-D))(D) \simeq F(D) \oplus E'.$$

In nice cases, a short exact sequence of vector bundles induces a short exact sequence of modifications. To make this more precise, consider a short exact sequence

$$0 \to S \to E \to Q \to 0. \tag{3.1}$$

For example, first suppose that $F \cap S$ is flat over the base X. Then equation (3.1) induces the short exact sequence

$$0 \to S[D \xrightarrow{+} F \cap S] \to E[D \xrightarrow{+} F] \to Q[D \xrightarrow{+} F/(F \cap S)] \to 0. \tag{3.2}$$

A more interesting example is when the base is a curve $X = C$, and $F \subset E$ is a line subbundle, and $D = np$, where $p \in C$ is a smooth point. Then we obtain an induced sequence for the modification $E[np \xrightarrow{+} F]$, as follows. Define k' to be the order to which F is contained in S in a neighborhood of p. In other words, if F is not contained in S, this is the length of the subscheme $\mathbb{P}S \cap \mathbb{P}F$ in $\mathbb{P}E$; if F is contained in S, this is ∞. Let $k = \max(k', n)$. Then equation (3.1) induces the short exact sequence

$$0 \to S[kp \xrightarrow{+} F|_{kp}] \to E[np \xrightarrow{+} F] \to Q[(n-k)p \xrightarrow{+} \overline{F}] \to 0, \tag{3.3}$$

where $\overline{F}$ is the saturation of the image of F in Q. When $k' = \infty$ or $k' = 0$, this agrees with equation (3.2).

3.2. Pointing bundles

Given an unramified map $f\colon C \to \mathbb{P}^r$, the sheaf $N_f = \ker(f^*\Omega_{\mathbb{P}^r} \to \Omega_X)^\vee$ is a vector bundle, which we refer to as the normal bundle of the map f. In almost all cases that we shall consider, f will be an embedding, in which case $N_f = N_C$ coincides with the normal bundle of the image.

We will primarily deal with modifications of N_f towards pointing subbundles $N_{f\to\Lambda}$, whose definition we now recall. Let $\Lambda \subset \mathbb{P}^r$ be a linear space of dimension λ. Let $\pi_\Lambda \circ f$ denote the composition of f with the projection map

$$\pi_\Lambda\colon \mathbb{P}^r \dashrightarrow \mathbb{P}^{r-\lambda-1}.$$

Let $U = U_\Lambda$ denote the open locus of $C\backslash(\Lambda \cap C)$ where $\pi_\Lambda \circ f$ is unramified; explicitly this is the locus of points of C whose tangent space does not meet Λ. Assuming that U is dense and contains the singular locus of C, we may define $N_{f\to\Lambda}$ as the unique subbundle of N_f whose restriction to U is

$$\ker(N_f|_U \to N_{\pi_\Lambda \circ f}|_U).$$

The notation $N_{f\to\Lambda}$ is evocative of the geometry of sections of $N_{f\to\Lambda}$: Informally speaking, they 'point towards' the subspace $\Lambda \subset \mathbb{P}^r$. When f is an embedding, we write $N_{C\to\Lambda} = N_{f\to\Lambda}$. If the projection $(\pi_\Lambda \circ f)\colon C \to \mathbb{P}^{r-\lambda-1}$ is unramified, then $N_{f\to\Lambda}$ sits in the pointing bundle exact sequence

$$0 \to N_{f\to\Lambda} \to N_f \to N_{\pi_\Lambda \circ f}(\Lambda \cap C) \to 0. \tag{3.4}$$

The same definitions work for families of curves in a projective bundle. For a treatment in this more general setting, see [2, Section 5].

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The simplest case, and our primary interest, is when $\Lambda = p$ is a point in $\mathbb{P}^r$. In this case, by [2, Propositions 6.2 and 6.3], we have the following explicit descriptions:

- If $p \in \mathbb{P}^r$ is a general point (in which case $U_p = C$), then $N_{f\to p} \simeq f^*\mathcal{O}_{\mathbb{P}^r}(1)$.
- If $p \in C$ is a general point (in which case $U_p = C\backslash p$), then $N_{f\to p} \simeq f^*\mathcal{O}_{\mathbb{P}^r}(1)(2p)$.

When modifying towards a pointing bundle, we use the simpler notation

$$N_f[D \xrightarrow{+} \Lambda] := N_f[D \xrightarrow{+} N_{f\to\Lambda}].$$

For two points $p, q \in C$, we also define the more compact notation

$$N_f[p \xleftrightarrow{+} q] := N_f[p \xrightarrow{+} q][q \xrightarrow{+} p].$$

We now restate a foundational result of Hartshorne–Hirschowitz, which describes the normal bundle of a nodal curve in projective space, in this language of pointing bundles. Let $X \cup_\Gamma Y$ be a reducible nodal curve. For each point $p_i \in \Gamma$, let q_i denote any point on $T_{p_i}Y\backslash p_i$. For simplicity, we introduce the following notation. For any subset $\Gamma' = \{p_1, \dots p_n\} \subseteq \Gamma$, we write

$$N_X[\Gamma' \overset{+}{\rightsquigarrow} Y] := N_X[p_1 \xrightarrow{+} q_1] \dots [p_n \xrightarrow{+} q_n].$$

When $\Gamma' = \Gamma$ is the full set of points where X and Y meet, we simplify further and write

$$N_X[\overset{+}{\rightsquigarrow} Y] := N_X[\Gamma \overset{+}{\rightsquigarrow} Y].$$

(We analogously define $N_X[\Gamma' \overset{-}{\rightsquigarrow} Y]$ and $N_X[\overset{-}{\rightsquigarrow} Y]$.)

Proposition 3.5 [18, Corollary 3.2]. *As above, let $X \cup Y \subseteq \mathbb{P}^r$ be a reducible nodal curve. Then*

$$N_{X\cup Y}|_X \simeq N_X[\overset{+}{\rightsquigarrow} Y].$$

3.3. Interpolation for vector bundles

3.3.1. Interpolation for bundles on nodal curves

Let C be a nodal curve, and let E be a nonspecial vector bundle on C. Twisting down by a point can only decrease h^0 and can only increase h^1. Therefore, E satisfies interpolation provided that there exist two divisors D_+ and D_- for which

$$h^0(E(-D_+)) = 0, \quad h^1(E(-D_-)) = 0, \quad \text{and} \quad \deg D_+ - \deg D_- \leq 1. \tag{3.5}$$

Alternatively, if E has rank r, then twisting down by a point either decreases h^0 by r or increases h^1. Thus, E satisfies interpolation if and only if, for every $n > 0$, some collection of n points impose the expected number of conditions:

$$h^0(E(-p_1 - \dots - p_n)) = \max(0, h^0(E) - rn) \qquad \text{for some } p_1, \dots, p_n.$$

More generally, we can use this idea to define interpolation for a space of sections of a vector bundle. Given $V \subseteq H^0(E)$, write

$$V(-p_1 - \dots - p_n) := \{\sigma \in V : \sigma|_{p_1} = \dots = \sigma|_{p_n} = 0\}.$$

We say that $V \subseteq H^0(E)$ satisfies interpolation if $H^1(E) = 0$ and, for every $n > 0$, there are n points $p_1, \dots, p_n$ such that

$$\dim V(-p_1 - \dots - p_n) = \max(0, \dim V - rn).$$

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The following basic result allows us to reduce interpolation for a vector bundle E on a reducible curve $X \cup_\Gamma Y$ to interpolation for a space of sections on one component.

Lemma 3.6 [23, Lemma 2.10]. *Let E be a vector bundle on $X \cup_\Gamma Y$. If the restriction map on Y*

$$\mathrm{res}_{Y,\Gamma} \colon H^0(Y, E|_Y) \to E|_\Gamma$$

is injective and the space of sections $\{\sigma \in H^0(X, E|_X) : \sigma|_\Gamma \in \mathrm{Im}(\mathrm{res}_{Y,\Gamma})\}$ has dimension $\chi(E)$ and satisfies interpolation, then E satisfies interpolation.

The main case of interest in this paper is when $Y = R$ is a rational curve and $E|_R$ is perfectly balanced, that is, $E|_R \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus r}$ for some $a \in \mathbb{Z}$.

Lemma 3.7. *Let $C \cup_\Gamma R$ be a nodal curve with R rational, and let E be a vector bundle on $C \cup R$ with $E|_R$ perfectly balanced of slope at least $\#\Gamma - 1$. If $E|_C$ satisfies interpolation, then E satisfies interpolation.*

Proof. Let D be an effective divisor of degree $\mu(E|_R) - \#\Gamma + 1$ supported on $R \backslash \Gamma$. It suffices to prove that $E(-D)$ satisfies interpolation, as we now explain. Let D_+ and D_- be the two divisors satisfying equation (3.5) for the vector bundle $E(-D)$. Then the divisors $D_+ + D$ and $D_- + D$ satisfy equation (3.5) for the vector bundle E. The bundle $E(-D)|_R$ is perfectly balanced of slope $\#\Gamma - 1$, and so $\mathrm{res}_{R,\Gamma}$ is an isomorphism. The result now follows from Lemma 3.6. □

3.3.2. Interpolation and twists

If E satisfies interpolation, then as in the proof of Lemma 3.7, the twist $E(D)$ by any effective divisor D also satisfies interpolation. Conversely, we have the following.

Lemma 3.8 [2, Proposition 4.12]. *Suppose that E is a vector bundle on a genus g curve such that*

$$\chi(E) \geq \mathrm{rk}(E) \cdot g.$$

If there exists an effective divisor D for which $E(D)$ satisfies interpolation, then E also satisfies interpolation.

3.3.3. Interpolation and modifications

Consider a vector bundle E and its modification $E[p \xrightarrow{+} F]$. Given sufficient generality of either p or F, or if the slope $\mu(E) \in \mathbb{Z}$, it is sometimes possible to deduce that $E[p \xrightarrow{+} F]$ satisfies interpolation from the assumption that E satisfies interpolation.

Lemma 3.9. *Let E be a vector bundle on C, let $p \in C$ be a smooth point and let $F \subseteq E|_p$. If E satisfies interpolation and $\mu(E) \in \mathbb{Z}$, then $E[p \xrightarrow{+} F]$ satisfies interpolation.*

Proof. Since $\mu(E) \in \mathbb{Z}$, there is an effective divisor D with

$$H^0(E(-D)) = 0 \quad \text{and} \quad H^1(E(-D)) = 0.$$

Then

$$H^0(E[p \xrightarrow{+} F](-D - p)) = 0 \quad \text{and} \quad H^1(E[p \xrightarrow{+} F](-D)) = 0.$$

□

Definition 3.10. We say that a collection of subspaces $\{W_b\}_{b \in B}$ of a vector space V are linearly general if, for any subspace $U \subset V$, there is some $b \in B$ so that W_b is transverse to U.

Lemma 3.11 [2, Proposition 4.10]. *Let E be a vector bundle on C, and let $p \in C$ be a smooth point. Let $\{F_b\}_{b \in B}$ be a collection of subspaces of $E|_p$ that all contain a fixed subspace F_0. Suppose that both E*

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and $E[p \to F_0]$ satisfy interpolation. If the collection $\{F_b/F_0\}_{b\in B}$ is linearly general in $E|_p/F_0$, then for some $b \in B$, the positive modification $E[p \xrightarrow{+} F_b]$ satisfies interpolation.

Lemma 3.12 [2, Proposition 4.21 for $n = 1$]. *Suppose E satisfies interpolation, $L \subset E$ is a nonspecial line subbundle, and the quotient $Q = E/L$ also satisfies interpolation. If $\mu(L) \leq \mu(H)$, then $E[p \xrightarrow{+} L]$ satisfies interpolation.*

Remark 3.13. While [2] assumes characteristic zero, none of the specific results we quote from [2] use this assumption – except for Proposition 4.21. This proposition states that if $\mu(L) \leq \mu(H) + n - 1$, then $E[np \xrightarrow{+} L]$ satisfies interpolation. The proof uses that vanishing at np imposes n conditions on sections of any linear series. This is true in *any* characteristic when $n = 1$, but *fails in positive characteristic for* $n > 1$. Since we will use Proposition 4.21 of [2] only when $n = 1$, we do not need a restriction on the characteristic.

Lemma 3.14. *Let E be a vector bundle on an irreducible curve C. Let $p_1, \ldots, p_n \in C$ be points, and let $L_i \subseteq E|_{p_i}$ be one-dimensional subspaces. Suppose that both E and $E[p_1 \xrightarrow{+} L_1] \cdots [p_n \xrightarrow{+} L_n]$ satisfy interpolation. Then for any $0 < m < n$, there is a collection of distinct indices $i_1, \ldots, i_m$ such that*

$$E[p_{i_1} \xrightarrow{+} L_{i_1}] \cdots [p_{i_m} \xrightarrow{+} L_{i_m}]$$

satisfies interpolation.

Proof. By induction on n we reduce to the case $m = n - 1$. Write $E' = E[p_1 \xrightarrow{+} L_1] \cdots [p_n \xrightarrow{+} L_n]$ and $N = \lceil \chi(E')/\mathrm{rk}\, E' \rceil$. Let D_N and D_{N-1} be general divisors of degrees N and $N - 1$, respectively. Since E and E' both satisfy interpolation and $\chi(E) < \chi(E')$, we have

$$h^0(E'(-D_N)) = 0, \quad h^0(E'(-D_{N-1})) \neq 0, \quad \text{and} \quad h^0(E(-D_{N-1})) < h^0(E'(-D_{N-1})).$$

Let $E_i = E[p_1 \xrightarrow{+} L_1] \cdots [p_{i-1} \xrightarrow{+} L_{i-1}][p_{i+1} \xrightarrow{+} L_{i+1}] \cdots [p_n \xrightarrow{+} L_n]$. Since $\chi(E_i) = \chi(E') - 1$, it suffices to show $h^0(E_i(-D_{N-1})) < h^0(E'(-D_{N-1}))$ for some i. This follows from the fact that

$$\bigcap_i H^0(E_i(-D_{N-1})) = H^0(E(-D_{N-1})) \subsetneq H^0(E'(-D_{N-1})). \qquad \square$$

3.3.4. Interpolation and short exact sequences

Lemma 3.15. *Consider an exact sequence*

$$0 \to S \to E \to Q \to 0$$

of vector bundles on an irreducible curve C. Suppose that S and Q satisfy interpolation and

$$\mu(S) \leq \lfloor \mu(Q) \rfloor + 1 \quad \textit{and} \quad \mu(Q) \leq \lfloor \mu(S) \rfloor + 1. \tag{3.6}$$

Then E also satisfies interpolation.

Proof. Since S and Q are nonspecial, E is nonspecial. By (3.6), there exists an integer $n \in \mathbb{Z}$ such that $\mu(S)$ and $\mu(Q)$ are contained in the closed interval $[n, n + 1]$. Since (3.5) is satisfied for D_+ a general divisor of degree $n + 1$, and D_- a general divisor of degree n, we conclude that E satisfies interpolation as desired. $\square$

We will most often use this result in the special case in which S is a line subbundle of E.

Corollary 3.16. *Suppose that $S \subset E$ is a nonspecial line subbundle and $|\mu(S) - \mu(E)| < 1$. If the quotient $Q = E/S$ satisfies interpolation, then E does as well.*

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Proof. By assumption $-1 < \frac{\deg(Q)-\mathrm{rk}(Q)\deg(S)}{\mathrm{rk}(Q)+1} < 1$. Hence, we have strict inequalities

$$\deg(S) < \mu(Q) + \frac{\mathrm{rk}(Q)+1}{\mathrm{rk}(Q)} \quad \text{and} \quad \mu(Q) < \deg(S) + \frac{\mathrm{rk}(Q)+1}{\mathrm{rk}(Q)},$$

which imply the required inequalities in Lemma 3.15, since $\mu(Q)$ is a $[1/\mathrm{rk}(Q)]$-integer. □

4. Overview

4.1. Base cases

We can reduce the number of base cases by extending Theorem 1.4 to $r = 1$ and $r = 2$. For $r = 2$, we replace N_C with the normal sheaf N_f, where $f\colon C \to \mathbb{P}^2$ is a general BN-curve. In this case, adjunction implies that $N_f = K_C \otimes f^*\mathcal{O}_{\mathbb{P}^2}(3)$ is a nonspecial line bundle, and therefore satisfies interpolation. For $r = 1$, we only consider the case where $f\colon C \to \mathbb{P}^1$ is an isomorphism, so $N_f = 0$ satisfies interpolation.

4.2. first strategy: degeneration of C

The first inductive strategy we will use is degeneration of C to reducible curves $X \cup Y$. In Section 5 we will study certain such degenerations, for which Y has a prescribed form, and we can thus relate interpolation for N_C to interpolation for certain modifications of N_X.

Since the sectional monodromy group of a general BN-curve always contains the alternating group [20], and in particular is $(r+1)$-transitive, it makes sense to talk of a general $(r+1)$-secant rational curve of degree $r-1$ in a hyperplane. While the following hypothesis does not encompass *all* modifications that might appear using this method, it includes those modifications that will play the most central role in our inductive argument:

Hypothesis 4.1 ($I(d,g,r,\ell,m)$). Let $C \subset \mathbb{P}^r$ be a general BN-curve of degree d and genus g. Let $u_1, v_1, \ldots, u_\ell, v_\ell$ be ℓ pairs of general points on C. Let $R_1, \ldots, R_m$ be m general $(r+1)$-secant rational curves of degree $r-1$ (contained in hyperplanes transverse to C). Then the modification

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1]\cdots[\overset{+}{\rightsquigarrow} R_m]$$

of the normal bundle of C satisfies interpolation.

A central complicating factor is that the inductive hypothesis $I(d,g,r,\ell,m)$ is not always true. The following definition describes a set of tuples (d,g,r,ℓ,m) for which we will prove that it holds.

Definition 4.2. A tuple (d,g,r,ℓ,m) is called good if it satisfies all the following conditions:

- The following inequalities hold:

$$d \geq g + r, \quad 0 \leq \ell \leq \frac{r}{2}, \quad \text{and} \quad 0 \leq m \leq \rho(d,g,r).$$

- If $g = m = 0$, then

$$2\ell \geq (1-d)\%(r-1),$$

where for integers a and b we write $a\%b$ for the reduced residue of a modulo b, and
- It is not the following set:

$$\left\{\begin{array}{llll} (5,2,3,0,0), & (4,1,3,1,0), & (4,1,3,0,1), & (4,1,3,1,1), \\ (6,2,4,0,0), & (5,1,4,1,0), & (5,1,4,1,1), & (5,1,4,2,1), \quad (6,2,4,1,1), \\ (7,2,5,0,0), & (6,1,5,0,1), & (6,1,5,1,1). & \end{array}\right\} \quad \text{(XEx)}$$

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Remark 4.3. By Clifford's theorem, the first inequality $d \geq g + r$ follows from $g \leq r$.

We conclude Section 5 by using this first strategy to show that if $I(d, g, r, 0, m)$ holds for every good tuple $(d, g, r, 0, m)$, then Theorem 1.4 holds except possibly for rational curves or canonical curves of even genus.

4.3. Second Strategy: Limits of Modifications and Projection

The basic issue with the first strategy described above is that every time we apply it, we get more modifications. In order to make an inductive argument work, we need a second inductive strategy that *decreases* the number of modifications.

Hypothesis $I(d, g, r, \ell, m)$ asserts that $N'_C := N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1] \cdots [\overset{+}{\rightsquigarrow} R_m]$ satisfies interpolation. Let p be a general point on C. The pointing bundle exact sequence induces the exact sequence

$$0 \to N_{C \to p} \to N'_C \to N_{\overline{C}}(p)[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{R}_1 \cup \cdots \cup \overline{R}_m] \to 0, \tag{4.1}$$

where $\overline{C}$ and $\overline{R}_j$ denote the images of C and R_j, respectively under projection from p. In order to apply Corollary 3.16 to relate interpolation for the original bundle N'_C to interpolation for the quotient bundle, the sequence must be close to balanced. The failure of the sequence to be balanced is related to the quantity

$$\delta = \delta(d, g, r, \ell, m) := \mu(N'_C) - \mu(N_{C \to p}) = \frac{2d + 2g - 2r + 2\ell + (r+1)m}{r-1}.$$

We first apply these ideas in Section 6 to treat the family of good tuples $(d, g, r, 0, 0)$ with $\delta(d, g, r, 0, 0) = 1$, which are difficult from the perspective of our more uniform inductive arguments.

More generally, in order to make this sequence sufficiently close to balanced, we will appropriately specialize the points on C at which the modifications occur. To illustrate this idea in the simplest possible case, assume here that $\ell \geq \lfloor \delta \rfloor$. Since the points $v_1, \ldots, v_{\lfloor \delta \rfloor}$ are general on C, we may specialize them all to the point p. This induces the specialization of N'_C to

$$N''_C := N_C[u_1 + \cdots + u_{\lfloor \delta \rfloor} \overset{+}{\leftrightarrow} p][u_{\lfloor \delta \rfloor + 1} \overset{+}{\leftrightarrow} v_{\lfloor \delta \rfloor + 1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m].$$

Using equation (3.1), the pointing bundle exact sequence becomes

$$\begin{aligned} 0 \to N_{C \to p}(u_1 + \cdots + u_{\lfloor \delta \rfloor}) \to N''_C \\ \to N_{\overline{C}}(p)[p \overset{+}{\to} u_1 + \cdots + u_{\lfloor \delta \rfloor}][u_{\lfloor \delta \rfloor + 1} \overset{+}{\leftrightarrow} v_{\lfloor \delta \rfloor + 1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{R}_1 \cup \cdots \cup \overline{R}_m] \to 0. \end{aligned}$$

By our auspicious choice to specialize exactly $\lfloor \delta \rfloor$ points to p, this sequence is now close enough to balanced to reduce to proving interpolation for the quotient bundle. Furthermore, since $u_1, \ldots, u_{\lfloor \delta \rfloor}$ are general, the modification at p in the quotient is linearly general and we can erase it by Lemma 3.12. It therefore suffices to prove interpolation for

$$N_{\overline{C}}[u_{\lfloor \delta \rfloor + 1} \overset{+}{\leftrightarrow} v_{\lfloor \delta \rfloor + 1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{R}_1 \cup \cdots \cup \overline{R}_m],$$

which evidently has fewer modifications. However, there are two basic issues with this argument:

1. In general, we might not have $\ell \geq \lfloor \delta \rfloor$.
2. Since $\overline{R}_i$ is still an $(r+1)$-secant curve of degree $r-1$, the argument does not reduce to another case of our inductive hypothesis.

To surmount both of these two difficulties, we will need to specialize the R_i as well.

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This second strategy will be fleshed out in Sections 7 and 8: In Section 7, we will study how to specialize the R_i so that they can also contribute modifications to $N_{C\to p}$. Then in Section 8, we will refine the basic argument outlined above to use these degenerations of the R_i as well, and also explain further degenerations that will be necessary to reduce to another case of our inductive hypothesis.

4.4. Outline of the remainder of paper

Section 9 is a brief interlude in which we use the inductive arguments of Section 8 to treat the case of rational curves not implied by $I(d,g,r,\ell,m)$ for good tuples. We also explain the counterexamples to Theorem 1.4 that are not counterexamples to Theorem 1.2. At this point, we will have reduced both Theorem 1.4 and Theorem 1.2 to $I(d,g,r,\ell,m)$ for good tuples, as well as Theorem 1.4 for canonical curves of even genus, which we treat at the end of the paper in Section 13. The intervening sections 10–12 inductively prove $I(d,g,r,\ell,m)$ for good tuples.

In Section 10, we complete a purely combinatorial analysis, in which we show that the inductive arguments of Section 8 can be applied to reduce $I(d,g,r,\ell,m)$ for *all* good tuples to

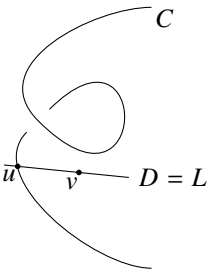
- $I(d,g,r,\ell,m)$ for a certain large but *finite* list of sporadic cases (d,g,r,ℓ,m) with $r \le 13$.
- The infinite family of tuples $(d,g,r,0,0)$ with $\delta = 1$, which was already treated in Section 6.

In Section 11, we give a more complicated, yet more flexible, inductive argument in the style of those in Section 8 and verify by exhaustive computer search that it reduces the finitely many sporadic cases identified above to a managable list of 30 base cases. These base cases are treated by ad-hoc techniques in Section 12.

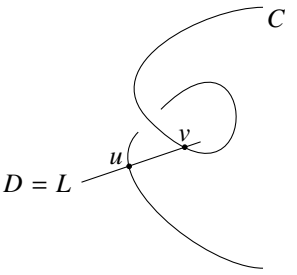
5. Basic degenerations

In this section, we discuss the three basic degenerations of BN-curves, to reducible curves $C \cup D$, that we will use in the proof of Theorem 1.4. In each subsection, we will first show that these degenerations lie in the Brill–Noether component. We will then relate interpolation for $N_{C\cup D}$, or a modification thereof, to interpolation for a particular modification of N_C.

In what follows, write $N'_{C\cup D}$ for a modification of $N_{C\cup D}$ away from D. In other words, $N'_{C\cup D}$ is a vector bundle on $C \cup D$, equipped with an isomorphism to $N_{C\cup D}$ over a dense open subset of $C \cup D$ containing the entire curve D, and in particular containing a neighborhood U of $C \cap D$ in C. Write N'_C for the bundle obtained by making the same modifications to N_C. In other words, N'_C is obtained from $N'_{C\cup D}|_{C\setminus(C\cap D)}$ by gluing along $U\setminus(C\cap D)$ via our given isomorphism to $N_C|_U$.



Peeling off a one-secant line.



Peeling off a one-secant line.

5.1. Peeling off one-secant lines

Our most basic degeneration will be when $D = L$ is a quasitransverse one-secant line. (Recall that two subschemes X and Y of a scheme Z are transverse (respectively, quasitransverse) at a point $p \in X \cap Y$ if the natural map of tangent spaces $T_pX \oplus T_pY \to T_pZ$ is surjective (respectively, either injective or

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surjective).) If C has degree d and genus g, then $C \cup L$ has degree $d+1$ and genus g. Write $v \in L \backslash \{u\}$ for any other point on the line L.

Lemma 5.1. *If C is a BN-curve, then $C \cup L$ is also a BN-curve.*

Proof. Generalizing C, we may suppose C is a general BN-curve. We will show $H^1(T_{\mathbb{P}^r}|_{C\cup L}) = 0$, which implies that the map $C \cup L \to \mathbb{P}^r$ may be lifted as $C \cup L$ is deformed to a general curve.

Since C is a general BN-curve, $H^1(T_{\mathbb{P}^r}|_C) = 0$ by the Gieseker–Petri theorem. Furthermore, we have $H^1(T_{\mathbb{P}^r}|_L(-u)) = 0$ because $T_{\mathbb{P}^r}|_L \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(r-1)}$. This implies $H^1(T_{\mathbb{P}^r}|_{C\cup L}) = 0$ as desired, using

$$0 \to T_{\mathbb{P}^r}|_L(-u) \to T_{\mathbb{P}^r}|_{C\cup L} \to T_{\mathbb{P}^r}|_C \to 0. \qquad \square$$

Lemma 5.2 (Lemma 8.5 of [2]). *If $N'_C(u)[2u \xrightarrow{-} v]$ satisfies interpolation, then so does $N'_{C\cup L}$.*

When C is nonspecial with genus small relative to r, we can combine Lemma 5.2 with Lemma 3.8 to reduce to a positive modification of N'_C.

Corollary 5.3. *Suppose that N'_C is a positive modification of N_C. If $d \geq g + r$ and $g \leq r + 6$ and $N'_C[2u \xrightarrow{+} v]$ satisfies interpolation, then $N'_{C\cup L}$ satisfies interpolation.*

Proof. Since $N'_C[2u \xrightarrow{+} v] \simeq \left(N'_C(u)[2u \xrightarrow{-} v]\right)(u)$, it suffices by Lemma 3.8 to show that

$$\chi\left(N'_C(u)[2u \xrightarrow{-} v]\right) \geq (r-1)g. \tag{5.1}$$

We have

$$\begin{aligned}\chi(N'_C(u)[2u \xrightarrow{-} v]) &\geq \chi(N_C(u)[2u \xrightarrow{-} v]) \\ &= (r+1)d - (r-3)(g-1) - (r-3) \\ &\geq (r+1)(g+r) - (r-3)g \\ &= 4g + r(r+1).\end{aligned}$$

If $g \leq r+6$, then

$$(r-1)g - 4g = (r-5)g \leq (r-5)(r+6) \leq r(r+1),$$

and the desired inequality (5.1) holds. $\square$

5.2. *Peeling off one-secant lines*

Our next basic degeneration will be to the union of a curve C and a quasitransverse one-secant line L, meeting C at points u and v. If C has degree d and genus g, then $C \cup L$ has degree $d+1$ and genus $g+1$.

Lemma 5.4. *If C is a BN-curve, then $C \cup L$ is also a BN-curve.*

Proof. As in the proof of Lemma 5.1, we have that $H^1(T_{\mathbb{P}^r}|_{C\cup L}) = 0$ by combining $H^1(T_{\mathbb{P}^r}|_C) = 0$ (from the Gieseker–Petri theorem) and $H^1(T_{\mathbb{P}^r}|_L(-u-v)) = 0$ (from $T_{\mathbb{P}^r}|_L \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}$). $\square$

We generalize Lemma 5.2 to one-secant lines. In slightly greater generality, let $p \in L\backslash\{u, v\}$ be a point, and Λ be a linear space disjoint from the span of the tangent lines to C at u and v.

Lemma 5.5 (Slight generalization of Lemma 8.8 of [2]). *If $N'_C(u+v)[u \xrightarrow{-} v][v \xrightarrow{-} u][v \xrightarrow{-} 2u + \Lambda]$ satisfies interpolation, then so does $N'_{C\cup L}[p \xrightarrow{+} \Lambda]$.*

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Proof. Imitate the proof Lemma 8.8 of [2], mutatis mutandis.

(In the notation of [2]: Take T instead to have dimension $r - 5 - \dim \Lambda$, where by convention $\dim \emptyset = -1$, and use instead the decomposition $N_L \simeq N_{L\to x} \oplus N_{L\to y} \oplus N_{L\to\Lambda} \oplus N_{L\to T}$.) □

As in Corollary 5.3, we can reduce to a positive modification of N'_C.

Corollary 5.6. *Suppose that N'_C is a positive modification of N_C. If $d \geq g + r$ and $g \leq r + 6$ and $N'_C[u \xrightarrow{+} v][v \xrightarrow{+} u][v \xrightarrow{+} 2u]$ satisfies interpolation, then $N'_{C\cup L}$ satisfies interpolation.*

Proof. We have

$$\begin{aligned}\chi\left(N'_C(u+v)[u \xrightarrow{-} v][v \xrightarrow{-} u][v \xrightarrow{-} 2u]\right) &\geq \chi\left(N_C(u+v)[u \xrightarrow{-} v][v \xrightarrow{-} u][v \xrightarrow{-} 2u]\right)\\ &= (r+1)d - (r-3)(g-1) - (r-5)\\ &\geq 4g + r(r+1) + 2.\end{aligned}$$

As in the proof of Corollary 5.3, if $g \leq r + 6$, then $4g + r(r + 1) + 2 \geq (r - 1)g$, and so

$$\chi\left(N'_C(u+v)[u \xrightarrow{-} v][v \xrightarrow{-} u][v \xrightarrow{-} 2u]\right) \geq (r-1)g. \tag{5.2}$$

Hence, by Lemma 3.8, $N'_C(u + v)[u \xrightarrow{-} v][v \xrightarrow{-} u][v \xrightarrow{-} 2u]$ satisfies interpolation. □

5.3. Peeling off rational normal curves in hyperplanes

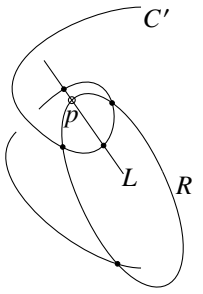
The final of our basic degenerations is to the union of a BN-curve C of degree at least $r + 1$ in $\mathbb{P}^r$, and an $(r + 1)$-secant rational curve R of degree $r - 1$ contained in a hyperplane H. If C has degree d and genus g, then $C \cup R$ has degree $d + r - 1$ and genus $g + r$. Observe that

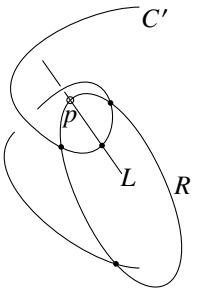
$$\rho(d, g, r) = \rho(d + r - 1, g + r, r) + 1.$$

Lemma 5.7. *If $\rho(d, g, r) \geq 1$, and C is a BN-curve, then $C \cup R$ is also a BN-curve.*

Proof. Generalizing C, we may suppose C is a general BN-curve.

If $g > 0$, then we can specialize C to the union $C' \cup L$, where C' is a general BN-curve of degree $d - 1$ and genus $g - 1$, and L is a general one-secant line. Otherwise, if $g = 0$, then we can specialize C to the union $C' \cup L$, where C' is a general BN-curve of degree $d - 1$ and genus g, and L is a general one-secant line. Either way, we can arrange for one of the points p of $C \cap R$ to specialize onto L, and the rest to specialize onto C'.





Write $\Gamma := (L \cup R) \cap C$. Note that this is a set of $r+1$ or $r+2$ points on C'. We will show the following:

(a) The curve $C' \cup L \cup R$ is a smooth point of the Hilbert scheme.
(b) The curve $L \cup R$ can be smoothed to a rational normal curve M while preserving the points of incidence with C'.
(c) The curve $C' \cup M$ is in the Brill–Noether component.

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By part (a), the curves $C \cup R$ and $C' \cup M$ are both generalizations of a smooth point of the Hilbert scheme. Hence, they are in the same component, which must be the Brill–Noether component by part (c).

Beginning with part (b), write N for the subsheaf of $N_{L\cup R}$ whose sections fail to smooth the node p. It suffices to show that $H^1(N(-\Gamma)) = 0$. This follows from the exact sequence

$$0 \to N_{R\cup L}|_R(-p-\Gamma) \to N(-\Gamma) \to N|_L(-\Gamma) \to 0,$$

together with the isomorphisms

$$\begin{aligned} N_{R\cup L}|_R &\simeq N_{R/H} \oplus \mathcal{O}_R(1)(p) \simeq \mathcal{O}_{\mathbb{P}^1}(r+2)^{\oplus r-2} \oplus \mathcal{O}_{\mathbb{P}^1}(r), \\ N|_L &\simeq N_L \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r-1}. \end{aligned}$$

Since $N_{L\cup R}(-\Gamma)$ is a positive modification of $N(-\Gamma)$, we also have $H^1(N_{L\cup R}(-\Gamma)) = 0$. Similarly, $N_{C'\cup L\cup R}|_{C'}$ is a positive modification of $N_{C'}$. As C' is a general BN-curve, the Gieseker–Petri theorem implies that $H^1(N_{C'\cup L\cup R}|_{C'}) = 0$. Hence, using the exact sequence

$$0 \to N_{L\cup R}(-\Gamma) \to N_{C'\cup L\cup R} \to N_{C'\cup L\cup R}|_{C'} \to 0,$$

we see that $H^1(N_{C'\cup L\cup R}) = 0$, and part (a) follows.

Finally, for part (c), we will show that $H^1(T_{\mathbb{P}^r}|_{C'\cup M}) = 0$, and hence the map from $C' \cup M$ to $\mathbb{P}^r$ can be lifted as $C' \cup M$ is smoothed to a general curve. This vanishing follows from the exact sequence

$$0 \to T_{\mathbb{P}^r}|_M(-\Gamma) \to T_{\mathbb{P}^r}|_{C'\cup M} \to T_{\mathbb{P}^r}|_{C'} \to 0,$$

the isomorphism $T_{\mathbb{P}^r}|_M \simeq \mathcal{O}_{\mathbb{P}^r}(r+1)^{\oplus r}$, and the Gieseker–Petri theorem ($H^1(T_{\mathbb{P}^r}|_{C'}) = 0$). □

Our next goal is to study the restricted normal bundle $N_{C\cup R}|_R \simeq N_R[\overset{+}{\rightsquigarrow} C]$, which is of slope $r+2$. In most cases, this bundle is perfectly balanced (equivalently semistable):

Lemma 5.8. *Unless r is odd and C is an elliptic normal curve, $N_R[\overset{+}{\rightsquigarrow} C]$ is perfectly balanced, that is*

$$N_R[\overset{+}{\rightsquigarrow} C] \simeq \mathcal{O}_{\mathbb{P}^1}(r+2)^{\oplus(r-1)}. \tag{5.3}$$

If r is odd and C is an elliptic normal curve, then $N_R[\overset{+}{\rightsquigarrow} C]$ is 'almost balanced', that is, is isomorphic to one of the two bundles:

$$\mathcal{O}_{\mathbb{P}^1}(r+2)^{\oplus(r-1)} \quad \text{or} \quad \mathcal{O}_{\mathbb{P}^1}(r+3) \oplus \mathcal{O}_{\mathbb{P}^1}(r+2)^{\oplus(r-3)} \oplus \mathcal{O}_{\mathbb{P}^1}(r+1).$$

Proof. Write d and g for the degree and genus of C. First, we reduce to the cases where C is nonspecial, that is, where $d \geq g+r$. To do this, when $d < g+r$, we inductively specialize C to a union $C' \cup D$, where C' is a general BN-curve of degree $d' \geq r+2$. Since in particular $d' \geq r+1$, we may specialize the points where R meets C onto C', thereby replacing C by C', which is not an elliptic normal curve since $d' \geq r+2$. To find such a specialization, we break into cases as follows:

1. If $d < g+r$ and $\rho(d,g,r) > 0$ (which forces $d \geq 2r+1 \geq r+3$), we apply Lemma 5.4 to degenerate C to the union of a general BN-curve C' of degree $d-1$ and genus $g-1$, with a one-secant line D.
2. If $d < g+r$ and $\rho(d,g,r) = 0$, but C is not a canonical curve (which forces $d \geq 3r \geq 2r+2$), we claim that we may specialize C to the union of a general BN-curve C' of degree $d-r$ and genus $g-r-1$, with an $(r+2)$-secant rational normal curve D. Indeed, as in the proof of Lemma 5.1, we have that $H^1(T_{\mathbb{P}^r}|_{C'\cup D}) = 0$ by combining $H^1(T_{\mathbb{P}^r}|_{C'}) = 0$ (from the Gieseker–Petri theorem) and $H^1(T_{\mathbb{P}^r}|_D(-D \cap C')) = 0$ (from $T_{\mathbb{P}^r}|_D \simeq \mathcal{O}_{\mathbb{P}^1}(r+1)^{\oplus r}$).

