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Asymptotic Stability of Landau Solutions to Navier–Stokes System Under L^p-Perturbations

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Abstract. In this paper, we show that Landau solutions to the Navier–Stokes system are asymptotically stable under L^3-perturbations. We give the local well-posedness of solutions to the perturbed system with the initial data in the L^3_σ space and the global well-posedness with the small initial data in the L^3_σ space, together with a study of the L^q decay for all $q > 3$. Moreover, we have also studied the local well-posedness, the global well-posedness and the stability in L^p spaces for $3 < p < \infty$.

Keywords. Navier–Stokes system, Landau solutions, Global well-posedness, Asymptotic stability.

1. Introduction

The Cauchy problem for the incompressible Navier–Stokes system in $\{(x,t)|x \in \mathbb{R}^3, t \geq 0\}$ with the given initial data and the external force has the form

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ and p denote the velocity field and pressure respectively.

Note that when we consider the construction of solutions to the Cauchy problem (1.1), there are essentially two methods: the energy method and the perturbation theory. The energy method is based on *a-priori* energy estimate

$$\int_{\mathbb{R}^3} |u(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} 2|\nabla u(x,s)|^2 dxds \leq \int_{\mathbb{R}^3} |u_0(x)|^2 dx + \int_0^t \int_{\mathbb{R}^3} 2(f \cdot u)(x,s) dxds.$$

The global existence of weak solutions was established by Leray [27] for the divergence-free initial data $u_0 \in L^2(\mathbb{R}^3)$ and $f = 0$. The energy method gives the existence, but the uniqueness and regularity for solutions still remain open, see *e.g.* [1,5,8,16,25,26,45] and references therein.

As for the perturbation theory, we treat the nonlinear term $(u \cdot \nabla)u$ as a perturbation and use the scaling property to choose function spaces. As we know, the system (1.1) has the natural scaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t).$$

Therefore, the space $L^3(\mathbb{R}^3)$ is a well-known simple example of the scaling-invariant space. By the Duhamel principle, we can rewrite the solution into an integral formulation

$$u(x,t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}(f - u \cdot \nabla u) ds, \tag{1.2}$$

where $\mathbb{P}$ denotes the Leray projector which projects on divergence-free vector fields. Solutions constructed in this way are called mild solutions [25,26]. Usually, by the contraction mapping principle, we can obtain the global well-posedness of mild solutions to the system (1.1) with small initial data in appropriate scaling-invariant spaces. We refer readers to [7,8,20,22,25,26,37] for additional background and references.

There are many results on the existence of weak solutions and the L^2-decay property of weak solutions of the Navier–Stokes system, see e.g. [20,33,39,48], also for the convection-diffusion equations [11] and references therein. When $f = 0$, the L^2-decay property of weak solutions to the system (1.1) can be viewed as the global asymptotic stability in L^2 of the trivial solution $(u, p) = (0, 0)$. Later, Borchers and Miyakawa [4] addressed similar questions on the global asymptotic stability of a family of stationary solutions.

The stationary Navier–Stokes system in $\mathbb{R}^3$ has the form

$$\begin{cases} -\Delta v + (v \cdot \nabla)v + \nabla p = f, \\ \nabla \cdot v = 0. \end{cases} \tag{1.3}$$

When $f = (b(c)\delta_0, 0, 0)$ with $b(c) = \frac{8\pi c}{3(c^2-1)}\left(2 + 6c^2 - 3c(c^2-1)\ln\left(\frac{c+1}{c-1}\right)\right)$ and δ_0 the Dirac measure, (v_c, p_c) given by the following formulas

$$\begin{aligned} v_c^1(x) &= 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|\,(c|x| - x_1)^2}, \quad v_c^2(x) = 2\frac{x_2\,(cx_1 - |x|)}{|x|\,(c|x| - x_1)^2}, \\ v_c^3(x) &= 2\frac{x_3\,(cx_1 - |x|)}{|x|\,(c|x| - x_1)^2}, \quad p_c(x) = 4\frac{cx_1 - |x|}{|x|\,(c|x| - x_1)^2}, \end{aligned} \tag{1.4}$$

with $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and a constant $|c| > 1$, are distributional solutions to the system (1.3) in $\mathbb{R}^3$. We note that $b(c)$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$, $\lim_{c\to 1} b(c) = +\infty$, $\lim_{c\to -1} b(c) = -\infty$ and $\lim_{|c|\to\infty} b(c) = 0$. The explicit stationary solutions (1.4) were discovered by Landau [24]. These solutions have been called Landau solutions. Tian and Xin [46] proved that all $(-1)-$homogeneous, axisymmetric nonzero solutions of the system (1.3) in $C^2(\mathbb{R}^3\backslash\{0\})$ are Landau solutions. Šverák [43] proved that Landau solutions are the only $(-1)-$homogeneous solutions in $C^2(\mathbb{R}^3\backslash\{0\})$. More details can be found in [9,24,41,43,46].

Karch and Pilarczyk [18] showed that Landau solutions are asymptotically stable under any L^2-perturbations. The crucial role played in their paper is an application of the Hardy-type inequality

$$\left|\int_{\mathbb{R}^3} w \cdot (w \cdot \nabla)v_c dx\right| \le K(c)\|\nabla \otimes w\|_2^2, \tag{1.5}$$

where the positive function $K(c) = 12\max_{j,k\in\{1,2,3\}} K_{j,k}(c)$,

$$\left|\partial_{x_j} v_c^k(x)\right| \le \frac{K_{j,k}(c)}{|x|^2}, \ j, k \in \{1, 2, 3\}, \ \forall\, x \in \mathbb{R}^3\backslash\{0\}. \tag{1.6}$$

Moreover, $K_{j,k}(c)$ satisfies

$$\lim_{|c|\to 1} K_{j,k}(c) = +\infty \text{ and } \lim_{|c|\to\infty} K_{j,k}(c) = 0. \tag{1.7}$$

In 2017, Karch, Pilarczyk and Schonbek [19] generalized the work of [18]. They gave a new method to show the L^2-asymptotic stability of a large class of global-in-time solutions including the Landau solutions. Their work also generalizes results in a series of articles on the L^2-asymptotic stability either of the zero solution [3,17,35,38,39,48] or nontrivial stationary solutions [4] to the system (1.1). The above results give the existence in the L^2_σ space [18,19], while the uniqueness in the L^2_σ space is a major open problem. We will consider the stability of Landau solutions to the Navier–Stokes system in L^p_σ spaces with $3 \le p < \infty$.

We denote $(u,p)(x,t)$ the solution to the Navier–Stokes system (1.1) with the given external force $f = (b(c)\delta_0, 0, 0)$ and the initial data $u_0 = v_c + w_0$. By a direct calculation, functions $w(x,t) = u(x,t) - v_c(x)$ and $\pi(x,t) = p(x,t) - p_c(x)$ satisfy the following system

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + (w\cdot\nabla)v_c + (v_c\cdot\nabla)\, w + \nabla\pi = 0, \\ \nabla\cdot w = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{1.8}$$

We will consider the well-posedness problem of the system (1.8) in L^p spaces with $3 \le p < \infty$. We will obtain the global well-posedness of solutions to the system (1.8) with the small initial data in the L^3_σ space, and the local well-posedness with the general initial data in the L^3_σ space, see Theorem 1.1. For the initial data $w_0 \in L^p_\sigma$ with $3 < p < \infty$, we get the local well-posedness results, see Theorem 1.2. In addition, for the general initial data in L^3_σ, we have the global existence of $L^2 + L^3$ weak solutions, see Definition 1.3 and Theorem 1.5.

Karch and Pilarczyk [18] denote the linear operator $\mathcal{L}$

$$\mathcal{L}u = -\Delta u + \mathbb{P}\left((u\cdot\nabla)v_c\right) + \mathbb{P}\left((v_c\cdot\nabla)\, u\right). \tag{1.9}$$

For the system (1.8), we can rewrite the solution in the following formula

$$w(x,t) = e^{-t\mathcal{L}}w_0 - \int_0^t e^{-(t-s)\mathcal{L}}\mathbb{P}\nabla\cdot(w\otimes w)ds := a + N(w,w). \tag{1.10}$$

Karch and Pilarczyk [18] showed that $-\mathcal{L}$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on $L^2_\sigma(\mathbb{R}^3)$. We show that for $1 < q < \infty$, $-\mathcal{L}$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on $L^q_\sigma(\mathbb{R}^3)$, see Theorem 3.1 in Sect. 3.

1.1. L^p Mild Solutions, $3 \le p < \infty$

Let us give the following standard definition of the L^p mild solution, $3 \le p < \infty$.

Definition 1.1. Let $3 \le p < \infty$ and $T > 0$, a function w is a L^p mild solution of the system (1.8) with the initial data $w_0 \in L^p_\sigma(\mathbb{R}^3)$ on $[0,T]$, if

$$w \in C([0,T]; L^p_\sigma(\mathbb{R}^3)) \cap L^{\frac{4p}{3}}([0,T]; L^{2p}_\sigma(\mathbb{R}^3)), \tag{1.11}$$

and

$$w(x,t) = e^{-t\mathcal{L}}w_0 - \int_0^t e^{-(t-s)\mathcal{L}}\mathbb{P}\nabla\cdot(w\otimes w)ds. \tag{1.12}$$

This solution is global if (1.11) and (1.12) hold for any $0 < T < \infty$.

In the above, $e^{-t\mathcal{L}}$ denotes the analytic semigroup of bounded linear operators on $L^p_\sigma(\mathbb{R}^3)$ generated by $-\mathcal{L}$, see Lemma 2.4 and Theorem 3.1. Properties of $\int_0^t e^{-(t-s)\mathcal{L}}\mathbb{P}\nabla\cdot(w\otimes w)ds$ can be seen in Lemma 2.5.

Now, we give the following theorem which shows the well-posedness results in the L^3_σ space and the L^q-decay rates of solutions to the system (1.8).

Theorem 1.1. *There exist positive universal constants c_3, ε_0 and C with the following properties (i) For every $|c| > c_3$ and $w_0 \in L^3_\sigma(\mathbb{R}^3)$, there exists a positive constant T depending only on w_0 such that the system (1.8) has a unique L^3 mild solution w on $[0,T]$. Moreover, $\nabla(|w|^{\frac{3}{2}}) \in L^2([0,T]; L^2(\mathbb{R}^3))$.*

(ii) If in addition, $\|w_0\|_{L^3(\mathbb{R}^3)} < \varepsilon_0$, then the system (1.8) has a unique global L^3 mild solution w. Moreover, $\nabla(|w|^{\frac{3}{2}}) \in L^2([0,\infty); L^2(\mathbb{R}^3))$,

$$\|w\|_{C_t(L^3_x)\cap L^4_t(L^6_x)} + \|\nabla(|w|^{\frac{3}{2}})\|^{\frac{2}{3}}_{L^2_t L^2_x} \le C\|w_0\|_{L^3(\mathbb{R}^3)}, \tag{1.13}$$

and

$$\lim_{t\to\infty}\|w(t)\|_{L^3(\mathbb{R}^3)}=0. \tag{1.14}$$

(iii) For any $q>3$, there exists a positive constant $\tilde{c}_q$ depending only on q such that when $|c|>\tilde{c}_q$, the solution in (ii) satisfies

$$w\in L^\infty([\tau,\infty),L^q_\sigma(\mathbb{R}^3)),\quad forall\quad \tau>0,$$

and

$$\|w(t)\|_{L^q(\mathbb{R}^3)}\le(\frac{1}{3}-\frac{1}{q})^{\frac{3}{2}(\frac{1}{3}-\frac{1}{q})}t^{\frac{3}{2q}-\frac{1}{2}}\|w_0\|_{L^3(\mathbb{R}^3)},\quad forall\quad t>0. \tag{1.15}$$

Remark 1.1. From (2.15) with $p=3$, (2.42), (2.53) and (2.56) in this paper, we see a more detailed dependence of $\tilde{c}_q$. On the other hand, we tend to believe that c can be chosen as a constant independent of q, and we plan to investigate the L^∞ decay property in our future work.

Remark 1.2. It follows from Theorem 1.1 that the flow described by the Landau solution is asymptotically stable under L^3-perturbations.

Remark 1.3. For the two-dimensional Navier–Stokes system, Carlen and Loss [10] gave the decay rate of solutions to the vorticity equation. We adapt the method in [10] to give the decay rate of solutions to the system (1.8), and we treat the pressure term π by using the A_p weight inequalities for the Riesz transforms [12,42].

Then, we have the following local well-posedness results with the initial data $w_0\in L^p_\sigma$, $3<p<\infty$.

Theorem 1.2. *For $p\in(3,\infty)$ and $w_0\in L^p_\sigma(\mathbb{R}^3)$, there exist two constants c_p and T, where c_p depends only on p while T depends only on p and $\|w_0\|_{L^p}$, such that for all $|c|>c_p$, the system (1.8) has a unique L^p mild solution w on $[0,T]$, satisfying $\nabla(|w|^{\frac{p}{2}})\in L^2_t([0,T];L^2_x(\mathbb{R}^3))$. If in addition, $w_0\in L^p_\sigma\cap L^3_\sigma(\mathbb{R}^3)$, $\|w_0\|_{L^3}<\varepsilon_0$, where ε_0 is as in Theorem 1.1, there exists a unique global L^p mild solution w to the system (1.8), satisfying*

$$\|w\|_{C_t(L^p_x)\cap L_t^{\frac{4p}{3}}(L^{2p}_x)}+\|\nabla(|w|^{\frac{p}{2}})\|^{\frac{2}{p}}_{L^2_tL^2_x}\le C\|w_0\|_{L^p}. \tag{1.16}$$

Remark 1.4. Under conditions of Theorem 1.2, from Theorem 1.1, we have that (1.13)-(1.14) hold, and (1.15) holds if $|c|>\tilde{c}_q$.

Remark 1.5. Note that $w_0\in L^p$ with $p>3$ implies $w_0\in L^2_{uloc}$. For the Navier–Stokes system with $u_0\in L^2_{uloc}$, several authors [2,21,25,26] gave the local existence of the weak solution u. Moreover, the global weak solution exists for the decaying initial data $u_0\in E_2=\{f\in L^2_{\rm uloc}:\lim_{|x_0|\to\infty}\|f\|_{L^2(B(x_0,1))}=0\}$. Kown and Tsai [23] generalized the global existence for the non-decaying initial data with slowly decaying oscillation. Very recently, J.J. Zhang and T. Zhang [49] have given the local existence of solutions to the system (1.8) with the initial data $w_0\in L^p_{uloc}$, $p\ge2$. Because of these results, we plan to study the global existence of weak solutions to the system (1.8) with the initial data $w_0\in L^p_\sigma$ for $p>3$ in our future work.

Remark 1.6. L. Li, Y.Y. Li and X. Yan investigated homogeneous solutions of the stationary Navier–Stokes system with isolated singularities on the unit sphere [28–31]. For a subclass of (-1)-homogeneous axisymmetric no-swirl solutions on the unit sphere minus north and south poles classified in [29], Y.Y. Li and X. Yan have proved in [32] the asymptotic stability under L^2-perturbations. We will focus on these homogeneous solutions in our future work.

Results in Theorems 1.1 and 1.2 show the existence and uniqueness of the solution w to the system (1.8) in the corresponding space. Actually the solution depends continuously on the initial data.

Theorem 1.3. *For every $|c| > c_p$ and $u_0 \in L^p_\sigma(\mathbb{R}^3)$ with $3 \le p < \infty$, assume that u is the unique mild solution to the system (1.8) on $[0, T_{max})$. Then, for any $T \in (0, T_{max})$, there exists $\varepsilon > 0$ such that for any $v_0 \in L^p_\sigma(\mathbb{R}^3)$, $\|u_0 - v_0\|_{L^p} < \varepsilon$, there exists a unique L^p mild solution v on $[0, T]$ with the initial data $v|_{t=0} = v_0$. Moreover,*

$$\lim_{u_0 \to v_0 \text{ in } L^p} \left(\|u - v\|_{C_T L^p_x \cap L^{\frac{4p}{3}}_T L^{2p}_x} + \left\| \nabla \left(|u - v|^{\frac{p}{2}} \right) \right\|^{\frac{2}{p}}_{L^2_T L^2_x} \right) = 0. \tag{1.17}$$

The constant c_p in the above theorem is the one given in Theorems 1.1 and 1.2.

