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Nonlinear speed-ups in ultracold quantum gases

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Abstract – Quantum mechanics is an inherently linear theory. However, collective effects in many body quantum systems can give rise to effectively nonlinear dynamics. In the present work, we analyze whether and to what extent such nonlinear effects can be exploited to enhance the rate of quantum evolution. To this end, we compute a suitable version of the quantum speed limit for numerical and analytical examples. We find that the quantum speed limit grows with the strength of the nonlinearity, yet it does not trivially scale with the "degree" of nonlinearity. This is numerically demonstrated for the parametric harmonic oscillator obeying Gross-Pitaevskii and Kolomeisky dynamics, and analytically for expanding boxes under Gross-Pitaevskii dynamics.

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In public perception, the prophesied *quantum advantage* of novel technologies has become almost synonymous with anticipated *quantum speed-ups*. This impression is driven by quantum computing, which indeed can solve certain problems faster than any classical computer could [1]. At least superficially this expectation seems to be at variance with the so-called *quantum speed limits* (QSLs), which are fundamental bounds on the maximal rate with which a quantum system can evolve [2,3]. In fact, diverging QSLs can be interpreted as a herald of classicality [4,5], as they are deeply rooted in more rigorous formulations of Heisenberg's uncertainty relation for energy and time [6].

The apparent contradiction quickly dissolves, once one realizes that in the lingo of computer science a "speed-up" simply refers to a smaller number of required single gate operations, whereas in quantum physics the QSL refers to the maximal rate with which such a gate operation can be applied [7]. Thus it also becomes rather obvious why so much research activity has been dedicated to the study of QSL in virtually all areas of quantum physics, including, *e.g.*, quantum communication [8–13], quantum computation [14,15], quantum control [16–18], many body physics [19,20], and quantum metrology [21,22]. See somewhat recent reviews on the topic [23,24].

The original QSLs were formulated for standard quantum mechanics [25], whose dynamics is described by the Schrödinger equation. However, over the last decade it has become obvious that there is a variety of "quantum resources" that can be employed to speed-up quantum dynamics. For instance, it has been established that judiciously designed open system dynamics permit environment driven speed-ups [26–31]. Similarly, the QSL is larger for non-Hermitian quantum dynamics [32,33], for which the quantum states can find literal shortcuts through Hilbert space.

In the present analysis, we will be focusing on another type of quantum dynamics, which has not received much attention in the study of the QSL. Processing information requires some form of interaction between signals. In photonic systems, such interactions can be enabled by nonlinear optical processes [34]. Nonlinear optics has become almost ubiquitous in fundamental science and technological applications [35] and can be realized, *e.g.*, with single atoms in cavities [36], in atomic ensembles [37], and through atom-atom interactions [38]. It is expected that nonlinear systems will be able to outperform linear systems in optical quantum information processing [34]. The reason is that nonlinear quantum optics can provide both linear operations such as storage and manipulation of quantum states, as well as nonlinear operations such as the generation of quanta and quantum logic between photonic quantum bits. Recent experimental developments in nonlinear quantum optics hold the promise to enable the development of universal quantum computers supporting conditional quantum logic operations, which are now within technological reach [34]. The natural question arises whether nonlinear interactions can be exploited as a quantum resource to enhance the QSL. Some evidence can be found in the literature [39,40], yet a comprehensive analysis within the framework of QSL appears to be lacking.

We show that there is indeed a speed-up of the rate of quantum evolution in nonlinear systems. To this end, we begin with a numerical investigation of a cold bosonic

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quantum gas in a harmonic trap, which is described by the Gross-Pitaevskii equation [41,42]. We find that the resulting QSL grows with the strength of the scattering length, meaning that the stronger the nonlinearity the faster the quantum state can evolve. While such numerical evidence is interesting, it is not as insightful as analytical treatments. Therefore, we then solve the Gross-Pitaevskii equation for a 1-dimensional box, whose volume expands at constant rate. For these dynamics, we obtain an analytical expression for the QSL, and we find our numerical evidence rigorously confirmed. Thus, as main result, we show that nonlinear quantum effects can quantifiably enhance the rate of quantum evolution. We conclude the analysis by also exploring the QSL for the parametric harmonic oscillator evolving under Kolomeisky dynamics. Remarkably, we find that the QSL does not trivially grow with the order of the nonlinearity.

