



A Liouville Theorem for Möbius Invariant Equations

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Abstract

In this paper, we classify Möbius invariant differential operators of second order in two-dimensional Euclidean space, and establish a Liouville type theorem for general Möbius invariant elliptic equations. The equations are naturally associated with a continuous family of convex cones Γ_p in $\mathbb{R}^2$, with parameter $p \in [1, 2]$, joining the half plane $\Gamma_1 := \{(\lambda_1, \lambda_2) : \lambda_1 + \lambda_2 > 0\}$ and the first quadrant $\Gamma_2 := \{(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0\}$. Chen and C. M. Li established in 1991 a Liouville type theorem corresponding to Γ_1 under an integrability assumption on the solution. The uniqueness result does not hold without this assumption. The Liouville type theorem we establish in this paper for Γ_p , $1 < p \leq 2$, does not require any additional assumption on the solution as for Γ_1 . This is reminiscent of the Liouville type theorems in dimensions $n \geq 3$ established by Caffarelli, Gidas and Spruck in 1989 and by A. B. Li and Y. Y. Li in 2003–2005, where no additional assumption was needed either. On the other hand, there is a striking new phenomena in dimension $n = 2$ that Γ_p for $p = 1$ is a sharp dividing line for such uniqueness result to hold without any further assumption on the solution. In dimensions $n \geq 3$, there is no such dividing line.

Keywords Liouville theorem · Möbius invariant · Fully nonlinear elliptic equations

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1 Introduction

For $n \geq 3$, consider the equation

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{R}^n. \quad (1.1)$$

The Liouville type theorem of Caffarelli et al. [5] asserts that positive C^2 solutions of (1.1) are of the form

$$u(x) = \left(\frac{a}{1 + a^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

where $a > 0$ and $\bar{x} \in \mathbb{R}^n$. Under an additional hypothesis $u(x) = O(|x|^{2-n})$ for large $|x|$, the result was established earlier by Obata [47] and Gidas et al. [19].

Geometrically, Eq. (1.1) means that the scalar curvature of the Riemannian metric $u^{\frac{4}{n-2}}dx^2$ is equal to $4n(n-1)$. An analogous equation in dimension two is

$$-\Delta u = e^u \quad \text{on } \mathbb{R}^2. \quad (1.2)$$

Geometrically, it means that the Gaussian curvature of $e^u dx^2$ is equal to $\frac{1}{2}$. The above equation has plenty of solutions according to a classical theorem of Liouville [45]: Let Ω be a simply connected domain in $\mathbb{R}^2$, then all C^2 real solutions u of $-\Delta u = e^u$ in Ω are of the form

$$u(x_1, x_2) = \ln \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2}, \quad (1.3)$$

where $z = x_1 + \sqrt{-1}x_2$, and $f(z)$ is a locally univalent meromorphic function, i.e., a meromorphic function in Ω which has zeros or poles of order at most 1. In particular, let $f(z)$ be any holomorphic function satisfying $f'(z) \neq 0$ in $\mathbb{C}$, the u given by the representation formula (1.3) is a solution of (1.2). For instance, if we take $f(z) = e^z$, then we obtain a solution $u(x_1, x_2) = \log(8e^{2x_1}(1 + e^{2x_1})^{-2})$.

On the other hand, Chen and C. M. Li proved in [15] that C^2 solutions of (1.2) satisfying

$$\int_{\mathbb{R}^2} e^u dx < +\infty \quad (1.4)$$

are of the form

$$u(x) = 2 \ln \frac{8a}{8a^2 + |x - x_0|^2} \quad \text{in } \mathbb{R}^2,$$

where $a > 0$ and $x_0 \in \mathbb{R}^2$.

Eq. (1.2) is conformally invariant. For a C^2 function u , let

$$u_\psi := u \circ \psi + \ln |J_\psi|, \quad (1.5)$$

where $\psi(z)$ is a holomorphic function with nonzero Jacobian determinant $|J_\psi|$. Then we have

$$-e^{-u_\psi} \Delta u_\psi = (-e^{-u} \Delta u) \circ \psi \quad \text{on } \mathbb{R}^2.$$

Here we consider the holomorphic function ψ as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

In particular, if u is a solution of (1.2), then u_ψ is also a solution in the corresponding domain. In fact, Eq. (1.2) is in a sense the only conformally invariant equation as explained below.

Let $\mathcal{S}^{2 \times 2}$ denote the set of 2×2 real symmetric matrices.

Definition 1.1 Let H be a function from $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}^{2 \times 2}$ to $\mathbb{R}$. We say that a second-order differential operator $H(\cdot, u, \nabla u, \nabla^2 u)$ is conformally invariant if for any meromorphic function ψ on $\mathbb{C}$, and any function $u \in C^2(\mathbb{R}^2)$, it holds that:

$$H(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi) \equiv H(\cdot, u, \nabla u, \nabla^2 u) \circ \psi. \quad (1.6)$$

Note that (1.6) is understood to hold at any point $z \in \mathbb{C}$ which is not a pole of ψ or zero of ψ' .

The following proposition classifies all conformally invariant second-order differentiable operators.

Proposition 1.2 $H(\cdot, u, \nabla u, \nabla^2 u)$ is conformally invariant in $\mathbb{R}^2$ if and only if it is of the form:

$$H(\cdot, u, \nabla u, \nabla^2 u) = g(-e^{-u} \Delta u),$$

where g is a function from $\mathbb{R}$ to $\mathbb{R}$.

In this paper, we study a larger class of invariant operators, namely those invariant under Möbius transformations.

Recall that a Möbius transformation is a transformation generated by a finite composition of

$$\phi(x) = x + x_0, \quad \phi(x) = \lambda x, \quad \phi(x) = O x, \quad \text{and} \quad \phi(x) = \frac{x}{|x|^2} \quad \text{in } \mathbb{R}^2,$$

where λ is a nonzero constant, and O is an orthogonal matrix. In complex variables $z = x_1 + \sqrt{-1}x_2$, they are given by

$$\phi(z) = \frac{az + b}{cz + d} \quad \text{or} \quad \frac{a\bar{z} + b}{c\bar{z} + d}, \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0.$$

Definition 1.3 Let H be a function from $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}^{2 \times 2}$ to $\mathbb{R}$. We say $H(\cdot, u, \nabla u, \nabla^2 u)$ is Möbius invariant if for any Möbius transformation ψ and any function $u \in C^2(\mathbb{R}^2)$, it holds that:

$$H(\cdot, u_\psi, \nabla u_\psi, \nabla^2 u_\psi) = H(\cdot, u, \nabla u, \nabla^2 u) \circ \psi \quad \text{on } \mathbb{R}^2. \quad (1.7)$$

We prove that all Möbius invariant operators can be expressed as $F(A^u)$, where

$$A^u := -e^{-u} \nabla^2 u + \frac{1}{2} e^{-u} du \otimes du - \frac{1}{4} e^{-u} |\nabla u|^2 I \quad (1.8)$$

is a 2×2 symmetric matrix operator of second order.

Notice that $\text{tr}(A^u) = -e^{-u} \Delta u$, conformally invariant operators are, as shown in Proposition 1.2, a function of the trace of A^u in this setting.

The operator A^u has the following invariance property: for any Möbius transformation ψ and any $x \in \mathbb{R}^2$, $A^{u_\psi}(x) = O^T(A^u \circ \psi)O$, where $O = |J_\psi|^{-1/2} J_\psi \in O(2)$, the set of all real orthogonal matrices.

Moreover, for a Möbius transformation ψ , denote $y = \psi(x)$ as the coordinate change, and $v = u_\psi$, then

$$e^u(A_{kl}^u dy_k \otimes dy_l) = e^v(A_{ij}^v dx_i \otimes dx_j).$$

Hence, $F(A^{u_\psi}) \equiv F(A^u \circ \psi)$ for any $F : \mathcal{S}^{2 \times 2} \rightarrow \mathbb{R}$ which is invariant under orthogonal conjugation. We say that F is invariant under orthogonal conjugation if

$$F(O^{-1}MO) = F(M), \quad \forall M \in \mathcal{S}^{2 \times 2}, O \in O(2). \quad (1.9)$$

For $M \in \mathcal{S}^{2 \times 2}$, let $\lambda(M) = (\lambda_1(M), \lambda_2(M))$ with $\lambda_1(M)$ and $\lambda_2(M)$ being the eigenvalues of M . Then a function F on $\mathcal{S}^{2 \times 2}$ satisfying (1.9) corresponds to a symmetric function f on $\mathbb{R}^2$ satisfying $F(M) = f(\lambda(M))$ for all $M \in \mathcal{S}^{2 \times 2}$.

We classify all Möbius invariant operators in the following proposition.

Proposition 1.4 *Let $H(\cdot, u, \nabla u, \nabla^2 u)$ be Möbius invariant in $\mathbb{R}^2$, then H is of the form:*

$$H(\cdot, u, \nabla u, \nabla^2 u) = F(A^u)$$

where $F : \mathcal{S}^{2 \times 2} \rightarrow \mathbb{R}$ is invariant under orthogonal conjugation, i.e., F satisfies (1.9).

