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# Asymptotic stability of homogeneous solutions of incompressible stationary Navier-Stokes equations

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**Abstract**

It was proved by Karch and Pilarczyk that Landau solutions are asymptotically stable under any $L^2$-perturbation. In our earlier work with L. Li, we have classified all $(-1)$-homogeneous axisymmetric no-swirl solutions of incompressible stationary Navier-Stokes equations in three dimension which are smooth on the unit sphere minus the south and north poles. In this paper, we study the asymptotic stability of the least singular solutions among these solutions other than Landau solutions, and prove that such solutions are asymptotically stable under any $L^2$-perturbation.


## 1. Introduction

Consider the incompressible stationary Navier-Stokes Equations in $\mathbb{R}^3$,

$$\begin{cases} -\Delta u + (u\cdot\nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

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These equations are invariant under the scaling $u(x) \to \lambda u(\lambda x)$ and $p(x) \to \lambda^2 p(\lambda x)$, $\lambda > 0$ and it is natural to study solutions which are invariant under this scaling. These solutions are referred to as $(-1)$-homogeneous solutions (although $p$ is $(-2)$-homogeneous).

Let $x = (x_1, x_2, x_3)$ be Euclidean coordinates and $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ be the corresponding unit normal vectors. Denote $x' = (x_1, x_2)$. Let $(r, \theta, \phi)$ be the spherical coordinates, where $r$ is the radial distance from the origin, $\theta$ is the angle between the radius vector and the positive $x_3$-axis, and $\phi$ is the meridian angle about the $x_3$-axis. A vector field $u$ can be written as

$$u = u_r e_r + u_\theta e_\theta + u_\phi e_\phi,$$

where

$$e_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad e_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \qquad e_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}.$$

A vector field $u$ is called axisymmetric if $u_r$, $u_\theta$ and $u_\phi$ are independent of $\phi$, and is called *no-swirl* if $u_\phi = 0$.

In 1944, L.D. Landau [7] discovered a 3-parameter family of explicit $(-1)$-homogeneous solutions of the stationary NSE in $C^\infty(\mathbb{R}^3 \setminus \{0\})$. These solutions, now called Landau solutions, are axisymmetric with no-swirl and have exactly one singularity at the origin. Tian and Xin proved in [23] that all $(-1)$-homogeneous, axisymmetric nonzero solutions of (1) in $C^\infty(\mathbb{R}^3 \setminus \{0\})$ are Landau solutions. Šverák proved in [21] that all (-1)-homogeneous nonzero solutions of (1) in $C^\infty(\mathbb{R}^3 \setminus \{0\})$ are Landau solutions. There have also been works on $(-1)$-homogeneous solutions of (1), see [2,14–17,19,20,24,25]. In [10–12], the $(-1)$-homogeneous axisymmetric solutions of (1) in $C^\infty(\mathbb{R}^3 \setminus \{(x_1, x_2) = 0\})$ with a possible singular ray $\{(x_1, x_2) = 0\}$ was studied, where such solutions with no-swirl were classified in [10] and [11], and existence of such solutions with nonzero swirl was proved in [10] and [12].

There has been much work in literature on the existence of weak solutions and $L^2$-decay of weak solutions of the evolutionary Navier-Stokes equations, see e.g. [1,3,6,8,9,13,18,22] and the references therein. Such $L^2$-decay of weak solutions can be viewed as the asymptotically stability of the zero stationary solution of (1). The asymptotic stability problem has been studied for other nonzero stationary solutions of (1) with some possible singularities in $\mathbb{R}^3$. Karch and Pilarczyk proved in [4] that small Landau solutions are asymptotically stable under $L^2$-perturbations. The $L^2$ asymptotic stability of other solutions with singularities is also studied in [5]. With special $(-1)$-homogeneous solutions which are different from Landau solutions obtained in [10–12], it is worth to explore the asymptotic stability or instability of these solutions. In this paper, we start this study for a family of solutions which are the simplest and least singular solutions among the solutions found in [10–12].

Denote $U = u \cdot r\sin\theta$ and $y = \cos\theta$. By the divergence free property of $u$ we have $u_r = \frac{1}{r}U_\theta'$. For (-1)-homogeneous axisymmetric no-swirl solutions, (1) can be reduced to

$$(1 - y^2)U_\theta' + 2yU_\theta + \frac{1}{2}U_\theta^2 = c_1(1 - y) + c_2(1 + y) + c_3(1 - y^2). \tag{2}$$

For $c_1 \ge -1$ and $c_2 \ge -1$, let

$$\bar{c}_3(c_1,c_2) := -\frac{1}{2}\left(\sqrt{1+c_1}+\sqrt{1+c_2}\right)\left(\sqrt{1+c_1}+\sqrt{1+c_2}+2\right),$$

where $c_1, c_2, c_3$ are real numbers. Denote $c = (c_1, c_2, c_3)$ and

$$J := \{c \in \mathbb{R}^3 \mid c_1 \geq -1, c_2 \geq -1, c_3 \geq \bar{c}_3(c_1,c_2)\}.$$

In [11], it was proved that there exist $\gamma^-, \gamma^+ \in C^0(J, \mathbb{R})$, satisfying $\gamma^-(c) < \gamma^+(c)$ if $c_3 > \bar{c}_3(c_1,c_2)$, and $\gamma^-(c) = \gamma^+(c)$ if $c_3 = \bar{c}_3(c_1,c_2)$, such that equation (2) has a unique solution $U_\theta^{c,\gamma}$ in $C^\infty(-1,1) \cap C^0[-1,1]$ satisfying $U_\theta^{c,\gamma}(0) = \gamma$ for every $c$ in $J$ and $\gamma^-(c) \leq \gamma \leq \gamma^+(c)$. In particular, $\gamma^+(0) > 0$ and $\gamma^-(0) < 0$. Moreover, let

$$\begin{aligned} u^{c,\gamma} &\equiv u_r^{c,\gamma} e_r + u_\theta^{c,\gamma} e_\theta = (U_\theta^{c,\gamma})' e_r + \frac{U_\theta^{c,\gamma}}{\sin\theta} e_\theta, \\ p^{c,\gamma} &= \frac{1}{r^2}(u_r^{c,\gamma} - \frac{1}{2}(u_\theta^{c,\gamma})^2) = \frac{1}{r^2}((U_\theta^{c,\gamma})' - \frac{(U_\theta^{c,\gamma})^2}{2\sin^2\theta}). \end{aligned} \tag{3}$$

$\{(u^{c,\gamma}, p^{c,\gamma}) \mid c \in J, \gamma^-(c) \leq \gamma \leq \gamma^+(c)\}$ are all $(-1)$-homogeneous axisymmetric no-swirl solutions of (1) in $C^\infty(\mathbb{R}^3 \setminus \{(x_1,x_2) = 0\})$. It was also obtained in [11] that

$$U_\theta^{c,\gamma}(-1) = \begin{cases} 2 + 2\sqrt{1+c_1}, & \text{when } \gamma = \gamma^+(c), \\ 2 - 2\sqrt{1+c_1}, & \text{otherwise}, \end{cases}$$

$$U_\theta^{c,\gamma}(1) = \begin{cases} -2 - 2\sqrt{1+c_2}, & \text{when } \gamma = \gamma^-(c), \\ -2 + 2\sqrt{1+c_2}, & \text{otherwise}. \end{cases}$$

