



# On the Parameterized Complexity of Reconfiguration of Connected Dominating Sets

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## Abstract

In a reconfiguration version of a decision problem  $\mathcal{Q}$  the input is an instance of  $\mathcal{Q}$  and two feasible solutions  $S$  and  $T$ . The objective is to determine whether there exists a step-by-step transformation between  $S$  and  $T$  such that all intermediate steps also constitute feasible solutions. In this work, we study the parameterized complexity of the CONNECTED DOMINATING SET RECONFIGURATION problem (CDS- R). It was shown in previous work that the DOMINATING SET RECONFIGURATION problem (DS- R) parameterized by  $k$ , the maximum allowed size of a dominating set in a reconfiguration sequence, is fixed-parameter tractable on all graphs that exclude a biclique  $K_{d,d}$  as a subgraph, for some constant  $d \geq 1$ . We show that the additional connectivity constraint makes the problem much harder, namely, that CDS- R is W[1]-hard parameterized by  $k + \ell$ , the maximum allowed size of a dominating set plus the length of the reconfiguration sequence, already on 5-degenerate graphs. On the positive side, we

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show that CDS- R parameterized by  $k$  is fixed-parameter tractable, and in fact admits a polynomial kernel on planar graphs.

**Keywords** Reconfiguration · Parameterized complexity · Connected dominating set · Graph structure theory

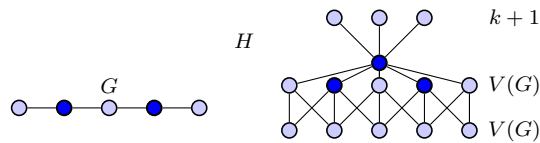
## 1 Introduction

In a decision problem  $\mathcal{Q}$ , we are usually asked to determine the existence of a feasible solution for an instance  $\mathcal{I}$  of  $\mathcal{Q}$ . In a *reconfiguration version* of  $\mathcal{Q}$ , we are instead given a source feasible solution  $S$  and a target feasible solution  $T$  and we are asked to determine whether it is possible to transform  $S$  into  $T$  by a sequence of step-by-step transformations such that after each intermediate step we also maintain feasible solutions. Formally, we consider a graph, called the *reconfiguration graph*, that has one vertex for each feasible solution and where two vertices are connected by an edge if we allow the transformation between the two corresponding solutions. We are then asked to determine whether  $S$  and  $T$  are connected in the reconfiguration graph, or even to compute a shortest path between them. Historically, the study of reconfiguration questions predates the field of computer science, as many classic one-player games can be formulated as such reachability questions [21,23], e.g., the 15-puzzle and Rubik’s cube. More recently, reconfiguration problems have emerged from computational problems in different areas such as graph theory [2,19,20], constraint satisfaction [13,28] and computational geometry [6,22,26], and even quantum complexity theory [12]. Reconfiguration problems have been receiving considerable attention in recent literature, we refer the reader to [18,27,31] for an extensive overview.

In this work, we consider the CONNECTED DOMINATING SET RECONFIGURATION problem (CDS- R) in undirected graphs. A *dominating set* in a graph  $G$  is a set  $D \subseteq V(G)$  such that every vertex of  $G$  lies either in  $D$  or is adjacent to a vertex in  $D$ . A dominating set  $D$  is a *connected dominating set* if the graph induced by  $D$  is connected. The DOMINATING SET problem and its connected variant have many applications, including the modeling of facility location problems, routing problems, and many more [1,15,35].

We study CDS- R under the *Token Addition/Removal* model (TAR model). Suppose we are given a connected dominating set  $D$  of a graph  $G$ , and imagine that a token is placed on each vertex in  $D$ . The TAR rule allows either the addition or removal of a single token at a time from  $D$ , if this results in a connected dominating set of size at most a given bound  $k \geq 1$ . A sequence  $D_1, \dots, D_\ell$  of connected dominating sets of a graph  $G$  is called a *reconfiguration sequence* between  $D_1$  and  $D_\ell$  under TAR if the change from  $D_i$  to  $D_{i+1}$  respects the TAR rule, for  $1 \leq i < \ell$ . The *length* of the reconfiguration sequence is  $\ell - 1$ .

The (CONNECTED) DOMINATING SET RECONFIGURATION problem for TAR gets as input a graph  $G$ , two (connected) dominating sets  $S$  and  $T$  and an integer  $k \geq 1$ , and the task is to decide whether there exists a reconfiguration sequence between  $S$  and  $T$  under TAR using at most  $k$  tokens.



**Fig. 1** A graph  $G$  with a minimum dominating set of size  $k = 2$  marked in dark blue and the graph  $H$  obtained in the standard reduction from DOMINATING SET to CONNECTED DOMINATING SET.  $G$  has a dominating set of size  $k$  if and only if  $H$  has a connected dominating set of size  $k + 1$ . If  $p$  is equal to the pathwidth of  $G$  then the pathwidth of  $H$  is bounded by  $2p + 1$

Structural properties of the reconfiguration graph for  $k$ -dominating sets were studied in [16,34]. The DOMINATING SET RECONFIGURATION problem was shown to be PSPACE-complete in [17], even on split graphs, bipartite graphs, planar graphs and graphs of bounded bandwidth. Both the pathwidth and the treewidth of a graph are bounded by its bandwidth, hence the DOMINATING SET RECONFIGURATION problem is PSPACE-complete on graphs of bounded pathwidth and treewidth. These hardness results motivated the study of the parameterized complexity of the problem. It was shown in [29] that the DOMINATING SET RECONFIGURATION problem is  $W[2]$ -hard when parameterized by  $k + \ell$ , where  $k$  is the bound on the number of tokens and  $\ell$  is the length of the reconfiguration sequence. However, the problem becomes fixed-parameter tractable (when parameterized by  $k$ ) on graphs that exclude a fixed complete bipartite graph  $K_{d,d}$  as a subgraph, as shown in [25]. Such so-called *biclique-free* classes are very general sparse graph classes, including in particular the planar graphs, which are  $K_{3,3}$ -free.

In this work we study the complexity of CDS- R. The standard reduction from DOMINATING SET to CONNECTED DOMINATING SET shows that CDS- R is also PSPACE-complete, even on graphs of bounded pathwidth (Fig. 1). We hence turn our attention to the parameterized complexity of the problem.<sup>1</sup> We first show that the additional connectivity constraint makes the problem much harder, namely, that CDS- R parameterized by  $k + \ell$  is  $W[1]$ -hard already on 5-degenerate graphs. As 5-degenerate graphs exclude the biclique  $K_{6,6}$  as a subgraph, DOMINATING SET RECONFIGURATION is fixed-parameter tractable on much more general graph classes than its connected variant. To prove hardness we first introduce an auxiliary problem that we believe is of independent interest. In the COLORED CONNECTED SUBGRAPH problem we are given a graph  $G$ , an integer  $k$ , and a (not necessarily proper) coloring  $c: V(G) \rightarrow C$ , for some color set  $C$  with  $|C| \leq k$ . The question is whether  $G$  contains a vertex subset  $H$  on at most  $k$  vertices such that  $G[H]$  is connected and  $H$  contains at least one vertex of every color in  $C$  (i.e.,  $c(H) = C$ ). The reconfiguration variant COLORED CONNECTED SUBGRAPH RECONFIGURATION (CCS- R) is defined as expected. We first prove that CCS- R reduces to CDS- R by a parameter preserving reduction (where  $k + \ell$  is the parameter) and the degeneracy of the reduced to graph is at most the degeneracy of the input graph plus one. We then prove that the known  $W[1]$ -hard problem MULTI-COLORED CLIQUE (see [4] for definitions) reduces to CCS- R on 4-degenerate graphs. The last reduction has the additional property that for an input  $(G, c, k)$  of MULTI-

<sup>1</sup> We note that the problem is easily shown to be slice-wise polynomial parameterized for parameter  $k + \ell$  as one can guess each set in the reconfiguration sequence.

COLORED CLIQUE the resulting instance of CCS- R admits either a reconfiguration sequence of length  $\mathcal{O}(k^3)$ , or no reconfiguration sequence at all. Hence, we derive that both CDS- R and CCS- R are  $W[1]$ -hard parameterized by  $k + \ell$  on 5-degenerate and 4-degenerate graphs, respectively.

The existence of a reconfiguration sequence of length at most  $\ell$  with connected dominating sets of size at most  $k$  can be expressed by a first-order formula of length depending only on  $k$  and  $\ell$ . It follows from [14] that the problem is fixed-parameter tractable parameterized by  $k + \ell$  on every nowhere dense graph class and the same is implied by [3] for every class of bounded cliquewidth. Nowhere dense graph classes are very general classes of uniformly sparse graphs, in particular the class of planar graphs is nowhere dense. Nowhere dense classes are themselves biclique-free, but are not necessarily degenerate. Hence, our hardness result on degenerate graphs essentially settles the question of fixed-parameter tractability for the parameter  $k + \ell$  on sparse graph classes. It remains an interesting open problem to find dense graph classes beyond classes of bounded cliquewidth on which the problem is fixed-parameter tractable.

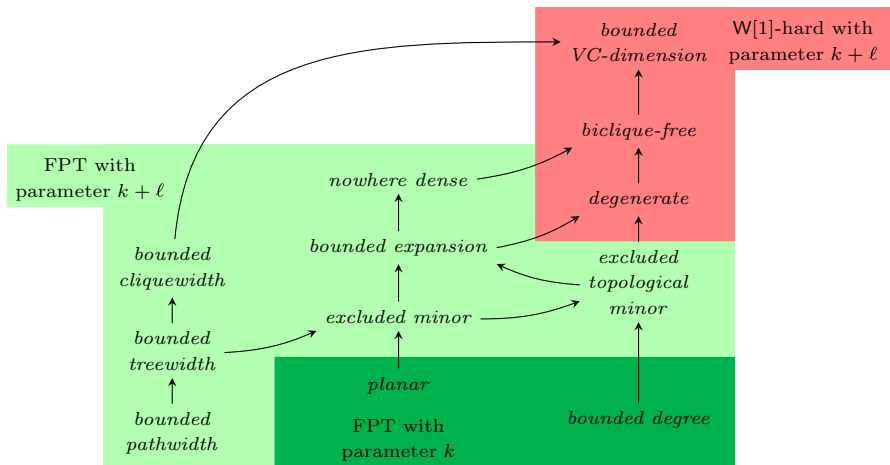
We then turn our attention to the smaller parameter  $k$  alone. We show that CDS- R parameterized by  $k$  is fixed-parameter tractable on the class of planar graphs. Our approach is as follows. We first compute a small *domination core* for  $G$ , a set of vertices that captures exactly the domination properties of  $G$  for dominating sets of sizes not larger than  $k$ . The notion of a domination core was introduced in the study of the DISTANCE-  $r$  DOMINATING SET problem on nowhere dense graph classes [5]. While the classification of interactions with the domination core would suffice to solve DOMINATING SET RECONFIGURATION on nowhere dense classes, additional difficulties arise for the connected variant. In a second step we use planarity to identify large subgraphs that have very simple interactions with the domination core and prove that they can be replaced by constant size gadgets such that the reconfiguration properties of  $G$  are preserved.

Observe that CDS- R parameterized by  $k$  is trivially fixed-parameter tractable on every class of bounded degree. The existence of a connected dominating set of size  $k$  implies that the diameter of  $G$  is bounded by  $k + 2$ , which in every bounded degree class implies a bound on the size of the graph depending only on the degree and  $k$ . We conjecture that CDS- R is fixed-parameter tractable parameterized by  $k$  on every nowhere dense graph class. However, resolving this conjecture remains open for future work (see Fig. 2).

The rest of the paper is organized as follows. We give background on graph theory and fix our notation in Sect. 2. We show hardness of CDS- R on degenerate graphs in Sect. 3 and show how to handle the planar case in Sect. 4.

## 2 Preliminaries

We denote the set of natural numbers by  $\mathbb{N}$ . For  $n \in \mathbb{N}$ , we let  $[n] = \{1, 2, \dots, n\}$ . We assume that each graph  $G$  is finite, simple, and undirected. We let  $V(G)$  and  $E(G)$  denote the vertex set and edge set of  $G$ , respectively. An edge between two vertices  $u$  and  $v$  in a graph is denoted by  $\{u, v\}$  or  $uv$ . The *open neighborhood* of a vertex  $v$  is denoted by  $N_G(v) = \{u \mid \{u, v\} \in E(G)\}$  and the *closed neighborhood* by



**Fig. 2** The map of tractability for CONNECTED DOMINATING SET RECONFIGURATION. The classes colored in dark green admit an FPT algorithm with parameter  $k$ , the classes colored in light green admit an FPT algorithm with parameter  $k + \ell$ . On the classes colored in red the problem is W[1]-hard with respect to the parameter  $k + \ell$

$N_G[v] = N_G(v) \cup \{v\}$ . The degree of a vertex  $v$ , denoted by  $d_G(v)$ , is  $|N_G(v)|$ . For a set of vertices  $S \subseteq V(G)$ , we define  $N_G(S) = \{v \notin S \mid \{u, v\} \in E(G), u \in S\}$  and  $N_G[S] = N_G(S) \cup S$ . The subgraph of  $G$  induced by  $S$  is denoted by  $G[S]$ , where  $G[S]$  has vertex set  $S$  and edge set  $\{\{u, v\} \in E(G) \mid u, v \in S\}$ . We let  $G - S = G[V(G) \setminus S]$ . A graph  $G$  is  $d$ -degenerate if every subgraph  $H \subseteq G$  has a vertex of degree at most  $d$ . For a set  $C$ , we use  $K[C]$  to denote the complete graph on vertex set  $C$ . For an integer  $r \in \mathbb{N}$ , an  $r$ -independent set in a graph  $G$  is a subset  $U \subseteq V(G)$  such that for any two distinct vertices  $u, v \in U$ , the distance between  $u$  and  $v$  in  $G$  is more than  $r$ . An independent set in a graph is a 1-independent set. A subset of vertices  $U$  in  $G$  is called a separator in  $G$  if  $G - U$  has more than one connected component. For  $s, t \in V(G)$ , we say  $U$  is an  $(s, t)$ -separator in  $G$  if there is no path from  $s$  to  $t$  in  $G - U$ .