In dimension $n \geq 3$, a classical theorem of Liouville states that any local conformal diffeomorphism in $\mathbb{R}^n$ is the restriction of a Möbius transformation. Therefore, unlike Definitions 1.1 and 1.3 for operators in $\mathbb{R}^2$, conformally invariant operators and Möbius invariant operators in $\mathbb{R}^n$ are the same for $n \geq 3$. The classification of conformally invariant operators of second order was given by A. B. Li and Y. Y. Li in [32]. Our proof of Proposition 1.4 follows the arguments there.

The main result in this paper is a Liouville type theorem for Möbius invariant elliptic equations $F(A^u) = 1$ in $\mathbb{R}^2$.

From now on, let

$$\Gamma \text{ be an open convex symmetric cone in } \mathbb{R}^2 \text{ with vertex at the origin,} \quad (1.10)$$

and

$$\Gamma_2 \subset \Gamma \subset \Gamma_1, \quad (1.11)$$

where $\Gamma_1 := \{(\lambda_1, \lambda_2) : \lambda_1 + \lambda_2 > 0\}$ and $\Gamma_2 := \{(\lambda_1, \lambda_2) : \lambda_1 > 0, \lambda_2 > 0\}$. Here, Γ being symmetric means that $(\lambda_1, \lambda_2) \in \Gamma$ implies $(\lambda_2, \lambda_1) \in \Gamma$. Also, a function f defined on Γ is said to be symmetric if $f(\lambda_1, \lambda_2) \equiv f(\lambda_2, \lambda_1)$.

It is not difficult to see that Γ satisfies (1.10) and (1.11) if and only if $\Gamma = \Gamma_p$ for some $1 \leq p \leq 2$ where

$$\Gamma_p = \{\lambda = (\lambda_1, \lambda_2) : \lambda_2 > (p-2)\lambda_1, \lambda_1 > (p-2)\lambda_2\}.$$

Note that the above definition of Γ_1 and Γ_2 is consistent with earlier definitions.

Theorem 1.5 *Let $\Gamma = \Gamma_p$ for some $1 < p \leq 2$, and let $f \in C^1(\Gamma)$ be symmetric and satisfy $\partial_{\lambda_i} f > 0$ in Γ , $i = 1, 2$. Assume that u is a C^2 solution of*

$$f(\lambda(A^u)) = 1 \quad \text{in } \mathbb{R}^2, \quad (1.12)$$

where $\lambda(A^u) \in \Gamma$ are the eigenvalues of A^u . Then, for some $x_0 \in \mathbb{R}^2$ and some constants $a, b > 0$ satisfying $(2^{-1}a^{-2}b, 2^{-1}a^{-2}b) \in \Gamma$ and $f(2^{-1}a^{-2}b, 2^{-1}a^{-2}b) = 1$,

$$u(x) \equiv 2 \ln \frac{8a}{8|x - x_0|^2 + b}. \quad (1.13)$$

Remark 1.6 Theorem 1.5 still holds when replacing $f \in C^1(\Gamma)$ and $\partial_{\lambda_i} f > 0$ in Γ , $i = 1, 2$ by: For any compact subset $K \subset \Gamma$, there exists constant $C(K) > 1$ such that

$$\frac{1}{C(K)} \|\mu\| \leq f(\lambda + \mu) - f(\lambda) \leq C(K) \|\mu\|, \quad \forall \lambda, \lambda + \mu \in K, \mu = (\mu_1, \mu_2), \mu_1, \mu_2 > 0.$$

Remark 1.7 For u given in (1.13), $A^u = 2^{-1}a^{-2}bI$, where I is the 2×2 identity matrix.

In the above theorem, no additional assumption on u near infinity is made. In particular, we do not assume a priori the integrability condition (1.4). The hypothesis $\partial_{\lambda_i} f > 0$ means that the equation $f(\lambda(A^u)) = 1$ is elliptic. For $\Gamma = \Gamma_1$ and $f(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$, the equation is $-\Delta u = e^u$. As mentioned earlier, solutions were classified in [15] under the assumption (1.4). In this case, the conclusion does not hold without the assumption. When $f(\lambda_1, \lambda_2) = \sigma_2(\lambda) := \lambda_1 \lambda_2$, defined on Γ_2 , the equation becomes $\det(A^u) = 1$.

Γ_p is a continuous family of convex cones in parameter $p \in [1, 2]$. When $p = 1$, without the integrability condition (1.4), there are abundant solutions to Eq. (1.12), as shown in (1.3). If $p > 1$, the uniqueness result holds with no additional assumption on u . We emphasize the curious fact that for Eq. (1.12), $p = 1$ is a sharp dividing line for such uniqueness result. There is no such dividing line in dimensions $n \geq 3$.

A Liouville type theorem of the form $\sigma_2(\lambda(A^u)) = 1$ in $\mathbb{R}^4$ was proved by S.-Y. A. Chang et al. in [12], where σ_2 denotes the second elementary symmetric function in $\mathbb{R}^4$. For $n \geq 3$, a Liouville type theorem for general elliptic equation

$f(\lambda(A^u)) = 1$ in $\mathbb{R}^n$ was proved by A. B. Li and Y. Y. Li in [32] and [33]. The latter is an extension of the above-mentioned Liouville type theorem for Eq. (1.1).

The crucial point in the proof of the uniqueness result Theorem 1.5 is to handle the possible singularity of the solution at infinity. Our proof requires new novel ingredients.

Theorem 1.5 is proved using the moving sphere method. The first step of the moving sphere method is to prove that for every $x \in \mathbb{R}^2$, there exists $\lambda_0(x) > 0$ such that $u_{x,\lambda}(y) \leq u(y)$ for all $0 < \lambda < \lambda_0(x)$ and $|y - x| \geq \lambda$, where

$$u_{x,\lambda}(y) := u\left(x + \frac{\lambda^2(y - x)}{|y - x|^2}\right) - 4 \ln \frac{|y - x|}{\lambda}.$$

The main difficulty we need to overcome to prove the theorem is to establish this step. When $n \geq 3$, this step can be achieved from the estimate $u(x) \geq (\min_{\partial B_1} u)|x|^{2-n}$ for $|x| \geq 1$, which follows from the superharmonicity and positivity of u ; see Lemma 2.1 in [43] and Lemma 2.1 in [32]. For Γ_1 in dimension $n = 2$, this step simply fails without the integrability condition (1.4) on u . The integrability condition was used in [15] to assure a uniform convergence of $\frac{u(x)}{\ln|x|}$ to a finite number as $|x|$ tends to infinity, and such uniform convergence has played a crucial role in the proof of the two dimensional Liouville type theorem there. The situation here is much more delicate since we do not have any additional assumption on u near infinity like (1.4), and it is difficult to establish a uniform limit of $\frac{u(x)}{\ln|x|}$ as $|x|$ tends to infinity. Instead, we establish the following asymptotic behavior of the solution near infinity which is enough to allow the moving sphere procedure to get started. We then prove the theorem without first knowing that $\frac{u(x)}{\ln|x|}$ has a finite limit at infinity or that the integrability condition (1.4) holds.

Proposition 1.8 *Let $\Gamma = \Gamma_p$ for some $1 < p \leq 2$, and let u be a viscosity supersolution of*

$$\lambda(A^u) \in \partial\Gamma \quad \text{in } \mathbb{R}^2 \setminus B_{\frac{r_0}{2}} \quad (1.14)$$

for some $r_0 > 0$. Then there exists $K_0 > 0$, such that

$$\inf_{\partial B_r} u(r) + 4 \ln r \quad \text{is monotonically nondecreasing for } r > K_0. \quad (1.15)$$

Consequently, $\liminf_{x \rightarrow \infty} (u(x) + 4 \ln |x|) > -\infty$.

Chow and Wan [16, Corollary 2] gave a complex analysis proof of the above-mentioned Liouville type theorem for Eq. (1.2) using Liouville's representation formula (1.3).

Question 1.9 *Is there a complex analysis proof of Theorem 1.5?*

In connection with the above question, we bring attention to a simple complex analysis proof of the following Jörgen's theorem by Nitsche [46]: Any smooth solution of $\det(D^2u) = 1$ in $\mathbb{R}^2$ must be a quadratic polynomial.