Numerical case study: parametric harmonic oscillator. – In the following, we consider quantum dynamics that are described by the Gross-Pitaevskii equation,

$$i\hbar\dot{\Psi}_t(x) = \left[-\frac{\hbar^2}{2m}\,\partial_x^2 + U\left(x,t\right) + \kappa\,\left|\Psi_t(x)\right|^2\right]\,\Psi_t(x), \quad (1)$$

where, as usual, we denote a derivative with respect to time by a dot. Further, $U(x,t)$ is a time-dependent, external potential and κ measures the "strength" of the nonlinearity.

Equation (1) was first discovered in the description of the propagation of light in nonlinear optical fibers and planar waveguides [35], and shortly after in the mean-field description of Bose-Einstein condensates [41,42]. In cold quantum gases the nonlinearity arises from an effective treatment of the interaction of the bosonic particles, whereas in nonlinear optics $\kappa\,\left|\Psi_t(x)\right|^2$ describes the polarization-dependent amplitude in the paraxial field equations. Since its first description, the nonlinear Schrödinger equation (1) has appeared also in many other areas of physics, such as in hydrodynamics [43], in plasma physics [44], and the propagation of Davydov's alpha-helix solitons, which are responsible for energy transport along molecular chains [45]. Thus, eq. (1) has become one of the best studied equations in quantum physics, from a conceptual as well as a numerical point of view [46–49].

The Gross-Pitaevskii equation (1) has also found some attention in quantum control [50–53], yet a genuinely nonlinear QSL appears to be lacking in the literature. In a first rigorous derivation of a QSL for simple, undriven Schrödinger dynamics, Mandelstam and Tamm [2] showed that the minimal time a quantum system needs to evolve between orthogonal states is bounded from below by the variance of the energy, ΔE, and $\tau_{\mathrm{QSL}} = \pi\hbar/2\Delta E$. Since, however, the variance of an operator is not necessarily a good quantifier for undriven dynamics [54], Margolus and Levitin [55] revisited the problem and derived a second bound on the quantum evolution time in terms of the average energy $E = \langle H\rangle - E_g$ over the ground state with energy E_g, $\tau_{\mathrm{QSL}} = \pi\hbar/2E$.

More recently, it has been recognized that the QSLs are bounds on the rate with which quantum states become distinguishable [5]. Therefore, one typically considers different geometric measures of distinguishability [56,57] in the derivation of QSL. In our work, we have shown that these different treatments become equivalent when considering the metric properties of the quantum dynamics [4,7], and that for purity preserving dynamics the QSL is simply determined by the rate with which a quantum state evolves [13]. Therefore, we define

$$v_{\mathrm{QSL}} \equiv \int \mathrm{d}x|\dot{\Psi}_t(x)|^2. \quad (2)$$

For linear dynamics, $\kappa = 0$, the quantum state evolution is simply given by the standard Schrödinger equation, $i\hbar|\dot{\Psi}\rangle = H(t)|\Psi\rangle$, where $H(t)$ is the time-dependent Hamiltonian. Thus we have

$$v_{\mathrm{QSL}}^0 \equiv v_{\mathrm{QSL}}(\kappa = 0) = \langle H^2(t)\rangle/\hbar^2. \quad (3)$$

Equation (2) is nothing else but a generalization of the Mandelstam-Tamm bound [2,13,58], which remains applicable for any purity preserving quantum dynamics, such as the Gross-Pitaevskii equation (1).