### 3 Hardness on Degenerate Graphs

In this section we prove that CDS- R and CCS- R are W[1]-hard when parameterized by  $k + \ell$  even on 5-degenerate and 4-degenerate graphs, respectively. Towards that, we first give a polynomial-time reduction from the W[1]-hard MULTICOLORED CLIQUE problem to CCS- R on 4-degenerate graphs with the property that for an input  $(G, c, k)$  of MULTICOLORED CLIQUE the resulting instance of CCS- R admits either a reconfiguration sequence of length  $\mathcal{O}(k^3)$  or no reconfiguration sequence at all. As a result, we conclude that CCS- R is W[1]-hard when parameterized by  $k + \ell$  on 4-degenerate graphs. Then, we give a parameter-preserving polynomial-time reduction from CCS- R to CDS- R.

Let us first formally define the CCS and CCS- R problems.

**COLORED CONNECTED SUBGRAPH (CCS)**

**Parameter:**  $k$

**Input:** A graph  $G$ , a vertex-coloring  $c: V(G) \rightarrow C$ , and  $k \in \mathbb{N}$  such that  $|C| \leq k$

**Question:** Is there a vertex subset  $S \subseteq V(G)$  of at most  $k$  vertices with at least one vertex from every color class such that  $G[S]$  is connected?

**COLORED CONNECTED SUBGRAPH RECONF (CCS- R)**

**Parameter:**  $k$

**Input:** A graph  $G$ , a vertex-coloring  $c: V(G) \rightarrow C$ , two sets  $Q_s, Q_t \subseteq V(G)$ , and  $k \in \mathbb{N}$  such that  $|C|, |Q_s|, |Q_t| \leq k$ ,  $c(Q_s) = c(Q_t) = C$ , and  $G[Q_s], G[Q_t]$  are connected

**Question:** Is there a reconfiguration sequence from  $Q_s$  to  $Q_t$ ?

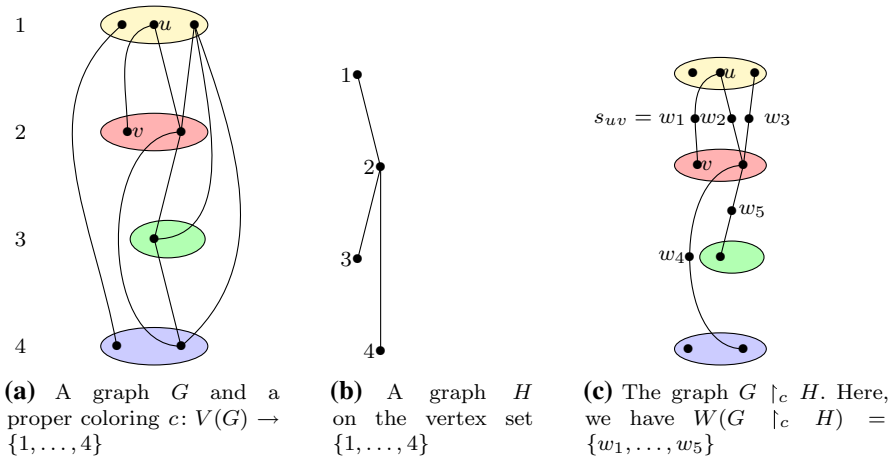
### 3.1 Reduction from Multicolored Clique to CCS-R

We now present the reduction from MULTICOLORED CLIQUE to CCS- R, which we believe to be of independent interest. We can assume, without loss of generality, that for an input  $(G, c, k)$  of MULTICOLORED CLIQUE,  $G$  is connected and  $c$  is a proper vertex-coloring, i.e., for any two distinct vertices  $u, v \in V(G)$  with  $c(u) = c(v)$  we have  $\{u, v\} \notin E(G)$ . Before we proceed let us define a graph operation.

**Definition 1** Let  $G$  be a graph and let  $c: V(G) \rightarrow \{1, \dots, k\}$  be a proper vertex coloring of  $V(G)$ . Let  $H$  be a graph on the vertex set  $\{1, \dots, k\}$ . We define the graph  $G \downarrow_c H$  as follows. We remove all edges  $\{u, v\} \in E(G)$  such that  $c(u) = i$  and  $c(v) = j$  and  $\{i, j\} \notin E(H)$ . We subdivide every remaining edge, i.e., for every remaining edge  $\{u, v\}$  we introduce a new vertex  $s_{uv}$ , remove the edge  $\{u, v\}$  and introduce instead the two edges  $\{u, s_{uv}\}$  and  $\{v, s_{uv}\}$ . We write  $W(G \downarrow_c H)$  for the set of all subdivision vertices  $s_{uv}$  (see Fig. 3).

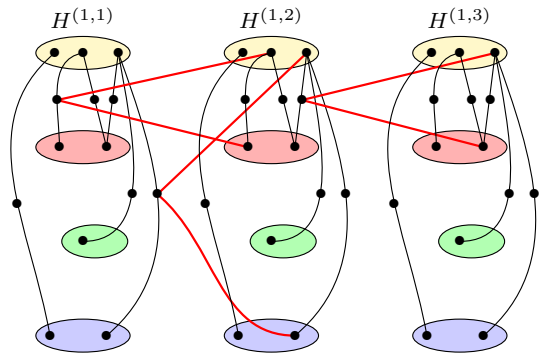
That is, to construct  $G \downarrow_c H$ , we first make a subgraph of  $G$  by deleting the edges between different color classes if there are no edges between the “corresponding” vertices in  $H$ , and then subdivide the remaining edges. Let  $(G, c, k)$  be the input instance of MULTICOLORED CLIQUE, where  $G$  is a connected graph and  $c$  is a proper  $k$ -vertex-coloring of  $G$ . We construct an instance  $(H, \hat{c}: V(H) \mapsto [k+1], Q_s, Q_t, 2k)$  of CCS- R ( $Q_s$  and  $Q_t$  are the source and target sets that we describe later). Note that the bound on the sizes of the solutions in the reconfiguration sequence is at most  $2k$ .

We first construct a routing gadget. For  $1 \leq i \leq k$ , let  $T^i$  be the star with vertex set  $\{1, \dots, k\}$  having vertex  $i$  as the center. For any  $1 \leq i \leq k$  and  $1 \leq r \leq 20k$ , we let  $H^{(i,r)}$  be a copy of the graph  $G \downarrow_c T^i$ . We let  $c_{(i,r)}$  be the partial vertex-coloring of  $H^{(i,r)}$  that is naturally inherited from  $G$ . For an illustration, consider the input instance  $(G, c, k)$  of MULTICOLORED CLIQUE depicted in Fig. 3a. Then,  $T^2$  is identical to the graph  $H$  in Fig. 3b and  $c$  represents  $H^{(2,r)} = G \downarrow_c T^2$ , for any  $1 \leq r \leq 20k$ . Now, for  $1 \leq i \leq k$  we define a graph  $H^i$  as follows. We use  $W(H^{(i,r)})$  to denote the set of subdivision vertices in  $H^{(i,r)}$ . For  $1 \leq r < 20k$  and all vertices  $u, v$  in  $V(H^{(i,r)}) \setminus W(H^{(i,r)})$ , we connect the copy of the subdivision vertex  $s_{uv}$  in  $H^{(i,r)}$  (if it exists) with the copies of the vertices  $u$  and  $v$  in  $H^{(i,r+1)}$  (see Fig. 4 for an illustration of a portion of  $H^1$  and Fig. 5 for an illustration of a portion of  $H^2$ ). We use  $W(H^i)$  to denote the set of subdivision vertices  $\bigcup_{r \in [20k]} W(H^{(i,r)})$ .

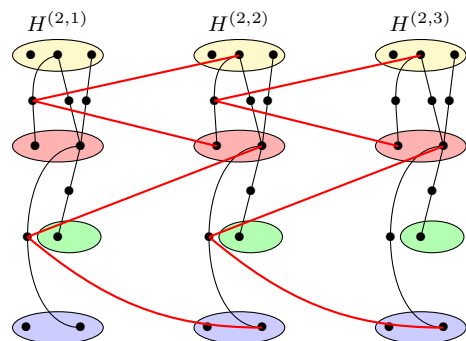


**Fig. 3** Construction of  $G \downharpoonright_c H$

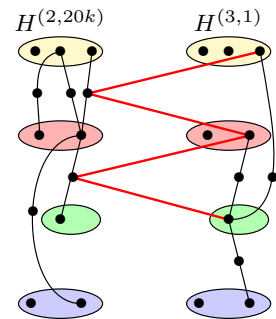
**Fig. 4** Construction of  $H^1$  from the instance  $(G, c, k)$  depicted in Fig. 3a. The red edges are some of the “crossing” edges but not all of them



**Fig. 5** Construction of  $H^2$  from the instance  $(G, c, k)$  depicted in Fig. 3a. The red edges are some of the “crossing” edges but not all of them



**Fig. 6** Illustration of the subgraph of  $R$  induced on  $V(H^{(2,20k)}) \cup V(H^{(3,1)})$  constructed from the instance  $(G, c, k)$  depicted in Fig. 3a. The red edges are some of the “crossing edges”



For each  $1 \leq i \leq k$ , we use  $c_i$  to denote a coloring on  $V(H^i)$  that is the union of  $c_{(i,1)}, c_{(i,2)}, \dots, c_{(i,20k)}$  and we color all the copies of the subdivision vertices using a new color  $k+1$ . In other words, we know that for each  $u \in V(H^i)$  we have  $u \in V(H^{(i,r)})$ , for some  $r \in \{1, \dots, 20k\}$ . Hence, if  $u \in V(H^{(i,r)}) \setminus W(H^{(i,r)})$  then we set  $c_i(u) = c_{(i,r)}(u)$ . For all  $s_{uv} \in W(H^i)$ , we set  $c_i(s_{uv}) = k+1$ .

Now, define a graph  $R$ , which is supergraph of  $H^1 \cup \dots \cup H^k$ , as follows. For  $1 \leq i < k$  and all vertices  $u$  and  $v$ , we connect the copy of the subdivision vertex  $s_{uv}$  in  $H^{(i,20k)}$  (if it exists) with the copies of the vertices  $u$  and  $v$  in  $H^{(i+1,1)}$  (see Fig. 6 for an illustration).

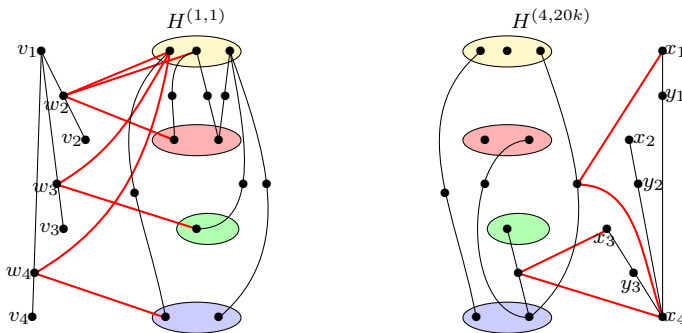
We additionally introduce two subgraphs  $H^0$  and  $H^{k+1}$ . The graph  $H^0$  is obtained by subdividing each edge of a star on vertex set  $\{v_1, \dots, v_k\}$  centered at  $v_1$ . Here we use  $w_2, \dots, w_k$  to denote the subdivision vertices. Similarly, the graph  $H^{k+1}$  is obtained by subdividing each edge of star on  $\{x_1, \dots, x_k\}$  centered at  $x_k$ . Here  $y_1, \dots, y_{k-1}$  denote the subdivision vertices. Let  $c_0$  and  $c_{k+1}$  be the colorings on  $\{v_1, \dots, v_k, w_2, \dots, w_k\}$  and  $\{x_1, \dots, x_k, y_1, \dots, y_{k-1}\}$ , respectively, defined as follows. For all  $1 \leq i \leq k$ ,  $c_0(v_i) = i$  and  $c_{k+1}(x_i) = i$ . For all  $2 \leq i \leq k$ ,  $c_0(w_i) = k+1$  and for all  $1 \leq i \leq k-1$ ,  $c_{k+1}(y_i) = k+1$ . Observe that we may interpret  $H^0$  as  $K[\{v_1, \dots, v_k\}] \upharpoonright_{c_0} T^0$  and  $H^{k+1}$  as  $K[\{x_1, \dots, x_k\}] \upharpoonright_{c_{k+1}} T^k$ , where  $T^0$  and  $T^k$  are two stars on vertex set  $\{1, \dots, k\}$ , with  $E(T^0) = \{\{1, i\} : 2 \leq i \leq k\}$  and  $E(T^k) = \{\{k, i\} : 1 \leq i \leq k-1\}$  (as previously defined).