We write $A_{ij}^u dx_i \otimes dx_j$ in complex variables. Denote $z = x_1 + \sqrt{-1}x_2$ and $\bar{z} = x_1 - \sqrt{-1}x_2$. It is straightforward to check that

$$A_{ij}^u dx_i \otimes dx_j = B_{zz}^u dz \otimes dz + B_{z\bar{z}}^u dz \otimes d\bar{z} + B_{\bar{z}z}^u d\bar{z} \otimes dz + B_{\bar{z}\bar{z}}^u d\bar{z} \otimes d\bar{z},$$

where

$$B^u = \begin{pmatrix} B_{zz}^u & B_{z\bar{z}}^u \\ B_{\bar{z}z}^u & B_{\bar{z}\bar{z}}^u \end{pmatrix} = e^{-u} \begin{pmatrix} -u_{z\bar{z}} & -u_{zz} + \frac{1}{2}u_z^2 \\ -u_{z\bar{z}} + \frac{1}{2}u_{\bar{z}}^2 & -u_{\bar{z}\bar{z}} \end{pmatrix}.$$

If u is a real function, then $-u_{z\bar{z}} + \frac{1}{2}u_z^2$ is the complex conjugate of $-u_{\bar{z}\bar{z}} + \frac{1}{2}u_{\bar{z}}^2$, and B^u is a Hermitian matrix.

We have the following ways to describe the Möbius invariance in complex variables.

For any Möbius transformation ψ and any $z \in \mathbb{C}$, $B^u_\psi(z) = U^*(B^u \circ \psi)U$, where

$$U = \frac{1}{\psi'(z)} \begin{pmatrix} \frac{\partial \psi}{\partial z} & \frac{\partial \psi}{\partial \bar{z}} \\ \frac{\partial \bar{\psi}}{\partial z} & \frac{\partial \bar{\psi}}{\partial \bar{z}} \end{pmatrix} \in U(2),$$

the set of all unitary matrices.

Moreover, for a Möbius transformation ψ , denote $z = \psi(w)$ as the coordinate change, and $v = u_\psi$. Then

$$\begin{aligned} & e^u (B_{zz}^u dz \otimes dz + B_{z\bar{z}}^u dz \otimes d\bar{z} + B_{\bar{z}z}^u d\bar{z} \otimes dz + B_{\bar{z}\bar{z}}^u d\bar{z} \otimes d\bar{z}) \\ &= e^v (B_{ww}^v dw \otimes dw + B_{w\bar{w}}^v dw \otimes d\bar{w} + B_{\bar{w}w}^v d\bar{w} \otimes dw + B_{\bar{w}\bar{w}}^v d\bar{w} \otimes d\bar{w}). \end{aligned}$$

The second-order differential matrix operator A^u in (1.8) corresponds to a $(0, 2)$ -tensor on the standard Euclidean 2-sphere $(\mathbb{S}^2, g_0)$ in $\mathbb{R}^3$. For a C^2 function u on $\mathbb{S}^2$, we associate with the conformal metric $g_u := e^u g_0$ a $(0, 2)$ -tensor

$$A_{g_u} := -\nabla_{g_0}^2 u + \frac{1}{2} du \otimes du - \frac{1}{4} |\nabla_{g_0} u|^2 g_0 + K_{g_0} g_0,$$

where $K_{g_0} \equiv 1$ is the Gaussian curvature of the metric g_0 .

Theorem 1.5 is the starting point of our proof of the following results in a subsequent paper [36] on the existence and compactness of solutions to the σ_2 -Nirenberg problem.

For K satisfying the nondegeneracy condition

$$|\nabla K|_{g_0} + |\Delta K|_{g_0} > 0 \quad \text{on } \mathbb{S}^2, \quad (1.16)$$

we define the sets

$$\begin{aligned}\text{Crit}_+(K) &= \{x \in \mathbb{S}^2 : \nabla_{g_0} K(x) = 0, \Delta_{g_0} K(x) > 0\}, \\ \text{Crit}_-(K) &= \{x \in \mathbb{S}^2 : \nabla_{g_0} K(x) = 0, \Delta_{g_0} K(x) < 0\}.\end{aligned}$$

Set $\deg(\nabla K, \text{Crit}_-(K)) := \deg(\nabla K, O, 0)$, where O is any open subset of $\mathbb{S}^2$ containing $\text{Crit}_-(K)$ and disjoint from $\text{Crit}_+(K)$. By (1.16), this is well defined.

Theorem 1.10 ([36]) *Assume that $K \in C^2(\mathbb{S}^2)$ is a positive function satisfying the nondegeneracy condition (1.16). Then there exists a positive constant C depending only on K , such that*

$$\|u\|_{C^2(\mathbb{S}^2)} \leq C,$$

for all C^2 solutions u of the equation

$$\sigma_2(\lambda(g_u^{-1}A_{g_u})) = K(x), \quad \lambda(A_{g_u}) \in \Gamma_2 \text{ on } \mathbb{S}^2. \quad (1.17)$$

Moreover, if $\deg(\nabla K, \text{Crit}_-(K)) \neq 1$, then (1.17) admits a solution.

Remark 1.11 If $K \in C^{2,\alpha}(\mathbb{S}^2)$, $0 < \alpha < 1$, and $\mathcal{O}$ is an open subset of $C^{4,\alpha}(\mathbb{S}^2)$ which contains all solutions of (1.17), then

$$\deg(\sigma_2(\lambda(A^v)) - K, \mathcal{O}, 0) = -1 + \deg(\nabla K, \text{Crit}_-(K)).$$

Here the degree on the left hand side is as defined in [34].

Remark 1.12 For any K satisfying (1.16) and having only isolated nondegenerate critical points,

$$\deg(\nabla K, \text{Crit}_-(K)) = \sum_{\bar{x} \in \mathbb{S}^2, \nabla K(\bar{x})=0, \Delta K(\bar{x})<0} (-1)^{i(\bar{x})},$$

where $i(\bar{x})$ denotes the number of negative eigenvalues of $\nabla^2 K(\bar{x})$.

Theorem 1.10 is related to the Nirenberg problem, which amounts to solving $\sigma_1(\lambda(A_{g_u})) = K$ on $\mathbb{S}^2$. There has been much work on the Nirenberg problem and related ones, see, e.g., K. C. Chang and Liu [10], S.-Y. A. Chang and Yang [14], Z.-C. Han [26], Jin et al. [31], and the references therein. For $n \geq 3$ and $k \geq 2$, the σ_k -Nirenberg problem was studied by S.-Y. A. Chang et al. [13] and Y. Y. Li et al. [41].

The above-mentioned Liouville type theorem for general conformally invariant equations $f(\lambda(A^u)) = 1$ in [33] was stimulated by the study of fully nonlinear elliptic equations involving the Schouten tensor, and in particular by the study of the σ_k -Yamabe problem.

The existence of solutions of the σ_k -Yamabe problem has been proved for $k \geq \frac{n}{2}$, $k = 2$ or when (M, g) is locally conformally flat, and the compactness of the set of solutions has been proved for $k \geq \frac{n}{2}$ when the manifold is not conformally equivalent to the standard sphere—they were established in [11, 20, 22, 24, 32, 37, 49]. For

more recent works on σ_k -Yamabe type problems, see, for example, [1–3, 7–9, 17, 18, 21, 23, 25, 27–30, 38, 39, 42, 48, 50, 51] and references therein. On the other hand, the existence of solutions remains open when $2 < k < \frac{n}{2}$, and the compactness of solutions remains open when $2 \leq k < \frac{n}{2}$.

One of our motivations in studying the Möbius invariant equations in dimension two is to gain insights and inspirations into solving the above-mentioned open problems on the existence and compactness of the σ_k -Yamabe problem, for $2 \leq k < \frac{n}{2}$.

The rest of our paper is organized as follows. In Sect. 2, we recall the definition of viscosity solutions for $\lambda(A'') \in \partial\Gamma$ and a regularization lemma. We then give a proof of the crucial asymptotic behavior of viscosity supersolutions Proposition 1.8. Theorem 1.5 is proved in Sect. 3 using the method of moving spheres, a variant of the method of moving planes. Propositions 1.2 and 1.4 are proved in Sect. 4. Two calculus lemmas are given in “Appendix” for the reader’s convenience.

2 Asymptotic Behavior

In this section, we establish an estimate on the asymptotic behavior for viscosity supersolution of $\lambda(A'') \in \partial\Gamma$ on $\mathbb{R}^2 \setminus \{\text{compact set}\}$. This estimate allows the method of moving spheres argument to get started in the proof of Theorem 1.5 in Sect. 3.

We start with the definition of viscosity solutions for $\lambda(A'') \in \partial\Gamma$, see [35, 40] for details.

Let us first define the set of upper semi-continuous and lower semi-continuous functions.

For any set $S \subset \mathbb{R}^2$, we use $\text{USC}(S)$ to denote the set of functions $u : S \rightarrow \mathbb{R} \cup \{-\infty\}$, $u \neq -\infty$ in S , satisfying

$$\limsup_{x \rightarrow x_0} u(x) \leq u(x_0), \quad \forall x_0 \in S.$$

Similarly, we use $\text{LSC}(S)$ to denote the set of functions $u : S \rightarrow \mathbb{R} \cup \{+\infty\}$, $u \neq +\infty$ in S , satisfying

$$\liminf_{x \rightarrow x_0} u(x) \geq u(x_0), \quad \forall x_0 \in S.$$

Definition 2.1 Let Γ satisfy (1.10) and (1.11) and Ω be an open subset in $\mathbb{R}^2$. We say $u \in \text{USC}(\Omega)$ is a viscosity subsolution of

$$\lambda(A'') \in \partial\Gamma \quad \text{in } \Omega \tag{2.1}$$

if for any point $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(u - \varphi)(x_0) = 0$ and $u - \varphi \leq 0$ near x_0 , we have

$$\lambda(A^\varphi(x_0)) \in \mathbb{R}^2 \setminus \Gamma.$$

Similarly, we say $u \in \text{LSC}(\Omega)$ is a viscosity supersolution of (2.1), if for any point $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $(u - \varphi)(x_0) = 0$ and $u - \varphi \geq 0$ near x_0 , we have

$$\lambda(A^\varphi(x_0)) \in \bar{\Gamma}.$$

We say u is a viscosity solution of (2.1), if it is both a subsolution and a supersolution.