As mentioned earlier, we would like to study the asymptotic stability or instability of the $(-1)$-homogeneous axisymmetric stationary solutions found in [10–12]. Different from Landau solutions, these solutions are singular at the north pole $N$ and/or south pole $S$. These solutions $u$ satisfy either $0 < \limsup_{|x|=1, x'\to 0} |x||x'||\nabla u(x)| < \infty$ or $\limsup_{|x|=1, x'\to 0} |x'|^2|\nabla u(x)| > 0$, while Landau solutions satisfy $\sup_{|x|=1} |x|^2|\nabla u| < \infty$. In this paper, we study the stability of $(-1)$-homogeneous axisymmetric no-swirl solutions satisfying $0 < \limsup_{x'\to 0} |x||x'||\nabla u(x)| < \infty$. These solutions are the family $\{(u^{c,\gamma}, p^{c,\gamma}) \mid (c,\gamma) \in M\}$, where

$$M := \{(c,\gamma) \mid c_1 = c_2 = 0, c_3 > -4, \gamma^-(c) < \gamma < \gamma^+(c)\}. \tag{4}$$

For any $(c,\gamma) \in M$, $U_\theta^{c,\gamma}$ satisfies

$$\begin{cases} (1-y^2)(U_\theta^{c,\gamma})' + 2yU_\theta^{c,\gamma} + \frac{1}{2}(U_\theta^{c,\gamma})^2 = c_3(1-y^2), -1 < y < 1, \\ U_\theta(0) = \gamma. \end{cases} \tag{5}$$

**Proposition 1.1.** *Let $(c,\gamma) \in M$, then $(u^{c,\gamma}(x), p^{c,\gamma}(x))$ satisfies*

$$\begin{cases} -\Delta u^{c,\gamma} + u^{c,\gamma}\cdot\nabla u^{c,\gamma} + \nabla p^{c,\gamma} = (4\pi c_3 \ln|x_3|\partial_{x_3}\delta_{(0,0,x_3)} - b^{c,\gamma}\delta_0)e_3, \quad x \in \mathbb{R}^3, \\ div\, u^{c,\gamma} = 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{6}$$

*where*

$$b^{c,\gamma} = \int_{-1}^{1} \left( y|U_\theta'|^2 - \frac{2-y^2}{1-y^2} U_\theta - \frac{y}{1-y^2} U_\theta^2 \right) dy. \tag{7}$$

*Equations (6) and (7) are understood in the following distribution sense: for any $\varphi \in C_c^\infty(\mathbb{R}^3)$, $j = 1, 2, 3$,*

$$\int_{\mathbb{R}^3} (\nabla u_j \nabla \varphi - u_i u_j \partial_{x_i} \varphi - p \partial_{x_j} \varphi) = [4\pi c_3 \int_{-\infty}^{\infty} \ln|x_3| \partial_{x_3} \varphi(0,0,x_3) dx_3 - b^{c,\gamma} \varphi(0)] \delta_{j3} e_3, \tag{8}$$

*and*

$$\int_{\mathbb{R}^3} u^{c,\gamma} \cdot \nabla \varphi = 0. \tag{9}$$

We now study the stability of the family of solutions $\{u^{c,\gamma} \mid (c,\gamma) \in M\}$. Let $\dot{H}^1(\mathbb{R}^3)$ denote the closure of $C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ under the norm $\|\nabla u\|_{L^2(\mathbb{R}^3)}$, and for $1 \le p < \infty$,

$$L_\sigma^p(\mathbb{R}^3) = \{u \in L^p(\mathbb{R}^3) \mid \operatorname{div} u = 0\}, \quad \dot{H}_\sigma^1(\mathbb{R}^3) = \{u \in \dot{H}^1(\mathbb{R}^3) \mid \operatorname{div} u = 0\},$$

and

$$\|u\|_{L_\sigma^p(\mathbb{R}^3)} := \|u\|_{L^p(\mathbb{R}^3)}, \quad \|u\|_{\dot{H}_\sigma^1(\mathbb{R}^3)} = \|\nabla u\|_{L^2(\mathbb{R}^3)}.$$

For a given solution $(u^{c,\gamma}, p^{c,\gamma})$ of (1), let $u = u(x,t)$ denote a solution of

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla p \\ \quad = (4\pi c_3 \ln|x_3| \partial_{x_3} \delta_{(0,0,x_3)} - b^{c,\gamma} \delta_0) e_3, \ (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} u = 0, \ (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u^{c,\gamma} + w_0, \end{cases} \tag{10}$$

where $w_0 \in L_\sigma^2(\mathbb{R}^3)$ and $b^{c,\gamma}$ is given by (7). Then $w(x,t) = u(x,t) - u^{c,\gamma}$ and $\pi(x) = p(x) - p^{c,\gamma}(x)$ satisfy the initial value problem

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla) w + (w \cdot \nabla) u^{c,\gamma} + (u^{c,\gamma} \cdot \nabla) w + \nabla \pi = 0, \ (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} w = 0, \ (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ w(x,0) = w_0(x). \end{cases} \tag{11}$$

We study the existence and asymptotic behavior of global-in-time weak solutions of (11). Let the energy space

$$X := L^\infty([0,\infty), L^2_\sigma(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1_\sigma),$$

and for $w$ in $X$

$$\|w\|_X := \|w\|_{L^\infty([0,\infty),L^2_\sigma(\mathbb{R}^3))} + \|w\|_{L^2([0,\infty),\dot{H}^1_\sigma)}.$$

Let $(\cdot,\cdot)$ denote the $L^2$-inner product, i.e. $(f,g) = \int_{\mathbb{R}^3} fg dx$. A vector $w \in X$ is a weak solution of (11) if for any $0 \le s \le t < \infty$ and $\varphi \in C([0,\infty), H^1_\sigma(\mathbb{R}^3) \cap C^1([0,\infty), L^2_\sigma(\mathbb{R}^3))$,

$$(w(t),\varphi(t)) + \int_s^t [(\nabla w, \nabla\varphi) + (w\cdot\nabla w,\varphi) + (w\cdot\nabla u^{c,\gamma},\varphi) + (u^{c,\gamma}\cdot\nabla w,\varphi)]d\tau$$
$$= (w(s),\varphi(s)) + \int_s^t (w,\varphi_\tau)d\tau.$$

**Theorem 1.1.** *There exists some $\mu_0 > 0$, such that for any $c = (0,0,c_3)$, $|(c,\gamma)| < \mu_0$, $w_0 \in L^2_\sigma(\mathbb{R}^3)$, there exists a weak solution $w$ of (11) in the energy space $X$. Moreover, $w$ is weakly continuous from $[0,\infty)$ to $L^2_\sigma(\mathbb{R}^3)$, and satisfies that*

$$\|w(t)\|_2^2 + \int_s^t \|\nabla\otimes w(\tau)\|_2^2 d\tau \le \|w(s)\|_2^2 \tag{12}$$

*for almost all $s \ge 0$, including $s = 0$ and all $t \ge s$.*

Recall that $\gamma^+(0) > 0$ and $\gamma^-(0) < 0$. So there is some $\mu_0'$, such that $\{(c,\gamma) \mid c_1 = c_2 = 0, |(c_3,\gamma)| \le \mu_0'\} \subset M$. We also have

**Theorem 1.2.** *There exists some $\mu_0 > 0$, such that for any $c = (0,0,c_3)$, $|(c,\gamma)| < \mu_0$ and weak solution $w \in X$ of (11) satisfying (12),*

$$\lim_{t\to\infty} \|w(t)\|_2 = 0.$$

*Moreover, if $w_0 \in L^p(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$ for some $\frac{6}{5} < p < 2$, then there exists some constant $C > 0$, depending only on $(c,\gamma)$, $n$, $p$ and $\|w_0\|_p$, such that $\|w(t)\|_2 \le Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}$, for all $t > 0$.*