Finally, for each  $2 \leq i \leq k$ , we connect the “subdivision vertex”  $w_i$  (adjacent to  $v_1$  and  $v_i$ ) to all vertices  $v \in V(H^{(1,1)})$  colored 1 or  $i$ , i.e., with  $c_{(1,1)}(v) \in \{1, i\}$ . For each subdivision vertex  $s_{ab} \in W(H^{(k,20k)})$ , we connect  $s_{ab}$  to  $x_k$  and  $x_i$ , where  $k = c_k(a) = c_{(k,20k)}(a)$  and  $i = c_k(b) = c_{(k,20k)}(b)$ . Recall that  $s_{ab}$  is adjacent to a vertex of color  $k$  and a vertex of color  $i$ , for some  $i < k$ . This completes the construction of  $H$  (see Fig. 7). We define  $\widehat{c}: V(H) \mapsto [k+1]$  to be the union of  $c_0, \dots, c_{k+1}$ . We define the starting configuration  $Q_s$  as the set  $\{v_1, \dots, v_k, w_2, \dots, w_k\}$  and the target configuration  $Q_t$  as the set  $\{x_1, \dots, x_k, y_1, \dots, y_{k-1}\}$ .

**Proposition 1** *The sets  $Q_s$  and  $Q_t$  are solutions of size  $2k-1$  of the CCS instance  $(H, \widehat{c}, 2k)$ .*

We now consider the instance  $(H, \widehat{c}, Q_s, Q_t, 2k)$  of the CCS-R problem. Let us give some high-level intuition about the construction before proceeding to formal proofs. Assuming that  $(G, c, k)$  is a yes-instance of MULTICOLORED CLIQUE, we





**Fig. 7** Illustration of connections between  $H^0$  and  $R$ , and  $H^{k+1}$  and  $R$  from the instance  $(G, c, k)$  depicted in Fig. 3a. The red edges are some of the “crossing edges” between  $H^0$  and  $H^1$ , and  $H^k$  and  $H^{k+1}$

show how to construct a reconfiguration sequence from  $Q_s$  to  $Q_t$  as follows. Our goal is to shift the connected vertices of  $Q_s$  through the subgraphs  $H^1, \dots, H^k$  (in that order) while maintaining connectivity and eventually reaching  $Q_t$ . To do so, we use the corresponding vertices of the clique in each  $H^{i,j}$  to maintain colorful sets and we use the vertices corresponding to subdivided edges to maintain connectivity. In the reverse direction, we shall show that in any reconfiguration sequence, each part of the constructed graph, i.e., each  $H^i$ , will allow us to guarantee that there exists a vertex colored  $i$  that is connected to vertices of every other color (while maintaining the choice of vertices along the way).

Before we analyze the reconfiguration properties of  $H$ , let us first verify that  $H$  is 4-degenerate.

**Lemma 1** *The graph  $H$  is 4-degenerate.*

**Proof** We iteratively remove minimum degree vertices and show that we can always remove a vertex of degree at most 4 in each step.

- Every subdivision vertex  $w \in W(H^i)$  for  $1 \leq i \leq k$  has degree at most 4; it has 4 neighbors in  $V(H^i) \cup V(H^{i+1})$ .
- After removal of all subdivision vertices the degree of the remaining vertices of each  $H^i$  is at most one. That is, a vertex in  $H^{(1,1)}$  may have a neighbor in  $\{w_2, \dots, w_k\}$ .
- After the removal of  $V(H^1) \cup \dots \cup V(H^k)$ , the degree of all vertices except  $v_1$  and  $x_k$  is at most 2.
- Finally we remove  $v_1$  and  $x_k$ .

This completes the proof.  $\square$

**Lemma 2** *Let  $T_1, T_2$  be two trees on vertex set  $\{1, \dots, k\}$  and let  $f_1, \dots, f_{k-1}$  be an ordering of the edges in  $T_2$ . Then, in polynomial time, we can find an ordering  $e_1, \dots, e_{k-1}$  of the edges in  $T_1$  such that the following holds. In the sequence of graphs  $T'_0, T'_1, \dots, T'_{k-1}$  on vertex set  $\{1, \dots, k\}$ , where for each  $0 \leq i \leq k-2$ ,  $T'_{i+1} = T'_i + f_i - e_i$  and  $T'_0 = T_1$ , we have that  $T'_i$  is a tree, for all  $i \in [k-1]$ , and  $T'_{k-1} = T_2$ .*

**Proof** We proceed by induction on  $\ell = |E(T_1) \setminus E(T_2)|$ . In the base case, we have  $\ell = 0$  and  $E(T_1) = E(T_2)$ . In this case  $f_1, \dots, f_{k-1}$  is also the required ordering of the edges in  $T_1$  (note that the sequence of graphs consists of only  $T_1 = T_2$  in this case).

Now consider the induction step,  $\ell > 1$ . Let  $j$  be the first index in  $\{1, \dots, k-1\}$  such that  $f_j \notin E(T_1)$ . We add  $f_j$  to  $T_1$  and this creates a cycle in  $T_1$ . Hence, there exists an edge  $e_j \in E(T_1) \setminus E(T_2)$  whose removal results in a tree. That is,  $T'_1 = T_1 + f_j - e_j$  is a tree. Notice that  $|E(T'_1) \setminus E(T_2)| = \ell - 1$ . By the induction hypothesis, there is a sequence  $g_1, \dots, g_{k-1}$  of edges in  $E(T'_1)$  such that for the sequence of graphs  $T'_1 = T''_0, T''_1, \dots, T''_{k-1}$  on vertex set  $\{1, \dots, k\}$ , we have  $T''_{i+1} = T''_i + f_i - g_i$ , each  $T''_i$  is a tree, and  $T_2 = T''_{k-1}$ ,  $0 \leq i < k$ . Since  $j$  is the first index in  $\{1, \dots, k-1\}$  such that  $f_j \notin E(T_1)$ ,  $T'_1 = T_1 + f_j - e_j$ , and  $T''_0, T''_1, \dots, T''_{k-1}$  are trees, we have that  $g_i = f_i$  for all  $i < j$ . Notice that  $f_j \in E(T'_1)$  and  $E(T_1) = (E(T'_1) \setminus \{f_j\}) \cup \{e_j\}$ .

We claim that  $e_1, \dots, e_{j-1}, e_j, e_{j+1}, \dots, e_{k-1}$ , where  $e_i = g_i$  for all  $i < j$ , is the required sequence of edges in  $T_1$ . Let  $T'_0, T'_1, \dots, T'_{k-1}$  be the sequence where, for each  $0 \leq i \leq k-2$ ,  $T'_{i+1} = T'_i + f_i - e_i$  and  $T'_0 = T_1$ . Since  $g_i = f_i = e_i$  for all  $i < j$ , we have that  $T_1 = T'_0 = T'_1 = \dots = T'_{j-1}$ . Moreover,  $T'_j = T_1 + \{f_1, \dots, f_j\} - \{e_1, \dots, e_j\} = T_1 + \{f_1, \dots, f_j\} - \{g_1, \dots, g_j\} = T''_j$  because  $E(T_1) = (E(T'_1) \setminus \{f_j\}) \cup \{e_j\}$  and  $e_i = g_i$  for all  $i < j$ . Then, the sequence  $T'_j, \dots, T'_{k-1}$  is the same as the sequence  $T''_j, \dots, T''_{k-1}$ . Therefore, the sequence  $e_1, \dots, e_{j-1}, e_j, e_{j+1}, \dots, e_{k-1}$  of edges in  $T_1$  satisfies the conditions of the lemma.  $\square$

**Lemma 3** *If there exists a  $k$ -colored clique in  $G$  then there is reconfiguration sequence of length  $\mathcal{O}(k^3)$  from  $Q_s$  to  $Q_t$  in  $(H, \widehat{c}, 2k)$ .*

**Proof** We aim to shift the connected vertices of  $Q_s$  through the subgraphs  $H^1, \dots, H^k$  (in that order) to maintain connectivity and eventually shift to  $Q_t$ . For each  $u_i \in V(G)$ ,  $1 \leq j \leq k$  and  $1 \leq r \leq 20k$ , we use  $u_i^{(j,r)}$  to denote the copy of  $u_i$  in  $H^{(j,r)}$ .

Let  $C = \{u_1, \dots, u_k\}$  be a  $k$ -colored clique in  $G$  such that  $c(u_i) = i$ , for all  $1 \leq i \leq k$ . To prove the lemma, we need to define a reconfiguration sequence starting from  $Q_s$  and ending at  $Q_t$  such that the cardinality of any solution in the sequence is at most  $2k$ . First we define  $k$  “colored” trees  $\widehat{T}_1, \dots, \widehat{T}_k$  each on  $2k-1$  vertices, and then prove that there are reconfiguration sequences from  $Q_s$  to  $V(\widehat{T}_1)$ ,  $V(\widehat{T}_1)$  to  $V(\widehat{T}_{i+1})$  for all  $1 \leq i < k$ , and  $V(\widehat{T}_k)$  to  $Q_t$ .

We start by defining  $\widehat{T}_1, \dots, \widehat{T}_k$ . For each  $1 \leq i \leq k$ ,  $C_i = \{u_1^{(i,1)}, \dots, u_k^{(i,1)}\}$  and  $S_i = \{z \in V(H^{(i,1)}) : |N_{H^{(i,1)}}(z) \cap C_i| = 2\}$ . That is, for each  $1 \leq j \leq k$  and  $j \neq i$ ,  $s_{u_i^{(i,1)}} u_j^{(i,1)} \in S_i$  (the subdivision vertex on the edge  $u_i^{(i,1)} u_j^{(i,1)}$  is in  $S_i$ ), and  $|S_i| = k-1$ . In other words,  $C_i$  contains the copies of the vertices of the clique  $C$  in  $H^{(i,1)}$  and  $S_i$  contains subdivision vertices corresponding to  $k-1$  edges in the clique incident on the  $i$ th colored vertex of the clique, such that  $H[C_i \cup S_i]$  is a tree. Now, define  $\widehat{T}_i = H[C_i \cup S_i]$ . It is easy to verify that  $\widehat{c}(C_i \cup S_i) = \{1, \dots, k+1\}$  and hence  $C_i \cup S_i = V(\widehat{T}_i)$  is a solution to the CCS instance  $(H, \widehat{c}, 2k)$ . Let  $T_s = H[Q_s]$  and  $T_t = H[Q_t]$ . Note that  $T_s$  and  $T_t$  are trees on  $2k-1$  vertices each.

**Case 1 Reconfiguration from  $Q_s$  to  $V(\widehat{T}_1)$ .** Informally, we move to  $\widehat{T}_1$  by adding a token on  $u_i^{(1,1)}$  and then removing a token from  $v_i$  for  $i$  in the order  $2, \dots, k, 1$

(for a total of  $2k$  token additions/removals). Finally, we move the tokens from  $\{w_2, \dots, w_{k-1}\}$  to  $S_1$  in  $2(k-1)$  steps. The length of the reconfiguration sequence is  $2k + 2(k-1) = 4k - 2$ .

Formally, we define  $Z_0 = Q_s$  and for each  $1 \leq j \leq k-1$ ,  $Z_{2j-1} = Z_{2j-2} \cup \{u_{j+1}^{(1,1)}\}$  and  $Z_{2j} = Z_{2j-1} \setminus \{v_{j+1}\}$ . That is, for each  $1 \leq j \leq k-1$ ,

$$\begin{aligned} Z_{2j-1} &= \{u_2^{(1,1)}, \dots, u_{j+1}^{(1,1)}\} \cup \{v_{j+1}, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\}, \text{ and} \\ Z_{2j} &= \{u_2^{(1,1)}, \dots, u_{j+1}^{(1,1)}\} \cup \{v_{j+2}, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\}. \end{aligned}$$

Next, we define  $Z_{2k-1}$  and  $Z_{2k}$  as

$$\begin{aligned} Z_{2k-1} &= \{u_2^{(1,1)}, \dots, u_k^{(1,1)}, u_1^{(1,1)}\} \cup \{v_1\} \cup \{w_1, \dots, w_{k-1}\}, \text{ and} \\ Z_{2k} &= \{u_2^{(1,1)}, \dots, u_k^{(1,1)}, u_1^{(1,1)}\} \cup \{w_1, \dots, w_{k-1}\}. \end{aligned}$$

In other words, the first five sets in the reconfiguration sequence look as follows:

$$\begin{aligned} Z_0 &= \{v_2, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\} \\ Z_1 &= \{u_2^{(1,1)}\} \cup \{v_2, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\} \\ Z_2 &= \{u_2^{(1,1)}\} \cup \{v_3, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\} \\ Z_3 &= \{u_2^{(1,1)}, u_3^{(1,1)}\} \cup \{v_3, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\} \\ Z_4 &= \{u_2^{(1,1)}, u_3^{(1,1)}\} \cup \{v_4, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\} \\ Z_5 &= \{u_2^{(1,1)}, u_3^{(1,1)}, u_4^{(1,1)}\} \cup \{v_4, \dots, v_k, v_1\} \cup \{w_1, \dots, w_{k-1}\}. \end{aligned}$$

It is easy to verify that  $Z_1, \dots, Z_{2k}$  are solutions to the CCS instance  $(H, \hat{c}, 2k)$ . Thus, we now have a reconfiguration sequence  $Z_0, Z_1, \dots, Z_{2k}$ , where  $Z_0 = Q_s$ .

Next, we explain how to get a reconfiguration sequence from  $Z_{2k}$  to  $V(\hat{T}_1)$ . Recall that  $Z_{2k} = C_1 \cup \{w_1, \dots, w_{k-1}\}$  and  $V(\hat{T}_1) = C_1 \cup S_1$ . Let  $s_j = s_{u_1^{(1,1)}u_j^{(1,1)}}$ , for all  $2 \leq j \leq k$ . Notice that  $S_1 = \{s_2, \dots, s_k\}$ . To obtain a reconfiguration sequence from  $Z_{2k}$  to  $V(\hat{T}_1)$ , we add  $s_j$  and then remove  $w_j$  for  $j$  in the order  $2, \dots, k$ . Since  $w_j$  and  $s_j$  connect the same two vertices from  $C_1$ , this reconfiguration sequence will maintain connectivity. Moreover, it is easy to verify that each set in the reconfiguration sequence uses all the colors  $\{1, \dots, k+1\}$ . Therefore, there exists a reconfiguration sequence of length  $4k - 2$  from  $Q_s$  to  $V(\hat{T}_1)$ .