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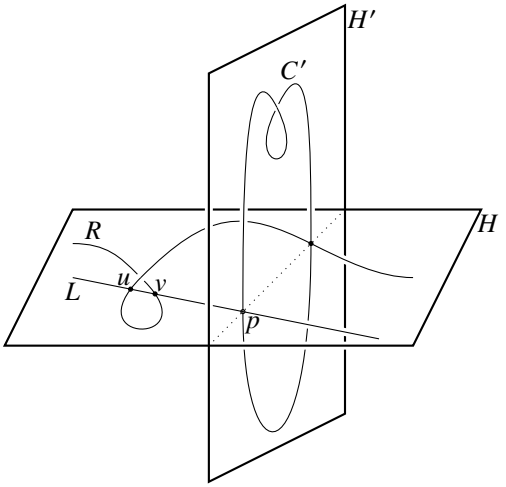
3. If C is a canonical curve, we claim that we may specialize C to the union of a general BN-curve C' of degree $r+2$ and genus 2, with a r-secant rational curve of degree $r-2$. Indeed, we glue an abstract curve of genus 2 to $\mathbb{P}^1$ at r general points, and map it to projective space via the complete linear series for the dualizing sheaf.

For C nonspecial, we will prove the lemma by induction on r. Let Γ be a collection of $r+1$ points where C meets R; for $r \geq 3$, this is exactly the intersection $C \cap R$, and so $N_R[\Gamma \overset{+}{\rightsquigarrow} C] = N_R[\overset{+}{\rightsquigarrow} C]$. However, we can extend the lemma to cover the case $r = 2$ as well, by replacing equation (5.3) with the assertion that $N_R[\Gamma \overset{+}{\rightsquigarrow} C] \simeq \mathcal{O}_{\mathbb{P}^1}(r+2)^{\oplus(r-1)}$. With this formulation, the base case of $r = 2$ is clear since $N_R[\Gamma \overset{+}{\rightsquigarrow} C]$ is a line bundle of degree 4. The base case $r = 3$ is [9, Lemma 4.2].

For the inductive step, we suppose $r \geq 4$. Let H' be a general hyperplane transverse to H. We will degenerate C to a union $C' \cup L$, where $C' \subset H'$ is a general BN-curve of degree $d-1$ and genus g. By hypothesis, $d \geq r+1$, and so C' meets H in at least r points, which are in linear general position in $H \cap H' \simeq \mathbb{P}^{r-2}$ since the sectional monodromy group of a general curve always contains the alternating group [20]. Since $\mathrm{Aut}\,\mathbb{P}^{r-2}$ acts transitively on r-tuples of points in linear general position, we can apply an automorphism so that $r-1$ of these points are on R and the final point p lies on a one-secant line L to R.



We claim that $C' \cup L$ is a BN-curve of degree d and genus g, as $H^1(T_{\mathbb{P}^r}|_{C' \cup L}) = 0$. Indeed, $H^1(T_{\mathbb{P}^r}|_{C'}) = 0$ because C' is nonspecial (and $T_{\mathbb{P}^r}$ is a quotient of $\mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1}$). Furthermore, since $T_{\mathbb{P}^r}|_L \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r}$, we have $H^1(T_{\mathbb{P}^r}|_L(-p)) = 0$. The result now follows by considering

$$0 \to T_{\mathbb{P}^r}|_L(-p) \to T_{\mathbb{P}^r}|_{C' \cup L} \to T_{\mathbb{P}^r}|_{C'} \to 0.$$

Call u and v the points on R where L is one-secant. Projecting from L induces an exact sequence

$$0 \to N_{R \to L}[u \overset{+}{\to} v][v \overset{+}{\to} u] \simeq \mathcal{O}_{\mathbb{P}^1}(r+2)^{\oplus 2} \to N_R[\overset{+}{\rightsquigarrow} C' \cup L] \to N_{\overline{R}}[\overset{+}{\rightsquigarrow} \overline{C'}](u+v) \to 0.$$

The curve $\overline{R}$ is again a rational curve of degree $r-3$ in a hyperplane that is incident to $\overline{C'}$ at $r-1$ points. Furthermore, if C is not an elliptic normal curve, then neither is $\overline{C'}$. Applying our inductive hypothesis (for $\mathbb{P}^{r-2}$), in combination with the above exact sequence, completes the proof. □

It is natural to ask what happens when C is an elliptic normal curve and r is odd. This case is only necessary to treat the special family of canonical curves of even genus. When we treat that case in Section 13, we will show that $N_R[\overset{+}{\rightsquigarrow} C]$ is not perfectly balanced in this case. Moreover, we will give a geometric construction of its Harder–Narasimhan filtration.

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5.4. Reduction to good tuples

In this section, we show that, apart from rational curves and canonical curves of even genus, all other cases of Theorem 1.4 follow from $I(d,g,r,0,m)$ for good tuples with $\ell = 0$.

Lemma 5.9. *Suppose that $\rho(d,g,r) \geq 0$ and that $(d,g,r) \neq (2r, r+1, r)$ if r is odd. If $g \geq r$ and $I(d-r+1, g-r, r, \ell, m+1)$ holds, then $I(d,g,r,\ell,m)$ holds.*

Proof. Let C be a general BN-curve of degree d and genus g in $\mathbb{P}^r$ with $g \geq r$. Let $u_1, v_1, \ldots, u_\ell, v_\ell$ be general points on C. Let $R_1, \ldots, R_m$ be general $(r+1)$-secant rational curves of degree $r-1$. The statement $I(d,g,r,\ell,m)$ asserts that

$$N_C[\overset{+}{\rightsquigarrow} R_1 + \cdots + R_m][u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell]$$

satisfies interpolation. Combining the assumptions that $\rho(d,g,r) \geq 0$ and $g \geq r$, we see that

$$d \geq r + \frac{rg}{r+1} \geq r + \frac{r^2}{r+1} > 2r - 1.$$

We may therefore prove $I(d,g,r,\ell,m)$ by peeling off an additional $(r+1)$-secant rational curve R_{m+1} of degree $r-1$. That is, we specialize the curve C as in Lemma 5.7 to the union of a general BN-curve C' of degree $d-r+1$ and genus $g-r$ and a rational curve R_{m+1}, in such a way that the points of $C \cap (R_1 \cup \cdots \cup R_m)$ and the u_i and v_i specialize onto C'. By virtue of specializing the auxilliary points onto C', we have

$$\left(N_{C' \cup R_{m+1}}[\overset{+}{\rightsquigarrow} R_1 + \cdots + R_m][u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell]\right)\big|_{R_{m+1}} \simeq N_{C' \cup R_{m+1}}|_{R_{m+1}}.$$

Since we assume that $(d,g,r) \neq (2r, r+1, r)$ if r is odd, Lemma 5.8 implies that $N_{C' \cup R_{m+1}}|_{R_{m+1}}$ is perfectly balanced. By Lemma 3.7, it suffices to prove that $N_{C' \cup R_{m+1}}|_{C'}$ satisfies interpolation. This restriction is

$$N_{C'}[\overset{+}{\rightsquigarrow} R_1 + \cdots + R_{m+1}][u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell],$$

which satisfies interpolation by our assumption that $I(d-r+1, g-r, r, \ell, m+1)$ holds. □

Proposition 5.10. *Suppose that $I(d,g,r,0,m)$ holds for all good $(d,g,r,0,m)$. Then $I(d,g,r,0,0)$ holds whenever $\rho(d,g,r) \geq 0$, except if*

- *(d,g,r) is in the list (1.4), or*
- *$(d,g,r) = (2r, r+1, r)$ and r is odd, or*
- *$g = 0$ and $d \not\equiv 1 \pmod{r-1}$.*

Proof. When $g = 0$, the tuple $(d,g,r,0,0)$ is good when $d \equiv 1 \bmod r-1$, so the result is a tautology. We therefore suppose $g \geq 1$.

We will prove by induction on g that $I(d,g,r,0,m)$ holds for $g \geq 1$ subject to the conditions that $m \leq \rho(d,g,r)$ and $(d,g,r,0,m)$ is not in the list (XEx). (If $(d,g,r,0,0)$ is in the list (XEx), then (d,g,r) is in the list (1.4), so this is sufficient.) Our base cases will be $g \leq r$; in these cases, $d \geq g+r$ by Remark 4.3, and so $(d,g,r,0,m)$ is good. For the inductive step, we apply Lemma 5.9 to reduce from $I(d,g,r,0,m)$ to $I(d-r+1, g-r, r, 0, m+1)$. □

6. The family with $\delta = 1$ and $\ell = m = 0$

In this section, we establish $I(d,g,r,0,0)$ for good tuples with $\delta = 1$. When $\ell = m = 0$, the condition $\delta = 1$ is equivalent to

$$2d + 2g = 3r - 1, \tag{6.1}$$

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and $I(d,g,r,0,0)$ asserts interpolation for N_C (with no modifications). Our argument will be by induction on $d-g-r$.

When $d-g-r=0$, then by equation (6.1), we have $(d,g,r)=(5g+1,g,4g+1)$. We may therefore conclude interpolation by [2, Lemma 11.3].

To complete the inductive step, observe that if $g=0$, then $d=(3r-1)/2$ by equation (6.1), and so such tuples $(d,0,r,0,0)$ are never good. It thus suffices to prove the following proposition in the case $g>0$. (The $g=0$ case is included here too since it will be useful later, when establishing Theorem 1.4 for rational curves, and it will follow via the same argument.)

Proposition 6.1. *Suppose that $\ell=m=0$, and $d>g+r$, and equation (6.1) holds. If $I(d-3,g,r-2,0,0)$ holds, and $g>0$ or the characteristic is not 2, then $I(d,g,r,0,0)$ also holds.*

In order to prove this proposition, we first establish the following lemmas.

Lemma 6.2. *Let L be a line meeting C quasitransversely at a smooth point x. For any points $y,y'\in L\backslash x$, the sections of $\mathbb{P}N_C$ corresponding to $N_{C\to y}$ and $N_{C\to y'}$ are tangent over x.*

Proof. We prove this by a calculation in local coordinates. We may choose an affine neighborhood of x in $\mathbb{P}^r$, and a local coordinate t on C so that $x=C(0)=0$, and C is given parametrically by the power series $C(t)=tC_1+t^2C_2+O(t^3)$. By assumption, $y'=a\cdot y$ for some invertible scalar a. It suffices to show that the three vectors $C(t)-y$, and $C(t)-y'$, and $C'(t)$, are dependent mod t^2. The explicit dependence is

$$-a(C(t)-y)+(C(t)-y')+t(a-1)C'(t)\equiv -a(tC_1-y)+(tC_1-y')+t(a-1)C_1=0(\text{mod } t^2).$$

□

Lemma 6.3. *Let u, v, and x be general points on a BN-curve C of degree d and genus g with*

$$2(d+1)+2g=3r-1 \quad \textit{and} \quad d\geq g+r. \tag{6.2}$$

Let y be a general point on the one-secant line $\overline{uv}$. Then the bundle

$$(N_{C\to u}\oplus N_{C\to v})\,[2x\xrightarrow{+}y]$$

satisfies interpolation if and only if $g>0$ or the characteristic is not 2.

Proof. Subtracting the first equation in equation (6.2) from 6 times the second inequality implies $2d+4-2g\geq 2g$. By Lemma 3.8, interpolation for $(N_{C\to u}\oplus N_{C\to v})\,[2x\xrightarrow{+}y]$ is thus equivalent to interpolation for

$$N:=(N_{C\to u}\oplus N_{C\to v})\,[2x\xrightarrow{-}y].$$

By Lemma 6.2, the subbundles $N_{C\to y}$ and $N_{C\to v}$ agree to second order at u. Therefore, the composition $N_{C\to y}\to N_{C\to u}\oplus N_{C\to v}\to N_{C\to u}$ vanishes to order 2 at u. Similarly, $N_{C\to y}\to N_{C\to v}$ vanishes to order 2 at v. Therefore, under the isomorphisms

$$N_{C\to y}\simeq \mathcal{O}_C(1),\quad N_{C\to u}\simeq \mathcal{O}_C(1)(2u),\quad \text{and}\quad N_{C\to v}\simeq \mathcal{O}_C(1)(2v), \tag{6.3}$$

the map $N_{C\to y}\to N_{C\to u}\oplus N_{C\to v}$ is a diagonal inclusion

$$\mathcal{O}_C(1)\to \mathcal{O}_C(1)(2u)\oplus\mathcal{O}_C(1)(2v)$$

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given by constant sections of $\mathcal{O}_C(2u)$ and $\mathcal{O}_C(2v)$ that vanish at $2u$ and $2v$, respectively. Such a section is indexed by two nonzero constants c and d. In other words,

$$N \simeq (\mathcal{O}_C(1)(2u) \oplus \mathcal{O}_C(1)(2v))\,[2x \xrightarrow{-} \mathcal{O}_C(1)],$$

where $O_C(1)$ is the diagonal subbundle identified above.

Since $\mu(N) = d+1$, it satisfies interpolation if and only if it has no cohomology when twisted down by a general line bundle of degree $d+2-g$. Write such a line bundle as $L^\vee(1)(2u+2v)$, where L is a general line bundle of degree $g+2$. We therefore want

$$N \otimes L(-1)(-2u-2v) = (L(-2v) \oplus L(-2u))\,[2x \xrightarrow{+} L(-2u-2v)] \tag{6.4}$$

to have no global sections, where the diagonal subbundle $L(-2u-2v)$ is indexed by $[c:d] \in \mathbb{G}_m$ as above.

As L is a general line bundle of degree $g+2$ and u and v are general points, $h^0(L(-2v)) = h^0(L(-2u)) = 1$; write σ and τ for the unique (up to scaling) sections of L vanishing to order 2 at u and v, respectively. Every section of $L(-2u) \oplus L(-2v)$ is a linear combination $a\sigma \oplus b\tau$, viewed as a section of $L \oplus L$. Such a global section comes from the subsheaf $(L(-2v) \oplus L(-2u))\,[2x \xrightarrow{+} L(-2u-2v)]$ when it is dependent with the constant diagonal section $c \oplus d$ at $2x$, that is, when the section $ad\sigma - bc\tau$ of L vanishes at $2x$. Hence, equation (6.4) has no cohomology if x is not a ramification point of the map $\varphi\colon C \to \mathbb{P}^1$ determined by $\langle \sigma, \tau \rangle \subseteq H^0(L)$.

As x was a general point, this holds if and only if φ is separable. If φ is not separable, then the characteristic p of the ground field is positive and φ factors through the Frobenius morphism F:

$$C \xrightarrow{F} C' \to \mathbb{P}^1.$$

In this case, L and the linear system $\langle \sigma, \tau \rangle$ are necessarily pulled back under F. In other words, $L \simeq F^*M$ for a line bundle M (necessarily general because L is general), of degree $(g+2)/p$ with $h^0(M) \geq 2$. Therefore,

$$\frac{g+2}{p} + 1 - g = h^0(M) \geq 2,$$

or upon rearrangement,

$$p \leq \frac{g+2}{g+1}.$$

Thus $g = 0$ and $p = 2$.

Conversely, when $g = 0$, there is a choice of coordinates $[t:s]$ on C so that $\langle \sigma, \tau \rangle = \langle t^2, s^2 \rangle$. If in addition $p = 2$, then the map φ is inseparable. □

Proof of Proposition 6.1. Specialize C to the union $C' \cup L$, where C' is a general BN-curve of degree $d-1$ and genus g in $\mathbb{P}^r$, and L is a one-secant line $\overline{xy}$ meeting C' at x. It suffices to show interpolation for

$$N_{C'}[2x \xrightarrow{+} y].$$

Let $u, v \in C'$ be general points, and specialize y to a general point on the line $\overline{uv}$. Projection from $\overline{uv}$ induces a pointing bundle exact sequence

$$0 \to (N_{C' \to u} \oplus N_{C' \to v})\,[2x \xrightarrow{+} y] \to N_{C'}[2x \xrightarrow{+} y] \to N_{\overline{C'}}(u+v) \to 0. \tag{6.5}$$

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By Lemma 6.3, the subbundle $(N_{C'\to u} \oplus N_{C'\to v})\,[2x \xrightarrow{+} y]$ satisfies interpolation. By hypothesis, $N_{\overline{C'}}$, and hence the quotient $N_{\overline{C'}}(u+v)$, also satisfies interpolation. Finally, by equation (6.1),

$$\mu\left((N_{C'\to u} \oplus N_{C'\to v})\,[2x \xrightarrow{+} y]\right) = d+2 = \frac{(r-1)d+2g-r-5}{r-3} = \mu(N_{\overline{C'}}(u+v)).$$

Thus, $N_{C'}[2x \xrightarrow{+} y]$ satisfies interpolation by Lemma 3.15. □

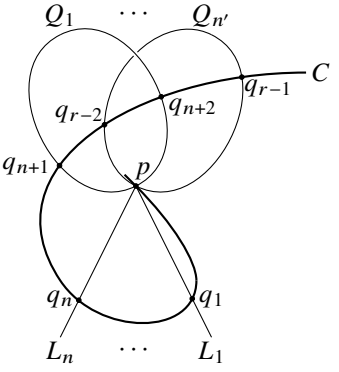
7. Specializations of the R_i

In this section, for some integers $n = n_i$, we construct a specialization of one of the rational curves $R = R_i$ so that exactly n modifications point towards a point p on C. We then show that this specialization plays well with projection from p.

7.1. Setup

Let n be an integer satisfying $0 \le n \le r-1$ and $n \equiv r-1 \bmod 2$. Let $p, q_1, q_2, \ldots, q_{r-1} \in C$ be points such that $2p + q_1 + \cdots + q_{r-1}$ lies in a hyperplane H. Assume that $2p + q_1 + \cdots + q_{r-1}$ is otherwise in linear general position, that is, $p + q_1 + \cdots + q_{r-1}$ and each $2p + q_1 + \cdots + q_{i-1} + q_{i+1} + \cdots + q_{r-1}$ spans H. When C is a general nonspecial BN-curve, we claim this assumption is satisfied if p is general and H is a general hyperplane containing $2p$. Indeed, in this case, the projection $\overline{C}$ from p is a general BN-curve, and p remains a general point on $\overline{C}$. Since the sectional monodromy group of a general curve always contains the alternating group [20], the corresponding points of $\overline{C}$ are in linear general position in the projection of H.

For i between 1 and n, write L_i for the line joining p and q_i. For j from 1e to $n' := (r-1-n)/2$, let Q_j be a plane conic passing through p, q_{n+2j-1} and q_{n+2j}. The following diagram illustrates the L_i and Q_j:



Define

$$R^\circ := L_1 \cup L_2 \cup \cdots \cup L_n \cup Q_1 \cup Q_2 \cup \cdots \cup Q_{n'}.$$

We will study when R° is a limit of $(r+1)$-secant rational normal curves R^t in hyperplanes that are $(r+1)$-secant to C, in such a way that exactly two points of secancy limit together to p while the remaining points of secancy limit to $q_1, q_2, \ldots, q_{r-1}$. For this, it is evidently necessary to have a containment of Zariski tangent spaces $T_pC \subset T_pR^\circ$. In what follows, we will show that this condition is sufficient.

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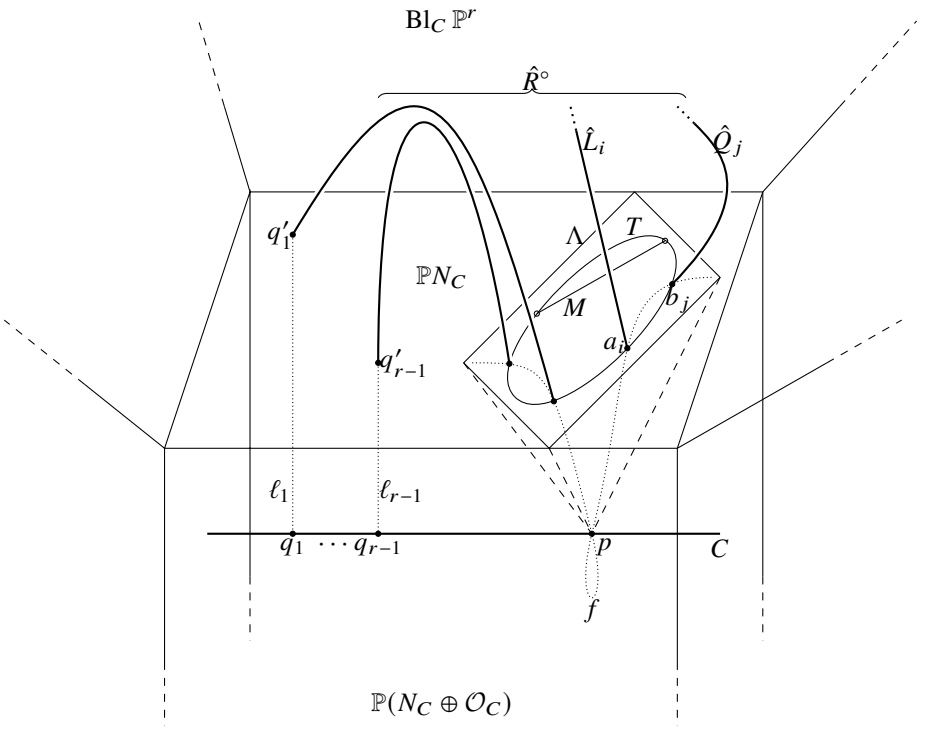
Suppose that $T_pC \subset T_pR^\circ$ and $m := n + n' = (r - 1 + n)/2 \geq 3$. Then the tangent line to L_i (respectively to Q_j) at p gives a distinguished point a_i (respectively b_j) in

$$\Lambda := \mathbb{P}(T_pR^\circ/T_pC) \simeq \mathbb{P}^{m-2} \subset \mathbb{P}N_C|_p.$$

Write $\Gamma = \{a_1, \ldots, a_n, b_1, \ldots, b_{n'}\}$, which is a collection of m points in Λ. Our linear generality assumption on $2p + q_1 + \cdots + q_{r-1}$ implies that Γ is linearly general in Λ. Indeed, a linear dependence between $a_{i_1}, \ldots, a_{i_\alpha}, b_{j_1}, \ldots, b_{j_\beta}$ in Λ implies a linear dependence between $2p, q_{i_1}, \ldots, q_{i_\alpha}, q_{n+2j_1-1}, q_{n+2j_1}, \ldots, q_{n+2j_\beta-1}, q_{n+2j_\beta}$ in H. Let $T \subset \Lambda$ be a general rational normal curve in Λ passing through Γ, and let M be a general one-secant line to T. Our argument will furthermore show that we can choose the family R^t so that the modifications along R^t at the points approaching p limit to M, that is, so that $N_C[\overset{+}{\rightsquigarrow} R^t]$ fits into a flat family whose central fiber is $N_C[q_1 + \cdots + q_{r-1} \overset{+}{\rightsquigarrow} R^\circ][p \overset{+}{\to} M]$.

7.2. *The construction*

To construct the desired family R^t, let B be the spectrum of a DVR, with special point $t = 0$. Consider the blowup of $\mathbb{P}^r \times B$ along $C \times 0$. The special fiber X over 0 has two components: The first is isomorphic to the blowup $\mathrm{Bl}_C \mathbb{P}^r$, and contains the proper transform $\hat{R}^\circ = \hat{L}_1 \cup \cdots \cup \hat{L}_n \cup \hat{Q}_1 \cup \cdots \cup \hat{Q}_{n'}$ of R°. The second is isomorphic to the normal cone $\mathbb{P}(N_C \oplus \mathcal{O}_C)$, and contains the special fiber of the proper transform of $C \times B$, which coincides with $\mathbb{P}\mathcal{O}_C \subset \mathbb{P}(N_C \oplus \mathcal{O}_C)$ and is isomorphic to C. The two components meet along $\mathbb{P}N_C$, and the intersection $\hat{R}^\circ \cap \mathbb{P}N_C$ is the finite set of points $\Gamma \cup \Gamma'$, where $\Gamma = \{a_1, \ldots, a_n, b_1, \ldots, b_{n'}\}$ lies in the fiber over p, and $\Gamma' = \{q'_1, q'_2, \ldots, q'_{r-1}\}$ contains one point q'_i in the fiber over each q_i. The following diagram illustrates the central fiber X:



Let $\ell_k \subset \mathbb{P}(N_C \oplus \mathcal{O}_C)|_{q_k}$ denote the line joining q_k to q'_k. These lines are pictured as dotted vertical lines in the above diagram.

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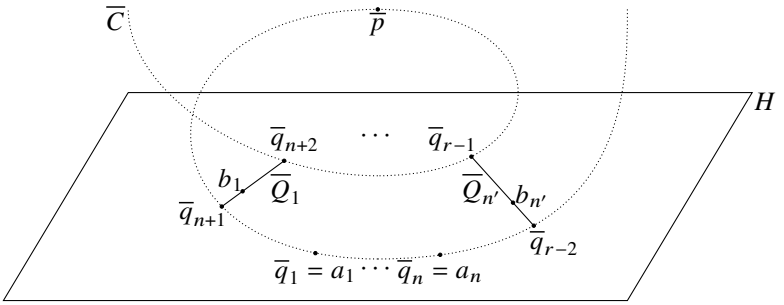
Write $\Delta = M \cap T$. The linear series $V := H^0(\mathcal{O}_T(1)) \oplus H^0(\mathcal{O}_T(1)(\Delta - \Gamma)) \subset H^0(\mathcal{O}_T(1)(\Delta))$ defines a map $\mathbb{P}^1 \simeq T \to \mathbb{P}V \simeq \mathbb{P}^{m-1}$ of degree m, which identifies the two points of Δ to a common point. Fix an embedding $\mathbb{P}V \hookrightarrow \mathbb{P}(N_C \oplus \mathcal{O}_C)$, which agrees with the identification $\mathbb{P}H^0(\mathcal{O}_T(1)) \simeq \Lambda$, and sends this common point to $p := \mathbb{P}\mathcal{O}_C|_p \in \mathbb{P}(N_C \oplus \mathcal{O}_C)|_p$. Composing these maps, we obtain a map $f\colon \mathbb{P}^1 \simeq T \to \mathbb{P}(N_C \oplus \mathcal{O}_C)$, which is pictured as the dotted curve in the above diagram. By construction, f passes through Γ and is nodal at p. Moreover, composing projection from p with f is the identity on T, and projection from p sends the Zariski tangent space of the image of f at p to M.

We now glue $\hat{R}^\circ$ to f and the ℓ_i, that is, we consider the map $F\colon \hat{R}^\circ \cup T \cup \ell_1 \cup \cdots \cup \ell_{r-1} \to X$ defined by the natural inclusions on $\hat{R}^\circ$ and the ℓ_i, and by f on T. To complete the argument, it suffices to deform F to the general fiber in a way that preserves its incidence to C. To check that this is possible, we just need to check that the corresponding obstruction space vanishes, that is, that $H^1(N_F[\tilde{\to} C]) = 0$.

7.3. *the normal space* $\mathbb{P}N_C|_p$

One tool that we will use – both to show in the next section that $H^1(N_F[\tilde{\to} C]) = 0$, and in the following section to analyze the transformation $[p \xrightarrow{+} M]$ – is the natural identification of $\mathbb{P}N_C|_p$ with the projection of $\mathbb{P}^r$ from T_pC. Under this projection, $\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_{r-1}$ are a collection of $r-1$ points, which are general subject to the condition of being contained in a hyperplane H. The conics Q_j project to the lines through $\overline{q}_{n+2j-1}$ and $\overline{q}_{n+2j}$. The image $\overline{p}$ of p is identified with the osculating 2-plane, which coincides with $\mathbb{P}N_{C\to p}|_p \in \mathbb{P}N_C|_p$. Under this identification, the points a_i are identified with $\overline{q}_i$, and the points b_j lie on $\overline{Q}_j$. The following diagram illustrates this setup:



7.4. *Vanishing of* $H^1(N_F[\tilde{\to} C])$

Given a vector bundle E on a reducible curve $X \cup_\Gamma Y$, recall that the Mayer–Vietoris sequence is

$$0 \to E \to E|_X \oplus E|_Y \to E|_\Gamma \to 0.$$

Applying this to $E = N_F[\tilde{\to} C]$, we obtain

$$0 \to N_F[\tilde{\to} C] \to N_{\hat{R}^\circ/\mathrm{Bl}_C \mathbb{P}^r} \oplus N_f[\tilde{\to} C] \oplus \bigoplus_{k=1}^{r-1} N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)}[\tilde{\to} C] \to T_{\mathbb{P}N_C}|_{\Gamma\cup\Gamma'} \to 0. \quad (7.1)$$

For each of the direct summands in the middle term, we both show that H^1 vanishes, and extract information about the image of its global sections in $T_{\mathbb{P}N_C}|_{\Gamma\cup\Gamma'}$. We then combine this information to show that the rightmost map is surjective on global sections.

Lemma 7.1. *We have* $H^1(N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)}[\tilde{\to} C]) = 0$, *and* $H^0(N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)}[\tilde{\to} C])$ *surjects onto* $T_{\mathbb{P}N_C}|_{q'_k}$.

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Proof. Both statements follow from $H^1(N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)}[\bar{\rightsquigarrow} C](-q_k')) = 0$, which can in turn be deduced from

$$\begin{aligned} 0 \to N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)|_{q_k}}(-q_k - q_k') \to N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)}[\bar{\rightsquigarrow} C](-q_k') \\ \to N_{\mathbb{P}(N_C\oplus\mathcal{O}_C)|_{q_k}/\mathbb{P}(N_C\oplus\mathcal{O}_C)}|_{\ell_k}(-q_k') \to 0, \end{aligned}$$

using the isomorphisms

$$N_{\ell_k/\mathbb{P}(N_C\oplus\mathcal{O}_C)|_{q_k}} \simeq \mathcal{O}_{\ell_k}(1)^{r-1} \quad \text{and} \quad N_{\mathbb{P}(N_C\oplus\mathcal{O}_C)|_{q_k}/\mathbb{P}(N_C\oplus\mathcal{O}_C)} \simeq \mathcal{O}_{\mathbb{P}(N_C\oplus\mathcal{O}_C)|_{q_k}}. \qquad \square$$

Lemma 7.2. *We have $H^1(N_f[\bar{\rightsquigarrow} C]) = 0$. Moreover, the image $H^0(N_f[\bar{\rightsquigarrow} C]) \to T_{\mathbb{P}N_C}|_\Gamma$ consists of those deformations of Γ that can be lifted to deformations of Λ, that is, this image coincides with the full preimage in $T_{\mathbb{P}N_C}|_\Gamma$ of the image of $H^0(N_{\Lambda/\mathbb{P}N_C}) \to N_{\Lambda/\mathbb{P}N_C}|_\Gamma$.*

Proof. The map $\pi_p \circ f\colon T \to T$ is the identity map, so the pointing bundle exact sequence yields a surjection

$$N_f[\bar{\rightsquigarrow} C] \to N_T,$$

which we may further compose with the surjection $N_T \to N_{\Lambda/\mathbb{P}N_C}|_T$ coming from the inclusion $T \subset \Lambda$. Define K via the exact sequence

$$0 \to K \to N_f[\bar{\rightsquigarrow} C] \to N_{\Lambda/\mathbb{P}N_C}|_T \to 0.$$

By considering the diagram

$$\begin{array}{ccccccccc} & & & & & & & & H^1(K(-\Gamma)) \\ & & & & & & & & \downarrow \\ H^0(K) & \longrightarrow & H^0(N_f[\bar{\rightsquigarrow} C]) & \twoheadrightarrow & H^0(N_{\Lambda/\mathbb{P}N_C}|_T) & \longrightarrow & H^1(K) & \longrightarrow & H^1(N_f[\bar{\rightsquigarrow} C]) \twoheadrightarrow H^1(N_{\Lambda/\mathbb{P}N_C}|_T) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ K|_\Gamma & \longrightarrow & T_{\mathbb{P}N_C}|_\Gamma & \longrightarrow & N_{\Lambda/\mathbb{P}N_C}|_\Gamma & \longrightarrow & 0 & & \\ \downarrow & & & & & & & & \\ H^1(K(-\Gamma)) & & & & & & & & \end{array}$$

and noting that $H^0(N_\Lambda) \simeq H^0(N_\Lambda|_T)$, it suffices to show that

$$H^1(K(-\Gamma)) = H^1(N_{\Lambda/\mathbb{P}N_C}|_T) = 0.$$

The first vanishing statement follows from the pointing bundle sequence

$$0 \to N_{f\to C}(-\Delta) \simeq \mathcal{O}_T(\Delta+\Gamma) \to K \to N_{T/\Lambda} \simeq \mathcal{O}_T(\Gamma)^{\oplus(m-3)} \to 0.$$

The second vanishing statement follows from the sequence:

$$0 \to N_{\Lambda/\mathbb{P}N_C|_p}|_T \simeq \mathcal{O}_T(1)^{r-m} \to N_{\Lambda/\mathbb{P}N_C}|_T \to N_{\mathbb{P}N_C|_p/\mathbb{P}N_C}|_T \simeq \mathcal{O}_T \to 0. \qquad \square$$

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We finally consider the bundle

$$N_{\hat{R}^\circ/\mathrm{Bl}_C\,\mathbb{P}^r} \simeq \bigoplus_{i=1}^{n} N_{\hat{L}_i/\mathrm{Bl}_C\,\mathbb{P}^r} \oplus \bigoplus_{j=1}^{n'} N_{\hat{Q}_j/\mathrm{Bl}_C\,\mathbb{P}^r}.$$

To describe the images of

$$H^0(N_{\hat{L}_i/\mathrm{Bl}_C\,\mathbb{P}^r}) \to T_{\mathbb{P}N_C}|_{a_i} \quad \text{and} \quad H^0(N_{\hat{Q}_j/\mathrm{Bl}_C\,\mathbb{P}^r}) \to T_{\mathbb{P}N_C}|_{b_j},$$

we define

$$A_i = T_{q_i}C \quad \text{and} \quad B_j = \langle T_{q_{n+2j-1}}C, T_{q_{n+2j}}C\rangle,$$

for the tangent line or span of tangent lines, and $\overline{A}_i$ and $\overline{B}_j$ for their projections from T_pC.

Lemma 7.3. *We have $H^1(N_{\hat{L}_i/\mathrm{Bl}_C\,\mathbb{P}^r}) = 0$. Moreover, the image of $H^0(N_{\hat{L}_i/\mathrm{Bl}_C\,\mathbb{P}^r}) \to T_{\mathbb{P}N_C}|_{a_i}$ has the following two properties:*

1. *It surjects onto $T_C|_p$.*
2. *The kernel of the map from the image to $T_C|_p$ is precisely $T_{a_i}\overline{A}_i$.*

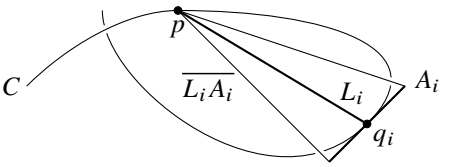
Proof. We have an exact sequence

$$0 \to N_{L_i/\mathbb{P}^r}[q_i \tilde{\to} C](-p) \to N_{\hat{L}_i/\mathrm{Bl}_C\,\mathbb{P}^r} \to T_C|_p \to 0.$$

Using this, the kernel in equation (2) is the image of $H^0(N_{L_i/\mathbb{P}^r}[q_i \tilde{\to} C](-p))$ in $T_{\mathbb{P}N_C}|_{a_i}$. Moreover, the normal bundle exact sequence for L_i in the span $\overline{L_iA_i}$ gives

$$0 \to N_{L_i/\overline{L_iA_i}}(-p) \to N_{L_i/\mathbb{P}^r}[q_i \tilde{\to} C](-p) \to N_{\overline{L_iA_i}/\mathbb{P}^r}|_{L_i}(-q_i - p) \to 0,$$

and $N_{L_i/\overline{L_iA_i}}(-p)|_p$ is identified with $T_{a_i}\overline{A}_i$ under projection from T_pC.



The same diagram chase as in Lemma 7.2 implies that it suffices to show:

$$H^1(N_{L_i/\overline{L_iA_i}}(-2p)) = H^0(N_{\overline{L_iA_i}/\mathbb{P}^r}|_{L_i}(-q_i - p)) = H^1(N_{\overline{L_iA_i}/\mathbb{P}^r}|_{L_i}(-q_i - p)) = 0.$$

These statements follow from the isomorphisms

$$N_{L_i/\overline{L_iA_i}}(-2p) \simeq \mathcal{O}_{L_i}(-1) \quad \text{and} \quad N_{\overline{L_iA_i}/\mathbb{P}^r}|_{L_i}(-p - q_i) \simeq \mathcal{O}_{L_i}(-1)^{r-2}.$$

□

Lemma 7.4. *We have $H^1(N_{\hat{Q}_j/\mathrm{Bl}_C\,\mathbb{P}^r}) = 0$. Moreover, the image of $H^0(N_{\hat{Q}_j/\mathrm{Bl}_C\,\mathbb{P}^r}) \to T_{\mathbb{P}N_C}|_{b_j}$ has the following two properties:*

1. *It surjects onto $T_C|_p$.*
2. *The kernel of the map from the image to $T_C|_p$ is precisely $T_{b_j}\overline{B}_j$.*

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Proof. We will imitate the proof of Lemma 7.3. We have an exact sequence

$$0 \to N_{Q_j/\mathbb{P}^r}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-p) \to N_{\hat{Q}_j/\mathrm{Bl}_C \mathbb{P}^r} \to T_C|_{b_j} \to 0.$$

Moreover, the normal bundle exact sequence for Q_j in the span $\overline{Q_j B_j}$ gives

$$\begin{aligned} 0 &\to N_{Q_j/\overline{Q_j B_j}}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-p) \to N_{Q_j/\mathbb{P}^r}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-p) \\ &\to N_{\overline{Q_j B_j}/\mathbb{P}^r}|_{Q_j}(-q_{n+2j-1} - q_{n+2j} - p) \to 0, \end{aligned}$$

and $N_{Q_j/\overline{Q_j B_j}}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-p)|_p$ is identified with $T_{b_j}\overline{B}_j$ under projection from $T_p C$. Our goal is therefore to show both

$$H^1(N_{Q_j/\overline{Q_j B_j}}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-2p)) = 0$$

and

$$H^0(N_{\overline{Q_j B_j}/\mathbb{P}^r}|_{Q_j}(-q_{n+2j-1} - q_{n+2j} - p)) = H^1(N_{\overline{Q_j B_j}/\mathbb{P}^r}|_{Q_j}(-q_{n+2j-1} - q_{n+2j} - p)) = 0.$$

The first vanishing statement follows from the exact sequence:

$$\begin{aligned} 0 &\to \left[N_{Q_j/\overline{Q_j}}(-q_{n+2j-1} - q_{n+2j} - 2p) \simeq \mathcal{O}_{\mathbb{P}^1}\right] \to N_{Q_j/\overline{Q_j B_j}}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-2p) \\ &\to \left[N_{\overline{Q_j}/\overline{Q_j B_j}}|_{Q_j}[q_{n+2j-1} + q_{n+2j} \tilde{\to} C](-2p) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\right] \to 0. \end{aligned}$$

The second vanishing statement follows from the isomorphism

$$N_{\overline{Q_j B_j}/\mathbb{P}^r}|_{Q_j}(-q_{n+2j-1} - q_{n+2j} - p) \simeq O_{\mathbb{P}^1}(-1)^{\oplus(r-4)}.$$

□

Combining these lemmas, we immediately see that H^1 of the middle terms in the sequence (7.1) vanish. We now see that the rightmost map of equation (7.1) is surjective on global sections, as follows. First, we apply Lemma 7.1 to handle the points of Γ'; this reduces our problem to showing the surjectivity of

$$H^0(N_{\hat{R}^\circ/\mathrm{Bl}_C \mathbb{P}^r}) \oplus H^0(N_f[\tilde{\to} C]) \to T_{\mathbb{P}N_C}|_\Gamma.$$

Applying Lemmas 7.3 and 7.4, we see that the composition to $(T_C|_p)^m$ is surjective. It thus suffices to show that the image contains the kernel of $T_{\mathbb{P}N_C}|_\Gamma \to (T_C|_p)^m$, that is, $T_{\mathbb{P}N_C|_p}|_\Gamma$.

