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Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two

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Abstract

We study the insulated conductivity problem with inclusions embedded in a bounded domain in $\mathbb{R}^n$. The gradient of solutions may blow up as ε, the distance between inclusions, approaches to 0. An upper bound for the blow up rate was proved to be of order $\varepsilon^{-1/2}$. The upper bound was known to be sharp in dimension $n = 2$. However, whether this upper bound is sharp in dimension $n \geq 3$ has remained open. In this paper, we improve the upper bound in dimension $n \geq 3$ to be of order $\varepsilon^{-1/2+\beta}$, for some $\beta > 0$.

1 Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and let D_1^* and D_2^* be two open sets whose closure belongs to Ω, touching only at the origin with the inner normal vector of ∂D_1^* pointing in the positive x_n-direction. Translating D_1^* and D_2^* by $\frac{\varepsilon}{2}$ along x_n-axis, we obtain

$$D_1^\varepsilon := D_1^* + \left(0', \frac{\varepsilon}{2}\right), \quad \text{and} \quad D_2^\varepsilon := D_2^* - \left(0', \frac{\varepsilon}{2}\right).$$

When there is no confusion, we drop the superscripts ε and denote $D_1 := D_1^\varepsilon$ and $D_2 := D_2^\varepsilon$. Denote $\widetilde{\Omega} := \Omega \setminus \overline{(D_1 \cup D_2)}$, we consider the following elliptic equation

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with Dirichlet boundary data:

$$\begin{cases} \operatorname{div}\Big(a_k(x)\nabla u_k\Big) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0, \infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \widetilde{\Omega}. \end{cases}$$

The equation above can be considered as a simple model for electric conduction, where a_k refers to conductivities, which can be assumed to be 1 in the matrix after normalization, and the solution u_k gives the voltage potential. From an engineering point of view, it is very important to estimate ∇u_k, which represents the electric fields, in the narrow region between the inclusions. This problem is analogous to a linear system of elasticity studied by Babuška, Andersson, Smith and Levin [5], where they analyzed numerically that, when the ellitpicity constants are bounded away from 0 and infinity, gradient of solutions remain bounded independent of ε, the distance between inclusions. Bonnetier and Vogelius [12] proved that for a fixed k, $|\nabla u_k|$ is bounded when $\varepsilon = 0$, for circular inclusions D_1 and D_2 in dimension $n = 2$. This result was extended by Li and Vogelius [28] to general second order elliptic equation of divergence form with piecewise Hölder coefficients and general shape of inclusions D_1 and D_2 in any dimension. Furthermore, they established a stronger $C^{1,\alpha}$ control of u_k, which is independent of ε, in each region. Li and Nirenberg [27] further extended this $C^{1,\alpha}$ result to general second order elliptic systems of divergence form.

When k equals to ∞ (perfect conductor) or 0 (insulator), it was shown in [13, 22, 32] that the gradient of solutions generally becomes unbounded, as $\varepsilon \to 0$. When k goes to ∞, u_k converges to the solution of the following perfect conductivity problem:

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ u = C_i \text{ (Constants)} & \text{on } \partial D_i,\ i = 1, 2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu} = 0 & i = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

When k goes to 0, u_k converges to the solution of the following insulated conductivity problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i,\ i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

See, e.g., Appendix of [6, 7] for derivations of the above equations. Here ν denotes the inward unit normal vectors on ∂D_i, $i = 1, 2$.

Ammari et al. proved in [3, 4], among other things, the following. Let D_1^* and D_2^* be unit balls in $\mathbb{R}^2$, and let H be a harmonic function in $\mathbb{R}^2$. They considered the perfect and insulated conductivity problems in $\mathbb{R}^2$:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}, \\ u = C_i \text{ (Constants)} & \text{on } \partial D_i, \ i = 1, 2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu} = 0 & i = 1, 2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

and

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \backslash \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \ i = 1, 2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases}$$

In both cases, they proved that for some C independent of ε,

$$\|\nabla u\|_{L^\infty(B_4)} \le C\varepsilon^{-1/2}.$$

They also showed that the upper bounds are optimal in the sense that for appropriate H,

$$\|\nabla u\|_{L^\infty(B_4)} \ge \varepsilon^{-1/2}/C.$$

Yun extended in [34, 35] the results allowing D_1^* and D_2^* to be any bounded strictly convex smooth domains.

The above gradient estimates were localized and extended to higher dimensions by Bao, Li and Yin in [6, 7]. For the perfect conductor case, they considered problem (1.2) and proved in [6] that

$$\begin{cases} \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 2, \\ \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C|\varepsilon \ln \varepsilon|^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 3, \\ \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n \ge 4. \end{cases}$$

The above bounds were shown to be optimal in the paper. For further works on the perfect conductivity problem and closely related ones, see e.g. [1, 2, 8–11, 14, 16, 17, 19–21, 23–26, 31] and the references therein.

For the insulated problem (1.3), it was proved in [7] that

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \le C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)} \quad \text{when } n \ge 2. \tag{1.4}$$

The upper bound is optimal for $n = 2$ as mentioned earlier, while it was not known whether it is optimal in dimensions $n \ge 3$.

Yun [36] considered the following insulated problem in $\mathbb{R}^3$ minus 2 balls: Let H be a harmonic function in $\mathbb{R}^3$, $D_1 = B_1\left(0, 0, 1 + \frac{\varepsilon}{2}\right)$, and $D_2 = B_1\left(0, 0, -1 - \frac{\varepsilon}{2}\right)$,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^3 \backslash \overline{(D_1 \cup D_2)}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D_i, \ i = 1, 2, \\ u(x) - H(x) = O(|x|^{-2}) & \text{as } |x| \to \infty. \end{cases}$$

He proved that for some positive constant C independent of ε,

$$\max_{|x_3| \leq \varepsilon/2} |\nabla u(0, 0, x_3)| \leq C \varepsilon^{\frac{\sqrt{2}-2}{2}}.$$

He also showed that this upper bound of $|\nabla u|$ on the ε-segment connecting D_1 and D_2 is optimal for $H(x) \equiv x_1$. Although this result does not provide an upper bound of $|\nabla u|$ in the complement of the ε-segment, it has added support to a long time suspicion that the upper bound $\varepsilon^{-1/2}$ obtained for dimension n=3 in [7] is not optimal.

In this paper, we focus on the insulated conductivity problem (1.3) in dimension $n \geq 3$, and improve the upper bound (1.4) to the rate $\varepsilon^{-1/2+\beta}$, for some $\beta > 0$. Analogous questions for elliptic system are still open, and we give some discussions in Sect. 4. We point out that the insulator case for Lamé systems in dimension $n = 2$ was studied by Lim and Yu [30].