To build intuition, and to gain some insight into possible nonlinear speed-ups, we now consider the parametric harmonic oscillator, with potential

$$U(x,t) = \frac{1}{2}m\omega_t^2 x^2. \quad (4)$$

To keep things as simple as possible, we further choose ω_t^2 to be a linear function of time,

$$\omega_t^2 = \omega_0^2 - \left(\omega_0^2 - \omega_1^2\right)t/\tau, \quad (5)$$

for which the linear Schrödinger dynamics is analytically solvable [59]. For this protocol, we solved eq. (1) numerically for several values of κ. For each of the realizations, we choose the ground state of the linear problem as initial state, namely

$$\Psi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\,\exp\left(-\frac{m\omega_0}{2\hbar}x^2\right). \quad (6)$$

The resulting solution is depicted in fig. 1. We observe that for moderate values of κ the solution of the nonlinear dynamics remains close to the linear solution, whereas for large values the dynamics is strongly distinguishable.

In fig. 2 we plot the corresponding QSL (2). As expected, we observe that the stronger the nonlinearity, *i.e.*, the larger the value of κ, the faster the evolution of the quantum state. While this numerical result is interesting and confirms our intuition, more analytical insight is desirable. In particular, it would be useful to be able to "design" optimal nonlinearities to meet, *e.g.*, speed requirements.

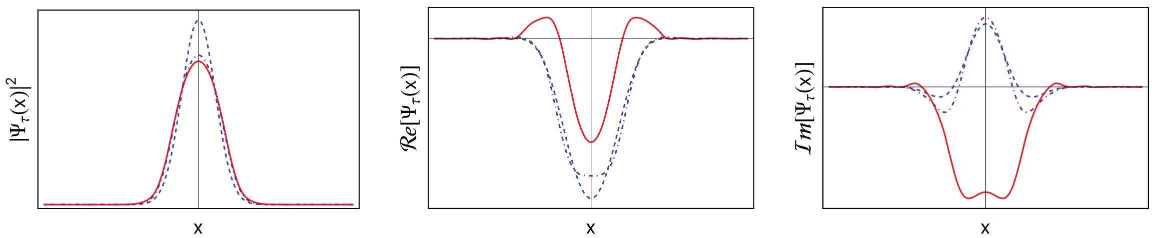

Fig. 1: From left to right: absolute value, real part, and imaginary part of the solution $\Psi_\tau(x)$ of eq. (1), for the parametric harmonic oscillator (4) and $\kappa = 0$ (blue, dashed line), $\kappa = 5$ (purple, dot-dashed line), and $\kappa = 10$ (red, solid line). Other parameters are $\hbar\omega_0 = 5$, $\hbar\omega_1 = 1$, $m = 1$, and $\tau = 2$.

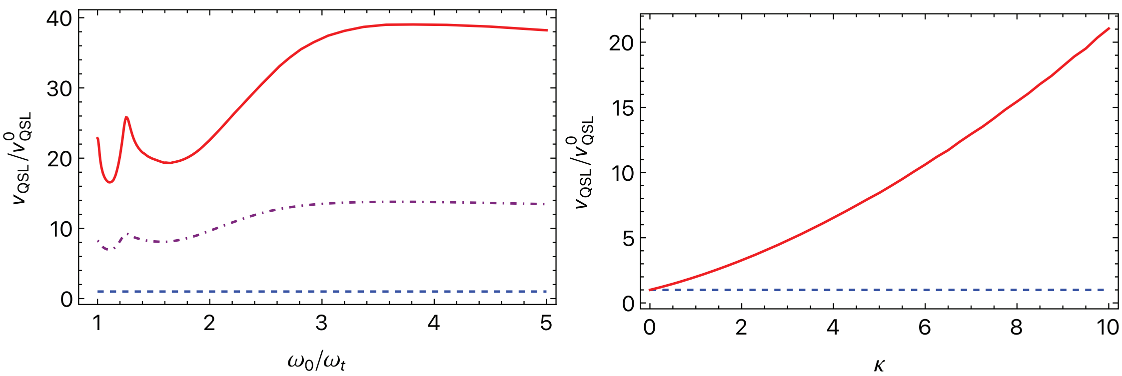

Fig. 2: Left: QSL (2) for the parametric harmonic oscillator (4) for $\kappa = 0$ (blue, dashed line), $\kappa = 5$ (purple, dot-dashed line), and $\kappa = 10$ (red, solid line). Right: QSL (2) for the parametric harmonic oscillator (4) evaluated at $t = \tau/2$ as a function of κ and with the same color coding as on the left. Other parameters are $\hbar\omega_0 = 5$, $\hbar\omega_1 = 1$, $m = 1$, and $\tau = 2$.