Theorem 1.1 and Theorem 1.2 can be established using the same arguments as [4], as long as the special stationary solutions $u^{c,\gamma}$ satisfy the following condition

$$\left|\int_{\mathbb{R}^3} (v\cdot\nabla u^{c,\gamma})\cdot w dx\right| \le K\|\nabla w\|_{L^2}\|\nabla v\|_{L^2}, \tag{13}$$

for some constant $K$ small enough, for any divergence free $v, w \in C_c^\infty(\mathbb{R}^3)$. In [4], (13) is proved by Hardy's inequality when $u^{c,\gamma}$ is replaced by small Landau solutions. In this paper, we analyze the solutions $u^{c,\gamma}$ where $(c, \gamma) \in M$, and obtain $|\nabla u^{c,\gamma}| \le C(|c| + |\gamma|)/(|x||x'|)$. So (13) is true if we have

$$\int_{\mathbb{R}^3} \frac{|v|^2}{|x||x'|} dx \le K \|\nabla v\|_{L^2}^2, \tag{14}$$

for any $v \in C_c^\infty(\mathbb{R}^3)$. Notice (14) cannot be proved by the classical Hardy's inequality. In Section 4, we prove the following extended Hardy-type inequality, which includes (14).

**Theorem 1.3.** *Let $n \ge 2$, $1 \le p < n$, $u \in C_c^1(\mathbb{R}^n)$, $\alpha p > 1 - n$, $(\alpha + \beta)p > -n$, then there exists some constant $C$, depending on $p$, $\alpha$ and $\beta$, such that*

$$\| |x|^\beta |x'|^\alpha u\|_{L^p(\mathbb{R}^n)} \le C \| |x|^{\beta+\alpha-\alpha'} |x'|^{\alpha'+1} \nabla u\|_{L^p(\mathbb{R}^n)}, \tag{15}$$

*for all $\alpha' \le \alpha$. Moreover, for any $\alpha' > \alpha$ and any $C > 0$, (15) fails in general.*

Estimate (14) is the special case of (15) with $p = 2$, $\alpha = \alpha' = \beta = -\frac{1}{2}$. Then we also have (13). Given (13), Theorem 1.1 and Theorem 1.2 can be proved by the same arguments used in [4], see also [5]. So in this paper we will only prove Theorem 1.3 and (13).

**Remark 1.1.** In [5], Karch, Pilarczyk and Schonbek proved the asymptotic stability of a class of general time-dependent solutions $u$ of (10) using Fourier analysis, where (13) with $u^{c,\gamma}$ replaced by $u$ is an essential assumption. A list of spaces were given in [5] where (13) is true if $u^{c,\gamma}$ is in one of those spaces. But the solutions $u^{c,\gamma}$ we discuss here are not in those spaces.

We will analyze in Section 2 the singular behaviors of $u^{c,\gamma}$, $(c, \gamma) \in M$. In Section 3 we study the force of $u^{c,\gamma}$, $(c, \gamma) \in M$. Theorem 1.3 will be proved in Section 4. Then as stated above, Theorem 1.1 and Theorem 1.2 follow with the same arguments as in [4].

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## 2. Estimate of the special solutions $u^{c,\gamma}$

**Lemma 2.1.** *Let $K$ be a compact subset of $M$. Then there exists some positive constant $C$, depending only on $K$, such that for any $(c, \gamma)$ in $K$ and $-1 \le y \le 1$,*

$$U_\theta^{c,\gamma}(y) = -\frac{c_3}{2}\operatorname{sgn}(y)(1 - y^2)\ln(1 - y^2) + O(1)(|c| + |\gamma|)(1 - y^2), \tag{16}$$

$$(U_\theta^{c,\gamma})'(y) = c_3 \ln(1 - y^2) + O(1)(|c| + |\gamma|), \tag{17}$$

*and*

$$(U_\theta^{c,\gamma})''(y) = -\frac{2c_3 y}{1 - y^2} + O(1)(|c| + |\gamma|)(|\ln(1 - y^2)|^2), \tag{18}$$

*where $O(1)$ denotes some quantity satisfying $|O(1)| \le C$ for some positive constant $C$ depending only on $K$.*

**Proof.** For convenience, let $C$ be a constant depending only on $K$, $O(1)$ be a function satisfying $|O(1)| \le C$ for all $-1 \le y \le 1$, and $C$ and $O(1)$ may vary from line to line. It is easy to see from (5) that $U_\theta^{(0,0,c_3),\gamma}(y)$ and $-U_\theta^{(0,0,c_3),-\gamma}(-y)$ satisfy the same equation and have the same value at $y = 0$ and therefore they are identically the same. So we only need to prove (16)-(18) for $-1 < y \le 0$.

By Theorem 1.5 in [11], there exists some constant $C$, such that

$$|\nabla_{c,\gamma} U_\theta^{c,\gamma}(y)| \le C, \quad \forall -1 < y < 1.$$

Using this and the fact that $U_\theta^{0,0} = 0$, we have that for all $-1 < y < 1$,

$$|U_\theta^{c,\gamma}(y)| = |U_\theta^{c,\gamma}(y) - U_\theta^{0,0}(y)| \le \sup_{(c,\gamma)\in K, -1<y<1} |\nabla_{c,\gamma} U_\theta^{c,\gamma}(y) \cdot (c,\gamma)| \le C(|c| + |\gamma|).$$

Thus

$$\|U_\theta^{c,\gamma}\|_{L^\infty(-1,1)} \le C(|c| + |\gamma|). \tag{19}$$

For simplicity we use $U_\theta$ to denote $U_\theta^{c,\gamma}$. By (5), we have

$$U_\theta' + \frac{U_\theta - 4}{2(1-y^2)} U_\theta = c_3 - \frac{2U_\theta}{1-y}.$$

Let

$$w(y) := \int_0^y \frac{U_\theta - 4}{2(1-s^2)} ds = \ln \frac{1-y}{1+y} + \int_0^y \frac{U_\theta(s)}{2(1-s^2)} ds. \tag{20}$$

Since $U_\theta(0) = \gamma$, we have

$$U_\theta(y) = \gamma e^{-w} + e^{-w} \int_0^y e^w (c_3 - \frac{2U_\theta}{1-s}) ds. \tag{21}$$

By (19), for any fixed $0 < \epsilon < 1/2$,

$$\|U_\theta\|_{L^\infty(-1,1)} < 4\epsilon, \quad \forall |(c,\gamma)| < \epsilon / C.$$

By the above and (20), we have that for $-1 < y < 0$,

$$|w(y) + \ln \frac{1+y}{1-y}| \le 2\epsilon \int_y^0 \frac{ds}{1-s^2} = \epsilon \ln \frac{1-y}{1+y}.$$

So

$$e^{w} \le 4(1+y)^{-1-\epsilon}, \quad e^{-w} \le (1+y)^{1-\epsilon}, \quad -1 < y \le 0.$$

By (19), (21) and the above, we have

$$|U_\theta| \le C\gamma(1+y)^{1-\epsilon} + C(|c|+|\gamma|)e^{-w(y)}\int_y^0 e^{w(s)}ds \le \frac{C}{\epsilon}(|c|+|\gamma|)(1+y)^{1-2\epsilon}. \tag{22}$$