From now on, we assume that ∂D_1^* and ∂D_2^* are C^2, and they are relatively convex near the origin. That is, for some positive constants R_0, κ, we assume that when $0 < |x'| < R_0$, ∂D_1^* and ∂D_2^* are respectively the graphs of two C^2 functions f and g in terms of x', and

$$\begin{aligned} f(x') > g(x'), \quad & \text{for } 0 < |x'| < R_0, \\ f(0') = g(0') = 0, \quad & \nabla_{x'} f(0') = \nabla_{x'} g(0') = 0, \quad (1.5) \\ \nabla^2_{x'}(f - g)(x') \geq \kappa I_{n-1}, \quad & \text{for } 0 < |x'| < R_0, \quad (1.6) \end{aligned}$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. Let $a(x) \in C^\alpha(\overline{\widetilde{\Omega}})$, for some $\alpha \in (0, 1)$, be a symmetric, positive definite matrix function satisfying

$$\lambda \leq a(x) \leq \Lambda, \quad \text{for } x \in \widetilde{\Omega},$$

for some positive constants λ, Λ. Let $\nu = (\nu_1, \cdots, \nu_n)$ denote the unit normal vector on ∂D_1 and ∂D_2, pointing towards the interior of D_1 and D_2. We consider the following insulated conductivity problem:

$$\begin{cases} -\partial_i(a^{ij} \partial_j u) = 0 & \text{in } \widetilde{\Omega}, \\ a^{ij} \partial_j u \nu_i = 0 & \text{on } \partial(D_1 \cup D_2), \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.7)$$

where $\varphi \in C^2(\partial\Omega)$ is given.

For $0 < r \leq R_0$, we denote

$$\Omega_{x_0,r} := \left\{(x', x_n) \in \widetilde{\Omega} \,\Big|\, -\frac{\varepsilon}{2} + g(x') < x_n < \frac{\varepsilon}{2} + f(x'),\ |x' - x_0'| < r\right\},$$
$$\Gamma_+ := \left\{x_n = \frac{\varepsilon}{2} + f(x'),\ |x'| < R_0\right\},$$
$$\Gamma_- := \left\{x_n = -\frac{\varepsilon}{2} + g(x'),\ |x'| < R_0\right\}. \tag{1.8}$$

Since the blow-up of gradient can only occur in the narrow region between D_1 and D_2, we will focus on the following problem near the origin:

$$\begin{cases} -\partial_i(a^{ij}\partial_j u) = 0 & \text{in } \Omega_{0,R_0}, \\ a^{ij}\partial_j u \nu_i = 0 & \text{on } \Gamma_+ \cup \Gamma_-, \end{cases} \tag{1.9}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ denotes the unit normal vector on Γ_+ and Γ_-, pointing upward and downward respectively.

Here is our main result in the paper.

Theorem 1.1 *Let f, g, a, α be as above, and let $u \in H^1(\Omega_{0,R_0})$ be a solution of (1.9) in dimension $n \geq 3$. There exist positive constants r_0, β and C depending only on n, λ, Λ, R_0, κ, α, $\|a\|_{C^\alpha(\Omega_{0,R_0})}$, $\|f\|_{C^2(\{|x'|\leq R_0\})}$ and $\|g\|_{C^2(\{|x'|\leq R_0\})}$, such that*

$$|\nabla u(x_0)| \leq C\|u\|_{L^\infty(\Omega_{0,R_0})}\left(\varepsilon + |x_0'|^2\right)^{-1/2+\beta}, \tag{1.10}$$

for all $x_0 \in \Omega_{0,r_0}$ and $\varepsilon \in (0, 1)$.

Remark 1.2 After submitting this work, Weinkove [33] gave another proof of (1.10) in dimensions $n \geq 4$ with a more explicit β, for $a^{ij} = \delta_{ij}$ and D_1, D_2 being unit balls. The authors of this paper extended Theorem 1.1 to include flatter insulators in [29].

Let $u \in H^1(\widetilde{\Omega})$ be a weak solution of (1.7). By the maximum principle and the gradient estimates of solutions of elliptic equations,

$$\|u\|_{L^\infty(\widetilde{\Omega})} \leq \|\varphi\|_{L^\infty(\partial\Omega)}, \tag{1.11}$$

and

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega}\setminus\Omega_{0,r_0})} \leq C\|\varphi\|_{C^2(\partial\Omega)}.$$

Therefore, a corollary of Theorem 1.1 is as follows.

Corollary 1.3 *Let $u \in H^1(\widetilde{\Omega})$ be a weak solution of (1.7) in dimension $n \geq 3$. There exist positive constants β and C depending only on n, λ, Λ, R_0, $\|a\|_{C^\alpha}$, $\|\partial D_1\|_{C^2}$, $\|\partial D_2\|_{C^2}$, $\|\partial\Omega\|_{C^2}$, and the principal curvatures of ∂D_1 and ∂D_2, such that*

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq C\|\varphi\|_{C^2(\partial\Omega)}\varepsilon^{-\frac{1}{2}+\beta}. \tag{1.12}$$

Remark 1.4 If there are more than two inclusions, estimate (1.12) still holds, with ε being the minimal distance between inclusions.

The rest of this paper will be organized as follows. In Sect. 2, we prove a lemma which is used in the proof of Theorem 1.1. Theorem 1.1 is proved in Sect. 3. In Sect. 4, we give a gradient estimate to a problem for elliptic systems analogous to problem (1.9).

2 A regularity lemma

In this section, we give a regularity lemma for elliptic systems (elliptic equations when $N = 1$). Let us first describe the nature of domains and operators. We define S to be a cylinder

$$S = \{(x', x_n) \in \mathbb{R}^n \,\big|\, |x'| < 1, |x_n| < 1\},$$

and some constants c_m, with $0 \le m \le l$, such that

$$-1 = c_0 < c_1 < \cdots < c_l = 1,$$

and denote the integer m_0 to be the integer such that

$$c_{m_0 - 1} \le 0 < c_{m_0}.$$

We divide the domain S into l parts by setting

$$\Omega_m = \{x \in S \,\big|\, c_{m-1} < x_n < c_m\}, \quad \text{for } 1 \le m \le l.$$

For $1 \le \alpha, \beta \le n$, $1 \le i, j \le N$, let $A_{ij}^{\alpha\beta}(x)$ be a function such that

$$\begin{aligned}
&\|A_{ij}^{\alpha\beta}\|_{L^\infty(S)} \le \Lambda,\\
&\int_S A_{ij}^{\alpha\beta}(x)\partial_\alpha\varphi_i(x)\partial_\beta\varphi_j(x) \ge \lambda \int_S |\nabla\varphi|^2, \quad \forall \varphi \in H_0^1(S; \mathbb{R}^N),
\end{aligned}$$

for some $\lambda, \Lambda > 0$, and for each $1 \le m \le l$, $A_{ij}^{\alpha\beta}(x) \in C^\mu(\overline{\Omega}_m)$, for some $0 < \mu < 1$. We denote $(A_{ij}^{\alpha\beta}(x))$ by $A(x)$.