**Case 2 Reconfiguration from  $V(\hat{T}_i)$  to  $V(\hat{T}_{i+1})$ .** First we define  $20k$  trees  $P_1, \dots, P_{20k}$ , each on  $2k-1$  vertices such that for all  $1 \leq r \leq 20k$ , (i)  $V(P_r) \subseteq V(H^{(i,r)})$ , and (ii)  $\hat{T}_i = P_1$ . Then we give a reconfiguration sequence from  $V(P_r)$  to  $V(P_{r+1})$  for all  $r \in [20k-1]$  and a reconfiguration sequence from  $V(P_{20k})$  to  $V(\hat{T}_{i+1})$ .

Recall that  $C = \{u_1, \dots, u_k\}$  is a  $k$ -colored clique in  $G$  such that  $c(u_i) = i$  for all  $1 \leq i \leq k$ . For each  $1 \leq r \leq 20k$ , let  $C_i^r = \{u_1^{(i,r)}, \dots, u_k^{(i,r)}\}$  and  $S_i^r = \{z \in V(H^{(i,r)}): N_{H^{(i,r)}}(z) \cap C_i^r = 2\}$ . That is, for each  $1 \leq j \leq k$  and  $j \neq i$ ,  $s_{u_i^{(i,r)}u_j^{(i,r)}} \in S_i^r$ .

(i.e., the subdivision vertex on the edge  $u_i^{(i,r)}u_j^{(i,r)}$  is in  $S_i^r$ ) and  $|S_i^r| = k - 1$ . Let  $P_r = H[C_i^r \cup S_i^r]$ . Notice that for all  $r \in [20k]$ ,  $P_r$  is a tree on  $2k - 1$  vertices. Moreover, for each  $1 \leq r \leq 20k$ ,  $V(P_r)$  is a solution to the CCS instance  $(H, \widehat{c}, 2k)$ .

**Case 2(a) Reconfiguration from  $V(P_r)$  to  $V(P_{r+1})$ .** By arguments similar to those given for Case 1, one can prove that there is a reconfiguration sequence of length  $4k - 2$  from  $V(P_r)$  to  $V(P_{r+1})$ , for all  $1 \leq r < 20k$ . For completeness we give the details here. Fix an integer  $1 \leq r < 20k$ . Let  $s_j = s_{u_i^{(i,r)}u_j^{(i,r)}}$  and  $s'_j = s_{u_i^{(i,r+1)}u_j^{(i,r+1)}}$  for all  $j \in \{1, \dots, k\} \setminus \{i\}$ . Notice that  $S_i^r = \{s_j : j \in \{1, \dots, k\} \setminus \{i\}\}$  and  $S_i^{r+1} = \{s'_j : j \in \{1, \dots, k\} \setminus \{i\}\}$ . Now we define  $Z_0 = V(P_r) = C_i^r \cup S_i^r$  and for each  $1 \leq j \leq i - 1$ ,  $Z_{2j-1} = Z_{2j-2} \cup \{u_j^{(i,r+1)}\}$  and  $Z_{2j} = Z_{2j-1} \setminus \{u_j^{(i,r)}\}$ . That is, for each  $1 \leq j \leq i - 1$ ,

$$\begin{aligned} Z_{2j-1} &= \{u_1^{(i,r+1)}, \dots, u_j^{(i,r+1)}\} \cup \{u_j^{(i,r)}, \dots, u_k^{(i,r)}\} \cup S_i^r, \text{ and} \\ Z_{2j} &= \{u_1^{(i,r+1)}, \dots, u_j^{(i,r+1)}\} \cup \{u_{j+1}^{(i,r)}, \dots, u_k^{(i,r)}\} \cup S_i^r. \end{aligned}$$

For each  $i \leq j \leq k - 1$ ,  $Z_{2j-1} = Z_{2j-2} \cup \{u_{j+1}^{(i,r+1)}\}$  and  $Z_{2j} = Z_{2j-1} \setminus \{u_{j+1}^{(i,r)}\}$ . That is, for each  $i \leq j \leq k - 1$ ,

$$\begin{aligned} Z_{2j-1} &= \{u_1^{(i,r+1)}, \dots, u_{i-1}^{(i,r+1)}, u_{i+1}^{(i,r+1)}, \dots, u_{j+1}^{(i,r+1)}\} \\ &\quad \cup \{u_{j+1}^{(i,r)}, \dots, u_k^{(i,r)}, u_i^{(i,r)}\} \cup S_i^r \end{aligned}$$

and

$$\begin{aligned} Z_{2j} &= \{u_1^{(i,r+1)}, \dots, u_{i-1}^{(i,r+1)}, u_{i+1}^{(i,r+1)}, \dots, u_{j+1}^{(i,r+1)}\} \\ &\quad \cup \{u_{j+2}^{(i,r)}, \dots, u_k^{(i,r)}, u_i^{(i,r)}\} \cup S_i^r. \end{aligned}$$

Next, we define  $Z_{2k-1}$  and  $Z_{2k}$  as

$$\begin{aligned} Z_{2k-1} &= \{u_1^{(i,r+1)}, \dots, u_k^{(i,r+1)}\} \cup \{u_i^{(i,r)}\} \cup S_i^r, \text{ and} \\ Z_{2k} &= \{u_1^{(i,r+1)}, \dots, u_k^{(i,r+1)}\} \cup S_i^r. \end{aligned}$$

Next, for each  $1 \leq j \leq k - 1$ , let  $Z_{2k+2j-1} = Z_{2k+2j-2} \cup \{s'_j\}$  and  $Z_{2k+2j} = Z_{2k+2j-1} \setminus \{s_j\}$ . It is easy to verify that  $Z_1, \dots, Z_{4k-2}$  are solutions to the CCS instance  $(H, \widehat{c}, 2k)$  and  $Z_0, \dots, Z_{4k-2}$  is a reconfiguration sequence where  $Z_0 = V(P_r)$  and  $Z_{4k-2} = V(P_{r+1})$ .

**Case 2(b) Reconfiguration from  $V(P_{20k})$  to  $V(\widehat{T}_{i+1})$ .** Next, we explain how to get a reconfiguration sequence from  $V(P_{20k})$  to  $V(\widehat{T}_{i+1})$  using Lemma 2. Recall that we have

$$\begin{aligned} C_i^{20k} &= \{u_1^{(i,20k)}, \dots, u_k^{(i,20k)}\} \text{ and} \\ S_i^{20k} &= \{z \in V(H^{(i,20k)}) : |N_{H^{(i,20k)}}(z) \cap C_i^{20k}| = 2\}. \end{aligned}$$

Let  $C_{i+1} = \{u_1^{(i+1,1)}, \dots, u_k^{(i+1,1)}\}$  and  $S_{i+1} = \{z \in V(H^{(i+1,1)}): N_{H^{(i+1,1)}}(z) \cap C_{i+1} = 2\}$ . For ease of presentation, let  $s_j = s_{u_i^{(i,20k)} u_j^{(i,20k)}}$  for all  $j \in \{1, \dots, k\} \setminus \{i\}$ . Also, let  $s'_j = s_{u_i^{(i+1,1)} u_j^{(i+1,1)}}$  for all  $j \in \{1, \dots, k\} \setminus \{i+1\}$ . That is,  $S_i^{20k} = \{s_j: j \in \{1, \dots, k\} \setminus \{i\}\}$  and  $S_{i+1} = \{s'_j: j \in \{1, \dots, k\} \setminus \{i+1\}\}$ . Notice that  $V(P_{20k}) = C_i^{20k} \cup S_i^{20k}$  and  $V(\widehat{T}_{i+1}) = C_{i+1} \cup S_{i+1}$ .

Towards proving the required reconfiguration sequence, we give a reconfiguration sequence from  $C_i^{20k} \cup S_i^{20k}$  to  $C_{i+1} \cup S_{i+1}$  and then from  $C_{i+1} \cup S_{i+1}$  to  $C_{i+1} \cup S_{i+1}$ . The reconfiguration sequence from  $C_i^{20k} \cup S_i^{20k}$  to  $C_{i+1} \cup S_{i+1}$  is similar to the one in Case 1. That is, we add  $u_j^{(i+1,1)}$  and delete  $u_j^{(i,20k)}$  for  $j$  in the order  $1, \dots, i-1, i+1, \dots, k, i$ . This gives a reconfiguration sequence from  $C_i^{20k} \cup S_i^{20k}$  to  $Z = C_{i+1} \cup S_{i+1}$  of length  $2k$ .

Next we explain how to get a reconfiguration sequence from  $Z = C_{i+1} \cup S_{i+1}$  to  $C_{i+1} \cup S_{i+1}$ . Notice that  $H[Z]$  and  $\widehat{T}_{i+1} = H[C_{i+1} \cup S_{i+1}]$  are trees. Recall that  $T^i$  is the star on  $\{1, \dots, k\}$  with vertex  $i$  being the center, and  $T^{i+1}$  is the star on  $\{1, \dots, k\}$  with vertex  $i+1$  being the center. Also,  $c_j$  is a coloring on  $H^j$  which is inherited from the coloring  $c$  of  $G$ . That is,  $c_{i+1}(u_j^{(i+1,1)}) = j$  for all  $1 \leq j \leq k$ . Then,  $H[Z] = K[C_{i+1}] \upharpoonright_{c_{i+1}} T^i$  and  $\widehat{T}_{i+1} = H[C_{i+1} \cup S_{i+1}] = K[C_{i+1}] \upharpoonright_{c_{i+1}} T^{i+1}$ .

Let  $e_1^{i+1}, \dots, e_{k-1}^{i+1}$  be an arbitrary ordering of the edges in  $T^{i+1}$ . By Lemma 2, we have a sequence  $e_1^i, \dots, e_{k-1}^i$  of edges in  $T^i$  such that for the sequence  $T_0^i, T_1^i, \dots, T_{k-1}^i$  on vertex set  $\{1, \dots, k\}$ , where for each  $0 \leq j \leq k-2$ ,  $T_{j+1}^i = T_j^i + e_j^{i+1} - e_j^i$  and  $T_0^i = T^i$ , the following holds.

- (i)  $T_j^i$  is a tree for all  $0 \leq j \leq k-1$ , and
- (ii)  $T_{k-1}^i = T^{i+1}$ .

This implies that, from the sequences  $e_1^i, \dots, e_{k-1}^i$  and  $e_1^{i+1}, \dots, e_{k-1}^{i+1}$ , we get a sequence  $f_1, \dots, f'_{k-1}$  on  $S_i^{20k}$  and a sequence  $f'_1, \dots, f'_{k-1}$  on  $S_{i+1}$  such that the for the sequence  $L_0, \dots, L_{2(k-1)}$ , where  $L_0 = C_{i+1} \cup \{f_1, \dots, f_{k-1}\}$  and for all  $1 \leq j \leq k-1$   $L_{2j-1} = (L_{2j-2} \cup \{f'_j\})$ ,  $L_{2j} = L_{2j-1} \setminus \{f_j\}$  the following holds.

- (1)  $H[L_i]$  is connected for all  $0 \leq i \leq k-1$ , and
- (2)  $L_{k-1} = S_{i+1} \cup C_{i+1}$ .

Here, conditions (1) and (2) follow from conditions (i) and (ii), respectively. Moreover,  $\widehat{c}(L_i) = [k+1]$  for all  $0 \leq i \leq 2(k-1)$  and  $L_0 = Z$ . Thus,  $L_0, \dots, L_{2(k-1)}$  is a valid reconfiguration sequence from  $Z$  to  $V(\widehat{T}_{i+1})$ . Note that the ordering on the edges implies an ordering by which we can move the subdivision vertices from  $S_i$  to  $S_{i+1}$  without violating connectivity. This implies that there is a reconfiguration sequence from  $V(P_{20k})$  to  $V(\widehat{T}_{i+1})$ , of length  $4k-2$ . Therefore, we have a reconfiguration sequence from  $V(\widehat{T}_i)$  to  $V(\widehat{T}_{i+1})$  of length  $\mathcal{O}(k^2)$ .

**Case 3 Reconfiguration from  $V(\widehat{T}_k)$  to  $V(T_t)$ .** The arguments for this case are similar to those given in Case 1, we therefore omit the details. By summing up the lengths of reconfiguration sequences, we get that if  $(G, c, k)$  is a yes-instance of MULTICOLORED CLIQUE then there is a reconfiguration sequence from  $Q_s$  to  $Q_t$ , of length  $\mathcal{O}(k^3)$ .  $\square$

**Lemma 4** *If there is a reconfiguration sequence from  $Q_s$  to  $Q_t$  then there is a  $k$ -colored clique in  $G$ .*

**Proof** For each  $1 \leq i \leq k+1$ , let  $R_i$  be the set of vertices colored by the color  $i$ . That is,  $R_i = \widehat{c}^{-1}(i)$ . First, we prove some auxiliary claims. The proofs of the following two claims follow from the construction of  $H$  and the definition of  $\widehat{c}$ .