Scale-invariant Gross-Pitaevskii dynamics. – Therefore, we now continue with a special class of time-dependent problems. Consider the scale-invariant, nonlinear Schrödinger equation

$$i\hbar\,\dot{\Psi}_t(x) = \left[-\frac{\hbar^2}{2m}\,\partial_x^2 + \frac{1}{\lambda_t^2}\,U\left(\frac{x}{\lambda_t}\right) + \frac{\kappa}{\lambda_t}\,|\Psi_t(x)|^2\right]\,\Psi_t(x), \tag{7}$$

where λ_t is an external control parameter, such as the volume of an optical trap confining the BEC. Then, if λ_t is given for some arbitrary a, b, and c by $\lambda_t = \sqrt{at^2 + 2bt + c}$, a solution of eq. (7) can be written as [51,52,60]

$$\Psi_t(x) = \exp\left(-\frac{i}{\hbar}\mu_0\,\tau(t)\right)\exp\left(\frac{im\dot{\lambda}_t}{2\hbar\lambda_t}\,x^2\right)\,\Phi(x,\lambda_t), \tag{8}$$

where $\Phi(x,\lambda_t) = \Phi(x/\lambda_t)/\lambda_t^2$ is a scale-invariant solution of the instantaneous Gross-Pitaevskii equation (7), μ_0 is the initial chemical potential, and $\tau(t) = \int_0^t \mathrm{d}s/\lambda_s^2$. Note that for linear dynamics $\kappa = 0$, the chemical potential reduces to the average energy of the initial state, ϵ_0.

A few lines of simply algebra reveal that the QSL (2) can be written as

$$v_{\rm QSL} = \frac{\mu_t^2}{\hbar^2} + \frac{m\mu_t}{\hbar^2}\left(\frac{\dot{\lambda}_t^2}{\lambda_t^2} - \frac{\ddot{\lambda}_t}{\lambda_t}\right)\langle x^2\rangle$$
$$+\frac{m^2}{4\hbar^2}\left(\frac{\dot{\lambda}_t^4}{\lambda_t^4} - \frac{2\dot{\lambda}_t^2\ddot{\lambda}_t}{\lambda_t^3} + \frac{\ddot{\lambda}_t^2}{\lambda_t^2}\right)\langle x^4\rangle + \frac{\dot{\lambda}_t^2}{\hbar^2}\,\langle p^2\rangle, \tag{9}$$

where we introduced the instantaneous chemical potential $\mu_t = \mu_0/\lambda_t^2$, and $\langle x^n\rangle = \int \mathrm{d}x\, x^n |\Phi\left(x,\lambda_t\right)|^2$.

Example: infinite square well. As a case study, we continue with an analytically solvable example. Arguably, the simplest, nontrivial problem in quantum mechanics is a particle trapped in an infinite square well. The potential simply is

$$U(x) = \begin{cases} 0, & \forall x \in [0,\lambda], \\ \infty, & \text{otherwise}. \end{cases} \tag{10}$$

Obviously the one-dimensional box with time-varying length is scale invariant. This has been exploited, for instance, in standard thermodynamic problems such as analyzing the validity of the quantum Jarzynski equality [61,62], and to construct explicit expressions for the counterdiabatic field [50,52,63].