For $1 \le \alpha \le n$, $1 \le i \le N$, let

$$\begin{aligned}
H(x) &= \{H_i\} \in L^\infty(S),\\
G(x) &= \{G_i^\alpha\} \in C^\mu(\overline{\Omega}_m),
\end{aligned}$$

for all $m = 1, \cdots, l$. Then we have the following interior gradient estimate.

Lemma 2.1 *Let $A(x)$, $H(x)$ and $G(x)$ be as above. There exists a positive constant C, depending only on n, μ, λ, Λ and an upper bound of $\{\|A\|_{C^\mu(\overline{\Omega}_m)}\}_{m=1}^{l}$, such that if $u \in H^1(S; \mathbb{R}^N)$ is a weak solution to*

$$\partial_\alpha (A_{ij}^{\alpha\beta}(x)\partial_\beta u_j) = H_i + \partial_\alpha G_i^\alpha \quad \text{in } S,$$

then

$$\|u\|_{L^\infty(\frac{1}{2}S)} + \|\nabla u\|_{L^\infty(\frac{1}{2}S)} \le C\left(\|u\|_{L^2(S)} + \|H\|_{L^\infty(S)} + \max_{1\le m\le l} \|G\|_{C^\mu(\overline{\Omega}_m)}\right).$$

Remark 2.2 When $A(x)$, $H(x)$ and $G(x)$ are constants on each Ω_m, the estimate was first proved by Chipot, Kinderlehrer and Vergara-Caffarelli [15, Theorem 2]. See also [28, Proposition 2.1] and [27, Proposition 1.6].

Remark 2.3 We point out that the constant C in the Lemma is independent of l.

Proof The proof of Lemma 2.1 is a modification of the proof of Proposition 4.1 in [27]. Even though the constant C in [27, Proposition 4.1] depends on l, the number of subdomains we divide in the domain S, this dependence only enters in estimating the quantities $\|A - \bar{A}\|_{Y^{1+\alpha,2}}$, $\|G - \bar{G}\|_{Y^{1+\alpha,2}}$, and $\|H - \bar{H}\|_{Y^{\alpha,2}}$ which will be defined below. We will show that such quantity is independent of l due to the nature of our domain S, and hence the constant C in Lemma 2.1 is independent of l. □

For $s > 0$, $1 < p < \infty$, we define the norm

$$\|f\|_{Y^{s,p}} := \sup_{0<r\le 1} r^{1-s}\left(\fint_{rS} |f|^p\right)^{1/p}.$$

We define a piecewise-constant coefficients $\bar{A}$ associated to A by setting

$$\bar{A}(x) := \begin{cases} \lim\limits_{x\in\Omega_m, x\to(0',c_{m-1})} A(x), & \text{if } x \in \Omega_m, m > m_0; \\ A(0), & \text{if } x \in \Omega_{m_0}; \\ \lim\limits_{x\in\Omega_m, x\to(0',c_m)} A(x), & \text{if } x \in \Omega_m, m < m_0. \end{cases}$$

Similarly, we can define piecewise-constant tensor $\bar{G}$ associated to G. We also define a constant vector $\bar{H}$ associated to H by

$$\bar{H} := \fint_S H.$$

Lemma 2.4 *Let A, $\bar{A}$, H, $\bar{H}$, G, $\bar{G}$ be as above. Then there exists a positive constant C, depending only on n, such that*

$$\|A-\bar{A}\|_{Y^{1+\mu,2}} \le C \max_{1\le m\le l} \|A\|_{C^\alpha(\overline{S}_m)},$$
$$\|G-\bar{G}\|_{Y^{1+\mu,2}} \le C \max_{1\le m\le l} \|G\|_{C^\alpha(\overline{S}_m)},$$
$$\|H-\bar{H}\|_{Y^{\mu,2}} \le C\|H\|_{L^\infty(S)}.$$

Proof The last inequality follows immediately from the definition of $Y^{\mu,2}$ and $\bar{H}$:

$$\|H-\bar{H}\|_{Y^{\mu,2}} \le \sup_{0<r\le 1} r^{1-\mu}\left(⨍_{rS}|H-\bar{H}|^2\right)^{1/2} \le C\|H\|_{L^\infty(S)}.$$

By a direct computation, we will have

$$\begin{aligned}
\left(⨍_{rS}|A-\bar{A}|^2\right)^{1/2} &\le \left(\frac{1}{|rS|}\sum_{m=1}^{l}\int_{rS\cap S_m}|A(x)-\bar{A}(x)|^2\,dx\right)^{1/2}\\
&\le \Bigg[\frac{1}{|rS|}\Bigg(\sum_{m=1}^{m_0-1}\|A\|^2_{C^\mu(\overline{S}_m)}\int_{rS\cap S_m}|x-(0',c_m)|^{2\mu}\,dx\\
&\quad+\|A\|^2_{C^\mu(\overline{S}_{m_0})}\int_{rS\cap S_{m_0}}|x|^{2\mu}\,dx\\
&\quad+\sum_{m=m_0+1}^{l}\|A\|^2_{C^\mu(\overline{S}_{m-1})}\int_{rS\cap S_{m-1}}|x-(0',c_{m-1})|^{2\mu}\,dx\Bigg)\Bigg]^{1/2}\\
&\le \max_{1\le m\le l}\|A\|_{C^\mu(\overline{S}_m)}\left(⨍_{rS}|x|^{2\mu}\,dx\right)^{1/2}\\
&\le C\max_{1\le m\le l}\|A\|_{C^\mu(\overline{S}_m)}r^\mu.
\end{aligned}$$

This proves the first inequality. The second inequality follows similarly. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. For a small r_0 independent of ε, and any $x_0 \in \Omega_{0,r_0}$, we estimate $|\nabla u(x_0)|$ as follows: First we establish a Harnack inequality in $\Omega_{x_0,r}\setminus\Omega_{x_0,r/2}$, for $r>0$ in a suitable range. Together with the maximum principle, this gives the oscillation of u in $\Omega_{x_0,\delta}$ a decay $\delta^{2\beta}$, for some positive ε-independent β, where

$$\delta := (\varepsilon+|x_0'|^2)^{1/2}.$$

Then we perform a suitable change of variables in $\Omega_{x_0,\delta/4}$, and apply Lemma 2.1 to obtain the desired estimate on $|\nabla u(x_0)|$.