**Claim 1** *(i)  $R_1 \cup \dots \cup R_k$  is an independent set in  $H$ , and (ii) every vertex in  $R_{k+1}$  is connected to vertices of at most two distinct colors.*

**Claim 2** *Let  $v, w \in V(H) \setminus (V(H^0) \cup V(H^{k+1}))$  be two distinct vertices such that  $\widehat{c}(v) = \widehat{c}(w)$  and  $\widehat{c}(v) \in \{1, \dots, k\}$ . If  $v$  and  $w$  have a common neighbor in  $V(H) \setminus V(H^0)$ , then  $v$  and  $w$  are copies of same vertex  $z \in V(G)$ .*

**Claim 3** *Let  $Y \subseteq V(H)$  be a vertex subset such that  $\widehat{c}(Y) = \{1, \dots, k+1\}$  and  $H[Y]$  is connected. Then,  $|Y| \geq 2k-1$ .*

**Proof** Let  $B = Y \cap \widehat{c}^{-1}(k+1) = Y \cap (R_1 \cup \dots \cup R_k)$ . Since  $\widehat{c}(Y) = \{1, \dots, k+1\}$ ,  $|B| \geq k$  and by Claim 1(i),  $B$  is an independent set in  $H$ . By Claim 1(ii), each vertex in  $R_{i+1}$  is connected to vertices of at most two distinct colors. Thus, since  $H[Y]$  is connected, the claim follows.  $\square$

Suppose  $(H, \widehat{c}, Q_s, Q_t, 2k)$  is a yes-instance of CCS- R. Then, there is a reconfiguration sequence  $D_1, \dots, D_\ell$  for  $\ell \in \mathbb{N}$ , where  $D_1 = Q_s$  and  $D_\ell = Q_t$ . Without loss of generality, we assume that the sequence  $D_1, \dots, D_\ell$  is a minimal reconfiguration sequence. Then, by Claim 3, for each  $i \in [\ell]$ ,  $2k-1 \leq |D_i| \leq 2k$ .

Moreover, since  $|D_1| = |D_\ell| = 2k-1$ , we have that for each even  $i$ ,  $D_i$  is obtained from  $D_{i-1}$  by a token addition, and for each odd  $i$ ,  $D_i$  is obtained from  $D_{i-1}$  by a token removal. This also implies that for each even  $i$ ,  $|D_i| = 2k$ , for each odd  $i$ ,  $|D_i| = 2k-1$ , and  $\ell$  is odd.

**Claim 4** *Let  $i \in [\ell]$  and  $|D_i| = 2k-1$ . Then, for all  $1 \leq j \leq k$ ,  $|D_i \cap R_j| = 1$ , and  $|D_i \cap R_{k+1}| = k-1$ . Moreover, each vertex in  $D_i \cap R_{k+1}$  will be adjacent to exactly two vertices in  $H[D_i]$  and these vertices will be of different colors from  $\{1, \dots, k\}$ .*

**Proof** By Claim 1,  $R_1 \cup \dots \cup R_k$  is independent and every vertex of  $R_{k+1}$  is adjacent to vertices of at most two different color classes. Hence, we need at least  $k-1$  vertices from  $R_{k+1}$  that make the connections between the vertices of  $D_i$  colored with  $\{1, \dots, k\}$ . The above statement along with the assumption  $|D_i| = 2k-1$  imply the claim.  $\square$

**Claim 5** *Let  $i \in \{2, \dots, \ell-1\}$ . Let  $v \in D_i$  and  $w \in D_{i+1}$  such that  $v, w \notin V(H^0) \cup V(H^{k+1})$ , at most one vertex in  $\{v, w\}$  is in  $V(H^{(1,1)})$ , and  $\widehat{c}(v) = \widehat{c}(w) \in \{1, \dots, k\}$ . Then,  $v$  and  $w$  are copies of the same vertex in  $G$ . Moreover,  $v, w \in V(H^j) \cup V(H^{j+1})$  for some  $j \in [k-1]$ .*

**Proof** Suppose  $v$  and  $w$  are not copies of the same vertex  $z \in V(G)$ . We know that  $|D_i| = 2k-1$  or  $|D_i| = 2k$ .

**Case 1**  $|D_i| = 2k - 1$ . Since  $D_i$  is a solution,  $D_i$  induces a connected subgraph in  $H$ . By Claim 4,  $|D_i \cap R_j| = 1$  for all  $j \in \{1, \dots, k\}$  and  $|D_i \cap R_{k+1}| = k - 1$ . Also, by Claim 1, (i)  $R_1 \cup \dots \cup R_k$  is an independent set in  $H$ , and (ii) every vertex in  $R_{k+1}$  is connected to vertices of at most two distinct colors. Statements (i) and (ii), and the fact that  $|D_i| = 2k - 1$  imply that (iii)  $H[D_i]$  is a tree and each vertex in  $D_i \cap R_{k+1}$  is incident to exactly two vertices in  $D_i$ . Since  $|D_{i+1}| = |D_i| + 1$ , in reconfiguration step  $i + 1$ , we add a vertex to obtain  $D_{i+1}$ . We know that  $v \in D_i$ . Since, for any color  $q \in [k]$ , there is exactly one vertex in  $D_i$  of color  $q$  (i.e.,  $|D_i \cap R_q| = 1$ ), we have that  $D_{i+1} = D_i \cup \{w\}$ . Moreover, in step  $i + 2$ , the vertex removed from  $D_{i+1}$  will be from  $\{v, w\}$  and that vertex will be  $v$  (because of the minimality assumption of the length of the reconfiguration sequence). That is,  $D_{i+2} = (D_i \cup \{w\}) \setminus \{v\}$ . Notice that  $|D_i| = |D_{i+2}| = 2k - 1$ . Let  $b$  a vertex in  $D_{i+2}$  which is adjacent to  $w$  in  $H[D_{i+2}]$ . Since  $R_{k+1} \cap D_i = R_{k+1} \cap D_{i+2}$  and  $|D_i| = |D_{i+2}| = 2k - 1$ , by Claim 1, the neighbors of  $b$  in  $H[D_i]$  and  $H[D_{i+2}]$  are of the same color. This implies that  $b$  is adjacent to  $v$  in  $H[D_i]$ . Thus, we proved that  $\{b, w\}, \{b, v\} \in E(H)$ . If  $b \in V(H^0)$ , then  $v, w \in V(H^{(1,1)})$  which is a contradiction to the assumption. Otherwise, by Claim 2, we conclude that  $v$  and  $w$  are copies of same vertex.

**Case 2**  $|D_i| = 2k$ . In this case  $D_{i+1}$  is obtained by removing a vertex from  $D_i$ . Moreover,  $i \geq 3$ , because we have two vertices in  $D_i$  from  $V(H) \setminus D_1$ . Since  $|D_{i+1}| = 2k - 1$ , because of Claim 4,  $D_{i+1}$  is obtained by removing the vertex  $v$  from  $D_i$ . That is,  $D_{i+1} = D_i \setminus \{v\}$  and  $v, w \in D_i$ . Then, again by Claim 4, there is  $v' \in \{v, w\}$  such that  $D_{i-1} \uplus \{v'\} = D_i$ . Let  $w' = \{v, w\} \setminus \{v'\}$ . Since  $i \geq 3$ , we now apply Case 1 with respect to  $w' \in D_{i-1}$  and  $v' \in D_i$  to complete the proof.  $\square$

**Claim 6** For any index  $j \in \{1, \dots, k\}$  and color  $q \in \{1, \dots, k\}$ , there exist an odd  $i \in \{3, \dots, \ell\}$  and  $r \in \{5k, \dots, 15k\}$  such that  $D_i$  contains a vertex of color  $q$  from  $V(H^{j,r})$ .

**Proof** Without loss of generality, assume that  $k \geq 2$ . Moreover, for any odd  $i \in [\ell - 2]$ , there is a vertex common in  $D_i$  and  $D_{i+2}$  (since  $k \geq 2$ ). This implies that  $H[D_1 \cup D_3 \dots D_\ell]$  is a connected subgraph of  $H$ . Notice that for any  $j \in \{1, \dots, k\}$  and  $r \in [20k]$ ,  $V(H^{(j,r)})$  is a  $(v_1, x_1)$ -separator in  $H$ . Therefore, since  $H[D_1 \cup D_3 \dots D_\ell]$  is connected and  $v_1, x_1 \in D_1 \cup D_\ell$ , (i) for any  $j \in [k]$  and  $r \in [20k]$ , there is an odd  $i \in [\ell]$  such that  $D_i$  contains a vertex from  $V(H^{(j,r)})$ . Now fix an index  $j \in \{1, \dots, k\}$  and a color  $q \in \{1, \dots, k\}$ . By statement (i), there is an odd  $i \in \{1, \dots, \ell\}$  such that  $D_i$  contains a vertex from  $V(H^{(j,10k)})$ . Since  $H[D_i]$  is connected,  $|D_i| = 2k - 1$ ,  $D_i \cap V(H^{(j,10k)}) \neq \emptyset$ , and any vertex in  $V(H) \setminus \bigcup_{r=5k}^{15k} V(H^{(j,r)})$  is at distance more than  $5k$  (by the construction of  $H$ ), we have that all the vertices in  $D_i$  belong to  $\bigcup_{r=5k}^{15k} V(H^{(j,r)})$ . Moreover, by Claim 4,  $D_i$  contains a vertex colored  $q$  and that will also be present in  $\bigcup_{r=5k}^{15k} V(H^{(j,r)})$ . This completes the proof of the claim.  $\square$

**Claim 7** For any color  $q \in \{1, \dots, k\}$ , the vertices of color  $q$  from  $\bigcup_{i=2}^k V(H^i)$  used in the reconfiguration sequence  $D_1, \dots, D_\ell$  are copies of the same vertex  $z \in V(G)$ . Moreover, exactly one vertex from  $V(H^j)$  of color  $q$  is used in the reconfiguration for all  $2 \leq j \leq k$ .

**Proof** Fix a color  $q \in \{1, \dots, k\}$ . By Claim 6, there are vertices of color  $q$  from  $V(H^j)$  for all  $j$  is used in the reconfiguration sequence. By Claim 5, all these vertices are copies of the same vertex  $z \in V(G)$ .  $\square$

Now we define a  $k$ -size vertex subset  $C \subseteq V(G)$  and prove that  $C$  is a clique in  $G$ . We let  $C = \{a_i \in V(G) : 1 \leq i \leq k, c(a_i) = i, \text{ and the copy of } a_i \text{ in } V(H^2) \text{ is used in } D_1, \dots, D_\ell\}$ . Because of Claim 7, we have that  $|C| = k$  and  $C$  contains a vertex of each color in  $c$ .  $C = \{a_1, \dots, a_k\} \subseteq V(G)$  and for each  $q \in [k]$ ,  $c(a_q) = q$ . We now prove that  $C$  is indeed a clique in  $G$ . Towards that, we need to prove that for each  $1 \leq q < j \leq k$ ,  $\{a_q, a_j\} \in E(G)$ .

**Claim 8** Let  $1 \leq q < j \leq k$ . Then,  $\{a_q, a_j\} \in E(G)$ .

**Proof** By Claim 6, we know that there exist an odd  $i \in [\ell]$  and  $r \in \{5k, \dots, 15k\}$  such that  $D_i$  contains a vertex of color  $q$  in  $V(H^{(j,r)})$ . Thus, by Claim 7, a copy of  $a_j$  and a copy of  $a_q$  are present in  $D_i$ . Let  $u_j$  and  $u_q$  be the vertices in  $D_i$  colored with  $j$  and  $q$ , respectively. By Claim 7,  $u_j$  is a copy of  $a_j$  and  $u_q$  is a copy of  $a_q$ . Any vertex  $b$  in  $V(H^j)$  colored  $k+1$  is adjacent to vertices of exactly two colors, out of which one color is  $j$ . Moreover, by the construction of  $H$ , (a) if  $b$  is adjacent to  $x$  and  $y$  in  $V(H^j)$ , and  $x$  and  $y$  are copies of  $x'$  and  $y'$  in  $G$ , respectively, then  $\{x', y'\} \in E(G)$ . We know that  $H[D_i]$  is connected,  $|R_s \cap D_i| = 1$  for all  $1 \leq s \leq k$ ,  $D_i \setminus R_{k+1}$  is an independent set in  $H$ , and each vertex in  $D_i$  colored with  $k+1$  is adjacent to exactly two vertices in  $D_i \setminus R_{k+1}$  with one of them being  $u_j$  (see Claims 1 and 4). This implies that there is common neighbor  $b$  for  $u_q$  and  $u_j$  and hence  $\{a_q, a_j\} \in E(G)$ , by statement (a) above. This completes the proof of the claim.  $\square$

This completes the proof of the lemma.  $\square$

**Theorem 1** CCS- R parameterized by  $k + \ell$  is  $W[1]$ -hard on 4-degenerate graphs.

### 3.2 Reduction from CCS-R to CDS-R

We give a polynomial-time parameter-preserving reduction from CCS- R to CDS- R that is fairly straightforward. Let  $(G, c, Q_s, Q_t, k)$  be an instance of CCS- R. Let  $c: V(G) \mapsto \{1, \dots, k'\}$ , where  $k' \leq k$ . We construct a graph  $H$  as follows. For each  $1 \leq i \leq k'$ , we add a vertex  $d_i$  and connect  $d_i$  to all the vertices in  $c^{-1}(i)$ . Next, for each  $1 \leq i \leq k'$ , we add a pendant vertex  $x_i$  (i.e.,  $\{d_i, x_i\}$  is an edge). Let  $D = \{d_1, \dots, d_{k'}\}$ . We output  $(H, Q_s \cup D, Q_t \cup D, k + k')$  as the new CDS- R instance.