Remark 1.7. Karch, Pilarczyk and Schonbek [19] showed the L^2-asymptotic stability of a large class of global-in-time solutions including the Landau solutions. Based on similar proof of Theorem 1.3, we can obtain the L^3-asymptotic stability of a class of solutions $v_c + w$, where w is as in Theorem 1.1. More precisely, letting V as perturbation of $v_c + w$, when $\|V_0\|_{L^3} \le (4C^2 e^{2C \int_0^\infty \|w\|^4_{L^6} dt})^{-1}$, using the similar proof of (7.5), we obtain

$$\|V\|_{C_t L^3_x \cap L^4_t L^6_x} + \left\| \nabla \left(|V|^{\frac{3}{2}} \right) \right\|^{\frac{2}{3}}_{L^2_t L^2_x} \le 2C \|V_0\|_{L^3} e^{C \int_0^\infty \|w\|^4_{L^6} dt}. \tag{1.18}$$

1.2. Weak Solution

Karch and Pilarczyk [18,19] proved the following results: for every $w_0 \in L^2_\sigma\left(\mathbb{R}^3\right)$, there exists a global weak solution

$$w \in C_w\left([0,T], L^2_\sigma\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T], \dot{H}^1_\sigma\left(\mathbb{R}^3\right)\right)$$

for every $T > 0$ which

$$\|w(t)\|_2^2 + 2(1 - K(c)) \int_s^t \|\nabla \otimes w(\tau)\|_2^2 \mathrm{d}\tau \leqq \|w(s)\|_2^2 \tag{1.19}$$

for almost all $s \ge 0$, including $s = 0$ and all $t \ge s$. The definition of the weak solution is as follows.

Definition 1.2. (L^2-weak solution) For $w_0 \in L^2_\sigma(\mathbb{R}^3)$, a function w is a L^2-weak solution of the system (1.8) on $[0, T]$, if

i) $w \in C_w([0,T]; L^2_\sigma(\mathbb{R}^3)) \cap L^2([0,T]; \dot{H}^1_\sigma(\mathbb{R}^3))$.

ii) For all $t \ge s \ge 0$, all $\varphi \in C\left([0,\infty), H^1_\sigma\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,\infty), L^2_\sigma\left(\mathbb{R}^3\right)\right)$,

$$\begin{aligned}(w(t), \varphi(t)) + \int_s^t \left[(\nabla w, \nabla \varphi) + (w \cdot \nabla w, \varphi) + (w \cdot \nabla v_c, \varphi) + (v_c \cdot \nabla w, \varphi) \right] \mathrm{d}\tau \\ = (w(s), \varphi(s)) + \int_s^t (w, \varphi_\tau)\, \mathrm{d}\tau.\end{aligned}$$

iii) For all $\phi \in C^\infty_c(\mathbb{R}^3)$, $\lim_{t \to 0} \int_{\mathbb{R}^3} w \cdot \phi dx = \int_{\mathbb{R}^3} w_0 \cdot \phi(0) dx$.

iv) w satisfies the energy inequality

$$\begin{aligned}&\int_{\mathbb{R}^3} |w|^2 \xi(x,t) dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla w|^2 \xi dx ds \\ &\quad \le \int_{\mathbb{R}^3} |w_0|^2 \xi(x,0) dx + \int_0^t \int_{\mathbb{R}^3} (2 v_c \otimes w : \nabla w \xi \\ &\qquad + (\partial_s \xi + \Delta \xi) |w|^2 + (|w|^2 + 2\pi + 2 v_c \cdot w)(w \cdot \nabla) \xi + |w|^2 v_c \cdot \nabla \xi) dx ds,\end{aligned}$$

for any $t \in [0, T]$ and for all non-negative smooth functions $\xi \in C^\infty_c([0,T] \times \mathbb{R}^3)$.

The following is a weak-strong uniqueness theorem that is analogous to the one for the Navier–Stokes system (Theorem 4.4 in [47]).

Theorem 1.4. *Let $|c| > 8\sqrt{2}+1$, $w_0 \in L^2_\sigma(\mathbb{R}^3)$. Assume that u, v are L^2-weak solutions of the system (1.8) on $[0,T]$ with the initial data $u|_{t=0} = v|_{t=0} = w_0$. Suppose $u \in L^s([0,T]; L^q(\mathbb{R}^3))$, $\frac{3}{q}+\frac{2}{s} = 1$, $q, s \in [2,\infty]$. When $(q,s) = (3,\infty)$, assume that $\|u\|_{L^\infty([0,T];L^3_\sigma(\mathbb{R}^3))}$ is sufficiently small. Then $u \equiv v$.*

We give the following proposition for which the detailed proof can be seen in Sect. 4.

Proposition 1.1. *For $w_0 \in L^p_\sigma(\mathbb{R}^3) \cap L^2_\sigma(\mathbb{R}^3)$, $p \geq 3$, $|c| > c_p$, where c_p is in Theorem 1.2, let w be the L^p mild solution of the system (1.8) on $[0,T]$. Then w is a L^2-weak solution of the system (1.8) on $[0,T]$.*

According to (1.19), there exists $t_0 > 0$ such that $w(t_0) \in L^p_\sigma \cap L^3_\sigma(\mathbb{R}^3)$, $3 < p \leq 6$ and $\|w(t_0)\|_{L^3} < \varepsilon_0$. According to Theorem 1.2, when $|c| > c_p$, there exists a unique L^p mild solution on $[t_0, \infty)$ to the system (1.8) with the initial data $w(t_0)$.

Corollary 1.1. *For $w_0 \in L^2_\sigma(\mathbb{R}^3)$, let w be a L^2-weak solution of the system (1.8). Then for every $3 \leq p \leq 6$ and $|c| > c_p$, there exists $T > 0$ such that $w(\cdot + T)$ is a L^p mild solution to the system (1.8) with the initial data $w(T) \in L^p_\sigma \cap L^2_\sigma(\mathbb{R}^3)$.*

Remark 1.8. Under conditions of Corollary 1.1, we have $\nabla(|w|^{\frac{p}{2}}) \in L^2([T,\infty); L^2(\mathbb{R}^3))$, and

$$\lim_{t\to\infty} \|w(t)\|_{L^2(\mathbb{R}^3)} = 0. \tag{1.20}$$

Furthermore, for $q \geq 3$, $|c| > \tilde{c}_q$, where $\tilde{c}_q$ is as in Theorem 1.1,

$$\|w(t)\|_{L^q(\mathbb{R}^3)} \leq (\frac{1}{2} - \frac{1}{q})^{\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}(t-T)^{\frac{3}{2}(\frac{1}{q}-\frac{1}{2})}\|w(T)\|_{L^2(\mathbb{R}^3)}, \quad \text{for all} \quad t > T. \tag{1.21}$$

For the general initial data $w_0 \in L^3_\sigma(\mathbb{R}^3)$, we will give the global existence of the $L^2 + L^3$ weak solution to the system (1.8). Inspired by the method in [6,19,40], for any $w_0 \in L^3(\mathbb{R}^3)$, we make a decomposition

$$w_0 = v_{10} + v_{20}, \tag{1.22}$$

with $\|v_{10}\|_{L^3} < \varepsilon_0$, where ε_0 is as in Theorem 1.1 and $v_{20} \in L^2 \cap L^3(\mathbb{R}^3)$. According to Theorem 1.1, there exists a unique global L^3 mild solution v_1 to the system

$$\begin{cases} \partial_t v_1 - \Delta v_1 + (v_1 \cdot \nabla)v_1 + (v_1 \cdot \nabla)v_c + (v_c \cdot \nabla)\, v_1 + \nabla \pi_1 = 0, \\ \nabla \cdot v_1 = 0, \\ v_1(x,0) = v_{10}. \end{cases} \tag{1.23}$$

Then $v_2 = w - v_1$ satisfies

$$\begin{cases} \partial_t v_2 - \Delta v_2 + (v_2 \cdot \nabla)v_2 + (v_2 \cdot \nabla)(v_c + v_1) + ((v_c + v_1) \cdot \nabla)\, v_2 + \nabla \pi_2 = 0, \\ \nabla \cdot v_2 = 0, \\ v_2(x,0) = v_{20}. \end{cases} \tag{1.24}$$

We can get the global existence of w by investigating the global existence of v_2. From Theorem 2.7 in [19], the system (1.24) has a weak solution

$$v_2 \in C_w\left([0,T]; L^2_\sigma\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T]; \dot{H}^1_\sigma\left(\mathbb{R}^3\right)\right) \tag{1.25}$$

for each $T > 0$, satisfying the strong energy inequality

$$\|v_2(t)\|_2^2 + 2\left(1 - K \sup_{t>0} \|v_c + v_1\|_{L^3_w}\right) \int_s^t \|\nabla v_2(\tau)\|_2^2 \mathrm{d}\tau \leqslant \|v_2(s)\|_2^2 \tag{1.26}$$

for a constant $K > 0$, almost all $s \geq 0$ and all $t \geq s$, and

$$\lim_{t\to\infty} \|v_2(t)\|_2 = 0. \tag{1.27}$$

In the spirit of the notion of the weak L^3-solution introduced by Seregin and Šverák [40], we give the following definition of the $L^2 + L^3$ weak solution of the system (1.8).

Definition 1.3. Let $T>0$, $w_0\in L^3_\sigma(\mathbb{R}^3)$, $w_0=v_{10}+v_{20}$. A vector field w is called a L^2+L^3 weak solution to the system (1.8) in $\mathbb{R}^3\times(0,T)$, if $w=v_1+v_2$ for some $v_1\in C((0,T);L^3_\sigma(\mathbb{R}^3))\cap L^4((0,T);L^6_\sigma(\mathbb{R}^3))$, and $v_2\in C_w\left((0,T);L^2_\sigma\left(\mathbb{R}^3\right)\right)\cap L^2\left((0,T);\dot{H}^1_\sigma\left(\mathbb{R}^3\right)\right)$ such that v_1 is a L^3 mild solution of (1.23) and v_2 satisfies the following conditions:

i) v_2 satisfies (1.24) in the sense of distributions,

ii)

$$\lim_{t\to 0}\|v_2(\cdot,t)-v_{20}\|_{L^2}=0, \tag{1.28}$$

iii) for all $t\in(0,T)$

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}|v_2(x,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla v_2|^2(x,s)dxds\\&\quad\le\frac{1}{2}\int_{\mathbb{R}^3}|v_{20}(x)|^2dx+\int_0^t\int_{\mathbb{R}^3}v_2\otimes(v_c+v_1):\nabla v_2dxds,\end{aligned} \tag{1.29}$$

iv) for a.a. $t\in(0,T)$ and any non-negative function $\varphi\in C_c^\infty\left(\mathbb{R}^3\times(0,T)\right)$,

$$\begin{aligned}&\int_{\mathbb{R}^3}|v_2(x,t)|^2\varphi(x,t)dx+2\int_0^t\int_{\mathbb{R}^3}|\nabla v_2|^2\varphi dxds\\&\le\int_0^t\int_{\mathbb{R}^3}(2(v_c+v_1)\otimes v_2:\nabla v_2\varphi+(\partial_s\varphi+\Delta\varphi)|v_2|^2\\&\quad+(|v_2|^2+2\pi_2+2(v_c+v_1)\cdot v_2)(v_2\cdot\nabla)\varphi+|v_2|^2(v_c+v_1)\cdot\nabla\varphi)dxds.\end{aligned} \tag{1.30}$$

We say w is a global L^2+L^3 weak solution to the system (1.8) if it is a L^2+L^3 weak solution to the system (1.8) in $\mathbb{R}^3\times(0,T)$ for all $0<T<\infty$. Hence, we give the existence of global L^2+L^3 weak solutions to the system (1.8) as follows.

Theorem 1.5. *Assume that $w_0\in L^3_\sigma(\mathbb{R}^3)$ has a decomposition $w_0=v_{10}+v_{20}$ with $v_{10}\in L^3_\sigma(\mathbb{R}^3)$, $\|v_{10}\|_{L^3(\mathbb{R}^3)}<\varepsilon_0$ and $v_{20}\in L^2_\sigma\cap L^3_\sigma(\mathbb{R}^3)$ where ε_0 is as in Theorem 1.1. Then, there exists a global L^2+L^3 weak solution w to the system (1.8) with $w=v_1+v_2$, $v_1(\cdot,0)=v_{10}$ and $v_2(\cdot,0)=v_{20}$.*

The proof of Theorem 1.5 is based on the proof of Theorem 2.1 in [19]. For the convenience of the reader, we will give details in Sect. 5.

Remark 1.9. When the initial data $w_0\in L^p_\sigma(\mathbb{R}^3)$, $2<p\le 3$, by interpolation theory, w_0 has a decomposition $w_0=v_{10}+v_{20}$ with $v_{10}\in L^3_\sigma(\mathbb{R}^3)$, $\|v_{10}\|_{L^3(\mathbb{R}^3)}<\varepsilon_0$ and $v_{20}\in L^2_\sigma$ where ε_0 is as in Theorem 1.1. Then, we can easily obtain the global existence of the L^2+L^3 weak solution to the system (1.8).

Scheme of the proof and organization of the paper. In Sect. 2, we give the proof of Theorem 1.1. In other words, we prove the local well-posedness of solutions to the system (1.8) with the general initial data, and the global well-posedness of solutions to the system (1.8) with the small initial data in the L^3_σ space. Also, we investigate the decay rate of solutions to the system (1.8). In Sect. 3, some properties of the linear operator $\mathcal{L}$ on L^p, $1<p<\infty$, are studied. In Sect. 4, we prove Theorem 1.4, Proposition 1.1 and illustrate Corollary 1.1 briefly. In Sect. 5, we illustrate Theorem 1.5, i.e. the global existence of the L^2+L^3 weak solution to the system (1.8). In Sects. 6 and 7, we give the proofs of Theorems 1.2 and 1.3, respectively.

Notations.

- We denote $\|\cdot\|_p$ (or $\|\cdot\|_{L^p}$), $\|\cdot\|_{L^p_tL^q_x}$, $\|\cdot\|_{L^p_TL^q_x}$, $\|\cdot\|_{C_tL^q_x}$, $\|\cdot\|_{C_TL^q_x}$ the norms of the Lebesgue spaces $L^p_x(\mathbb{R}^3)$, $L^p_t([0,\infty);L^q_x(\mathbb{R}^3))$, $L^p_t([0,T];L^q_x(\mathbb{R}^3))$, $C([0,\infty);L^q_x(\mathbb{R}^3))$, $C_t([0,T];L^q_x(\mathbb{R}^3))$, respectively, with $p,q\in[1,\infty]$.
- $C_0^\infty(\mathbb{R}^3)$ denotes the set of smooth and compactly supported functions.

• $C_w([0,T];L^q_x(\mathbb{R}^3))$ with $q\in[1,\infty)$ denotes the set of weakly continuous $L^q(\mathbb{R}^3)$-valued functions in t, i.e. for any $t_0\in[0,T]$ and $w\in L^{q'}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} v(x,t)\cdot w(x)dx\to\int_{\mathbb{R}^3} v\,(x,t_0)\cdot w(x)dx\quad\text{ as } t\to t_0.$$

• For each space Y, we set $Y_\sigma=\{u\in Y:\ \operatorname{div} u=0\}$.
• We denote u_i the ith coordinate $(i=1,2,3)$ of a vector u.
• Constants independent of solutions may change from line to line and will be denoted by C.

2. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. Our method is based on the following contraction mapping theorem (cf. [1], Theorem 1.72).

Lemma 2.1. *Let E be a Banach space, N be a continuous bilinear map from $E\times E$ to E, and α be a positive real number such that*

$$\alpha<\frac{1}{4\|N\|}\ \text{ with }\ \|N\|:=\sup_{\|u\|,\|v\|\le1}\|N(u,v)\|. \tag{2.1}$$

Then for any a in a ball $B(0,\alpha)$ (i.e., with center 0 and radius α) in E, there exists a unique x in ball $B(0,2\alpha)$ such that

$$x=a+N(x,x). \tag{2.2}$$

We will also use a property of Landau solution v_c which can be obtained by a direct calculation.