Denote $\mu_1 = \int_0^{-1} \frac{U_\theta(s)}{2(1-s^2)}ds$. By (22) we have $\mu_1 = O(1)(|c|+|\gamma|)/\epsilon$. Then by (20) and (22), we have

$$w(y) = \ln\frac{1-y}{1+y} + \mu_1 + \int_{-1}^{y} \frac{U_\theta(s)}{2(1-s^2)}ds = \ln\frac{1-y}{1+y} + \mu_1 + O(1)\frac{|c|+|\gamma|}{\epsilon}(1+y)^{1-2\epsilon}.$$

Then we have

$$e^{w} = \frac{1-y}{1+y}e^{\mu_1}(1+O(1)\frac{|c|+|\gamma|}{\epsilon}(1+y)^{1-2\epsilon}),$$
$$e^{-w} = \frac{1+y}{1-y}e^{-\mu_1}(1+O(1)\frac{|c|+|\gamma|}{\epsilon}(1+y)^{1-2\epsilon}).$$

Using the above, (21) and (22), we have that for $-1 < y \le 0$,

$$\begin{aligned} U_\theta(y) &= \gamma e^{-w(y)} + c_3 e^{-w(y)}\int_0^y e^{w(s)}ds - e^{-w(y)}\int_0^y e^{w(s)}\frac{2U_\theta}{1-s}ds \\ &= c_3(1+y)\ln(1+y) + O(1)\frac{|c|+|\gamma|}{\epsilon}(1+y). \end{aligned}$$

Estimate (16) is established.

Next, we make the estimate of $U_\theta'$ and prove (17). By (5) and (16), we have that for $-1 < y \le 0$,

$$U_\theta' = c_3 - \frac{1}{1-y^2}(\frac{1}{2}U_\theta^2 + 2yU_\theta) = c_3\ln(1+y) + O(1)(|c|+|\gamma|).$$

Estimate (17) is established.

Differentiating (5), and using (16) and (17), we have for $-1 < y \le 0$ that

$$(1-y^2)U_\theta'' = -2c_3 y - U_\theta U_\theta' - 2U_\theta = 2c_3 + O(1)(|c|+|\gamma|)(1+y)|\ln(1+y)|^2.$$

Estimate (18) follows immediately. The lemma is proved. □

**Corollary 2.1.** *Let $K$ be a compact subset of $M$. Then there exists some positive constant $C$, depending only on $K$, such that for all $(c,\gamma)$ in $K$, and $x$ in $\mathbb{R}^3 \setminus \{x'=0\}$.*

$$u_\theta^{c,\gamma}(x) = -\frac{c_3 \operatorname{sgn}(x_3)|x'|}{|x|^2}\ln\frac{|x'|}{|x|} + \frac{O(1)(|c|+|\gamma|)|x'|}{|x|^2}, \tag{23}$$

$$u_r^{c,\gamma}(x) = \frac{2c_3}{|x|}\ln\frac{|x'|}{|x|} + \frac{O(1)(|c|+|\gamma|)}{|x|}, \tag{24}$$

*and*

$$|\nabla u(x)| = \frac{2|c_3|}{|x||x'|} + \frac{O(1)(|c|+|\gamma|)}{|x|^2}\ln\frac{|x|}{|x'|}. \tag{25}$$

**Proof.** For convenience write $u^{c,\gamma} = u$. By definition, $u = u_r e_r + u_\theta e_\theta$, where $u_r = \frac{1}{r}U_\theta'$, $u_\theta = \frac{1}{r\sin\theta}U_\theta$. Denote $y = \cos\theta$, by Lemma 2.1, we have

$$U_\theta^{c,\gamma}(y) = -c_3 \operatorname{sgn}(\cos\theta)\sin^2\theta \ln\sin\theta + O(1)(|c|+|\gamma|)\sin^2\theta.$$

Since $r = |x|$ and $|x'| = |x|\sin\theta$, estimate (23) follows from the above. Estimate (24) follows from (17).

Next, we compute

$$\nabla u = \nabla u_r e_r + u_r \nabla e_r + \nabla u_\theta e_\theta + u_\theta \nabla e_\theta.$$

By (16) and (17), we have

$$\begin{aligned}
|\nabla u_r| &= \left|\frac{\partial u_r}{\partial r}e_r + \frac{1}{r}\frac{\partial u_r}{\partial\theta}e_\theta\right| = \left|-\frac{1}{r^2}U_\theta'(y)e_r + \frac{1}{r^2}U_\theta''(y)(-\sin\theta)e_\theta\right| \\
&= \frac{2|c_3|}{r^2\sin\theta} + O(1)\frac{(|c|+|\gamma|)}{r^2}\ln\sin\theta = \frac{2|c_3|}{|x||x'|} + O(1)\frac{|c|+|\gamma|}{|x|^2}\ln\frac{|x|}{|x'|},
\end{aligned}$$

and

$$\begin{aligned}
|\nabla u_\theta| &= \left|\frac{\partial u_\theta}{\partial r}e_r + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}e_\theta\right| = \left|-\frac{1}{r^2}U_\theta e_r + \frac{1}{r^2}U_\theta'(-\sin\theta)e_\theta\right| \\
&\le \frac{C(|c|+|\gamma|)}{|x|^2}\frac{|x'|}{|x|}\ln\frac{|x|}{|x'|}.
\end{aligned}$$

Since $|\nabla e_r| + |\nabla e_\theta| \le C/r$, estimate (25) follows from the above. □

## 3. Force of $u^{c,\gamma}$, $(c,\gamma) \in M$

In this section, we study the force of the special solutions $u^{c,\gamma}$ and prove Proposition 1.1, where $(c,\gamma)$ in $M$ and $M$ is the set defined by (4). Recall that $(r,\theta,\phi)$ are the polar coordinates, let $\rho = r\sin\theta$, $(\rho,\phi,z)$ be the cylindrical coordinates, $y = \cos\theta$. Recall $(u^{c,\gamma}, p^{c,\gamma})$ are given by (3), where $U_\theta^{c,\gamma}(y)$ is a solution of (5). For convenience, denote $u = u^{c,\gamma}$, $p = p^{c,\gamma}$ and $U_\theta = U_\theta^{c,\gamma}$. In Euclidean coordinates, $x = (x_1,x_2,x_3)$ and $u = (u_1,u_2,u_3)$.

**Proof of Proposition 1.1.** Let $(c, \gamma) \in M$. For any $R > 0$, let

$$\Omega := \{x \in \mathbb{R}^3 | |x'| \le R, -R < x_3 < R\}. \tag{26}$$

We prove (8) and (9) for any $\varphi \in C_c^\infty(\Omega)$. Throughout the proof we denote $O(1)$ as some quantity satisfying $|O(1)| \le C$ for some $C > 0$ depending only on $(c, \gamma)$, $R$ and $\varphi$.