As a point of reference, and to build intuition we begin with the linear case $\kappa = 0$. For completeness, recall that the eigensystem is given by

$$\psi_n(x,\lambda) = \sqrt{\frac{2}{\lambda}}\,\sin\left(\frac{n\pi}{\lambda}\,x\right) \quad\text{and}\quad E_n(\lambda) = \frac{\hbar^2\pi^2}{2m\lambda^2}\,n^2. \tag{11}$$

Moreover, we choose a parameterization in which the box is expanded at constant rate v, and hence $\lambda_t = \lambda_0 + vt$. Accordingly, the QSL (9) simplifies to read

$$v_{\rm QSL} = \frac{\epsilon_t^2}{\hbar^2} + \frac{m\epsilon_t}{\hbar^2}\,\frac{v^2}{\lambda_t^2}\,\langle x^2\rangle + \frac{m^2}{4\hbar^2}\,\frac{v^4}{\lambda_t^4}\,\langle x^4\rangle + \frac{v^2}{\hbar^2}\,\langle p^2\rangle, \tag{12}$$

where $\epsilon_t = \epsilon_0/\lambda_t^2$ and ϵ_0 is the initial energy. We immediately observe that $v_{\rm QSL}$ is a sum of positive terms, and hence we can continue comparing the nonlinear case to the linear result term by term.

The corresponding nonlinear problem can also be solved analytically [64], and the stationary states read

$$\psi_n(x,\lambda) = \sqrt{\frac{\nu K(\nu)}{\lambda\,(K(\nu)-E(\nu))}}\,\mathrm{sn}\left(\frac{2nK(\nu)}{\lambda}x\middle|\nu\right). \tag{13}$$

Here $\mathrm{sn}(x|\nu)$ is the Jacobian elliptic function [65], and $K(\nu)$ and $E(\nu)$ are elliptic integrals. The parameter ν is directly related to the strength of the nonlinearity κ. It it is implicitly determined by

$$K(\nu)\,\left(K(\nu)-E(\nu)\right) = \frac{m\lambda\,\kappa}{4n^2\hbar^2}, \tag{14}$$

which follows from substituting eq. (13) into eq. (7) for $U(x)$ as given in eq. (10). Accordingly, the generalized eigenvalues can be written as

$$\mu_n(\lambda) = \frac{\hbar^2}{2m\lambda^2}\,n^2\,(2K(\nu))^2\,(1+\nu). \tag{15}$$

With eqs. (13)–(15) the QSL (12) can be computed exactly. However, the resulting expression is rather involved, and hence does not permit to gain much insight.

Therefore, we now continue to compute corrections to the quantum speed limit up to linear order in the strength of the nonlinearity κ. The linear eigensystem (11) is recovered for $\nu = 0$. Thus, we begin by expanding the left side of eq. (14) up to linear order in ν (see footnote [1]), from which we obtain

$$\nu \simeq \frac{2m\lambda\,\kappa}{n^2\hbar^2\pi^2}. \tag{16}$$

Similarly, the generalized eigenvalues [64] can be written in terms of the linear energy eigenvalue $E_n(\lambda)$ (11) as

$$\mu_n(\lambda) \simeq E_n(\lambda) + \frac{3}{2}\frac{\kappa}{\lambda}. \tag{17}$$

Hence, for the first term in eq. (12) we have already obtained the leading order correction, *i.e.*, "the nonlinear speed-up".

The other terms in eq. (12) require a little more work. We continue with the second moment of the momentum, which can in fact be computed in closed form. To further simplify the analysis, we now assume that the system was initially prepared in its ground state, $n = 1$. In this case we have

$$\langle p^2\rangle = \frac{(2K(\nu))^2}{3\lambda^2(K(\nu)-E(\nu))}\,\left[K(\nu)(\nu-1)+E(\nu)(\nu+1)\right]. \tag{18}$$

Using eq. (16) for $n = 1$ we obtain in leading order of κ

$$\langle p^2\rangle = \langle p^2\rangle_{\rm lin} + \frac{m^2\kappa^2}{8\hbar^4\pi^2}, \tag{19}$$

[1]See ref. [65] for mathematical details of the elliptic integrals.

which is again larger than the expression corresponding to linear dynamics, $\langle p^2\rangle_{\rm lin}$.