Lemma 2.2. *Let v_c be the Landau solution given by (1.4), then we have*

$$\||x|v_c\|_{L^\infty}\le\frac{2\sqrt{2}}{|c|-1}:=K_c. \tag{2.3}$$

The next lemma is a fundamental inequality with the singular weight in Sobolev spaces: the so-called Hardy inequality which goes back to the pioneering work by G.H. Hardy [13,14].

Lemma 2.3. *For any f in $\dot H^1(\mathbb{R}^3)$, there holds*

$$\left(\int_{\mathbb{R}^3}\frac{|f(x)|^2}{|x|^2}dx\right)^{\frac12}\le2\|\nabla f\|_{L^2(\mathbb{R}^3)}. \tag{2.4}$$

To complete the proof of Theorem 1.1, we need Lemmas 2.4 and 2.5 which give the results for the linear part a and the nonlinear part N in (1.10), respectively. The linear part a satisfies the following Cauchy problem

$$\begin{cases} a_t-\Delta a+(a\cdot\nabla)v_c+(v_c\cdot\nabla)\,a+\nabla\pi_1=0,\\ \nabla\cdot a=0,\\ a(x,0)=w_0(x). \end{cases} \tag{2.5}$$

Namely, $a(x,t)$ satisfies

$$\int_{\mathbb{R}^3}w_0\phi dx+\int_0^\infty\int_{\mathbb{R}^3}\left\{w\,(-\partial_t\phi-\Delta\phi)-w\otimes w:\nabla\phi-(w\otimes v_c+v_c\otimes w):\nabla\phi\right\}dxdt=0, \tag{2.6}$$

for all $\phi\in C_c^\infty([0,\infty)\times\mathbb{R}^3)$ with $\nabla\cdot\phi=0$.

Lemma 2.4. *Let $p \in (1,\infty)$. For every c satisfies (2.15) and (2.21), there exists a unique global-in-time solution $a(x,t) \in C_t L^p_x \cap L_t^{\frac{4p}{3}} L^{2p}_x$ to the system (2.5) with the initial data $w_0 \in L^p_\sigma(\mathbb{R}^3)$, satisfying*

$$\|a(\cdot,t)\|_{L^p} \le \|a(\cdot,s)\|_{L^p}, \tag{2.7}$$

for any $0 \le t \le s < \infty$,,

$$\|a\|_{C_t L^p_x \cap L_t^{\frac{4p}{3}} L^{2p}_x} + \left\|\nabla\left(|a|^{\frac{p}{2}}\right)\right\|^{\frac{2}{p}}_{L^2_t L^2_x} \le C\|w_0\|_{L^p}, \tag{2.8}$$

for a universal constant C.

Proof. By the classical approximation method, it is easy to get the global existence of solutions a. For simplicity, we omit the detailed proof and give some *a-prioi* estimates for a. Suppose a is sufficiently smooth, multiplying the equation $(2.5)_1$ by $|a|^{p-2}a$ and integrate it on $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \partial_t a \cdot (|a|^{p-2}a)dx = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|^p_{L^p}, \tag{2.9}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3} -\Delta a \cdot (|a|^{p-2}a)dx &= (p-2)\int_{\mathbb{R}^3} |a|^{p-4}\sum_i [(\partial_i a_l)a_l]^2 + \int_{\mathbb{R}^3} |\nabla a|^2|a|^{p-2} \\ &= \frac{4(p-2)}{p^2}\|\nabla(|a|^{\frac{p}{2}})\|^2_{L^2} + \||\nabla a||a|^{\frac{p-2}{2}}\|^2_{L^2}.\end{aligned} \tag{2.10}$$

When $p \ge 2$, we obtain

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|^p_{L^p} + \frac{4(p-2)}{p^2}\|\nabla(|a|^{\frac{p}{2}})\|^2_{L^2} + \||\nabla a||a|^{\frac{p-2}{2}}\|^2_{L^2} \\ &\quad = -\int_{\mathbb{R}^3} \mathrm{div}(a\otimes v_c + v_c \otimes a)\cdot(|a|^{p-2}a)dx - \int_{\mathbb{R}^3} \nabla\pi_1 \cdot (|a|^{p-2}a)dx.\end{aligned} \tag{2.11}$$

Using integration by parts, Hölder's inequality, Lemma 2.2 and the classical Hardy inequality in Lemma 2.3, we have

$$\begin{aligned}-\int_{\mathbb{R}^3} \mathrm{div}(a\otimes v_c + v_c\otimes a)\cdot(|a|^{p-2}a)dx &= \int_{\mathbb{R}^3} (a\otimes v_c + v_c\otimes a)\cdot\nabla(|a|^{p-2}a)dx \\ &= \int_{\mathbb{R}^3} a_i (v_c)_j \partial_i(|a|^{p-2}a_j)dx \\ &\le C\int_{\mathbb{R}^3} |\nabla(|a|^{\frac{p}{2}})||a|^{\frac{p}{2}}|v_c|dx \\ &\le C\,\||x|v_c\|_{L^\infty}\left\|\nabla(|a|^{\frac{p}{2}})\right\|_{L^2}\left\|\frac{|a|^{\frac{p}{2}}}{|x|}\right\|_{L^2} \\ &\le CK_c\left\|\nabla(|a|^{\frac{p}{2}})\right\|^2_{L^2}.\end{aligned} \tag{2.12}$$

Thanks to the system (2.5), we get

$$\pi_1 = -\Delta^{-1}\partial_i\partial_j\,(a\otimes v_c + v_c\otimes a)\,.$$

The operator $\Delta^{-1}\partial_i\partial_j$ is Calderón-Zygmund operator. According to Example 9.1.7 in [12], there holds $|x|^{p-2} \in A_p$ with $1 < p < \infty$. By Hölder's inequality, Hardy inequality in Lemma 2.3, Sobolev embedding and boundedness of the Riesz transforms on weighted L^p spaces (see Theorem 9.4.6 in [12]), we have

$$\int_{\mathbb{R}^3} \nabla\pi_1 \cdot (|a|^{p-2}a)dx \le C\||x|^{\frac{p-2}{p}}\,(a\otimes v_c + v_c\otimes a)\,\|_{L^p}\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}\left\|\frac{|a|^{\frac{p}{2}-1}}{|x|^{\frac{p-2}{p}}}\right\|_{L^{\frac{2p}{p-2}}}$$
$$\le C\||x|v_c\|_{L^\infty}\left\|\frac{a}{|x|^{\frac{2}{p}}}\right\|_{L^p}\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}\left\|\nabla(|a|^{\frac{p}{2}})\right\|_{L^2}^{\frac{p-2}{p}}$$
$$\le CK_c\left\|\frac{a}{|x|^{\frac{2}{p}}}\right\|_{L^p}\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^{\frac{2p-2}{p}}$$
$$\le CK_c\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^2. \tag{2.13}$$

Combining (2.11)-(2.13), we deduce

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^p}^p + \frac{4(p-2)}{p^2}\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^2 \le CK_c\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^2. \tag{2.14}$$

Under the assumption,

$$\frac{4(p-2)}{p^2} - CK_c > 0, \quad \text{if } p \ge 2, \tag{2.15}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^p}^p + C\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^2 \le 0, \tag{2.16}$$

for a positive constant C. Therefore, (2.7) holds and we have

$$\sup_t \|a(t)\|_{L^p}^p + C\|\nabla(|a|^{\frac{p}{2}})\|_{L_t^2L_x^2}^2 \le \|w_0\|_{L^p}. \tag{2.17}$$

By the interpolation theory, we deduce

$$\|a\|_{L_t^\infty L_x^p \cap L_t^{\frac{4p}{3}} L_x^{2p}} + \left\|\nabla\left(|a|^{\frac{p}{2}}\right)\right\|_{L_t^2L_x^2}^{\frac{2}{p}} \le C\|w_0\|_{L^p}. \tag{2.18}$$

When $1 < p < 2$, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^p}^p + (p-2)\int_{\mathbb{R}^3}|a|^{p-4}\sum_i[(\partial_i a_l)a_l]^2 + \int_{\mathbb{R}^3}|\nabla a|^2|a|^{p-2}$$
$$= -\int_{\mathbb{R}^3}\mathrm{div}(a\otimes v_c + v_c\otimes a)\cdot(|a|^{p-2}a)dx - \int_{\mathbb{R}^3}\nabla\pi\cdot(|a|^{p-2}a)dx.$$

Thanks to Lemma 3.1, there holds

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^p}^p + (p-1)\int_{\mathbb{R}^3}|\nabla a|^2|a|^{p-2}$$
$$\le \left|\int_{\mathbb{R}^3}\mathrm{div}(a\otimes v_c + v_c\otimes a)\cdot(|a|^{p-2}a)dx\right| + \left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|a|^{p-2}a)dx\right|.$$

Thanks to (2.12) and (2.13), there holds

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^p}^p + (p-1)\int_{\mathbb{R}^3}|\nabla a|^2|a|^{p-2}dx \le CK_c\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^2. \tag{2.19}$$

Moreover,

$$CK_c\|\nabla(|a|^{\frac{p}{2}})\|_{L^2}^2 = CK_c\frac{p^2}{4}\int_{\mathbb{R}^3}|a|^{p-4}\sum_i[(\partial_i a_l)a_l]^2dx$$
$$\le C(p)K_c\int_{\mathbb{R}^3}|a|^{p-2}|\nabla a|^2dx, \tag{2.20}$$

where $C(p)$ is a constant depending on p. Under the assumption,

$$p-1-C(p)K_c>0, \quad \text{if } 1<p<2, \tag{2.21}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^p}^p + C\int_{\mathbb{R}^3}|a|^{p-2}|\nabla a|^2dx \le 0, \tag{2.22}$$

for a positive constant C. Hence we deduce (2.7) and (2.18), since that

$$\left\|\nabla\left(|a|^{\frac{p}{2}}\right)\right\|_{L^2_x}^2 \le C\int_{\mathbb{R}^3}|a|^{p-2}|\nabla a|^2dx.$$

Then, we consider the continuity of the solution a over time t. Because of the translational invariance in time, we only consider time around 0. For any sequence $t_n \to 0$, according to (2.18), there holds

$$\|a(t_n,\cdot)\|_{L^p_x} \le \|w_0\|_{L^p_x}. \tag{2.23}$$

Therefore, there exists subsequence $\{t_{n_j}\}_{j\in\mathbb{Z}^+}$, such that $t_{n_j}\to 0$ as $j\to\infty$, and

$$a(\cdot,t_{n_j}) \rightharpoonup w_0(\cdot) \text{ weakly in } L^p.$$

Therefore, we have

$$\|w_0\|_{L^p} \le \underline{\lim}_{j\to\infty}\|a(\cdot,t_{n_j})\|_{L^p}. \tag{2.24}$$

By the energy inequalities (2.16) and (2.22), there holds

$$\overline{\lim}_{j\to\infty}\|a(\cdot,t_{n_j})\|_{L^p} \le \|w_0\|_{L^p}. \tag{2.25}$$

From (2.24) and (2.25), we get

$$\lim_{j\to\infty}\|a(\cdot,t_{n_j})\|_{L^p} = \|w_0\|_{L^p}, \tag{2.26}$$

and

$$a(\cdot,t)\to w_0(\cdot) \text{ in } L^p \text{ as } t\to 0. \tag{2.27}$$

Hence, we have $a\in C([0,\infty);L^p_x)$ and finish the proof of (2.8). □

Remark 2.1. Indeed, more strictly, we can prove the existence of a by the approximation theory. Assume $a_0=0$, we construct the iterative sequence $\{a_k\}$ as follows

$$\begin{cases} \partial_t a_k - \Delta a_k = -(a_{k-1}\cdot\nabla)v_c - (v_c\cdot\nabla)a_{k-1} - \nabla\pi_{k-1}, \ \text{ for } k=1,2,\dots,\\ \nabla\cdot a_k = 0,\\ \pi_{k-1} = (-\Delta)^{-1}\partial_i\partial_j(v_c\otimes a_{k-1} + a_{k-1}\otimes v_c),\\ a_k|_{t=0} = w_0. \end{cases}$$

We claim that $\{a_k\}\in C([0,\infty);L^p_x)\cap L_t^{\frac{4p}{3}}([0,\infty);L^{2p}_x)$ and $\nabla(|a_k|^{\frac{p}{2}})\in L^2_t([0,\infty);L^2_x)$. By Duhamel principle, a_k also satisfy the integral formulation $e^{t\Delta}a_k - \int_0^t e^{(t-s)\Delta}\mathbb{P}\mathrm{div}(a_{k-1}\otimes v_c + v_c\otimes a_{k-1})ds$. Since semigroup $e^{t\Delta}: L^p\to L^p$, we have $a_k\in C([0,\infty);L^p_x)$. By the energy estimate, we have that $\{a_k\}$ is a Cauchy sequence in $L_t^{\frac{4p}{3}}([0,\infty);L^{2p}_x)$. The limit of $\{a_k\}$ is a which satisfies Lemma 2.4. We omit the details.

For $w_0\in L^p_\sigma(\mathbb{R}^3)$, $1<p<\infty$, and $0\le t<\infty$, let

$$T(t)w_0 := a(x,t), \tag{2.28}$$

where $a(x,t)$ is the unique solution of (2.5) given by Lemma 2.4. Then $T(t)$, $0\le t<\infty$, is a one parameter family of bounded linear operators from $L^p_\sigma(\mathbb{R}^3)$ into $L^p_\sigma(\mathbb{R}^3)$ satisfying $T(0)=I$, the identity operator of $L^p_\sigma(\mathbb{R}^3)$, $T(t+s)=T(t)T(s)$ for every $t,s\ge 0$, $\|T(t)\|\le 1$ for every $t\ge 0$, and $\lim_{t\to 0+}T(t)w_0 = w_0$ in

$L^p_\sigma(\mathbb{R}^3)$. Therefore, $T(t)$ is a strongly continuous semigroup of the contraction, see Definition 1.2.1 and Section 1.3 in [36].

The linearized operator $-\mathcal{L}$, given in (1.9), with the domain of definition

$$D(-\mathcal{L}) := \left\{ w_0 \in L^3_\sigma(\mathbb{R}^3) : \lim_{t\to 0+} \frac{T(t)w_0 - w_0}{t} \text{ exists in } L^p_\sigma(\mathbb{R}^3) \right\}, \tag{2.29}$$

is the infinitesimal generator of the semigroup $T(t)$, see Section 1.1 of [36]. By Corollary 1.2.5 in [36], $D(-\mathcal{L})$ is dense in $L^p_\sigma(\mathbb{R}^3)$ and $-\mathcal{L}$ is a closed linear operator in $L^p_\sigma(\mathbb{R}^3)$. We also denote $T(t)$ as $e^{-t\mathcal{L}}$. We will prove that $e^{-t\mathcal{L}}$ is an analytic semigroup in Sect. 3.

Next, we will estimate the nonlinear part $N(w_1, w_2)$, where

$$N(w_1, w_2) = -\int_0^t e^{-(t-s)\mathcal{L}} \mathbb{P}\nabla \cdot (w_1 \otimes w_2) ds$$

for any $w_1, w_2 \in L^4([0,T]; L^6(\mathbb{R}^3))$. Denote $z = N(w_1, w_2)$, it's obvious that z satisfies the following system

$$\begin{cases} z_t - \Delta z + (z\cdot\nabla)v_c + (v_c\cdot\nabla)z + \nabla\pi_2 = -\mathrm{div}(w_1 \otimes w_2), \\ \nabla \cdot z = 0, \\ z(x,0) = 0. \end{cases} \tag{2.30}$$

Lemma 2.5. *For every c satisfies (2.42), there exists a unique solution $z(x,t) \in C([0,T], L^3_\sigma(\mathbb{R}^3)) \cap L^4([0,T], L^6_\sigma(\mathbb{R}^3))$ to the system (2.30) with $w_1, w_2 \in L^4([0,T]; L^6(\mathbb{R}^3))$, satisfying*

$$\|z(t)\|_{C([0,T];L^3_x)\cap L^4_t([0,T];L^6_x)} + \left\| \nabla\left(|z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L^2_t([0,T];L^2_x)} \le C_2 \|w_1\|_{L^4_t([0,T];L^6_x)} \|w_2\|_{L^4_t([0,T];L^6_x)}, \tag{2.31}$$

for a universal constant C_2 which is independent of T.