Since removing any point from Γ yields a linearly independent collection of points, any deformation in $\mathbb{P}N_C|_p$ of all but one point of Γ lifts to a deformation of Λ. Combining Lemma 7.2 with Lemma 7.3, we therefore conclude that the image contains each $T_{\langle \Lambda, \overline{A}_i\rangle}|_\Gamma$. Similarly, combining Lemma 7.2 with Lemma 7.4, we conclude that the image contains each $T_{\langle \Lambda, \overline{B}_j\rangle}|_\Gamma$. Since the $T_{\langle \Lambda, \overline{A}_i\rangle}|_\Gamma$ and $T_{\langle \Lambda, \overline{B}_j\rangle}|_\Gamma$ span $T_{\mathbb{P}N_C|_p}|_\Gamma$, the desired result follows.

7.5. *The transformation* $[p \xrightarrow{+} M]$

We next show that M is 'suitably generic' in $\mathbb{P}N_C|_p$.

Lemma 7.5. *Fix a general BN-curve C and a general point $p \in C$.*

1. *If $n \geq 2$: As $q_1, q_2, \ldots, q_{r-1}$ vary, M is linearly general in $\mathbb{P}N_C|_p$.*
2. *If $n \geq 3$: This remains true if we fix $q_{n+1}, q_{n+2}, \ldots, q_{r-1} \in C$ to be general. In other words, as just the remaining points $q_1, q_2, \ldots, q_n$ vary, M is still linearly general in $\mathbb{P}N_C|_p$.*

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Proof. Fix $\Lambda \subset \mathbb{P}N_C|_p$ of codimension 2; we want to show that M can be disjoint from Λ. In the first case, $\overline{q}_1$ and $\overline{q}_2$ are general points on $\overline{C}$. In the second case, since any $(r-1-n)+2 = r+1-n \leq r-2$ points in $\mathbb{P}^{r-2}$ lie in a hyperplane, the points $\overline{q}_1$ and $\overline{q}_2$ remain general even as $q_{n+1}, q_{n+2}, \ldots, q_{r-1} \in C$ are fixed. In either case, the line between $\overline{q}_1$ and $\overline{q}_2$ is therefore disjoint from Λ. This completes the proof because M can be specialized to the line between $\overline{q}_1$ and $\overline{q}_2$. □

Lemma 7.5(2) is sharp, in the sense that the conclusion is always false if $n = 2$ (the subspace M is never transverse to $\Lambda = \overline{q_3 q_4 \cdots q_{r-1}}$). Nevertheless, there is a variant that does hold for $n = 2$. By part (1), the general such M is disjoint from $\overline{p} = \mathbb{P}N_{C\to p}|_p \in \mathbb{P}N_C|_p$. We may therefore ask for the weaker conclusion that the image of M is linearly general in the quotient $\mathbb{P}(N_C/N_{C\to p})|_p$, that is, that M is transverse to any Λ containing $\overline{p}$. In this case, the analog of Lemma 7.5(2) holds apart from a single counterexample.

Lemma 7.6. *Suppose $n = 2$, and fix a general BN-curve C and general points $p, q_3, q_4, \ldots, q_{r-1} \in C$. If C is not an elliptic normal curve, then as q_1, q_2 vary, M is linearly general in $\mathbb{P}(N_C/N_{C\to p})|_p$.*

Proof. By assumption, $r - 1 \equiv n = 2 \bmod 2$; hence, r is odd. If C is not an elliptic curve, then since $d \geq r+1$, either $d \geq r+2$ or $(d,g) = (r+1, 0)$. We consider these two cases separately.

Case 1: $d \geq r+2$. Let $\Lambda \subset \mathbb{P}N_C|_p$ be any codimension 2 plane containing $\overline{p}$. We will show that M can be chosen disjoint from Λ. Since any $(r-1-n)+1 = r-n = r-2$ points lie in a hyperplane, $\overline{q}_1$ is a general point on $\overline{C}$ and is therefore not contained in Λ. Let $H \simeq \mathbb{P}^{r-3}$ be a general hyperplane containing $\overline{q}_3, \overline{q}_4, \ldots, \overline{q}_{r-1}$. Since $\overline{p} \notin H$ and $\overline{p} \in \Lambda$, it follows that Λ is transverse to H. As $d \geq r+2$, the hyperplane section $H \cap \overline{C}$ contains two points $\{x, y\}$ distinct from $\overline{q}_1, \overline{q}_3, \overline{q}_4, \ldots, \overline{q}_{r-1}$. Since the sectional monodromy group of a general curve always contains the alternating group [20], the points $\{x, y, \overline{q}_1, \overline{q}_3, \overline{q}_4, \ldots, \overline{q}_{r-1}\}$ are in linear general position. For any $k \geq 0$, there is a unique k-plane in $\mathbb{P}^{2k+2}$ meeting each of $k+2$ lines in linear general position. Applying this with $k = (r-7)/2$, we see that there is a unique $[(r-3)/2]$-plane $\Lambda_x \subset H$ containing $\overline{q}_1$ and x, and meeting each of the lines $\overline{Q}_j$. If $\overline{q}_2 = x$, then by this uniqueness, Λ_x coincides with the projection of $T_p R^\circ$. Similarly define Λ_y. Because M can be linearly general in either Λ_x or Λ_y, it suffices to show that one of Λ_x or Λ_y contains a line disjoint from Λ.

Note that $\Lambda_x \cap \overline{Q}_1$ is the projection of x from $\langle \overline{q}_1, \overline{Q}_2, \ldots, \overline{Q}_{n'} \rangle$ onto $\overline{Q}_1$, and similarly for $\Lambda_y \cap \overline{Q}_1$. It follows that $\Lambda_x \cap \overline{Q}_1 \neq \Lambda_y \cap \overline{Q}_1$, and thus that $\langle \Lambda_x, \Lambda_y \rangle$ contains $\overline{Q}_1$. Similarly, $\langle \Lambda_x, \Lambda_y \rangle$ contains all other $\overline{Q}_i$. By inspection, $\langle \Lambda_x, \Lambda_y \rangle$ contains $\overline{q}_1, x$, and y. Therefore, $\langle \Lambda_x, \Lambda_y \rangle = H$. In particular, $\langle \Lambda_x, \Lambda_y \rangle$ is a distinct hyperplane from $\langle \Lambda, \overline{q}_1 \rangle$. Without loss of generality, Λ_x contains a point $z \notin \langle \Lambda, \overline{q}_1 \rangle$. Then $\langle z, \overline{q}_1 \rangle$ gives the desired line contained in Λ_x and disjoint from Λ.

Case 2: $(d, g) = (r+1, 0)$. Since $\overline{f}\colon \mathbb{P}^1 \simeq C \to \mathbb{P}^{r-2}$ is a general rational curve of degree $r-1$, it suffices to verify that M is linearly general for a particular choice of $\overline{f}$. We may therefore take

$$\overline{f}(t) = \left[t^2 + 1 : t : \frac{t-p_3}{t-q_3} : \frac{t-p_4}{t-q_4} : \cdots : \frac{t-p_{r-1}}{t-q_{r-1}} \right],$$

where $p_i \in \mathbb{P}^1$ are general. For $3 \leq i \leq r-1$, we have $\overline{f}(q_i) = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, where the 1 occurs in the ith position. (The interested reader may verify that this is not actually a specialization, that is, the general rational curve of degree $r-1$ in $\mathbb{P}^{r-2}$ is of this form after applying automorphisms of the source and target.)

Let $H = H_s$ be a generic hyperplane passing through $\overline{q}_3, \overline{q}_4, \ldots, \overline{q}_{r-1}$, defined by the ratio of the first two coordinates being equal to s. Note that H meets $\overline{f}(\mathbb{P}^1)$ at two other points $\overline{q}_1 = \overline{f}(q_1)$ and $\overline{q}_2 = \overline{f}(q_2)$. The parameters q_1 and q_2 are the solutions of the equation $t + t^{-1} = (t^2+1)/t = s$. The projection Λ_s of $T_p R^\circ$ is the unique $[(r-3)/2]$-plane Λ_s containing $\overline{q}_1$ and $\overline{q}_2$ and meeting each of the lines $\overline{Q}_i$. We will show that, for $s \in \mathbb{P}^1$ generic, Λ_s is transverse to any fixed subspace Λ of codimension 2 containing $\overline{p}$. Hence, a general line $M \subseteq \Lambda_s$ is disjoint from Λ.

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To show this, we calculate Λ_s explicitly. Since Λ_s is unique, it suffices to exhibit a particular $[(r-3)/2]$-plane containing $\overline{q}_1$ and $\overline{q}_2$ and meeting each of the lines $\overline{Q}_i$. We claim that we may take:

$$\Lambda_s = \langle \alpha(s), \beta_1(s), \beta_2(s), \dots, \beta_{(r-3)/2}(s) \rangle,$$

where

$$\alpha(s) = \left[s : 1 : \frac{p_3q_4s - p_3 - q_4}{q_3q_4 - 1} : \frac{p_4q_3s - p_4 - q_3}{q_3q_4 - 1} : \right.$$
$$\left. \cdots : \frac{p_{r-2}q_{r-1}s - p_{r-2} - q_{r-1}}{q_{r-2}q_{r-1} - 1} : \frac{p_{r-1}q_{r-2}s - p_{r-1} - q_{r-2}}{q_{r-2}q_{r-1} - 1} \right]$$

$$\beta_i(s) = \left[0 : 0 : \cdots : 0 : 0 : \frac{p_{2i+1}}{q_{2i+1}} \cdot \frac{s - q_{2i+1} - p_{2i+1}^{-1}}{s - q_{2i+1} - q_{2i+1}^{-1}} : \frac{p_{2i+2}}{q_{2i+2}} \cdot \frac{s - q_{2i+2} - p_{2i+2}^{-1}}{s - q_{2i+2} - q_{2i+2}^{-1}} : 0 : 0 : \cdots : 0 : 0 \right].$$

Here, the nonzero entries of $\beta_i(s)$ occur in the $(2i+1)$st and $(2i+2)$nd coordinates. Indeed, Λ_s meets $\overline{Q}_i$ at $\beta_i(s)$, so it suffices to check that Λ_s contains $\overline{f}(t)$ when $s = t + t^{-1}$. This follows from the following identity, which may be verified by separately considering the first coordinate, the second coordinate, the $(2i+1)$st coordinate, and the $(2i+2)$nd coordinate:

$$\overline{f}(t) = t \cdot \alpha(t + t^{-1}) - \sum_i \frac{(q_{2i+1}t - 1)(q_{2i+2}t - 1)}{q_{2i+1}q_{2i+2} - 1} \cdot \beta_i(t + t^{-1}).$$

This establishes that Λ_s is given by the above explicit formula, as claimed.

From the above explicit formulas for α and the β_i, it is evident that α is an isomorphism from $\mathbb{P}^1$ onto a line L, and the β_i are quadratic maps from $\mathbb{P}^1$ onto lines M_i such that $L, M_1, M_2, \dots, M_{(r-3)/2}$ are linearly independent and span $\mathbb{P}^{r-2}$. In fact, the above formulas for the β_i imply that, up to changing coordinates on the M_i, the β_i are *independently general* quadratic maps – so in particular distinct (from themselves and from α). Since the image of $\overline{f}$ does not lie in any union of proper linear subspaces, and Λ must meet $\overline{p}$ (which is a general point on the image of $\overline{f}$), all that remains is to prove Lemma 7.7 below. □

Lemma 7.7. *Let $L_1, L_2, \dots, L_k \subset \mathbb{P}^{2k-1}$ be linearly independent lines, and $\beta_i : \mathbb{P}^1 \to L_i$ be maps which are pairwise distinct (under every possible identification of L_i with L_j).*

If $\Lambda \subset \mathbb{P}^{2k+1}$ is a fixed codimension 2 subspace that is not transverse to $\langle \beta_1(s), \beta_2(s), \dots, \beta_k(s) \rangle$ for $s \in \mathbb{P}^1$ general, then Λ is the span of $k-1$ of the k given lines $L_1, L_2, \dots, L_k$.

Proof. We argue by induction on k. For the base case, we take $k = 1$, which is vacuous.

For the inductive step, we suppose $k \geq 2$, and we divide into cases based on how Λ meets $\langle L_1, L_2, \dots, L_{k-1} \rangle$. If $\Lambda = \langle L_1, L_2, \dots, L_{k-1} \rangle$, then the desired conclusion evidently holds.

Next, consider the case when Λ meets $\langle L_1, L_2, \dots, L_{k-1} \rangle$ in codimension 1. Fix $s \in \mathbb{P}^1$ general. Then the intersection $\Lambda \cap \langle L_1, L_2, \dots, L_{k-1} \rangle$ does not contain, and hence is transverse to, $\langle \beta_1(s), \beta_2(s), \dots, \beta_{k-1}(s) \rangle$ inside of $\langle L_1, L_2, \dots, L_{k-1} \rangle \simeq \mathbb{P}^{2k-3}$. Also, we have $\beta_k(s) \notin \Lambda + \langle L_1, L_2, \dots, L_{k-1} \rangle$, since $\langle L_1, L_2, \dots, L_k \rangle = \mathbb{P}^{2k-1}$. Combining these, Λ is transverse to $\langle \beta_1(s), \beta_2(s), \dots, \beta_k(s) \rangle$ in violation of our assumption.

Finally, consider the case when Λ is transverse to $\langle L_1, L_2, \dots, L_{k-1} \rangle$. Applying our inductive hypothesis, $\Lambda \cap \langle L_1, L_2, \dots, L_{k-1} \rangle$ is the span of $k-2$ of the $k-1$ given lines $L_1, L_2, \dots, L_{k-1}$. If $k \geq 3$, then Λ contains some L_i, and projecting from this L_i and applying our inductive hypothesis completes the proof.

It thus remains only to rule out the case when $k = 2$ and Λ is transverse to L_1; exchanging the roles of L_1 and L_2, we may also suppose Λ is transverse to L_2. Projection from Λ then defines an isomorphism $L_1 \simeq L_2$. By assumption, $\beta_1 \neq \beta_2$ with respect to this identification of L_1 with L_2, that is, Λ is disjoint from $\langle \beta_1(s), \beta_2(s) \rangle$ for $s \in \mathbb{P}^1$ generic, in violation of our assumption. □

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8. Inductive arguments

In this section, we suppose that (d, g, r, ℓ, m) is good and give several inductive arguments that reduce $I(d, g, r, \ell, m)$ to cases where d is smaller or where d is the same and m is smaller. In the next section, we will show that these arguments reduce all allowed instances $I(d, g, r, \ell, m)$ to the already considered infinite family of cases with $(\delta, \ell, m) = (1, 0, 0)$, plus finitely many sporadic base cases in small projective spaces.

8.1. Outline of inductive arguments

In order to indicate the specializations and projections of the original BN-curve C, we introduce the following notation. Write $C(0, 0; 0) = C$ for our original general BN-curve of degree d and genus g in $\mathbb{P}^r$. More generally, the notation $C(a, b; c)$ will denote a curve obtained from $C(0, 0; 0)$ by peeling off a one-secant lines (as described in equation (1) below), peeling off b two-secant lines (as described in (2) below), and projecting from c general points on the curve (as described in equation (4) below). In particular, $C(a, b; c)$ is a BN-curve of degree $d - a - b - c$ and genus $g - b$ in $\mathbb{P}^{r-c}$. The inductive arguments we will give will make use of the following six key ingredients:

1. (cf. Section 5.1) We peel off a one-secant line, that is, we degenerate $C(a, b; c)$ to $C(a + 1, b; c) \cup L$, where L is a one-secant line to $C(a + 1, b; c)$, meeting $C(a + 1, b; c)$ at a point we will call x. In this case, we write y for some point in $L\backslash\{x\}$. We always do this specialization so that all marked points determining the modification data specialize onto $C(a + 1, b; c)\backslash\{x\}$.
2. (cf. Section 5.2) We peel off a one-secant line, that is, we degenerate $C(a, b; c)$ to $C(a, b + 1; c) \cup L$, where L is one-secant to $C(a, b + 1; c)$, meeting $C(a, b + 1; c)$ at points we will denote $\{z, w\}$. We always do this specialization so that all marked points determining the modification data specialize onto $C(a, b + 1; c)\backslash\{z, w\}$.
3. We specialize the modification data. For the modifications $[\overset{+}{\rightsquigarrow} R_i]$, we use the technology developed in Section 7. For the remaining modifications, we specialize the marked points determining the modification data (which start out general).
4. We project from a point $p \in C(a, b; c)$. Namely, if we write $C(a, b; c + 1)$ for the projection of $C(a, b; c)$ from p, then the pointing bundle exact sequence induces (cf. equation (3.3)) an exact sequence

$$0 \to N_{C(a,b;c)\to p}(\text{mods to } p) \to N_{C(a,b;c)}[\text{mods}] \to N_{C(a,b;c+1)}[\text{residual mods}](p) \to 0.$$

   If the number n of modifications towards p satisfies $|n - \delta| < 1$, then by Corollary 3.16 interpolation for $N_{C(a,b;c)}[\text{mods}]$ follows from interpolation for $N_{C(a,b;c+1)}[\text{residual mods}]$. More generally, if n satisfies $|n - \delta| \leq 1 - \frac{\epsilon}{r-1}$, then we may iterate this construction (i.e., first specialize as desired and then project) a total of ϵ times.
5. We erase modifications that are linearly general. Namely, suppose that one of our modifications $[p \overset{+}{\to} M]$ is linearly general. Then interpolation for $N[p \overset{+}{\to} M]$ follows from interpolation for N by Lemma 3.11. More generally, if M is not linearly general, but contains some subspace M_0 and is linearly general in the quotient $N|_p/M_0$, then interpolation for $N[p \overset{+}{\to} M]$ follows from interpolation for N and $N[p \overset{+}{\to} M_0]$ by Lemma 3.11.
6. We specialize any remaining R_i to pass through the center of projection. In more detail, suppose that we projected from a point p, and that prior to this step, R_i remains general; write $\overline{R}_i$ for the projection of R_i from p. Specializing R_i to pass through p then induces the specialization of $\overline{R}_i$ to a union $R_i' \cup L$, where R_i' is an r-secant rational normal curve in a hyperplane (the projection from p of the hyperplane containing R_i), and L is a line passing through p and a point $t \in R_i'$. This has the effect of replacing the modification $[\overset{+}{\rightsquigarrow} \overline{R}_i]$ with the modifications $[\overset{+}{\rightsquigarrow} R_i'][p \overset{+}{\to} t]$. By Lemma 8.1 below, R_i' is a general r-secant rational normal curve in a hyperplane, and $t \in R_i'$ is a general point. The modification $[p \overset{+}{\to} t]$ is therefore in a linearly general direction, and can be erased as above. In

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other words, at least when no other modifications are made at p, the combined effect of these steps is to replace $[\overset{+}{\rightsquigarrow} \overline{R}_i]$ with $[\overset{+}{\rightsquigarrow} R_i']$ (which fits well with our inductive hypothesis).

Lemma 8.1. *Let $p, q_1, \dots, q_r \in \mathbb{P}^{r-1}$ be a general set of points, and write $\overline{q}_i \in \mathbb{P}^{r-2}$ for the projection of q_i from p. Let $\overline{R} \subset \mathbb{P}^{r-2}$ be a general rational normal curve passing through $\overline{q}_1, \overline{q}_2, \dots, \overline{q}_r$, and $x \in \overline{R}$ be a general point. Then there exists a rational normal curve R through $p, q_1, \dots, q_r$ whose tangent direction at p corresponds to x, and whose projection from p is $\overline{R}$.*

Proof. Such a rational curve, if it exists, is unique. We can therefore simply compare the dimension of the space of rational curves through $p, q_1, \dots, q_r$, to the dimension of the space of rational curves through $\overline{q}_1, \overline{q}_2, \dots, \overline{q}_r$ together with a choice of point on that rational curve. Visibly both are equal to $r-2$. □

8.2. *Main inductive arguments*

We begin with the following proposition, which applies this method without utilizing specialization (2) (peeling off a one-secant line) and which specializes the R_i as in Section 7. Since this is the first application of the method described above, we include some additional explanations which serve to clarify this method, and will be omitted in subsequent applications.

Proposition 8.2. *Let ℓ' and m' be integers satisfying $0 \le \ell' \le \ell$ and $0 \le m' \le m$, with $m' = 0$ if $r = 3$. Let d' be an integer satisfying $g+r \le d' \le d$, with $d' > g+r$ if both $g = 0$ and $m \ne 0$. For $1 \le i \le m'$, let n_i be an integer satisfying $n_i \equiv r-1 \bmod 2$ and $2 \le n_i \le r-1$, with $n_i \ne 2$ if $(d', g) = (r+1, 1)$. Define*

$$\overline{\ell} = \ell - \ell' + \frac{(r-1)m' - \sum n_i}{2} \quad \text{and} \quad \overline{m} = m - m'.$$

If

$$2m' + \ell' \le r-2 \quad \text{and} \quad \left|\delta - \left[\ell' + 2(d-d') + \sum n_i\right]\right| \le 1 - \frac{1}{r-1},$$

and $I(d'-1, g, r-1, \overline{\ell}, \overline{m})$ holds, then so does $I(d, g, r, \ell, m)$.

Proof. Our goal is to establish $I(d, g, r, \ell, m)$, which asserts interpolation for

$$N_{C(0,0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m].$$

Our assumption that $g + r \le d' \le d$, with $d' > g + r$ if both $g = 0$ and $m \ne 0$, implies that we may peel off $d - d'$ one-secant lines. (Recall from the discussion at the beginning of the section that this means we specialize $C(0,0;0)$ to the union of a BN-curve $C(d-d', 0; 0) \subset \mathbb{P}^r$ of degree d' and genus g, with $d - d'$ one-secant lines, in such a way that all u_i and v_i, and all points of intersection with the R_i, specialize onto $C(d-d', 0; 0)$.) This reduces our problem to showing interpolation for

$$N := N_{C(d-d',0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m][2x_1 \overset{+}{\to} y_1] \cdots [2x_{d-d'} \overset{+}{\to} y_{d-d'}].$$

For $1 \le i \le m'$, write $n_i' = (r-1-n_i)/2$, and degenerate R_i as in Section 7 to the union R_i°, of n_i lines $L_{i,j}$ meeting C at p_i and $q_{i,j}$, and n_i' conics $Q_{i,j}$ meeting C at p_i and q_{i,n_i+2j-1} and q_{i,n_i+2j}. This induces a specialization of N to

$$N_{C(d-d',0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_{m'+1} \cup \cdots \cup R_m][2x_1 \overset{+}{\to} y_1] \cdots [2x_{d-d'} \overset{+}{\to} y_{d-d'}]$$
$$[q_{1,1} + \cdots + q_{1,r-1} \overset{+}{\rightsquigarrow} R_1^\circ] \cdots [q_{m',1} + \cdots + q_{m',r-1} \overset{+}{\rightsquigarrow} R_{m'}^\circ][p_1 \overset{+}{\to} M_1] \cdots [p_{m'} \overset{+}{\to} M_{m'}].$$

Now, fix a general point $p \in C$, and specialize $p_1, p_2, \dots, p_{m'}, v_1, v_2, \dots, v_{\ell'}, y_1, y_2, \dots, y_{d-d'}$ all to p. Because $2m' + \ell' \le r-2 \le r-1$ by assumption, the limiting directions $M_1, \dots, M_{m'}, u_1, \dots, u_{\ell'}$ are

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linearly independent in $\mathbb{P}N_C|_p$, and the limit is therefore treelike (cf. Defintion 3.3). Hence, this induces a further specialization of N to

$$\begin{aligned}&N^\circ := N_{C(d-d',0;0)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_{m'+1}\cup\cdots\cup R_m]\\&[u_1+\cdots+u_{\ell'}+2x_1+\cdots+2x_{d-d'} \overset{+}{\to} p]\\&[q_{1,1}+\cdots+q_{1,r-1} \overset{+}{\rightsquigarrow} R_1^\circ]\cdots[q_{m',1}+\cdots+q_{m',r-1} \overset{+}{\rightsquigarrow} R_{m'}^\circ][p \overset{+}{\to} M],\end{aligned}$$

where $M = \mathrm{Span}(M_1,\dots,M_{m'},u_1,\dots,u_{\ell'}) \subset \mathbb{P}N_C|_p$. Furthermore, M is disjoint from $\mathbb{P}N_{C\to p}|_p$ by combining the assumption $2m'+\ell' \le r-2$ with Lemma 7.5(1). Finally, M is linearly general in $\mathbb{P}(N_C/N_{C\to p})|_p$ by Lemmas 7.5(2) and 7.6.

It remains to see that N° satisfies interpolation. For this, we project from p. In other words, as described at the beginning of the section, we use the following pointing bundle exact sequence:

$$\begin{aligned}0 \to N_{C(d-d',0;0)\to p}(u_1+\cdots+u_{\ell'}+2x_1+\cdots+2x_{d-d'}+(q_{1,1}+\cdots+q_{1,n_1})\\+\cdots+(q_{m',1}+\cdots+q_{m',n_{m'}})) \to N^\circ \to Q(p) \to 0,\end{aligned} \tag{8.1}$$

where

$$\begin{aligned}&Q := N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{R}_{m'+1}\cup\cdots\cup\overline{R}_m][p \overset{+}{\to} \overline{M}]\\&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}]\cdots[q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}]\cdots[q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}]\cdots[q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].\end{aligned}$$

The number of transformations towards p is $\ell'+2(d-d')+\sum n_i$. (These transformations occur at $u_1,\dots,u_{\ell'},x_1,\dots,x_{d-d'},q_{1,1},\dots,q_{1,n_1},\dots,q_{m',1},\dots,q_{m',n_{m'}}$, c.f. equation (8.1). In particular, this specialization does *not* produce a positive transformation at p in the direction of p, because M is disjoint from $\mathbb{P}N_{C\to p}|_p$ as explained above.) Our assumption that $|\delta-[\ell'+2(d-d')+\sum n_i]| \le 1-\frac{1}{r-1}$ therefore implies that interpolation for N° follows from interpolation for Q by Corollary 3.16.

We next erase the transformation at p. In other words, the only way that Q depends on the points $u_1,\dots,u_{\ell'},q_{1,1},\dots,q_{1,n_1},\dots,q_{m',1},\dots,q_{m',n_{m'}}$ is via the dependence of $\overline{M}$ on these points. As only these points vary, $\overline{M}$ is linearly general. Thus interpolation for Q follows, by Lemma 3.11, from interpolation for

$$\begin{aligned}&N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{R}_{m'+1}\cup\cdots\cup\overline{R}_m]\\&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}]\cdots[q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}]\cdots[q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}]\cdots[q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].\end{aligned}$$

Finally, we specialize the remaining R_i, for $m'+1 \le i \le m$, to pass through p. Namely, we first specialize $R_{m'+1}$ to pass through p, which induces the specialization of $\overline{R}_{m'+1}$ to a union $R'_{m'+1}\cup L$ as described in Subsection 8.1(6). The effect of this specialization on the above bundle is to replace the modification $[\overset{+}{\rightsquigarrow} \overline{R}_{m'+1}]$ with the modifications $[\overset{+}{\rightsquigarrow} R'_{m'+1}][p \overset{+}{\to} t]$, where t is a general point on $R'_{m'+1}$. In other words, the above bundle specializes to

$$\begin{aligned}&N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R'_{m'+1}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+2}\cup\cdots\cup\overline{R}_m][p \overset{+}{\to} t]\\&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}]\cdots[q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}]\cdots[q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}]\cdots[q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].\end{aligned}$$

The modification $[p \overset{+}{\to} t]$ is in a linearly general direction and may therefore be erased by Lemma 3.11. In other words, interpolation for this bundle follows from interpolation for

$$\begin{aligned}&N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R'_{m'+1}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+2}\cup\cdots\cup\overline{R}_m]\\&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}]\cdots[q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}]\cdots[q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}]\cdots[q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].\end{aligned}$$

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Similarly specializing $R_{m'+2}$, then $R_{m'+3}$, and so on until R_m, we reduce to interpolation for the bundle

$$N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R'_{m'+1} \cup \cdots \cup R'_m]$$
$$[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}],$$

which is just the assertion $I(d'-1, g, r-1, \overline{\ell}, \overline{m})$. □

The n_i appearing in Proposition 8.2 are constrained mod 2. It is thus often difficult to apply Proposition 8.2 in situations where δ is an integer with the 'wrong' parity. We introduce the following variant, which has the advantage that its difficult parity is the opposite of the difficult parity for Proposition 8.2.

Proposition 8.3. *Let ℓ', m', d', the n_i, $\overline{\ell}$ and $\overline{m}$ be as in Proposition 8.2. If*

$$m' < m, \quad 2m' + \ell' < r - 2, \quad \text{and} \quad \left|\delta - \left[1 + \ell' + 2(d-d') + \sum n_i\right]\right| \leq 1 - \frac{1}{r-1},$$

and

$$I(d'-1, g, r-1, \overline{\ell}, \overline{m}), \quad I(d'-1, g, r-1, \overline{\ell}, \overline{m}-1), \quad \text{and} \quad I(d'-2, g, r-2, \overline{\ell}, \overline{m})$$

all hold, then so does $I(d, g, r, \ell, m)$.

Proof. As in the proof of Proposition 8.2, it suffices to show that N° satisfies interpolation, where

$$N^\circ := N_{C(d-d',0;0)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_{m'+1} \cup \cdots \cup R_m]$$
$$[u_1 + \cdots + u_{\ell'} + 2x_1 + \cdots + 2x_{d-d'} \overset{+}{\rightarrow} p]$$
$$[q_{1,1} + \cdots + q_{1,r-1} \overset{+}{\rightsquigarrow} R_1^\circ] \cdots [q_{m',1} + \cdots + q_{m',r-1} \overset{+}{\rightsquigarrow} R_{m'}^\circ][p \overset{+}{\rightarrow} M].$$

Write $R_{m'+1} \cap C = \{s_0, s_1, s_2, \ldots, s_{r-1}, s_r\}$. We first specialize $R_{m'+1}$ to a union $R \cup L$, where L is the line through s_0 and s_r, and R is a rational curve of degree $r-2$ passing through $s_1, s_2, \ldots, s_{r-1}$ and meeting L at a single point. We then specialize s_r to p. These specializations induce a specialization of N° to

$$N_{C(d-d',0;0)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R][\overset{+}{\rightsquigarrow} R_{m'+2} \cup \cdots \cup R_m]$$
$$[s_0 + u_1 + \cdots + u_{\ell'} + 2x_1 + \cdots + 2x_{d-d'} \overset{+}{\rightarrow} p]$$
$$[q_{1,1} + \cdots + q_{1,r-1} \overset{+}{\rightsquigarrow} R_1^\circ] \cdots [q_{m',1} + \cdots + q_{m',r-1} \overset{+}{\rightsquigarrow} R_{m'}^\circ][p \overset{+}{\rightarrow} M'],$$

where $M' = \langle M, \mathbb{P}N_{C \to s_0}|_p \rangle$. We then project from p, thereby reducing to interpolation for

$$N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][s_1 + \cdots + s_{r-1} \overset{+}{\rightsquigarrow} \overline{R}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+2} \cup \cdots \cup \overline{R}_m][p \overset{+}{\rightarrow} \overline{M}']$$
$$[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].$$

Here, we have written out the modification $[s_1 + \cdots + s_{r-1} \overset{+}{\rightsquigarrow} \overline{R}]$ because $\overline{R}$ also meets $C(d-d', 0; 1)$ at s_0. Note that $\overline{M}'$ is not linearly general since it contains the fixed direction $\mathbb{P}N_{C(d-d',0;1) \to s_0}|_p$; however, it is linearly general in the quotient by $\mathbb{P}N_{C(d-d',0;1) \to s_0}|_p$. Using Lemma 3.11, we reduce to interpolation for the pair of bundles

$$Q := N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][s_1 + \cdots + s_{r-1} \overset{+}{\rightsquigarrow} \overline{R}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+2} \cup \cdots \cup \overline{R}_m]$$
$$[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}]$$

and $Q[p \overset{+}{\rightarrow} s_0]$.

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Specializing $R_{m'+2}, R_{m'+3}, \dots, R_m$ to pass through p, interpolation for these two bundles follows from interpolation for the two bundles

$$\begin{aligned} Q^- &:= N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][s_1 + \cdots + s_{r-1} \overset{+}{\rightsquigarrow} \overline{R}][\overset{+}{\rightsquigarrow} R'_{m'+2} \cup \cdots \cup R'_m] \\ &\quad [q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}] \\ Q^+ &:= Q^-[p \overset{+}{\rightarrow} s_0]. \end{aligned}$$

By Lemma 3.14, interpolation for Q^- follows from interpolation for the two closely related vector bundles where all (respectively none) of the transformations along $\overline{R}$ are performed:

$$\begin{aligned} &N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{R} \cup R'_{m'+2} \cup \cdots \cup R'_m] \\ &[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}] \end{aligned}$$

$$\begin{aligned} &N_{C(d-d',0;1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R'_{m'+2} \cup \cdots \cup R'_m] \\ &[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}]. \end{aligned}$$

But these are the assertions $I(d'-1, g, r-1, \overline{\ell}, \overline{m})$ and $I(d'-1, g, r-1, \overline{\ell}, \overline{m}-1)$, respectively, which hold by assumption.

It remains to see that Q^+ satisfies interpolation. Applying Lemma 3.12 and noting that we have already established interpolation for Q^- above, it suffices to check interpolation for $Q^-/N_{C(d-d',0;1)\to s_0}$, which after twisting down by s_0 is isomorphic to

$$\begin{aligned} &N_{C(d-d',0;2)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{\overline{R}} \cup \overline{R}'_{m'+2} \cup \cdots \cup \overline{R}'_m] \\ &[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}]. \end{aligned}$$

Specializing $R'_{m'+2}, R'_{m'+3}, \dots, R'_m$ to pass through s_0, we reduce to interpolation for

$$\begin{aligned} &N_{C(d-d',0;2)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} \overline{\overline{R}} \cup R''_{m'+2} \cup \cdots \cup R''_m] \\ &[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}]. \end{aligned}$$

But this is just the assertion $I(d'-2, g, r-2, \overline{\ell}, \overline{m})$, which holds by assumption. □

8.3. Large parameters

Both of the main inductive arguments above impose upper bounds on $2m' + \ell'$ (depending on r). It is thus difficult to apply them when any of the remaining parameters, that is, d, g, or m, is large. (Note that ℓ is already bounded in terms of r by construction.) We therefore next give three inductive arguments that apply for large values of d, g and m, respectively.

Proposition 8.4. *Suppose that $d \geq g+2r-1$. If $I(d-(r-1), g, r, \ell, m)$ holds, then so does $I(d, g, r, \ell, m)$.*

Proof. We want to show interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m].$$

Peeling off $r-1$ one-secant lines, it suffices to show interpolation for

$$N_{C(r-1,0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m][2x_1 \overset{+}{\rightarrow} y_1] \cdots [2x_{r-1} \overset{+}{\rightarrow} y_{r-1}].$$

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Specializing $x_1, x_2, \ldots, x_{r-1}$ to a common point $x \in C$ (while leaving $y_1, y_2, \ldots, y_{r-1}$ general) reduces to interpolation for

$$N_{C(r-1,0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m](2x).$$

Removing the twist, this bundle satisfies interpolation provided that

$$N_{C(r-1,0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m]$$

satisfies interpolation, which is the assertion $I(d-(r-1), g, r, \ell, m)$ that holds by assumption. □

Proposition 8.5. *Suppose that $g \geq r$. If $I(d-(r-1), g-r, r, \ell, m+1)$ holds, then so does $I(d, g, r, \ell, m)$.*

Proof. Since (d, g, r, ℓ, m) is good, $(d, g, r) \neq (2r, r+1, r)$, and so this is a special case of Lemma 5.9. □

Proposition 8.6. *Suppose that $m \geq r-1$. If $I(d, g, r, \ell, m-(r-1))$ holds, then so does $I(d, g, r, \ell, m)$.*

Proof. We want to show interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m].$$

Fix points $q_1, q_2, \ldots, q_{r+1}$ lying in a general hyperplane section of C. For $m-(r-2) \leq i \leq m$, specialize R_i to a general rational curve of degree $r-1$ meeting C at $q_1, q_2, \ldots, q_{r+1}$. This induces a specialization of the above bundle to

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_{m-(r-1)}](q_1 + \cdots + q_{r+1}).$$

Removing the twist, this bundle satisfies interpolation provided that

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_{m-(r-1)}]$$

satisfies interpolation, which is the assertion $I(d, g, r, \ell, m-(r-1))$ that holds by assumption. □

8.4. Small parameters

The remaining cases where our main inductive arguments do not apply are when various parameters are small (which deprives us of flexibility in choosing ℓ' and the n_i). Some of the arguments we give here readily generalize to larger values of various parameters, but since we will not need them in that regime, we opt to simplify the exposition as far as possible. We first consider two cases where $\ell = 0$ and $m = 1$.

Proposition 8.7. *Suppose that $\ell = 0$ and $m = 1$. Let ϵ be an integer satisfying $0 \leq \epsilon \leq (d-g-r)/2$, with $\epsilon < (d-g-r)/2$ if $g = 0$. If*

$$|\delta - (2\epsilon + 1)| \leq 1 - \frac{2}{r-1},$$

and $I(d-2\epsilon-2, g, r-2, 0, 1)$ holds, then so does $I(d, g, r, 0, 1)$.