By Lemma 2.1,

$$|U_\theta(y)| = O(1)\sin^2\theta |\ln \sin\theta|, \; |(U_\theta)'(y)| = O(1)|\ln\sin\theta|, \; |(U_\theta)''(y)| = \frac{O(1)}{\sin^2\theta}. \tag{27}$$

Recall that here "′" denotes the derivative with respect to $y$. By Corollary 2.1 and (3), we have

$$|u_\theta| = \frac{O(1)\sin\theta|\ln\sin\theta|}{r}, \; |u_r| = \frac{O(1)|\ln\sin\theta|}{r}, \; |\nabla u| = \frac{O(1)}{r^2\sin\theta}, \; |p| = \frac{O(1)|\ln\sin\theta|}{r^2}. \tag{28}$$

We first prove (9). For any $\epsilon > 0$, denote

$$\Omega_\epsilon := \{x \in \mathbb{R}^3 | |x'| \le \epsilon, -R < x_3 < R\}.$$

Let $o_\epsilon(1)$ be a function where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. Since $u \in C^\infty(\mathbb{R}^3 \setminus \{x' = 0\})$, we have $\operatorname{div} u = 0$ in $\mathbb{R}^3 \setminus \{x' = 0\}$. Therefore

$$\int_{\mathbb{R}^3} u\cdot\nabla\varphi dx = \int_{\Omega\setminus\Omega_\epsilon} u\cdot\nabla\varphi dx + \int_{\Omega_\epsilon} u\cdot\nabla\varphi dx = -\int_{\partial\Omega_\epsilon\cap\{|x'|=\epsilon\}} u\cdot\nabla\varphi dx + \int_{\Omega_\epsilon} u\cdot\nabla\varphi dx.$$

By (28), we have $|u| \le C/|x|$. So

$$\int_{\partial\Omega_\epsilon} |u\cdot\nabla\varphi| dx \le \int_{\partial\Omega_\epsilon} \frac{C}{|x|} d\sigma(x) \le C\epsilon|\ln\epsilon|, \quad \int_{\Omega_\epsilon} |u\cdot\nabla\varphi| dx \le \int_{\Omega_\epsilon} \frac{C}{|x|} dx \le C\epsilon^2|\ln\epsilon|.$$

Sending $\epsilon$ to 0 in the above leads to (9).

Next, we prove (8). Denote the stress tensor

$$T_{ij}(u) := p\delta_{ij} + u_i u_j - (\partial_{x_j} u_i + \partial_{x_i} u_j).$$

Then (8) is equivalent to

$$\int_\Omega T_{ij}(u)\partial_{x_i}\varphi dx = [4\pi c_3 \int_{-R}^{R} \ln|x_3|\partial_{x_3}\varphi(0,0,x_3)dx_3 - b\varphi(0)]\delta_{j3}e_3, \quad \forall\varphi \in C_c^\infty(\Omega), \tag{29}$$

where $b = b^{c,\gamma}$ is given by (7).

**Claim 1.** $T_{ij}(u) \in L^q_{loc}(\mathbb{R}^3)$, for any $q < \frac{3}{2}$.

To prove the Claim, notice that by (28), we have that

$$|T_{ij}| \leq |p| + |u|^2 + 2|\nabla u| \leq \frac{C}{r^2 \sin\theta}. \tag{30}$$

So for any $R > 0$ and $\Omega$ defined by (26), we have, using $q < \frac{3}{2}$,

$$\int_\Omega |T_{ij}|^q \leq C \int_{B_{2R}} \frac{1}{r^{2q}|\sin\theta|^q} dx = C \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{r^{2q-2}|\sin\theta|^{q-1}} r^2 \sin\theta d\phi d\theta dr \leq C.$$

The Claim is proved.

Using Claim 1 and the fact that $\partial_{x_i} T_{ij} = 0$ in $\mathbb{R}^3 \setminus \{x' = 0\}$ for any $1 \leq j \leq 3$, we have that

$$-\int_\Omega T_{ij}\partial_{x_i}\varphi dx = -\int_{\Omega\setminus\Omega_\epsilon} T_{ij}\partial_{x_i}\varphi dx - \int_{\Omega_\epsilon} T_{ij}\partial_{x_i}\varphi dx = \int_{\partial\Omega_\epsilon} T_{ij}\cdot\nu_i\varphi dx - \int_{\Omega_\epsilon} T_{ij}\partial_{x_i}\varphi dx.$$

Let

$$L_j := \int_{\partial\Omega_\epsilon} T_{ij}\cdot\nu_i\varphi dx.$$

Since $T_{ij} \in L^1(\Omega)$, we have $\int_{\Omega_\epsilon} T_{ij}\partial_{x_i}\varphi dx = o_\epsilon(1)$. So for each $j = 1, 2, 3$,

$$-\int_\Omega T_{ij}\partial_{x_i}\varphi dx = L_j + o_\epsilon(1). \tag{31}$$

By computation

$$L_j = \int_{\partial\Omega_\epsilon\cap\{|x'|=\epsilon\}} T_{ij}\cdot\nu_i\varphi(0,0,x_3) + O(1)\epsilon \int_{\partial\Omega_\epsilon\cap\{|x'|=\epsilon\}} |T_{ij}| =: L_j^{(1)} + L_j^{(2)}.$$

By (30), we have that for $j = 1, 2, 3$,

$$|L_j^{(2)}| \leq C\epsilon \int_{\partial\Omega_\epsilon\cap\{|x'|=\epsilon\}} |T_{ij}| d\sigma \leq C \int_{-R}^R \frac{\epsilon}{\sqrt{\epsilon^2 + x_3^2}} dx_3 \leq C\epsilon|\ln\epsilon| \to 0, \text{ as } \epsilon \to 0. \tag{32}$$

**Lemma 3.1.**

$$L_j^{(1)} = 0, \quad j = 1, 2.$$

**Proof.** We will show that $T_{ij} \cdot \nu_i = F(|x'|, x_3)x_j$ for some function $F(|x'|, x_3)$, $j = 1, 2$, so its integral on any cylinder $\{|x'| = \epsilon\}$ vanishes. Let $x' = (x_1, x_2)$, $u' = (u_1, u_2)$, $\nabla' = (\partial_1, \partial_2)$, $(\rho, \phi, z)$ be the cylindrical coordinates, and the unit normal $e_\rho = (\cos\phi, \sin\phi, 0)$, $e_\phi = (-\sin\phi, \cos\phi, 0)$, $e_z = (0, 0, 1)$. So we have $x = \rho e_\rho + ze_z$, $x' = \rho e_\rho$. Notice $u$ is axisymmetric no-swirl, we can write $u = u_\rho e_\rho + u_z e_z$, where $u_\rho$ and $u_z$ are both independent of $\phi$. By computation,

$$x' \cdot u' = \rho u_\rho, \quad x' \cdot \nabla' u = \rho \frac{\partial u_\rho}{\partial \rho} e_\rho + \rho \frac{\partial u_z}{\partial \rho} e_z,$$

and

$$\nabla(x' \cdot u') = \nabla(\rho u_\rho) = \frac{\partial(\rho u_\rho)}{\partial \rho} e_\rho + \frac{\partial(\rho u_\rho)}{\partial z} e_z.$$

On $\partial\Omega_\epsilon \cap \{|x'| = \epsilon\}$, the outer-normal $\nu = \frac{1}{\rho}(x_1, x_2, 0)$. Since $u$ is axisymmetric, $u_\rho$ is independent of $\phi$, so $u_1 = u_\rho(\rho, z)\cos\phi$, and we have

$$\begin{aligned} T_{i1} \cdot \nu_i &= \frac{1}{\rho}\left(px_1 + x' \cdot u' u_1 - x' \cdot \nabla' u_1 - \partial_1(x' \cdot u') + u_1\right) \\ &= \frac{1}{\rho}\left(p\rho\cos\phi + \rho u_\rho u_\rho \cos\phi - \rho\frac{\partial u_\rho}{\partial \rho}\cos\phi - \frac{\partial(\rho u_\rho)}{\partial \rho}\cos\phi + u_\rho\cos\phi\right) \\ &= G(\rho, z)\cos\phi, \end{aligned}$$

where

$$G(\rho, z) = \frac{1}{\rho}\left(p\rho + \rho u_\rho u_\rho - \rho\frac{\partial u_\rho}{\partial \rho} - \frac{\partial(\rho u_\rho)}{\partial \rho} + u_\rho\right).$$