Let us recall the well-known regularization of semi-continuous functions which will be used in the paper, see [4, 40] for details.

Lemma 2.2 *Let Ω be an open set, $\Omega' \subset\subset \Omega$, and $\Omega'' \subset\subset \Omega'$. Let $u \in \text{LSC}(\bar{\Omega})$ be a viscosity supersolution of (2.1). We define the ϵ -lower envelope of u by*

$$u_\epsilon(x) := \inf_{y \in \Omega'} \left\{ u(y) + \frac{1}{\epsilon} |y - x|^2 \right\}, \quad \forall x \in \bar{\Omega'}.$$

Then there exists $\epsilon_0 = \epsilon_0(\Omega', \Omega'') > 0$, such that for $0 < \epsilon < \epsilon_0$,

- (i) u_ϵ is monotonically decreasing in ϵ , and $u_\epsilon \rightarrow u$ pointwise, as $\epsilon \rightarrow 0^+$.
- (ii) $u_\epsilon(x) - \frac{1}{\epsilon} |x|^2$ is concave in Ω' . Consequently, u_ϵ is second-order differentiable almost everywhere in Ω'' and u_ϵ is Lipschitz in Ω'' .
- (iii) u_ϵ is a viscosity supersolution of (2.1) in Ω'' , and $\lambda(A^{u_\epsilon}) \in \bar{\Gamma}$ at the points of Ω'' where u is second-order differentiable.

Proof We will prove it for $\Omega = B_{2R}$, $\Omega' = B_R$, $\Omega'' = B_{R/2}$; the general case can be proved in the same way. Let $x_0, x_1 \in \bar{B}_R$. Obviously there exists x_0^* , such that $u_\epsilon(x_0) = u(x_0^*) + \frac{1}{\epsilon} |x_0^* - x_0|^2$; moreover, $u_\epsilon(x_0) \leq u(x_0)$. So we obtain

$$\frac{1}{\epsilon} |x_0^* - x_0|^2 = u_\epsilon(x_0) - u(x_0^*) \leq u(x_0) - u(x_0^*);$$

hence,

$$|x_0^* - x_0|^2 \leq \epsilon \operatorname{osc} u,$$

and we also know $u(x_0^*) - u(x_0) \leq u_\epsilon(x_0) - u(x_0) \leq 0$. Now by the lower semi-continuity of u , we obtain (i).

Take $x \in \bar{B}_R$,

$$\begin{aligned} u_\epsilon(x_0) &\leq u(x) + \frac{1}{\epsilon} |x - x_0|^2 \\ &\leq u(x) + \frac{1}{\epsilon} |x - x_1|^2 + \frac{2}{\epsilon} |x - x_1| |x_1 - x_0| + \frac{1}{\epsilon} |x_1 - x_0|^2 \\ &\leq u(x) + \frac{1}{\epsilon} |x - x_1|^2 + \frac{3}{\epsilon} \cdot 2R |x_1 - x_0|. \end{aligned}$$

Take supremum over $x \in \bar{B}_R$, then we obtain $u_\epsilon(x_0) \leq u_\epsilon(x_1) + \frac{6R}{\epsilon} |x_1 - x_0|$. Hence $|u_\epsilon(x_0) - u_\epsilon(x_1)| \leq \frac{6R}{\epsilon} |x_1 - x_0|$, so u_ϵ is locally Lipschitz.

Define

$$P_0(x) = u(x_0^*) + \frac{1}{\epsilon} |x - x_0^*|,$$

so it has the property

$$\begin{aligned} P_0(x_0) &= u_\epsilon(x_0), \\ P_0(x) &\geq u_\epsilon(x), \quad x \in \bar{B}_R. \end{aligned}$$

That is, P_0 touches u_ϵ from the above at x_0 in $\bar{B}_R$. So for small h , $\Delta_h^2(u_\epsilon(x_0) - \frac{1}{\epsilon} |x_0|^2) \leq 0$, where

$$\Delta_h^2 u(x_0) := \frac{u(x_0 + h) + u(x_0 - h) - 2u(x_0)}{|h|^2}.$$

This implies that $u_\epsilon(x) - \frac{1}{\epsilon} |x|^2$ is concave in B_R , so we have obtained (ii).

Let $x_0 \in B_{R/2}$, and let $P(x)$ be a paraboloid touches u_ϵ by below at x_0 . Consider the paraboloid

$$Q(x) = P(x + x_0 - x_0^*) - \frac{1}{\epsilon} |x_0 - x_0^*|^2.$$

Since

$$|x_0^* - x_0|^2 \leq \epsilon \operatorname{osc} u,$$

we can pick ϵ_0 such that $\forall \epsilon \leq \epsilon_0$, if $x_0 \in B_{R/2}$, then $x_0^* \in B_R$. Take any x sufficient closed to x_0^* so that $x + x_0 - x_0^* \in B_R$.

By the definition of u_ϵ ,

$$u_\epsilon(x + x_0 - x_0^*) \leq u(x) + \frac{1}{\epsilon} |x_0 - x_0^*|^2.$$

If x is sufficiently close to x_0^* , then by the assumption on P , we obtain

$$u_\epsilon(x + x_0 - x_0^*) \geq P(x + x_0 - x_0^*).$$

Therefore, we have

$$Q(x) = P(x + x_0 - x_0^*) - \frac{1}{\epsilon} |x_0 - x_0^*|^2 \leq u(x).$$

At x_0^* , we have, by definition,

$$Q(x_0^*) = P(x_0) - \frac{1}{\epsilon} |x_0 - x_0^*|^2 = u_\epsilon(x_0) - \frac{1}{\epsilon} |x_0 - x_0^*|^2 = u(x_0^*).$$

This shows Q is a paraboloid touching u from below at x_0^* . Since u is a viscosity supersolution,

$$\lambda(A^{Q(x_0^*)}) \in \bar{\Gamma}.$$

However, $\nabla Q(x_0^*) = \nabla P(x_0)$, $\nabla^2 Q(x_0^*) = \nabla^2 P(x_0)$, so we obtain

$$\lambda(A^{P(x_0)}) \in \bar{\Gamma}.$$

This implies that u_ϵ is a viscosity supersolution. (iii) is proved. $\square$

In the remaining part of this section, we prove the asymptotic behavior Proposition 1.8.

Proof of Proposition 1.8 Let

$$v(r) := \inf_{|x|=r} u(x).$$

In the following, we will make the identification $v(y) := v(|y|)$.

Since $u \in \text{LSC}(\mathbb{R}^2 \setminus B_{r_0/2})$, we have $v \in \text{LSC}(\mathbb{R}^2 \setminus B_{r_0/2})$. Being the infimum over a family of viscosity supersolutions, v is a viscosity supersolution of (1.14).

Define

$$\tilde{v}_{\epsilon,R}(x) = \inf_{y \in \bar{B}_{2R} \setminus B_{\frac{3r_0}{4}}} \left\{ v(|y|) + \frac{1}{\epsilon} |y - x|^2 \right\}, \quad \forall x \in \bar{B}_{2R} \setminus B_{\frac{3r_0}{4}},$$

for $R > r_0 + 2$, $0 < \epsilon < \epsilon_0(R)$, where $\epsilon_0(R) = \epsilon_0(\Omega', \Omega'') > 0$ is as defined in Lemma 2.2 with $\Omega' = B_{2R} \setminus \bar{B}_{\frac{3r_0}{4}}$, $\Omega'' = B_R \setminus \bar{B}_{r_0}$. Clearly, $\tilde{v}_{\epsilon,R}(x)$ is radially symmetric, and we will use $\tilde{v}_{\epsilon,R}(r)$ to denote $\tilde{v}_{\epsilon,R}(x)$, for $|x| = r$.