Proof. We omit the detailed proof of the existence of the solution z since it can be obtained by the classical approximation method. Then, we give some *a-prioi* estimates for z. Suppose z is sufficiently smooth, multiplying the equation $(2.30)_1$ by $|z|z$ and integrating it on $\mathbb{R}^3$, we have

$$\begin{aligned} &\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^3_{L^3} + \frac{8}{9}\|\nabla(|z|^{\frac{3}{2}})\|^2_{L^2} \\ &\quad = -\int_{\mathbb{R}^3} \mathrm{div}(z\otimes v_c + v_c \otimes z)\cdot(|z|z)dx - \int_{\mathbb{R}^3} \mathrm{div}(w_1\otimes w_2)\cdot(|z|z)dx - \int_{\mathbb{R}^3} \nabla\pi_2\cdot(|z|z)dx. \end{aligned} \tag{2.32}$$

The estimate for the first term on the right-hand side of (2.32) is the same as (2.12) with $p = 3$. Hence, there holds

$$-\int_{\mathbb{R}^3} \mathrm{div}(z\otimes v_c + v_c\otimes z)\cdot(|z|z)dx \le \frac{16}{3}K_c\|\nabla(|z|^{\frac{3}{2}})\|^2_{L^2}. \tag{2.33}$$

For the second term on the right-hand side of (2.32), by integration by parts, Hölder's inequality and Young's inequality, we get

$$\begin{aligned} -\int_{\mathbb{R}^3} \mathrm{div}(w_1\otimes w_2)\cdot(|z|z)dx &= \int_{\mathbb{R}^3} (w_1\otimes w_2)\cdot\nabla(|z|z)dx \\ &\le \frac{4}{3}\int_{\mathbb{R}^3} |w_1\otimes w_2| \left|\nabla\left(|z|^{\frac{3}{2}}\right)\right| |z|^{\frac{1}{2}} dx \\ &\le \frac{4}{3}\left\|\nabla\left(|z|^{\frac{3}{2}}\right)\right\|_{L^2} \|z\|^{\frac{1}{2}}_{L^3} \|w_1\otimes w_2\|_{L^3} \\ &\le \frac{2}{15}\left\|\nabla\left(|z|^{\frac{3}{2}}\right)\right\|^2_{L^2} + \frac{10}{3}\|z\|_{L^3}\|w_1\otimes w_2\|^2_{L^3} \\ &\le \frac{2}{15}\int_0^T \left\|\nabla\left(|z|^{\frac{3}{2}}\right)\right\|^2_{L^2} dt + \frac{10}{3}\|z\|_{L^\infty_T L^3_x}\|w_1\|^2_{L^4_T L^6_x}\|w_2\|^2_{L^4_T L^6_x}. \end{aligned} \tag{2.34}$$

For the third term on the right-hand side of (2.32), according to the system (2.30), we obtain

$$\pi_2 = -\Delta^{-1}\partial_i\partial_j \left(z \otimes v_c + v_c \otimes z + w_1 \otimes w_2\right). \tag{2.35}$$

Using integration by parts and Hölder's inequality, we have the following estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla\pi_2 \cdot (|z|z)dx \\
&= \int_{\mathbb{R}^3} \Delta^{-1}\partial_i\partial_j \left(z \otimes v_c + v_c \otimes z + w_1 \otimes w_2\right) \nabla(|z|) \cdot z dx \\
&\le \frac{2}{3}\int_{\mathbb{R}^3} \left|\Delta^{-1}\partial_i\partial_j \left(z \otimes v_c + v_c \otimes z + w_1 \otimes w_2\right)\right| \left|\nabla\left(|z|^{\frac{3}{2}}\right)\right| |z|^{\frac{1}{2}} dx \\
&\le \frac{2}{3}\int_{\mathbb{R}^3} \left|\Delta^{-1}\partial_i\partial_j \left(z \otimes v_c + v_c \otimes z\right)\right| \left|\nabla\left(|z|^{\frac{3}{2}}\right)\right| |z|^{\frac{1}{2}} dx \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^3} \left|\Delta^{-1}\partial_i\partial_j \left(w_1 \otimes w_2\right)\right| \left|\nabla\left(|z|^{\frac{3}{2}}\right)\right| |z|^{\frac{1}{2}} dx.
\end{aligned} \tag{2.36}$$

Thanks to (2.13) with $p = 3$, we have

$$\frac{2}{3}\int_{\mathbb{R}^3} \left|\Delta^{-1}\partial_i\partial_j \left(z \otimes v_c + v_c \otimes z\right)\right| \left|\nabla\left(|z|^{\frac{3}{2}}\right)\right| |z|^{\frac{1}{2}} dx \le C_3 K_c \|\nabla(|z|^{\frac{3}{2}})\|_{L^2}^2. \tag{2.37}$$

For the second term on the right-hand side of (2.36), by Hölder's inequality, the property of Riesz operator and Young's inequality, we have

$$\begin{aligned}
&\frac{2}{3}\int_{\mathbb{R}^3} \left|\Delta^{-1}\partial_i\partial_j \left(w_1 \otimes w_2\right)\right| \left|\nabla\left(|z|^{\frac{3}{2}}\right)\right| |z|^{\frac{1}{2}} dx \\
&\le \frac{2}{3}\|\Delta^{-1}\partial_i\partial_j \left(w_1 \otimes w_2\right)\|_{L^3} \|\nabla(|z|^{\frac{3}{2}})\|_{L^2} \||z|^{\frac{1}{2}}\|_{L^6} \\
&\le \frac{2}{3} H_3 \|w_1 \otimes w_2\|_{L^3} \|\nabla(|z|^{\frac{3}{2}})\|_{L^2} \|z\|_{L^3}^{\frac{1}{2}} \\
&\le \frac{2}{15} H_3 \left\|\nabla\left(|z|^{\frac{3}{2}}\right)\right\|_{L^2}^2 + \frac{10}{3} H_3 \|z\|_{L^3} \|w_1 \otimes w_2\|_{L^3}^2,
\end{aligned} \tag{2.38}$$

where H_3 is a constant and origins from the following inequality

$$\|\Delta^{-1}\partial_i\partial_j f\|_{L^r} \le H_r \|f\|_{L^r}, \tag{2.39}$$

for $1 < r < \infty$. For the scalar Riesz transforms, Iwaniec and Martin [15] showed that the norm $\|R_l\|_{L^r}$ of the Riesz operator $R_l : L^r(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ is equal to

$$\begin{cases} \tan(\frac{\pi}{2r}), \text{if } 1 < r \le 2, \\ \cot(\frac{\pi}{2r}), \text{if } 2 \le r < \infty. \end{cases} \tag{2.40}$$

Combining with (2.36), (2.37) and (2.38), we have the following estimate

$$\int_{\mathbb{R}^3} \nabla\pi_2 \cdot (|z|z)dx \le \left(C_3 K_c + \frac{2}{15} H_3\right) \left\|\nabla(|z|^{\frac{3}{2}})\right\|_{L^2}^2 + \frac{10}{3} H_3 \|z\|_{L^3} \|w_1 \otimes w_2\|_{L^3}^2.$$

Therefore, we deduce

$$\begin{aligned}
&\|z\|_{L_T^\infty L_x^3}^3 + \left(\frac{8}{3} - 16K_c - \frac{2}{5} - 3C_3K_c - \frac{2}{5}H_3\right) \left\|\nabla(|z|^{\frac{3}{2}})\right\|_{L_T^2 L_x^2}^2 \\
&\le (10 + 10H_3)\|z\|_{L_T^\infty L_x^3} \|w_1\|_{L_T^4 L_x^6}^2 \|w_2\|_{L_T^4 L_x^6}^2.
\end{aligned} \tag{2.41}$$

Choosing $|c|$ big enough such that

$$\frac{8}{3} - 16K_c - \frac{2}{5} - 3C_3K_c - \frac{2}{5}H_3 > 0, \tag{2.42}$$

we have

$$\|z\|_{L_T^\infty L_x^3}+\left\|\nabla(|z|^{\frac{3}{2}})\right\|^{\frac{2}{3}}_{L_T^2L_x^2}\le C\|w_1\|_{L_T^4L_x^6}\|w_2\|_{L_T^4L_x^6}, \tag{2.43}$$

for a universal constant C. By the interpolation theory, there holds

$$\|z(t)\|_{L_T^\infty L_x^3\cap L_T^4L_x^6}+\left\|\nabla\left(|z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L_T^2L_x^2}\le C_2\|w_1\|_{L_T^4L_x^6}\|w_2\|_{L_T^4L_x^6}, \tag{2.44}$$

for a universal constant C_2. By similar argument as (2.26), we can obtain the continuity of z over time t. Combining with (2.44), we deduce (2.31). □

Proof of Theorem 1.1.. For every c satisfies (2.15) with $p=3$ and (2.42), according to Lemma 2.4 with $p=3$ and Lemma 2.5, we have

$$\|a(t)\|_{L_t^4L_x^6}\le C_1\|w_0\|_{L^3}, \tag{2.45}$$

$$\|N(w_1,w_2)\|_{L_t^4([0,T];L_x^6)}\le C_2\|w_1\|_{L_t^4([0,T];L_x^6)}\|w_2\|_{L_t^4([0,T];L_x^6)}. \tag{2.46}$$

Using Lemma 2.1 with $E=L_t^4L_x^6$, we have that there exists $T>0$ such that $\|a\|_{L_T^4L_x^6}<\frac{1}{4C_2}$, and the system (1.8) exists a unique local solution $w\in L_T^4L_x^6$ on $[0,T]$. According to Lemma 2.4 with $p=3$ and Lemma 2.5, the solution $w\in C([0,T];L^3(\mathbb{R}^3))$, $\nabla(|w|^{\frac{3}{2}})\in L^2([0,T];L^2(\mathbb{R}^3))$, and (i) holds.

Also, using Lemma 2.1 with $E=L_t^4L_x^6$, when $\|w_0\|_{L^3}<\frac{1}{4C_1C_2}:=\varepsilon_0$, there exists a global unique solution $w\in L_t^4L_x^6$ and

$$\|w\|_{L_t^4L_x^6}\le C\|w_0\|_{L^3}. \tag{2.47}$$

According to Lemma 2.4 with $p=3$ and Lemma 2.5, the solution $w\in C([0,\infty);L^3(\mathbb{R}^3))$, $\nabla(|w|^{\frac{3}{2}})\in L^2([0,\infty);L^2(\mathbb{R}^3))$.

Next, we will investigate the decay rate of the solution w, i.e. (1.15). Our method is inspired by [10]. Let $T>0$ and $3\le q<\infty$. Denote $r(t)=\frac{1}{\frac{1}{T}(\frac{1}{q}-\frac{1}{3})t+\frac{1}{3}}$. First, we give *a-prioi* estimate of $\|w(\cdot,t)\|_{r(t)}$.

Claim.

$$\frac{d}{dt}\|w(\cdot,t)\|_{r(t)}\le\frac{3}{2}\frac{1}{T}\left(\frac{1}{3}-\frac{1}{q}\right)\ln\frac{1}{T}\left(\frac{1}{3}-\frac{1}{q}\right)\|w(\cdot,t)\|_{r(t)},\ \forall\ t\in[0,T]. \tag{2.48}$$

Then, from Gronwall inequality, we obtain

$$\|w(\cdot,T)\|_{L^q}\le C_qT^{\frac{3}{2}\left(\frac{1}{q}-\frac{1}{3}\right)}\|w(\cdot,0)\|_{L^3},\ \forall\ T>0, \tag{2.49}$$

with $C_q=(\frac{1}{3}-\frac{1}{q})^{\frac{3}{2}(\frac{1}{3}-\frac{1}{q})}$, *and (1.15) holds.*

To prove **Cliam** (2.48), from $(1.8)_1$, by the direct computation, we have

$$\begin{aligned}
&r(t)^2\|w(\cdot,t)\|_{r(t)}^{r(t)-1}\frac{\mathrm{d}}{\mathrm{d}t}\|w(\cdot,t)\|_{r(t)}\\
&\quad=\dot r(t)\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)}\ln\left(|w(\cdot,t)|^{r(t)}/\|w(\cdot,t)\|_{r(t)}^{r(t)}\right)\mathrm{d}x\\
&\qquad+r(t)^2\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}w_i\left(\partial_j\partial_jw_i-\partial_j(w_iw_j+w_iv_j+v_iw_j)-\partial_i\pi\right)\mathrm{d}x.
\end{aligned}\tag{2.50}$$

Using integration by parts, we get

$$\begin{aligned}
&r(t)^2\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}w_i\partial_j\partial_jw_i\mathrm{d}x\\
&=-r(t)^2\int_{\mathbb{R}^3}\partial_j(|w(t)|^{r(t)-2}w_i)\partial_jw_i dx
\end{aligned}$$

$$
\begin{aligned}
&= -r(t)^2 \int_{\mathbb{R}^3} \frac{1}{2}\partial_j |w(t)|^{r(t)-2} \partial_j |w|^2 dx - r(t)^2 \int_{\mathbb{R}^3} |w(t)|^{r(t)-2} |\nabla w|^2 dx \\
&= -r(t)^2 \int_{\mathbb{R}^3} \frac{4(r(t)-2)}{r(t)^2} \left| \nabla (|w(\cdot,t)|^{r(t)/2}) \right|^2 dx - r(t)^2 \int_{\mathbb{R}^3} |w(t)|^{r(t)-2} |\nabla w|^2 dx \\
&= -4r(t)(r(t)-2) \| \nabla (|w(\cdot,t)|^{\frac{r(t)}{2}}) \|_{L^2}^2 - r(t)^2 \int_{\mathbb{R}^3} |w(t)|^{r(t)-2} |\nabla w|^2 dx.
\end{aligned}
$$

Combining with (2.50), we have

$$
\begin{aligned}
& r(t)^2 \|w(\cdot,t)\|_{r(t)}^{r(t)-1} \frac{\mathrm{d}}{\mathrm{d}t} \|w(\cdot,t)\|_{r(t)} \\
&= \dot{r}(t) \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)} \ln \left(|w(\cdot,t)|^{r(t)} / \|w(\cdot,t)\|_{r(t)}^{r(t)} \right) \mathrm{d}x - 4r(t)(r(t)-2) \| \nabla (|w(\cdot,t)|^{\frac{r(t)}{2}}) \|_{L^2}^2 \\
&\quad -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} |\nabla w(\cdot,t)|^2 dx \\
&\quad -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \left(\partial_j (w_i w_j + w_i v_j + v_i w_j) + \partial_i \pi \right) \mathrm{d}x.
\end{aligned} \tag{2.51}
$$

Next, we will estimate the fourth term on the right-hand side of (2.51) in the following lemma, and give the proof in "Appendix".