Proof. We want to show that $N_C[\overset{+}{\rightsquigarrow} R_1]$ satisfies interpolation. Peeling off 2ϵ one-secant lines, we reduce to interpolation for

$$N_{C(2\epsilon,0;0)}[\overset{+}{\rightsquigarrow} R_1][2x_1 \overset{+}{\to} y_1] \cdots [2x_{2\epsilon} \overset{+}{\to} y_{2\epsilon}].$$

Write $R_1 \cap C = \{s_0, s_1, s_2, \ldots, s_{r-1}, s_r\}$. As in the proof of Proposition 8.3, specialize R_1 to a union $R \cup L$, where L is the line through s_0 and s_r, and R is a rational curve of degree $r-2$ passing

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through $s_1, s_2, \dots, s_{r-1}$ and meeting L at a single point. Then specialize $y_1, y_2, \dots, y_\epsilon$ to s_0, and $y_{\epsilon+1}, y_{\epsilon+2}, \dots, y_{2\epsilon}$ to s_r. This reduces our problem to interpolation for

$$N_{C(2\epsilon,0;0)}[\overset{+}{\hookrightarrow} R][s_r + 2x_1 + \cdots + 2x_\epsilon \overset{+}{\to} s_0][s_0 + 2x_{\epsilon+1} + \cdots + 2x_{2\epsilon} \overset{+}{\to} s_r].$$

Projecting from s_0 and then from s_r, we reduce to interpolation for $N_{C(2\epsilon,0;2)}[\overset{+}{\hookrightarrow} \overline{R}]$, which is the assertion $I(d-2\epsilon-2, g, r-2, 0, 1)$ that holds by assumption. □

Proposition 8.8. *If $I(4k-3, 2k-2, 2k-1, k-3, 0)$ holds, then so does $I(4k+1, 2k-1, 2k+1, 0, 1)$, provided that $k \geq 3$.*

Proof. Note that $\delta(4k+1, 2k-1, 2k+1, 0, 1) = 5$. Our goal is to show interpolation for $N_C[\overset{+}{\hookrightarrow} R_1]$. Peeling off a one-secant line and a one-secant line, we reduce to interpolation for

$$N_{C(1,1;0)}[2x \overset{+}{\to} y][z \overset{+}{\leftrightarrow} w][z \overset{+}{\to} 2w][\overset{+}{\hookrightarrow} R_1].$$

Degenerate R_1 as in Section 7 to the union R_1°, of four lines L_j meeting C at p and q_j, and $k-2$ conics Q_j meeting C at p and q_{2j+3} and q_{2j+4}. This induces a specialization of the above bundle to

$$N_{C(1,1;0)}[2x \overset{+}{\to} y][z \overset{+}{\leftrightarrow} w][z \overset{+}{\to} 2w][q_1 + q_2 + q_3 + q_4 \overset{+}{\to} p][q_5 + \cdots + q_{2k} \overset{+}{\rightsquigarrow} R_1^\circ][p \overset{+}{\to} M],$$

where M is linearly general as q_1, q_2, q_3, q_4 vary. Specializing z to p, we reduce to interpolation for

$$N_{C(1,1;0)}[2x \overset{+}{\to} y][p \overset{+}{\to} w][p \overset{+}{\to} 2w][w + q_1 + q_2 + q_3 + q_4 \overset{+}{\to} p][q_5 + \cdots + q_{2k} \overset{+}{\rightsquigarrow} R_1^\circ][p \overset{+}{\to} M].$$

Projecting from p, we reduce to interpolation for

$$N_{C(1,1;1)}[2x \overset{+}{\to} y][q_5 \overset{+}{\leftrightarrow} q_6] \cdots [q_{2k-1} \overset{+}{\leftrightarrow} q_{2k}][p \overset{+}{\to} w][p \overset{+}{\to} \overline{M} + 2w].$$

Specializing y and q_5 to w, we reduce to interpolation for

$$N_{C(1,1;1)}[q_7 \overset{+}{\leftrightarrow} q_8] \cdots [q_{2k-1} \overset{+}{\leftrightarrow} q_{2k}][p + q_6 + 2x \overset{+}{\to} w][p \overset{+}{\to} \overline{M} + 2w][w \overset{+}{\to} q_6].$$

Projecting from w, we reduce to interpolation for

$$N_{C(1,1;2)}[q_7 \overset{+}{\leftrightarrow} q_8] \cdots [q_{2k-1} \overset{+}{\leftrightarrow} q_{2k}][p \overset{+}{\to} \overline{M} + w][w \overset{+}{\to} q_6].$$

Erasing the transformation $[w \overset{+}{\to} q_6]$, and then $[p \overset{+}{\to} \overline{M} + w]$, we reduce to interpolation for

$$N_{C(1,1;2)}[q_7 \overset{+}{\leftrightarrow} q_8] \cdots [q_{2k-1} \overset{+}{\leftrightarrow} q_{2k}],$$

which is the assertion $I(4k-3, 2k-2, 2k-1, k-3, 0)$ that holds by assumption. □

We finally consider several arguments that are adapted to the case $m = 0$.

Proposition 8.9. *Suppose $m = 0$, and $g \geq 3$, and $r \geq 6$. Let ϵ be an integer with $0 \leq \epsilon \leq (d-g-r)/3$. If*

$$|\delta - (2\epsilon + 3)| \leq 1 - \frac{3}{r-1},$$

and $I(d-3\epsilon-6, g-3, r-3, \ell+1, 0)$ and $I(d-3\epsilon-6, g-3, r-3, \ell, 0)$ hold, then so does $I(d, g, r, \ell, 0)$.

Proof. Our goal is to show interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell].$$

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Peeling off three two-secant lines and 3ϵ one-secant lines, we reduce to interpolation for

$$N_{C(3\epsilon,3;0)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][2x_1 \overset{+}{\to} y_1]\cdots[2x_{3\epsilon} \overset{+}{\to} y_{3\epsilon}]$$
$$[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\to} 2w_3].$$

Specializing $y_1, y_2, \ldots, y_\epsilon, w_1$ to z_2, and $y_{\epsilon+1}, y_{\epsilon+2}, \ldots, y_{2\epsilon}, w_2$ to z_3, we reduce to interpolation for

$$N_{C(3\epsilon,3;0)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][2x_{2\epsilon+1} \overset{+}{\to} y_{2\epsilon+1}]\cdots[2x_{3\epsilon} \overset{+}{\to} y_{3\epsilon}]$$
$$[z_3 \overset{+}{\to} w_3][z_3 \overset{+}{\to} 2w_3][z_2 \overset{+}{\to} z_1]$$
$$[2x_1 + \cdots + 2x_\epsilon + z_1 + z_3 \overset{+}{\to} z_2][2x_{\epsilon+1} + \cdots + 2x_{2\epsilon} + z_2 + w_3 \overset{+}{\to} z_3][z_1 \overset{+}{\to} 2z_2][z_2 \overset{+}{\to} 2z_3].$$

Projecting from z_2, and then z_3, we reduce to interpolation for

$$N_{C(3\epsilon,3;2)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][2x_{2\epsilon+1} \overset{+}{\to} y_{2\epsilon+1}]\cdots[2x_{3\epsilon} \overset{+}{\to} y_{3\epsilon}]$$
$$[z_3 \overset{+}{\to} w_3][z_3 \overset{+}{\to} 2w_3][z_1 \overset{+}{\leftrightarrow} z_2][z_2 \overset{+}{\to} z_3].$$

Specializing $y_{2\epsilon+1}, y_{2\epsilon+2}, \ldots, y_{3\epsilon}, w_3$ to z_2, we reduce to interpolation for

$$N_{C(3\epsilon,3;2)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][z_2 \overset{+}{\to} z_1 + z_3][2x_{2\epsilon+1} + \cdots + 2x_{3\epsilon} + z_1 + z_3 \overset{+}{\to} z_2][z_3 \overset{+}{\to} 2z_2].$$

Projecting from z_2 (again), we reduce to interpolation for

$$N_{C(3\epsilon,3;3)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][z_3 \overset{+}{\to} z_2][z_2 \overset{+}{\to} z_1 + z_3].$$

Erasing the transformation $[z_2 \overset{+}{\to} z_1 + z_3]$, we reduce to interpolation for the pair of bundles

$$N_{C(3\epsilon,3;3)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][z_3 \overset{+}{\leftrightarrow} z_2] \quad \text{and} \quad N_{C(3\epsilon,3;3)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][z_3 \overset{+}{\to} z_2].$$

The first is our assumption $I(d-3\epsilon-6, g-3, r-3, \ell+1, 0)$. For the second, we erase the transformation $[z_3 \overset{+}{\to} z_2]$ to reduce to interpolation for

$$N_{C(3\epsilon,3;3)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell],$$

which is our assumption $I(d-3\epsilon-6, g-3, r-3, \ell, 0)$. □

Proposition 8.10. *Suppose* $m = 0$ *and* $g \geq 1$ *and that*

$$|\delta - 2| \leq 1 - \frac{1}{r-1}.$$

If $I(d-2, g-1, r-1, \ell+1, 0)$ *holds, then so does* $I(d, g, r, \ell, 0)$.

Proof. Our goal is to show interpolation for $N_C[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell]$. Peeling off a one-secant line, we reduce to interpolation for

$$N_{C(0,1;0)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][z \overset{+}{\leftrightarrow} w][z \overset{+}{\to} 2w].$$

Projecting from w, we reduce to interpolation for

$$N_{C(0,1;1)}[u_1 \overset{+}{\leftrightarrow} v_1]\cdots[u_\ell \overset{+}{\leftrightarrow} v_\ell][z \overset{+}{\leftrightarrow} w],$$

which is our assumption $I(d-2, g-1, r-1, \ell+1, 0)$. □

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Proposition 8.11. *Suppose* $m = 0$ *and* $g \geq 3$ *and* $r \geq 6$ *and that*

$$|\delta - 4| \leq 1 - \frac{2}{r-1}.$$

If $I(d-5, g-3, r-2, \ell+1, 0)$ *and* $I(d-5, g-3, r-2, \ell, 0)$ *hold, then so does* $I(d, g, r, \ell, 0)$.

Proof. Our goal is to show interpolation for $N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell]$. Peeling off three two-secant lines, we reduce to interpolation for

$$N_{C(0,3;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell]$$
$$[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\rightarrow} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\rightarrow} 2w_2][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\rightarrow} 2w_3].$$

Specializing w_2 to w_1, we reduce to interpolation for

$$N_{C(0,3;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][z_1 + z_2 \overset{+}{\leftrightarrow} w_1][z_1 + z_2 \overset{+}{\rightarrow} 2w_1][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\rightarrow} 2w_3].$$

Projecting from w_1, we reduce to interpolation for

$$N_{C(0,3;1)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][z_1 + z_2 \overset{+}{\leftrightarrow} w_1][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\rightarrow} 2w_3].$$

Specializing w_3 to w_1, we reduce to interpolation for

$$N_{C(0,3;1)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][z_1 + z_2 + z_3 \overset{+}{\leftrightarrow} w_1][z_3 \overset{+}{\rightarrow} 2w_1].$$

Projecting from w_1 (again), we reduce to interpolation for

$$N_{C(0,3;2)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][z_3 \overset{+}{\rightarrow} w_1][w_1 \overset{+}{\rightarrow} z_1 + z_2 + z_3].$$

Erasing the transformation $[w_1 \overset{+}{\rightarrow} z_2 + z_2 + z_3]$, we reduce to interpolation for the pair of bundles

$$N_{C(0,3;2)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][z_3 \overset{+}{\leftrightarrow} w_1] \quad \text{and} \quad N_{C(0,3;2)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][z_3 \overset{+}{\rightarrow} w_1].$$

The first is our assumption $I(d-5, g-3, r-2, \ell+1, 0)$. For the second, we erase the transformation $[z_3 \overset{+}{\rightarrow} w_1]$ to reduce to interpolation for

$$N_{C(0,3;2)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell],$$

which is our assumption $I(d-5, g-3, r-2, \ell, 0)$. □

9. Interlude: some cases not implied by I(d, g, r,ℓ, m)

As explained in Section 4, our main inductive argument will establish $I(d, g, r, \ell, m)$ for all good tuples. We have already seen that:

- $I(d, g, r, \ell, m)$ for all good tuples implies Theorem 1.4 except in a couple of cases.
- Theorem 1.4 implies Theorem 1.2 except in a couple of cases.

Of course, we must also check Theorem 1.4 and Theorem 1.2, respectively, in these couple of cases. The most difficult of these is Theorem 1.4 for canonical curves of even genus $g \geq 8$, which we defer to Section 13. Here, we quickly take care of all the others.

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9.1. Theorem 1.4 for rational curves

Consulting Proposition 5.10, we may assume $d \not\equiv 1 \bmod r-1$. By assumption, this implies the characteristic is distinct from 2. It thus suffices to argue that Theorem 1.4 holds for rational curves in characteristic distinct from 2, which we do by induction on d as follows:

If $\delta < 1$: We apply Proposition 8.2 with $d' = d$.
If $\delta = 1$: We apply Proposition 6.1 (the characteristic assumption enters here).
If $1 < \delta < 2$: We apply Proposition 8.2 with $d' = d - 1$.
If $2 \leq \delta$: Upon rearrangement this implies $d \geq 2r - 1$. We may thus apply Proposition 8.4.

9.2. Theorem 1.2 for rational curves

Using Theorem 1.4, we deduce Theorem 1.2 for rational curves when the characteristic is distinct from 2. Here, we show that Theorem 1.2 also holds for rational curves in characteristic 2.

Lemma 9.1. *Suppose the evaluation map $\overline{M}_{g,n}(\mathbb{P}^r, d) \to (\mathbb{P}^r)^n$ is dominant in characteristic 0. Then it is dominant in all characteristics.*

Proof. Because $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is proper over $\operatorname{Spec}\mathbb{Z}$, and the evaluation map is dominant in characteristic 0, the evaluation map is therefore surjective over $\operatorname{Spec}\mathbb{Z}$. □

If $g = 0$, then $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is *irreducible* in any characteristic, and so we conclude the truth of Theorem 1.2 in characteristic 2 from the truth of Theorem 1.2 in characteristic 0.

Remark 9.2. The reader might hope to apply Lemma 9.1 to higher genus curves. Unfortunately, all we learn is that *some* component of $\overline{M}_{g,n}(\mathbb{P}^r, d)$ dominates $(\mathbb{P}^r)^n$ in positive characteristic. This is a fatal flaw when the genus is positive, because there are other components, not corresponding to BN-curves, which would tell us nothing about the interpolation problem for positive-genus curves. For example, consider the component containing those stable maps which contract a smooth curve of genus g to a point and map a rational tail to $\mathbb{P}^r$ with degree d.

9.3. Theorem 1.2 for $(d, g, r) = (6, 2, 4)$

We want to show such a BN-curve can pass through nine general points. It suffices to show $H^1(N_C(-D)) = 0$ when D is a general divisor of degree 9 on C. Peeling off a one-secant line and specializing one of the points of D onto the one-secant line, this reduces to

$$H^1(N_{C(0,1;0)}[u \xrightarrow{+} v][v \xrightarrow{+} u][v \xrightarrow{+} 2u](-v - D')) = 0,$$

where D' is now a general divisor of degree 8 on $C(0, 1; 0)$. This follows in turn from

$$H^1(N_{C(0,1;0)}[2v \xrightarrow{+} u](-v - D')) = 0$$

because this is a subsheaf with punctual quotient. Since $v + D'$ is a general divisor of degree 9 on $C(0, 1; 0)$, this follows from interpolation for

$$N_{C(0,1;0)}[2v \xrightarrow{+} u].$$

Projecting from u, we reduce to interpolation for $N_{C(0,1;1)}$, which is Theorem 1.4 for $(d, g, r) = (4, 1, 3)$.

10. Combinatorics

In this section, we show, by a purely combinatorial argument, that the inductive arguments in Section 8 apply to all good tuples (d, g, r, ℓ, m) except for:

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- The infinite family $(d, g, r, 0, 0)$ with $\delta = 1$ already treated in Section 6;
- A *finite* number of other cases.

We begin by showing that these inductive arguments apply to all but finitely many tuples for each projective space, that is, for each value of r. To reduce casework, define:

$$\epsilon_0 = \epsilon_0(g) = \begin{cases} 1 & \text{if } g = 0; \\ 0 & \text{if } g \neq 0. \end{cases} \quad \text{and} \quad \epsilon_1 = \epsilon_1(d, g) = \begin{cases} 1 & \text{if } d > g + r; \\ 0 & \text{if } d = g + r. \end{cases}$$

Proposition 10.1. *Let (d, g, r, ℓ, m) be a good tuple. Then one of the arguments of Section 8.3 may be applied unless*

$$d \leq g + 2r - 1, \quad g \leq r - 1, \quad \textit{and} \quad m \leq r - 2 + \epsilon_0, \tag{10.1}$$

or unless

$$(d, g, r, \ell, m - (r - 1)) \textit{ lies in equation (XEx).} \tag{10.2}$$

Proof. If $g \geq r$, then we may apply Lemma 8.5. If $m \geq r - 1 + \epsilon_0$, then we may apply Lemma 8.6, unless $(d, g, r, \ell, m - (r - 1))$ lies in equation (XEx).

We may thus assume $m \leq r - 2 + \epsilon_0 \leq r - 1$. For any $d' > g + r$, this implies $\rho(d', g, r) \geq r + 1 \geq m$. Therefore, if $d \geq g + 2r$, we may apply Lemma 8.4. □

For any *fixed* r, conditions (10.1) and (10.2) describe a *finite* set of tuples (d, g, r, ℓ, m) as promised. It therefore suffices to prove:

Theorem 10.2. *If $r \geq 14$, one of the arguments in Section 8 may be applied, unless $\ell = m = 0$ and $\delta = 1$.*

The remainder of this section is devoted to a proof of Theorem 10.2, which is a purely combinatorial exercise. Since all tuples in equation (XEx) have $r \leq 5$, by Proposition 10.1, we may suppose equation (10.1) is satisfied.

10.1. The cases with $m \neq 0$

Our first step will be to show that Proposition 8.2 by itself handles the majority of these cases. This consists of showing that we may assign integer values to the various parameters appearing in Proposition 8.2 that satisfy the desired inequalities. We shall accomplish this using the following lemma, which gives a sufficient criterion for a system of inequalities to have an integer solution.

Lemma 10.3. *Let a_i/b_i and c_j/d_j be rational numbers. There is an integer n satisfying*

$$n \geq \frac{a_i}{b_i} \textit{ for all } i \quad \textit{and} \quad n \leq \frac{c_j}{d_j} \textit{ for all } j,$$

provided that, for all i and j, we have

$$\frac{a_i}{b_i} \leq \frac{c_j}{d_j} - \frac{(b_i - 1)(d_j - 1)}{b_i d_j}.$$

Proof. The collection of intervals $[a_i/b_i, c_j/d_j]$ is closed under intersection, so it suffices to check that there is an integer n satisfying

$$\frac{a}{b} \leq n \leq \frac{c}{d} \tag{10.3}$$

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provided that

$$\frac{a}{b} \leq \frac{c}{d} - \frac{(b-1)(d-1)}{bd}. \tag{10.4}$$

For this, we note that equation (10.3) is equivalent to

$$\frac{a-1}{b} < n < \frac{c+1}{d}.$$

Since any interval of length greater than 1 contains an integer, it suffices to have

$$\frac{c+1}{d} - \frac{a-1}{b} > 1,$$

or equivalently,

$$\frac{c+1}{d} - \frac{a-1}{b} \geq 1 + \frac{1}{bd}.$$

Upon rearrangement this yields equation (10.4) as desired. □

The following simple observations will be used repeatedly in what follows.

Lemma 10.4. *If r is even and δ is an integer, then $\delta \equiv m \bmod 2$.*

Proof. This follows directly from examining the formula

$$\delta = \frac{2d + 2g - 2r + 2\ell + (r+1)m}{r-1}.$$

□

Lemma 10.5. *In Propositions 8.2 and 8.3, suppose that $d' \neq g + r$ if $d \neq g + r$. Then*

$$\overline{m} \leq \rho(d'-1, g, r-1) \quad \text{and} \quad \overline{m} \leq \rho(d'-2, g, r-2).$$

Proof. We divide into cases as follows.

Case 1: $d = g+r$. This implies $d' = d = g+r$; thus, $g = \rho(d,g,r) = \rho(d'-1,g,r-1) = \rho(d'-2,g,r-2)$. On the other hand, because $m \leq \rho(d,g,r)$, we have $\overline{m} = m - m' \leq m \leq \rho(d,g,r)$.

Case 2: $d > g+r$. This implies $d' \geq g+r+1$; thus, $\rho(d'-1,g,r-1) \geq g+r$ and $\rho(d'-2,g,r-2) \geq g+r-1$. On the other hand, because $m \leq r - 2 + \epsilon_0$, we have $\overline{m} = m - m' \leq m \leq r-1$. □

The first main step of our combinatorial analysis is the following.

Proposition 10.6. *Let (d,g,r,ℓ,m) be a good tuple satisfying (10.1) with $m \neq 0$ and $r \geq 14$. Then the conditions of Proposition 8.2 can be satisfied unless one of the following holds:*

1. *$\ell = 0$, and δ is an integer with the same parity as r, and $\delta < r$ if r is even.*
2. *$\ell < \delta < \ell + 2$ and $g > 0$.*
3. *$(d,g,r,\ell,m) = (3k+1, k, 2k, 0, 2k-3)$ for some k.*
4. *$(d,g,r,\ell,m) = (k+1, 0, k, 0, 1)$ for some k.*

Proof. We will show a slightly stronger statement: The conditions of Proposition 8.2 can be satisfied, together with the additional conditions that

$$m' \neq m \text{ if } g = 0, \quad \text{and} \quad d' \neq g + r \text{ if } d \neq g + r,$$

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unless either one of the above-mentioned conditions holds or

$$(d, g, r, \ell, m) = (4k-2, 0, 2k, 0, 1) \text{ or } (4k+1, 2k-1, 2k, 0, 2k-3) \quad \text{for some } k.$$

This is indeed a stronger statement because if $(d, g, r, \ell, m) = (4k-2, 0, 2k, 0, 1)$, then the conditions of Proposition 8.2 can be satisfied by taking:

$$\ell' = 0, \quad m' = m = 1, \quad d' = d = 4k-2, \quad n_1 = 3,$$

and if $(d, g, r, \ell, m) = (4k+1, 2k-1, 2k, 0, 2k-3)$, then the conditions of Proposition 8.2 can be satisfied by taking:

$$\ell' = 0, \quad m' = 1, \quad d' = 4k-1, \quad n_1 = 2k-1.$$

The advantage of this first additional condition is that $m' \neq m$ implies $\overline{m} \neq 0$. In combination with Lemma 10.5 (which applies because of the second additional condition), these conditions therefore imply that $(d'-1, g, r-1, \overline{\ell}, \overline{m})$ is good provided only that

$$\overline{\ell} \leq \frac{r-1}{2}.$$

A further advantage of this second additional condition is that $\sum n_i$ can be any integer of the form $(r-1)m' - 2n$ where

$$0 \leq n \leq \kappa m' \quad \text{where} \quad \kappa = \kappa(d, g, r) := \begin{cases} \frac{r-4}{2} & \text{if } r \text{ is even;} \\ \frac{r-5}{2} & \text{if } r \text{ is odd and } (d, g) = (r+1, 1); \\ \frac{r-3}{2} & \text{if } r \text{ is odd and } (d, g) \neq (r+1, 1). \end{cases}$$

We next write down a system of inequalities such that an integer solution (for ℓ', m', d', and n) to this system guarantees that the conditions of Proposition 8.2 can be satisfied:

$$0 \leq m' \leq m - \epsilon_0 \tag{10.5}$$

$$2m' + \ell' \leq r - 2 \tag{10.6}$$

$$g + r + \epsilon_1 \leq d' \leq d \tag{10.7}$$

$$0 \leq n \leq \kappa m' \tag{10.8}$$

$$0 \leq \ell' \leq \ell \tag{10.9}$$

$$|\delta - [\ell' + 2(d - d') + (r-1)m' - 2n]| \leq 1 - \frac{1}{r-1} \tag{10.10}$$

$$\ell - \ell' + n \leq \frac{r-1}{2}. \tag{10.11}$$

Using equation (10.9), the inequality (10.6) follows from $2m' + \ell \leq r - 2$. We introduce a new variable $s = d' + n$. Replacing equation (10.6) with $2m' + \ell \leq r - 2$ and rewriting the resulting system in terms of s and n, we obtain:

$$0 \leq m' \leq m - \epsilon_0 \tag{10.12}$$

$$2m' + \ell \leq r - 2 \tag{10.13}$$

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$$s - d \leq n \leq s - g - r - \epsilon_1 \tag{10.14}$$

$$0 \leq n \leq \kappa m' \tag{10.15}$$

$$0 \leq \ell' \leq \ell \tag{10.16}$$

$$\delta - [2d - 2s + (r-1)m'] - \frac{r-2}{r-1} \leq \ell' \leq \delta - [2d - 2s + (r-1)m'] + \frac{r-2}{r-1} \tag{10.17}$$

$$n \leq \frac{r-1}{2} + \ell' - \ell. \tag{10.18}$$

We use Lemma 10.3 to eliminate the variable n. In other words, equations (10.14), (10.15) and (10.18) involve n. Applying Lemma 10.3, there is such an integer n provided that:

$$s - d \leq s - g - r - \epsilon_1 \tag{10.19}$$

$$s - d \leq \kappa m' \tag{10.20}$$

$$s - d \leq \frac{r-1}{2} + \ell' - \ell \tag{10.21}$$

$$0 \leq s - g - r - \epsilon_1 \tag{10.22}$$

$$0 \leq \kappa m' \tag{10.23}$$

$$0 \leq \frac{r-1}{2} + \ell' - \ell. \tag{10.24}$$

Inequalities (10.19) and (10.23) are immediate (they follow from $d \geq g+r+\epsilon_1$ and $m' \geq 0$ respectively). Rearranging the remaining inequalities, and including the inequalities (10.12), (10.13), (10.16) and (10.17) that do not involve n, it therefore suffices to show that there is an integer solution to the following system:

$$\begin{aligned}
0 \leq m' &\leq m - \epsilon_0 \\
2m' + \ell &\leq r - 2 \\
0 \leq \ell' &\leq \ell \\
\delta - [2d - 2s + (r-1)m'] - \frac{r-2}{r-1} &\leq \ell' \leq \delta - [2d - 2s + (r-1)m'] + \tfrac{r-2}{r-1} \\
s &\leq d + \kappa m' \\
s - d + \ell - \frac{r-1}{2} &\leq \ell' \\
g + r + \epsilon_1 &\leq s \\
\ell - \frac{r-1}{2} &\leq \ell'.
\end{aligned}$$

Using Lemma 10.3 to eliminate the variable ℓ' replaces the inequalities involving ℓ' with:

$$0 \leq \ell, \tag{10.25}$$

$$0 \leq \delta - [2d - 2s + (r-1)m'] + \frac{r-2}{r-1} \tag{10.26}$$

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$$\delta - [2d - 2s + (r-1)m'] - \frac{r-2}{r-1} \le \ell \tag{10.27}$$

$$\delta - [2d - 2s + (r-1)m'] - \frac{r-2}{r-1} \le \delta - [2d - 2s + (r-1)m'] + \frac{r-2}{r-1} - \frac{(r-2)^2}{(r-1)^2} \tag{10.28}$$

$$s - d + \ell - \frac{r-1}{2} \le \ell \tag{10.29}$$

$$s - d + \ell - \frac{r-1}{2} \le \delta - [2d - 2s + (r-1)m'] + \frac{r-2}{r-1} - \frac{r-2}{2r-2} \tag{10.30}$$

$$\ell - \frac{r-1}{2} \le \ell \tag{10.31}$$

$$\ell - \frac{r-1}{2} \le \delta - [2d - 2s + (r-1)m'] + \frac{r-2}{r-1} - \frac{r-2}{2r-2}. \tag{10.32}$$

Inequalities (10.25), (10.28) and (10.31) are immediate. Simplifying the remaining ones and including the inequalities that do not involve ℓ', we obtain:

$$s \le d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} + \frac{r-2}{2r-2} \tag{10.33}$$

$$s \le d + \frac{r-1}{2}v \tag{10.34}$$

$$s \le d + \kappa m' \tag{10.35}$$

$$s \ge d + \frac{(r-1)m' - \delta}{2} - \frac{r-2}{2r-2} \tag{10.36}$$

$$s \ge d + (r-1)m' - \delta + \ell - \frac{r^2 - r - 1}{2r-2} \tag{10.37}$$

$$s \ge d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r-4} \tag{10.38}$$

$$s \ge g + r + \epsilon_1 \tag{10.39}$$

$$0 \le m' \le m - \epsilon_0 \tag{10.40}$$

$$2m' + \ell \le r - 2. \tag{10.41}$$

We now eliminate the variable s. Mostly, we will accomplish this by using Lemma 10.3, except we will compare equations (10.33) and (10.36) by ad-hoc methods. Namely, for equations (10.33) and (10.36), we want there to be an integer between

$$d + \frac{(r-1)m' - \delta}{2} - \frac{r-2}{2r-2} \quad \text{and} \quad d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} + \frac{r-2}{2r-2}.$$

By direct inspection, such an integer exists if and only if

$$\ell \ne 0 \quad \text{or} \quad (r-1)m' - \delta \text{ is not an odd integer.} \tag{10.42}$$

Eliminating s, we therefore have condition (10.42) plus the following system of inequalities:

$$d + \frac{(r-1)m' - \delta}{2} - \frac{r-2}{2r-2} \le d + \frac{r-1}{2} - \frac{2r-3}{4r-4} \tag{10.43}$$

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$$d + \frac{(r-1)m' - \delta}{2} - \frac{r-2}{2r-2} \leq d + \kappa m' \tag{10.44}$$

$$d + (r-1)m' - \delta + \ell - \frac{r^2 - r - 1}{2r-2} \leq d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} + \frac{r-2}{2r-2} - \frac{(2r-3)^2}{(2r-2)^2} \tag{10.45}$$

$$d + (r-1)m' - \delta + \ell - \frac{r^2 - r - 1}{2r-2} \leq d + \frac{r-1}{2} - \frac{2r-3}{4r-4} \tag{10.46}$$

$$d + (r-1)m' - \delta + \ell - \frac{r^2 - r - 1}{2r-2} \leq d + \kappa m' \tag{10.47}$$

$$d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r-4} \leq d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} + \frac{r-2}{2r-2} - \frac{(4r-5)(2r-3)}{(4r-4)(2r-2)} \tag{10.48}$$

$$d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r-4} \leq d + \frac{r-1}{2} - \frac{4r-5}{8r-8} \tag{10.49}$$

$$d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r-4} \leq d + \kappa m' \tag{10.50}$$

$$g + r + \epsilon_1 \leq d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} + \frac{r-2}{2r-2} \tag{10.51}$$

$$g + r + \epsilon_1 \leq d + \frac{r-1}{2} \tag{10.52}$$

$$g + r + \epsilon_1 \leq d + \kappa m' \tag{10.53}$$

$$0 \leq m' \leq m - \epsilon_0 \tag{10.54}$$

$$2m' + \ell \leq r - 2. \tag{10.55}$$

Inequalities (10.48), (10.52) and (10.53) are immediate. Moreover, equations (10.43), (10.45), (10.46) and (10.49) all follow from

$$(r-1)m' - \delta + \ell \leq \frac{r^2 - 2r}{r-1},$$

and equations (10.47) and (10.50) follow from

$$(r-1)m' - \delta + \ell \leq \kappa m' + \frac{r^2 - r - 2}{2r-2}.$$

The above system of inequalities therefore follows from the following system:

$$(r-1)m' - \delta \leq 2\kappa m' + \frac{r-2}{r-1} \tag{10.56}$$

$$(r-1)m' - \delta + \ell \leq \frac{r^2 - 2r}{r-1} \tag{10.57}$$

$$(r-1)m' - \delta + \ell \leq \kappa m' + \frac{r^2 - r - 2}{2r-2} \tag{10.58}$$

$$g + r + \epsilon_1 \leq d + \frac{(r-1)m' - \delta}{2} + \frac{\ell}{2} + \frac{r-2}{2r-2} \tag{10.59}$$

$$0 \leq m' \leq m - \epsilon_0 \tag{10.60}$$

$$2m' + \ell \leq r - 2. \tag{10.61}$$

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All that remains is therefore to show that there is an integer m' satisfying equations10.56–10.61 plus condition (10.42). For this, we divide into three cases as follows.

Case 1: $\ell = 0$ and r is even and δ is an even integer. In this case, we will take $m' = 2$, which evidently satisfies equation (10.42). Substituting $\ell = 0$ and $m' = 2$ into equations 10.56–10.61, it remains only to verify:

$$2(r-1) - \delta \leq \frac{2r^2 - 9r + 6}{r-1} \tag{10.62}$$

$$2(r-1) - \delta \leq \frac{r^2 - 2r}{r-1} \tag{10.63}$$

$$2(r-1) - \delta \leq \frac{3r^2 - 11r + 6}{2r-2} \tag{10.64}$$

$$g + r + \epsilon_1 \leq d + \frac{2(r-1) - \delta}{2} + \frac{r-2}{2r-2} \tag{10.65}$$

$$0 \leq 2 \leq m - \epsilon_0 \tag{10.66}$$

$$4 \leq r - 2. \tag{10.67}$$

Note that $\delta \geq r$ by our exclusion of the cases $\delta < r$ in Proposition 10.6(1). This implies equations (10.62), (10.63) and (10.64). Since $d \geq g + r + \epsilon_1$, inequality (10.65) follows from $\delta \leq 2r - 2$. Inequality (10.66) follows from $m \geq 2 + \epsilon_0$, and equation (10.67) is immediate. All that remains is therefore to check the following pair of inequalities:

$$\delta \leq 2r - 2 \tag{10.68}$$

$$m \geq 2 + \epsilon_0. \tag{10.69}$$

For equation (10.68), we note that

$$\delta = \frac{2(d - g - 2r + 1) + 4g + (r+1)m + 2r - 2}{r-1} \leq \frac{4(r-1) + (r+1)(r-1) + 2r - 2}{r-1} \leq 2r - 2.$$

For equation (10.69), we note that m is even by Lemma 10.4; in particular, $m \geq 2$. Inequality (10.69) thus holds unless $g = 0$ and $m = 2$. But in this case,

$$\delta = \frac{2d + 2}{r-1} \leq \frac{2(2r-1) + 2}{r-1} < r,$$

contradicting our assumption that $\delta \geq r$.

Case 2: $\ell = 0$ and r is even and δ is an odd integer. In this case, we will take $m' = 1$, which again evidently satisfies equation (10.42). Substituting $\ell = 0$ and $m' = 1$ into equations 10.56–10.61, it remains only to verify:

$$(r-1) - \delta \leq \frac{r^2 - 4r + 2}{r-1} \tag{10.70}$$

$$(r-1) - \delta \leq \frac{r^2 - 2r}{r-1} \tag{10.71}$$

$$(r-1) - \delta \leq \frac{r^2 - 3r + 1}{r-1} \tag{10.72}$$

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$$g + r + \epsilon_1 \leq d + \frac{(r-1) - \delta}{2} + \frac{r-2}{2r-2} \tag{10.73}$$

$$0 \leq 1 \leq m - \epsilon_0 \tag{10.74}$$

$$2 \leq r - 2. \tag{10.75}$$

Inequalities (10.70), (10.71) and (10.72) follow from $\delta \geq 3$, inequality (10.73) from the inequality $\delta \leq 2(d - g - r - \epsilon_1) + (r-1)$, inequality (10.74) from $m \geq 1 + \epsilon_0$, and equation (10.75) is immediate. All that remains is therefore to check the following system of inequalities:

$$\delta \geq 3 \tag{10.76}$$

$$\delta \leq 2(d - g - r - \epsilon_1) + (r-1) \tag{10.77}$$

$$m \geq 1 + \epsilon_0. \tag{10.78}$$

For equation (10.76), since $m \geq 1$, we have $\delta > 1$. Since δ is an odd integer, $\delta \geq 3$ as desired. For equation (10.78), since $m \geq 1$, the inequality holds unless $g = 0$ and $m = 1$. But in this case,

$$\delta = \frac{2d}{r-1} - 1 \leq \frac{2(2r-1)}{r-1} - 1 < 5,$$

and so $\delta = 3$, and so $d = 2r - 2$. In other words, writing $r = 2k$, we have

$$(d, g, r, \ell, m) = (4k-2, 0, 2k, 0, 1),$$

which is one of the cases excluded by assumption.

All that remains is to verify equation (10.77). Note that m is odd by Lemma 10.4; in particular, since $m \leq r - 2 + \epsilon_0$ by equation (10.1), we have one of:

$$m \leq r-5, \quad m = r-3, \quad \text{or} \quad m = r-1,$$

where the final case can only occur if $g = 0$. Our argument will be via casework as follows.

Subcase 2.1: $d \geq g + r + 2$. By separately considering the cases $g = 0$ (in which case $m \leq r-1$) and $g > 0$ (in which case $m \leq r-3$), we have $4g + (r+1)m \leq r^2 + 2r - 7$, with equality only if $g = r-1$ and $m = r-3$. Therefore,

$$\begin{aligned}\delta &= 2(d - g - r - 1) + (r+1) - \frac{(r^2 + 2r - 7) - 4g - (r+1)m + (2r-4)(d - g - r - 2)}{r-1} \\ &\leq 2(d - g - r - \epsilon_1) + (r+1),\end{aligned}$$

with equality only if $g = r-1$ and $m = r-3$ and $d = g + r + 2 = 2r + 1$. Since δ is an odd integer, we therefore have $\delta \leq 2(d - g - r - \epsilon_1) + (r-1)$ unless, writing $r = 2k$, we have

$$(d, g, r, \ell, m) = (2r+1, r-1, r, 0, r-3) = (4k+1, 2k-1, 2k, 0, 2k-3),$$

which is again one of the cases excluded by assumption.

Subcase 2.2: $d \leq g + r + 1$ and $m \leq r - 5$. We have

$$\delta = \frac{2(d - g - r - 1) + 4g + 2 + (r+1)m}{r-1} \leq \frac{4(r-1) + 2 + (r+1)(r-5)}{r-1} = r + 1 - \frac{6}{r-1} < r + 1.$$

Since δ is an odd integer, this implies $\delta \leq r - 1$ as desired.

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Subcase 2.3: $d = g + r$ and $m = r - 3$. We have

$$\delta = r - 1 + \frac{4(g-1)}{r-1}.$$

Since δ is an integer, and $r - 1$ is odd, this implies $g \equiv 1 \bmod r - 1$, which since $0 \leq g \leq r - 1$ in turn implies $g = 1$, and so $\delta = r - 1$.