Since $\tilde{v}_{\epsilon,R}(r) - \frac{1}{\epsilon} r^2$ is concave, by the Rademacher's theorem and the Alexandroff's theorem, $\tilde{v}_{\epsilon,R}$ is differentiable almost everywhere, $\tilde{v}'_{\epsilon,R}(r) - \frac{2}{\epsilon} r$ is monotonically nonincreasing, and $\tilde{v}_{\epsilon,R}$ is second-order differentiable almost everywhere. More precisely, there exists $E_{\epsilon,R} \subset (r_0, R)$ with $|E_{\epsilon,R} \cap (r_0, R)| = R - r_0$, satisfying:

For $r \in E_{\epsilon,R}$,

- (i) $\tilde{v}'_{\epsilon,R}(r) := \lim_{h \rightarrow 0} \frac{\tilde{v}_{\epsilon,R}(r+h) - \tilde{v}_{\epsilon,R}(r)}{h}$ is finite,
- (ii) $\tilde{v}''_{\epsilon,R}(r) := \lim_{h \rightarrow 0} \frac{\tilde{v}'_{\epsilon,R}(r+h) - \tilde{v}'_{\epsilon,R}(r)}{h}$ is finite, where $\tilde{v}'_{\epsilon,R}(\cdot)$ is the extension to (r_0, R) of $\tilde{v}'_{\epsilon,R}$ defined on $E_{\epsilon,R}$, given by

$$\tilde{v}'_{\epsilon,R}(t) := \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \int_{t-\delta < s < t+\delta} \tilde{v}'_{\epsilon,R}(s) ds, \quad \forall t \in (r_0, R).$$

- (iii) $\lim_{h \rightarrow 0} \frac{|\tilde{v}_{\epsilon,R}(r+h) - \tilde{v}_{\epsilon,R}(r) - \tilde{v}'_{\epsilon,R}(r)h - \frac{1}{2}\tilde{v}''_{\epsilon,R}(r)h^2|}{h^2} = 0$.

In the following, when there is no ambiguity, we use $\tilde{v}$ to denote $\tilde{v}_{\epsilon,R}$. Denote $\lambda(A^{\tilde{v}}) = (\lambda_1, \lambda_2)$, then

$$\lambda_1(r) = \frac{1}{e^{\tilde{v}}} \left(-\tilde{v}'' + \frac{1}{4}(\tilde{v}')^2 \right), \quad \lambda_2(r) = \frac{1}{e^{\tilde{v}}} \left(-\frac{\tilde{v}'}{r} - \frac{1}{4}(\tilde{v}')^2 \right) \quad \text{in } E_{\epsilon,R}.$$

By Lemma 2.2 (ii)–(iii), $\tilde{v}' - \frac{2}{\epsilon}r$ is monotonically nonincreasing, and $\lambda(A^{\tilde{v}}) \in \bar{\Gamma}$ in $E_{\epsilon,R}$. We distinguish into two cases.

Case 1: $\Gamma = \Gamma_2$.

By Lemma 2.2 (iii), $(\lambda_1, \lambda_2) \in \bar{\Gamma}_2$ in $E_{\epsilon,R}$, so

$$\lambda_2 = \frac{1}{e^{\tilde{v}}} \left(-\frac{\tilde{v}'}{r} - \frac{1}{4}(\tilde{v}')^2 \right) = -\frac{\tilde{v}'}{e^{\tilde{v}}} \left(\frac{1}{r} + \frac{1}{4}\tilde{v}' \right) \geq 0 \quad \text{in } E_{\epsilon,R}.$$

It follows that

$$(\tilde{v} + 4 \ln r)' \geq 0 \quad \text{in } E_{\epsilon,R}.$$

Since $\tilde{v}$ is locally Lipschitz, $\tilde{v} + 4 \ln r$ is monotonically nondecreasing in (r_0, R) . Sending $\epsilon \rightarrow 0$, using Lemma 2.2 (i), we obtain (1.15).

Case 2: $\Gamma_2 \subsetneq \Gamma \subsetneq \Gamma_1$.

Define

$$\tilde{E}_{\epsilon,R} := \left\{ r \in E_{\epsilon,R} : \tilde{v}'(r) < -\frac{4}{r} \right\}.$$

When there is no confusion, denote

$$E = E_{\epsilon,R}, \quad \tilde{E} = \tilde{E}_{\epsilon,R}.$$

It is clear that

$$\lambda_2 = \frac{1}{e^{\tilde{v}}} \left(-\frac{\tilde{v}'}{r} - \frac{1}{4}(\tilde{v}')^2 \right) = -\frac{(\tilde{v}')^2 \left(\frac{1}{\tilde{v}'} + \frac{r}{4} \right)}{e^{\tilde{v}} r} < 0 \quad \text{on } \tilde{E}. \quad (2.2)$$

Since $(\lambda_1, \lambda_2) \in \Gamma$, we also have $\lambda_1 > 0$ on $\tilde{E}$, so

$$\lambda_1 = \frac{1}{e^{\tilde{v}}} \left(-\tilde{v}'' + \frac{1}{4}(\tilde{v}')^2 \right) = \frac{(\tilde{v}')^2}{e^{\tilde{v}}} \left(\frac{1}{\tilde{v}'} + \frac{r}{4} \right)' > 0 \quad \text{on } \tilde{E}. \quad (2.3)$$

Denote

$$g(r) = \frac{1}{\tilde{v}'(r)} + \frac{r}{4}.$$

From (2.2) and (2.3), we have

$$g > 0, \quad g' > 0 \quad \text{in } \tilde{E}.$$

We start with a lemma:

Lemma 2.3 *For $R > r_0 + 2$, $0 < \epsilon < \epsilon_0(R)$, if $a \in \tilde{E}_{\epsilon,R}$, then*

$$(a, R) \cap \tilde{E}_{\epsilon, R} = (a, R) \cap E_{\epsilon, R}.$$

Moreover, $\forall a < \alpha < \beta < R, \alpha, \beta \in E$, we have

$$0 < \int_{\alpha}^{\beta} g'(r) dr \leq g(\beta) - g(\alpha). \quad (2.4)$$

Proof Let $\bar{r} \in \tilde{E}$. By Lemma 2.2 (ii), $\tilde{v}' - \frac{2}{\epsilon}r$ is monotonically nonincreasing, so

$$\lim_{r \rightarrow \bar{r}^+} \tilde{v}'(r) \leq \tilde{v}'(\bar{r}) \leq \lim_{r \rightarrow \bar{r}^-} \tilde{v}'(r). \quad (2.5)$$

Therefore,

$$\lim_{r \rightarrow \bar{r}^+} \left(\tilde{v}'(r) + \frac{4}{r} \right) \leq \tilde{v}'(\bar{r}) + \frac{4}{\bar{r}} < 0.$$

Hence, there exists $\delta > 0$, such that $\tilde{v}'(r) + \frac{4}{r} < 0$ for any $\bar{r} < r < \bar{r} + \delta$, i.e., $(\bar{r}, \bar{r} + \delta) \cap E \subset \tilde{E}$.

Since $a \in \tilde{E}$, $b := \sup\{c : (a, c) \cap E \subset \tilde{E}\}$ is well defined.

For $a < s < \beta < b, \alpha, s, \beta \in E$, since $\tilde{v}'(r) - \frac{2}{\epsilon}r$ is monotonically nonincreasing, we have

$$\tilde{v}'(s) - \tilde{v}'\left(s + \frac{1}{m}\right) \geq -\frac{2}{\epsilon} \frac{1}{m}.$$

By the definition of $\tilde{E}$, we have, using $-\tilde{v}'(r) \geq \frac{4}{r}$, for any $a < r < b, r \in E$,

$$0 \leq \frac{1}{\tilde{v}'(r + \frac{1}{m})\tilde{v}'(r)} \leq \frac{r(r + \frac{1}{m})}{16}.$$

Therefore,

$$m \left(\frac{1}{\tilde{v}'(s + \frac{1}{m})} - \frac{1}{\tilde{v}'(s)} \right) = \frac{m(\tilde{v}'(s) - \tilde{v}'(s + \frac{1}{m}))}{\tilde{v}'(s + \frac{1}{m})\tilde{v}'(s)} \geq -\frac{s(s + \frac{1}{m})}{8\epsilon} \geq -\frac{b^2}{8\epsilon}.$$

So let

$$h_m(s) = m \left(\frac{1}{\tilde{v}'(s + \frac{1}{m})} - \frac{1}{\tilde{v}'(s)} \right) \geq -\frac{b^2}{8\epsilon}, \quad \alpha < s < \beta.$$

We have

$$\lim_{m \rightarrow \infty} h_m(s) = \left(\frac{1}{\tilde{v}'(s)} \right)' \quad \text{in } E.$$

By Fatou's lemma,

$$\begin{aligned} \int_{\alpha}^{\beta} \left(\frac{1}{\tilde{v}'(s)} \right)' ds &\leq \liminf_{m \rightarrow \infty} \int_{\alpha}^{\beta} h_m(s) ds \\ &= \liminf_{m \rightarrow \infty} \left(-m \int_{\alpha}^{\alpha + \frac{1}{m}} \frac{1}{\tilde{v}'(s)} ds + m \int_{\beta}^{\beta + \frac{1}{m}} \frac{1}{\tilde{v}'(s)} ds \right) \\ &= \frac{1}{\tilde{v}'(\beta)} - \frac{1}{\tilde{v}'(\alpha)}. \end{aligned}$$

Thus, using $g' > 0$ in $\tilde{E}$, (2.4) follows.