Lemma 2.6.

$$
\begin{aligned}
& \left| r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \left(\partial_j (w_i w_j + w_i v_j + v_i w_j) + \partial_i \pi \right) \mathrm{d}x \right| \\
&\quad \le \left(4r(t)(r(t)-2)(\mu - 1) - 2r(t)^2 K_c \right) \| \nabla (|w(t)|^{\frac{r(t)}{2}}) \|_{L^2}^2,
\end{aligned} \tag{2.52}
$$

with

$$
\mu = \inf_{3 \le r \le q} \left\{ 1 - \frac{1}{4} C\varepsilon_0 - \frac{1}{2} K_c - K_c - 2C_r K_c - H_{\frac{3r}{r+1}} C\varepsilon_0 \right\} > \frac{1}{2}. \tag{2.53}
$$

From (2.51) and (2.52), we get

$$
\begin{aligned}
& r(t)^2 \|w(t)\|_{r(t)}^{r(t)-1} \frac{\mathrm{d}}{\mathrm{d}t} \|w(t)\|_{r(t)} \\
&\quad \le \dot{r}(t) \int_{\mathbb{R}^3} |w(t)|^{r(t)} \ln \left(|w(t)|^{r(t)} / \|w(t)\|_{r(t)}^{r(t)} \right) \mathrm{d}x \\
&\qquad - \left(4r(t)(r(t)-2)\mu - 2r(t)^2 K_c \right) \| \nabla (|w(t)|^{\frac{r(t)}{2}}) \|_{L^2}^2.
\end{aligned} \tag{2.54}
$$

Applying the sharp logarithmic Sobolev inequality in [10,34], we have

$$
2 \int |u|^2 \ln \left(\frac{|u|}{\|u\|_{L^2}} \right) dx + 3(1 + \ln a) \|u\|_{L^2}^2 \le \frac{a^2}{\pi} \int |\nabla u|^2 dx. \tag{2.55}
$$

Using $a = \left(\pi \frac{4r(t)(r(t)-2)\mu - 2K_c r(t)^2}{\dot{r}(t)} \right)^{\frac{1}{2}}$ and $u = |w|^{\frac{r(t)}{2}}$, we obtain

$$
\begin{aligned}
& r(t)^2 \|w(t)\|_{r(t)}^{r(t)-1} \frac{\mathrm{d}}{\mathrm{d}t} \|w(t)\|_{r(t)} \\
&\quad \le -\dot{r}(t) \left(3 + \frac{3}{2} \ln \frac{(4\pi\mu - 2\pi K_c) r(t)^2 - 8\pi\mu r(t)}{\dot{r}(t)} \right) \|w\|_{L^r}^r \\
&\quad \le \frac{r(t)^2}{T} \left(\frac{1}{q} - \frac{1}{3} \right) \left[3 + \frac{3}{2} \ln \left(\frac{-8\pi\mu}{r(t)} + 4\pi\mu - 2K_c \pi \right) + \frac{3}{2} \ln \frac{1}{T} \left(\frac{1}{3} - \frac{1}{q} \right) \right] \|w\|_{L^r}^r \\
&\quad \le \frac{3}{2} \frac{r(t)^2}{T} \left(\frac{1}{3} - \frac{1}{q} \right) \ln \frac{1}{T} \left(\frac{1}{3} - \frac{1}{q} \right) \|w\|_{L^r}^r,
\end{aligned}
$$

when

$$\frac{-8\pi\mu}{r(t)}+4\pi\mu-2K_c\pi\geq\frac{-8\pi\mu}{3}+4\pi\mu-2K_c\pi=\frac{4\pi\mu}{3}-2K_c\pi>1, \tag{2.56}$$

and **Claim** (2.48) holds.

Finally, we will prove (1.14). Since $w_0\in L^3$ and $\|w_0\|_{L^3}<\varepsilon_0$, there exists a subsequence denoted by $\{w_{0,n}\}$ such that $w_{0,n}\in L^2\cap L^3$ and

$$w_{0,n}\to w_0 \text{ in } L^3 \text{ as } n\to\infty.$$

According to Theorem 1.4, Corollary 1.1 and Remark 1.8, we have

$$\lim_{t\to\infty}\|w_n(\cdot,t)\|_{L^3}=0.$$

Based on similar proof of (7.5), when $\|w_{0,n}-w_0\|_{L^3}\leq(4C^2e^{2C\int_0^T\|w\|_{L^6}^4dt})^{-1}$, we have

$$\|w_n-w\|_{L_t^\infty([0,\infty);L_x^3)}\leq 2C\|w_{0,n}-w_0\|_{L^3}e^{C\int_0^\infty\|w\|_{L^6}^4dt},$$

for a positive constant C. From (2.47), we have

$$\lim_{n\to\infty}\|w_n-w\|_{L_t^\infty([0,\infty);L_x^3)}=0,$$

and (1.14) holds. □

Remark 2.2. To give strict proof of (1.15), we consider the approximation scheme. Using the method in [23,25,26], the mollified system in $\mathbb{R}^3\times(0,\infty)$ is as follows

$$\begin{cases} w_t^\epsilon-\Delta w^\epsilon+\left(\mathcal{J}_\epsilon\left(w^\epsilon\right)\cdot\nabla\right)w^\epsilon+\left(\mathcal{J}_\epsilon(w^\epsilon)\cdot\nabla\right)v_c+\left(v_c\cdot\nabla\right)\mathcal{J}_\epsilon(w^\epsilon)+\nabla\pi^\epsilon=0,\\ \nabla\cdot w^\epsilon=0,\\ w^\epsilon(x,0)=w_0(x), \end{cases} \tag{2.57}$$

where $\mathcal{J}_\epsilon(v)=v*\eta_\epsilon,\epsilon>0$, the mollifier $\eta_\epsilon(x)=\varepsilon^{-3}\eta\left(\frac{x}{\epsilon}\right)$ with positive $\eta\in C_c^\infty(B(0,1))$, $\int\eta dx=1$. By the classical approximation method, the solution w^ϵ satisfies (1.15). Similar as (2.50), we have

$$\begin{aligned} &r(t)^2\|w^\epsilon(\cdot,t)\|_{r(t)}^{r(t)-1}\frac{\mathrm{d}}{\mathrm{d}t}\|w^\epsilon(\cdot,t)\|_{r(t)}\\ &\quad=\dot r(t)\int_{\mathbb{R}^3}|w^\epsilon(\cdot,t)|^{r(t)}\ln\left(|w^\epsilon(\cdot,t)|^{r(t)}/\|w^\epsilon(\cdot,t)\|_{r(t)}^{r(t)}\right)\mathrm{d}x\\ &\qquad+r(t)^2\int_{\mathbb{R}^3}|w^\epsilon(\cdot,t)|^{r(t)-2}w_i^\epsilon\Big(\partial_j\partial_jw_i^\epsilon-\partial_j(\mathcal{J}_\epsilon(w^\epsilon)_iw_j^\epsilon+\mathcal{J}_\epsilon(w^\epsilon)_iv_j+v_i\mathcal{J}_\epsilon(w^\epsilon)_j)-\partial_i\pi^\epsilon\Big)\mathrm{d}x. \end{aligned}$$

Similar as (2.54), we have

$$\begin{aligned} &r(t)^2\|w^\epsilon(t)\|_{r(t)}^{r(t)-1}\frac{\mathrm{d}}{\mathrm{d}t}\|w^\epsilon(t)\|_{r(t)}\\ &\leq\dot r(t)\int_{\mathbb{R}^3}|w^\epsilon(t)|^{r(t)}\ln\left(|w^\epsilon(t)|^{r(t)}/\|w^\epsilon(t)\|_{r(t)}^{r(t)}\right)\mathrm{d}x-\left(4r(t)(r(t)-2)\mu-2r(t)^2K_c\right)\|\nabla(|w^\epsilon(t)|^{\frac{r(t)}{2}})\|_{L^2}^2. \end{aligned}$$

Similar to the procedure in the proof of (2.49), we obtain

$$\|w^\epsilon(\cdot,t)\|_{L^q}\leq(\frac{1}{3}-\frac{1}{q})^{\frac{3}{2}(\frac{1}{3}-\frac{1}{q})}t^{\frac{3}{2}(\frac{1}{q}-\frac{1}{3})}\|w^\epsilon(\cdot,0)\|_{L^3}. \tag{2.58}$$

By the classical compactness theory, we have that the solution w satisfies (1.15).

3. The Linear Operator $\mathcal{L}$

In this section, using the idea of Borchers and Miyakawa [3], we prove that $e^{-t\mathcal{L}}$ is an analytic semigroup.

We consider the following system

$$\begin{cases} \lambda u - \Delta u + (u\cdot\nabla)v_c + (v_c\cdot\nabla)\,u + \nabla p = f, \\ \nabla\cdot u = 0. \end{cases} \tag{3.1}$$

For $\delta > 0$ small, set $\Sigma_\delta = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| < \frac{\pi}{2} + \delta\}$. It is easy to see that for $\lambda = \sigma + \sqrt{-1}\tau \in \Sigma$, σ, τ real, if $\sigma < 0$, then

$$|\sigma| < \delta|\tau|. \tag{3.2}$$

Theorem 3.1. *For $1 < q < \infty$, there exist two positive constants δ and $\bar{c}_q$ which depend only on q such that for any $|c| > \bar{c}_q$, $\lambda \in \Sigma_\delta$, and $u \in C^\infty_{c,\sigma}(\mathbb{R}^3)$ satisfying the system (3.1), we have*

$$\|u\|_{L^q} \le \frac{C}{|\lambda|}\|f\|_{L^q}, \tag{3.3}$$

where C is a constant depending only on q and δ. Consequently, $e^{-t\mathcal{L}}$ is an analytic semigroup of bounded linear operators on $L^q_\sigma(\mathbb{R}^3)$ in the sector $\{\lambda : |\arg\lambda| < \delta\}$.

The last statement in the above theorem follows from the estimate (3.3), together with the fact that $e^{-t\mathcal{L}}$ is a strongly continuous semigroup of the contraction on $L^q_\sigma(\mathbb{R}^3)$ for $1 < q < \infty$ which are established in Sect. 2, see Theorem 1.5.2 in [36].

To prove Theorem 3.1, we need the following lemma.

Lemma 3.1. *The vector field u has the following property*

$$|\nabla(|u|^2)|^2 \le 4|\nabla u|^2|u|^2. \tag{3.4}$$

Consequently, for $1 \le q \le 2$, we have

$$\frac{q-2}{4}\int_{\mathbb{R}^3}|u|^{q-4}|\nabla(|u|^2)|^2dx + \int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx \ge (q-1)\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx. \tag{3.5}$$

Proof. By a direct calculation, we have

$$|\nabla(|u|^2)|^2 = \sum_j |\partial_j(|u|^2)|^2 = \sum_j |\partial_j < u, u >|^2 \le 4\sum_j | < \partial_j u, u >|^2. \tag{3.6}$$

For fixed j, using Cauchy-Schwartz inequality, we obtain

$$|\nabla(|u|^2)|^2 \le 4\sum_j(|\partial_j u|^2|u|^2) = 4|\nabla u|^2|u|^2. \tag{3.7}$$

□

Proof of Theorem 3.1. The value of δ will be chosen in the proof below. Multiplying the equation $(3.1)_1$ by $|u|^{q-2}\overline{u}$, and integrating it on $\mathbb{R}^3$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\nabla u\cdot\nabla(|u|^{q-2}\overline{u})dx + \lambda\int_{\mathbb{R}^3}|u|^q dx + \int_{\mathbb{R}^3}(v_c\cdot\nabla u)\cdot(|u|^{q-2}\overline{u})dx \\ &\quad + \int_{\mathbb{R}^3}(u\cdot\nabla v_c)\cdot(|u|^{q-2}\overline{u})dx + \int_{\mathbb{R}^3}\nabla p\cdot(|u|^{q-2}\overline{u})dx = \int_{\mathbb{R}^3} f\cdot(|u|^{q-2}\overline{u})dx. \end{aligned} \tag{3.8}$$

Set $I_1 + I_2 + I_3 + I_4 + I_5 = \int_{\mathbb{R}^3} f\cdot(|u|^{q-2}\overline{u})dx$. For the first part, we have

$$I_1 = \int_{\mathbb{R}^3}\partial_j u_i\partial_j\left((u_m\overline{u}_m)^{\frac{q-2}{2}}\overline{u}_i\right)dx$$

$$= \int_{\mathbb{R}^3} (\partial_j u_i)\overline{u}_i \frac{q-2}{2}|u|^{q-4}\left[(\partial_j u_m)\overline{u}_m + \overline{(\partial_j u_m)\overline{u}_m}\right] dx + \int_{\mathbb{R}^3} |\nabla u|^2 |u|^{q-2} dx.$$

Denote $\xi_j = (\partial_j u_m)\overline{u}_m = a_j + \sqrt{-1}b_j$, then

$$\partial_j(|u|^2) = (\partial_j u_m)\overline{u}_m + \overline{(\partial_j u_m)\overline{u}_m} = 2\ \mathrm{Re}\ \xi_j = 2a_j. \tag{3.9}$$

Therefore, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} \frac{q-2}{2}|u|^{q-4}\xi_j\left[\xi_j + \overline{\xi}_j\right] + |\nabla u|^2|u|^{q-2}dx \\ &= \int_{\mathbb{R}^3} \frac{q-2}{2}|u|^{q-4}(a_j + \sqrt{-1}b_j)\,(2a_j) + |\nabla u|^2|u|^{q-2}dx \\ &= \int_{\mathbb{R}^3} \frac{q-2}{4}|u|^{q-4}|\nabla(|u|^2)|^2 dx + \sqrt{-1}\int_{\mathbb{R}^3}(q-2)|u|^{q-4}a_j b_j dx + \int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx. \end{aligned}$$

For the second part I_2

$$I_2 = (\sigma + \sqrt{-1}\tau)\int_{\mathbb{R}^3}|u|^q dx. \tag{3.10}$$

It is easy to see that

$$2\sum_j |a_j||b_j| \le |\nabla u|^2|u|^2. \tag{3.11}$$

Then, using Lemma 3.1, we have

$$\mathrm{Re}(I_1 + I_2) \ge \min\{q-1, 1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx + \sigma\int_{\mathbb{R}^3}|u|^q dx \tag{3.12}$$

$$\mathrm{Im}(I_1 + I_2) = \tau\int_{\mathbb{R}^3}|u|^q dx + \int_{\mathbb{R}^3}(q-2)|u|^{q-4}a_j b_j dx. \tag{3.13}$$

We distinguish into two cases:
Case 1. $\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx \ge 8\delta|\tau|\int_{\mathbb{R}^3}|u|^q dx$.
Case 2. $\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx < 8\delta|\tau|\int_{\mathbb{R}^3}|u|^q dx$.

In Case 1, we deduce from (3.12), using (3.2) and requiring $0 < \delta \le 1$, that

$$\begin{aligned} |I_1 + I_2| &\ge \mathrm{Re}(I_1 + I_2) \\ &\ge \frac{1}{2}\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx + (4\delta|\tau| + \sigma)\int_{\mathbb{R}^3}|u|^q dx \\ &\ge \frac{1}{2}\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx + \delta(|\tau| + |\sigma|)\int_{\mathbb{R}^3}|u|^q dx. \end{aligned} \tag{3.14}$$

In Case 2, we derive from (3.11) and (3.13) that

$$\begin{aligned} |\mathrm{Im}(I_1 + I_2)| &\ge |\tau|\int_{\mathbb{R}^3}|u|^q dx - \frac{|q-2|}{2}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx \\ &\ge (1 - \frac{4\delta|q-2|}{\min\{q-1,1\}})|\tau|\int_{\mathbb{R}^3}|u|^q dx \\ &\ge \frac{|\tau|}{2}\int_{\mathbb{R}^3}|u|^q dx, \end{aligned} \tag{3.15}$$

when $8|q-2|\delta < \min\{q-1,1\}$, and

$$\begin{aligned} |I_1 + I_2| &\ge \frac{1}{\sqrt{2}}(\mathrm{Re}(I_1 + I_2) + |\mathrm{Im}(I_1 + I_2)|) \\ &\ge \frac{\min\{q-1,1\}}{\sqrt{2}}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx + \frac{\frac{|\tau|}{2} + \sigma}{\sqrt{2}}\int_{\mathbb{R}^3}|u|^q dx. \end{aligned}$$

$$\geq \frac{1}{2}\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx+\frac{1}{6}(|\tau|+|\sigma|)\int_{\mathbb{R}^3}|u|^q dx,$$

when $\delta \leq \frac{1}{6}$. So in both cases, we have proved that

$$|I_1+I_2| \geq \frac{1}{2}\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx+\delta(|\tau|+|\sigma|)\int_{\mathbb{R}^3}|u|^q dx. \tag{3.16}$$

By integration by parts, Hölder's inequality, Hardy inequality and Lemma 3.1, we have

$$\begin{aligned}|I_3+I_4| &\leq CK_c\int_{\mathbb{R}^3}\frac{|\nabla u|}{|x|}|u|^{q-1}dx\\ &\leq CK_c\left\||\nabla u||u|^{\frac{q-2}{2}}\right\|_{L^2}\left\|\frac{|u|^{\frac{q}{2}}}{|x|}\right\|_{L^2}\\ &\leq CK_c\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx.\end{aligned} \tag{3.17}$$