So

$$L_1^{(1)} = \int_{\rho=\epsilon} T_{i1}\varphi_1(0, 0, z)\nu_i d\sigma = \epsilon \int_{-R}^{R} G(\epsilon, z)\varphi_1(0, 0, z)dz \int_0^{2\pi} \cos\phi d\phi = 0$$

With similar argument we also have $L_2^{(1)} = 0$. The lemma is proved. □

**Lemma 3.2.**

$$\lim_{\epsilon\to 0} L_3^{(1)} = 4\pi c_3 \int_{-R}^{R} \ln|x_3|\partial_{x_3}\varphi(0, 0, x_3)dx_3 - b\varphi(0),$$

*where b is the constant defined by (7).*

**Proof.** Recall

$$L_3^{(1)} = \frac{1}{\epsilon} \int\limits_{\rho=\epsilon} (T_{13}x_1 + T_{23}x_2)\varphi(0,0,x_3),$$

and for $i = 1, 2$,

$$T_{i3} = u_i u_3 - \frac{\partial u_i}{\partial x_3} - \frac{\partial u_3}{\partial x_i}.$$

Since $u = (u_1, u_2, u_3) = \frac{1}{r} U'_\theta e_r + \frac{1}{r\sin\theta} U_\theta e_\theta$, we have

$$u_1(x_1, x_2, x_3) = \frac{x_1}{r^2} U'_\theta(y) + \frac{x_1 x_3}{r\rho^2} U_\theta(y),$$
$$u_2(x_1, x_2, x_3) = \frac{x_2}{r^2} U'_\theta(y) + \frac{x_2 x_3}{r\rho^2} U_\theta(y),$$
$$u_3(x_1, x_2, x_3) = \frac{x_3}{r^2} U'_\theta(y) - \frac{1}{r} U_\theta(y).$$

Recall that $r^2 = x_1^2 + x_2^2 + x_3^2$, $\rho^2 = x_1^2 + x_2^2$, $y = \cos\theta = \frac{x_3}{r} = \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}$. By computation we have

$$\frac{\partial u_i}{\partial x_3} = \frac{x_i}{r^3} U_\theta - \frac{x_i x_3}{r^4} U'_\theta + \frac{x_i \rho^2}{r^5} U''_\theta, \quad \frac{\partial u_3}{\partial x_i} = \frac{x_i}{r^3} U_\theta - \frac{x_i x_3}{r^4} U'_\theta - \frac{x_i x_3^2}{r^5} U''_\theta.$$

So

$$\sum_{i=1}^{2} T_{i3} x_i = \frac{\rho^2 x_3}{r^4} |U'_\theta|^2 + \frac{x_3^2 - \rho^2}{r^3} U_\theta U'_\theta - \frac{x_3}{r^2} U_\theta^2 + \frac{2\rho^2 x_3}{r^4} U'_\theta - \frac{2\rho^2}{r^3} U_\theta - \frac{\rho^2(\rho^2 - x_3^2)}{r^5} U''_\theta. \quad (33)$$

Since $U_\theta$ satisfy (5), take derivative of the first equation of (5) both sides with respect to $y$, we have

$$(1 - y^2) U''_\theta + 2U_\theta + U_\theta U'_\theta = -2c_3 y.$$

Plug in $U''_\theta = -\frac{1}{1-y^2}(2U_\theta + U_\theta U'_\theta + 2c_3 y)$ in (33), we have

$$\sum_{i=1}^{2} T_{i3} x_i = -\frac{2c_3 x_3}{r^2} + \left( \frac{\rho^2 x_3}{r^4} (|U'_\theta|^2 + 2U'_\theta + 4c_3) - \frac{x_3}{r^2} U_\theta^2 - \frac{x_3^2}{r^3} U_\theta \right).$$

Let

$$G(x) = \frac{\rho^2 x_3}{r^4} (|U'_\theta|^2 + 2U'_\theta + 4c_3) - \frac{x_3}{r^2} U_\theta^2 - \frac{x_3^2}{r^3} U_\theta. \quad (34)$$

Now we have

$$
\begin{aligned}
L_3^{(1)} &= \frac{1}{\epsilon}\int\limits_{\rho=\epsilon}(T_{13}x_1+T_{23}x_2)\varphi(0,0,x_3)d\sigma \\
&= -\frac{1}{\epsilon}\int\limits_{\rho=\epsilon}\frac{2c_3x_3}{r^2}\varphi(0,0,x_3)d\sigma+\frac{1}{\epsilon}\int\limits_{\rho=\epsilon}G(x)\varphi(0,0,x_3)d\sigma \\
&=: A+B.
\end{aligned}
$$

Since $\varphi(0,0,R)=\varphi(0,0,-R)=0$, we have

$$
A=-4\pi c_3\int\limits_{-R}^{R}\frac{x_3}{\epsilon^2+x_3^2}\varphi(0,0,x_3)dx_3=2\pi c_3\int\limits_{-R}^{R}\ln(\epsilon^2+x_3^2)\partial_{x_3}\varphi(0,0,x_3)dx_3.
$$

So

$$
\lim_{\epsilon\to 0}A=4\pi c_3\int\limits_{-R}^{R}\ln|x_3|\partial_{x_3}\varphi(0,0,x_3)dx_3. \tag{35}
$$

Next, write

$$
\begin{aligned}
B &= \frac{1}{\epsilon}\int\limits_{\rho=\epsilon}G(x)\varphi(0,0,x_3)d\sigma \\
&= \frac{1}{\epsilon}\int\limits_{\rho=\epsilon}G(x)\varphi(0)d\sigma+\frac{1}{\epsilon}\int\limits_{\rho=\epsilon}G(x)(\varphi(0,0,x_3)-\varphi(0))d\sigma \\
&=: B_1+B_2.
\end{aligned}
$$

We have $|B_2|\le C\int_{-R}^{R}|G(x)x_3|dx_3$ By (27) and (34), we have that for $|x'|=\epsilon$, $-R\le x_3\le R$,

$$
|G(x)x_3|\le C\frac{\rho^2x_3^2}{r^4}(|\ln\frac{\rho}{r}|^2+|\ln\frac{\rho}{r}|)\le\frac{\epsilon^2}{\epsilon^2+x_3^2}(|\ln\frac{\epsilon}{\sqrt{\epsilon^2+x_3^2}}|^2+|\ln\frac{\epsilon}{\sqrt{\epsilon^2+x_3^2}}|).
$$

So $\lim_{\epsilon\to 0}|G(x)x_3|=0$ a.e. $x_3\in[-R,R]$, and $|G(x)x_3|\le C$ for $-R\le x_3\le R$. By the dominated convergence theorem, we have

$$
\lim_{\epsilon\to 0}B_2=0. \tag{36}
$$

Next, let $b_\epsilon=\frac{1}{\epsilon}\int_{\rho=\epsilon}G(x)dx$. We have $B_1=b_\epsilon\varphi(0)$. Let $\delta=R/\sqrt{\epsilon^2+R^2}$, we have $0<\delta<1$. On $\{\rho=\epsilon\}$, $r=\sqrt{\epsilon^2+x_3^2}$, therefore $y=\cos\theta=x_3/r=x_3/\sqrt{\epsilon^2+R^2}$, we have

$$
dy=\frac{\epsilon^2}{(\epsilon^2+x_3^2)^{\frac{3}{2}}}dx_3,\ \text{so}\ dx_3=\frac{r^3}{\epsilon^2}dy. \tag{37}
$$