Now we will prove $b = R$.

By (2.4), using also $\tilde{v}' < 0$ on $\tilde{E}$, there exists $\mu > 0$, such that

$$\tilde{v}' + \frac{4}{r} < -\mu < 0 \quad \text{on} \quad \left(\frac{a+b}{2}, b \right) \cap E.$$

If $b \neq R$, then by Inequality (2.5), there exists $\delta > 0$, such that

$$\tilde{v}' + \frac{4}{r} < -\frac{\mu}{2} < 0 \quad \text{on} \quad [b, b + \delta) \cap E.$$

This violates the definition of b .

Thus, $b = R$. Lemma 2.3 is now proved. $\square$

For given $R > r_0 + 2$ and $0 < \epsilon < \epsilon_0(R)$, define $a_0^{\epsilon, R}$ to be:

$$a_0^{\epsilon, R} = \inf \{ a : a \in \tilde{E}_{\epsilon, R}, \text{ if } \tilde{E}_{\epsilon, R} \neq \emptyset \}.$$

If $\tilde{E}_{\epsilon, R} = \emptyset$ then define $a_0^{\epsilon, R} = R$.

If $\tilde{E}_{\epsilon, R} \neq \emptyset$, then $r_0 \leq a_0^{\epsilon, R} < R$, and, by Lemma 2.3, $(a_0^{\epsilon, R}, R) \cap \tilde{E}_{\epsilon, R} = (a_0^{\epsilon, R}, R) \cap E_{\epsilon, R}$. Since $\Gamma_2 \subsetneq \Gamma \subsetneq \Gamma_1$, there exists a unique constant $0 < p < 1$ such that $(1, -p) \in \partial\Gamma$. Note that

$$\lambda_2 > -p\lambda_1 \quad \text{in} \quad (a_0^{\epsilon, R}, R) \cap \tilde{E}_{\epsilon, R}. \quad (2.6)$$

We will prove

$$\alpha^{-\frac{1}{p}} g(\alpha) < \beta^{-\frac{1}{p}} g(\beta) \quad \text{for} \quad a_0^{\epsilon, R} < \alpha < \beta < R. \quad (2.7)$$

Let $k(r) = r^{-\frac{1}{p}} g(r)$. We have, using (2.2), (2.3) and (2.6),

$$k'(r) > 0 \quad \text{in} \quad (a_0^{\epsilon, R}, R) \cap \tilde{E}_{\epsilon, R}.$$

Set $\tilde{h}_m(s) = m(k(s + \frac{1}{m}) - k(s))$, $a_0^{\epsilon, R} < \alpha < s < \beta < R$, $\alpha, s, \beta \in \tilde{E}_{\epsilon, R}$, then using Lemma 2.3, we have

$$\tilde{h}_m(s) = m \left(\left(s + \frac{1}{m} \right)^{-\frac{1}{p}} g \left(s + \frac{1}{m} \right) - s^{-\frac{1}{p}} g(s) \right) \geq m \left(\left(s + \frac{1}{m} \right)^{-\frac{1}{p}} - s^{-\frac{1}{p}} \right) g(s) \geq -C$$

and

$$\tilde{h}_m(s) \rightarrow k'(s) \quad \text{as } m \rightarrow \infty, \text{ for } a_0^{\epsilon, R} < \alpha < s < \beta < R, s \in \tilde{E}_{\epsilon, R}.$$

By Fatou's lemma, we have

$$\begin{aligned} k(\beta) - k(\alpha) &= \liminf_{m \rightarrow \infty} \left(m \int_{\beta}^{\beta + \frac{1}{m}} k(s) ds - m \int_{\alpha}^{\alpha + \frac{1}{m}} k(s) ds \right) \\ &= \liminf_{m \rightarrow \infty} \int_{\alpha}^{\beta} \tilde{h}_m(s) ds \geq \int_{\alpha}^{\beta} k'(s) ds > 0, \quad \alpha, \beta \in \tilde{E}_{\epsilon, R}. \end{aligned}$$

Hence, (2.7) follows.

By Lemma 2.2 (i), for any $R > r_0 + 2$, there exists $\epsilon'(R) \rightarrow 0^+$ as $R \rightarrow \infty$, such that $\forall 0 < \epsilon < \epsilon'(R)$,

$$|\tilde{v}_{\epsilon, R}(x) - v(x)| \leq e^{-R}, \quad \forall r_0 \leq x \leq R. \quad (2.8)$$

We will have two cases:

Case 1: There exists $R_i \rightarrow \infty$, and $0 < \epsilon_i < \epsilon'(R_i)$, such that $a_0^{\epsilon_i, R_i} \rightarrow \infty$.

Case 2: There exists $R_i \rightarrow \infty$, and $0 < \epsilon_i < \epsilon'(R_i)$, such that $a_0^{\epsilon_i, R_i} \rightarrow K_0 < \infty$.

For Case 1:

For any $r_0 < \alpha < \beta$, $\beta < a_0^{\epsilon_i, R_i}$ for large i . So

$$\frac{d}{dr} (\tilde{v}_{\epsilon_i, R_i}(r) + 4 \ln r) = \tilde{v}'_{\epsilon_i, R_i}(r) + \frac{4}{r} \geq 0, \quad \forall \alpha \leq r \leq \beta.$$

Integrating on $[\alpha, \beta]$, by Lemma 2.2 (ii), we obtain

$$\tilde{v}_{\epsilon_i, R_i}(\beta) + 4 \ln \beta \geq \tilde{v}_{\epsilon_i, R_i}(\alpha) + 4 \ln \alpha.$$

Sending i to ∞ , using (2.8), we obtain

$$v(\beta) + 4 \ln \beta \geq v(\alpha) + 4 \ln \alpha.$$

For Case 2:

For any $K_0 < \alpha < \beta < \infty$, $[\alpha, \beta] \subset (a_0^{\epsilon_i, R_i}, \frac{R_i}{2})$ for large i .

Using (2.7), we have

$$\frac{1}{4} \cdot \frac{R_i}{2} > \frac{1}{\tilde{v}'_{\epsilon_i, R_i}(\frac{R_i}{2})} + \frac{1}{4} \cdot \frac{R_i}{2} > \frac{g(x)}{x^p} \left(\frac{R_i}{2} \right)^{\frac{1}{p}} \quad \text{for } x \in [\alpha, \beta].$$

Since $0 < p < 1$, it follows that

$$\frac{1}{\tilde{v}'_{\epsilon_i, R_i}(x)} + \frac{x}{4} < CR_i^{1-\frac{1}{p}} = o(1) \quad \text{uniform for } x \in [\alpha, \beta].$$

Therefore,

$$\frac{d}{dx}(\tilde{v}_{\epsilon_i, R_i}(x) + 4 \ln x) = \tilde{v}'_{\epsilon_i, R_i}(x) + \frac{4}{x} > o(1) \quad \text{uniform for } x \in [\alpha, \beta].$$

By Lemma 2.2 (ii), integrating on x , we obtain

$$\tilde{v}_{\epsilon_i, R_i}(\beta) + 4 \ln \beta > \tilde{v}_{\epsilon_i, R_i}(\alpha) + 4 \ln \alpha + o(1).$$

Sending $i \rightarrow \infty$, using (2.8) and $R_i \rightarrow \infty$, we obtain

$$v(\beta) + 4 \ln \beta \geq v(\alpha) + 4 \ln \alpha.$$

Proposition 1.8 is now proved. $\square$

3 Liouville Theorem

In this section, we prove Theorem 1.5. Given the asymptotic behavior established in Sect. 2, we can handle the possible singularity of u at infinity either by following the proof in [33, Theorem 1.3], or by invoking a general result in [6]. Here we give the latter.

Recall:

Theorem A [6, Theorem 1.1] *For $n \geq 1$, let Ω be a domain in $\mathbb{R}^n$ containing the origin, and let $F \in C^1(\Omega, \mathbb{R}, \mathbb{R}^n, \mathcal{S}^{n \times n})$ satisfy*

$$-\frac{\partial F}{\partial M_{ij}}(x, s, p, M) > 0, \quad \forall (x, s, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n \times n}.$$

Assume that $u \in C^2(\Omega \setminus \{0\})$ satisfies

$$\text{For any } V \in \mathbb{R}^n, w(x) := u(x) + V \cdot x \text{ satisfies } \inf_{B_r \setminus \{0\}} w = \min_{\partial B_r} w, \quad \forall 0 < r < \bar{r}, \quad (3.1)$$

for some $\bar{r} > 0$, $v \in C^2(\Omega)$, and

$$\begin{aligned} u &> v \quad \text{in } \Omega \setminus \{0\}, \\ F(x, u, \nabla u, \nabla^2 u) &\geq F(x, v, \nabla v, \nabla^2 v) \quad \text{in } \Omega \setminus \{0\}. \end{aligned}$$

Then

$$\liminf_{x \rightarrow 0} (u - v)(x) > 0.$$

Remark 3.1 In $n \geq 2$, a superharmonic function $u \in C^0(B_1 \setminus \{0\})$ satisfying $\inf_{B_1 \setminus \{0\}} u > -\infty$ has the above property (3.1).