We fix a $\gamma \in (0,1)$, and let $r_0 > 0$ denote a constant depending only on $n, \kappa, \gamma, R_0, \|f\|_{C^2}$ and $\|g\|_{C^2}$, whose value will be fixed in the proof. We will always consider $0 < \varepsilon \leq r_0^2$. First, we require r_0 small so that for $|x_0'| < r_0$,

$$10\delta < \delta^{1-\gamma} < R_0/4.$$

Lemma 3.1 *There exists a small r_0, depending only on $n, \kappa, \gamma, R_0, \|f\|_{C^2}$ and $\|g\|_{C^2}$, such that for any $x_0 \in \Omega_{0,r_0}$, $5\delta < r < \delta^{1-\gamma}$, if $u \in H^1(\Omega_{x_0,2r} \setminus \Omega_{x_0,r/4})$ is a positive solution to the equation*

$$\begin{cases} -\partial_i(a^{ij}(x)\partial_j u(x)) = 0 & \text{in } \Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}, \\ a^{ij}(x)\partial_j u(x)\nu_i(x) = 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}}, \end{cases}$$

then,

$$\sup_{\Omega_{x_0,r}\setminus\Omega_{x_0,r/2}} u \leq C \inf_{\Omega_{x_0,r}\setminus\Omega_{x_0,r/2}} u, \tag{3.1}$$

for some constant $C > 0$ depending only on $n, \kappa, \lambda, \Lambda, R_0, \|f\|_{C^2}$ and $\|g\|_{C^2}$, but independent of r and u.

Proof We only need to prove (3.1) for $|x_0'| > 0$, since the $|x_0'| = 0$ case follows from the result for $|x_0'| > 0$ and then sending $|x_0'|$ to 0. We denote

$$h_r := \varepsilon + f\left(x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right) - g\left(x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right),$$

and perform a change of variables by setting

$$\begin{cases} y' = x' - x_0', \\ y_n = 2h_r\left(\dfrac{x_n - g(x') + \varepsilon/2}{\varepsilon + f(x') - g(x')} - \dfrac{1}{2}\right), \end{cases} \quad (x', x_n) \in \Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}. \tag{3.2}$$

This change of variables maps the domain $\Omega_{x_0,2r}\setminus\Omega_{x_0,r/4}$ to an annular cylinder of height h_r, denoted by $Q_{2r,h_r}\setminus Q_{r/4,h_r}$, where

$$Q_{s,t} := \{y = (y', y_n) \in \mathbb{R}^n \mid |y'| < s, |y_n| < t\}, \tag{3.3}$$

for $s, t > 0$. We will show that the Jacobian matrix of the change of variables (3.2), denoted by $\partial_x y$, and its inverse matrix $\partial_y x$ satisfy

$$|(\partial_x y)^{ij}| \leq C, \quad |(\partial_y x)^{ij}| \leq C, \quad \text{for } y \in Q_{2r,h_r}\setminus Q_{r/4,h_r}, \tag{3.4}$$

where $C > 0$ depends only on $n, \kappa, R_0, \|f\|_{C^2}$ and $\|g\|_{C^2}$.

Let $v(y) = u(x)$, then v satisfies

$$\begin{cases} -\partial_i (b^{ij}(y)\partial_j v(y)) = 0 & \text{in } Q_{2r,h_r} \backslash Q_{r/4,h_r}, \\ \qquad b^{nj}(y)\partial_j v(y) = 0 & \text{on } \{y_n = -h_r\} \cup \{y_n = h_r\}, \end{cases} \tag{3.5}$$

where the matrix $(b^{ij}(y))$ is given by

$$(b^{ij}(y)) = \frac{(\partial_x y)(a^{ij})(\partial_x y)^t}{\det(\partial_x y)}, \tag{3.6}$$

$(\partial_x y)^t$ is the transpose of $\partial_x y$.

It is easy to see that (3.4) implies, using $\lambda \le (a^{ij}) \le \Lambda$,

$$\frac{\lambda}{C} \le (b^{ij}(y)) \le C\Lambda, \quad \text{for } y \in Q_{2r,h_r} \backslash Q_{r/4,h_r}, \tag{3.7}$$

for some constant $C > 0$ depending only on n, R_0, κ, $\|f\|_{C^2}$ and $\|g\|_{C^2}$.

In the following and throughout this section, we will denote $A \sim B$, if there exists a positive universal constant C, which might depend on $n, \lambda, \Lambda, R_0, \kappa, \|f\|_{C^2}$, and $\|g\|_{C^2}$, but not depend on ε, such that $C^{-1}B \le A \le CB$.

From (3.2), one can compute that

$$\begin{aligned}
(\partial_x y)^{ii} &= 1, \quad \text{for } 1 \le i \le n-1, \\
(\partial_x y)^{nn} &= \frac{2h_r}{\varepsilon + f(x_0' + y') - g(x_0' + y')}, \\
(\partial_x y)^{ni} &= -\frac{2h_r \partial_i g(x_0' + y') + 2y_n[\partial_i f(x_0' + y') - \partial_i g(x_0' + y')]}{\varepsilon + f(x_0' + y') - g(x_0' + y')}, \\
&\quad \text{for } 1 \le i \le n-1, \\
(\partial_x y)^{ij} &= 0, \quad \text{for } 1 \le i \le n-1, j \ne i.
\end{aligned}$$

By (1.5) and (1.6), one can see that

$$h_r \sim \varepsilon + \left| x_0' - \frac{r}{4} \frac{x_0'}{|x_0'|} \right|^2.$$

Since $|y_n| \le h_r$, by using (1.5) and (1.6), we have that, for $1 \le i \le n-1$,

$$\begin{aligned}\left|(\partial_x y)^{ni}\right| &\le C\frac{h_r|\partial_i g(x_0'+y')| + h_r[|\partial_i f(x_0'+y')| + |\partial_i g(x_0'+y')|]}{\varepsilon + f(x_0'+y') - g(x_0'+y')}\\ &\le C\frac{h_r}{\varepsilon + f(x_0'+y') - g(x_0'+y')}\left[|\partial_i f(x_0'+y')| + |\partial_i g(x_0'+y')|\right]\\ &\le C\frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^2}{\varepsilon + |x_0'+y'|^2}|x_0'+y'|.\end{aligned}$$

Since $r/4 < |y'| < 2r < 2\delta^{1-\gamma}$ and $|x_0'| < \delta$, we can estimate

$$\left|(\partial_x y)^{ni}\right| \le C|x_0'+y'| \le C(|x_0'| + |y'|) \le C\delta^{1-\gamma}.$$

Next, we will show that

$$(\partial_x y)^{nn} \sim 1, \quad \text{for } y \in Q_{2r,h_r} \setminus Q_{r/4,h_r}. \tag{3.8}$$

Indeed, by (1.5) and (1.6), we have

$$(\partial_x y)^{nn} = \frac{2h_r}{\varepsilon + f(x_0'+y') - g(x_0'+y')} \sim \frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^2}{\varepsilon + |x_0'+y'|^2}.$$

Since $|y'| > r/4$, it is easy to see

$$(\partial_x y)^{nn} \le C\frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^2}{\varepsilon + |x_0'+y'|^2} \le C.$$

On the other hand, since $|y'| < 2r$ and $|x_0'| < \delta < r/5$, we have

$$\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^2 \ge \varepsilon + \left(\left|\frac{r}{4}\frac{x_0'}{|x_0'|}\right| - |x_0'|\right)^2 \ge \varepsilon + \left(\frac{r}{4} - \frac{r}{5}\right)^2 = \varepsilon + \frac{1}{400}r^2,$$

and

$$\varepsilon + |x_0'+y'|^2 \le \varepsilon + 2|x_0'|^2 + 2|y'|^2 \le \varepsilon + \frac{2}{25}r^2 + 8r^2 < \varepsilon + 9r^2.$$

Therefore,

$$(\partial_x y)^{nn} \ge \frac{1}{C}\frac{\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^2}{\varepsilon + |x_0'+y'|^2} \ge \frac{1}{C}\frac{\varepsilon + r^2/400}{\varepsilon + 9r^2} \ge \frac{1}{C},$$

and (3.8) is verified.