**Lemma 5** If  $G$  is a  $d$ -degenerate graph then  $H$  is a  $(d + 1)$ -degenerate graph.

**Proof** For each vertex  $v \in V(G)$ ,  $d_H(v) = d_G(v) + 1$ . Thus, after removing  $V(G)$  and  $\{x_i : 1 \leq i \leq k'\}$ , the remaining graph is edgeless.  $\square$

It is easy to verify that for any reconfiguration sequence  $Q_s = R_1, \dots, R_\ell = Q_t$  of the instance  $(G, c, Q_s, Q_t, k)$  of CCS-R,  $Q_s \cup D = R_1 \cup D, \dots, R_\ell \cup D = Q_t \cup D$  is a reconfiguration sequence of the instance  $(H, Q_s \cup D, Q_t \cup D, k + k')$  of CDS-R. Now we prove the reverse direction.



**Lemma 6** *If  $(H, Q_s \cup D, Q_t \cup D, k + k')$  is a yes-instance then  $(G, c, Q_s, Q_t, k)$  is a yes-instance.*

**Proof** Notice that the set  $D$  is contained in any connected dominating set of  $H$ . Moreover for any minimal connected dominating set  $Z$  in  $H$ ,  $Z \cap \{x_i : 1 \leq i \leq k'\} = \emptyset$ ,  $H[Z \setminus D]$  is connected, and  $Z \setminus D$  contains a vertex from  $c^{-1}(i)$  for all  $1 \leq i \leq k'$  (recall that  $G$  is a subgraph of  $H$ ). Therefore, by deleting  $D$  from each set in a reconfiguration sequence of  $(H, Q_s \cup D, Q_t \cup D, k + k')$ , we get a valid reconfiguration sequence of  $(G, c, Q_s, Q_t, k)$ . This completes the proof.  $\square$

Thus, by Theorem 1, we have the following theorem.

**Theorem 2** *CDS- R parameterized by  $k + \ell$  is  $W[1]$ -hard on 5-degenerate graphs.*

## 4 Fixed-Parameter Tractability on Planar Graphs

This section is devoted to proving that CDS- R under TAR parameterized by  $k$  is fixed-parameter tractable on planar graphs. In fact, we show that the problem admits a polynomial kernel. Recall that a kernel for a parameterized problem  $\mathcal{Q}$  is a polynomial-time algorithm that computes for each instance  $(I, k)$  of  $\mathcal{Q}$  an equivalent instance  $(I', k')$  with  $|I'| + k' \leq f(k)$  for some computable function  $f$ . The kernel is polynomial if the function  $f$  is polynomial. We prove that for every instance  $(G, S, T, k)$  of CDS- R, with  $G$  planar, we can compute in polynomial time an instance  $(G', S, T, k)$  where  $|V(G')| \leq h(k)$  for some polynomial  $h$ ,  $G'$  planar, and where there exists a reconfiguration sequence under TAR from  $S$  to  $T$  in  $G$  (using at most  $k$  tokens) if and only if such a sequence exists in  $G'$ .

Our approach is as follows. We first compute a small *domination core* for  $G$ , that is, a set of vertices that captures exactly the domination properties of  $G$  for dominating sets of sizes not larger than  $k$ . While the classification of interactions with the domination core would suffice to solve DOMINATING SET RECONFIGURATION, additional difficulties arise for the connected variant. In a second step we use planarity to identify large subgraphs that have very simple interactions with the domination core and prove that they can be replaced by constant size gadgets such that the reconfiguration properties of  $G$  are preserved.

### 4.1 Domination Cores

**Definition 2** Let  $G$  be a graph and let  $k \geq 1$  be an integer. A  $k$ -*domination core* is a subset  $C \subseteq V(G)$  of vertices such that every set  $X \subseteq V(G)$  of size at most  $k$  that dominates  $C$  also dominates  $G$ .

It is not difficult to see that DOMINATING SET is fixed-parameter tractable on all graphs that admit a  $k$ -domination core of size at most  $f(k)$  that is computable in time  $g(k) \cdot n^c$ , for any computable functions  $f, g$  and constant  $c$ . This approach was first used (implicitly) in [5] to solve DISTANCE- $r$  DOMINATING SET on nowhere dense graph classes. In case  $k$  is the size of a minimum (distance- $r$ ) dominating set, one can establish

the existence of a linear size  $k$ -domination core on classes of bounded expansion [7] (including the class of planar graphs) and a polynomial size (in fact an almost linear size)  $k$ -domination core on nowhere dense graph classes [9,24]. If  $k$  is not minimum, there exist classes of bounded expansion such that a  $k$ -domination core must have at least quadratic size [8]. The most general graph classes that admit  $k$ -domination cores are given in [10]. Moreover, DOMINATING SET RECONFIGURATION and DISTANCE- $r$  DOMINATING SET RECONFIGURATION are fixed-parameter tractable on all graphs that admit small (distance- $r$ )  $k$ -domination cores [25,33].

**Lemma 7** *There exists a polynomial  $h$  such that for all  $k \geq 1$ , every planar graph  $G$  admits a polynomial-time computable  $k$ -domination core of size at most  $h(k)$ .*

The lemma is implied by Theorem 1.6 of [24] by the fact that planar graphs are nowhere dense. We want to stress again that the polynomial size of the  $k$ -domination core results from the fact that  $k$  may not be the size of a minimum dominating set, if  $k$  is minimum we can find a linear size core. Explicit bounds on the degree of the polynomial can be derived from [30,32], but we refrain from doing so to not disturb the flow of ideas.

The following lemma is immediate from the definition of a  $k$ -domination core.

**Lemma 8** *If  $C$  is a  $k$ -domination core and  $D$  is a dominating set of size at most  $k$  that contains a vertex set  $W \subset D$  such that  $N[D] \cap C = N[D \setminus W] \cap C = C$ , then  $D \setminus W$  is also a dominating set.*

**Definition 3** Let  $G$  be a graph and let  $A \subseteq V(G)$ . The *projection* of a vertex  $v \in V(G) \setminus A$  into  $A$  is the set  $N(v) \cap A$ . If two vertices  $u, v$  have the same projection into  $A$  we write  $u \sim_A v$ .

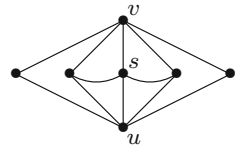
Obviously, the relation  $\sim_A$  is an equivalence relation. The following lemma is folklore, one possible reference is [11].

**Lemma 9** *Let  $G$  be a planar graph and let  $A \subseteq V(G)$ . Then there exists a constant  $c$  such that there are at most  $c \cdot |A|$  different projections to  $A$ , that is, the equivalence relation  $\sim_A$  has at most  $c \cdot |A|$  equivalence classes.*

## 4.2 Reduction Rules

Let  $G$  be an embedded planar graph. We say that a vertex  $v$  *touches* a face  $f$  if  $v$  is drawn inside  $f$  or belongs to the boundary of  $f$  or is adjacent to a vertex on the boundary of  $f$ . We fix two connected dominating sets  $S$  and  $T$  of size at most  $k$ . We will present a sequence of lemmas, each of which implies a polynomial-time computable reduction rule that allows us to transform  $G$  to a planar graph  $G'$  that inherits its embedding from  $G$ , with  $S, T \subseteq V(G')$  and that has the same reconfiguration properties with respect to  $S$  and  $T$  as  $G$ . To not overload notation, after stating a lemma with a reduction rule, we assume that the reduction rule is applied until this is no longer possible and call the resulting graph again  $G$ . We also assume that whenever one or more of our reduction rules are applicable, then they are applied in the order presented. We will guarantee that  $S$  and  $T$  will always be connected dominating sets of size at most  $k$ , hence, after each

**Fig. 8** A vertex  $s \in S_i$  can dominate at most 3 vertices of  $N(u) \cap N(v)$



application of a reduction rule, we can recompute a  $k$ -domination core in polynomial time. This yields only polynomial overhead and allows us to assume that we always have marked a  $k$ -domination core  $C$  of size at most  $h(k)$  as described in Lemma 7. This allows us to state the lemmas as if  $G$  and  $C$  are fixed. Without loss of generality we assume that  $C$  contains  $S$  and  $T$ .

**Definition 4** A set  $W$  of vertices or edges is *irrelevant* if there is a reconfiguration sequence from  $S$  to  $T$  in  $G$  if and only if there is a reconfiguration sequence from  $S$  to  $T$  in  $G - W$ .

**Definition 5** Let  $u, v \in V(G)$  be distinct vertices. We call the set  $D(u, v) := (N(u) \cap N(v)) \cup \{u, v\}$  the *diamond* induced by  $u$  and  $v$ . We call  $|N(u) \cap N(v)|$  the *thickness* of  $D(u, v)$ .

**Lemma 10** If  $G$  contains a diamond  $D(u, v)$  of thickness greater than  $3k$ , then at least one of  $u$  or  $v$  must be occupied by a token in every step of every reconfiguration sequence from  $S$  to  $T$ .

**Proof** Assume  $S = S_1, \dots, S_t = T$  is a reconfiguration sequence from  $S$  to  $T$  and  $u, v \notin S_i$  for some  $1 \leq i \leq t$ .

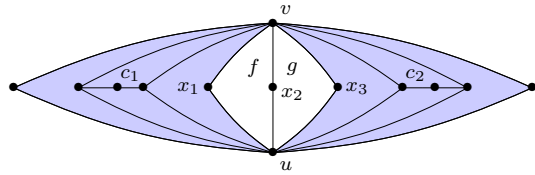
Then every  $s \in S_i$  can dominate at most 3 vertices of  $N(u) \cap N(v)$ : otherwise  $u, v, s$  together with 3 vertices of  $N(u) \cap N(v)$  different from  $u, v$  and  $s$  would form a complete bipartite graph  $K_{3,3}$  (Fig. 8).  $\square$

**Lemma 11** If  $G$  contains a diamond  $D(u, v)$  of thickness greater than  $3k$ , then we can remove all internal edges in  $D(u, v)$ , i.e., edges with both endpoints in  $N(u) \cap N(v)$ .

**Proof** Assume  $S = S_1, \dots, S_t = T$  is a reconfiguration sequence from  $S$  to  $T$ . According to Lemma 10, for each  $1 \leq i \leq t$ ,  $S_i \cap \{u, v\} \neq \emptyset$ . Hence all vertices of  $N(u) \cap N(v)$  are always dominated by at least one of  $u$  or  $v$ , say by  $u$ . Moreover, having tokens on more than one vertex of  $N(u) \cap N(v)$  will never create connectivity via internal edges that is not already there via edges incident on  $u$ . In other words, for any connected dominating set  $S$  of  $G$ , if an edge  $yz$  is used for connectivity, where  $y, z \in N(u) \cap N(v)$ , then the edge can be replaced by the path  $yuz$  or the path  $yvz$  (depending on which of  $u$  or  $v$  is in  $S$ ).  $\square$

As described earlier, we now apply the reduction rule of Lemma 11 until this is no longer possible, and name the resulting graph again  $G$ . As we did not make use of the properties of a  $k$ -domination core in the lemma, it is sufficient to recompute a  $k$ -domination core  $C$  after applying the reduction rule exhaustively. In the following it may be necessary to recompute it after each application of a reduction rule. We will not mention these steps explicitly in the following.

**Fig. 9** Every vertex of  $C \setminus \{u, v\}$  can touch at most 3 consecutive faces of  $H$ . In the figure we assume the vertices  $c_1$  and  $c_2$  are in  $C \setminus \{u, v\}$ . The faces that are touched by  $c_1$  or  $c_2$  are colored in blue. The uncolored faces  $f$  and  $g$  are not touched by vertices of  $C \setminus \{u, v\}$



**Lemma 12** *If  $G$  contains a diamond  $D(u, v)$  of thickness greater than  $4|C| + 3k + 1$  then  $G$  contains an irrelevant vertex.*

**Proof** Let  $H$  be the subgraph of  $G$  induced by  $D(u, v)$ . We enumerate the vertices of  $N(u) \cap N(v)$  consecutively as  $x_1, \dots, x_t$  for some  $t > 4|C| + 3k + 1$ . We let  $X = \{x_1, \dots, x_t\}$ . Note that since we have  $t$  vertex-disjoint paths between  $u$  and  $v$  in  $H$ , these paths define the boundaries of  $t$  faces in the plane embedding of  $H$  (after applying the reduction rule of Lemma 11,  $H$  has all the edges  $\{u, x\}$  and  $\{v, x\}$  for  $x \in N(u) \cap N(v)$  and no other edges). Each vertex in  $C \setminus \{u, v\}$  can be adjacent in  $H$  to at most two vertices in  $X$ , hence each vertex in  $C \setminus \{u, v\}$  can touch at most 3 consecutive faces of  $H$  (Fig. 9).

This leaves  $|C| + 3k + 1$  faces of  $H$  that are not touched by a vertex of  $C \setminus \{u, v\}$ . By the pigeonhole principle we can find 2 adjacent faces  $f$  and  $g$  of  $H$  that are not touched by a vertex of  $C \setminus \{u, v\}$ .

We let  $x_1$  and  $x_2$  denote the two vertices on the boundary of face  $f$  different from  $u$  and  $v$  and we let  $x_2$  and  $x_3$  denote the two vertices on the boundary of face  $g$  different from  $u$  and  $v$ . Recall that, due to Lemma 11, we know that there are no edges between those three vertices. Let  $W$  denote the set of all vertices contained in the face of the cycle  $u, x_1, v, x_3, u$ . In particular,  $W$  contains  $x_2$ . We claim that the vertices of  $W$  can be removed from  $G$  without changing the reconfiguration properties of  $G$ , i.e.,  $W$  is a set of irrelevant vertices. Let  $G' = G - W$ . First observe that, since  $S, T \subseteq C$ ,  $W \cap (S \cup T) = \emptyset$ , hence  $S, T \subseteq V(G')$ . We show that reconfiguration from  $S$  to  $T$  is possible in  $G$  if and only if reconfiguration from  $S$  to  $T$  is possible in  $G'$ .