Subcase 2.4: $d = g + r + 1$ and $m = r - 3$. We have

$$\delta = r - 1 + \frac{2(2g-1)}{r-1}.$$

Since δ is an integer, and $r - 1$ is odd, this implies $2g \equiv 1 \equiv r \bmod r - 1$, which since $0 \leq g \leq r - 1$ in turn implies $g = r/2$. Writing $r = 2k$, we therefore have $g = k$ and $d = g + r + 1 = 3k + 1$, that is, we have

$$(d, g, r, \ell, m) = (3k + 1, k, 2k, 0, 2k - 3),$$

which is again one of the cases excluded by assumption.

Subcase 2.5: $d \leq g + r + 1$ and $m = r - 1$. Since $m \leq r - 2 + \epsilon_0$, we would have $g = 0$, and thus $d = r + 1$. But this would imply $\delta = (r^2 + 1)/(r - 1) \notin \mathbb{Z}$, in contradiction to our assumption that δ is an odd integer.

Case 3: $\ell \neq 0$ or r is odd or δ is not an integer. If $\ell \neq 0$ or δ is not an integer, then equation (10.42) holds. Otherwise, the current assumption implies r is odd, so δ is even (the cases where δ is also odd are excluded), and so equation (10.42) again holds. We conclude that equation (10.42) is automatic.

All that remains is therefore to check that there exists an integer m' satisfying equations 10.56–10.61. Rearranging to make the bounds on m' explicit, this is the system:

$$m' \leq \frac{1}{r - 1 - 2\kappa} \cdot \left(\delta + \frac{r-2}{r-1}\right) \tag{10.79}$$

$$m' \leq \frac{1}{r-1} \cdot \left(\delta - \ell + \frac{r^2 - 2r}{r-1}\right) \tag{10.80}$$

$$\leq \frac{1}{r - 1 - \kappa} \cdot \left(\delta - \ell + \frac{r^2 - r - 2}{2r - 2}\right) \tag{10.81}$$

$$m' \leq m - \epsilon_0 \tag{10.82}$$

$$m' \leq \frac{r - 2 - \ell}{2} \tag{10.83}$$

$$m' \geq \frac{1}{r-1} \cdot \left(\delta - 2(d - g - r - \epsilon_1) - \ell - \frac{r-2}{r-1}\right) \tag{10.84}$$

$$m' \geq 0. \tag{10.85}$$

We will compare the inequalities (10.79) and (10.81) to equation (10.84) using ad-hoc methods. But first we handle all of the other comparisons using Lemma 10.3 by verifying the following system of inequalities:

$$0 \leq \frac{1}{r - 1 - 2\kappa} \cdot \left(\delta + \frac{r-2}{r-1}\right)$$

$$0 \leq \frac{1}{r-1} \cdot \left(\delta - \ell + \frac{r^2 - 2r}{r-1}\right)$$

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$$0 \le \frac{1}{r-1-\kappa} \cdot \left(\delta - \ell + \frac{r^2-r-2}{2r-2}\right)$$
$$0 \le m - \epsilon_0$$
$$0 \le \frac{r-2-\ell}{2}$$
$$\frac{1}{r-1} \cdot \left(\delta - 2(d-g-r-\epsilon_1) - \ell - \frac{r-2}{r-1}\right) \le \frac{1}{r-1} \cdot \left(\delta - \ell + \frac{r^2-2r}{r-1}\right) - \frac{(r-2)^4}{(r-1)^4}$$
$$\frac{1}{r-1} \cdot \left(\delta - 2(d-g-r-\epsilon_1) - \ell - \frac{r-2}{r-1}\right) \le m - \epsilon_0$$
$$\frac{1}{r-1} \cdot \left(\delta - 2(d-g-r-\epsilon_1) - \ell - \frac{r-2}{r-1}\right) \le \frac{r-2-\ell}{2} - \frac{(r-1)^2-1}{2(r-1)^2}.$$

Substituting in the definition of δ and rearranging, these inequalities are equivalent to:

$$2(d-g-r) + 4g + 2\ell + (r+1)m \ge -r+2 \tag{10.86}$$
$$4(d-g-r) + 8g + (r-3)(r-2\ell) + 2(r+1)m \ge -r^2+r \tag{10.87}$$
$$4(d-g-r) + 8g + (r-3)(r-2\ell) + 2(r+1)m \ge -2r+2 \tag{10.88}$$
$$m - 1 \ge \epsilon_0 - 1 \tag{10.89}$$
$$r - 2\ell \ge -r+4 \tag{10.90}$$
$$2(r-1)^3(d-g-r-\epsilon_1) \ge -5r^3+23r^2-35r+18 \tag{10.91}$$
$$(2r-4)(d-g-r-\epsilon_1) + 4(r-1-g) + (r-3)\ell + (r^2-3r)(m-1) + 2(1-\epsilon_1) \ge (r-1)^2\epsilon_0 - r^2 + 6r \tag{10.92}$$
$$(8r-16)(d-g-r) + 16(r-1-g) + (r^2-4r+7)(r-2\ell) + (4r+4)(r-1-m) + 8(1-\epsilon_1) \ge -r^3+10r^2+5r. \tag{10.93}$$

From these expressions, we see that all but equation (10.92) is immediate and that equation (10.92) holds when $\epsilon_0 = 0$. But when $\epsilon_0 = 1$, then $g = 0$ and $\epsilon_1 = 1$, and so equation (10.92) becomes

$$(2r-4)(d-r-1) + (r^2-3r)(m-1) + (r-3)\ell \ge 5,$$

which holds unless $d = r+1$ and $m = 1$ and $\ell = 0$, or equivalently unless

$$(d, g, r, \ell, m) = (k+1, 0, k, 0, 1),$$

which is again one of the cases excluded by assumption.

All that remains is our promised ad-hoc comparison of equations (10.79) and (10.81) to equation (10.84). That is, we want to show that there are integers between:

$$\frac{1}{r-1} \cdot \left(\delta - 2(d-g-r-\epsilon_1) - \ell - \frac{r-2}{r-1}\right) \quad \text{and} \quad \frac{1}{r-1-2\kappa} \cdot \left(\delta + \frac{r-2}{r-1}\right)$$
$$\frac{1}{r-1} \cdot \left(\delta - 2(d-g-r-\epsilon_1) - \ell - \frac{r-2}{r-1}\right) \quad \text{and} \quad \frac{1}{r-1-\kappa} \cdot \left(\delta - \ell + \frac{r^2-r-2}{2r-2}\right).$$

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Remark $*$: *If* $(d,g)=(r+1,1)$ *then* $\delta<\ell+2$. Indeed, if $(d,g)=(r+1,1)$, then $m\le\rho(d,g,r)=1$, and so

$$\delta=\frac{4+2\ell+(r+1)m}{r-1}\le\frac{r+5+2\ell}{r-1}<\ell+2.$$

Subcase 3.1: $\delta<\ell+1+2(d-g-r-\epsilon_1)$. In this case, the lower bound is nonpositive. We have already shown that both upper bounds are nonnegative above, so there is nothing more to check.

Subcase 3.2: $\delta\ge\ell+3$. By Remark $*$ above, $\kappa\ge\frac{r-4}{2}$. It suffices to show that there are integers between

$$\frac{1}{r-1}\cdot\left(\delta-\ell-\frac{r-2}{r-1}\right)\quad\text{and}\quad\frac{1}{r-1-2\cdot\frac{r-4}{2}}\cdot\left(\delta-\ell+\frac{r-2}{r-1}\right)=\frac{1}{3}\cdot\left(\delta-\ell+\frac{r-2}{r-1}\right)$$
$$\frac{1}{r-1}\cdot\left(\delta-\ell-\frac{r-2}{r-1}\right)\quad\text{and}\quad\frac{1}{r-1-\frac{r-4}{2}}\cdot\left(\delta-\ell+\frac{r^2-r-2}{2r-2}\right)=\frac{1}{r+2}\cdot\left(2(\delta-\ell)+\frac{r^2-r-2}{r-1}\right).$$

By Lemma 10.3, this follows from the following inequalities:

$$\frac{1}{r-1}\cdot\left(\delta-\ell-\frac{r-2}{r-1}\right)\le\frac{1}{3}\cdot\left(\delta-\ell+\frac{r-2}{r-1}\right)-\frac{(3(r-1)-1)((r-1)^2-1)}{3(r-1)^3}$$
$$\frac{1}{r-1}\cdot\left(\delta-\ell-\frac{r-2}{r-1}\right)\le\frac{1}{r+2}\cdot\left(2(\delta-\ell)+\frac{r^2-r-2}{r-1}\right)-\frac{((r+2)(r-1)-1)((r-1)^2-1)}{(r+2)(r-1)^3}$$

But these are immediate for $\delta-\ell\ge3$, using the assumption $r\ge14$.

Subcase 3.3: $\ell+1+2(d-g-r-\epsilon_1)\le\delta<\ell+3$. These inequalities force $d=g+r+\epsilon_1$ or equivalently

$$d=g+r\quad\text{or}\quad d=g+r+1.\tag{10.94}$$

The inequality $\delta<\ell+3$ also implies

$$\frac{1}{r-1}\cdot\left(\delta-\ell-\frac{r-2}{r-1}\right)\le1.$$

It therefore suffices to show

$$\frac{1}{r-1-2\kappa}\cdot\left(\delta+\frac{r-2}{r-1}\right)\ge1\quad\text{and}\quad\frac{1}{r-1-\kappa}\cdot\left(\delta-\ell+\frac{r^2-r-2}{2r-2}\right)\ge1,$$

or upon rearrangement

$$\delta\ge r-2-2\kappa+\frac{1}{r-1}\quad\text{and}\quad\delta\ge\ell+\frac{r}{2}-1-\kappa+\frac{1}{r-1}.$$

Subsubcase 3.3.1: $g=0$. By equation (10.94), we have $d=r+1$. Since $g=0$, we have $(d,g)\ne(r+1,1)$. Our goal is thus to show

$$\delta\ge\begin{cases}1+\frac{1}{r-1} & \text{if } r \text{ is odd},\\ 2+\frac{1}{r-1} & \text{if } r \text{ is even};\end{cases}\quad\text{and}\quad\delta\ge\ell+\begin{cases}\frac{1}{2}+\frac{1}{r-1} & \text{if } r \text{ is odd},\\ 1+\frac{1}{r-1} & \text{if } r \text{ is even}.\end{cases}$$

When $\ell=0$, the first inequality implies the second. In this case, recall that $(d,g,r,\ell,m)=(k+1,0,k,0,1)$ is excluded by assumption. Therefore,$m\ge2$, which implies the first inequality because

$$\delta=\frac{2+(r+1)m}{r-1}\ge\frac{2+2(r+1)}{r-1}\ge2+\frac{1}{r-1}.$$

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Now, suppose $\ell \geq 1$. Note that $\delta \geq \ell + 1 \geq 2$, which implies the first inequality unless r is even and $\delta = \ell + 1$. Similarly, $\delta \geq \ell + 1$ implies the second inequality unless r is even and $\delta = \ell + 1$. It thus remains only to show that it is impossible to have $\delta = \ell + 1$ when r is even. To see this, observe that

$$\ell + 1 = \delta = \frac{2 + 2\ell + (r+1)m}{r-1}$$

implies

$$(r-3)(\ell+1) = (r+1)m.$$

But if r were even, then this would imply $(r+1) \mid (\ell+1)$, which forces $\ell + 1 \geq r + 1$, contradicting our assumption that $\ell \leq r/2$.

Subsubcase 3.3.2: $g > 0$. We excluded the cases $\ell < \delta < \ell + 2$ in Proposition 10.6(2). Since we have $\ell+1 \leq \delta < \ell+3$, we therefore have $\ell+2 \leq \delta < \ell+3$. Moreover, by Remark $*$, we have $(d, g) \neq (r+1, 1)$. As in the previous subsubcase, our goal thus is to show

$$\delta \geq \begin{cases} 1 + \frac{1}{r-1} & \text{if } r \text{ is odd,} \\ 2 + \frac{1}{r-1} & \text{if } r \text{ is even;} \end{cases} \quad \text{and} \quad \delta \geq \ell + \begin{cases} \frac{1}{2} + \frac{1}{r-1} & \text{if } r \text{ is odd,} \\ 1 + \frac{1}{r-1} & \text{if } r \text{ is even.} \end{cases}$$

Since $\delta \geq \ell + 2$, the second inequality is immediate in all cases. Also, the first inequality is immediate if r is odd. To see the first inequality when r is even, note that $\delta \geq \ell + 2 \geq 2$, so the first inequality holds unless we have equality everywhere, that is, unless $\ell = 0$ and $\delta = 2$. But this possibility is excluded by assumption (recall that in Case 3 we have $\ell \neq 0$ or r odd or δ is not an integer). □

The majority of the remaining cases are handled by Proposition 8.3. More precisely:

Proposition 10.7. *Let (d, g, r, ℓ, m) be a good tuple satisfying (10.1) with $m \neq 0$ and $r \geq 14$. Suppose in addition that either condition (1) or (2) of Lemma 10.6 is satisfied. Then the conditions of Proposition 8.3 can be satisfied unless one of the following holds:*

1. $(d, g, r, \ell, m) = (4k, 0, 2k + 1, 0, 1)$ *for some k.*
2. $(d, g, r, \ell, m) = (4k + 1, 2k - 1, 2k + 1, 0, 1)$ *for some k.*

Proof. We separately consider the following three cases.

Case 1: $\ell < \delta < \ell + 2$ and $g > 0$. We take $m' = 0$, $\ell' = \ell$, and $d' = d$, which satisfy the conditions of Proposition 8.3 by Lemma 10.5.

Case 2: $\ell = 0$, and δ and r are even integers with $\delta < r$. We take

$$m' = 1, \quad \ell' = 0, \quad d' = d, \quad \text{and} \quad n_1 = \delta - 1.$$

Applying Lemma 10.5, the conditions of Proposition 8.3 are satisfied provided that

$$m > 1 \quad \text{and}$$

$$2 \leq n_1 = \delta - 1 \leq r - 1 \quad \text{with } n_1 = \delta - 1 \neq 2 \text{ if } (d', g) = (r+1, 1).$$

Since δ is an even integer, $\delta - 1 \neq 2$; since $\delta < r$, we have $\delta - 1 \leq r - 1$. All that remains to check is therefore that $m \geq 2$ and that $2 \leq \delta - 1$, which since δ is an even integer is equivalent to $\delta > 2$.

To see this, we first apply Lemma 10.4 to conclude that m is even. Since $m \neq 0$, this implies $m \geq 2$ as desired. This in turn implies $\delta > 2$ because

$$\delta = \frac{2d + 2g - 2r + 2\ell + (r+1)m}{r-1} \geq \frac{2(r+1)}{r-1} > 2.$$

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Case 3: $\ell = 0$, and δ and r are odd integers. As in the proof of Proposition 10.6, we show a slightly stronger statement: The conditions of Proposition 8.3 can be satisfied, together with the additional conditions that

$$m' \neq m-1 \text{ if } g=0, \quad \text{and} \quad d' \neq g+r \text{ if } d \neq g+r,$$

unless either one of the above-mentioned conditions holds or

$$(d,g,r,\ell,m) = (3k+1, k-1, 2k+1, 0, 1) \quad \text{for some } k.$$

This is indeed a stronger statement because if $(d,g,r,\ell,m) = (3k+1, k-1, 2k+1, 0, 1)$, then the conditions of Proposition 8.3 can be satisfied by taking:

$$\ell' = 0, \quad m' = 0, \quad \text{and} \quad d' = d-1 = 3k.$$

Again as in the proof of Proposition 10.6, Lemma 10.5 guarantees that the tuples $(d'-1, g, r-1, \overline{\ell}, \overline{m})$, $(d'-1, g, r-1, \overline{\ell}, \overline{m}-1)$, and $(d'-2, g, r-2, \overline{\ell}, \overline{m})$, are all good provided only that

$$\overline{\ell} \leq \frac{r-3}{2}.$$

With κ as in the proof of Proposition 10.6, our task is thus to show that the following system of inequalities can be satisfied for integers ℓ', m', d', and n:

$$0 \leq m' \leq m-1-\epsilon_0 \tag{10.95}$$

$$2m'+\ell' \leq r-3 \tag{10.96}$$

$$g+r+\epsilon_1 \leq d' \leq d \tag{10.97}$$

$$0 \leq n \leq \kappa m' \tag{10.98}$$

$$0 \leq \ell' \leq \ell = 0 \tag{10.99}$$

$$|\delta - [1+\ell'+2(d-d')+(r-1)m'-2n]| \leq 1-\frac{1}{r-1} \tag{10.100}$$

$$\ell - \ell' + n \leq \frac{r-3}{2}. \tag{10.101}$$

Inequalities (10.99) and (10.100) are satisfied by taking

$$\ell' = 0 \quad \text{and} \quad n = d-d'+\frac{r-1}{2}m' - \frac{\delta-1}{2}.$$

Substituting these into the remaining inequalities and rearranging, we reduce to the system of inequalities:

$$0 \leq m' \leq m-1-\epsilon_0 \tag{10.102}$$

$$m' \leq \frac{r-3}{2} \tag{10.103}$$

$$g+r+\epsilon_1 \leq d' \leq d \tag{10.104}$$

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$$d + \frac{r-1}{2}m' - \frac{\delta-1}{2} - \kappa m' \leq d' \leq d + \frac{r-1}{2}m' - \frac{\delta-1}{2} \tag{10.105}$$

$$d + \frac{r-1}{2}m' - \frac{\delta-1}{2} - \frac{r-3}{2} \leq d'. \tag{10.106}$$

All bounds on d' are integers because r and δ are both odd integers. Using Lemma 10.3 to eliminate d' replaces equations 10.104–10.106 with

$$g + r + \epsilon_1 \leq d \tag{10.107}$$

$$g + r + \epsilon_1 \leq d + \frac{r-1}{2}m' - \frac{\delta-1}{2} \tag{10.108}$$

$$d + \frac{r-1}{2}m' - \frac{\delta-1}{2} - \kappa m' \leq d \tag{10.109}$$

$$d + \frac{r-1}{2}m' - \frac{\delta-1}{2} - \kappa m' \leq d + \frac{r-1}{2}m' - \frac{\delta-1}{2} \tag{10.110}$$

$$d + \frac{r-1}{2}m' - \frac{\delta-1}{2} - \frac{r-3}{2} \leq d \tag{10.111}$$

$$d + \frac{r-1}{2}m' - \frac{\delta-1}{2} - \frac{r-3}{2} \leq d + \frac{r-1}{2}m' - \frac{\delta-1}{2}. \tag{10.112}$$

Inequalities (10.107) and (10.112) always hold, while equation (10.110) is implied by $m' \geq 0$. Rearranging the others, and including equations (10.102) and (10.103), we arrive at the system

$$\begin{aligned}
m' &\geq 0 \\
m' &\geq \frac{\delta - 1 - 2(d - g - r - \epsilon_1)}{r-1} \\
m' &\leq \frac{\delta - 1}{r - 1 - 2\kappa} \\
m' &\leq \frac{\delta + r - 4}{r-1} \\
m' &\leq m - 1 - \epsilon_0 \\
m' &\leq \frac{r-3}{2}.
\end{aligned}$$

Applying Lemma 10.3 to eliminate m', we reduce to the system

$$0 \leq \frac{\delta - 1}{r - 1 - 2\kappa} \tag{10.113}$$

$$0 \leq \frac{\delta + r - 4}{r-1} \tag{10.114}$$

$$0 \leq m - 1 - \epsilon_0 \tag{10.115}$$

$$0 \leq \frac{r-3}{2} \tag{10.116}$$

$$\frac{\delta - 1 - 2(d - g - r - \epsilon_1)}{r-1} \leq \frac{\delta - 1}{r - 1 - 2\kappa} - \frac{\left(\frac{r-1}{2} - 1\right)\left(\frac{r-1-2\kappa}{2} - 1\right)}{\left(\frac{r-1}{2}\right)\left(\frac{r-1-2\kappa}{2}\right)} \tag{10.117}$$

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$$\frac{\delta - 1 - 2(d-g-r-\epsilon_1)}{r-1} \leq \frac{\delta + r - 4}{r-1} - \frac{\left(\frac{r-1}{2} - 1\right)^2}{\left(\frac{r-1}{2}\right)^2} \tag{10.118}$$

$$\frac{\delta - 1 - 2(d-g-r-\epsilon_1)}{r-1} \leq m - 1 - \epsilon_0 \tag{10.119}$$

$$\frac{\delta - 1 - 2(d-g-r-\epsilon_1)}{r-1} \leq \frac{r-3}{2}. \tag{10.120}$$

Since δ is an odd integer, we have $\delta \geq 1$, which implies equations (10.113) and (10.114). The inequality (10.116) is immediate. Inequality (10.118) follows from $d \geq g + r + \epsilon_1$, which holds by construction. For the remaining inequalities (10.115), (10.117), (10.119) and (10.120), we use the inequality $\epsilon_1 \leq 1$ to reduce to the system

$$m \geq 1 + \epsilon_0 \tag{10.121}$$

$$\frac{\delta - 1 - 2(d-g-r-1)}{r-1} \leq \frac{\delta - 1}{r-1-2\kappa} - \frac{(r-3)(r-3-2\kappa)}{(r-1)(r-1-2\kappa)} \tag{10.122}$$

$$\frac{\delta - 1 - 2(d-g-r-1)}{r-1} \leq m - 1 - \epsilon_0 \tag{10.123}$$

$$\frac{\delta - 1 - 2(d-g-r-1)}{r-1} \leq \frac{r-3}{2}. \tag{10.124}$$

We divide our analysis as follows.

Inequality (10.121): This inequality asserts that we do not simultaneously have $g = 0$ (hence, $\epsilon_0 = 1$) and $m = 1$. So assume $g = 0$ and $m = 1$. Then

$$\delta = \frac{2d + 2g - 2r + 2\ell + (r+1)m}{r-1} = \frac{2d - r + 1}{r-1}.$$

Since $r = g + r \leq d \leq g + 2r - 1 = 2r - 1$, we would have

$$1 < \frac{2r - r + 1}{r-1} \leq \delta \leq \frac{2(2r-1) - r + 1}{r-1} < 5.$$

Since δ is an odd integer, $\delta = 3$, and so $d = 2r - 2$. In other words, writing $r = 2k + 1$, we would have $(d, g, r, \ell, m) = (4k, 0, 2k + 1, 0, 1)$. But this case is excluded by assumption.

Inequality (10.122) when $(d, g) = (r + 1, 1)$: In this case, $\kappa = (r - 5)/2$ so upon rearrangement, equation (10.122) becomes

$$\delta \geq \frac{3r - 3}{r - 5}.$$

However,

$$\delta = \frac{2d + 2g - 2r + 2\ell + (r+1)m}{r-1} = \frac{(r+1)m + 4}{r-1};$$

since δ is an integer, this implies $2m + 4 \equiv (r + 1)m + 4 \equiv 0 \bmod r - 1$, and so $m \equiv -2 \bmod (r - 1)/2$, which implies $m \geq (r - 1)/2 - 2 = (r - 5)/2$. Therefore,

$$\delta = \frac{(r+1)m + 4}{r-1} \geq \frac{(r+1) \cdot (r-5)/2 + 4}{r-1} = \frac{r-3}{2}.$$

As $r \geq 14$, this implies $\delta \geq (3r - 3)/(r - 5)$ as desired.

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Inequality (10.123) when $m \leq 1+\epsilon_0$: As we have already established $m \geq 1+\epsilon_0$, this implies $m = 1+\epsilon_0$. In this case, equation (10.123) asserts

$$\delta \leq 2d - 2g - 2r - 1.$$

By definition,

$$\begin{aligned}\delta &= \frac{2d+2g-2r+(r+1)(1+\epsilon_0)}{r-1} \\ &= 2d-2g-2r+5-\frac{(2r-4)(d-g-r)+(4r-6)-(4g+(r+1)\epsilon_0)}{r-1}.\end{aligned}$$

Since $d \geq g+r$, we have $(2r-4)(d-g-r) \geq 0$, with $(2r-4)(d-g-r) \geq 2r-4 \geq 4$ unless equality holds. Similarly, since $g \leq r-1$, we have $4g+(r+1)\epsilon_0 \leq 4r-4$, with $4g+(r+1)\epsilon_0 \leq 4r-8$ unless equality holds. Putting this together, we have $\delta \leq 2d-2g-2r+5+2/(r-1)$, with $\delta \leq 2d-2g-2r+5-2/(r-1)$ unless equality holds. As δ is an odd integer, this implies $\delta \leq 2d-2g-2r+3$.

If $g = 0$, then $4g+(r+1)\epsilon_0 = r+1$. Therefore $\delta \leq 2d-2g-2r+5-(3r-7)/(r-1)$, with $\delta \leq 2d-2g-2r+5-(5r-11)/(r-1) < 2d-2g-2r+1$ unless equality holds. As δ is an odd integer, this implies $\delta \leq 2d-2g-2r-1$ as desired.

It thus remains only to rule out the cases where $g > 0$ and $\delta = 2d-2g-2r+3$ or $\delta = 2d-2g-2r+1$. Upon rearrangement, this is equivalent to

$$(r-2)(d-g-r)-2g = -(r-2) \text{ or } 1.$$

Since r is odd, considering the above equation mod 2 implies that $d-g-r$ must also be odd. If $d-g-r \geq 3$, then since $g \leq r-1$, the left-hand side is at least $3(r-2)-2(r-1) = r-4$, which is impossible. Therefore,in this case we must have $d-g-r = 1$. Solving for g we obtain $g = r-2$ or $g = (r-3)/2$. In other words, writing $r = 2k+1$, we would have $(d,g,r,\ell,m) = (4k+1, 2k-1, 2k+1, 0, 1)$ or $(d,g,r,\ell,m) = (3k+1, k-1, 2k+1, 0, 1)$. But these cases are excluded by assumption.

Inequalities (10.122) when $(d,g) \neq (r+1,1)$, and (10.123) when $m \geq 2+\epsilon_0$, and (10.124) (in all cases): Since $(d,g) \neq (r+1,1)$ for equation (10.122), we may substitute $\kappa = (r-3)/2$. Substituting in the definition of δ and rearranging equations (10.122), (10.123) and (10.124), we obtain

$$\begin{aligned}(6r-10)(d-g-r)+(r^2-2r-3)(m-1-\epsilon_0)+(4r-12)(g+\epsilon_0-1) \\ +(r^2-6r+9)\epsilon_0+2r-14 \geq 0 \\ (2r-4)(d-g-r)+(r^2-3r)(m-2-\epsilon_0)+4(r-1-g)+(r+1)(1-\epsilon_0)+r^2-10r+3 \geq 0 \\ (4r-8)(d-g-r)+(2r+2)(r-1-m)-8(r-1-g)+r^3-7r^2-3r+9 \geq 0.\end{aligned}$$

This establishes the desired inequalities (using that $m \geq 2+\epsilon_0$ for equation (10.123)). □

Finally, we complete our analysis of the case $m \neq 0$ by verifying the desired result in the four remaining one-parameter infinite families of cases:

Case (3) of Lemma 10.6: This follow from Proposition 8.3 with the following parameters:

$$\ell' = 0, \quad m' = 2, \quad d' = d = 3k+1, \quad \text{and} \quad (n_1, n_2) = (3, 2k-3).$$

Case (4) of Lemma 10.6: This follows from Proposition 8.7 with $\epsilon = 0$.
Case (1) of Lemma 10.7: This follows from Proposition 8.7 with $\epsilon = 1$.
Case (2) of Lemma 10.7: This follows from Proposition 8.8.

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10.2. The cases with $m = 0$ and $g \neq 0$

As in the case $m \neq 0$, we will begin by showing that Proposition 8.2 handles 'most' of the cases by itself.

Proposition 10.8. *Let $(d, g, r, \ell, 0)$ be a good tuple (with $m = 0$) satisfying equation (10.1), such that $g \neq 0$ and $r \geq 14$. Then the conditions of Proposition 8.2 can be satisfied unless one of the following holds:*

1. $\delta \geq \ell + 1 + 2(d - g - r)$.
2. $\ell = 0$ *and* δ *is an odd integer.*

Proof. Our goal is to show the existence of integers d' and ℓ', such that $(d'-1, g, r-1, \ell-\ell', 0)$ is good (which is equivalent to $\ell - \ell' = \overline{\ell} \leq (r-1)/2$), and the inequalities of Proposition 8.2 are satisfied:

$$\ell - \ell' \leq \frac{r-1}{2} \tag{10.125}$$

$$0 \leq \ell' \leq \ell \tag{10.126}$$

$$g + r \leq d' \leq d \tag{10.127}$$

$$|\delta - [\ell' + 2(d - d')]| \leq 1 - \frac{1}{r-1} \tag{10.128}$$

$$\ell' \leq r - 2. \tag{10.129}$$

Inequality (10.129) follows from equation (10.126) and the hypothesis $\ell \leq r/2$. Rewriting equations 10.125–10.128, we obtain the system

$$\begin{aligned} \ell - \frac{r-1}{2} &\leq \ell' \\ 0 &\leq \ell' \leq \ell \\ \delta - 2(d-d') - \frac{r-2}{r-1} \leq \ell' &\leq \delta - 2(d-d') + \frac{r-2}{r-1} \\ g + r &\leq d' \leq d. \end{aligned}$$

Applying Lemma 10.3 to eliminate ℓ', it suffices to show there is an integer solution d' to the system:

$$\ell - \frac{r-1}{2} \leq \ell \tag{10.130}$$

$$\ell - \frac{r-1}{2} \leq \delta - 2(d-d') + \frac{r-2}{r-1} - \frac{r-2}{2r-2} \tag{10.131}$$

$$0 \leq \ell \tag{10.132}$$

$$0 \leq \delta - 2(d-d') + \frac{r-2}{r-1} \tag{10.133}$$

$$\delta - 2(d-d') - \frac{r-2}{r-1} \leq \ell \tag{10.134}$$

$$\delta - 2(d-d') - \frac{r-2}{r-1} \leq \delta - 2(d-d') + \frac{r-2}{r-1} - \frac{(r-2)^2}{(r-1)^2} \tag{10.135}$$

$$g + r \leq d' \leq d. \tag{10.136}$$

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Inequalities (10.130), (10.132) and (10.135) are immediate. Rearranging the remaining inequalities, we obtain

$$d' \geq d - \frac{\delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r - 4} \tag{10.137}$$

$$d' \geq d - \frac{\delta}{2} - \frac{r - 2}{2r - 2} \tag{10.138}$$

$$d' \geq g + r \tag{10.139}$$

$$d' \leq d - \frac{\delta}{2} + \frac{\ell}{2} + \frac{r - 2}{2r - 2} \tag{10.140}$$

$$d' \leq d. \tag{10.141}$$

We next eliminate d'. Comparing equation (10.138) to equation (10.140), we want there to be an integer between

$$d - \frac{\delta}{2} - \frac{r - 2}{2r - 2} \quad \text{and} \quad d - \frac{\delta}{2} + \frac{\ell}{2} + \frac{r - 2}{2r - 2}.$$

By inspection, such an integer exists unless δ is an odd integer and $\ell = 0$, which is excluded by assumption Proposition 10.8(2). Applying Lemma 10.3 for the remaining pairs of inequalities, we reduce to verifying

$$d - \frac{\delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r - 4} \leq d - \frac{\delta}{2} + \frac{\ell}{2} + \frac{r - 2}{2r - 2} - \frac{(2r - 3)(4r - 5)}{(2r - 2)(4r - 4)} \tag{10.142}$$

$$d - \frac{\delta}{2} + \frac{\ell}{2} - \frac{r^2 - r - 1}{4r - 4} \leq d \tag{10.143}$$

$$d - \frac{\delta}{2} - \frac{r - 2}{2r - 2} \leq d \tag{10.144}$$

$$g + r \leq d - \frac{\delta}{2} + \frac{\ell}{2} + \frac{r - 2}{2r - 2} \tag{10.145}$$

$$g + r \leq d. \tag{10.146}$$

Upon rearrangement, equation (10.145) is equivalent to

$$\delta \leq \ell + 1 + 2(d - g - r) - \frac{1}{r - 1},$$

which holds by our assumption Proposition 10.8(1). The remaining inequalities rearrange to

$$\begin{aligned} 2r^3 - 8r^2 + 10r - 5 &\geq 0 \\ (r - 3)(r - 2\ell) + 4(d - g - r) + 8g + 2r - 1 &\geq 0 \\ 2\ell + 2(d - g - r) + 4g + r - 2 &\geq 0 \\ d - g - r &\geq 0, \end{aligned}$$

which hold because $(d, g, r, \ell, 0)$ is good and $r \geq 14$. □

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Lemma 10.9. *Suppose that condition (1) or (2) of Proposition 10.8 is satisfied, but* $(\ell, \delta) \neq (0, 1)$ *and* $r \geq 14$. *Then one of the following two conditions holds:*

$$\ell \leq 3, \quad g \geq 4, \quad \text{and} \quad 1 + \frac{1}{r-1} \leq \delta \leq 5 - \frac{2}{r-1} \tag{10.147}$$

or

$$\ell = 0, \quad g \geq 4, \quad d \geq g + r + 3, \quad \text{and} \quad \delta = 5. \tag{10.148}$$

Proof. We divide into cases according to whether equation (1) or equation (2) is satisfied.

Case 1: equation (1) holds. In this case, we establish equation (10.147). Because ℓ and g are integers, the first two inequalities follow from $\ell < 4$ and $g > 3$ respectively. Upon rearrangement, these inequalities become

$$\begin{aligned}(r-1)(\delta - (\ell + 1 + 2(d-g-r))) + (2r-4)(d-g-r) + 4(r-1-g) + (r-9) > 0\\ (r-1)(\delta - (\ell + 1 + 2(d-g-r))) + (2r-4)(d-g-r) + (r-3)\ell + (r-13) > 0,\end{aligned}$$

and therefore hold for $r \geq 14$ as desired. Since $\ell \geq 0$ and $d \geq g + r$, we have

$$\delta \geq \ell + 1 + 2(d-g-r) \geq 1,$$

with equality only if $\ell = 0$. But equality is excluded by assumption (as $(\ell, \delta) \neq (0, 1)$). Finally, the inequality $\delta \leq 5 - 2/(r-1)$ becomes upon rearrangement

$$(2r-2)(\delta - (\ell + 1 + 2(d-g-r))) + (2r-2)(d-g-r) + (4r-4)(r-1-g) + (r^2 - 12r + 15) \geq 0.$$

Case 2: equation (2) holds. In this case,

$$(r-1)(7-\delta) = 2(g + 2r - 1 - d) + 4(r-1-g) + (r-1) > 0.$$

Since δ is an odd integer, but $\delta \neq 1$ (because $\ell = 0$ so $\delta = 1$ is excluded by assumption), we therefore have $\delta = 3$ or $\delta = 5$. In particular,

$$4(g-3) = (r-1)(\delta - 3) + 2(g + 2r - 1 - d) + (r - 13) > 0;$$

since g is an integer, this implies $g \geq 4$.

Subcase 2.1: $\delta = 3$. Then equation (10.147) is satisfied.

Subcase 2.2: $\delta = 5$. In this case, $2(d - g - r - 3) = (r-1)(\delta - 5) + 4(r - 1 - g) + (r - 7) \geq 0$, so equation (10.148) is satisfied. □

Recall that the case $\delta = 1$ and $\ell = m = 0$ is excluded in Theorem 10.2. Therefore, to complete our analysis of the case $m = 0$ and $g \neq 0$, we just have to handle the following two cases:

If equation (10.147) holds Then we apply one of the following propositions according to the value of δ:

- If $1 + \frac{1}{r-1} \leq \delta \leq 3 - \frac{1}{r-1}$: Proposition 8.10.
- If $2 + \frac{3}{r-1} \leq \delta \leq 4 - \frac{3}{r-1}$: Proposition 8.9 with $\epsilon = 0$.
- If $3 + \frac{2}{r-1} \leq \delta \leq 5 - \frac{2}{r-1}$: Proposition 8.11.

Note that the union of these intervals covers the entire interval for δ given by the final inequality of equation (10.147). Moreover, the conditions $\ell \leq 3$ and $g \geq 4$ imply that all tuples appearing in these lemmas are good (they have positive genus and at most $\ell + 1 \leq 5$ lines in a projective space of dimension at least $r - 3 \geq 11$).

If equation (10.148) holds Then we apply Proposition 8.9 with $\epsilon = 1$.

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10.3. The cases with $m = g = 0$

Since $m = 0$, Lemma 8.4 can be applied unless $d \leq 2r - 2$. In the remaining cases, we will show that Proposition 8.2 *always* applies.

Proposition 10.10. *Let $(d, 0, r, \ell, 0)$ be a good tuple (with $m = g = 0$) satisfying $d \leq 2r-2$ and $r \geq 14$. Then the conditions of Proposition 8.2 can be satisfied.*

Proof. As in the proof of Lemma 10.8, our goal is to show the existence of certain integers d' and ℓ' which in particular must satisfy:

$$\ell - \frac{r-1}{2} \leq \ell' \tag{10.149}$$

$$0 \leq \ell' \leq \ell \tag{10.150}$$

$$\delta - 2(d-d') - \frac{r-2}{r-1} \leq \ell' \leq \delta - 2(d-d') + \frac{r-2}{r-1} \tag{10.151}$$

$$r \leq d' \leq d, \tag{10.152}$$

plus possibly some additional conditions to guarantee that $(d'-1, 0, r-1, \ell-\ell', 0)$ is good. For this, we divide into cases as follows:

Case 1: $d = r$. In this case, we take $d' = r$. With this choice $(1-(d'-1))\%((r-1)-1) = 0$, and so equations (10.149)–(10.152) are sufficient for $(d'-1, g, r-1, \ell-\ell', 0)$ to be good. Substituting $d = d' = r$ and $\delta = 2\ell/(r-1)$, our goal is thus to show that there is an integer ℓ' satisfying

$$\begin{aligned} \ell - \frac{r-1}{2} &\leq \ell' \\ 0 &\leq \ell' \leq \ell \\ \frac{2\ell - r + 2}{r-1} \leq \ell' &\leq \frac{2\ell + r - 2}{r-1}. \end{aligned}$$

Applying Lemma 10.3, it suffices to verify

$$0 \leq \ell \tag{10.153}$$

$$0 \leq \frac{2\ell + r - 2}{r-1} \tag{10.154}$$

$$\frac{2\ell - r + 2}{r-1} \leq \ell \tag{10.155}$$

$$\frac{2\ell - r + 2}{r-1} \leq \frac{2\ell + r - 2}{r-1} - \frac{(r-2)^2}{(r-1)^2} \tag{10.156}$$

$$\ell - \frac{r-1}{2} \leq \ell \tag{10.157}$$

$$\ell - \frac{r-1}{2} \leq \frac{2\ell + r - 2}{r-1} - \frac{r-2}{2r-2}. \tag{10.158}$$

Inequalities (10.153)–(10.155) follow from $\ell \geq 0$, and equations (10.156) and (10.157) are automatic, and equation (10.158) follows from $\ell \leq r/2$.