Proof of Theorem 1.5 We use the method of moving spheres. The proof is similar to that of [33, Theorem 1.3], see also the proofs of [43, Theorem 1.1] and [44, Theorem 1.1].

To begin with, we recall the notation

$$u_{x,\lambda}(y) = u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 4 \ln \frac{|y-x|}{\lambda}.$$

Denote $F(A^u) = f(\lambda(A^u))$. Notice that if $F(A^u) = 1$, then $F(A^{u,\lambda}) = 1$, where we have used the invariance property of A^u and F .

For the sake of convenience, denote

$$u_\lambda(y) = u\left(\frac{\lambda^2 y}{|y|^2}\right) - 4 \ln \frac{|y|}{\lambda}.$$

Lemma 3.2 *For every $x \in \mathbb{R}^2$, there exists $\lambda_0(x) > 0$ such that $u_{x,\lambda}(y) \leq u(y)$, for all $0 < \lambda < \lambda_0(x)$ and $|y-x| \geq \lambda$.*

Proof Without loss of generality. We assume $x = 0$. Consider the function $u(r, \theta) + 2 \ln r$. We have

$$\frac{\partial}{\partial r}(u(r, \theta) + 2 \ln r) = \frac{\partial u}{\partial r} + \frac{2}{r}.$$

Thus, there exists $r_0 > 0$, such that for all $0 < r < r_0$, we have

$$\frac{\partial}{\partial r}(u(r, \theta) + 2 \ln r) > 0.$$

It follows that for $0 < r_1 \leq r_2 < r_0$, we have

$$u(r_1, \theta) + 2 \ln r_1 \leq u(r_2, \theta) + 2 \ln r_2.$$

Choose $r_2 = |y|$, $r_1 = \frac{\lambda^2}{|y|}$, then for all $0 < \lambda \leq |y| < r_0$, we have

$$u\left(\frac{\lambda^2}{|y|}, \theta\right) + 2 \ln \frac{\lambda^2}{|y|} \leq u(|y|, \theta) + 2 \ln |y|,$$

i.e.,

$$u\left(\frac{\lambda^2}{|y|}, \theta\right) - 4 \ln \frac{|y|}{\lambda} \leq u(|y|, \theta).$$

It follows that for all $0 < \lambda \leq |y| < r_0$, we have

$$u_\lambda(y) \leq u(y).$$

Now let us consider $|y| \geq r_0$. By Proposition 1.8, we have

$$\liminf_{|y| \rightarrow \infty} (u(y) + 4 \ln |y|) > -\infty.$$

Thus, there exists a constant a , such that for all $|y| \geq r_0$, we have

$$u(y) + 4 \ln |y| \geq a.$$

Then for $|y| \geq r_0 \geq \lambda$, we have

$$\begin{aligned} u(y) - u_\lambda(y) &= u(y) + 4 \ln |y| - u\left(\frac{\lambda^2 y}{|y|^2}\right) - 4 \ln \lambda \\ &\geq a - u\left(\frac{\lambda^2 y}{|y|^2}\right) - 4 \ln \lambda. \end{aligned}$$

It follows that there exists $0 < \lambda_0 \leq r_0$, such that for all $|y| \geq r_0$, $0 < \lambda \leq \lambda_0$, we have

$$u(y) \geq u_\lambda(y).$$

The lemma is now proved. $\square$

By Lemma 3.2, we can define for $x \in \mathbb{R}^2$,

$$\bar{\lambda}(x) := \sup\{\mu : u_{x,\mu}(y) \leq u(y), |y - x| \geq \lambda, 0 < \lambda < \mu\} \in (0, +\infty].$$

By Proposition 1.8,

$$\alpha := \liminf_{|x| \rightarrow \infty} (u(x) + 4 \ln |x|) \in (-\infty, +\infty].$$

We have two cases:

Case 1: $\alpha = +\infty$.

We will prove that this case does not occur. We first prove that

$$\bar{\lambda}(x) = +\infty, \quad \forall x \in \mathbb{R}^2. \quad (3.2)$$

Once (3.2) is proved, by Lemma A.1, we obtain that u must be constant and therefore $A^u = 0$. This violates the condition $\lambda(A^u) \in \Gamma$. Hence, Case 1 does not occur. Without loss of generality, we only need to prove (3.2) for $x = 0$. Suppose $\bar{\lambda} := \bar{\lambda}(0) < \infty$. For each $\lambda > 0$ fixed, we have

$$u(y) - u_\lambda(y) = u(y) + 4 \ln |y| - u\left(\frac{\lambda^2 y}{|y|^2}\right) - 4 \ln \lambda, \quad y \in \mathbb{R}^2 \setminus \{0\}.$$

Since $\alpha = +\infty$, we have, for every $\lambda > 0$, $u(y) - u_\lambda(y) \rightarrow +\infty$ as $|y| \rightarrow \infty$. It follows that there exists $M > 0$, such that

$$u_\lambda(y) < u(y), \quad 0 \leq \lambda \leq \bar{\lambda} + 1, \quad |y| \geq M. \quad (3.3)$$

Since $F(A^u) = 1$ is Möbius invariant, we have $F(A^{u_\lambda}) = 1$. Therefore, by the condition $\partial_{\lambda_i} f > 0$ in Γ , there exists a linear second-order elliptic operator L , such that

$$\begin{aligned} L(u - u_{\bar{\lambda}}) &= 0 \quad \text{on } \mathbb{R}^2 \setminus B_{\bar{\lambda}}, \\ u - u_{\bar{\lambda}} &= 0 \quad \text{on } \partial B_{\bar{\lambda}}. \end{aligned}$$

By the maximum principle and the Hopf lemma, we have

$$u - u_{\bar{\lambda}} > 0 \quad \text{on } \mathbb{R}^2 \setminus B_{\bar{\lambda}}, \quad (3.4)$$

$$\frac{\partial}{\partial r}(u - u_{\bar{\lambda}}) > 0 \quad \text{on } \partial B_{\bar{\lambda}}. \quad (3.5)$$

By the compactness of $\partial B_{\bar{\lambda}}$ and (3.5), we have

$$\frac{\partial}{\partial r}(u - u_{\bar{\lambda}})|_{\partial B_{\bar{\lambda}}} \geq b > 0.$$

By the continuity of ∇u , there exists $\delta > 0$, such that

$$\frac{\partial}{\partial r}(u - u_{\bar{\lambda}})(y) \geq \frac{b}{2}, \quad \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \delta, \quad \lambda \leq |y| \leq \lambda + \delta.$$

Since $u = u_{\lambda}$ on ∂B_{λ} , we have

$$u_{\bar{\lambda}}(y) \leq u(y), \quad \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \delta, \quad \lambda \leq |y| \leq \lambda + \delta. \quad (3.6)$$

On the other hand, (3.4) implies

$$u_{\bar{\lambda}}(y) \leq u(y), \quad \bar{\lambda} + \delta \leq y \leq M.$$

By the continuity of u , there exists $0 < \epsilon < \delta$, such that

$$u_{\bar{\lambda}}(y) \leq u(y), \quad \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \epsilon, \quad \bar{\lambda} + \delta \leq y \leq M. \quad (3.7)$$

By (3.3), (3.6) and (3.7), we have proved

$$u_{\bar{\lambda}}(y) \leq u(y), \quad \bar{\lambda} \leq \lambda \leq \bar{\lambda} + \epsilon, \quad |y| \geq \lambda.$$

This violates the definition of $\bar{\lambda}$, and (3.2) is proved.

Case 2: $\alpha \in \mathbb{R}$.

We first claim that $\bar{\lambda}(x) < \infty$, $\forall x \in \mathbb{R}^2$.

In fact, for all $|y - x| \geq \lambda$, $0 < \lambda < \bar{\lambda}(x)$, we have

$$u_{x,\lambda}(y) \leq u(y),$$

i.e.,

$$u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 4 \ln \frac{|y-x|}{\lambda} \leq u(y).$$

Fix $x \in \mathbb{R}^2$, $\lambda > 0$, let $y \rightarrow \infty$, and then let $\lambda \rightarrow \bar{\lambda}(x)$. It follows that

$$\alpha \geq u(x) + 4 \ln \bar{\lambda}(x), \quad \forall x \in \mathbb{R}^2.$$

Therefore, $\bar{\lambda}(x) < \infty$, $\forall x \in \mathbb{R}^2$. The claim is proved.

Next, we will prove

$$u_{x, \bar{\lambda}(x)} \equiv u \quad \text{on } \mathbb{R}^2 \setminus \{x\}, \quad \forall x \in \mathbb{R}^2. \quad (3.8)$$

Once (3.8) is proved, Theorem 1.5 follows from Lemma A.2.