We also have $\sqrt{1-y^2} = \sin\theta = \epsilon/r$. By (27), (34) and (37), we have

$$\begin{aligned} b_\epsilon &= \int_{-\delta}^{\delta} \left( \frac{\epsilon^2 y}{r^3}(|U_\theta'|^2 + 2U_\theta' + 4c_3) - \frac{y}{r}(U_\theta^2 + yU_\theta) \right) \frac{r^3}{\epsilon^2} dy \\ &= \int_{-\delta}^{\delta} \left( y(|U_\theta'|^2 + 2U_\theta') - \frac{y}{1-y^2}(U_\theta^2 + yU_\theta) \right) dy \\ &= 2\delta(U_\theta(\delta) + U_\theta(-\delta)) + \int_{-\delta}^{\delta} \left( y|U_\theta'|^2 - \frac{2-y^2}{1-y^2}U_\theta - \frac{y}{1-y^2}U_\theta^2 \right) dy. \end{aligned}$$

As $\epsilon \to 0$, $\delta \to 1$, so $\delta(U_\theta(\delta) + U_\theta(-\delta)) \to 2(U_\theta(1) + U_\theta(-1)) = 0$. By (27),

$$b := \lim_{\epsilon\to 0} b_\epsilon = \int_{-1}^{1} \left( y|U_\theta'|^2 - \frac{2-y^2}{1-y^2}U_\theta - \frac{y}{1-y^2}U_\theta^2 \right) dy.$$

Recall $B_1 = b_\epsilon \varphi(0)$, we have

$$\lim_{\epsilon\to 0} B_1 = b\varphi(0). \tag{38}$$

Lemma 3.2 follows from (35), (36) and (38). □

Proposition 1.1 follows from (29), (31), (32), Lemma 3.1 and Lemma 3.2. □

## 4. Proof of Theorem 1.3

**Proof of Theorem 1.3.** For convenience, let $C$ denote a constant depending only on $p, \alpha, \beta, \alpha'$ and $n$, which may vary from line to line. We first prove that if (15) holds for some $C$, then $\alpha' \le \alpha$.

Let $0 < \delta < 1$, $f_\delta(x')$ be a smooth function of $x'$, such that

$$f_\delta(x') := \begin{cases} 1, & 2\delta \le |x'| \le 3\delta, \\ 0, & |x'| \le \delta \text{ or } |x'| \ge 4\delta, \end{cases}$$

and $|\nabla_{x'} f| \le C/\delta$. Let $g(x_n)$ be a smooth function such that

$$g(x_n) := \begin{cases} 1, & 2 \le |x_n| \le 3, \\ 0, & |x_n| \le 1 \text{ or } |x_n| \ge 4, \end{cases}$$

and $|g'(x_n)| \le C$. Define $u_\delta(x) := f_\delta(x')g(x_n)$, then $u_\delta$ is in $C_c^1(\mathbb{R}^n)$. By computation,

$$\||x|^\beta|x'|^\alpha u_\delta\|^p_{L^p(\mathbb{R}^n)} \geq \int_2^3 \int_{2\delta\leq|x'|\leq 3\delta} |x|^{\beta p}|x'|^{\alpha p}dx'dx_n \geq \delta^{\alpha p+n-1}/C.$$

On the other hand, since $\delta \leq 1$,

$$\begin{aligned}
&\||x|^{\beta+\alpha-\alpha'}|x'|^{\alpha'+1}\nabla u_\delta\|^p_{L^p(\mathbb{R}^n)} \\
&\leq \int_1^4 \int_{\delta\leq|x'|\leq 4\delta} |x|^{(\beta+\alpha-\alpha')p}|x'|^{(\alpha'+1)p}(|\nabla_{x'}f_\delta(x')|^p|g(x_n)|^p + |f_\delta(x')|^p|g'(x_n)|^p)dx'dx_n \\
&\leq C\int_{\delta\leq|x'|\leq 4\delta} |x'|^{(\alpha'+1)p}(|\nabla_{x'}f_\delta(x')|^p + |f_\delta(x')|^p)dx' \\
&\leq C\int_{\delta\leq|x'|\leq 4\delta} |x'|^{(\alpha'+1)p}\delta^{-p}dx' \leq C\delta^{\alpha' p+n-1}.
\end{aligned}$$

Since $u_\delta$ satisfies (15), we have $\delta^{\alpha p+n-1} \leq C\delta^{\alpha' p+n-1}$ for any $0 < \delta < 1$, therefore $\alpha' \leq \alpha$.

Next, we prove (15) for $\alpha' \leq \alpha$. Since $|x'| \leq |x|$, we only need to prove it for $\alpha' = \alpha$, i.e.

$$\||x|^\beta|x'|^\alpha u\|_{L^p(\mathbb{R}^n)} \leq C\||x|^\beta|x'|^{\alpha+1}\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{39}$$

We introduce the spherical coordinates in $\mathbb{R}^n$. Let $r > 0$, $\theta_1, ..., \theta_{n-2} \in [0,\pi]$ and $\theta_{n-1} \in [0, 2\pi]$. Denote

$$\begin{aligned}
x_1 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \\
x_2 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\
x_3 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-3}\cos\theta_{n-2}, \\
&\cdots \\
x_{n-1} &= r\sin\theta_1\cos\theta_2, \\
x_n &= r\cos\theta_1.
\end{aligned}$$

Then $|x'| = r\sin\theta_1$ and $dx = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}drd\theta_1\cdots d\theta_{n-1}$. Let $\omega = (\theta_1, ..., \theta_{n-1})$, $\omega' = (\theta_2, ..., \theta_{n-1})$, $\Omega = \sin^{n-2}\theta_1\cdots\sin\theta_{n-2}$, and $\Omega' = \sin^{n-3}\theta_2\cdots\sin\theta_{n-2}$, $E = \{\omega' \mid 0 \leq \theta_i \leq \pi, 2 \leq i \leq n-2, 0 \leq \theta_{n-1} \leq 2\pi\}$. Denote $d\omega = d\theta_1\cdots d\theta_{n-1}$ and $d\omega' = d\theta_2\cdots d\theta_{n-1}$. We can express

$$\int_{\mathbb{R}^n} (|x|^\beta|x'|^\alpha|u|)^p dx = \int_{\mathbb{R}^n} r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{\alpha p+n-2}|u|^p\Omega' drd\omega$$

By assumption, $\lambda := (\alpha+\beta)p + n > 0$. For each fixed $\omega \in [0,\pi]^{n-2} \times [0, 2\pi]$, let $\hat{u}(s) := u(s^{1/\lambda}, \omega)$, it is well-known that

$$\int_0^\infty |\hat{u}(s)|^p ds \le C(p) \int_0^\infty |\hat{u}'(s)|^p s^p ds.$$

Namely,

$$\int_0^\infty |u(r,\omega)|^p r^{(\alpha+\beta)p+n-1} dr \le C \int_0^\infty |\partial_r u|^p r^{(\alpha+\beta+1)p+n-1} dr, \quad \forall \omega \in [0,\pi]^{n-2} \times [0, 2\pi].$$

Let $0 < \epsilon < \pi/4$ be fixed, the constant $C$ also depends on $\epsilon$. By the above we have

$$\int_\epsilon^{\pi-\epsilon} \int_E \int_0^\infty |u|^p r^{(\alpha+\beta)p+n-1} \Omega' dr d\omega' d\theta_1 \le C \int_\epsilon^{\pi-\epsilon} \int_E \int_0^\infty |\nabla u|^p r^{(\alpha+\beta+1)p+n-1} \Omega' dr d\omega' d\theta_1. \quad (40)$$