Case 2: $d \geq r+1$. Since $r+1 \leq d \leq 2r-2$, we have $(1-d)\%(r-1) = (1-d)+2(r-1) = 2r-1-d$. Therefore, as $(d, g, r, \ell, 0)$ is good by assumption, ℓ satisfies

$$\frac{2r-1-d}{2} \leq \ell \leq \frac{r}{2}. \tag{10.159}$$

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Similarly, because $d'-1 \le 2r-3$, we have $(1-(d'-1))\%((r-1)-1) \le (2-d')+2(r-2) = 2r-2-d'$. Therefore, $(d'-1, g, r-1, \ell-\ell', 0)$ is good provided equations (10.149)–(10.152) are satisfied, and also $2(\ell-\ell') \ge 2r-2-d'$. In other words, we want to show that there are integers ℓ' and d' satisfying the following system (here we have substituted in $\delta = (2d-2r+2\ell)/(r-1)$):

$$\ell - \frac{r-1}{2} \le \ell' \tag{10.160}$$

$$0 \le \ell' \le \ell \tag{10.161}$$

$$\frac{2d-2r+2\ell}{r-1} - 2(d-d') - \frac{r-2}{r-1} \le \ell' \le \frac{2d-2r+2\ell}{r-1} - 2(d-d') + \frac{r-2}{r-1} \tag{10.162}$$

$$r \le d' \le d \tag{10.163}$$

$$\ell' \le \ell - \frac{2r-2-d'}{2}. \tag{10.164}$$

Subcase 2.1: $d = r+1$. In this case, equation (10.159) becomes $r/2 - 1 \le \ell \le r/2$. In other words, (d, r, ℓ) is of one of the following forms:

$$(d,r,\ell) = (2k+1, 2k, k-1), \quad (d,r,\ell) = (2k+1, 2k, k), \quad \text{or} \quad (d,r,\ell) = (2k+2, 2k+1, k).$$

These cases may satisfy equations 10.160–10.164 by taking

$$(d',\ell') = (2k, 0), \quad (d',\ell') = (2k+1, 1), \quad \text{respectively} \quad (d',\ell') = (2k+1, 0).$$

Subcase 2.2: $d \ge r+2$. In this case, we take $\ell' = 2-2(d-d')$, in which case equations 10.160–10.164 become

$$\begin{aligned} d' &\ge \frac{4d+2\ell-r-3}{4} \\ d-1 \le d' &\le d-1+\frac{\ell}{2} \\ 3r \le 2d+2\ell &\le 5r-4 \\ r \le d' &\le d \\ d' &\le \frac{4d+2\ell-2r-2}{3}. \end{aligned}$$

The inequality $d' \ge r$ follows from $d' \ge d-1$ since $d = r+2$. Deleting the inequality $d' \ge r$ and eliminating d' via Lemma 10.3, we reduce to the system

$$\frac{4d+2\ell-r-3}{4} \le d-1+\frac{\ell}{2}-\frac{3}{8} \tag{10.165}$$

$$\frac{4d+2\ell-r-3}{4} \le d \tag{10.166}$$

$$\frac{4d+2\ell-r-3}{4} \le \frac{4d+2\ell-2r-2}{3} - \frac{1}{2} \tag{10.167}$$

$$d-1 \le d-1+\frac{\ell}{2} \tag{10.168}$$

$$d-1 \le d \tag{10.169}$$

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$$d - 1 \leq \frac{4d + 2\ell - 2r - 2}{3} \tag{10.170}$$

$$3r \leq 2d + 2\ell \tag{10.171}$$

$$2d + 2\ell \leq 5r - 4. \tag{10.172}$$

Inequalities (10.165), (10.168) and (10.169) are immediate. Rearranging the others, we obtain

$$\begin{aligned}(r - 2\ell) + 3 &\geq 0\\ (2\ell - 2r + 1 + d) + 3(d - r - 2) &\geq 0\\ 2\ell - 2r + 1 + d &\geq 0\\ (2\ell - 2r + 1 + d) + (d - r - 2) + 1 &\geq 0\\ (r - 2\ell) + 2(2r - 2 - d) + 8 &\geq 0,\end{aligned}$$

which all follow from equation (10.159), our assumption $r + 2 \leq d \leq 2r - 2$, and the hypotheses in (10.1). □

11. Most of the sporadic cases

The finite set of sporadic cases identified in the previous section is unfortunately rather large. Our next task is to introduce an additional argument that, in combination with the arguments of Section 8, applies to handle most of the sporadic cases, that is, all but a list that is short enough to *write down explicitly*.

This argument will, essentially, be a variant on Proposition 8.2, but where we allow transformations to come together at p in a less restricted way. In particular, we will weaken the hypothesis '$2m' + \ell \leq r - 2$' in the statement of Proposition 8.2 by allowing more modifications to limit to the point p than the rank of the normal bundle. In this regime, the limiting bundle can depend on how the points are specialized into p. We will be able to give a description of some possible limits by limiting the marked points into p one at a time inductively. At each step, we will be able to identify what the limiting modifications are at p. Suppose that, after limiting some collection of marked points into p, we have a transformation at p of the form

$$(np)[p \xrightarrow{+} \Lambda_1][p \xrightarrow{+} \Lambda_2] \quad \text{where} \quad \Lambda_1 \supseteq \Lambda_2, \tag{†}$$

where Λ_1 and Λ_2 are linear spaces in $\mathbb{P}N_C|_p$. Let S and W be sets of parameters varying in irreducible bases with $S \subseteq W$. (For us, W will be the collection of all corresponding marked points, and S will be those marked points at which the projected normal bundle is *not* modified.) Assume that Λ_2 is linearly general as the parameters S vary, and assume that Λ_1 is either:

- Linearly general as the parameters W vary ('weakly general');
- Weakly general and its image in $\mathbb{P}(N_C/N_{C\to p})|_p$ is linearly general as only the parameters S vary ('strongly general').

We will summarize this situation by three pieces of data: the linear dimensions $t_1 = \operatorname{rk}\Lambda_1 = \dim\Lambda_1 + 1$ and $t_2 = \operatorname{rk}\Lambda_2 = \dim\Lambda_2 + 1$, and whether we are in the weak or strong case. Note that we always have $t_2 \leq t_1 \leq r - 2$. (We do not need to keep track of the integer n since this is can be deduced from the Euler characteristic of the limit bundle, which is the same as the original bundle.)

Modifications of type (†) occur naturally when considering the degenerations that make up the key inductive argument outlined in Section 8.1. We review them here (and add one additional argument that we will use in this section) in order to motivate the shape of equation (†).

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1. If we peel off a one-secant line $\overline{xy}$ and limit the point x to p, we obtain $[p \xrightarrow{+} y][p \xrightarrow{+} y]$ at p. This is of the form (†) with $\Lambda_1 = \Lambda_2 = y$; hence, we have $t_1 = t_2 = 1$. Since y is a general point, and no modifications occur at y in the quotient by projection from p, both Λ_1 and Λ_2 are strongly linearly general.
2. If we peel off a one-secant line $\overline{zw}$ and limit the point z to p, we obtain $[p \xrightarrow{+} 2w][p \xrightarrow{+} w]$ at p. This is of the form (†) with $\Lambda_1 = T_w C$ and $\Lambda_2 = w$; hence, we have $(t_1, t_2) = (2, 1)$. Since w is a general point on the curve, at which no modifications occur in the quotient by projection from p, both Λ_1 and Λ_2 are strongly linearly general.
3. If we specialize R as in Section 7 to contain n lines through p, then at p we obtain the modification $[p \xrightarrow{+} \Lambda]$, where $\operatorname{rk}\Lambda = 2$. This is of the form (†) with $(t_1, t_2) = (2, 0)$. By Lemmas 7.5 and 7.6, the subspace Λ is strongly linearly general if

$$n \geq \begin{cases} 3 & \text{if } C \text{ is an elliptic normal curve} \\ 2 & \text{otherwise.} \end{cases}$$

(3′) If we specialize one of the v_i to p, we obtain modification $[p \xrightarrow{+} u_i]$ at p. This is of the form (†) with $(t_1, t_2) = (1, 0)$ and is strongly linearly general since u_i is a general point, at which no modifications occur in the quotient by projection from p.

7. We allow ourselves one new degeneration in our more general inductive step, which is similar to equation (6) from Section 8.1 but crucially different in that we specialize R to pass through p *before* we project. We first specialize R to the union of a line L through 2 points s_0, s_r on C and a rational curve R' of degree $r-2$ through $r-1$ points on C and meeting L at one point. Then we specialize s_0 to p. This results in the modification $[p \xrightarrow{+} s_r]$ at p, which is of type $(t_1, t_2) = (1, 0)$. This modification is linearly general as *all* the points of contact between C and R' move. However, s_r is constrained to be one of the points at which the r-secant rational curve $\overline{R'}$ meets $\overline{C}$, and modifications occur at the remainder of these points, so it is only weakly linearly general.

Our first goal is to understand what happens when we limit into p another point p', at which we have another transformation $(n'p')[p' \xrightarrow{+} \Lambda_1'][p' \xrightarrow{+} \Lambda_2']$ of the same form (†) (depending on sets of parameters $S' \subseteq W'$ disjoint from W). In the following five cases, which we consider separately, we will see that in the limit we obtain another transformation of the form (†) (depending on parameters $S \cup S' \subseteq W \cup W'$).

Most of the subspaces whose generality we must assess are of the form $\Lambda + \Lambda'$ (the span of Λ and Λ'). If Λ and Λ' are both linearly general as independent parameters X and X' vary, then their span is linearly general, as we now show. Let M be a fixed subspace; there is a choice of the parameters X for which the corresponding subspace Λ meets M transversely. Then there is a choice of the parameters X' for which the corresponding subspace Λ' meets $M + \Lambda$ transversely. For this choice of $X \cup X'$, the subspace M meets $\Lambda + \Lambda'$ transversely. The only case where the resulting modification is not of this form is (c) below.

(a) If $t_1 + t_1' < r - 1$: In this case, the limiting transformation is

$$((n + n')p)[p \xrightarrow{+} \Lambda_1 + \Lambda_1'][p \xrightarrow{+} \Lambda_2 + \Lambda_2'].$$

This transformation is of the desired form. The subspace $\Lambda_1 + \Lambda_1'$ is strongly general if Λ_1 and Λ_1' are both strongly general and weakly general otherwise.

(b) If $t_2 + t_2' < t_1 + t_1' = r - 1$: In this case, the limiting transformation is

$$((n + n' + 1)p)[p \xrightarrow{+} \Lambda_2 + \Lambda_2'][p \xrightarrow{+} \emptyset].$$

This transformation is of the desired form, and the subspace $\Lambda_2 + \Lambda_2'$ is always strongly general.

(c) If $t_2' = 0$ and $t_1' + t_2 \leq r - 1 \leq t_1' + t_1$: In this case, the limiting transformation is

$$((n + n' + 1)p)[p \xrightarrow{+} \Lambda_2 + (\Lambda_1 \cap \Lambda_1')][p \xrightarrow{+} \emptyset].$$

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We now show that $\Lambda_2 + (\Lambda_1 \cap \Lambda_1')$ is strongly linearly general if both Λ_1 and Λ_1' are strongly general and weakly linearly general otherwise. Indeed, let M be any fixed subspace; we want to show that M is transverse to $\Lambda_2 + (\Lambda_1 \cap \Lambda_1')$. Both Λ_1 and Λ_2 are linearly general; since the parameters W vary in an irreducible base, there is a single choice of the parameters W for which M is simultaneously transverse to both Λ_1 and Λ_2. Since Λ_1 is transverse to M for our choice of parameters W, and $\Lambda_2 \subset \Lambda_1$, we can restrict to Λ_1 and consider transversality as subspaces of Λ_1. The subspace $M \cap \Lambda_1$ is transverse to Λ_2 for our choice of parameters W. The subspace $\Lambda_1' \cap \Lambda_1$ is transverse to Λ_2 since Λ_1' is linearly general, varying with independent parameters. Therefore,the transversality of $M \cap \Lambda_1$ and $\Lambda_2 + (\Lambda_1' \cap \Lambda_1)$ is equivalent to the transversality of $\Lambda_1 \cap \Lambda_1'$ and $\Lambda_2 + (M \cap \Lambda_1)$, which again follows by the linear generality of Λ_1'.

(d) If $t_1' + t_2 < t_1 + t_2' = r - 1$: In this case, the limiting transformation is

$$((n+n'+1)p)[p \xrightarrow{+} \Lambda_2 + \Lambda_1'][p \xrightarrow{+} \emptyset].$$

This transformation is of the desired form, with $\Lambda_2 + \Lambda_1'$ strongly general if Λ_1' is strongly general, and weakly general otherwise.

(e) If $t_1 + t_2' = t_1' + t_2 = r - 1$: In this case, the limiting transformation is

$$((n+n'+2)p)[p \xrightarrow{+} \emptyset][p \xrightarrow{+} \emptyset].$$

This transformation is of the desired form, with $\emptyset$ always strongly general.

Definition 11.1. For integers $0 \le i, j < r - 1$, let $\{s_{ij}\}$ and $\{w_{ij}\}$ be collections of nonnegative integers, and consider s_{ij} (respectively w_{ij}) marked points decorated with modifications of type (†) with $(t_1, t_2) = (i, j)$ and Λ_1 strongly (respectively weakly) general. Consider all ways of limiting these marked points into p, one at a time in some order, such that at every step of the process, we are in one of the five cases discussed above. If there is such an order for which the final resulting transformation at p satisfies $t_2 = 0$ and Λ_1 is strongly general, then we say that $\{s_{ij}\}$ and $\{w_{ij}\}$ is erasable.

We are now ready to state our more flexible variant on Proposition 8.2. The high-level overview is that, in some order, we do the following specializations:

- Peel off $g - g'$ two-secant lines. Specialize all of them into p as in equation (1).
- Peel off $\epsilon_{\text{in}} + \epsilon_{\text{out}} = d - g - d' + g'$ one-secant lines. Specialize ϵ_{in} of these into p as in the proof of Proposition 8.2 and the remaining ϵ_{out} of them into p as in equation (2).
- Specialize m' of the rational curves R_i as in Section 7 to lines and conics through p_i. Specialize all of the p_i to p as in equation (3).
- Specialize ℓ' of the points v_i to p as in (3′).
- Specialize m'' of the rational curves R_i to the union $L_i \cup R_i'$ as in equation (7). Specialize one of the points where L_i meets C into p.
- Specialize the remaining $m - m' - m''$ rational curves R_i to pass through p as in Section 8.1(6).

After projecting from p, we will reduce to a case of our inductive hypothesis plus a single linearly general modification at p precisely when the modifications at p above are erasable.

Proposition 11.2. *Let ℓ', m', and m'' be nonnegative integers satisfying $\ell' \le \ell$ and $m' + m'' \le m$, with $m' = 0$ if $r = 3$. Let d' and g' be integers satisfying $0 \le g' \le g$ and $g' + r \le d' \le d - g + g'$, with $d' > g' + r$ if both $g' = 0$ and $m \ne 0$. Let ϵ_{in} and ϵ_{out} be nonnegative integers with $\epsilon_{in} + \epsilon_{out} = d - g - d' + g'$. For $1 \le i \le m'$, let n_i be an integer satisfying $n_i \equiv r - 1 \bmod 2$ and $2 \le n_i \le r - 1$, with $n_i \ne 2$ if $(d', g') = (r + 1, 1)$. Define*

$$\overline{\ell} = \ell - \ell' + \frac{(r-1)m' - \sum n_i}{2} \quad \text{and} \quad \overline{m}_{max} = m - m' \quad \text{and} \quad \overline{m}_{min} = m - m' - m''.$$

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Suppose that the following collection is erasable:

$$\begin{aligned} s_{10} &= \ell' + m - m' - m'' \\ s_{11} &= \epsilon_{out} \\ s_{20} &= m' \\ s_{21} &= g - g' \\ w_{10} &= m''. \end{aligned} \tag{11.1}$$

If

$$\left|\delta - \left[2\epsilon_{in} + g - g' + m'' + \ell' + \left\lfloor \frac{2\epsilon_{out} + 3(g-g') + m + m' + \ell'}{r-1} \right\rfloor + \sum n_i\right]\right| \leq 1 - \frac{1}{r-1},$$

and $I(d'-1, g', r-1, \overline{\ell}, \overline{m})$ holds for all $\overline{m}$ with $\overline{m}_{min} \leq \overline{m} \leq \overline{m}_{max}$, then so does $I(d, g, r, \ell, m)$.

Proof. Our goal is to show interpolation for

$$N_{C(0,0;0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m].$$

Peeling off $\epsilon_{\text{in}} + \epsilon_{\text{out}} = d - g - d' + g'$ one-secant lines and $g - g'$ two-secant lines reduces to interpolation for

$$\begin{aligned} &N_{C(d-g-d'+g', g-g';0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m] \\ &[2x_1 \overset{+}{\rightarrow} y_1] \cdots [2x_{\epsilon_{\text{in}}+\epsilon_{\text{out}}} \overset{+}{\rightarrow} y_{\epsilon_{\text{in}}+\epsilon_{\text{out}}}] \\ &[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\rightarrow} 2w_1] \cdots [z_{g-g'} \overset{+}{\leftrightarrow} w_{g-g'}][z_{g-g'} \overset{+}{\rightarrow} 2w_{g-g'}]. \end{aligned}$$

For $1 \leq i \leq m'$, write $n_i' = (r - 1 - n_i)/2$, and degenerate R_i as in Section 7 to the union R_i°, of n_i lines $L_{i,j}$ meeting C at p_i and $q_{i,j}$, and n_i' conics $Q_{i,j}$ meeting C at p_i and q_{i,n_i+2j-1} and q_{i,n_i+2j}. For $m' + 1 \leq i \leq m$, write $R_i \cap C = \{s_0^i, s_1^i, s_2^i, \ldots, s_{r-1}^i, s_r^i\}$. For $m' + 1 \leq i \leq m' + m''$, specialize R_i to a union $R_i^- \cup L_i$, where L_i is the line through s_0^i and s_r^i, and R_i^- is a rational curve of degree $r - 2$ passing through $s_1^i, s_2^i, \ldots, s_{r-1}^i$ and meeting L_i at a single point. This induces a specialization of the above bundle to

$$\begin{aligned} &N_{C(d-g-d'+g', g-g';0)}[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_{m'+1}^- \cup \cdots \cup R_{m'+m''}^-][\overset{+}{\rightsquigarrow} R_{m'+m''+1} \cup \cdots \cup R_m] \\ &[q_{1,1} + \cdots + q_{1,r-1} \overset{+}{\rightsquigarrow} R_1^\circ] \cdots [q_{m',1} + \cdots + q_{m',r-1} \overset{+}{\rightsquigarrow} R_{m'}^\circ] \\ &[s_0^{m'+1} \overset{+}{\leftrightarrow} s_r^{m'+1}] \cdots [s_0^{m'+m''} \overset{+}{\leftrightarrow} s_r^{m'+m''}][p_1 \overset{+}{\rightarrow} M_1] \cdots [p_{m'} \overset{+}{\rightarrow} M_{m'}] \\ &[2x_1 \overset{+}{\rightarrow} y_1] \cdots [2x_{\epsilon_{\text{in}}+\epsilon_{\text{out}}} \overset{+}{\rightarrow} y_{\epsilon_{\text{in}}+\epsilon_{\text{out}}}] \\ &[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\rightarrow} 2w_1] \cdots [z_{g-g'} \overset{+}{\leftrightarrow} w_{g-g'}][z_{g-g'} \overset{+}{\rightarrow} 2w_{g-g'}]. \end{aligned}$$

Fix a general point $p \in C$, and specialize $p_1, p_2, \ldots, p_{m'}, v_1, v_2, \ldots, v_{\ell'}, y_1, y_2, \ldots, y_{\epsilon_{\text{in}}}, x_{\epsilon_{\text{in}}+1}, x_{\epsilon_{\text{in}}+2}, \ldots, x_{\epsilon_{\text{in}}+\epsilon_{\text{out}}}, z_1, z_2, \ldots, z_{g-g'}, s_0^{m'+1}, s_0^{m'+2}, \ldots, s_0^m$ all to p in some order. Our assumption that equation (11.1) is erasable implies that we may choose the order so that the limiting bundle is

$$\begin{aligned} &N_{C(d-g-d'+g', g-g';0)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_{m'+1}^- \cup \cdots \cup R_{m'+m''}^-] \\ &[s_1^{m'+m''+1} + s_2^{m'+m''+1} + \cdots + s_r^{m'+m''+1} \overset{+}{\rightsquigarrow} R_{m'+m''+1}] \cdots [s_1^m + s_2^m + \cdots + s_r^m \overset{+}{\rightsquigarrow} R_m] \\ &[q_{1,1} + \cdots + q_{1,r-1} \overset{+}{\rightsquigarrow} R_1^\circ] \cdots [q_{m',1} + \cdots + q_{m',r-1} \overset{+}{\rightsquigarrow} R_{m'}^\circ] \\ &[2x_1 + \cdots + 2x_{\epsilon_{\text{in}}} + w_1 + \cdots + w_{g-g'} + s_r^{m'+1} + \cdots + s_r^{m'+m''} + u_1 + \cdots + u_{\ell'} \overset{+}{\rightarrow} p](np)[p \overset{+}{\rightarrow} \Lambda], \end{aligned}$$

for some integer n and subspace $\Lambda \subset \mathbb{P}N_{C(d-g-d'+g', g-g';0)}|_p$, disjoint from $\mathbb{P}N_{C(d-g-d'+g', g-g';0)\to p}|_p$ and whose image $\overline{\Lambda}$ in $\mathbb{P}(N_{C(d-g-d'+g', g-g';0)}/N_{C(d-g-d'+g', g-g';0)\to p})|_p$ is linearly general. Computing

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the Euler characteristic, we obtain

$$(r-1)n + \operatorname{rk}\Lambda = 2\epsilon_{\text{out}} + 3(g-g') + m + m' + \ell',$$

and so

$$n = \left\lfloor \frac{2\epsilon_{\text{out}} + 3(g-g') + m + m' + \ell'}{r-1} \right\rfloor.$$

Projecting from p, we reduce to interpolation for

$$\begin{aligned}
&N_{C(d-g-d'+g',g-g';1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_{\ell} \overset{+}{\leftrightarrow} v_{\ell}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+m''+1} \cup \cdots \cup \overline{R}_m] \\
&[s_1^{m'+1} + \cdots + s_{r-1}^{m'+1} \overset{+}{\rightsquigarrow} \overline{R^-_{m'+1}}] \cdots [s_1^{m'+m''} + \cdots + s_{r-1}^{m'+m''} \overset{+}{\rightsquigarrow} \overline{R^-_{m'+m''}}] \\
&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \\
&\cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}](np)[p \overset{+}{\rightarrow} \overline{\Lambda}].
\end{aligned}$$

Erasing the transformation at p, we reduce to interpolation for

$$\begin{aligned}
&N_{C(d-g-d'+g',g-g';1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_{\ell} \overset{+}{\leftrightarrow} v_{\ell}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+m''+1} \cup \cdots \cup \overline{R}_m] \\
&[s_1^{m'+1} + \cdots + s_{r-1}^{m'+1} \overset{+}{\rightsquigarrow} \overline{R^-_{m'+1}}] \cdots [s_1^{m'+m''} + \cdots + s_{r-1}^{m'+m''} \overset{+}{\rightsquigarrow} \overline{R^-_{m'+m''}}] \\
&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].
\end{aligned}$$

By Lemma 3.14, this follows in turn from interpolation for the bundles

$$\begin{aligned}
&N_{C(d-g-d'+g',g-g';1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_{\ell} \overset{+}{\leftrightarrow} v_{\ell}][\overset{+}{\rightsquigarrow} \overline{R}_{m'+m''+1} \cup \cdots \cup \overline{R}_m][\overset{+}{\rightsquigarrow} \overline{R^-_{i_1}} \cup \overline{R^-_{i_2}} \cup \cdots \cup \overline{R^-_{i_j}}] \\
&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}],
\end{aligned}$$

with $m'+1 \le i_1 < i_2 < \cdots < i_j \le m'+m''$. Finally, specializing $R_{m'+m''+1}, R_{m'+m''+2}, \ldots, R_m$ to pass through p (as in the proof of Proposition 8.2), we reduce to interpolation for

$$\begin{aligned}
&N_{C(d-g-d'+g',g-g';1)}[u_{\ell'+1} \overset{+}{\leftrightarrow} v_{\ell'+1}] \cdots [u_{\ell} \overset{+}{\leftrightarrow} v_{\ell}][\overset{+}{\rightsquigarrow} R'_{m'+m''+1} \cup \cdots \cup R'_m][\overset{+}{\rightsquigarrow} \overline{R^-_{i_1}} \cup \overline{R^-_{i_2}} \cup \cdots \cup \overline{R^-_{i_j}}] \\
&[q_{1,n_1+1} \overset{+}{\leftrightarrow} q_{1,n_1+2}] \cdots [q_{1,r-2} \overset{+}{\leftrightarrow} q_{1,r-1}] \cdots [q_{m',n_{m'}+1} \overset{+}{\leftrightarrow} q_{m',n_{m'}+2}] \cdots [q_{m',r-2} \overset{+}{\leftrightarrow} q_{m',r-1}].
\end{aligned}$$

But these are precisely our assumptions $I(d'-1, g', r-1, \overline{\ell}, \overline{m})$ for $\overline{m}_{\min} \le \overline{m} \le \overline{m}_{\max}$. □

We then write a computer program in python [14] (see Appendix 1) which iterates over all of the finitely many sporadic cases identified in the previous section, that is, those tuples (d, g, r, ℓ, m) satisfying $r \le 13$ and equation (10.1) or (10.2), but excluding those tuples with $(\delta, \ell, m) = (1, 0, 0)$. In each case, all possible parameters for every inductive argument in Section 8, as well as all possible parameters for Proposition 11.2, are tried. In all but the following 30cases, one of these arguments applies:

(4, 0, 3, 0, 1)	(4, 0, 3, 0, 2)	(4, 0, 3, 1, 1)	(5, 0, 3, 0, 1)	(5, 1, 3, 0, 1)	(5, 1, 3, 1, 1)
(5, 2, 3, 0, 1)	(5, 2, 3, 0, 2)	(5, 2, 3, 1, 1)	(6, 2, 3, 0, 1)	(5, 0, 4, 0, 1)	(5, 0, 4, 2, 0)
(6, 2, 4, 0, 2)	(7, 3, 4, 0, 1)	(7, 3, 4, 1, 1)	(7, 1, 5, 0, 1)	(7, 2, 5, 0, 1)	(7, 2, 5, 2, 2)
(9, 2, 5, 0, 0)	(8, 3, 5, 2, 0)	(9, 4, 5, 0, 0)	(9, 4, 5, 1, 0)	(7, 0, 6, 0, 1)	(7, 1, 6, 2, 1)
(7, 1, 6, 3, 1)	(8, 2, 6, 2, 0)	(11, 5, 6, 0, 0)	(8, 1, 7, 0, 1)	(8, 1, 7, 1, 1)	(11, 4, 7, 1, 0)

Our remaining task is therefore to verify $I(d, g, r, \ell, m)$ in these 30 base cases (as well as to prove Theorem 1.4 for canonical curves of even genus).

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12. The remaining sporadic cases

12.1. The cases $(d,g,r,\ell,m)=(7,1,6,2,1)$, $(7,1,6,3,1)$, *and* $(8,2,6,2,0)$

In these three cases, our previous arguments apply provided that $I(6,1,5,3,0)$ holds. (Note that $(6,1,5,3,0)$ is, however, not good, which is why we were not able to deal with these cases in the previous section and need to separately consider them here.) Indeed:

If $(d,g,r,\ell,m)=(7,1,6,2,1)$ We apply Proposition 8.2 with parameters

$$\ell'=0,\quad m'=1,\quad d'=7,\quad \text{and}\quad n_1=3.$$

If $(d,g,r,\ell,m)=(7,1,6,3,1)$ We apply Proposition 8.2 with parameters

$$\ell'=1,\quad m'=1,\quad d'=7,\quad \text{and}\quad n_1=3.$$

If $(d,g,r,\ell,m)=(8,2,6,2,0)$ We apply Proposition 8.10.

It thus remains to check $I(6,1,5,3,0)$. For this, we simply apply Proposition 8.2 with parameters

$$\ell'=3,\quad m'=0,\quad \text{and}\quad d'=6,$$

thereby reducing $I(6,1,5,3,0)$ to $I(5,1,4,0,0)$, which suffices because $(5,1,4,0,0)$ is good.

12.2. The cases $(d,g,r,\ell,m)=(4,0,3,1,1)$, $(5,1,3,1,1)$, *and* $(5,2,3,1,1)$

In each of these cases, we want to show interpolation for

$$N_C[u \overset{+}{\leftrightarrow} v][\overset{+}{\rightsquigarrow} R].$$

Write $R\cap C=\{q_1,q_2,q_3,q_4\}$. Specializing u to q_1 and v to q_2 induces a specialization of this bundle to

$$N_C[q_3+q_4 \overset{+}{\rightsquigarrow} R](q_1+q_2).$$

Removing the twists at q_1 and q_2, we reduce to interpolation for

$$N_C[q_3+q_4 \overset{+}{\rightsquigarrow} R].$$

Specializing R to the union of the lines $\overline{q_1q_2}\cup\overline{q_3q_4}$ induces a specialization of this bundle to

$$N_C[q_3 \overset{+}{\leftrightarrow} q_4].$$

Interpolation for this bundle is the assertion $I(d,g,3,1,0)$, and $(d,g,3,1,0)$ is good in each of these cases.

12.3. The cases $(d,g,r,\ell,m)=(5,1,3,0,1)$ *and* $(6,2,3,0,1)$

In both of these cases, we we want to show interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$. Peeling off a one-secant line, we reduce to interpolation for

$$N_{C(1,0;0)}[\overset{+}{\rightsquigarrow} R][z \overset{+}{\leftrightarrow} w][z \overset{+}{\rightarrow} 2w] \simeq N_{C(1,0;0)}[\overset{+}{\rightsquigarrow} R][z \overset{+}{\leftrightarrow} w](z).$$

Removing the twist at z, we reduce to interpolation for

$$N_{C(1,0;0)}[\overset{+}{\rightsquigarrow} R][z \overset{+}{\leftrightarrow} w].$$

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Interpolation for this bundle is the assertion $I(d-1, g-1, 3, 1, 1)$, and $(d-1, g-1, 3, 1, 1)$ is good in both of these cases.

12.4. *The cases* $(d, g, r, \ell, m) = (4, 0, 3, 0, 2)$ *and* $(5, 2, 3, 0, 2)$

Let $x_1, y_1, x_2, y_2 \in C$ be four general points. Projection from $\overline{x_i y_i}$ defines a general map $\pi_i \colon C \to \mathbb{P}^1$ of degree $d-2$, which is in particular separable. Since x_2 and y_2 are general, $\overline{x_2 y_2}$ does not meet the tangent line to C at either x_1, y_1, or any of the ramification points of π_1. Thus, $(\pi_1, \pi_2) \colon C \to \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image, an isomorphism near x_1 and y_1 (and by symmetry near x_2 and y_2), and its image is nodal.

The number of nodes is the difference between the arithmetic and geometric genus, which is $(d-3)^2 - g \neq 0$. Therefore, there is a pair of points $z, w \in C$, distinct from each other and x_1, x_2, y_1, y_2, with $\pi_i(z) = \pi_i(w)$ for both i. Geometrically, x_i, y_i, z, w are four distinct coplanar points. Since x_i and y_i are general, $\overline{x_i y_i}$ is not a trisecant to C, so in particular, (x_i, y_i, z) and (x_i, y_i, w) are not collinear. Because x_i and y_i can be exchanged via monodromy, this implies no three of x_i, y_i, z, w are collinear.

Our goal is to show interpolation for $N_C[\overset{+}{\rightsquigarrow} R_1 \cup R_2]$. Specializing R_i to meet C at x_i, y_i, z, w, this bundle specializes to

$$N_C[x_1 + y_1 \overset{+}{\rightsquigarrow} R_1][x_2 + y_2 \overset{+}{\rightsquigarrow} R_2](z + w).$$

Removing the twists at z and w, we reduce to interpolation for

$$N_C[x_1 + y_1 \overset{+}{\rightsquigarrow} R_1][x_2 + y_2 \overset{+}{\rightsquigarrow} R_2].$$

Specializing R_i to the union of lines $\overline{x_i y_i} \cup \overline{zw}$, this bundle specializes to

$$N_C[x_1 \overset{+}{\leftrightarrow} y_1][x_2 \overset{+}{\leftrightarrow} y_2].$$

Interpolation for this bundle is the assertion $I(d, g, 3, 2, 0)$. Although the $(d, g, 3, 2, 0)$ are not good, our previous arguments still apply in these cases:

If $(d, g) = (4, 0)$ We apply Proposition 8.2 with parameters:

$$\ell' = 1, \quad m' = 0, \quad \text{and} \quad d' = 3.$$

If $(d, g) = (5, 2)$ We apply Proposition 11.2 with parameters:

$$\ell' = 1, \quad m' = m'' = \epsilon_{\text{in}} = \epsilon_{\text{out}} = 0, \quad d' = 3, \quad \text{and} \quad g' = 0.$$

(The required erasability of $(s_{10}, s_{11}, s_{20}, s_{21}, w_{10}) = (1, 0, 0, 2, 0)$ can be checked by specializing the points in any order.)

12.5. *The cases* $(d, g, r, \ell, m) = (4, 0, 3, 0, 1)$ *and* $(5, 0, 3, 0, 1)$

In both of these cases, we want to show interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$. Write $C \cap R = \{q_1, q_2, q_3, q_4\}$. Peel off a one-secant line, that is, degenerate C to $C(1, 0; 0) \cup L$ – but in such a way that q_4 specializes onto L, while q_1, q_2, and q_3 specialize onto $C(1, 0; 0)$. The restriction of the modified normal bundle to L is perfectly balanced of slope 2, so by Lemma 3.7, this reduces interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$ to interpolation for

$$N_{C(1,0;0)}[q_1 + q_2 + q_3 \overset{+}{\rightsquigarrow} R][z \overset{+}{\rightarrow} q_4].$$

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Erasing the transformation $[z \overset{+}{\to} q_4]$, we reduce to interpolation for

$$N_{C(1,0;0)}[q_1 + q_2 + q_3 \overset{+}{\rightsquigarrow} R].$$

Specializing R to the union of lines $\overline{q_1q_2} \cup \overline{q_3q_4}$ induces a specialization of this bundle to

$$N_{C(1,0;0)}[q_1 \overset{+}{\leftrightarrow} q_2][q_3 \overset{+}{\to} q_4].$$

Erasing the transformation at q_3, we reduce to interpolation for

$$N_{C(1,0;0)}[q_1 \overset{+}{\leftrightarrow} q_2],$$

which is the assertion $I(d-1, 0, 3, 1, 0)$. Both $(3, 0, 3, 1, 0)$ and $(4, 0, 3, 1, 0)$ are good.

12.6. The case $(d, g, r, \ell, m) = (5, 2, 3, 0, 1)$

In this case, we want to show interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$. Write $C \cap R = \{q_1, q_2, q_3, q_4\}$. Peel off a one-secant line, that is, degenerate C to $C(0, 1; 0) \cup L$ – but in such a way that q_3 and q_4 specialize onto L, while q_1 and q_2 specialize onto $C(0, 1; 0)$. The restriction of the modified normal bundle to L is perfectly balanced of slope 3, so by Lemma 3.7, this reduces interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$ to interpolation for

$$N_{C(0,1;0)}[q_1 + q_2 \overset{+}{\rightsquigarrow} R][z \overset{+}{\leftrightarrow} w].$$

Let Q be the unique quadric containing $C(0, 1; 0)$ and the line $\overline{zw}$. Then interpolation for this bundle follows from the balanced exact sequence

$$0 \to N_{C(0,1;0)/Q}(z + w) \to N_{C(0,1;0)}[q_1 + q_2 \overset{+}{\rightsquigarrow} R][z \overset{+}{\leftrightarrow} w] \to N_Q|_{C(0,1;0)}(q_1 + q_2) \to 0.$$

12.7. The case $(d, g, r, \ell, m) = (5, 0, 4, 2, 0)$

In this case, we want to show interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2].$$

Peel off a one-secant line, that is, degenerate C to $C(1, 0; 0) \cup L$ – but in such a way that v_1 and v_2 specialize onto L, while u_1 and u_2 specialize onto $C(1, 0; 0)$. By Lemma 3.7, this reduces to interpolation for

$$N_{C(1,0;0)}[u_1 \overset{+}{\to} v_1][u_2 \overset{+}{\to} v_2][z \overset{+}{\to} v_2].$$

Specializing v_1 to z, we reduce to interpolation for

$$N_{C(1,0;0)}[u_1 \overset{+}{\to} z][u_2 \overset{+}{\to} v_2][z \overset{+}{\to} v_2].$$

Projecting from z, we reduce to interpolation for

$$N_{C(1,0;1)}[u_2 \overset{+}{\to} v_2][z \overset{+}{\to} v_2].$$

Specializing v_2 onto the line $\overline{zu_2}$, we reduce to interpolation for

$$N_{C(1,0;1)}[z \overset{+}{\leftrightarrow} u_2].$$

This is the assertion $I(3, 0, 3, 1, 0)$, and $(3, 0, 3, 1, 0)$ is good.

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12.8. The cases $(d, g, r, \ell, m) = (6, 2, 4, 0, 2)$ and $(7, 3, 4, 0, 1)$

In these cases, we want to show interpolation for $N_C[\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m]$. Write $R_i \cap C = \{q_{i1}, q_{i2}, q_{i3}, q_{i4}, q_{i5}\}$. Note that $m \leq g$ in both cases, so we may peel off m two-secant lines, that is, degenerate C to $C(0, m; 0) \cup L_1 \cup \cdots \cup L_m$ – but in such a way that q_{i5} specializes onto L_i, while the remaining q_{ij} specialize onto $C(0, m; 0)$. By Lemma 3.7, this reduces interpolation for $N_C[\overset{+}{\rightsquigarrow} R_1 \cup \cdots \cup R_m]$ to interpolation for

$$N_{C(0,m;0)}[q_{11} + q_{12} + q_{13} + q_{14} \overset{+}{\rightsquigarrow} R_1] \cdots [q_{m1} + q_{m2} + q_{m3} + q_{m4} \overset{+}{\rightsquigarrow} R_m][z_1 \overset{+}{\leftrightarrow} w_1] \cdots [z_m \overset{+}{\leftrightarrow} w_m].$$

Specialize R_i to the union of lines $\overline{q_{i1}q_{i2}}$, $\overline{q_{i3}q_{i4}}$ and the unique line through q_{i5} meeting both of these two lines. This induces a specialization of the above bundle to

$$N_{C(0,m;0)}[q_{11} \overset{+}{\leftrightarrow} q_{12}][q_{13} \overset{+}{\leftrightarrow} q_{14}] \cdots [q_{m1} \overset{+}{\leftrightarrow} q_{m2}][q_{m3} \overset{+}{\leftrightarrow} q_{m4}][z_1 \overset{+}{\leftrightarrow} w_1] \cdots [z_m \overset{+}{\leftrightarrow} w_m].$$

In other words, all that remains is to check the assertion $I(d - m, g - m, 4, 3m, 0)$.