According to (3.1), we obtain

$$p=\frac{\mathrm{div}}{\Delta}f-\frac{\partial_i\partial_j}{\Delta}(v_c\otimes u+u\otimes v_c)_{ij}.$$

By integration by parts, we have

$$I_5=\int_{\mathbb{R}^3}\frac{\nabla\mathrm{div}}{\Delta}f\cdot(|u|^{q-2}\overline{u})dx+\int_{\mathbb{R}^3}\frac{\partial_i\partial_j}{\Delta}(v_c\otimes u+u\otimes v_c)_{ij}\nabla\cdot(|u|^{q-2}\overline{u})dx. \tag{3.18}$$

By Hölder's inequality, Hardy inequality, Sobolev embedding, the boundedness of the Riesz transforms on weighted L^p spaces (Theorem 9.4.6 in [12]) and Lemma 3.1, we have

$$\left|\int_{\mathbb{R}^3}\frac{\nabla\mathrm{div}}{\Delta}f\cdot(|u|^{q-2}\overline{u})dx\right| \leq C\|f\|_{L^q}\|u\|_{L^q}^{q-1}, \tag{3.19}$$

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\frac{\partial_i\partial_j}{\Delta}(v_c\otimes u+u\otimes v_c)_{ij}\nabla\cdot(|u|^{q-2}\overline{u})dx\right|\\ &\leq C\||x|^{\frac{q-2}{q}}(u\otimes v_c+v_c\otimes u)\|_{L^q}\|\nabla(|u|^{\frac{q}{2}})\|_{L^2}\left\|\frac{|u|^{\frac{q}{2}-1}}{|x|^{\frac{q-2}{q}}}\right\|_{L^{\frac{2q}{q-2}}}\\ &\leq C\||x|v_c\|_{L^\infty}\left\|\frac{u}{|x|^{\frac{2}{q}}}\right\|_{L^q}\|\nabla(|u|^{\frac{q}{2}})\|_{L^2}\left\|\nabla(|u|^{\frac{q}{2}})\right\|_{L^2}^{\frac{q-2}{q}}\\ &\leq CK_c\left\|\frac{u}{|x|^{\frac{2}{q}}}\right\|_{L^q}\|\nabla(|u|^{\frac{q}{2}})\|_{L^2}^{\frac{2q-2}{q}}\\ &\leq CK_c\|\nabla(|u|^{\frac{q}{2}})\|_{L^2}^2\\ &\leq CK_c\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx,\end{aligned} \tag{3.20}$$

and

$$|I_5| \leq C\|f\|_{L^q}\|u\|_{L^q}^{q-1}+CK_c\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx. \tag{3.21}$$

Combining with (3.8), (3.16)-(3.17), (3.21) and the condition of K_c small enough, by Hölder's inequality, we have

$$\frac{1}{2}\min\{q-1,1\}\int_{\mathbb{R}^3}|\nabla u|^2|u|^{q-2}dx+\delta(|\tau|+|\sigma|)\int_{\mathbb{R}^3}|u|^q dx \leq C\|f\|_{L^q}\|u\|_{L^q}^{q-1}. \tag{3.22}$$

Since $\lambda=\sigma+\sqrt{-1}\tau$, we can get (3.3) easily. □

4. Weak–Strong Uniqueness

In this section, we will prove Theorem 1.4, Proposition 1.1 and illustrate Corollary 1.1 briefly.

Proof of of Theorem 1.4. Following the proof of Theorem 4.4 in [47], setting $g = v - u$, we have

$$\begin{cases} \partial_t g - \Delta g + \nabla \pi = -((u+g)\cdot\nabla)g - (g\cdot\nabla)u - g\cdot\nabla v_c - v_c\cdot\nabla g, \\ \nabla\cdot g = 0, \\ g(x,0) = 0. \end{cases} \tag{4.1}$$

Using g itself as a test function and integrating this in time from 0 to t, we have

$$\int \frac{|g|^2}{2}dx + \int_0^t \int |\nabla g|^2 dxdt \le \int_0^t \int (u+v_c)\cdot(g\cdot\nabla)g dxdt. \tag{4.2}$$

Denote $E(t) = \operatorname{ess\,sup}_{s<t} \|g(s)\|_2^2 + \int_0^t \|\nabla g\|_2^2 d\tau$ and $t_0 = \sup\{t \in [0,T] : g(s) = 0 \text{ if } 0 < s < t\}$.

We claim that $t_0 = T$. Using the contradiction argument, we assume that $t_0 < T$. Since

$$\left| \iint uv\nabla w dxdt \right| \le C\|u\|_{L_t^s L_x^q} \|v\|^{2/s}_{L_t^\infty L_x^2} \|v\|^{3/q}_{L_t^2 L_x^6} \|\nabla w\|_{L^2_{t,x}}, \tag{4.3}$$

for $\frac{3}{q} + \frac{2}{s} = 1$ with $1 \le q, s \le \infty$, we get

$$\left| \int_{t_0}^t \int u\cdot(g\cdot\nabla)g dxd\tau \right| \le C\|u\|_{L_t^s L_x^q} E(t), \tag{4.4}$$

for $t \in [t_0, T]$. By Hölder inequality, Hardy inequality and Lemma 2.2, we have

$$\begin{aligned} \left| \int_{t_0}^t \int v_c\cdot(g\cdot\nabla)g dxd\tau \right| &\le \int_{t_0}^t \||x|v_c\|_{L^\infty} \left\| \frac{g}{|x|} \right\|_{L^2} \|\nabla g\|_{L^2} d\tau \\ &\le 2\int_{t_0}^t \||x|v_c\|_{L^\infty} \|\nabla g\|^2_{L^2} d\tau \\ &\le 2K_c \|\nabla g\|^2_{L^2_{t,x}} \\ &\le 2K_c E(t), \end{aligned} \tag{4.5}$$

for $t \in [t_0, T]$. Hence, there holds

$$E(t) \le C\|u\|_{L_t^s([t_0,t];L_x^q)} E(t) + 2K_c E(t). \tag{4.6}$$

If $s < \infty$, we have $C\|u\|_{L_t^s([t_0,t];L_x^q)} < \frac{1}{4}$ for t sufficiently close to t_0. If $s = \infty$, we need $C\|u\|_{L_t^\infty([t_0,t];L_x^3)} < \frac{1}{4}$. When $K_c < \frac{1}{4}$, we have that $E(s) = 0$ for all $s \in [t_0, t]$, which makes a contradiction to the definition of t_0. Hence, $t_0 = T$ and for all $t \in [0, T]$. □

Following is the proof of Proposition 1.1.

Proof of Proposition 1.1. For $p \ge 3$, our goal is to show that the L^p mild solution w is a L^2-weak solution. The crucial part is to prove that $w \in C_w([0,T]; L^2_x) \cap L^2_T(\dot{H}^1_x)$. Set $w = a + z$ as in Sect. 2. We will prove $a \in C_w([0,T]; L^2_x) \cap L^2_T(\dot{H}^1_x)$ and $z \in C_w([0,T]; L^2_x) \cap L^2_T(\dot{H}^1_x)$ as follows.

Multiplying $2.5)_1$ by a, then integrating it on $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|^2_{L^2} + \|\nabla a\|^2_{L^2} = -\int_{\mathbb{R}^3} \operatorname{div}(a\otimes v_c + v_c\otimes a)\cdot a dx. \tag{4.7}$$

By similar estimate as (2.12), using integration by parts, we obtain

$$-\int_{\mathbb{R}^3} \operatorname{div}(a\otimes v_c + v_c\otimes a)\cdot a dx \le 2K_c \|\nabla a\|^2_{L^2}, \tag{4.8}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\|_{L^2}^2+\|\nabla a\|_{L^2}^2\le 2K_c\|\nabla a\|_{L^2}^2. \tag{4.9}$$

Since $|c|>c_p$ where c_p is as in Theorem 1.2, we can guarantee $1-2K_c>0$. Combining with similar argument as (2.26), we have $a\in C([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$.

When $p\in[3,4]$, using $(w_1,w_2)=(w,w)$, multiplying $(2.30)_1$ by z and integrating it on $\mathbb{R}^3$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{L^2}^2+\|\nabla z\|_{L^2}^2\\&=-\int_{\mathbb{R}^3}\mathrm{div}(z\otimes v_c+v_c\otimes z)\cdot z dx-\int_{\mathbb{R}^3}\mathrm{div}(w\otimes w)\cdot z dx.\end{aligned} \tag{4.10}$$

By similar argument as (2.33), using integration by parts, we have

$$-\int_{\mathbb{R}^3}\mathrm{div}(z\otimes v_c+v_c\otimes z)\cdot z dx\le 2K_c\|\nabla z\|_{L^2}^2. \tag{4.11}$$

By integration by parts, Hölder's inequality and Cauchy inequality, we have

$$-\int_{\mathbb{R}^3}\mathrm{div}(w\otimes w)\cdot z dx=\int_{\mathbb{R}^3}(w\otimes w)\nabla z dx\le\|w\otimes w\|_{L^2}\|\nabla z\|_{L^2}\le C\|w\|_{L^4}^4+\frac{1}{10}\|\nabla z\|_{L^2}^2. \tag{4.12}$$

Then, from (4.10)-(4.12), we have

$$\frac{1}{2}\|z\|_{L_T^\infty L_x^2}^2+(\frac{9}{10}-2K_c)\|\nabla z\|_{L_T^2L_x^2}^2\le C\|w\|_{L_T^4L_x^4}^4. \tag{4.13}$$

Since the L^p mild solution $w\in L_T^\infty L_x^p\cap L_T^{\frac{4p}{3}}L_x^{2p}$, $p\in[3,4]$, by interpolation theory, we have $w\in L_T^{\frac{8p}{3(4-p)}}L_x^4$, and $z\in L^\infty([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$. Combining with similar argument as (2.26), we have $z\in C([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$. Then, $w\in C_w([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$. One can easily prove that w is a L^2-weak solution of the system (1.8) on $[0,T]$, and omit the details.

When $4<p\le 8$, the L^p mild solution $w\in L_T^\infty L_x^p\cap L_T^{\frac{4p}{3}}L_x^{2p}\subset L_T^{\frac{16p}{24-3p}}L_x^p\subset L_T^{\frac{16}{3}}L_x^8\subset L_T^{\frac{16}{5}}L_x^8$, from Lemma 6.2, we could obtain that

$$\|z\|_{C_tL^{\frac{4}{\cap}}L_t^{\frac{16}{3}}L_x^8}\le C\|w\|_{L_t^{\frac{16}{5}}L_x^8}\|w\|_{L_t^{\frac{16}{3}}L_x^8}, \tag{4.14}$$

and $z\in C([0,T];L_x^4)$. Combing $a\in L_T^\infty L_x^p\cap L_T^{\frac{4p}{3}}L_x^{2p}\cap C([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$, we have that $w\in C([0,T];L_x^4)$. From the argument in (4.13), we have that $z\in C([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$.

When $p>8$, the L^p mild solution $w\in L_T^\infty L_x^p\cap L_T^{\frac{4p}{3}}L_x^{2p}$, from Lemma 6.2, we could obtain that

$$\|z\|_{C_tL^{\frac{p}{2}}\cap L_t^{\frac{2p}{3}}L_x^p}\le C\|w\|_{L_t^{\frac{2p}{p-3}}L_x^p}\|w\|_{L_t^{\frac{2p}{3}}L_x^p}, \tag{4.15}$$

and $z\in C([0,T];L_x^{\frac{p}{2}})\cap L_t^{\frac{2p}{3}}L_x^p$. Combing $a\in L_T^\infty L_x^p\cap L_T^{\frac{4p}{3}}L_x^{2p}\cap C([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$, we have that $w\in C([0,T];L_x^{\frac{p}{2}})\cap L_T^{\frac{2p}{3}}L_x^p$. By the induction, we can get $w\in C([0,T];L_x^{\frac{p}{2^K}})\cap L_T^{\frac{p2^{2-K}}{3}}L_x^{p2^{1-K}}$, for some $K\in\mathbb{Z}^+$ such that $4<\frac{p}{2^K}\le 8$. From the argument in the case that $4<p\le 8$, we have that $z\in C([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$.

Therefore, the L^p mild solution $w\in C_w([0,T];L_x^2)\cap L_T^2(\dot{H}_x^1)$, one can easily prove that w is a L^2-weak solution of the system (1.8) on $[0,T]$, and omit the details.

Based on the proof of Proposition 1.1, we have the following results in the global time. For simplicity, we omit the detailed proof.

Corollary 4.1. *For $p\ge 3$, $T>0$, let c_p and ε_0 be as in Theorem 1.2, $|c|>c_p$. For $w_0\in L_\sigma^p(\mathbb{R}^3)\cap L_\sigma^2(\mathbb{R}^3)$ and $\|w_0\|_{L^3(\mathbb{R}^3)}<\varepsilon_0$, let w be a global L^p mild solution of the system (1.8). Then w is a global L^2-weak solution of the system (1.8).*

Combining with Theorem 1.2 and Corollary 4.1, we deduce Corollary 1.1.

5. Global $L^2 + L^3$ Weak Solution

In this section, we will illustrate Theorem 1.5, i.e. we will give the global existence of the $L^2 + L^3$ weak solution to the system (1.8).

Note that when we consider the existence of the weak solution to the Navier–Stokes system, there are essentially two methods: the energy method and the perturbation theory. The energy method gives the global existence for any initial data $v_0 \in L^2_\sigma(\mathbb{R}^3)$. We cannot use this method since the space L^2 doesn't contain the space L^3 in the whole space $\mathbb{R}^3$. In the perturbation theory, by the contraction mapping theorem, there exists a unique global weak solution to the Navier–Stokes system for the small initial data $v_0 \in L^3_\sigma(\mathbb{R}^3)$. Both methods cannot give direct results on the global existence for arbitrary $v_0 \in L^3_\sigma(\mathbb{R}^3)$.

Hence, many authors have developed various approaches to adapt the theory of the weak solutions so that it could allow $v_0 \in L^3_\sigma(\mathbb{R}^3)$. Calderón [6] raised a method such that the $L^3_\sigma(\mathbb{R}^3)$ initial data v_0 can be decomposed as

$$v_0 = v_0^1 + v_0^2, \tag{5.1}$$

where v_0^1 is small in $L^3_\sigma(\mathbb{R}^3)$ and v_0^2 belongs to $L^2_\sigma \cap L^3_\sigma(\mathbb{R}^3)$. Because of the smallness, the initial data v_0^1 generates a global smooth solution v_1 by the perturbation theory. Then the equation (1.24) for $v_2 = v - v_1$ can be solved by the energy method. Seregin and Šverák [40] used another method to obtain a global weak solution for $v_0 \in L^3_\sigma(\mathbb{R}^3)$. The main idea of [40] is as follows. Let v_1 be the solution of the linear version of the Navier–Stokes system, seek the solution v of the Navier–Stokes system as $v = v_1 + v_2$, write down the equation that v_2 satisfied, then get the property of v by investigating v_2. It's a general idea that the correction term v_2 might be easier to deal with than the full solution v. Related work can be referred to [26,27,40].

Inspired by above methods, we will decompose the initial data $w_0 = v_{10} + v_{20}$ and investigate the global existence of solutions $w = v_1 + v_2$ to the system (1.8). For $w_0 \in L^3_\sigma(\mathbb{R}^3)$, we have the following decomposition

$$w_0 = v_{10} + v_{20}, \tag{5.2}$$

with $\|v_{10}\|_{L^3} < \varepsilon_0$ and $v_{20} \in L^2_\sigma \cap L^3_\sigma(\mathbb{R}^3)$. According to Theorem 1.1, there exists a unique global L^3 mild solution v_1 to the system (1.23). The crucial part is the global existence of v_2. Since $v_{20} \in L^2_\sigma \cap L^3_\sigma$, this is the standard reasoning based on the Galerkin method (cf. [19] Proof of Theorem 2.7). We claim there exists a global weak solution $v_2 \in C_w\left([0,T]; L^2_\sigma\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T]; \dot{H}^1_\sigma\left(\mathbb{R}^3\right)\right)$ for any $T > 0$. According to Definition 1.3, there exists a global $L^2 + L^3$ weak solution to the system (1.8). Detailed proof of the global existence of v_2 can be seen below.