We have shown $(\partial_x y)^{ii} \sim 1$, for all $i = 1, \cdots, n$, and $|(\partial_x y)^{ij}| \leq C\delta^{1-\gamma}$, for $i \neq j$. We further require r_0 to be small enough so that off-diagonal entries of $\partial_x y$ are small. Therefore (3.4) follows. As mentioned earlier, (3.7) follows from (3.4).

Now we define, for any integer l,

$$A_l := \left\{ y \in \mathbb{R}^n \mid \frac{r}{4} < |y'| < 2r, \ (2l-1)h_r < z_n < (2l+1)h_r \right\}.$$

Note that $A_0 = Q_{2r,h_r} \backslash Q_{r/4,h_r}$. For any $l \in \mathbb{Z}$, we define a new function $\tilde{v}$ by

$$\tilde{v}(y) := v\left(y', (-1)^l (y_n - 2lh_r)\right), \quad \forall y \in A_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \cdots, n-1$,

$$\tilde{b}^{nk}(y) = \tilde{b}^{kn}(y) := (-1)^l b^{nk}\left(y', (-1)^l (y_n - 2lh_r)\right), \quad \forall y \in A_l,$$

and for other indices,

$$\tilde{b}^{ij}(y) := b^{ij}\left(y', (-1)^l (y_n - 2lh_r)\right), \quad \forall y \in A_l.$$

Therefore, $\tilde{v}(y)$ and $\tilde{b}^{ij}(y)$ are defined in the infinite cylinder shell $Q_{2r,\infty} \backslash Q_{r/4,\infty}$. By (3.5), $\tilde{v} \in H^1(Q_{2r,\infty} \backslash Q_{r/4,\infty})$ satisfies

$$-\partial_i (\tilde{b}^{ij}(y) \partial_j \tilde{v}(y)) = 0 \quad \text{in } Q_{2r,\infty} \backslash Q_{r/4,\infty}.$$

Note that for any $l \in \mathbb{Z}$ and $y \in A_l$, $\tilde{b}(y) = (\tilde{b}^{ij}(y))$ is orthogonally conjugated to $b\left(y', (-1)^l (y_n - 2lh_r)\right)$. Hence, by (3.7), we have

$$\frac{\lambda}{C} \leq \tilde{b}(y) \leq C\Lambda, \quad \text{for } y \in Q_{2r,\infty} \backslash Q_{r/4,\infty}.$$

We restrict the domain to be $Q_{2r,r} \backslash Q_{r/4,r}$, and make the change of variables $z = y/r$. Set $\bar{v}(z) = \tilde{v}(y)$, $\bar{b}^{ij}(z) = \tilde{b}^{ij}(y)$, we have

$$-\partial_i (\bar{b}^{ij}(z) \partial_j \bar{v}(z)) = 0 \quad \text{in } Q_{2,1} \backslash Q_{1/4,1},$$

and

$$\frac{\lambda}{C} \leq \bar{b}(z) \leq C\Lambda, \quad \text{for } z \in Q_{2,1} \backslash Q_{1/4,1}.$$

Then by the Harnack inequality for uniformly elliptic equations of divergence form, see e.g. [18, Theorem 8.20], there exists a constant C depending only on $n, \kappa, \lambda, \Lambda, R_0, \|f\|_{C^2}$ and $\|g\|_{C^2}$, such that

$$\sup_{Q_{1,1/2} \backslash Q_{1/2,1/2}} \bar{v} \leq C \inf_{Q_{1,1/2} \backslash Q_{1/2,1/2}} \bar{v}.$$

We further require r_0 to be small enough, so that

$$h_r \le C\left(\varepsilon + \left|x_0' - \frac{r}{4}\frac{x_0'}{|x_0'|}\right|^2\right) \le C(\varepsilon + r^2) \le C(\delta^2 + r^2) < \delta + \frac{r}{4} < \frac{r}{2}.$$

In particular, we have

$$\sup_{Q_{1,h_r/r}\setminus Q_{1/2,h_r/r}} \bar{v} \le C \inf_{Q_{1,h_r/r}\setminus Q_{1/2,h_r/r}} \bar{v},$$

which is (3.1) after reversing the change of variables. □

Remark 3.2 If dimension $n = 2$, Lemma 3.1 fails since $Q_{2,1}\setminus Q_{1/4,1} \subset \mathbb{R}^2$ is the union of two disjoint rectangular domains, and the Harnack inequality cannot be applied on it. In fact, in our proof of Theorem 1.1, Lemma 3.1 is the only ingredient where dimension $n \ge 3$ is used. As mentioned above, the conclusion of Theorem 1.1 does not hold in dimension $n = 2$.

For any domain $A \subset \widetilde{\Omega}$, we denote the oscillation of u in A by $\mathrm{osc}_A u := \sup_A u - \inf_A u$. Using Lemma 3.1, we obtain a decay of $\mathrm{osc}_{\Omega_{x_0,\delta}} u$ in δ as follows.

Lemma 3.3 *Let u be a solution of (1.9). For any $x_0 \in \Omega_{0,r_0}$, where r_0 is as in Lemma 3.1, there exist positive constants σ and C, depending only on n, R_0, κ, $\|f\|_{C^2}$ and $\|g\|_{C^2}$, such that*

$$osc_{\Omega_{x_0,\delta}} u \le C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma}. \tag{3.9}$$

Proof For simplicity, we drop the x_0 subscript and denote $\Omega_r = \Omega_{x_0,r}$ in this proof. Let $5\delta < r < \delta^{1-\gamma}$ and $u_1 = \sup_{\Omega_{2r}} u - u$, $u_2 = u - \inf_{\Omega_{2r}} u$. By Lemma 3.1, we have

$$\sup_{\Omega_r\setminus\Omega_{r/2}} u_1 \le C_1 \inf_{\Omega_r\setminus\Omega_{r/2}} u_1,$$
$$\sup_{\Omega_r\setminus\Omega_{r/2}} u_2 \le C_1 \inf_{\Omega_r\setminus\Omega_{r/2}} u_2,$$

where $C_1 > 1$ is a constant independent of r. Since both u_1 and u_2 satisfy equation (1.9), by the maximum principle,

$$\sup_{\Omega_r\setminus\Omega_{r/2}} u_i = \sup_{\Omega_r} u_i, \quad \inf_{\Omega_r\setminus\Omega_{r/2}} u_i = \inf_{\Omega_r} u_i,$$

for $i = 1, 2$. Therefore,

$$\sup_{\Omega_r} u_1 \le C_1 \inf_{\Omega_r} u_1,$$
$$\sup_{\Omega_r} u_2 \le C_1 \inf_{\Omega_r} u_2.$$