To compute the second and fourth moment of x, we now need to expand the stationary states (13). For small ν the Jacobi elliptic function can be approximated by [65]

$$\mathrm{sn}(x|\nu) \simeq \sin(x) - \frac{1}{4}\nu\,(x-\sin(x)\cos(x))\cos(x) \tag{20}$$

and thus we have

$$\begin{aligned}\psi_n(x,\lambda) \simeq\, & A_n\sin\left(\frac{2nK(\nu)}{\lambda}x\right)\\ & -\frac{1}{4}\nu\left(\frac{2nK(\nu)}{\lambda}x - \sin\left(\frac{2nK(\nu)}{\lambda}x\right)\cos\left(\frac{2nK(\nu)}{\lambda}x\right)\right)\\ & \times\cos\left(\frac{2nK(\nu)}{\lambda}x\right).\end{aligned} \tag{21}$$

Note that $K(0) = \pi/2$ and hence in leading order we recover the eigenfunctions of the linear case (11). Moreover, A_n is the new normalization coefficient, which can be written in leading order of ν as

$$A_n \simeq \sqrt{\frac{2}{\lambda}} - \frac{\nu}{8\sqrt{2\lambda}}. \tag{22}$$

Accordingly, the second moment of x becomes

$$\langle x^2\rangle \simeq \langle x^2\rangle_{\rm lin} + \frac{3m\lambda^3\,\kappa}{32\hbar^2\pi^4}. \tag{23}$$

where we again employed eq. (16). As before we observe that the leading order in κ is additive and positive. Similarly, we have for the fourth moment

$$\langle x^4\rangle \simeq \langle x^4\rangle_{\rm lin} + \frac{3(8\pi^2-15)m\lambda^3\,\kappa}{128\hbar^2\pi^6}\,, \tag{24}$$

and thus all terms in eq. (9) for nonlinear dynamics are larger than for the linear, $\kappa = 0$ case.

Hence, we immediately conclude that the quantum speed limit (9), *i.e.*, the maximal rate of quantum state evolution, for quantum boxes that are prepared in their corresponding ground states, and whose volume is changed at constant rate, is larger for nonlinear dynamics than for linear dynamics. These findings are illustrated in fig. 3.

In conclusion, the analytically solvable model confirms the numerically found result, namely that nonlinear quantum dynamics supports faster evolution than linear dynamics. However, from the analytical result we have also obtained exact, closed expressions for $v_{\rm QSL}$ in leading order of κ.

Beyond Gross-Pitaevskii dynamics. – In our analysis we have so far restricted ourselves to the 1-dimensional Gross-Pitaevskii equation (1). However, it has been noted [66] that in some situations the description

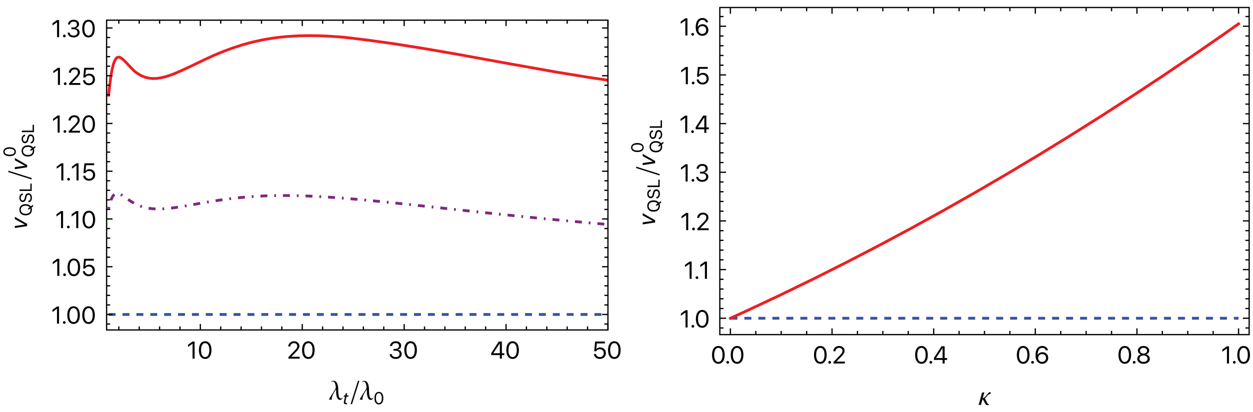

Fig. 3: Left: QSL (2) for the time-dependent box (10) for $\kappa = 0$ (blue, dashed line), $\kappa = 0.25$ (purple, dot-dashed line), and $\kappa = 0.5$ (red, solid line). Right: QSL (2) for the time-dependent box (10) as a function of κ and with the same color coding as on the left. Other parameters are $\hbar = 1$, $v = 1$, and $m = 1$.