If $(d, g, m) = (6, 2, 2)$ In this case, writing C for a curve of degree $d - m = 4$ and genus $g - m = 0$, we want to establish interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_6 \overset{+}{\leftrightarrow} v_6].$$

Specializing 'to a tetrahedron', that is, specializing u_1, u_2 to u_3, and u_4, v_6 to v_1, and v_4, u_5 to v_2, and v_5, u_6 to v_3, this bundle specializes to

$$N_C(u_3 + v_1 + v_2 + v_3).$$

Removing the twists at u_3, v_1, v_2, and v_3, we reduce to interpolation for N_C, which is the assertion $I(4, 0, 4, 0, 0)$. Note that $(4, 0, 4, 0, 0)$ is good.

If $(d, g, m) = (7, 3, 1)$ In this case, writing C for a curve of degree $d - m = 6$ and genus $g - m = 2$, we want to establish interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2][u_3 \overset{+}{\leftrightarrow} v_3].$$

Note that $\delta = 4\frac{2}{3}$. Peeling off two one-secant lines, we reduce to interpolation for

$$N_{C(0,2;0)}[u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2][u_3 \overset{+}{\leftrightarrow} v_3][z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\rightarrow} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\rightarrow} 2w_2].$$

Limiting w_1 to w_2, this bundle specializes to

$$N_{C(0,2;0)}[u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2][u_3 \overset{+}{\leftrightarrow} v_3][z_1 + z_2 \overset{+}{\leftrightarrow} w_2][z_1 + z_2 \overset{+}{\rightarrow} 2w_2].$$

Projecting from w_2, we reduce to interpolation for

$$N_{C(0,2;1)}[u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2][u_3 \overset{+}{\leftrightarrow} v_3][z_1 + z_2 \overset{+}{\rightarrow} w_2].$$

Limiting v_3 to w_2, and v_2 to v_1, we reduce to interpolation for

$$N_{C(0,2;1)}[u_1 + u_2 \overset{+}{\rightarrow} v_1][z_1 + z_2 + u_3 \overset{+}{\rightarrow} w_2][w_2 \overset{+}{\rightarrow} u_3].$$

Projecting from w_2 again, we reduce to interpolation for

$$N_{C(0,2;2)}(u_1 + u_2 + w_2),$$

which is a nonspecial line bundle and therefore satisfies interpolation.

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12.9. The cases $(d, g, r, \ell, m) = (5, 0, 4, 0, 1)$*, and* $(7, 3, 4, 1, 1)$

We want to show interpolation for a vector bundle of rank 3, and degree 28 and 46, respectively. By Lemma 3.9, it suffices to check interpolation for corresponding vector bundles of rank 27 and 45 where one positive transformation is omitted.

For $(d, g, r, \ell, m) = (5, 0, 4, 0, 1)$, the bundle of degree 28 for which we want to show interpolation is $N_C[\overset{+}{\rightsquigarrow} R]$. Write $C \cap R = \{q_1, q_2, q_3, q_4, q_5\}$. It suffices to establish interpolation for the degree 27 vector bundle $N_C[q_1 + q_2 + q_3 + q_4 \overset{+}{\rightsquigarrow} R]$. Specialize R to the union of the lines $\overline{q_1q_2}$, $\overline{q_3q_4}$ and the unique line through q_5 meeting both of these two lines. This induces a specialization of this bundle to $N_C[q_1 \overset{+}{\leftrightarrow} q_2][q_3 \overset{+}{\leftrightarrow} q_4]$, which is the assertion $I(5, 0, 4, 2, 0)$. Observe that $(5, 0, 4, 2, 0)$ was already considered above in Section 12.7.

For $(d, g, r, \ell, m) = (7, 3, 4, 1, 1)$, our bundle of degree 46 is $N_C[u \overset{+}{\leftrightarrow} v][\overset{+}{\rightsquigarrow} R]$, and we can reduce to interpolation for the degree 45 vector bundle $N_C[u \overset{+}{\rightarrow} v][\overset{+}{\rightsquigarrow} R]$. Erasing the transformation at u, this reduces to interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$, which is the assertion $I(7, 3, 4, 0, 1)$. Observe that $(7, 3, 4, 0, 1)$ was already considered above in Section 12.8.

12.10. The case $(d, g, r, \ell, m) = (7, 1, 5, 0, 1)$

In this case we want to show interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$. Write $R \cap C = \{q_1, q_2, q_3, q_4, q_5, q_6\}$. Peel off a one-secant line, that is, degenerate C to $C(0, 1; 0) \cup L$ — but in such a way that q_2 and q_4 specialize onto L and the remaining points specialize onto $C(0, 1; 0)$. By Lemma 3.7, this reduces interpolation for $N_C[\overset{+}{\rightsquigarrow} R]$ to interpolation for

$$N_{C(0,1;0)}[q_1 + q_3 + q_5 + q_6 \overset{+}{\rightsquigarrow} R][u \overset{+}{\leftrightarrow} v].$$

Specializing R to the union of the three lines $\overline{q_1q_2}$, $\overline{q_3q_4}$, $\overline{q_5q_6}$, and the unique fourth line in $\mathbb{P}^5$ meeting these three lines, we reduce to interpolation for

$$N_{C(0,1;0)}[q_1 \overset{+}{\rightarrow} q_2][q_3 \overset{+}{\rightarrow} q_4][q_5 \overset{+}{\leftrightarrow} q_6][u \overset{+}{\leftrightarrow} v].$$

Limiting q_2 to u and q_4 to v, we reduce to proving interpolation for

$$N_{C(0,1;0)}[q_1 + v \overset{+}{\rightarrow} u][q_3 + u \overset{+}{\rightarrow} v][q_5 \overset{+}{\leftrightarrow} q_6].$$

Projecting from u and then v, we reduce to interpolation for

$$N_{C(0,1;1)}[q_5 \overset{+}{\leftrightarrow} q_6].$$

This is the assertion $I(4, 0, 3, 1, 0)$, and $(4, 0, 3, 1, 0)$ is good.

12.11. The case $(d, g, r, \ell, m) = (7, 2, 5, 0, 1)$

We want to show that $N_C[\overset{+}{\rightsquigarrow} R]$ satisfies interpolation. Peeling off a one-secant line, we reduce to interpolation for

$$N_{C(0,1;0)}[\overset{+}{\rightsquigarrow} R][z \overset{+}{\leftrightarrow} w][z \overset{+}{\rightarrow} 2w].$$

We now specialize R as in Section 7 to the union of two lines $\overline{q_1p}$ and $\overline{q_2p}$, and a three-secant conic through q_3, q_4, p. Then limit w to p. This induces a specialization of our bundle to

$$N_{C(0,1;0)}[q_3 + q_4 \overset{+}{\rightsquigarrow} R^\circ][z + q_1 + q_2 \overset{+}{\rightarrow} p][z \overset{+}{\rightarrow} 2p][p \overset{+}{\rightarrow} z + q_1 + q_2].$$

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Projecting from p we reduce to interpolation for

$$N_{C(0,1;1)}[q_3 \overset{+}{\leftrightarrow} q_4][z \overset{+}{\to} p].$$

Limiting q_4 to p, we reduce to interpolation for

$$N_{C(0,1;1)}[q_3 + z \overset{+}{\to} p][p \overset{+}{\to} q_3].$$

Projecting from p, we reduce to interpolation for $N_{C(0,1;2)}[p \overset{+}{\to} q_3]$. Erasing the transformation $[p \overset{+}{\to} q_3]$, this reduces to $I(4,1,3,0,0)$, and $(4,1,3,0,0)$ is good.

12.12. The case $(d, g, r, \ell, m) = (7, 2, 5, 2, 2)$

This case asserts interpolation for

$$N_C[\overset{+}{\rightsquigarrow} R_1 + R_2][u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2].$$

We first specialize each R_i as in Section 7 to the union of two lines $\overline{q_{i1}p_i}, \overline{q_{i2}p_i}$ and a three-secant conic through $\{p_i, q_{i3}, q_{i4}\}$. We then specialize p_1 and p_2 together to a common point p. This induces a specialization of our bundle to

$$N_C(p)[q_{11} + q_{12} + q_{21} + q_{22} \overset{+}{\to} p][q_{13} + q_{14} \overset{+}{\rightsquigarrow} R_1^\circ][q_{23} + q_{24} \overset{+}{\rightsquigarrow} R_2^\circ][u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2].$$

Limiting u_1 to p and removing the overall twist at p reduces us to interpolation for

$$N_C[v_1 + q_{11} + q_{12} + q_{21} + q_{22} \overset{+}{\to} p][q_{13} + q_{14} \overset{+}{\to} Q_1][q_{23} + q_{24} \overset{+}{\to} Q_2][p \overset{+}{\to} v_1][u_2 \overset{+}{\leftrightarrow} v_2].$$

Projecting from p, we reduce to interpolation for

$$N_{C(0,0;1)}[q_{13} \overset{+}{\leftrightarrow} q_{14}][q_{23} \overset{+}{\leftrightarrow} q_{24}][p \overset{+}{\to} v_1][u_2 \overset{+}{\leftrightarrow} v_2].$$

Erasing the transformation $[p \overset{+}{\to} v_1]$ and peeling off two one-secant lines, we reduce to

$$N_{C(0,2;1)}[q_{13} \overset{+}{\leftrightarrow} q_{14}][q_{23} \overset{+}{\leftrightarrow} q_{24}][u_2 \overset{+}{\leftrightarrow} v_2][z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2].$$

Limiting w_1 and w_2 to p, we reduce to interpolation for

$$N_{C(0,2;1)}[q_{13} \overset{+}{\leftrightarrow} q_{14}][q_{23} \overset{+}{\leftrightarrow} q_{24}][u_2 \overset{+}{\leftrightarrow} v_2][z_1 + z_2 \overset{+}{\leftrightarrow} p][z_1 + z_2 \overset{+}{\to} 2p].$$

Projecting from p, we reduce to interpolation for

$$N_{C(0,2;2)}[q_{13} \overset{+}{\leftrightarrow} q_{14}][q_{23} \overset{+}{\leftrightarrow} q_{24}][u_2 \overset{+}{\leftrightarrow} v_2][z_1 + z_2 \overset{+}{\to} p].$$

Limiting q_{24} to q_{14}, and v_2 to p, and removing the resulting twist at q_{14}, we reduce to

$$N_{C(0,2;2)}[q_{13} + q_{23} \overset{+}{\to} q_{14}][u_2 + z_1 + z_2 \overset{+}{\to} p][p \overset{+}{\to} u_2].$$

Projecting from p, we reduce to interpolation for $N_{C(0,2;3)}$, which is a nonspecial line bundle.

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12.13. The case $(d, g, r, \ell, m) = (9, 2, 5, 0, 0)$

Peeling off two one-secant lines reduces to interpolation for

$$N_{C(0,2;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2].$$

Limit the points z_1 and w_2 to a common point p. This induces the specialization of our bundle to

$$N_{C(0,2;0)}[p \overset{+}{\leftrightarrow} w_1 + z_2][p \overset{+}{\to} 2w_1][z_2 \overset{+}{\to} 2p].$$

Projection from p reduces to interpolation for

$$N_{C(0,2;1)}[p \overset{+}{\to} w_1][z_2 \overset{+}{\to} p].$$

Erasing the transformation $[p \overset{+}{\to} w_1]$ and then projecting from p, we reduce to interpolation for $N_{C(0,2;2)}$. This is $I(5, 0, 3, 0, 0)$, and $(5, 0, 3, 0, 0)$ is good.

12.14. The cases $(d, g, r, \ell, m) = (9, 4, 5, 0, 0)$ *and* $(9, 4, 5, 1, 0)$

We want that both N_C and $N_C[u \overset{+}{\leftrightarrow} v]$ satisfy interpolation. Peeling off four one-secant lines, we reduce to interpolation for

$$N_{C(0,4;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\to} 2w_3][z_4 \overset{+}{\leftrightarrow} w_4][z_4 \overset{+}{\to} 2w_4]$$

and

$$N_{C(0,4;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\to} 2w_3] [z_4 \overset{+}{\leftrightarrow} w_4][z_4 \overset{+}{\to} 2w_4][u \overset{+}{\leftrightarrow} v].$$

Specializing w_2 to w_1, and w_4 to w_3, we reduce to interpolation for

$$N_{C(0,4;0)}[z_1 + z_2 \overset{+}{\leftrightarrow} w_1][z_1 + z_2 \overset{+}{\to} 2w_1][z_3 + z_4 \overset{+}{\leftrightarrow} w_3][z_3 + z_4 \overset{+}{\to} 2w_3] \text{ and}$$
$$N_{C(0,4;0)}[z_1 + z_2 \overset{+}{\leftrightarrow} w_1][z_1 + z_2 \overset{+}{\to} 2w_1][z_3 + z_4 \overset{+}{\leftrightarrow} w_3][z_3 + z_4 \overset{+}{\to} 2w_3][u \overset{+}{\leftrightarrow} v].$$

Projecting from w_1 and then w_3, we reduce to interpolation for

$$N_{C(0,4;2)}[z_1 + z_2 \overset{+}{\to} w_1][z_3 + z_4 \overset{+}{\to} w_3] \quad \text{and} \quad N_{C(0,4;2)}[z_1 + z_2 \overset{+}{\to} w_1][z_3 + z_4 \overset{+}{\to} w_3][u \overset{+}{\leftrightarrow} v].$$

Specializing v to w_1, we reduce to interpolation for

$$N_{C(0,4;2)}[z_1 + z_2 \overset{+}{\to} w_1][z_3 + z_4 \overset{+}{\to} w_3] \quad \text{and} \quad N_{C(0,4;2)}[z_1 + z_2 + u \overset{+}{\to} w_1][z_3 + z_4 \overset{+}{\to} w_3][w_1 \overset{+}{\to} u].$$

Projecting from w_1, we reduce to interpolation for $N_{C(0,4;3)}$, which is a nonspecial line bundle.

12.15. The cases $(d, g, r, \ell, m) = (8, 3, 5, 2, 0)$ *and* $(11, 5, 6, 0, 0)$

We first reduce interpolation for both of these bundles to the same statement.

For $(d, g, r, \ell, m) = (11, 5, 6, 0, 0)$ Note that $\delta = 4$. Our goal is to establish interpolation for N_C. We first peel off one two-secant lines, which reduces our problem to interpolation for

$$N_{C(2,0;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2].$$

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Limiting w_2 to w_1 induces a specialization of this bundle to

$$N_{C(2,0;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1][z_2 \overset{+}{\leftrightarrow} w_1][z_2 \overset{+}{\to} 2w_1].$$

Projecting from w_1, we reduce to interpolation for

$$N_{C(2,0;1)}[z_1 + z_2 \overset{+}{\leftrightarrow} w_1].$$

For $(d, g, r, \ell, m) = (8, 3, 5, 2, 0)$, our goal is to establish interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1][u_2 \overset{+}{\leftrightarrow} v_2].$$

Limiting v_2 to v_1, we reduce to interpolation for

$$N_C[u_1 + u_2 \overset{+}{\leftrightarrow} v_1].$$

To finish the argument, let C be a general BN-curve of degree 8 and genus 3 in $\mathbb{P}^5$, and $p, q_1, q_2 \in C$ be general points. Above, we have shown that both of the desired assertions reduce to interpolation for the modified normal bundle

$$N_C[q_1 + q_2 \overset{+}{\leftrightarrow} p].$$

We next peel off two one-secant lines, that is, degenerate C to $C \cup L_1 \cup L_2$, where L_1 and L_2 are one-secant lines to C – but in such a way that q_i limits onto L_i, and p limits onto C. Applying Lemma 5.5, we reduce to interpolation for

$$N_{C(0,2;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1 + p][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2 + p][p \overset{+}{\to} q_1 + q_2].$$

Over the function field of the moduli space of *unordered* pairs of triples $\{(z_1, w_1, q_1), (z_2, w_2, q_2)\}$, the transformation $[p \overset{+}{\to} q_1 + q_2]$ is linearly general as just q_1 and q_2 vary. Indeed, geometrically, it is transverse to any subspace of the normal space at p *except* for the two subspaces $N_{C \to L_1}|_p$ and $N_{C \to L_2}|_p$ – but neither of these subspaces is rational over this function field. Therefore, we may erase the transformation at p, thereby reducing to interpolation for

$$N_{C(0,2;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1 + p][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2 + p].$$

Note that $\delta = 3\frac{1}{2}$ for this bundle. Peeling off a one-secant line, we reduce to interpolation for

$$N_{C(0,3;0)}[z_1 \overset{+}{\leftrightarrow} w_1][z_1 \overset{+}{\to} 2w_1 + p][z_2 \overset{+}{\leftrightarrow} w_2][z_2 \overset{+}{\to} 2w_2 + p][z_3 \overset{+}{\leftrightarrow} w_3][z_3 \overset{+}{\to} 2w_3].$$

Specializing w_3 to p, and w_2 to w_1, we reduce to interpolation for

$$N_{C(0,3;0)}[z_1 + z_2 \overset{+}{\leftrightarrow} w_1][z_1 + z_2 \overset{+}{\to} 2w_1][z_1 + z_2 + z_3 \overset{+}{\to} p][z_3 \overset{+}{\to} 2p][p \overset{+}{\to} z_3].$$

Projecting from w_1 and then p, we reduce to interpolation for

$$N_{C(0,3;2)}[z_1 + z_2 \overset{+}{\to} w_1][z_3 \overset{+}{\leftrightarrow} p].$$

Finally, projecting from w_1 again, we reduce to interpolation for $N_{C(0,3;3)}$, which is a nonspecial line bundle.

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12.16. The case $(d, g, r, \ell, m) = (7, 0, 6, 0, 1)$

Arguing as in the proof of Proposition 8.3, it suffices to show that Q^- and Q^+ satisfy interpolation, where

$$Q^- = N_{C(0,0;1)}[s_1 + \cdots + s_5 \overset{+}{\rightsquigarrow} \overline{R}], \quad \text{and} \quad Q^+ = Q^-[p \overset{+}{\to} s_0].$$

As in the proof of Proposition 8.3, interpolation for Q^+ follows from interpolation for Q^- given the assertion $I(5, 0, 4, 0, 1)$. Since $(5, 0, 4, 0, 1)$ is good, it suffices to prove interpolation for Q^-. By Lemma 3.8, this follows in turn from interpolation for

$$Q^-(s_0) = Q^-[s_0 \overset{+}{\rightsquigarrow} \overline{R}][s_0 \overset{+}{\to} \Lambda],$$

where $\Lambda \subset Q^-|_{s_0}$ is codimension 1. By Lemma 3.9, since $\mu(Q^-[s_0 \overset{+}{\rightsquigarrow} \overline{R}]) \in \mathbb{Z}$, interpolation for $Q^-(s_0)$ follows from interpolation for

$$Q^-[s_0 \overset{+}{\rightsquigarrow} \overline{R}] = N_{C(0,0;1)}[\overset{+}{\rightsquigarrow} \overline{R}].$$

This is the assertion $I(6, 0, 5, 0, 1)$, and $(6, 0, 5, 0, 1)$ is good.

12.17. The cases $(d, g, r, \ell, m) = (8, 1, 7, 0, 1)$ *and* $(8, 1, 7, 1, 1)$

In both of these cases, our goal is to show interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][\overset{+}{\rightsquigarrow} R_1].$$

We specialize R_1 to the union of the lines q_1q_2, q_3q_4, q_5q_6, q_7q_8, together with a plane conic meeting each of these four lines. This induces a specialization of this bundle to

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][q_1 \overset{+}{\leftrightarrow} q_2][q_3 \overset{+}{\leftrightarrow} q_4][q_5 \overset{+}{\leftrightarrow} q_6][q_7 \overset{+}{\leftrightarrow} q_8].$$

Note that the points $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8$ are *not* general, as they are constrained to lie in a hyperplane.

Let $p_1, p_2, p_3, p_4 \in C$ be points with $\mathcal{O}_C(1) = 2p_1 + 2p_2 + 2p_3 + 2p_4$ (such points exist by Riemann–Roch because C is an elliptic curve). By construction, $H^0(\mathcal{O}_C(1)(-2p_1 - 2p_2 - 2p_3 - 2p_4)) = 1$; as C is embedded by a complete linear series, we conclude that the tangent lines to C at p_1, p_2, p_3, p_4 span a hyperplane H. Specializing the hyperplane containing $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8$ to H, in such a way that q_1 and q_8 specialize to p_1, and q_2 and q_3 specialize to p_2, and q_4 and q_5 specialize to p_3, and q_6 and q_7 specialize to p_4, we obtain a further specialization of the above bundle to

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1] \cdots [u_\ell \overset{+}{\leftrightarrow} v_\ell][p_1 + p_3 \overset{+}{\leftrightarrow} p_2 + p_4]. \tag{12.1}$$

Note that, since $H^0(\mathcal{O}_C(1)(-p_1 - p_2 - p_3 - p_4)) = 4$, the points p_1, p_2, p_3, p_4 are linearly independent. We claim interpolation for (12.1) reduces to interpolation for $N_{C(0,0;4)}$. To see this, we divide into cases as follows.

If $\ell = 0$ Note that $\delta = 2$. Our goal is to show interpolation for

$$N_C[p_1 + p_3 \overset{+}{\leftrightarrow} p_2 + p_4].$$

Projecting from each of p_1, p_2, p_3, p_4 in turn, we reduce to interpolation for $N_{C(0,0;4)}$.

If $\ell = 1$ Note that $\delta = 2\frac{1}{3}$. Our goal is to show interpolation for

$$N_C[u_1 \overset{+}{\leftrightarrow} v_1][p_1 + p_3 \overset{+}{\leftrightarrow} p_2 + p_4].$$

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Specializing u_1 to p_1 and v_1 to p_3, we reduce to interpolation for

$$N_C[p_1 \overset{+}{\leftrightarrow} p_3][p_1 + p_3 \overset{+}{\leftrightarrow} p_2 + p_4].$$

Projecting from p_1, p_2, p_3, and p_4, we reduce to interpolation for $N_{C(0,0;4)}$.

It remains to check interpolation for $N_{C(0,0;4)}$, which is the assertion $I(4,1,3,0,0)$, and $(4,1,3,0,0)$ is good.

12.18. The case $(d,g,r,\ell,m) = (11,4,7,1,0)$

Our goal is to show interpolation for $N_C[u \overset{+}{\leftrightarrow} v]$. Note that $\delta = 3$. Peeling off a one-secant line, we reduce to interpolation for

$$N_{C(0,1;0)}[z \overset{+}{\leftrightarrow} w][z \overset{+}{\rightarrow} 2w][u \overset{+}{\leftrightarrow} v].$$

Specializing v to w, we reduce to interpolation for

$$N_{C(0,1;0)}[z + u \overset{+}{\leftrightarrow} w][z \overset{+}{\rightarrow} 2w].$$

Projecting from w, we reduce to interpolation for

$$N_{C(0,1;1)}[z \overset{+}{\rightarrow} w][w \overset{+}{\rightarrow} z + u].$$

Erasing the transformation $[w \overset{+}{\rightarrow} z + u]$, we reduce to interpolation for the two bundles

$$N_{C(0,1;1)}[z \overset{+}{\leftrightarrow} w] \quad \text{and} \quad N_{C(0,1;1)}[z \overset{+}{\rightarrow} w].$$

The first of these statements is the assertion $I(9,3,6,1,0)$. For the second, erasing the transformation at z reduces it to interpolation for $N_{C(0,1;1)}$, which is the assertion $I(9,3,6,0,0)$. Note that both $(9,3,6,0,0)$ and $(9,3,6,1,0)$ are good.

13. Canonical curves of even genus

In this section, we prove interpolation for the normal bundle of a general canonical curve of even genus $g \geq 8$, which is the last remaining case (cf. Proposition 5.10 and Section 9). These cases are difficult, in part, because interpolation does not hold for canonical curves of genus 4 and 6, that is, when $r = 3$ or $r = 5$.

We will do this via degeneration to $E \cup R$, as in Section 5.3. That is, E is an elliptic normal curve in $\mathbb{P}^r$, and R is a general $(r+1)$-secant rational curve of degree $r-1$, where $r = g-1$ is odd.

13.1. Reduction to a bundle on E

Recall that, due to the exceptional case of elliptic normal curves in odd-dimensional projective spaces in Lemma 5.8, we cannot reduce interpolation for $N_{E\cup R}$ to interpolation for $N_{E\cup R}|_E$. Instead, we will reduce interpolation for $N_{E\cup R}$ to interpolation for a certain *modification* of $N_{E\cup R}|_E$. Our first step will be to show that $N_{E\cup R}|_R$ is not perfectly balanced, and give a geometric description of its Harder–Narasimhan (HN) filtration.

Lemma 13.1. *Let $q_1, \ldots, q_{2n+2}$ be a general collection of points on $\mathbb{P}^1$. Let $p_1, \ldots, p_{2n+2}$ be a general collection of points on a general elliptic curve E. Then there exist exactly two maps of degree $n+1$ from E to $\mathbb{P}^1$ that send p_i to q_i.*

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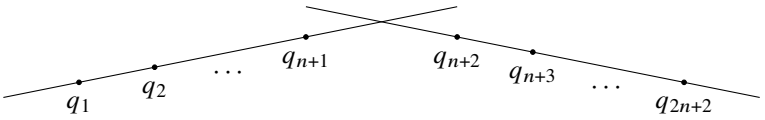
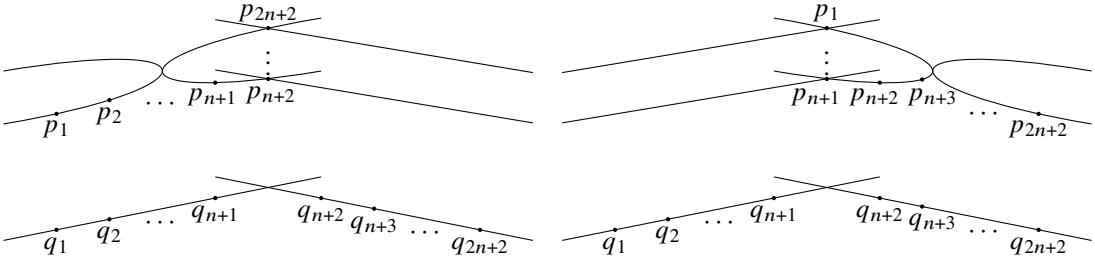
Proof. If $f\colon E \to \mathbb{P}^1$ is a general map of degree $n+1$, then $f^*T_{\mathbb{P}^1}(-p_1 - \cdots - p_{2n+2})$ has vanishing cohomology. Therefore, deformations of f are in bijection with deformations of the $f(p_i)$. The number of maps of degree $n+1$ from E to $\mathbb{P}^1$ that send p_i to q_i is therefore finite and nonzero.

To calculate this number, we degenerate the target $\mathbb{P}^1$ to a binary curve, with $q_1, q_2, \ldots, q_{n+1}$ on the left $\mathbb{P}^1$, and $q_{n+2}, q_{n+3}, \ldots, q_{2n+2}$ on the right $\mathbb{P}^1$. This degeneration is illustrated in the following diagram:



Such a map $E \to \mathbb{P}^1$ then degenerates to an admissible cover from a marked curve whose stable model is $(E, p_1, p_2, \ldots, p_{2n+2})$. One can construct two such admissible covers (both with no infinitesimal deformations sending p_i to q_i): In one such cover, E maps to the left component, with $p_1, p_2, \ldots, p_{n+1}$ mapping to $q_1, q_2, \ldots, q_{n+1}$, and $p_{n+2}, p_{n+3}, \ldots, p_{2n+2}$ mapping to the node; each of the points $p_{n+2}, p_{n+3}, \ldots, p_{2n+2}$ is then attached to a rational tail mapping isomorphically onto the right component. Similarly, in the other such cover, E maps to the right component, with $p_{n+2}, p_{n+3}, \ldots, p_{2n+2}$ mapping to $q_{n+2}, q_{n+3}, \ldots, q_{2n+2}$, and $p_1, p_2, \ldots, p_{n+1}$ mapping to the node and attached to a rational tail mapping isomorphically onto the left component. These covers are pictured in the following diagrams:



In fact, these are the only two such admissible covers. Indeed, the curve E must map to one of the two components of the above degeneration of the target $\mathbb{P}^1$, say without loss of generality to the left component. Then $p_{n+2}, \ldots, p_{2n+2}$ must map to the node, which we normalize to $[1:0]$. Hence, the map $E \to \mathbb{P}^1$ is given by $[s:1]$ for a section $s \in H^0(\mathcal{O}(p_{n+2} + \cdots + p_{2n+2}))$. Since $p_1, \ldots, p_{n+1}$ are general, the evaluation map $H^0(\mathcal{O}(p_{n+2} + \cdots + p_{2n+2})) \to \bigoplus_{i=1}^{n+1} \mathcal{O}(p_{n+2} + \cdots + p_{2n+2})|_{p_i}$ is an isomorphism. Hence, s is uniquely determined. We conclude that, when $q_1, q_2, \ldots, q_{2n+2} \in \mathbb{P}^1$ are general, there are exactly two such maps. □

Write $f_i\colon E \to \mathbb{P}^1$ (for $i \in \{1,2\}$) for these two maps, and $\overline{f} = (f_1, f_2)\colon E \to \mathbb{P}^1 \times \mathbb{P}^1$ for the resulting map.

Lemma 13.2. *In the setup of Lemma 13.1, the map $\overline{f}$ is a general map from E to $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(n+1, n+1)$. In particular:*

- *$\overline{f}$ is birational onto its image, and its image is nodal.*
- *$\overline{f}^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1) \in \mathrm{Pic}^0(E)$ is general (and thus nontrivial).*

Proof. Fix a general elliptic curve E. Let $f\colon E \to \mathbb{P}^1 \times \mathbb{P}^1$ be a general map. Write $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ for the diagonal. Then the $\{p_1, p_2, \ldots, p_{2n+2}\} = f^{-1}(\Delta) \subset E$, and their images $\{q_1, q_2, \ldots, q_{2n+2}\} \subset \mathbb{P}^1$

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under the composition of either projection with f, satisfy all the hypotheses of Lemma 13.1 except for possibly genericity.

To complete the proof, it remains to check that there is no obstruction to deforming f so that these points become general. In other words, we must check $H^1(f^*T_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta)) = 0$. But this is true because f is general and $f^*T_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta) \simeq f^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1) \oplus f^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1)$. □

Lemma 13.3. *In the setup of Lemma 13.1, we have*

$$\overline{f}^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \simeq \mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2}).$$

Proof. The isomorphism class of the line bundle $\overline{f}^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ is independent of the moduli of the points $q_1, \ldots, q_{2n+2}$ because they vary in a rational base. Hence, we may calculate it in the degeneration of Lemma 13.1, where the result clearly holds. □

Lemma 13.4. *In the setup of Lemma 13.1, the pushforward of $\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2})$ along either map is perfectly balanced, that is,*

$$(f_i)_*\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2}) \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n+1)}.$$

Proof. Since f_i is of degree $n+1$, the pushforward $(f_i)_*\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2})$ is a rank $n+1$ vector bundle on $\mathbb{P}^1$, that is, we can write

$$(f_i)_*\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2}) \simeq \bigoplus_{j=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(a_j).$$

The integers a_j satisfy

$$\sum a_j = \chi\left(\bigoplus_{j=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(a_j)\right) - (n+1) = \chi(\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2})) - (n+1)$$
$$= 2n + 2 - (n+1) = n + 1,$$

so to see that $a_j = 1$ for all j, it suffices to see that $a_j < 2$ for all j, that is, that

$$0 = H^0\left(\bigoplus_{j=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(a_j - 2)\right) = H^0(\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2}) \otimes f_i^*\mathcal{O}_{\mathbb{P}^1}(-2)),$$

or equivalently that

$$\mathcal{O}_E(p_1 + p_2 + \cdots + p_{2n+2}) \neq \overline{f}^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0) = \overline{f}^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \otimes \overline{f}^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1),$$

which follows from Lemmas 13.2 and 13.3. □

Write $r = 2n+1$. Applying Lemma 13.1, there are exactly two maps $f_i\colon E \to \mathbb{P}^1 \simeq R$, of degree $n+1$, sending $\Gamma|_E$ to $\Gamma|_R$, where $\Gamma = E \cap R$. Write $\overline{f} = (f_1, f_2)\colon E \to \mathbb{P}^1 \times \mathbb{P}^1$. Let S denote the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the nodes of the image of E under $\overline{f}$, so that the f_i give rise to an embedding $f\colon E \hookrightarrow S$. Writing $F_1, \ldots, F_{n^2-1}$ for the exceptional divisors, define

$$L := \mathcal{O}_S(n,n)\left(-\sum F_i\right) = K_S(1,1)(E).$$

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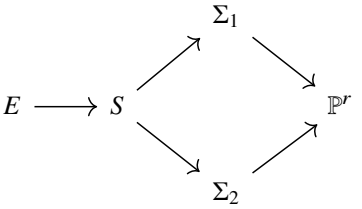
By adjunction on S and Lemma 13.3, we have $L|_E \simeq \mathcal{O}_S(1,1)|_E \simeq \mathcal{O}_E(1)$. Let $\pi_1 \colon S \to \mathbb{P}^1$ and $\pi_2 \colon S \to \mathbb{P}^1$ denote the two projections. By intersection theory, for $x \in \mathbb{P}^1$, the restriction of the line bundle $L(-E)$ to the corresponding fiber $\pi_i^{-1}(x)$ is the (unique) line bundle of total degree -1 that is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$ on any exceptional divisors lying over x. In particular, it has vanishing cohomology, so by the theorem on cohomology and base-change, $(\pi_i)_* L(-E) = R^1(\pi_i)_* L(-E) = 0$. Combining this with Lemma 13.4, we therefore have a natural identification

$$(\pi_i)_* L \simeq (f_i)_* \mathcal{O}_E(1) \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{n+1}. \tag{13.1}$$

The map $S \to \mathbb{P}^{2n+1}$ via $|L|$ thus factors through a (uniquely defined) embedding of $\mathbb{P}[(\pi_i)_* L] \simeq \mathbb{P}^1 \times \mathbb{P}^n$, via the complete linear system of the relative $\mathcal{O}(1)$ on $\mathbb{P}[(\pi_i)_* L]$, which corresponds to $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(1,1)|$ on $\mathbb{P}^1 \times \mathbb{P}^n$. In particular, since $S \to \mathbb{P}[(\pi_i)_* L]$ is an embedding, so is $S \to \mathbb{P}^{2n+1}$. Write $\Sigma_i \subset \mathbb{P}^{2n+1}$ for the scroll obtained as the image of the map $\mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^{2n+1}$.

Putting all of this together, we can summarize this situation with the following diagram of inclusions:



Lemma 13.5. *The intersection $\Sigma_1 \cap \Sigma_2$ coincides with S.*

Proof. Let $x_1, x_2 \in \mathbb{P}^1$ be any two points, and write Λ_i for the fiber of Σ_i over x_i. Note that Λ_i is the span of the divisor $f_i^{-1}(x_i)$.

First, suppose that (x_1, x_2) is not a node of $\overline{f}(E)$. If (x_1, x_2) does not lie on $\overline{f}(E)$, then the span of Λ_1 and Λ_2 is the span of the divisor $f_1^{-1}(x_1) + f_2^{-1}(x_2)$. Since this divisor is linearly equivalent to $O_E(1)$, the span is a hyperplane. Otherwise, if $(x_1, x_2) = \overline{f}(y)$ lies on $\overline{f}(E)$, then the span of Λ_1 and Λ_2 is the span of the divisor $f_1^{-1}(x_1) + f_2^{-1}(x_2) - y$. Since any $2n+1$ points on E are linearly general, this span is again a hyperplane. Either way, since Λ_1 and Λ_2 span a hyperplane, they must meet at the single point that is the image of (x_1, x_2) on S.

Next, suppose that (x_1, x_2) is a node of $\overline{f}(E)$, say $(x_1, x_2) = \overline{f}(y_1) = \overline{f}(y_2)$. Then the span of Λ_1 and Λ_2 is the span of the divisor $f_1^{-1}(x_1) + f_2^{-1}(x_2) - y_1 - y_2$. Since any $2n$ points on E are linearly general, this span is codimension 2, and so Λ_1 and Λ_2 meet along the line that is the image of the exceptional divisor over (x_1, x_2).

Combining these two cases, we see that $\Sigma_1 \cap \Sigma_2$ coincides with S set-theoretically. To upgrade this to a scheme-theoretic equality, we must show that Σ_1 and Σ_2 are quasi-transverse along S, meaning that the tangent spaces to Σ_1 and Σ_2 at points of S span a hyperplane. (They cannot span all of $\mathbb{P}^r$, because $\Sigma_1 \cap \Sigma_2$ is pure of dimension 2.) Away from the exceptional divisors, this is straightforward: The tangent space to Σ_i contains Λ_i, so it suffices to note that Λ_1 and Λ_2 span a hyperplane.