Adding up the above two inequalities, we have

$$\operatorname{osc}_{\Omega_r} u \leq \left(\frac{C_1 - 1}{C_1 + 1}\right) \operatorname{osc}_{\Omega_{2r}} u.$$

Now we take $\sigma > 0$ such that $2^{-\sigma} = \frac{C_1-1}{C_1+1}$, then

$$\operatorname{osc}_{\Omega_r} u \leq 2^{-\sigma} \operatorname{osc}_{\Omega_{2r}} u. \tag{3.10}$$

We start with $r = r_0 = \delta^{1-\gamma}/2$, and set $r_{i+1} = r_i/2$. Keep iterating (3.10) $k+1$ times, where k satisfies $5\delta \leq r_k < 10\delta$, we will have

$$\operatorname{osc}_{\Omega_\delta} u \leq \operatorname{osc}_{\Omega_{r_k}} u \leq 2^{-(k+1)\sigma} \operatorname{osc}_{\Omega_{2r_0}} u \leq 2^{1-(k+1)\sigma} \|u\|_{L^\infty(\Omega_{\delta^{1-\gamma}})}.$$

Since $10\delta > r^k = 2^{-k} r_0 = 2^{-(k+1)}\delta^{1-\gamma}$, we have $2^{-(k+1)} < 10\delta^\gamma$, and hence (3.9) follows immediately. □

Proof of Theorem 1.1 Let $u \in H^1(\Omega_{0,R_0})$ be a solution of (1.9). For $x_0 \in \Omega_{0,r_0}$, we have, using Lemma 3.3,

$$\|u - u_0\|_{L^\infty(\Omega_{x_0,\delta})} \leq C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma}, \tag{3.11}$$

for some constant u_0. We denote $v := u - u_0$, and v satisfies the same equation (1.9). We work on the domain $\Omega_{x_0,\delta/4}$, and perform a change of variables by setting

$$\begin{cases} y' = \delta^{-1}(x' - x_0'), \\ y_n = \delta^{-1}x_n. \end{cases} \tag{3.12}$$

The domain $\Omega_{x_0,\delta/4}$ becomes

$$\left\{ y \in \mathbb{R}^n \;\middle|\; |y'| \leq \frac{1}{4}, \delta^{-1}\left(-\frac{1}{2}\varepsilon + g(x_0'+\delta y')\right) < y_n < \delta^{-1}\left(\frac{1}{2}\varepsilon + f(x_0'+\delta y')\right)\right\}.$$

We make a change of variables again by

$$\begin{cases} z' = 4y', \\ z_n = 2\delta\left(\frac{\delta y_n - g(x_0'+\delta y')+\varepsilon/2}{\varepsilon + f(x_0'+\delta y') - g(x_0'+\delta y')} - \frac{1}{2}\right). \end{cases} \tag{3.13}$$

Now the domain in z-variables becomes a thin plate $Q_{1,\delta}$, where $Q_{s,t}$ is defined as in (3.3). Let $w(z) = v(x)$, then w satisfies

$$\begin{cases} -\partial_i(b^{ij}(z)\partial_j w(z)) = 0 \quad \text{in } Q_{1,\delta}, \\ b^{nj}(z)\partial_j w(z) = 0 \quad \text{on } \{z_n = -\delta\} \cup \{z_n = \delta\}, \end{cases} \tag{3.14}$$

where the matrix $b(z) = (b^{ij}(z))$ is given by

$$(b^{ij}(z)) = \frac{(\partial_y z)(a^{ij})(\partial_y z)^t}{\det(\partial_y z)}. \tag{3.15}$$

Similar to the proof of Lemma 3.1, we will show that the Jacobian matrix of the change of variables (3.13), denoted by $\partial_y z$, and its inverse matrix $\partial_z y$ satisfy

$$|(\partial_y z)^{ij}| \le C, \quad |(\partial_z y)^{ij}| \le C, \quad \text{for } z \in Q_{1,\delta}, \tag{3.16}$$

where $C > 0$ depends only on n, κ, R_0, $\|f\|_{C^2}$ and $\|g\|_{C^2}$. This leads to

$$\frac{\lambda}{C} \le b(z) \le C\Lambda, \quad \text{for } z \in Q_{1,\delta}. \tag{3.17}$$

From (3.13), one can compute that

$$\begin{aligned}
(\partial_y z)^{ii} &= 4, \quad \text{for } 1 \le i \le n-1,\\
(\partial_y z)^{nn} &= \frac{2\delta^2}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)},\\
(\partial_y z)^{ni} &= -\frac{2\delta^2 \partial_i g(x_0' + \delta z'/4) + 2z_n\delta[\partial_i f(x_0' + \delta z'/4) - \partial_i g(x_0' + \delta z'/4)]}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)}\\
&\qquad \text{for } 1 \le i \le n-1,\\
(\partial_y z)^{ij} &= 0, \quad \text{for } 1 \le i \le n-1,\ j \ne i.
\end{aligned}$$

First we will show that

$$(\partial_y z)^{nn} \sim 1, \quad \text{for } z \in Q_{1,\delta}. \tag{3.18}$$

Since $|z'| < 1$ and $|x_0'| < \delta$, it is easy to see that

$$(\partial_y z)^{nn} \sim \frac{\delta^2}{\varepsilon + |x_0' + \delta z'/4|^2} \ge \frac{\delta^2}{\varepsilon + C\delta^2} \ge \frac{1}{C}, \quad \text{for } z \in Q_{1,\delta},$$

due to (1.5) and (1.6). On the other hand,

$$
\begin{aligned}
(\partial_y z)^{nn} &\sim \frac{\delta^2}{\varepsilon + |x_0' + \delta z'/4|^2} \\
&= \frac{\delta^2}{\varepsilon + |x_0'|^2 + (1/4)^2\delta^2|z'|^2 + \delta x_0' \cdot z'/2} \\
&\leq \frac{\delta^2}{\delta^2 + (1/4)^2|z'|^2\delta^2 - |z'||x_0'|\delta/2} \\
&\leq \frac{\delta^2}{(1 + (1/4)^2|z'|^2 - 1/2)\delta^2} \leq C, \quad \text{for } z \in Q_{1,\delta}.
\end{aligned}
$$

Therefore, (3.18) is verified.