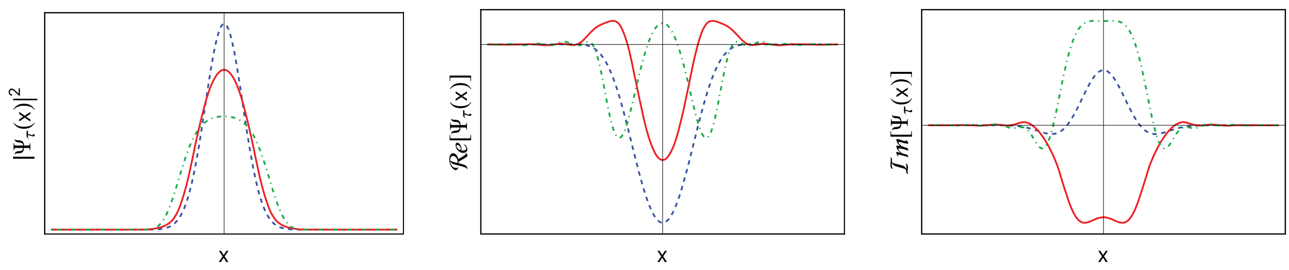

Fig. 4: From left to right: absolute value, real part, and imaginary part of the solution $\Psi_\tau(x)$ for the parametric harmonic oscillator (4) with linear dynamics (blue, dashed line), Gross-Pitaevskii dynamics (1) (red, solid line), and Kolomeisky dynamic (25) (green, dot-dashed line). Other parameters are $\hbar\omega_0 = 5$, $\hbar\omega_1 = 1$, $m = 1$, $\kappa = 10$, and $\tau = 2$.

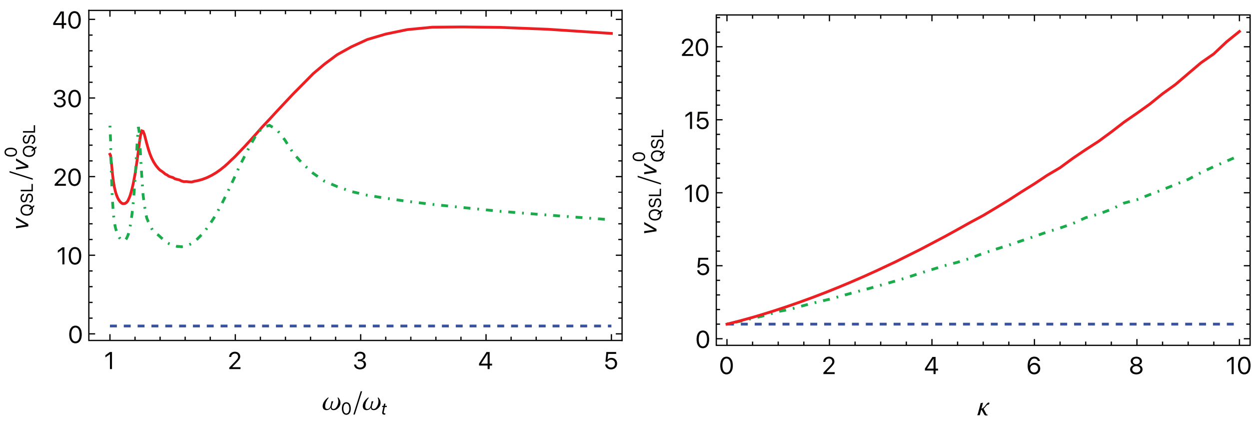

Fig. 5: Left: QSL (2) for the parametric harmonic oscillator (4) with linear dynamics (blue, dashed line), Gross-Pitaevskii dynamics (1) (red, solid line), and Kolomeisky dynamic (25) (green, dot-dashed line). Right: QSL (2) for the parametric harmonic oscillator (4) evaluated at $t = \tau/2$ as a function of κ and with the same color coding as on the left. Other parameters are $\hbar\omega_0 = 5$, $\hbar\omega_1 = 1$, $m = 1$, and $\tau = 2$.