Assume  $S = S_1, \dots, S_t = T$  is a reconfiguration sequence from  $S$  to  $T$  in  $G$ . Let  $S'_1, \dots, S'_t$ , where for  $1 \leq i \leq t$ ,  $S'_i := S_i$  if  $S_i$  does not contain a vertex of  $W$  and  $S'_i := (S_i \setminus W) \cup \{x_1\}$  otherwise. Note that this modification leaves  $S$  and  $T$  unchanged, hence,  $S'_1 = S_1$  and  $S'_t = S_t$ . We claim that  $S'_1, \dots, S'_t$  is a reconfiguration sequence from  $S$  to  $T$  in  $G'$ .

**Claim 9** *For  $1 \leq i \leq t$ ,  $S'_i$  is a dominating set of  $G$ , and hence also of  $G'$ .*

**Proof** No vertex of  $W$  is adjacent to a vertex of  $C \setminus \{u, v\}$  and  $W \cap C = \emptyset$  by construction. Hence, the only vertices of  $C$  that are possibly adjacent to a vertex of  $W$  are the vertices  $u$  and  $v$ . Whenever  $S_i$  contains a vertex of  $W$ , we have  $x_1 \in S'_i$ , which dominates both  $u$  and  $v$ . Hence,  $S'_i$  dominates at least the vertices of  $C$  that  $S_i$  dominates. We use Lemma 8 to conclude that  $S'_i$  is a dominating set of  $G$ .  $\square$

**Claim 10** *For  $1 \leq i \leq t$ ,  $S'_i$  is connected.*

**Proof** Let  $s_1, s_2 \in S_i \setminus W$  and let  $P$  be a shortest path between  $s_1$  and  $s_2$  in  $G[S_i]$ . We have to show that there exists a path between  $s_1$  and  $s_2$  in  $G[S'_i]$ . If  $P$  does not use a vertex of  $W$ , then there is nothing to show. Hence, assume  $P$  uses a vertex of  $W$ . By definition of  $W$ , both  $s_1$  and  $s_2$  lie outside or on the boundary of the face  $h$  of the cycle  $u, x_1, v, x_3$  that contains  $x_2$ . Hence,  $P$  must enter and leave the face  $h$ , and as  $P$  is a shortest path, it must enter and leave via opposite vertices, i.e., via  $u$  and  $v$ , or via  $x_1$  and  $x_3$  (as all other pairs are linked by an edge and we could find a shorter path). If  $P$  contains  $u$  and  $v$ , then we can replace the vertices of  $W$  on  $P$  by  $x_1$  and we are done.

Hence, assume  $P$  uses  $x_1$  and  $x_3$ . As  $D(u, v)$  is a diamond of thickness greater than  $4|C| + 3k + 1 > 3k$ , according to Lemma 10 at least one of the vertices  $u$  and  $v$ , say  $u$ , is contained in  $S_i$ , and by definition also in  $S'_i$ . Then we can replace the vertices of  $W$  on  $P$  by  $u$  and we are again done.  $\square$

Finally, the following claim is immediate from the definition of each  $S'_i$ . Combining Claims 9, 10, and 11, we conclude that  $S'_1, \dots, S'_t$  is a reconfiguration sequence from  $S$  to  $T$  in  $G'$ .

**Claim 11**  $S'_{i+1}$  is obtained from  $S'_i$  by the addition or removal of a single token for all  $1 \leq i < t$ .

To prove the opposite direction, assume  $S = S'_1, \dots, S'_t = T$  is a reconfiguration sequence from  $S$  to  $T$  in  $G'$ . We claim that this is also a reconfiguration sequence from  $S$  to  $T$  in  $G$ . All we have to show is that  $S'_i$  is a dominating set of  $G$  for all  $1 \leq i \leq t$ . This follows immediately from the fact that  $S'_i$  is a dominating set of  $G'$ , and hence, as  $W$  is not adjacent to  $C \setminus \{u, v\}$  and  $W \cap C = \emptyset$ , also a dominating set of  $C$  in  $G$ . Then according to Lemma 8,  $S'_i$  also dominates  $G$ . We conclude that there is a reconfiguration sequence from  $S$  to  $T$  in  $G$  if and only if there is a reconfiguration sequence from  $S$  to  $T$  in  $G' = G - W$ .  $\square$

We may in the following assume that  $G$  does not contain diamonds of thickness greater than  $4|C| + 3k + 1$ .

**Corollary 1** If a vertex  $v \in V(G)$  has degree greater than  $(4|C| + 3k + 1) \cdot k$ , then the token on  $v$  is never lifted throughout a reconfiguration sequence.

**Proof** Assume  $S = S_1, \dots, S_t = T$  is a reconfiguration sequence from  $S$  to  $T$  in  $G$  and assume there is  $S_i$  with  $v \notin S_i$ . The dominating set  $S_i$  has at most  $k$  vertices and must dominate  $N(v)$ . Hence, there must be one vertex  $u \in S_i$  that dominates at least a  $1/k$  fraction of  $N(v)$ , which is larger than  $4|C| + 3k + 1$ . Then there is a diamond  $D(u, v)$  of thickness greater than  $4|C| + 3k + 1$ , which does not exist after application of the reduction rule of Lemma 12.  $\square$

According to Corollary 1, the only vertices that can have high degree after applying the reduction rules are vertices that are never lifted throughout a reconfiguration sequence. This gives rise to another reduction rule that is similar to the rule of Lemma 11.

**Lemma 13** Assume  $v$  is a vertex of degree greater than  $(4|C| + 3k + 1) \cdot k$ . Then we may remove all edges with both endpoints in  $N(v)$ .

**Proof** Let  $G'$  be the graph obtained from  $G$  by removing all edges with both endpoints in  $N(v)$ . We claim that reconfiguration between  $S$  and  $T$  is possible in  $G$  if and only if it is possible in  $G'$ . The fact that  $S$  and  $T$  are in fact connected dominating sets in  $G'$  is implied by the argument below.

Assume  $S = S_1, \dots, S_t = T$  is a reconfiguration sequence from  $S$  to  $T$  in  $G$ . We claim that the same sequence is a reconfiguration sequence in  $G'$ . According to Corollary 1,  $v \in S_i$  for all  $1 \leq i \leq t$ . This implies that  $S_i$  is connected in  $G'$  for all  $1 \leq i \leq t$ , as all  $x, y \in S_i$  that are no longer connected by an edge in  $G'$  but were connected in  $G$  are connected via a path of length 2 using the vertex  $v$ . It is also easy to see that  $S_i$  is a dominating set in  $G'$ , as all vertices that are no longer dominated by  $s \in S_i$  in  $G$  are still dominated by  $v$ . Observe that this in particular implies that  $S$  and  $T$  are connected dominating sets in  $G'$ . Vice versa, if  $S = S_1, \dots, S_t = T$  is a reconfiguration sequence from  $S$  to  $T$  in  $G'$ , this is trivially also a reconfiguration sequence in  $G$ .  $\square$

The following reduction rule is obvious.

**Lemma 14** *If a vertex  $v$  has more than  $k + 1$  pendant neighbors, i.e., neighbors of degree exactly one, then it suffices to retain exactly  $k + 1$  of them in the graph.*

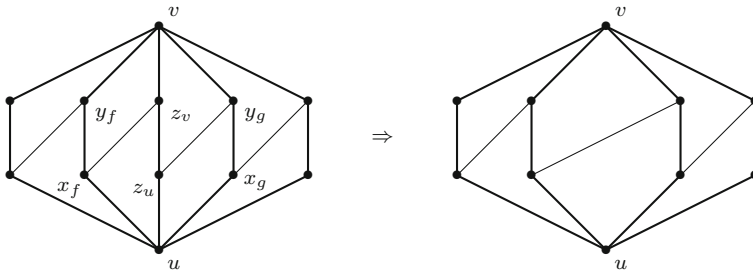
**Lemma 15** *There are at most  $c|C| \cdot (4|C| + 3k + 1)$  vertices of  $V(G) \setminus C$  that have 2 neighbors in  $C$ , where  $c$  is the constant of Lemma 9.*

**Proof** According to Lemma 9 there are at most  $c|C|$  different projections to  $C$ . Each projection class that has at least 3 representatives has size at most 2, as otherwise we would find a  $K_{3,3}$  as a subgraph, contradicting the planarity of  $G$ . Consider a class with a projection of size 2 into  $C$ . Denote these two vertices of  $C$  by  $u$  and  $v$ . If this class has more than  $4|C| + 3k + 1$  representatives, then  $D(u, v)$  is a diamond of thickness greater than  $4|C| + 3k + 1$ , which cannot exist after exhaustive application of the reduction rule of Lemma 12.  $\square$

We now come to the description of our final reduction rule. Let  $D$  denote the set of vertices containing both  $C$  and all vertices of  $V(G) \setminus C$  having at least two neighbors in  $C$ . In other words,  $V(G) \setminus D$  contains all those vertices in  $V(G) \setminus C$  that have exactly one neighbor in  $C$ . According to Lemma 15 at most  $c|C| \cdot (4|C| + 3k + 1)$  vertices have two neighbors in  $C$ , hence  $|D| \leq c|C| \cdot (4|C| + 3k + 1) + |C| =: p$ .

**Lemma 16** *Assume there are two vertices  $u$  and  $v$  with degree greater than  $4p + (4|C| + 3k + 1) \cdot k + 1$ . Let  $\mathcal{P}$  be a maximum set of vertex-disjoint paths of length at least 2 that run between  $u$  and  $v$  using only vertices in  $V(G) \setminus D$ . If  $|\mathcal{P}| > 4p + (4|C| + 3k + 1) \cdot k + 1$ , then there is  $G'$  such that the instances  $(G, S, T, k)$  and  $(G', S, T, k)$  are equivalent,  $G'$  is planar, and  $|V(G')| < |V(G)|$ .*

**Proof** We first show that we can essentially establish the situation depicted in Fig. 10. We may assume that the paths of  $\mathcal{P}$  are induced paths, otherwise we may replace them by induced paths. Let  $H$  be the graph induced on  $u, v$  and the vertices of  $\mathcal{P}$ . In the figure, the paths of  $\mathcal{P}$  are depicted by thick edges, while the diagonal edges do not belong to the paths. This situation is similar to the situation in the proof of Lemma 12. Just as in the proof of Lemma 12, we find two adjacent faces  $f, g$  of  $H$  that do not touch a vertex of  $D \setminus \{u, v\}$ .



**Fig. 10** An exemplary situation handled by Lemma 16

**Claim 12** *The paths bounding  $f$  and  $g$  have length 3, i.e., they have exactly two inner vertices.*

**Proof** First observe that  $P \in \mathcal{P}$  cannot have length exactly 2, as then  $P$  contains a vertex adjacent to both  $u$  and  $v$ . However, the vertices with this property lie in  $D$ , and hence by construction not on  $P$ .

Assume there is  $P \in \mathcal{P}$  of length greater than 3. Let  $M(u)$  denote the neighbors of  $u$  that are in  $V(G) \setminus D$  and are only adjacent to  $u$  and to no other vertex of  $C$ . Similarly, let  $M(v)$  denote the neighbors of  $v$  that are in  $V(G) \setminus D$  and are only adjacent to  $v$  and to no other vertex of  $C$ . By construction, the faces  $f$  and  $g$  do not contain vertices of  $D \setminus \{u, v\}$ . Furthermore,  $P$  contains exactly one vertex of  $M(u)$  and exactly one vertex of  $M(v)$ . It cannot contain two vertices of one of these sets, as otherwise  $P$  is not an induced path. Hence, assume that  $P$  contains another vertex  $x$  that is not in  $M(u) \cup M(v)$ . Then  $x$  must be dominated by a vertex different from  $u$  and from  $v$ . However, by construction, the faces  $f$  and  $g$  do not touch a vertex of  $D \setminus \{u, v\} \supseteq (S \cup T) \setminus \{u, v\}$ , a contradiction.  $\square$

Denote by  $x_f, y_f$  the two vertices that lie on the boundary of  $f$  and not on the boundary of  $g$  and by  $x_g, y_g$  the two vertices that lie on the boundary of  $g$  and not on the boundary of  $f$ . Assume that  $x_f, x_g \in M(u)$  and  $y_f, y_g \in M(v)$ . Denote by  $z_u, z_v$  the vertices shared by  $f$  and  $g$  different from  $u$  and  $v$  that are adjacent to  $u$  and  $v$ , respectively. Denote by  $W$  the set of all vertices that lie inside the face  $h$  of the cycle  $u, x_f, y_f, v, y_g, x_g, u$  that contains the vertices  $z_u$  and  $z_v$ . Hence  $W$  contains at least the vertices  $z_u$  and  $z_v$ . By Corollary 1, we know that  $u, v \in S_i$ , for all  $1 \leq i \leq t$  (both  $u$  and  $v$  can never be lifted). Consequently, by Lemma 13, we know that there are no edges with both endpoints in  $N(v)$  nor edges with both endpoints in  $N(u)$ . Combining the previous fact with the fact that all vertices of  $W$  are adjacent to either  $u$  or  $v$  (but not both) and to no other vertex of  $C \supseteq S \cup T$ , we conclude that  $W$  consists of exactly the two vertices  $z_u$  and  $z_v$  and that there are no edges between  $z_u$  and  $x_g, x_f$  and no edges between  $z_v$  and  $y_g, y_f$ . Note that we can safely assume that none of the degree-one neighbors of  $u$  or  $v$  are inside  $W$ . We claim that the vertices  $z_u$  and  $z_v$  are irrelevant and can be removed after possibly introducing an additional edge to the graph. Recall that  $S$  and  $T$  do not contain the vertices  $z_u$  and  $z_v$ . We define  $G'$  as follows.