First, we will construct weak solutions v_2 to the system (1.24). This is the standard reasoning based on the Galerkin method (cf. [19] Proof of Theorem 2.1). Since $H^1_\sigma\left(\mathbb{R}^3\right)$ is separable, there exists a sequence $\{g_m\}_{m=1}^\infty$ which is free and total in $H^1_\sigma\left(\mathbb{R}^3\right)$. For each $m = 1, 2, \ldots$ Define an approximate solution $w_m = \sum_{i=1}^m d_{im}(t) g_i$, which satisfies the following system of ordinary differential equations

$$\begin{aligned} &\langle w_m'(t), g_j\rangle + \langle \nabla w_m(t), \nabla g_j\rangle + \langle (w_m(t)\cdot\nabla)\, w_m(t), g_j\rangle \\ &+ \langle (w_m(t)\cdot\nabla)\,(v_c + v_1), g_j\rangle + \langle ((v_c + v_1)\cdot\nabla)\, w_m(t), g_j\rangle = 0 \text{ for } j = 1, \ldots, m, \end{aligned} \tag{5.3}$$

where the term corresponding to the pressure in (1.24) vanishes in (5.3) because of $\mathrm{div} g_j = 0$. The system (5.3) has a unique local solution $\{d_{im}(t)\}_{i=1}^m$. By *a-priori* estimates of the sequence $\{w_m\}_{m=1}^\infty$ obtained below in (5.7), the solution $d_{im}(t)$ is global.

We will prove terms $\langle (w_m(t)\cdot\nabla)\,(v_c + v_1), g_j\rangle$ and $\langle ((v_c + v_1)\cdot\nabla)\, w_m(t), g_j\rangle$ in (5.3) are convergent. By Hölder and Sobolev inequalities in the Lorentz $L^{p,q}$-spaces (see [19]), we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3} g_j (w_m\cdot\nabla)(v_c + v_1)\mathrm{d}x\right| &\le C\|(v_c + v_1) w_m\|_{L^2}\|\nabla g_j\|_{L^2} \\ &\le C\|v_c + v_1\|_{L^{3,\infty}}\|w_m\|_{L^{6,2}}\|\nabla g_j\|_{L^2} \end{aligned}$$

$$\leq C\|v_c+v_1\|_{L^{3,\infty}}\|\nabla w_m\|_{L^2}\|\nabla g_j\|_{L^2}, \tag{5.4}$$

and

$$\left|\int_{\mathbb{R}^3}((v_c+v_1)\cdot\nabla)w_m g_j \mathrm{d}x\right| \leq C\|v_c+v_1\|_{L^{3,\infty}}\|\nabla w_m\|_{L^2}\|\nabla g_j\|_{L^2}. \tag{5.5}$$

Multiplying the equation (5.3) by d_{jm} and sum up equations for $j=1,2,\ldots,m$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(t)\|_2^2+\|\nabla w_m(t)\|_2^2+\langle(w_m(t)\cdot\nabla)(v_c+v_1),w_m(t)\rangle=0. \tag{5.6}$$

Using the inequality (5.4) and integrating it from 0 to t, we obtain

$$\|w_m(t)\|_2^2+2\left(1-K\sup_{t>0}\|v_c+v_1\|_{L^{3,\infty}}\right)\int_0^t\|\nabla w_m(\tau)\|_2^2\,\mathrm{d}\tau\leq\|w_0\|_2^2. \tag{5.7}$$

Since $|c|$ big enough such that $K\sup_{t>0}\|v_c+v_1\|_{L^{3,\infty}}<1$. Thus we obtain a subsequence, also denoted by $\{w_m\}_{m=1}^{\infty}$, converging to $v_2\in C_w\left([0,T];L^2_\sigma\left(\mathbb{R}^3\right)\right)\cap L^2\left([0,T];\dot{H}^1_\sigma\left(\mathbb{R}^3\right)\right)$. Now, repeating the classical reasoning from [19], we obtain the existence of a weak solution in the energy space $C_w\left([0,T];L^2_\sigma\left(\mathbb{R}^3\right)\right)\cap L^2\left([0,T];\dot{H}^1_\sigma\left(\mathbb{R}^3\right)\right)$ for all $T>0$ which satisfies the strong energy inequality (5.7).

Hence we get a global L^2+L^3 weak solution w of the form $w=v_1+v_2$. Moreover, we have the asymptotic behavior of v_2 and omit the proof which can be referred to [19].

6. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2. Our method is based on the contraction mapping theorem, see Lemma 2.1, Lemmas 2.4, 6.1 and 6.2.

To get an *a-prioi* estimate of z, in which the crucial estimate is as follows:

$$\begin{aligned}
-\int_{\mathbb{R}^3}\mathrm{div}(w_1\otimes w_2)\cdot(|z|^{p-2}z)dx&=\int_{\mathbb{R}^3}(w_1\otimes w_2)\cdot\nabla(|z|^{p-2}z)dx\\
&\leq C\int_{\mathbb{R}^3}(w_1\otimes w_2)\cdot\nabla\left(|z|^{\frac{p}{2}}\right)|z|^{\frac{p}{2}-1}dx\\
&\leq C\left\|\nabla\left(|z|^{\frac{p}{2}}\right)\right\|_{L^2}\left\||z|^{\frac{p}{2}-1}\right\|_{L^{\frac{2p}{p-2}}}\|w_1\otimes w_2\|_{L^p}\\
&\leq C\left\|\nabla\left(|z|^{\frac{p}{2}}\right)\right\|_{L^2}\|z\|_{L^p}^{\frac{p}{2}-1}\|w_1\otimes w_2\|_{L^p}\\
&\leq\varepsilon\left\|\nabla\left(|z|^{\frac{p}{2}}\right)\right\|_{L^2}^2+C(\varepsilon)\|z\|_{L^p}^{p-2}\|w_1\otimes w_2\|_{L^p}^2.
\end{aligned}$$

We have

$$\sup_t\|z(t)\|_{L^p_x}+\left\|\nabla(|z|^{\frac{p}{2}})\right\|_{L^2_tL^2_x}^{\frac{2}{p}}\leq C\|w_1\|_{L^4_tL^{2p}_x}\|w_2\|_{L^4_tL^{2p}_x}\leq CT^{\frac{p-3}{2p}}\|w_1\|_{L_t^{\frac{4p}{3}}L^{2p}_x}\|w_2\|_{L_t^{\frac{4p}{3}}L^{2p}_x}.$$

Hence, we have the following *a-prioi* estimate and more detailed proof can be referred to in the proof of Lemma 2.5.

Lemma 6.1. *Let $p\in(3,\infty)$, c_p is as in Theorem 1.2. For every $|c|>c_p$, there exists a L^p mild solution $z(x,t)$ on $[0,T]$ to the system (2.30) with $w_1,w_2\in L_t^{\frac{4p}{3}}([0,T];L^{2p}(\mathbb{R}^3))$, satisfying*

$$\|z\|_{C_TL^p\cap L_T^{\frac{4p}{3}}L^{2p}_x}+\|\nabla(|z|^{\frac{p}{2}})\|_{L^2_tL^2_x}^{\frac{2}{p}}\leq CT^{\frac{p-3}{2p}}\|w_1\|_{L_T^{\frac{4p}{3}}L^{2p}_x}\|w_2\|_{L_T^{\frac{4p}{3}}L^{2p}_x}, \tag{6.1}$$

for a constant C.

When the initial data $w_0 \in L^p_\sigma \cap L^3_\sigma$ and $\|w_0\|_{L^3} < \varepsilon_0$, we have $w \in C_t L^3 \cap L^4_t L^6_x$, $\nabla\left(|w|^{\frac{p}{2}}\right) \in L^2_t L^2_x$ according Theorem 1.1. By the interpolation inequality, we have $w \in L_t^{\frac{4p}{2p-3}} L_x^{2p}$. The proof is very similar as the proof of Lemma 2.5, in which the crucial estimate is as follows:

$$\|w_1 \otimes w_2\|_{L^2_t L^p_x} \le \|w_1\|_{L_t^{\frac{4p}{2p-3}} L_x^{2p}} \|w_2\|_{L_t^{\frac{4p}{3}} L_x^{2p}}. \tag{6.2}$$

Hence we have the following *a-prioi* estimate and more detailed proof can be referred to in the proof of Lemma 2.5.

Lemma 6.2. *Let $p \in (3, \infty)$, c_p is as in Theorem 1.2. For every $|c| > c_p$, there exists a global-in-time L^p mild solution $z(x,t)$ to the system (2.30) with $w_1 \in L_t^{\frac{4p}{2p-3}}([0,\infty); L_x^{2p}(\mathbb{R}^3))$ and $w_2 \in L_t^{\frac{4p}{3}}([0,\infty); L_x^{2p}(\mathbb{R}^3))$, satisfying*

$$\|z\|_{C_t L^p \cap L_t^{\frac{4p}{3}} L_x^{2p}} + \|\nabla(|z|^{\frac{p}{2}})\|^{\frac{2}{p}}_{L^2_t L^2_x} \le C\|w_1\|_{L_t^{\frac{4p}{2p-3}} L_x^{2p}} \|w_2\|_{L_t^{\frac{4p}{3}} L_x^{2p}}, \tag{6.3}$$

for a constant C.

Proof of Theorem 1.2. For a constant $|c| > c_p$ where c_p depends only on p, according to Lemma 2.4, we have

$$\|a(t)\|_{L_t^{\frac{4p}{3}} L_x^{2p}} \le C\|w_0\|_{L^p}. \tag{6.4}$$

Applying Lemma 6.1, we have

$$\|N(w_1, w_2)\|_{L_t^{\frac{4p}{3}} L_x^{2p}} \le CT^{\frac{p-3}{2p}} \|w_1\|_{L_t^{\frac{4p}{3}} L_x^{2p}} \|w_2\|_{L_t^{\frac{4p}{3}} L_x^{2p}}. \tag{6.5}$$

Using Lemma 2.1 with $E = L_T^{\frac{4p}{3}} L_x^{2p}$, when $CT^{\frac{p-3}{2p}}\|w_0\|_{L^p} < 1$, the system (1.8) has a unique solution $w \in L_T^{\frac{4p}{3}} L_x^{2p}$ on $[0,T]$.

Then we will prove the global existence of w with the initial data $w_0 \in L^p_\sigma \cap L^3_\sigma$ and $\|w_0\|_{L^3} < \varepsilon_0$. Since $\|w_0\|_{L^3} < \varepsilon_0$, according to Theorem 1.1, there exists a global unique solution $w \in C_t L^3_x \cap L^4_t L^6_x$, $\nabla(|w|^{\frac{3}{2}}) \in L^2_t L^2_x$, and $\|w\|_{C_t L^3_x \cap L^4_t L^6_x} + \|\nabla|w|^{\frac{2}{3}}\|^{\frac{2}{3}}_{L^2_t L^2_x} \le C\|w_0\|_{L^3}$. By the interpolation inequality, $w \in C_t L^3_x$ and $\nabla(|w|^{\frac{3}{2}}) \in L^2_t L^2_x$ deduce $w \in L_t^{\frac{4p}{2p-3}} L_x^{2p}$, and

$$\|w\|_{L_t^{\frac{4p}{2p-3}} L_x^{2p}} \le C\|w_0\|_{L^3} < C\varepsilon_0. \tag{6.6}$$

Thanks to (2.8) and (6.3), we have

$$\|w\|_{C_t L^p_x \cap L_t^{\frac{4p}{3}} L_x^{2p}} \le C\|w_0\|_{L^p} + C\|w\|_{L_t^{\frac{4p}{2p-3}} L_x^{2p}} \|w\|_{C_t L^p_x \cap L_t^{\frac{4p}{3}} L_x^{2p}}. \tag{6.7}$$

Combining with (6.6) and the interpolation theory, we deduce (1.16). □

7. Proof of Theorems 1.3

In this section, we give the proof of Theorems 1.3.

Proof of Theorem 1.3. Setting $Z = u - v$, we have

$$\begin{cases} Z_t - \Delta Z + \operatorname{div}(-Z \otimes Z + Z \otimes u + u \otimes Z) + (Z \cdot \nabla) v_c + (v_c \cdot \nabla) z + \nabla \pi_z = 0, \\ \nabla \cdot Z = 0, \\ Z(x,0) = Z_0. \end{cases} \tag{7.1}$$

By the Duhamel principle, we can rewrite the solution z into an integral formulation

$$Z(x,t) = e^{-t\mathcal{L}} u_0 - \int_0^t e^{-(t-s)\mathcal{L}} \mathbb{P}\operatorname{div}(-Z \otimes Z + Z \otimes u + u \otimes Z) ds. \tag{7.2}$$

By the contraction mapping theorem, it's easy to give the existence of the solution. Next, we only give some *a-prioi* estimates.

When $p = 3$, by Lemma 2.4 and the method in Lemma 2.5, we get

$$\|Z\|_{C_T L^3_x \cap L^4_T L^6_x} + \left\|\nabla\left(|Z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L^2_T L^2_x} \le C_1\|Z_0\|_{L^3} + C_2\|Z\|^2_{L^4_T L^6_x} + C_2\left(\int_0^T (\|Z\|_{L^6}\|u\|_{L^6})^2 dt\right)^{\frac{1}{2}}. \tag{7.3}$$

By the interpolation inequality, Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\left(\int_0^T (\|Z\|_{L^6}\|u\|_{L^6})^2 dt\right)^{\frac{1}{2}} &\le \left(\int_0^T \left(\|Z\|^{\frac{1}{4}}_{L^3}\|Z\|^{\frac{3}{4}}_{L^9}\|u\|_{L^6}\right)^2 dt\right)^{\frac{1}{2}}\\ &\le \left(\int_0^T \|Z\|^{\frac{1}{2}}_{L^3}\|Z\|^{\frac{3}{2}}_{L^9}\|u\|^2_{L^6} dt\right)^{\frac{1}{2}}\\ &\le \|Z\|^{\frac{3}{4}}_{L^3_T L^9_x}\left(\int_0^T \|Z\|_{L^3}\|u\|^4_{L^6} dt\right)^{\frac{1}{4}}\\ &\le \varepsilon\|Z\|_{L^3_T L^9_x} + \frac{27}{256\varepsilon^3}\int_0^T \|Z\|_{L^3}\|u\|^4_{L^6} dt.\end{aligned}$$

Combining with (7.3), by Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$\begin{aligned}&\|Z\|_{C_T L^3_x \cap L^4_T L^6_x} + \left\|\nabla\left(|Z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L^2_T L^2_x}\\ &\le C\|Z_0\|_{L^3} + C\|Z\|^2_{L^4_T L^6_x} + C\varepsilon\|Z\|_{L^3_T L^9_x} + \frac{C}{\varepsilon^3}\int_0^T \|Z\|_{L^3}\|u\|^4_{L^6} dt\\ &\le C\|Z_0\|_{L^3} + C\|Z\|^2_{L^4_T L^6_x} + C\varepsilon\left\|\nabla\left(|Z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L^2_T L^2_x} + \frac{C}{\varepsilon^3}\int_0^T \|Z\|_{L^3}\|u\|^4_{L^6} dt.\end{aligned}$$

Taking $C\varepsilon = \frac{1}{2}$, using Gronwall's inequality, we get

$$\|Z\|_{C_T L^3_x \cap L^4_T L^6_x} + \left\|\nabla\left(|Z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L^2_T L^2_x} \le C(\|Z_0\|_{L^3} + \|Z\|^2_{L^4_T L^6_x})e^{C\int_0^T \|u\|^4_{L^6} dt}. \tag{7.4}$$

When $\|Z_0\|_{L^3} \le (4C^2 e^{2C\int_0^T \|u\|^4_{L^6} dt})^{-1}$, by the continuity method, we have

$$\|Z\|_{C_T L^3_x \cap L^4_T L^6_x} + \left\|\nabla\left(|Z|^{\frac{3}{2}}\right)\right\|^{\frac{2}{3}}_{L^2_T L^2_x} \le 2C\|Z_0\|_{L^3} e^{C\int_0^T \|u\|^4_{L^6} dt}. \tag{7.5}$$

Therefore, (1.17) holds with $p = 3$.