Since $|z_n| < \delta$, $|z'| < 1$ and $|x_0'| < \delta$, by (1.5) and (1.6), for $1 \leq i \leq n-1$,

$$
\begin{aligned}
|(\partial_y z)^{ni}| &\leq \frac{2\delta^2|\partial_i g(x_0' + \delta z'/4)| + 2\delta^2[|\partial_i f(x_0' + \delta z'/4)| + |\partial_i g(x_0' + \delta z'/4)|]}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)} \\
&\leq \frac{C\delta^2}{\varepsilon + f(x_0' + \delta z'/4) - g(x_0' + \delta z'/4)}[|\partial_i f(x_0' + \delta z'/4)| \\
&\quad + |\partial_i g(x_0' + \delta z'/4)|] \\
&\leq C\frac{\delta^2}{\varepsilon + |x_0' + \delta z'/4|^2}|x_0' + \delta z'/4| \\
&\leq C(|x_0'| + \delta|z'|) \leq C\delta,
\end{aligned}
$$

where in the last line, we have used the same arguments in showing $(\partial_y z)^{nn} \leq C$ earlier.

We have shown $(\partial_y z)^{ii} \sim 1$, for all $i = 1, \cdots, n$, and $|(\partial_y z)^{ij}| \leq C\delta$, for $i \neq j$. We further require r_0 to be small enough so that off-diagonal entries are small. Therefore (3.16) follows. As mentioned earlier, (3.17) follows from (3.16).

Next, we will show

$$
\|b\|_{C^\alpha(\overline{Q}_{1,\delta})} \leq C, \tag{3.19}
$$

for some $C > 0$ depending only on $n, \kappa, R_0, \|a\|_{C^\alpha}, \|f\|_{C^2}$ and $\|g\|_{C^2}$, by showing

$$
|\nabla_z(\partial_y z)^{ij}(z)| \leq C, \quad \left|\nabla_z \frac{1}{\det(\partial_y z)}\right| \leq C, \quad \text{for } z \in Q_{1,\delta}. \tag{3.20}
$$

Then (3.19) follows from (3.20), (3.15), and $\|a\|_{C^\alpha} \leq C$.

By a straightforward computation, we have, for any $i=1,\cdots,n-1$,

$$\begin{aligned}\left|\partial_{z_i}\frac{1}{\det(\partial_y z)}\right| &= \left|\partial_{z_i}\left(\frac{\varepsilon+f(x_0'+\delta z'/4)-g(x_0'+\delta z'/4)}{2\cdot 4^{n-1}\delta^2}\right)\right|\\ &= \left|\frac{\delta[\partial_i f(x_0'+\delta z'/4)-\partial_i g(x_0'+\delta z'/4)]}{2\cdot 4^{n-1}\delta^2}\right|\\ &\le \frac{C}{\delta}[|\partial_i f(x_0'+\delta z'/4)|+|\partial_i g(x_0'+\delta z'/4)|]\\ &\le \frac{C}{\delta}|x_0'+\delta z'/4|\le C,\quad \text{for } z\in Q_{1,\delta},\end{aligned}$$

where in the last inequality, (1.5) and (1.6) have been used. For any $i=1,\cdots,n-1$,

$$\begin{aligned}|\partial_{z_i}(\partial_y z)^{nn}| &= \left|\frac{2\delta^3[\partial_i f(x_0'+\delta z'/4)-\partial_i g(x_0'+\delta z'/4)]}{(\varepsilon+f(x_0'+\delta z'/4)-g(x_0'+\delta z'/4))^2}\right|\\ &\le \frac{C\delta^3}{(\varepsilon+|x_0'+\delta z'/4|^2)^2}|x_0'+\delta z'/4|\\ &\le \frac{C\delta^3}{\delta^4}(|x_0'|+|\delta z'|)\le C,\quad \text{for } z\in Q_{1,\delta},\end{aligned}$$

where in the last line, we have used the same arguments in showing $(\partial_y z)^{nn}\le C$ earlier. Similar computations apply to $\partial_{z_i}(\partial_y z)^{ni}$, for $i=1,\cdots,n-1$, and we have

$$|\partial_{z_i}(\partial_y z)^{ni}|\le C,\quad \text{for } z\in Q_{1,\delta}.$$

Finally, we compute, for $i=1,\cdots,n-1$,

$$\begin{aligned}|\partial_{z_n}(\partial_y z)^{ni}| &= \left|\frac{2\delta[\partial_i f(x_0'+\delta z'/4)-\partial_i g(x_0'+\delta z'/4)]}{\varepsilon+f(x_0'+\delta z'/4)-g(x_0'+\delta z'/4)}\right|\\ &\le \frac{C\delta|x_0'+\delta z'/4|}{\varepsilon+|x_0'+\delta z'/4|^2}\le C,\quad \text{for } z\in Q_{1,\delta}.\end{aligned}$$

Therefore, (3.20) is verified, and hence (3.19) follows as mentioned above.

Now we define

$$S_l := \left\{z\in\mathbb{R}^n \;\middle|\; |z'|<1,\ (2l-1)\delta<z_n<(2l+1)\delta\right\}$$

for any integer l, and

$$S := \left\{z\in\mathbb{R}^n \;\middle|\; |z'|<1,\ |z_n|<1\right\}.$$

Note that $Q_{1,\delta} = S_0$. As in the proof of Lemma 3.1, we define, for any $l \in \mathbb{Z}$, a new function $\tilde{w}$ by setting

$$\tilde{w}(z) := w\left(z', (-1)^l (z_n - 2l\delta)\right), \quad \forall z \in S_l.$$

We also define the corresponding coefficients, for $k = 1, 2, \cdots, n-1$,

$$\tilde{b}^{nk}(z) = \tilde{b}^{kn}(z) := (-1)^l b^{nk}\left(z', (-1)^l (z_n - 2l\delta)\right), \quad \forall z \in S_l,$$

and for other indices,

$$\tilde{b}^{ij}(z) := b^{ij}\left(z', (-1)^l (z_n - 2l\delta)\right), \quad \forall y \in S_l.$$

Then $\tilde{w}$ and $\tilde{b}^{ij}$ are defined in the infinite cylinder $Q_{1,\infty}$. By (3.14), $\tilde{w}$ satisfies the equation

$$-\partial_i(\tilde{b}^{ij}\partial_j\tilde{w}) = 0, \quad \text{in } Q_{1,\infty}.$$

Note that for any $l \in \mathbb{Z}$, $\tilde{b}(z)$ is orthogonally conjugated to $b\left(z', (-1)^l (z_n - 2l\delta)\right)$, for $z \in S_l$. Hence, by (3.17), we have

$$\frac{\lambda}{C} \le \tilde{b}(z) \le C\Lambda, \quad \text{for } z \in Q_{1,\infty},$$

and, by (3.19),

$$\|\tilde{b}\|_{C^\alpha(\overline{S}_l)} \le C, \quad \forall l \in \mathbb{Z}.$$

Apply Lemma 2.1 on S with $N = 1$, we have

$$\|\nabla\tilde{w}\|_{L^\infty(\frac{1}{2}S)} \le C\|\tilde{w}\|_{L^2(S)}.$$

It follows that

$$\|\nabla w\|_{L^\infty(Q_{1/2,\delta})} \le \frac{C}{\delta}\|w\|_{L^2(Q_{1,\delta})} \le C\|w\|_{L^\infty(Q_{1,\delta})},$$

for some positive constant C, depending only on n, α, R_0, κ, λ, Λ, $\|a\|_{C^\alpha}$, $\|f\|_{C^2}$ and $\|g\|_{C^2}$.