- If  $\{u, v\} \notin E(G)$  and  $(\{x_f, z_v\} \in E(G) \text{ or } \{y_f, z_u\} \in E(G))$  and  $(\{x_g, z_v\} \in E(G) \text{ or } \{y_g, z_u\} \in E(G))$  then  $G'$  is obtained from  $G$  by deleting  $z_u$  and  $z_v$  and introducing the edge  $\{x_f, y_g\}$ .
- Otherwise,  $G'$  is obtained from  $G$  by simply deleting  $z_u$  and  $z_v$ .

We claim that  $(G, S, T, k)$  and  $(G', S, T, k)$  are equivalent instances of CDS- R. Assume first that there exists a reconfiguration sequence  $S = S_1, \dots, S_t = T$  in  $G$ . We distinguish two cases. First assume that  $\{u, v\} \in E(G)$ . Hence,  $G'$  is obtained from  $G$  by simply deleting  $z_u$  and  $z_v$ . Let  $S'_1, \dots, S'_t$ , where for  $1 \leq i \leq t$ ,  $S'_i = S_i \setminus \{z_u, z_v\}$ . We claim that  $S'_1, \dots, S'_t$  is a reconfiguration sequence from  $S$  to  $T$  in  $G'$ .

**Claim 13** For  $1 \leq i \leq t$ ,  $S'_i$  is a dominating set of  $G$ , and hence also of  $G'$ .

**Proof** The vertices  $z_u$  and  $z_v$  are not adjacent to a vertex of  $C \setminus \{u, v\}$  and  $\{z_u, z_v\} \cap C = \emptyset$ . Hence, the only vertices of  $C$  that are possibly adjacent to  $z_u$  or  $z_v$  are the vertices  $u$  and  $v$ . According to Lemma 1,  $u, v \in S_i$ , and moreover  $u, v \in S'_i$ , for all  $1 \leq i \leq t$ . Hence,  $S'_i$  dominates at least the vertices of  $C$  that  $S_i$  dominates. We use Lemma 8 to conclude that  $S'_i$  is a dominating set of  $G$ .  $\square$

**Claim 14** For  $1 \leq i \leq t$ ,  $S'_i$  is connected.

**Proof** Let  $s_1, s_2 \in S'_i \setminus \{z_u, z_v\}$  and let  $P$  be a shortest path between  $s_1$  and  $s_2$  in  $G[S_i]$ . We have to show that there exists a path between  $s_1$  and  $s_2$  in  $G[S'_i]$ . If  $P$  does not use  $z_u$  nor  $z_v$  then there is nothing to prove. Hence, assume  $P$  uses  $z_u$  or  $z_v$  (or both). By definition of  $W$ , both  $s_1$  and  $s_2$  lie outside the face  $h$  of the cycle  $u, x_f, y_f, v, y_g, x_g, u$  that contains  $z_u, z_v$ . Hence,  $P$  must enter and leave the face  $h$ , say it enters at  $u$  and leaves at  $y_f$ . All other possibilities are handled analogously. Then we can avoid the vertices  $z_u$  and  $z_v$  by walking to  $v$  first, then  $u$  (or  $x_f$ ), and then to  $y_f$ .  $\square$

The next claim follows from the definition of  $S'_i$  and the fact that we can remove any duplicate consecutive sets in a reconfiguration sequence.

**Claim 15**  $S'_{i+1}$  is obtained from  $S'_i$  by the addition or removal of a single token for all  $1 \leq i < t$ .

This finishes the proof in case  $\{u, v\} \in E(G)$ . Hence, we assume now that  $\{u, v\} \notin E(G)$  and  $(\{x_f, z_v\} \in E(G) \text{ or } \{y_f, z_u\} \in E(G))$  and  $(\{x_g, z_v\} \in E(G) \text{ or } \{y_g, z_u\} \in E(G))$ . That is,  $G'$  is obtained from  $G$  by deleting  $z_u$  and  $z_v$  and introducing the edge  $\{x_f, y_g\}$ . We now obtain  $S'_i$  from  $S_i$ , for  $1 \leq i \leq t$ , by replacing

- $z_u$  by  $x_f$  and  $z_v$  by  $y_g$  if  $S_i \cap \{z_u, z_v\} = \{z_u, z_v\}$ ,
- $z_u$  by  $x_f$  if  $S_i \cap \{z_u, z_v\} = \{z_u\}$ , and
- $z_v$  by  $y_g$  if  $S_i \cap \{z_u, z_v\} = \{z_v\}$ .

We claim that  $S'_1, \dots, S'_t$  is a reconfiguration sequence from  $S$  to  $T$  in  $G'$ . We need no new arguments to prove that each  $S'_i$  is a dominating set of  $G$  and hence of  $G'$  and that each  $S'_{i+1}$  is obtained from  $S'_i$  by adding or removing one token. It remains to show that each  $S'_i$  is connected in  $G'$ .

**Claim 16** For  $1 \leq i \leq t$ ,  $S'_i$  is connected in  $G'$ .



**Proof** According to Lemma 1,  $u, v \in S_i$ , and also  $u, v \in S'_i$ , for all  $1 \leq i \leq t$ . If  $S_i \setminus \{z_u, z_v\}$  is connected,  $S'_i$  is also connected, hence assume  $S_i \setminus \{z_u, z_v\}$  is not connected. As  $X = \{u, x_f, z_u, x_g\}$  is connected via  $u$  and  $Y = \{v, y_f, z_v, y_g\}$  is connected via  $v$ , it suffices to show that our vertex exchange creates a connection in  $G'$  between any vertex of  $X$  and any vertex of  $Y$ . If  $S_i \cap \{z_u, z_v\} = \{z_u, z_v\}$  this is clear, as we shift the tokens to  $x_f$  and  $y_g$  and in  $G'$  we have introduced the edge  $\{x_f, y_g\}$ . If  $S_i \cap \{z_u, z_v\} = \{z_u\}$ , then  $\{z_u, y_g\} \in E(G)$  and  $y_g \in S_i$ , or  $\{z_u, y_f\} \in E(G)$  and  $y_f \in S_i$ . We move the token  $z_u$  to  $x_f$ . In the first case we have connectivity via the new edge  $\{x_f, y_g\} \in E(G')$ , and in the second case we have connectivity via the edge  $\{x_f, y_f\} \in E(G)$ . The case  $S_i \cap \{z_u, z_v\} = \{z_v\}$  is symmetric.  $\square$

This finishes the proof that if  $(G, S, T, k)$  is a positive instance then  $(G', S, T, k)$  is a positive instance. Now assume that there exists a reconfiguration sequence  $S = S'_1, \dots, S'_t = T$  in  $G'$ . In case we do not introduce the new edge to obtain  $G'$  from  $G$ , we do not need new arguments to see that  $S'_1, \dots, S'_t$  is a reconfiguration sequence also in  $G$ . Moreover, if  $G''[S'_i]$  is connected for all  $i$ , where  $G''$  is obtained from  $G'$  by removing the edge  $\{x_f, y_g\}$ , then again there is nothing to prove as  $G'$  is a subgraph of  $G$  and therefore  $S = S'_1, \dots, S'_t = T$  is a reconfiguration sequence in  $G$ . Hence, assume that there exists at least one contiguous subsequence  $\sigma$  starting at index  $s$  and ending at index  $f$  (with possibly  $s = f$ ) such that  $G''[S'_s], G''[S'_{s+1}], \dots, G''[S'_f]$  are not connected. In other words, there exists a subsequence of length one or more that uses the edge  $\{x_f, y_g\}$  for connectivity. Moreover, we assume, without loss of generality (the other case is symmetric), that  $S'_s$  is obtained from  $S'_{s-1}$  by adding a token on vertex  $y_g$ , i.e.,  $S'_s = S'_{s-1} \cup \{y_g\}$ , and  $S'_{f+1}$  is obtained from  $S'_f$  by removing the token on vertex  $x_f$ , i.e.,  $S'_{f+1} = S'_f \setminus \{x_f\}$ . We also assume that  $E(G)$  contains the edges  $\{x_f, z_v\}$  and  $\{z_u, y_g\}$  (the remaining cases are handled identically). It remains to show how to modify  $\sigma$  so that it does not use the edge  $\{x_f, y_g\}$  for connectivity and remains a valid reconfiguration sequence in  $G$ . By applying the same arguments for any such subsequence we obtain the required reconfiguration sequence in  $G$ . We modify  $\sigma$  as follows. We let  $S''_i = (S'_i \setminus \{y_g\}) \cup \{z_v\}$ , for  $s \leq i \leq f$ . Then we replace  $S'_{f+1}$  by four new sets  $A_1, A_2, A_3$ , and  $A_4$ , where  $A_1 = S'_f \setminus \{x_f\}$ ,  $A_2 = A_1 \cup \{z_u\}$ ,  $A_3 = A_2 \setminus \{z_v\}$ ,  $A_3 = A_3 \cup \{y_g\}$ , and  $A_4 = A_3 \setminus \{z_u\}$ . Using the fact that the vertices  $x_f, y_f, x_g, y_g$  are not adjacent to vertices of  $D \setminus \{u, v\}$ , it is easy to see that this yields a valid reconfiguration sequence, as both domination and connectivity are preserved. This completes the proof of the lemma.  $\square$

We are ready to state the final result.

**Theorem 3** CDS- R under TAR parameterized by  $k$  admits a polynomial kernel on planar graphs.

**Proof** Our kernelization algorithm starts by computing (in polynomial time) a  $k$ -domination core  $C$  of size at most  $h(k)$  as described in Lemma 7. Without loss of generality we assume that  $C$  contains  $S$  and  $T$ . After each application of a reduction rule, we recompute the core, giving a polynomial blow-up of the running time. We are left to prove that each reduction rule can be implemented in polynomial time and that we end up with a polynomial number of vertices. It is clear that the reduction rules of Lemmas 12, 13 and 14 can easily be implemented in polynomial time. The reduction

rule of Lemma 16 is slightly more involved, however, we can use a standard maximum-flow algorithm to compute in polynomial time a maximum set of vertex-disjoint paths in a subgraph of  $G$ . It remains to bound the size of  $G$ . Recall that we call  $D$  the set of all vertices  $C$  and of all vertices of  $V(G) \setminus C$  that have at least 2 neighbors in  $C$ . It follows from Lemma 15 that  $D$  has size at most  $c|C| \cdot (4|C| + 3k + 1) + |C| =: p$ , where  $c$  is the constant of Lemma 9. We are left to bound the number of vertices in  $V(G) \setminus C$  having exactly one neighbor in  $C$  (recall that each vertex in  $V(G) \setminus C$  has at least one neighbor in  $S \cup T \subseteq C$ ).

Let  $p' = (4p + (4|C| + 3k + 1) \cdot k + 1) \cdot (4|C| + 3k + 1) \cdot k + k + 1$ , which is still a polynomial in  $k$ . Towards a contradiction, assume that there exists an equivalence class  $Q$  in  $\sim_C$  with a projection of size one containing more than  $p'$  vertices. Let  $u \in C$  denote the projection of the aforementioned class. Due to Lemma 14, we know that at most  $k + 1$  of the vertices in  $Q$  are pendant, i.e., adjacent to only  $u$  in  $G$ . Since we cannot apply the reduction rule of Lemma 13 any more, we know that there are no edges with both endpoints in  $Q$ . Hence, all but  $k + 1$  vertices of  $Q$  must be adjacent to at least one other vertex in  $V(G) \setminus C$ . Let  $R = N_G(Q) \setminus \{u\}$  denote this set of neighbors. No vertex in  $R$  can be adjacent to more than  $4|C| + 3k + 1$  vertices of  $Q$ , as we cannot apply the reduction rule of Lemma 12. The vertices of  $R$  must be dominated by  $S$ , and cannot be dominated by  $u$ , as otherwise two neighbors of  $u$  would be connected. Hence, there is  $v \in S$  different from  $u$  that dominates at least a  $1/k$  fraction of  $R$ . This implies the existence of at least  $4p + (4|C| + 3k + 1) \cdot k + 1$  vertex-disjoint paths of length at least 2 that run between  $u$  and  $v$ . But in this case, the reduction rule of Lemma 16 is applicable. Therefore, we conclude that  $Q$  cannot exist, obtaining a bound on the size of all equivalence classes of  $\sim_C$ , as needed.  $\square$

## 5 Conclusion

We have shown that the CDS- R problem parameterized by  $k$  is fixed-parameter tractable for planar graphs and (trivially) for graphs of bounded degree. Moreover, a simple observation shows that the problem is fixed-parameter tractable parameterized by  $k + \ell$  on every nowhere dense graph class and the same holds for every class of bounded cliquewidth. On the negative side, our reduction shows that CDS- R parameterized by  $k + \ell$  is  $W[1]$ -hard on 5-degenerate graphs. It remains open to determine where exactly the boundary between tractable and intractable lies for CDS- R parameterized by  $k$ . We conjecture that CDS- R is fixed-parameter tractable parameterized by  $k$  on every nowhere dense graph class. However, resolving this conjecture remains open for future work (see Fig. 2). Towards proving that conjecture, we believe that the classes of graphs of bounded pathwidth or treewidth are the obvious next classes to study.

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