When $p > 3$, by Lemma 2.4 and the method in Lemma 6.1, we get

$$\begin{aligned}&\|Z\|_{C_T L^p_x \cap L^{\frac{4p}{3}}_T L^{2p}_x} + \left\|\nabla\left(|Z|^{\frac{p}{2}}\right)\right\|^{\frac{2}{p}}_{L^2_T L^2_x}\\ &\le C\|Z_0\|_{L^p} + C\left(\int_0^T (\|Z\|_{L^{2p}_x}\|Z\|_{L^{2p}_x})^2 dt\right)^{\frac{1}{2}} + C\left(\int_0^T (\|Z\|_{L^{2p}_x}\|u\|_{L^{2p}_x})^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{7.6}$$

By the interpolation inequality, Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\left(\int_0^T \left(\|Z\|_{L^{2p}_x}\|u\|_{L^{2p}_x}\right)^2 dt\right)^{\frac{1}{2}} &\le \left(\int_0^T \left(\|Z\|^{\frac{3}{4}}_{L^{3p}_x}\|Z\|^{\frac{1}{4}}_{L^p_x}\|u\|_{L^{2p}_x}\right)^2 dt\right)^{\frac{1}{2}}\\ &\le \left\|\|Z\|^{\frac{3}{4}}_{L^{3p}_x}\right\|_{L^{\frac{4p}{3}}_T}\left\|\|Z\|^{\frac{1}{4}}_{L^p_x}\|u\|_{L^{2p}_x}\right\|_{L^{\frac{4p}{2p-3}}_T}\end{aligned}$$

$$\leq \|Z\|^{\frac{3}{4}}_{L^p_T L^{3p}_x} \left\| \|Z\|^{\frac{1}{4}}_{L^p_x} \|u\|_{L^{2p}_x} \right\|_{L_T^{\frac{4p}{2p-3}}}$$
$$\leq \varepsilon \|Z\|_{L^p_T L^{3p}_x} + \frac{27}{256\varepsilon^3} \left\| \|Z\|^{\frac{1}{4}}_{L^p_x} \|u\|_{L^{2p}_x} \right\|^4_{L_T^{\frac{4p}{2p-3}}}.$$

Combining with (7.6), using Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
&\|Z\|_{C_T L^p_x \cap L_T^{\frac{4p}{3}} L^{2p}_x} + \left\| \nabla \left(|Z|^{\frac{p}{2}} \right) \right\|^{\frac{2}{p}}_{L^2_T L^2_x} \\
&\leq C\|Z_0\|_{L^p} + C \left(\int_0^T (\|Z\|_{L^{2p}_x} \|Z\|_{L^{2p}_x})^2 dt \right)^{\frac{1}{2}} + C\varepsilon \|Z\|_{L^p_T L^{3p}_x} + \frac{C}{\varepsilon^3} \left\| \|Z\|^{\frac{1}{4}}_{L^p_x} \|u\|_{L^{2p}_x} \right\|^4_{L_T^{\frac{4p}{2p-3}}} \\
&\leq C\|Z_0\|_{L^p} + C \left(\int_0^T (\|Z\|_{L^{2p}_x} \|Z\|_{L^{2p}_x})^2 dt \right)^{\frac{1}{2}} + C\varepsilon \left\| \nabla \left(|Z|^{\frac{p}{2}} \right) \right\|^{\frac{2}{p}}_{L^2_T L^2_x} + \frac{C}{\varepsilon^3} \left\| \|Z\|^{\frac{1}{4}}_{L^p_x} \|u\|_{L^{2p}_x} \right\|^4_{L_T^{\frac{4p}{2p-3}}}.
\end{aligned}$$

Taking $C\varepsilon = \frac{1}{2}$, using Gronwall's inequality, we have

$$\|Z\|_{C_T L^p_x \cap L_T^{\frac{4p}{3}} L^{2p}_x} + \left\| \nabla \left(|Z|^{\frac{p}{2}} \right) \right\|^{\frac{2}{p}}_{L^2_T L^2_x} \leq C \left(\|Z_0\|_{L^p} + T^{\frac{p-3}{2p}} \|Z\|^2_{L_T^{\frac{4p}{3}} L^{2p}_x} \right) \exp \left\{ C\|u\|^{\frac{4p}{2p-3}}_{L_T^{\frac{4p}{2p-3}} L^{2p}_x} \right\}.$$

By the continuity method, when $4C^2 T^{\frac{p-3}{2p}} \|Z_0\|_{L^p} \exp\{2C\|u\|^{\frac{4p}{2p-3}}_{L_T^{\frac{4p}{2p-3}} L^{2p}_x}\} < 1$, we deduce

$$\|Z\|_{C_T L^p_x \cap L_T^{\frac{4p}{3}} L^{2p}_x} + \left\| \nabla \left(|Z|^{\frac{p}{2}} \right) \right\|^{\frac{2}{p}}_{L^2_T L^2_x} \leq 2C\|Z_0\|_{L^p} \exp \left\{ C\|u\|^4_{L_T^{\frac{4p}{2p-3}} L^{2p}_x} \right\}. \tag{7.7}$$

When $\|Z_0\|_{L^p} \to 0$, (7.7) implies (1.17) with $p > 3$. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

8. Appendix

Proof of Lemma 2.6. Set

$$\begin{aligned} II &= -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \partial_j (w_i w_j) \mathrm{d}x, \\ III &= -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \partial_j (w_i v_j) \mathrm{d}x, \\ IV &= -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \partial_j (v_i w_j) \mathrm{d}x, \\ V &= -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \partial_i \pi \mathrm{d}x. \end{aligned} \tag{8.1}$$

Thanks to integration by parts, Hölder's inequality and Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ (the best constant can be seen in [44]), we have

$$\begin{aligned} II &= -\frac{r(t)^2}{2} \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_j \partial_j |w|^2 \mathrm{d}x \\ &= \frac{r(t)^2}{2} \int_{\mathbb{R}^3} \partial_j (|w(\cdot,t)|^{r(t)-2}) w_j |w|^2 \mathrm{d}x \\ &\le r(t)(r(t)-2) \int_{\mathbb{R}^3} \left| \nabla (|w(\cdot,t)|^{\frac{r(t)}{2}}) \right| |w|^{\frac{r(t)}{2}} |w| \mathrm{d}x \\ &\le r(t)(r(t)-2) \left\| \nabla \left(|w(t)|^{\frac{r(t)}{2}} \right) \right\|_{L^2} \left\| |w(t)|^{\frac{r(t)}{2}} \right\|_{L^6} \|w(t)\|_{L^3} \\ &\le r(t)(r(t)-2) \left\| \nabla \left(|w(t)|^{\frac{r(t)}{2}} \right) \right\|_{L^2}^2 \|w(t)\|_{L^3}. \end{aligned} \tag{8.2}$$

Combining (1.13) with $\|w_0\|_{L^3} \le \varepsilon_0$, there holds

$$II \le r(t)(r(t)-2) C \varepsilon_0 \left\| \nabla (|w(t)|^{\frac{r(t)}{2}}) \right\|_{L^2}^2. \tag{8.3}$$

According to Hölder's inequality, the Hardy inequality in Lemmas 2.2 and 2.3, we deduce

$$\begin{aligned} II &= \frac{r(t)^2}{2} \int_{\mathbb{R}^3} \partial_j (|w(\cdot,t)|^{r(t)-2}) |w|^2 v_j \mathrm{d}x \\ &= r(t)(r(t)-2) \int_{\mathbb{R}^3} v_c \cdot \nabla (|w(\cdot,t)|^{\frac{r(t)}{2}}) |w(\cdot,t)|^{\frac{r(t)}{2}} \mathrm{d}x \\ &\le r(t)(r(t)-2) \left\| \nabla \left(|w(t)|^{\frac{r(t)}{2}} \right) \right\|_{L^2} \left\| \frac{|w(t)|^{\frac{r(t)}{2}}}{|x|} \right\|_{L^2} \| |x| v_c \|_{L^\infty} \\ &\le 2r(t)(r(t)-2) K_c \left\| \nabla \left(|w(t)|^{\frac{r(t)}{2}} \right) \right\|_{L^2}^2. \end{aligned} \tag{8.4}$$

For the term II, using integration by parts, we have

$$\begin{aligned} IV &= -r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} w_i \partial_j (v_i w_j) \mathrm{d}x \\ &= r(t)^2 \int_{\mathbb{R}^3} \partial_j (|w(\cdot,t)|^{r(t)-2}) w_i v_i w_j \mathrm{d}x + r(t)^2 \int_{\mathbb{R}^3} |w(\cdot,t)|^{r(t)-2} \partial_j w_i v_i w_j \mathrm{d}x. \end{aligned}$$

The estimate of the first part is similar to (8.4). We have

$$r(t)^2 \int_{\mathbb{R}^3} \partial_j (|w(\cdot,t)|^{r(t)-2}) w_i v_i w_j \mathrm{d}x \le 4r(t)(r(t)-2) K_c \left\| \nabla \left(|w(t)|^{\frac{r(t)}{2}} \right) \right\|_{L^2}^2.$$

By Lemma 2.2, Cauchy inequality and the Hardy inequality, we can estimate the second part as follows

$$\begin{aligned}
&r(t)^2\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}\partial_j w_i v_i w_j \mathrm{d}x\\
&\le r(t)^2\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}|\partial_j w_i||x|v_i|\frac{|w_j|}{|x|}\mathrm{d}x\\
&\le r(t)^2K_c\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}|\partial_j w_i|\frac{|w_j|}{|x|}\mathrm{d}x\\
&\le \frac{r(t)^2}{2}K_c\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}|\nabla w(\cdot,t)|^2dx+\frac{r(t)^2}{2}K_c\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}\frac{|w(\cdot,t)|^2}{|x|^2}dx\\
&\le \frac{r(t)^2}{2}K_c\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}|\nabla w(\cdot,t)|^2dx+2r(t)^2K_c\left\|\nabla\left(|w(t)|^{\frac{r(t)}{2}}\right)\right\|_{L^2}^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
IV\le&\frac{r(t)^2}{2}K_c\int_{\mathbb{R}^3}|w(\cdot,t)|^{r(t)-2}|\nabla w(\cdot,t)|^2dx\\
&+(4r(t)(r(t)-2)+2r(t)^2)K_c\left\|\nabla\left(|w(t)|^{\frac{r(t)}{2}}\right)\right\|_{L^2}^2.
\end{aligned}\tag{8.5}$$

Note that the pressure $\pi=-\frac{\partial_i\partial_j}{\Delta}(w_iw_j+v_iw_j+w_iv_j)$, using integration by parts, we obtain

$$\begin{aligned}
V&=r(t)^2\int_{\mathbb{R}^3}\partial_i(|w(\cdot,t)|^{r(t)-2})w_i\pi\mathrm{d}x\\
&=r(t)^2\int_{\mathbb{R}^3}\partial_i(|w(\cdot,t)|^{r(t)-2})w_i\left(-\frac{\partial_i\partial_j}{\Delta}(v_iw_j+w_iv_j+w_iw_j)\right)\mathrm{d}x.
\end{aligned}\tag{8.6}$$

This term is more complex to deal with, we will estimate it more carefully. Set

$$V_1=r(t)^2\int_{\mathbb{R}^3}\partial_i(|w(\cdot,t)|^{r(t)-2})w_i\left(-\frac{\partial_i\partial_j}{\Delta}(v_iw_j+w_iv_j)\right)\mathrm{d}x,$$

and

$$V_2=r(t)^2\int_{\mathbb{R}^3}\partial_i(|w(\cdot,t)|^{r(t)-2})w_i\left(-\frac{\partial_i\partial_j}{\Delta}w_iw_j\right)\mathrm{d}x.$$

According to [12], there holds $|x|^{r-2}\in A_r$ with $1<r<\infty$. By Hölder's inequality, boundedness of the Riesz transforms on weighted L^p spaces (Theorem 9.4.6 in [12]), Lemma 2.2 and the Hardy inequality, there holds

$$\begin{aligned}
V_1&\le 2r(t)(r(t)-2)\int_{\mathbb{R}^3}\left|\nabla\left(|w(\cdot,t)|^{\frac{r(t)}{2}}\right)\right||w(\cdot,t)|^{\frac{r(t)}{2}-1}\left|\frac{\partial_i\partial_j}{\Delta}(v_iw_j+w_iv_j)\right|\mathrm{d}x\\
&\le 4r(t)(r(t)-2)C_r\||x|^{\frac{r-2}{r}}(v_c\otimes w)\|_{L^r}\left\|\frac{w^{\frac{r}{2}-1}}{x^{\frac{r-2}{r}}}\right\|_{L^{\frac{2r}{r-2}}}\|\nabla(|w(\cdot,t)|^{\frac{r}{2}})\|_{L^2}\\
&\le 4r(t)(r(t)-2)C_r\||x|v_c\|_{L^\infty}\||x|^{-\frac{2}{r}}w\|_{L^r}\left\|\frac{w}{|x|^{\frac{2}{r}}}\right\|_{L^r}^{\frac{r}{2}-1}\|\nabla(|w(\cdot,t)|^{\frac{r}{2}})\|_{L^2}\\
&\le 4r(t)(r(t)-2)C_rK_c\left\|\frac{|w|^{\frac{r}{2}}}{|x|}\right\|_{L^2}\|\nabla(|w(\cdot,t)|^{\frac{r}{2}})\|_{L^2}\\
&\le 8r(t)(r(t)-2)C_rK_c\|\nabla(|w(\cdot,t)|^{\frac{r}{2}})\|_{L^2}^2,
\end{aligned}\tag{8.7}$$

where C_r is as in Theorem 9.4.6 in [12]. Thanks to [15], we deduce that $\|\frac{\partial_i\partial_j}{\Delta}f\|_{L^r}\le H_r\|f\|_{L^r}$. Combining with Hölder's inequality and Sobolev embedding $\dot{H}^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, we have

$$V_2\le 2r(t)(r(t)-2)\|\nabla(|w(\cdot,t)|^{\frac{r(t)}{2}})\|_{L^2}\||w(\cdot,t)|^{\frac{r(t)}{2}-1}\|_{L^{\frac{6r}{r-2}}}\left\|\frac{\partial_i\partial_j}{\Delta}w_iw_j\right\|_{L^{\frac{3r}{r+1}}}$$

$$\le 2r(t)(r(t)-2)H_{\frac{3r}{r+1}}\|\nabla(|w(\cdot,t)|^{\frac{r(t)}{2}})\|_{L^2}\||w(\cdot,t)|^{\frac{r(t)}{2}-1}\|_{L^{\frac{6r}{r-2}}}\|w\otimes w\|_{L^{\frac{3r}{r+1}}}$$

$$\le 2r(t)(r(t)-2)H_{\frac{3r}{r+1}}\|\nabla(|w(\cdot,t)|^{\frac{r(t)}{2}})\|_{L^2}\|w(\cdot,t)\|^{\frac{r(t)}{2}-1}_{L^{3r}}\|w(\cdot,t)\|_{L^{3r}}\|w(\cdot,t)\|_{L^3}$$

$$\le 4r(t)(r(t)-2)H_{\frac{3r}{r+1}}\|\nabla(|w(\cdot,t)|^{\frac{r(t)}{2}})\|_{L^2}\||w(\cdot,t)|^{\frac{r(t)}{2}}\|_{L^6}\|w(\cdot,t)\|_{L^3}$$

$$\le 4r(t)(r(t)-2)H_{\frac{3r}{r+1}}\|\nabla(|w(\cdot,t)|^{\frac{r(t)}{2}})\|^2_{L^2}\|w(\cdot,t)\|_{L^3}.$$

According to (1.13), when $\|w_0\|_{L^3}\le\varepsilon_0$, there holds

$$V_2\le 4r(t)(r(t)-2)H_{\frac{3r}{r+1}}C\varepsilon_0\|\nabla(|w(\cdot,t)|^{\frac{r(t)}{2}})\|^2_{L^2}. \tag{8.8}$$

From the above estimates, we can finish the proof of Lemma 2.6. □

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