Since $\|(\partial_z y)\|_{L^\infty(Q_{1,\delta})} \le C$ by (3.16), where C depends only on R_0, κ, $\|f\|_{C^2}$ and $\|g\|_{C^2}$, and in particular, is independent of ε and δ. Reversing the change of variables (3.13) and (3.12), we have

$$\delta\|\nabla v\|_{L^\infty(\Omega_{x_0,\delta/8})} \le C\|v\|_{L^\infty(\Omega_{x_0,\delta/4})} \le C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{\gamma\sigma}$$

by (3.11). In particular, this implies

$$|\nabla u(x_0)| \leq C\|u\|_{L^\infty(\Omega_{x_0,\delta^{1-\gamma}})}\delta^{-1+\gamma\sigma},$$

and it concludes the proof of Theorem 1.1 after taking $\beta = \gamma\sigma/2$. □

4 Gradient estimates of elliptic systems

A natural question is whether the estimate in Theorem 1.1 can be extended to elliptic systems of divergence form. We tend to believe that the answer to this question is affirmative, and plan to pursue this in a subsequent paper. Following closely the proof of (1.4) in [7], we give a preliminary gradient estimate of elliptic systems in this section.

We consider the vector-valued function $u = (u_1, \cdots, u_N)$, and for $1 \leq \alpha, \beta \leq n$, $1 \leq i, j \leq N$, let $A_{ij}^{\alpha\beta}(x)$ be a function such that

$$\|A_{ij}^{\alpha\beta}\|_{L^\infty(\Omega_{0,R_0})} \leq \Lambda,$$
$$\int_{\Omega_{0,R_0}} A_{ij}^{\alpha\beta}(x)\partial_\alpha\varphi_i(x)\partial_\beta\varphi_j(x) \geq \lambda \int_{\Omega_{0,R_0}} |\nabla\varphi|^2, \quad \forall\varphi \in H_0^1(\Omega_{0,R_0};\mathbb{R}^N),$$

for some $\lambda, \Lambda > 0$, where Ω_{0,R_0} is defined as in (1.8). We assume $A_{ij}^{\alpha\beta}(x) \in C^\mu(\Omega_{0,R_0})$ for some $\mu \in (0,1)$, and consider the system

$$\begin{cases} -\partial_\alpha\left(A_{ij}^{\alpha\beta}(x)\partial_\beta u_j(x)\right) = 0 & \text{in } \Omega_{0,R_0}, \\ A_{ij}^{\alpha\beta}(x)\partial_\beta u_j(x)\nu_\alpha(x) = 0 & \text{on } \Gamma_+ \cup \Gamma_-, \end{cases} \tag{4.1}$$

for $i = 1, \cdots, N$, where Γ_+, Γ_- are defined as in (1.8), $\nu = (\nu_1, \cdots, \nu_n)$ denotes the unit normal vector on Γ_+ and Γ_-, pointing upward and downward respectively. We have the following gradient estimate by essentially following the proof of Theorem 1.2 in [7].

Theorem 4.1 *Let $u \in H_0^1(\Omega_{0,R_0};\mathbb{R}^N)$ be a solution to (4.1) in dimension $n \geq 2$, with the coefficient $A_{ij}^{\alpha\beta}$ defined as above. There exist positive constants r_0 and C depending only on n, λ, Λ, R_0, κ, μ, $\|A\|_{C^\mu(\Omega_{0,R_0})}$, $\|f\|_{C^2(\{|x'|\leq R_0\})}$ and $\|g\|_{C^2(\{|x'|\leq R_0\})}$, such that*

$$|\nabla u(x_0)| \leq C\|u\|_{L^\infty(\Omega_{0,R_0})}\left(\varepsilon + |x_0'|^2\right)^{-1/2}, \tag{4.2}$$

for all $\varepsilon \in (0,1)$, $x_0 \in \Omega_{0,r_0}$.

Remark 4.2 The elliptic systems we have considered include the linear systems of elasticity: $n = N$, and the coefficients $A_{ij}^{\alpha\beta}$ satisfy

$$A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} = A_{\alpha j}^{i\beta},$$

and for all $n \times n$ symmetric matrices $\{\xi_\alpha^i\}$,

$$\lambda|\xi|^2 \le A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \le \Lambda|\xi|^2.$$

Proof of Theorem 4.1 Let $u \in H^1(\Omega_{0,R_0}; \mathbb{R}^N)$ be a solution to (4.1). We perform the changes of variables (3.12) and (3.13). For any $1 \le i, j \le N$, we define

$$B_{ij}^{\alpha\beta}(z) = \frac{(\partial_y z)(A_{ij}^{\alpha\beta})(\partial_y z)^t}{\det(\partial_y z)},$$

and let $v(z) = u(x)$. Then v satisfies

$$\begin{cases} -\partial_\alpha \left(B_{ij}^{\alpha\beta}(z)\partial_\beta v_j(z) \right) = 0 & \text{in } Q_{1,\delta}, \\ B_{ij}^{n\beta}(z)\partial_\beta v_j(z) = 0 & \text{on } \{z_n = -\delta\} \cup \{z_n = \delta\}, \end{cases}$$

for $i = 1, \cdots, N$, where $Q_{s,t}$ is defined as in (3.3). As in the proof of Theorem 1.1, we can show that

$$\begin{aligned} &\|B_{ij}^{\alpha\beta}\|_{L^\infty(Q_{1,\delta})} \le C\Lambda, \quad \|B_{ij}^{\alpha\beta}\|_{C^\mu(\bar{Q}_{1,\delta})} \le C, \\ &\int_{Q_{1,\delta}} B_{ij}^{\alpha\beta}(z)\partial_\alpha\varphi_i(z)\partial_\beta\varphi_j(z) \ge \frac{\lambda}{C}\int_{Q_{1,\delta}} |\nabla\varphi|^2, \quad \forall \varphi \in H_0^1(Q_{1,\delta}; \mathbb{R}^N), \end{aligned}$$

where C is a positive constant that depends only on n, N, μ, R_0, κ, λ, Λ, $\|A\|_{C^\mu}$, $\|f\|_{C^2}$ and $\|g\|_{C^2}$. Then we argue as in the proof of Theorem 1.1 to obtain

$$|\nabla v(0)| \le C\|v\|_{L^\infty(Q_{1,\delta})},$$

which is (4.2) after reversing the changes of variables (3.12) and (3.13). $\square$

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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