It remains to consider an exceptional divisor $M = \Lambda_1 \cap \Lambda_2$. Write Λ for the span of Λ_1 and Λ_2. As in the previous case, the span of the tangent spaces to the Σ_i contains Λ; however, in this case Λ is codimension 2. It thus remains to show that the natural map $N_{M/\Sigma_1} \oplus N_{M/\Sigma_2} \to N_{\Lambda/\mathbb{P}^r}|_M$ is everywhere nonzero along M. Recall that we write y_i for the points on E lying over the node, that is, satisfying $\overline{f}(y_1) = \overline{f}(y_2) = (x_1, x_2)$ so that M is the line spanned by y_1 and y_2. Let y_i^j be nontrivial first-order deformations of the y_i satisfying $f_j(y_1^j) = f_j(y_2^j)$. Such deformations are pictured in the following diagram:

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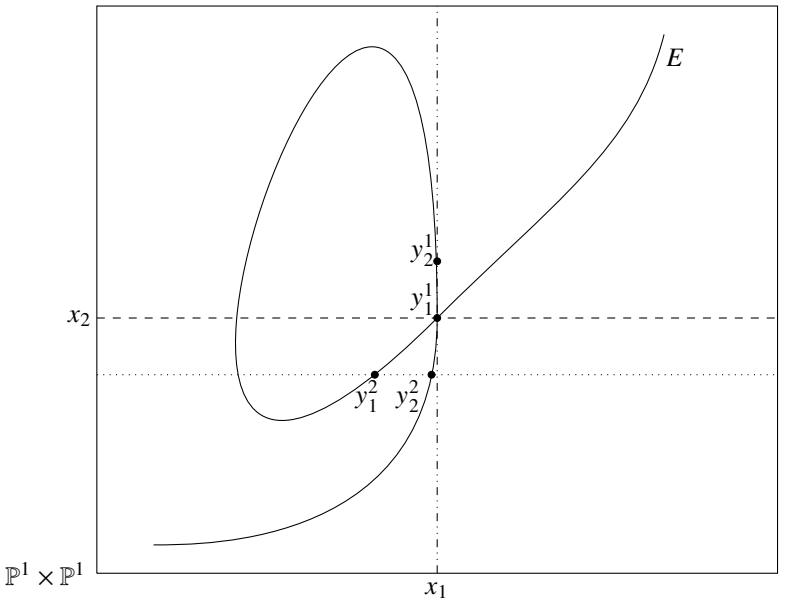

The lines joining y_1^j and y_2^j give first-order deformations of M in Σ_j, that is, sections σ_j of N_{M/Σ_j}. It suffices to see that the images of the σ_j in $N_{\Lambda/\mathbb{P}^r}|_M$ do not simultaneously vanish anywhere along M.

The span of Λ and the tangent line to E at y_1 is the span of the divisor $f_1^{-1}(x_1) + f_2^{-1}(x_2) - y_2$. Since any $2n+1$ points on E are linearly general, this span is a hyperplane. Similarly, the span of Λ and the tangent line to E at y_2 is a hyperplane. Moreover, the span of Λ and the tangent lines to E at *both* y_1 and y_2 is the span of the divisor $f_1^{-1}(x_1) + f_2^{-1}(x_2)$, which is linearly equivalent to $\mathcal{O}_E(1)$, and therefore again spans a hyperplane. Since this hyperplane contains the first two of these hyperplanes, all three of these hyperplanes must be equal. Write Λ' for this hyperplane.

By construction, the images of both σ_j are nonzero sections in the subspace $N_{\Lambda/\Lambda'}|_M \simeq \mathcal{O}_M(1)$. Because $\overline{f}(E)$ is nodal, the deformations $(y_1^1, y_2^1), (y_1^2, y_2^2)$ form a basis of $T_{y_1}E \oplus T_{y_2}E$. The images of these two sections thus form a basis of $H^0(N_{\Lambda/\Lambda'}|_M) = H^0(\mathcal{O}_M(1))$ and so do not simultaneously vanish anywhere along M as desired. □

The upshot of this is that we have a natural filtration of $N_{E\cup R}$, whose successive quotients are vector bundles of ranks 1, $2n-2$ and 1, respectively:

$$0 \subset N_{E\cup R/S} \subset N_{E\cup R/\Sigma_1} + N_{E\cup R/\Sigma_2} \subset N_{E\cup R}. \tag{13.2}$$

Lemma 13.6. *We have* $c_1(N_{E\cup R/\Sigma_j}|_R) = n(2n+3)+1$.

Proof. The Picard group of Σ_j is spanned by the class γ of one n-plane and the restriction of the hyperplane class h from $\mathbb{P}^{2n+1}$. One computes that $K_{\Sigma_j} = (n-1)\gamma - (n+1)h$ for such a scroll. By adjunction we have

$$\begin{aligned} c_1(N_{R/\Sigma_j}) &= c_1(K_R) - c_1(K_S|_R) \\ &= -2 - ((n-1)\gamma - (n+1)h)\cdot R \\ &= -2 - (n-1)(\gamma\cdot R) + (n+1)(h\cdot R) \\ &= -2 - (n-1)\cdot 1 + (n+1)\cdot 2n \\ &= 2n^2 + n - 1. \end{aligned}$$

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Therefore,

$$c_1(N_{E\cup R/\Sigma_j}|_R) = c_1(N_{R/S}) + \#\Gamma = (2n^2+n-1)+(2n+2) = n(2n+3)+1.$$

□

Proposition 13.7. *The vector bundle $N_{E\cup R}|_R \simeq N_R[\overset{+}{\rightsquigarrow} E]$ is isomorphic to*

$$\mathcal{O}_{\mathbb{P}^1}(2n+4)\oplus\mathcal{O}_{\mathbb{P}^1}(2n+3)^{\oplus(2n-2)}\oplus\mathcal{O}_{\mathbb{P}^1}(2n+2).$$

Moreover, its HN-filtration is precisely the restriction of the filtration (13.2) to R.

Proof. By Lemma 5.8, we have either $N_R[\overset{+}{\rightsquigarrow} E] \simeq \mathcal{O}_{\mathbb{P}^1}(2n+4)\oplus\mathcal{O}_{\mathbb{P}^1}(2n+3)^{\oplus(2n-2)}\oplus\mathcal{O}_{\mathbb{P}^1}(2n+2)$ or $N_R[\overset{+}{\rightsquigarrow} E] \simeq \mathcal{O}_{\mathbb{P}^1}(2n+3)^{\oplus 2n}$. Lemma 13.6 rules out the second case since $\mathcal{O}_{\mathbb{P}^1}(2n+3)^{\oplus 2n}$ admits no subbundle of rank n and first Chern class $n(2n+3)+1$.

Moreover, any subbundle of $\mathcal{O}_{\mathbb{P}^1}(2n+4)\oplus\mathcal{O}_{\mathbb{P}^1}(2n+3)^{\oplus(2n-2)}\oplus\mathcal{O}_{\mathbb{P}^1}(2n+2)$ of rank n and first Chern class $n(2n+3)+1$ contains $O_{\mathbb{P}^1}(2n+4)$ and is contained in $O_{\mathbb{P}^1}(2n+4)\oplus\mathcal{O}_{\mathbb{P}^1}(2n+3)^{\oplus(2n-2)}$. Since the graded pieces of equation (13.2) are the intersection and span of the $N_{E\cup R/\Sigma_j}$, they must therefore coincide with the HN-filtration. □

This provides the promised determination of $N_R[\overset{+}{\rightsquigarrow} E]$, and the promised geometric construction of its HN-filtration. This geometric description of the HN-filtration of $N_{E\cup R}|_R$ allows us to reduce interpolation for $N_{E\cup R}$ to interpolation for a modification of $N_{E\cup R}|_E \simeq N_E[\overset{+}{\rightsquigarrow} R]$ as follows.

Lemma 13.8. *Let p and q be two distinct points of $E\cap R$. Then $N_{E\cup R}$ satisfies interpolation provided that*

$$N_E[\overset{+}{\rightsquigarrow} R][p \overset{+}{\to} N_{E/S}][q \overset{-}{\to} N_{E/\Sigma_1}+N_{E/\Sigma_2}] \tag{13.3}$$

satisfies interpolation.

Proof. We imitate the basic idea of the proof of [2, Lemma 8.8]. Write $\Gamma := E\cap R$, which has size $r+1$. Write x, y, z for three general points on R. Twisting down, we have

$$N_{E\cup R}(-x-y-z)|_R \simeq \mathcal{O}_{\mathbb{P}^1}(r-2)\oplus\mathcal{O}_{\mathbb{P}^1}(r-1)^{\oplus r-3}\oplus\mathcal{O}_{\mathbb{P}^1}(r).$$

Therefore,the evaluation map

$$\mathrm{ev}_{R,\Gamma}\colon H^0(N_{E\cup R}|_R) \to N_{E\cup R}|_\Gamma$$

is injective when restricted to the subspace $H^0(N_{E\cup R}|_R(-x-y-z))$. Our aim is to suitably specialize the points x, y, z so as to be able to identify the subspace of sections of $N_{E\cup R}|_E$

$$V_{x,y,z} := \left\{\sigma\in H^0(E, N_{E\cup R}|_E) : \sigma|_\Gamma \in \mathrm{Im}\left(\mathrm{ev}_{R,\Gamma}|_{H^0(N_{E\cup R}|_R(-x-y-z))}\right)\right\}$$

that glues to the image of $H^0(N_{E\cup R}|_R(-x-y-z))$ under $\mathrm{ev}_{R,\Gamma}$. By Lemma 3.6, it suffices to show that this subspace of sections has the correct dimension and satisfies interpolation to conclude that $N_{E\cup R}(-x-y-z)$ satisfies interpolation, which implies that $N_{E\cup R}$ satisfies interpolation.

Since $\#\Gamma = r+1$, the evaluation map $\mathrm{ev}_{R,\Gamma}$ restricted to the sections of the largest factor $\mathcal{O}_{\mathbb{P}^1}(r)$ is already an isomorphism. The evaluation on the other factors is not an isomorphism: On the $\mathcal{O}_{\mathbb{P}^1}(r-1)$ factors, the image is a codimension 1 subspace of $\mathcal{O}_{\mathbb{P}^1}(r-1)|_\Gamma$, and on the $\mathcal{O}_{\mathbb{P}^1}(r-2)$ factor, the image is a codimension 2 subspace of $\mathcal{O}_{\mathbb{P}^1}(r-2)|_\Gamma$. We will appropriately specialize so as to force these subspaces to be 'coordinate' planes.

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First, limit x to p. The gluing data across the nodes (in particular at p) is fixed, and therefore the limiting codimension 1 subspace of $\mathcal{O}_{\mathbb{P}^1}(r-1)|_\Gamma$ contains the subspace $\mathcal{O}_{\mathbb{P}^1}(r-1)|_{\Gamma\setminus p} \oplus 0|_p$ of sections vanishing at p. Since this subspace has the correct dimension, it must be the flat limit. Then limit y to q. In the limit, the codimension 2 subspace of $\mathcal{O}_{\mathbb{P}^1}(r-2)|_\Gamma$ must contain the subspace $\mathcal{O}_{\mathbb{P}^1}(r-2)|_{\Gamma\setminus\{p,q\}} \oplus 0|_p \oplus 0|_q$ of sections vanishing at p and q. Since this has the correct dimension, it must be the flat limit. Since the HN-filtration on $N_{E\cup R}|_R$ is the restriction of equation (13.2) to R, this flat limit is

$$H^0\left(E, N_E[\overset{+}{\rightsquigarrow} R][p \overset{-}{\to} N_{E/S}][q \overset{-}{\to} N_{E/\Sigma_1} + N_{E/\Sigma_2}]\right).$$

Since this subspace is the space of sections of a vector bundle, it suffices to show that this bundle satisfies interpolation. To complete the proof, we note that $\mu(N_E[\overset{+}{\rightsquigarrow} R][p \overset{-}{\to} N_{E/S}][q \overset{-}{\to} N_{E/\Sigma_1}+N_{E/\Sigma_2}]) \geq 1$, and so it suffices by Lemma 3.8 to prove interpolation after twisting up by p. □

13.2. The case $r \geq 9$

By Lemma 13.8, it suffices to show equation (13.3) satisfies interpolation. On an elliptic curve, we can characterize which bundles satisfy interpolation in terms of the Atiyah classification.

Lemma 13.9. *Let $\mathscr{E}$ be a vector bundle on an elliptic curve E. Then $\mathscr{E}$ satisfies interpolation if and only if there is a nonnegative integer a for which every Jordan–Holder (JH) factor $\mathscr{F}$ of $\mathscr{E}$ satisfies*

$$a \leq \mu(\mathscr{F}) \leq a+1 \quad \textit{and} \quad \mathscr{F} \neq \mathcal{O}_E. \tag{13.4}$$

Proof. By the Atiyah classification, every JH-factor of $\mathscr{E}$ is both a subbundle and quotient of $\mathscr{E}$.

First, suppose $\mathscr{E}$ satisfies interpolation. Then $\mathscr{E}$ is nonspecial, so every JH-factor $\mathscr{F}$ is nonspecial or equivalently satisfies $\mu(\mathscr{F}) \geq 0$ and $\mathscr{F} \neq \mathcal{O}_E$. If no such nonnegative integer a exists, then there would be a positive integer b and JH-factors $\mathscr{F}_1$ and $\mathscr{F}_2$ with $\mu(\mathscr{F}_1) < b < \mu(\mathscr{F}_2)$. This is a contradiction, since for general points $p_1, p_2, \ldots, p_b \in E$, we would have

$$H^0(\mathscr{E}(-p_1 - \cdots - p_b)) \neq 0 \quad \text{and} \quad H^1(\mathscr{E}(-p_1 - \cdots - p_b)) \neq 0.$$

In the other direction, suppose there is a nonnegative integer a for which every JH-factor $\mathscr{F}$ satisfies equation (13.4). Then for general points $p_1, p_2, \ldots, p_{a+1} \in E$,

$$H^0(\mathscr{E}(-p_1 - \cdots - p_{a+1})) = 0 \quad \text{and} \quad H^1(\mathscr{E}(-p_1 - \cdots - p_a)) = 0.$$

Therefore,$\mathscr{E}$ satisfies interpolation. □

Lemma 13.10. *Let $\mathscr{E}$ be a vector bundle on an elliptic curve E, and let a and b be integers. For two points $p, q \in E$, consider subspaces $\Delta \subseteq \mathscr{E}|_p$ of rank 1 and $\Lambda \subseteq \mathscr{E}|_q$ of corank 1. If every JH-factor $\mathscr{F}$ of $\mathscr{E}$ satisfies $a < \mu(\mathscr{F}) < b$, then every JH-factor $\mathscr{F}'$ of $\mathscr{E}' := \mathscr{E}[p \overset{+}{\to} \Delta][q \overset{-}{\to} \Lambda]$ satisfies $a \leq \mu(\mathscr{F}') \leq b$.*

Proof. Up to replacing $\mathscr{E}'$ with its dual, it suffices to show that every JH-factor $\mathscr{F}'$ of $\mathscr{E}'$ satisfies $\mu(\mathscr{F}') \leq b$. Since every JH-factor is a subbundle, and $\mathscr{E}'$ is a subsheaf of $\mathscr{E}[p \overset{+}{\to} \Delta]$, it suffices to show that every subsheaf $\mathscr{F}'$ of $\mathscr{E}[p \overset{+}{\to} \Delta]$ satisfies $\mu(\mathscr{F}') \leq b$.

If $\mathscr{F}'$ is a subsheaf of $\mathscr{E}$, we are done by assumption. Otherwise, write x for the degree of $\mathscr{F}'$ and y for the rank of $\mathscr{F}'$ so that $\mu(\mathscr{F}') = x/y$. Write $\mathscr{F} = \mathscr{F}' \cap \mathscr{E}$ for the corresponding subsheaf of $\mathscr{E}$. Then $(x-1)/y = \mu(\mathscr{F}) < b$. Since b is an integer, $\mu(\mathscr{F}') = x/y \leq b$ as desired. □

Combining Lemmas 13.9 and 13.10, it suffices to prove:

Proposition 13.11. *If $r \geq 9$ is odd, every JH-factor $\mathscr{F}$ of $N_E[\overset{+}{\rightsquigarrow} R]$ satisfies $r+4 < \mu(\mathscr{F}) < r+5$.*

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Our proof of Proposition 13.11 will be by induction on r.

When $r = 7$, we have $\mu(N_E[\overset{+}{\rightsquigarrow} R]) = 12 = r+5$. Since $I(8,1,7,0,1)$ holds, we deduce from Lemma 13.9 that every JH-factor $\mathscr{F}$ of $N_E[\overset{+}{\rightsquigarrow} R]$ has slope exactly $r+5$ in this case. Although Proposition 13.11 does not hold in this case, it is close enough that we will be able to leverage it to establish the case $r = 9$.

In general, our strategy will be to use our inductive hypothesis to show that Proposition 13.11 is close enough to holding that naturality of the HN-filtration forces it to hold exactly.

Definition 13.12. Let E be a genus 1 curve. We say that a map

$$\operatorname{Pic}^a E \xrightarrow{f} \operatorname{Pic}^b E$$

is natural if for any automorphism $\theta\colon E \to E$, the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Pic}^a E & \xrightarrow{f} & \operatorname{Pic}^b E \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} \\ \operatorname{Pic}^a E & \xrightarrow{f} & \operatorname{Pic}^b E \end{array}$$

Proposition 13.13. *If* $\operatorname{Pic}^a E \to \operatorname{Pic}^b E$ *is natural, then* a *divides* b.

Proof. Translation by an a-torsion point acts as the identity on $\operatorname{Pic}^a E$, and so it must also act as the identity on $\operatorname{Pic}^b E$. □

Proof of Proposition 13.11. It suffices to prove that $N_E[\overset{+}{\rightsquigarrow} R]$ has no subbundles of slope $r+5$ or more and no quotient bundles of slope $r+4$ or less. We will prove this by induction on r, using the case $r = 7$ discussed above as our base case. Our argument will consist of two steps:

1. We *specialize* so that the statement of the proposition becomes *false* but still close enough to true that we can gather information about the possible limits of subbundles of large slope (respectively quotient bundles of small slope).
2. Leveraging this information, we apply Proposition 13.13 to the *general* fiber.

Our specialization will be of R to the union $R^\circ = \overline{pq} \cup R^-$, of a one-secant line $\overline{pq}$ and an $(r-1)$-secant rational curve R^- of degree $r-2$ meeting $\overline{pq}$ at a single point. Projection from $\overline{pq}$ induces an exact sequence

$$0 \to \left[S := \mathcal{O}_E(1)(2p+q) \oplus \mathcal{O}_E(1)(2q+p)\right] \to N_E[\overset{+}{\rightsquigarrow} R^\circ] \to \left[Q := N_{E(0,0;2)}(p+q)[\overset{+}{\rightsquigarrow} \overline{R^-}]\right] \to 0. \tag{13.5}$$

The bundle S is perfectly balanced of slope $r+4$. The bundle Q is a twist of another instance of our problem in $\mathbb{P}^{r-2}$. If $r \geq 11$, then by induction, every JH-factor of Q has slope strictly between $r+4$ and $r+5$; if $r = 9$, then every JH-factor of Q has slope exactly $r+5$.

We begin by showing $N_E[\overset{+}{\rightsquigarrow} R]$ has no quotient bundles of slope $r+4$ or less. Since Q has no quotient bundles of slope $r+4$ or less, any such quotient must specialize to a quotient of S, and therefore must have slope exactly $r+4$ and rank at most 2. Let G be the maximal such quotient of $N_E[\overset{+}{\rightsquigarrow} R]$ (i.e., on the general fiber). Our above specialization of R shows that $\mu(G) = r+4$ and $n := \operatorname{rk} G \leq \operatorname{rk} S = 2$. The determinant $\det G$ depends on the following data:

- A line bundle $\mathcal{O}_E(1)$.
- A basis for $H^0(\mathcal{O}_E(1))$.
- A hyperplane $H \subset \mathbb{P}H^0(\mathcal{O}_E(1))^\vee$.
- A rational curve $R \subset H$ of degree $r-1$ passing through the $r+1$ points of $E \cap H$.

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Except for the choice of line bundle $\mathcal{O}_E(1)$, all of these data vary in a rational family. Since any map from a rational variety to an abelian variety is constant, $\det G$ depends only on the choice of line bundle $\mathcal{O}_E(1)$. Extracting the determinant of G therefore gives a natural map

$$\mathrm{Pic}^{r+1} E \to \mathrm{Pic}^{n(r+4)} E.$$

Hence, by Proposition 13.13, we have $(r+1) \mid n(r+4)$, and therefore $(r+1) \mid 3n$. Since we have $3n \leq 6 < 10 \leq r+1$, we must have $3n = 0$, that is, $n = 0$ as desired.

We next show that $N_E[\overset{+}{\rightsquigarrow} R]$ has no subbundles of slope $r+5$ or more. If $r > 9$, then the specialization $N_E[\overset{+}{\rightsquigarrow} R^\circ]$ has no such subbundle because S and Q do not. It therefore remains only to consider the case $r = 9$, in which every JH-factor of S has slope 13 and every JH-factor of Q has slope 14. Let G be the maximal such subbundle of $N_E[\overset{+}{\rightsquigarrow} R]$ (i.e., on the general fiber). Our above specialization of R shows that $\mu(G) = 14$ and $n := \mathrm{rk}\, G \leq \mathrm{rk}\, Q = 6$. Extracting the determinant of G therefore gives a natural map

$$\mathrm{Pic}^{10} E \to \mathrm{Pic}^{14n} E.$$

Hence, by Proposition 13.13, we have that $10 \mid 14n$, so $5 \mid n$. If $n = 0$, we are done, so suppose that $n = 5$.

To obtain a contradiction, we analyze what happens in our specialization, in which G specializes to a subbundle G° of Q with slope 14 and rank 5. We consider the determinant $\det[G^\circ(-p-q)]$. A priori this depends only on $\mathcal{O}_E(1)$, p, and q (the remaining data vary in a rational family). In fact, we claim it depends only on

$$\mathcal{O}_{E(0,0;2)}(1) = \mathcal{O}_E(1) - p - q.$$

Indeed, $\det[G^\circ(-p-q)]$ is a product of JH-factors of $N_{E(0,0;2)}[\overset{+}{\rightsquigarrow} \overline{R^-}]$, which is a discrete set of possibilities once we fix $\mathcal{O}_{E(0,0;2)}(1)$ and some additional data varying in a rational family. As we fix $\mathcal{O}_{E(0,0;2)}(1)$ and these additional data, we may allow $\{p, q\}$ to vary arbitrarily in $E \times E$ by Lemma 8.1. In this way, we obtain a map from $E \times E$ to this discrete set, which must therefore be constant because $E \times E$ is connected. Therefore,$\det[G^\circ(-p-q)]$ depends only on $\mathcal{O}_{E(0,0;2)}(1)$ plus these additional data varying in a rational family and thus only on $\mathcal{O}_{E(0,0;2)}(1)$. The determinant of $G^\circ(-p-q)$ therefore gives a natural map

$$\mathrm{Pic}^{8} E \to \mathrm{Pic}^{60} E.$$

This is a contradiction by Proposition 13.13 since $8 \nmid 60$. □

13.3. The case r = 7

To handle this case, we will first have to study the restriction of equation (13.2) to E, which we do for arbitrary odd r. This is a filtration of $N_E[\overset{+}{\rightsquigarrow} R]$ whose successive quotients are:

$$N_{E/S}(1), \quad N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E, \quad \text{and} \quad \frac{N_E}{N_{E/\Sigma_1} + N_{E/\Sigma_2}}.$$

We write $H_i := f_i^* \mathcal{O}_{\mathbb{P}^1}(1)$ for the corresponding hyperplane class.

Proposition 13.14. *We have $N_{E/S} \simeq \mathcal{O}_E(3-n)$ (and so $N_{E/S}(1) \simeq \mathcal{O}_E(4-n)$).*

Proof. We consider the sequence (*not* exact) of maps

$$[N_{E/S} \simeq N_f] \to N_{\overline{f}} \to \overline{f}^* N_{\overline{f}(E)/\mathbb{P}^1 \times \mathbb{P}^1}.$$

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By inspection, both of these maps drop rank exactly at the points of E lying over the nodes of $\overline{f}$. Therefore,their Chern classes lie in a linear progression, that is,

$$\begin{aligned} c_1(N_{E/S}) &= 2 \cdot c_1(N_{\overline{f}}) - c_1(\overline{f}^* N_{\overline{f}(E)/\mathbb{P}^1\times\mathbb{P}^1}) \\ &= -2c_1(\overline{f}^* K_{\mathbb{P}^1\times\mathbb{P}^1}) - c_1(\overline{f}^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(n+1, n+1)) \\ &= -2(-2H_1 - 2H_2) - (n+1)(H_1 + H_2) \\ &= (3-n)(H_1 + H_2). \end{aligned}$$

Proposition 13.15. *We have* $c_1(N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E) = \mathcal{O}_E(3n-3)$. □

Proof. As in the proof of Proposition 13.7, define the classes γ_j and h in $\operatorname{Pic}\Sigma_j$ to be the class of one n-plane, and the restriction of the hyperplane class from $\mathbb{P}^{2n+1}$, respectively. By adjunction,

$$c_1(N_{E/\Sigma_j}) = -c_1(K_{\Sigma_j}|_E) = -\big((n-1)\gamma_j - (n+1)h\big) \cdot E.$$

Therefore,

$$\begin{aligned} c_1(N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E) &= c_1(N_{E/\Sigma_1}) + c_1(N_{E/\Sigma_2}) - 2c_1(N_{E/S}) \\ &= -(n-1)(\gamma_1+\gamma_2)\cdot E + 2(n+1)(h\cdot E) - 2(3-n)(H_1+H_2) \\ &= -(n-1)(H_1+H_2) + 2(n+1)(H_1+H_2) - 2(3-n)(H_1+H_2) \\ &= (3n-3)(H_1+H_2). \end{aligned}$$

□

Proposition 13.16. *We have* $\frac{N_E}{N_{E/\Sigma_1}+N_{E/\Sigma_2}} \simeq \mathcal{O}_E(2)$.

Proof. Since $K_{\mathbb{P}^{2n+1}} = \mathcal{O}_{\mathbb{P}^{2n+1}}(-(2n+2))$, we have $c_1(N_E) = \mathcal{O}_E(2n+2)$. Combined with the previous two propositions, this implies the statement of the proposition ($(2n+2) - (3-n) - (3n-3) = 2$). □

We now take $r = 7$ (equivalently $n = 3$) and let p and q be points of $E \cap R$. By Lemma 13.8, it suffices to show interpolation for

$$N_E[\tilde{\to} R][p \xrightarrow{+} N_{E/S}][q \xrightarrow{-} N_{E/\Sigma_1} + N_{E/\Sigma_2}].$$

This bundle has slope 12, so it suffices to show that for a general effective divisor D of degree 12,

$$H^0(N_E[\tilde{\to} R][p \xrightarrow{+} N_{E/S}][q \xrightarrow{-} N_{E/\Sigma_1} + N_{E/\Sigma_2}](-D)) = 0. \tag{13.6}$$

Furthermore, $I(8,1,7,0,1)$ holds, so $h^0(N_E[\tilde{\to} R](-D)) = h^1(N_E[\tilde{\to} R](-D)) = 0$, which implies that $h^0(N_E[\tilde{\to} R][p \xrightarrow{+} N_{E/S}](-D)) = 1$. Call the unique section σ. If there is any point $q \in E \cap R \setminus \{p\}$ for which $\sigma|_q \notin (N_{E/\Sigma_1} + N_{E/\Sigma_2})|_q$, then we have proved the desired vanishing (13.6). We may therefore assume that at all points of $E \cap R \setminus \{p\}$, the value of σ lies in the subbundle $N_{E/\Sigma_1} + N_{E/\Sigma_2}$. By Proposition 13.16, we have the exact sequence

$$0 \to \big(N_{E/\Sigma_1} + N_{E/\Sigma_2}\big)[\tilde{\to} R](-D) \to N_E[\tilde{\to} R](-D) \to \mathcal{O}_E(2)(-D) \to 0. \tag{13.7}$$

Since $\deg(\mathcal{O}_E(2)(-D)(-E \cap R + p)) = -3$, we must have that σ comes from a section of

$$\big(N_{E/\Sigma_1} + N_{E/\Sigma_2}\big)[\tilde{\to} R][p \xrightarrow{+} N_{E/S}](-D).$$

It therefore suffices to show that for some $p \in E \cap R$, this bundle has no global sections (or, equivalently since the degree is -3 such that $h^1 = 3$). Using equation (13.7), we have

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$h^1\left(\left(N_{E/\Sigma_1}+N_{E/\Sigma_2}\right)[\overset{+}{\rightsquigarrow} R](-D)\right) = 4$, so such a point $p \in E \cap R$ exists unless making positive modifications towards *all* points of $E \cap R$ does not decrease the h^1:

$$h^1\left(\left(N_{E/\Sigma_1}+N_{E/\Sigma_2}\right)[\overset{+}{\rightsquigarrow} R](-D)[E \cap R \overset{+}{\to} N_{E/S}]\right) = 4.$$

Equivalently, we are done unless

$$h^0\left(\left(N_{E/\Sigma_1}+N_{E/\Sigma_2}\right)[\overset{+}{\rightsquigarrow} R](-D)[E \cap R \overset{+}{\to} N_{E/S}]\right) = 8. \tag{13.8}$$

Taking the sum of the two normal bundle exact sequences for $S \hookrightarrow \Sigma_1$ and $S \hookrightarrow \Sigma_2$ along E yields

$$0 \to N_{E/S}(2(E \cap R)) \to \left(N_{E/\Sigma_1}+N_{E/\Sigma_2}\right)[\overset{+}{\rightsquigarrow} R][E \cap R \overset{+}{\to} N_{E/S}] \to N_{S/\Sigma_1}|_E \oplus N_{S/\Sigma_2}|_E \to 0. \tag{13.9}$$

Twisting down by D, the line subbundle $N_{E/S}(2(E \cap R))(-D)$ has degree 4, and hence four global sections and vanishing H^1. The quotient twisted down by D (which has degree 0) must therefore also have four global section in order for equation (13.8) to hold. Furthermore, the two scrolls Σ_1 and Σ_2 are exchanged by monodromy because the two maps $f_i\colon E \to \mathbb{P}^1$ in Lemma 13.1 have degree $n+1$, which is not a multiple of $2n+2$, and hence by Proposition 13.13 they cannot be individually naturally defined. Thus, the two rank 2 bundles $N_{S/\Sigma_i}|_E(-D)$ necessarily both have 2 sections. If $N_{S/\Sigma_i}|_E$ were indecomposable, then by the Atiyah classification, it would necessarily be an extension of a degree 12 line bundle M by itself. As long as $\mathcal{O}_E(D) \neq M$, we would have $h^0(N_{S/\Sigma_i}|_E(-D)) = 0$. Therefore,$N_{S/\Sigma_i}|_E$ is a direct sum of line bundles. Since $h^0(N_{S/\Sigma_i}|_E(-D)) = 2$:

$$N_{S/\Sigma_i}|_E \simeq L_{i1} \oplus L_{i2} \qquad \text{where} \qquad \deg(L_{i1}) = 14 \text{ and } \deg(L_{i2}) = 10.$$

Then $\det[L_{11} \oplus L_{21}]$ gives a natural map

$$\mathrm{Pic}^8 E \to \mathrm{Pic}^{28} E,$$

which is a contradiction by Proposition 13.13, since $8 \nmid 28$.

A. Code for Section 11

```
#!/usr/bin/python

XX = set([(5,2,3,0,0), (4,1,3,1,0), (4,1,3,0,1), (4,1,3,1,1),
          (6,2,4,0,0), (5,1,4,1,0), (5,1,4,1,1), (5,1,4,2,1), (6,2,4,1,1),
          (7,2,5,0,0), (6,1,5,0,1), (6,1,5,1,1)])

def good(d, g, r, l, m):
  if not (d >= g+r and 0 <= 2*l <= r and 0 <= m <= (r+1)*d - r*g - r*(r+1)):
    return False
  if m == g == 0 and 2 * l < (1 - d) % (r - 1):
    return False
  if (d,g,r,l,m) in XX:
    return False
  return True

def n_lower_bound(d, g, r):
  if r % 2 == 0:
    return 3
  elif (d, g) == (r + 1, 1):
    return 4
  else:
    return 2

def erasable(r, s10, s11, s20, s21, w10, t1=0, t2=0, strong=True):
  if s10 == s11 == s20 == s21 == w10 == 0:
```

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```
      return (strong and t2 == 0)

    if s10 > 0:
      if t1 + 1 < r - 1:
        if erasable(r, s10 - 1, s11, s20, s21, w10, t1 + 1, t2, strong):
          return True
      elif t2 < t1 + 1 == r - 1:
        if erasable(r, s10 - 1, s11, s20, s21, w10, t2, 0, True):
          return True

    if s11 > 0:
      if t1 + 1 < r - 1:
        if erasable(r, s10, s11 - 1, s20, s21, w10, t1 + 1, t2 + 1, strong):
          return True
      elif t2 + 1 < t1 + 1 == r - 1:
        if erasable(r, s10, s11 - 1, s20, s21, w10, t2 + 1, 0, True):
          return True

    if s20 > 0:
      if t1 + 2 < r - 1:
        if erasable(r, s10, s11, s20 - 1, s21, w10, t1 + 2, t2, strong):
          return True
      elif t2 < t1 + 2 == r - 1:
        if erasable(r, s10, s11, s20 - 1, s21, w10, t2, 0, True):
          return True
      elif t2 + 2 <= r - 1 <= t1 + 2:
        if erasable(r, s10, s11, s20 - 1, s21, w10, t2 + t1 + 2-(r-1), 0, strong):
          return True

    if s21 > 0:
      if t1 + 2 < r - 1:
        if erasable(r, s10, s11, s20, s21 - 1, w10, t1 + 2, t2 + 1, strong):
          return True
      elif t2 + 1 < t1 + 2 == r - 1:
        if erasable(r, s10, s11, s20, s21 - 1, w10, t2 + 1, 0, True):
          return True
      elif t2 + 2 < t1 + 1 == r - 1:
        if erasable(r, s10, s11, s20, s21 - 1, w10, t2 + 2, 0, True):
          return True
      elif t1 + 1 == t2 + 2 == r - 1:
        if erasable(r, s10, s11, s20, s21 - 1, w10, 0, 0, True):
          return True

    if w10 > 0:
      if t1 + 1 < r - 1:
        if erasable(r, s10, s11, s20, s21, w10 - 1, t1 + 1, t2, False):
          return True

    return False

  def can_induct(d, g, r, l, m):
    # Proposition 8.4
    if d >= g + 2 * r - 1 and good(d - (r - 1), g, r, l, m):
      return True

    # Proposition 8.6
    if m >= r - 1 and good(d, g, r, l, m - (r - 1)):
      return True

    deltanum = 2 * d + 2 * g - 2 * r + 2 * l + (r + 1) * m

    for lp in range(l + 1):
      for mp in range(m + 1):
        if (2 * mp + lp > r - 2) or (mp > 0 and r == 3):
          continue
        mbar = m - mp
```

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```
      for dp in range(g + r, d + 1):
        if g == 0 and m != 0 and dp == g + r:
          continue

        in0 = lp + 2 * (d - dp)

        for sum_n in range(n_lower_bound(dp, g, r) * mp, (r - 1) * mp + 1, 2):
          lbar = l - lp + ((r - 1) * mp - sum_n) // 2

          # Proposition 8.2
          if (r-1)*(in0 + sum_n - 1) + 1 <= deltanum <= (r-1)*(in0 + sum_n + 1) - 1:
            if good(dp - 1, g, r - 1, lbar, mbar):
              return True

          # Proposition 8.3
          if mp < m and 2 * mp + lp < r - 2:
            if (r-1)*(in0 + sum_n) + 1 <= deltanum <= (r-1)*(in0 + sum_n + 2) - 1:
              if good(dp - 1, g, r - 1, lbar, mbar):
                if good(dp - 1, g, r - 1, lbar, mbar - 1):
                  if good(dp - 2, g, r - 2, lbar, mbar):
                    return True

  # Proposition 8.7
  if l == 0 and m == 1:
    for epsilon in range(0, (d - g - r) // 2 + 1):
      if g > 0 or 2 * epsilon < d - g - r:
        if 2 * epsilon * (r - 1) + 2 <= deltanum <= (2 * epsilon + 2) * (r - 1) - 2:
          if good(d - 2 * epsilon - 2, g, r - 2, 0, 1):
            return True

  # Proposition 8.8
  k = r // 2
  if k >= 3 and (d, g, r, l, m) == (4 * k + 1, 2 * k - 1, 2 * k + 1, 0, 1):
    if good(4 * k - 3, 2 * k - 2, 2 * k - 1, k - 3, 0):
      return True


  # Proposition 8.9
  if m == 0 and g >= 3 and r >= 6:
    for epsilon in range((d - g - r) // 3 + 1):
      if ((2*epsilon+2)*(r-1) + 3 <= deltanum <= (2*epsilon+4)*(r-1) - 3):
        if good(d - 3 * epsilon - 6, g - 3, r - 3, l + 1, 0):
          if good(d - 3 * epsilon - 6, g - 3, r - 3, l, 0):
            return True

  # Proposition 8.10
  if m == 0 and g >= 1 and (r - 1) + 1 <= deltanum <= 3 * (r - 1) - 1:
    if good(d - 2, g - 1, r - 1, l + 1, 0):
      return True

  # Proposition 8.11
  if m == 0 and g >= 3 and r >= 6 and 3*(r-1) + 2 <= deltanum <= 5*(r-1) - 2:
    if good(d - 5, g - 3, r - 2, l + 1, 0) and good(d - 5, g - 3, r - 2, l, 0):
      return True

  ## Proposition 11.2... ##
  for lp in range(l + 1):
    for mp in range(m + 1):
      if r == 3 and mp > 0:
        continue
      for mpp in range(m - mp + 1):
        mbar_max = m - mp
        mbar_min = m - mp - mpp

        for gp in range(g + 1):
          for dp in range(gp + r, d - g + gp + 1):
            if gp == 0 and m != 0 and dp == gp + r:
```

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```
                continue

              for epsin in range(d - g - dp + gp + 1):
                epsout = d - g - dp + gp - epsin
                out = 2 * epsout + 3 * (g - gp) + m + mp + lp
                in0 = 2 * epsin + g - gp + mpp + lp + out // (r - 1)

                for sum_n in range(n_lower_bound(dp, g, r) * mp, (r - 1) * mp + 1, 2):
                  lbar = l - lp + ((r - 1) * mp - sum_n) // 2

                  if (r - 1) * (in0 + sum_n - 1) + 1 <= deltanum:
                    if deltanum <= (r - 1) * (in0 + sum_n + 1) - 1:
                      if erasable(r, lp + m - mp - mpp, epsout, mp, g - gp, mpp):
                        ok = True
                        for mbar in range(mbar_min, mbar_max + 1):
                          if not good(dp - 1, gp, r - 1, lbar, mbar):
                            ok = False
                            break
                        if ok:
                          return True

  return False

base_cases = []
for r in range(3, 14):
  for g in range(r):
    for d in range(g + r, g + 2 * r):
      for l in range(r//2 + 1):
        for m in range(r):
          if (l, m) == (0, 0) and 2 * d + 2 * g == 3 * r - 1: #cf. Section 6
            continue
          if good(d, g, r, l, m):
            if not can_induct(d, g, r, l, m):
              base_cases.append((d, g, r, l, m))

for (d, g, r, l, m) in XX:
  if good(d, g, r, l, m + r - 1):
    if not can_induct(d, g, r, l, m + r - 1):
      if (d, g, r, l, m + r - 1) not in base_cases:
        base_cases.append((d, g, r, l, m + r - 1))

print('␣&␣'.join([str(i) for i in base_cases]))
```

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Competing interest. The authors have no competing interest to declare.

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