of Bose systems requires a fundamental modification. This is, in particular, the case when accounting for three particles interactions [67–69]. Such scenarios are better described by

$$i\hbar\,\dot{\Psi}_t(x) = \left[-\frac{\hbar^2}{2m}\,\partial_x^2 + U\left(x,t\right) + \kappa\,\left|\Psi_t(x)\right|^4\right]\,\Psi_t(x), \tag{25}$$

where the nonlinear term is of fifth order, rather than third in eq. (1). For an in-depth discussion of the significance, properties, and experimental relevance of the Kolomeisky equation (25) we refer to a recent review article [70].

For our present purposes the immediate question arises whether making the dynamics "more nonlinear" allows for a further speed-up of the evolution. To systematically answer this question, we solved eq. (25) numerically for the parametric harmonic oscillator (4) with the linear protocol (5), and for the same initial state (11). In fig. 4 we depict the result together with linear evolution and the Gross-Pitaevskii dynamics.

We observe that the solution of the Kolomeisky dynamics (25) is markedly different from the linear and Gross-Pitaevskii case (1). Since the quantum speed limit is determined by the rate at which states become distinguishable [5], it appears intuitive that Kolomeisky dynamics evolve at different speeds than Gross-Pitaevskii dynamics. However, from plots like in fig. 4 it is not obvious whether the corresponding speed limit is quantifiably larger or smaller.

Therefore, we again computed the QSL (9), which is plotted in fig. 5. We immediately observe that for the present case, the QSL for Kolomeisky dynamics is almost always smaller than for the Gross-Pitaevskii case. Hence, we conclude that higher order nonlinearities do generally not enhance the rate of quantum evolution. However, now the obvious question arises whether there is an "optimal" nonlinearity for which the QSL becomes maximal. Since answering this question will require an in-depth, systematic explorations of physically relevant, nonlinear evolution equations, we leave the answer for future work.

Concluding remarks. – In the present work we have demonstrated, with numerical and analytical examples, that nonlinear quantum dynamics can enhance the rate of quantum evolution. These findings are not only of academic interest, but they may find application in a variety of practically relevant problems. For instance, we can foresee that nonlinear quantum speed-ups may be particularly relevant for quantum communication, quantum computation, and quantum thermodynamics.

Nonlinear effects in classical optics have been well-studied in the context of communication [71]. However, nonlinear effects in optical fibers are typically very weak [71], since they depend on the intensity of the transmitted signals and the interaction length. Thus, exploiting nonlinear effects in quantum optics appears more realistic than in classical optics [72]. Our findings clearly show that nonlinear effects may enable faster quantum communication.

Another area, where nonlinear quantum speed-ups may be relevant, is nonlinear quantum computation. In this framework, unitary gates are replaced with nonlinear quantum operations [73–75]. Our results indicate that nonlinear computing not only has a computational advantage, but also that single gate operations could be implemented faster than in linear computers. Hence, the overall processing time could be made shorter, and the quantum computer becomes less susceptible to environmental noise.

Finally, we have recently shown that Bose-Einstein condensation can boost the performance of quantum heat engines [76]. However, our previous analysis [76] was restricted to endoreversible cycles, *i.e.*, finite time operation that is still slow enough to allow the working medium to reach a state of local equilibrium. At this point, it is not obvious that the performance boost will persist for engines that operate far from equilibrium. However, our present results suggest that, indeed, the power output may be enhanced by collaborative quantum effects within the working medium, which would constitute a thermodynamic advantage arising from the effectively nonlinear quantum dynamics. See also ref. [77] for similar conclusions.

* * *

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Data availability statement: All data that support the findings of this study are included within the article (and any supplementary files).

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