Without loss of generality, we only need to prove (3.8) for $x = 0$. We still denote $\bar{\lambda} = \bar{\lambda}(0)$. It suffices to prove

$$u_{\bar{\lambda}} \equiv u \quad \text{on } \mathbb{R}^2 \setminus B_{\bar{\lambda}}. \quad (3.9)$$

From the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}} \leq u \quad \text{on } \mathbb{R}^2 \setminus B_{\bar{\lambda}}.$$

Suppose on the contrary that (3.9) does not hold, then by the strong maximum principle,

$$u_{\bar{\lambda}} < u \quad \text{on } \mathbb{R}^2 \setminus B_{\bar{\lambda}}.$$

This is equivalent to

$$u_{\bar{\lambda}} > u \quad \text{on } B_{\bar{\lambda}} \setminus \{0\}.$$

By Theorem A, we know

$$\liminf_{|y| \rightarrow 0} (u_{\bar{\lambda}}(y) - u(y)) > 0,$$

namely

$$\liminf_{|y| \rightarrow \infty} (u(y) - u_{\bar{\lambda}}(y)) > 0.$$

Hence, there exists $\epsilon_0 > 0$ and $M > 0$, such that

$$u(y) \geq u_{\bar{\lambda}}(y) + \epsilon, \quad |y| > M.$$

It follows that there exists $\delta > 0$, such that

$$u(y) \geq u_{\lambda}(y), \quad 0 \leq \lambda \leq \bar{\lambda} + \delta, \quad |y| \geq M.$$

Similar to Case 1, using the Hopf lemma, there exists $\epsilon > 0$, such that

$$u(y) \geq u_{\lambda}(y), \quad 0 \leq \lambda \leq \bar{\lambda} + \epsilon, \quad |y| \geq \lambda.$$

This violates the definition of $\bar{\lambda}$.

Therefore, we have proved (3.8). □

4 Conformal Invariance and Möbius Invariance

We first prove Proposition 1.4:

Proof of Proposition 1.4 Let $x \in \mathbb{R}^2$, $s \in \mathbb{R}$, $p \in \mathbb{R}^2$, $M \in \mathcal{S}^{2 \times 2}$, and $O \in O(2)$. Let u be a smooth function satisfying $u(x) = s$, $\nabla u(x) = p$, and $\nabla^2 u(x) = M$. Consider $\psi(z) = z + x$. Now evaluating (1.7) at the origin, we have

$$H(0, s, p, M) = H(x, s, p, M).$$

So H is independent of x . In the following, we denote $H(s, p, M) = H(0, s, p, M)$.

Now define another smooth function u satisfying $u(0) = s$, $\nabla u(0) = 0$, and $\nabla^2 u(0) = M$. Consider $\psi(z) = Oz$, then evaluating (1.7) at the origin gives:

$$H(s, 0, O^{-1}MO) = H(s, 0, M). \quad (4.1)$$

Next define a function u satisfying $u(0) = s$, $\nabla u(0) = p$, and $\nabla^2 u(0) = M$. Consider $\psi(z) = e^{-s/2}z$, so $u_\psi(z) = u(e^{-s/2}z) - s$. Evaluating (1.7) at the origin gives:

$$H(0, e^{-s/2}p, e^{-s}M) = H(s, p, M). \quad (4.2)$$

Finally, define $\alpha = -\frac{4p}{|p|^2} \in \mathbb{R}^2$, and pick a smooth function u satisfying $u(\alpha) = s$, $\nabla u(\alpha) = p$, and $\nabla^2 u(\alpha) = M$. Consider the Möbius transformation $\psi(z) = \frac{16z}{|p|^2|z|^2}$. By a direct computation, we have $u_\psi(z) = u(\frac{16z}{|p|^2|z|^2}) - 4 \ln |z| + 4 \ln 4 - 4 \ln |p|$ and $\psi^{-1}(\alpha) = -\frac{4p}{|p|^2}$.

We can easily check that

$$u_\psi \circ \psi^{-1}(\alpha) = u(\alpha) = s$$

and

$$\nabla u_\psi \circ \psi^{-1}(\alpha) = 0.$$

So

$$A^{u_\psi} \circ \psi^{-1}(\alpha) = -\frac{\nabla^2 u_\psi \circ \psi^{-1}(\alpha)}{e^{u_\psi}} = -\frac{\nabla^2 u_\psi \circ \psi^{-1}(\alpha)}{e^s}.$$

By a direct computation, we can check the property that for any $y \in \mathbb{R}^2$ and Möbius transformation ψ ,

$$A^{u_\psi} \circ \psi^{-1}(y) \sim A^u(y),$$

where the notation $A \sim B$ means A and B are orthogonally similar to each other. Hence,

$$-\frac{\nabla^2 u_\psi \circ \psi^{-1}(\alpha)}{e^s} \sim A^u(\alpha),$$

i.e.,

$$\nabla^2 u_\psi \circ \psi^{-1}(\alpha) \sim -e^s A^u(\alpha). \quad (4.3)$$

Evaluating (1.7) at $\psi^{-1}(\alpha)$, using also (4.1), (4.2) and (4.3),

$$\begin{aligned} H(s, p, M) &= H(s, 0, \nabla^2 u_\psi \circ \psi^{-1}(\alpha)) \\ &= H(s, 0, -e^s A^u(\alpha)) \\ &= H(0, 0, -A^u(\alpha)). \end{aligned}$$

The proposition follows. $\square$

Then we prove Proposition 1.2:

Proof of Proposition 1.2 It is easy to see that we still have

$$H(\cdot, u, \nabla u, \nabla^2 u) = F(A^u),$$

where F is invariant under orthogonal conjugation. However, we observe that for general meromorphic functions, the relation $A^{u_\psi} \sim A^u \circ \psi$ does not hold. Now we are going to prove that F must be a function of the trace.

Set $\psi(z) = iz^2$, i.e., $\psi(x, y) = (2xy, -x^2 + y^2)$. Set $u(x, y) = ax^2$. So

$$A^u \circ \psi = -\frac{1}{e^{4ax^2y^2}} \begin{pmatrix} 2a - 4a^2x^2y^2 & 0 \\ 0 & 4a^2x^2y^2 \end{pmatrix}.$$

Evaluating (1.6) at $(0, y)$, we have

$$A^u \circ \psi(0, y) = \begin{pmatrix} -2a & 0 \\ 0 & 0 \end{pmatrix}.$$

Similarly, by a direct computation,

$$\begin{aligned} u_\psi(x, y) &= 4ax^2y^2 + \ln 4 + \ln(x^2 + y^2), \\ A^{u_\psi}(0, y) &= \begin{pmatrix} -2a - \frac{3}{4y^4} & 0 \\ 0 & \frac{3}{4y^4} \end{pmatrix}. \end{aligned}$$

Taking appropriate a and y , the relation $F(A^u) \circ \psi = F(A^{u_\psi})$ implies that

$$F\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right) = F\left(\begin{pmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

for any $\{\lambda_1, \lambda_2\}$ satisfying that at least one of them is positive. Negative case comes from evaluating (1.6) at $(x, 0)$. Hence F must be a function of the trace, i.e., $F(A^u) = g(-e^{-u} \Delta u)$. Moreover, it is straightforward to check that $g(-e^{-u} \Delta u)$ is conformally invariant. $\square$

Appendix A: Two Calculus Lemmas

We now state two calculus lemmas for the reader's convenience.

Lemma A.1 *Let $u \in C^1(\mathbb{R}^2)$ satisfy*

$$u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 4 \ln \frac{|y-x|}{\lambda} \leq u(y), \quad \forall \lambda > 0, x \in \mathbb{R}^2, |y-x| \geq \lambda.$$

Then u must be constant.

Proof Let $f = e^u$, then we have

$$\left(\frac{\lambda}{|y-x|}\right)^4 f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \quad \forall \lambda > 0, x \in \mathbb{R}^2, |y-x| \geq \lambda.$$

By [43, Lemma 11.1] (see also [44, Lemma 3.3]), we conclude that f is a constant; hence, u is a constant. $\square$

Lemma A.2 *Let $u \in C^1(\mathbb{R}^2)$. Suppose that for every $x \in \mathbb{R}^2$, there exists $\lambda(x) > 0$ such that*

$$u_{x, \lambda(x)}(y) = u(y), \quad y \in \mathbb{R}^2 \setminus \{x\}.$$

Then for some $a > 0, b > 0, \bar{x} \in \mathbb{R}^2$,

$$u(x) = 2 \ln \frac{8a}{8|x-\bar{x}|^2 + b}.$$

Proof Let $f = e^u$, then we have, for every $x \in \mathbb{R}^2$, there exists $\lambda(x) > 0$ such that

$$\left(\frac{\lambda}{|y-x|}\right)^4 f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) = f(y), \quad y \in \mathbb{R}^2 \setminus \{x\}.$$

By [43, Lemma 11.1] (see also [44, Lemma 3.7]),

$$f(x) = \pm \left(\frac{a}{d + |x - \bar{x}|^2}\right)^2.$$

Lemma A.2 follows. $\square$

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