Similarly, we have

$$\int_\epsilon^{2\epsilon} \int_E \int_0^\infty |u|^p r^{(\alpha+\beta)p+n-1} \Omega' dr d\omega' d\theta_1 \le C \int_\epsilon^{2\epsilon} \int_E \int_0^\infty |\nabla u|^p r^{(\alpha+\beta+1)p+n-1} \Omega' dr d\omega' d\theta_1.$$

So there exists some $\bar{\theta}_1 \in [\epsilon, 2\epsilon]$, such that

$$\begin{aligned} &\int_E \int_0^\infty |u(r,\bar{\theta}_1,\omega')|^p r^{(\alpha+\beta)p+n-1} \Omega' dr d\omega' \\ &\le C \int_\epsilon^{2\epsilon} \int_E \int_0^\infty |\partial_r u|^p r^{(\alpha+\beta+1)p+n-1} \Omega' dr d\omega' d\theta_1. \end{aligned} \quad (41)$$

Notice for $\theta_1 \in [0, \frac{\pi}{2}]$, $\theta_1 \le \sin\theta_1 \le 2\theta_1$. By computation, using $\alpha p + n > 1$, for every fixed $r$ and $\omega'$,

$$\begin{aligned} &\int_0^{\bar{\theta}_1} |u(r,\theta_1,\omega')|^p |\sin\theta_1|^{\alpha p+n-2} d\theta_1 \\ &= \int_0^{\bar{\theta}_1} |\sin\theta_1|^{\alpha p+n-2} \left( |u(r,\bar{\theta}_1,\omega')|^p - \int_{\theta_1}^{\bar{\theta}_1} \partial_t |u(r,t,\omega')|^p dt \right) d\theta_1 \\ &\le C \int_0^{\bar{\theta}_1} \theta_1^{\alpha p+n-2} |u(r,\bar{\theta}_1,\omega')|^p d\theta_1 + C \int_0^{\bar{\theta}_1} |u(r,t,\omega')|^{p-1} |\partial_t u(r,t,\omega'|) \int_0^t \theta_1^{\alpha p+n-2} d\theta_1 dt \end{aligned}$$

$$\le C|u(r,\bar\theta_1,\omega')|^p + C\int_0^{\bar\theta_1} |\sin t|^{\alpha p+n-1}|u(r,t,\omega')|^{p-1}|\partial_t u|dt$$

$$\le C|u(r,\bar\theta_1,\omega')|^p + \frac{1}{2}\int_0^{\bar\theta_1} |u|^p|\sin t|^{\alpha p+n-2}dt + C\int_0^{\bar\theta_1} |\sin t|^{(\alpha+1)p+n-2}|\partial_t u|^p d\theta_1.$$

Thus

$$\int_0^{\bar\theta_1} |u(r,\theta_1,\omega')|^p|\sin\theta_1|^{\alpha p+n-2}d\theta_1$$

$$\le C|u(r,\bar\theta_1,\omega')|^p + C\int_0^{\bar\theta_1} |\sin\theta_1|^{(\alpha+1)p+n-2}|\partial_{\theta_1}u(r,\theta_1,\omega')|^p d\theta_1.$$

Multiply both sides of the above by $r^{(\alpha+\beta)p+n-1}\Omega'$, and take integral with respect to $r$ and $\omega'$. By (41), we have

$$\begin{aligned}
&\int_0^{\epsilon}\int_E\int_0^{\infty} |u|^p r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{\alpha p+n-2}\Omega' dr d\omega' d\theta_1\\
&\le \int_0^{\bar\theta_1}\int_E\int_0^{\infty} |u|^p r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{\alpha p+n-2}\Omega' dr d\omega' d\theta_1\\
&\le C\int_E\int_0^{\infty} |u(r,\bar\theta_1,\omega')|^p r^{(\alpha+\beta)p+n-1}\Omega' dr d\omega'\\
&+ C\int_0^{\bar\theta_1}\int_E\int_0^{\infty} r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{(\alpha+1)p+n-2}|\partial_{\theta_1}u|^p\Omega' dr d\omega' d\theta_1 \qquad (42)\\
&\le C\int_{\epsilon}^{2\epsilon}\int_E\int_0^{\infty} |\partial_r u|^p r^{(\alpha+\beta+1)p+n-1}\Omega' dr d\omega' d\theta_1\\
&+ C\int_0^{\bar\theta_1}\int_E\int_0^{\infty} r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{(\alpha+1)p+n-2}|\partial_{\theta_1}u|^p\Omega' dr d\omega' d\theta_1\\
&\le C\int_0^{2\epsilon}\int_E\int_0^{\infty} |\nabla u|^p r^{(\alpha+\beta+1)p+n-1}|\sin\theta_1|^{(\alpha+1)p+n-2}\Omega' dr d\omega' d\theta_1
\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_{\pi-\epsilon}^{\pi}\int_{E}\int_{0}^{\infty}|u|^{p}r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{\alpha p+n-2}\Omega' dr d\omega' d\theta_1\\&\leq C\int_{\pi-2\epsilon}^{\pi}\int_{E}\int_{0}^{\infty}|\nabla u|^{p}r^{(\alpha+\beta+1)p+n-1}|\sin\theta_1|^{(\alpha+1)p+n-2}\Omega' dr d\omega' d\theta_1\end{aligned}\tag{43}$$

By (40), (42) and (43), we have

$$\begin{aligned}&\int_{\mathbb{R}^n}|u|^{p}r^{(\alpha+\beta)p+n-1}|\sin\theta_1|^{\alpha p+n-2}\Omega' dr d\omega\\&\leq C\int_{\mathbb{R}^n}|\nabla u|^{p}r^{(\alpha+\beta+1)p+n-1}|\sin\theta_1|^{(\alpha+1)p+n-2}\Omega' dr d\omega,\end{aligned}$$

which is equivalent to (39). The theorem is proved. □

**Corollary 4.1.** *Let $K$ be a compact subset of $M$, $(c,\gamma)\in K$. Then there exists some positive constant $C$, depending only on $K$, such that for any $w\in\dot{H}^1(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3}|w|^2|\nabla u^{c,\gamma}|dx+\int_{\mathbb{R}^3}|w|^2|u^{c,\gamma}|^2dx\leq C(|c|+|\gamma|)\|\nabla w\|_{L^2}^2.$$

**Proof.** By Corollary 2.1, we have

$$|u^{c,\gamma}|\leq\frac{C(|c|+|\gamma|)}{\sqrt{|x||x'|}},\qquad |\nabla u^{c,\gamma}|\leq\frac{C(|c|+|\gamma|)}{|x||x'|}.$$

By Theorem 1.3 with $\alpha=\beta=-\frac{1}{2}$, $p=2$ and $n=3$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}|w|^2|u^{c,\gamma}|^2dx+\int_{\mathbb{R}^3}|w|^2|\nabla u^{c,\gamma}|dx\\&\leq C(|c|+|\gamma|)\int_{\mathbb{R}^3}\frac{|w|^2}{|x||x'|}dx\leq C(|c|+|\gamma|)\int_{\mathbb{R}^3}|x|^{-1}|x'||\nabla w|^2dx\\&\leq C(|c|+|\gamma|)\|\nabla w\|_{L^2}^2.\quad\square\end{aligned}$$

Notice (13) follows from Corollary 4.1.

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