Journal of Functional Analysis 281 (2021) 109198

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The axisymmetric σ_k-Nirenberg problem

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A R T I C L E   I N F O

Article history:
Received 8 June 2021
Accepted 9 July 2021
Available online 16 July 2021
Communicated by H. Brezis

Keywords:
σ_k-curvature
Compactness and non-compactness
Existence and non-existence
Axisymmetry

A B S T R A C T

We study the problem of prescribing σ_k-curvature for a conformal metric on the standard sphere $\mathbb{S}^n$ with $2 \le k < n/2$ and $n \ge 5$ in axisymmetry. Compactness, non-compactness, existence and non-existence results are proved in terms of the behaviors of the prescribed curvature function K near the north and the south poles. For example, consider the case when the north and the south poles are local maximum points of K of flatness order $\beta \in [2, n)$. We prove among other things the following statements. (1) When $\beta > n - 2k$, the solution set is compact, has a nonzero total degree counting and is therefore non-empty. (2) When $\beta = n-2k$, there is an explicit positive constant $C(K)$ associated with K. If $C(K) > 1$, the solution set is compact with a nonzero total degree counting and is therefore non-empty. If $C(K) < 1$, the solution set is compact but the total degree counting is 0, and the solution set is sometimes empty and sometimes non-empty. (3) When $\frac{2}{n-2k} \le \beta < n - 2k$, the solution set is compact, but the total degree counting is zero, and the solution set is sometimes empty and sometimes non-empty. (4) When $\beta < \frac{n-2k}{2}$, there exists K for which there exists a blow-up sequence of solutions

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[1] Partially supported by NSF Grants DMS-1501004, DMS-2000261, and Simons Fellows Award 677077.
[2] Partially supported by NSF of China (No. 11701027 and No. 11971061).

https://doi.org/10.1016/j.jfa.2021.109198
0022-1236/

with unbounded energy. In this same range of β, there exists also some K for which the solution set is empty.

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1. Introduction

We consider the σ_k-Nirenberg problem on the n-sphere $\mathbb{S}^n$ $(n \geq 3)$: Find a metric conformal to the standard metric on $\mathbb{S}^n$ such that its σ_k-curvature is equal to a prescribed positive function on $\mathbb{S}^n$.

Recall that, for a metric g on $\mathbb{S}^n$, the σ_k-curvature of g is defined as follows. Let Ric_g, R_g and A_g denote respectively the Ricci curvature, the scalar curvature and the Schouten tensor of g:

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right).$$

Let $\lambda(A_g)$ denote the eigenvalues of A_g with respect to g. For $1 \leq k \leq n$, the σ_k-curvature of g is then the function $\sigma_k(\lambda(A_g))$ where σ_k is the k-elementary symmetric function, $\sigma_k(\lambda) = \sum\limits_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k}$. Our equation of interest is thus

$$\sigma_k(\lambda(A_g)) = K \text{ and } \lambda(A_g) \in \Gamma_k \text{ on } \mathbb{S}^n \tag{1.1}$$

where g is the unknown metric which is conformal to the standard metric, K is a prescribed positive function on $\mathbb{S}^n$, and Γ_k is the connected component of $\{\lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0\}$ which contains the positive cone $\{\lambda \in \mathbb{R}^n : \lambda_1, \ldots, \lambda_n > 0\}$.

Let $\mathring{g}$ denote the standard metric on $\mathbb{S}^n$ and write the metric g as $g_v = v^{\frac{4}{n-2}}\mathring{g}$ for some positive function v. Note that

$$A_{g_v} = A_{\mathring{g}} - \frac{2}{n-2}v^{-1}\nabla^2_{\mathring{g}}v + \frac{2n}{(n-2)^2}v^{-2}dv \otimes dv - \frac{2}{(n-2)^2}v^{-2}|dv|^2_{\mathring{g}}\mathring{g}.$$

Therefore, for $2 \leq k \leq n$, (1.1) is a fully nonlinear elliptic equation for v. Similar equations involving eigenvalues of the Hessian of a function were first considered in [5].

In a recent paper [22], we started our study of the σ_k-Nirenberg problem. We proved an existence and compactness result in the case $k \geq n/2$ under the assumption that the prescribed curvature function K satisfies certain non-degeneracy condition at its critical points, which generalized a result of Chang, Han and Yang [6] for $k = 2$ in dimension 4. We refer the readers to [22] for a discussion of related works.

The compactness issue for the σ_k-Nirenberg problem as well as for the related σ_k-Yamabe problem on compact manifolds when $2 \leq k < n/2$ is a challenging open problem. In the present paper, we study this issue in the restrictive setting of axisymmetry. Namely, we view $(\mathbb{S}^n, \mathring{g}) = \{(x^1)^2 + \ldots + (x^{n+1})^2 = 1\}$ as the unit sphere embedded in $\mathbb{R}^{n+1}$ and suppose that the functions K and v depend only on $\theta = \arccos x^{n+1}$. In addition, we assume that K has the following behaviors at the north and south pole: there exist $a_1, a_2 \neq 0$ and $\beta_1, \beta_2 > 1$ such that if we write

$$K(\theta) = K(0) + a_1\theta^{\beta_1} + R_1(\theta) = K(\pi) + a_2(\pi - \theta)^{\beta_2} + R_2(\theta)$$

then

$$\lim_{\theta \to 0} \frac{|R_1(\theta)| + |\theta||R_1'(\theta)|}{|\theta|^{\beta_1}} = \lim_{\theta \to \pi} \frac{|R_2(\theta)| + |\pi - \theta||R_2'(\theta)|}{|\pi - \theta|^{\beta_2}} = 0. \tag{1.2}$$

Our study is motivated by earlier works in the case $k = 1$ by Bianchi and Egnell [2], Chen and Lin [8,9], and Li [17,18], where there is a qualitative difference in the analysis when the exponents β_1, β_2 belong to $(1, \frac{n-2}{2})$, $[\frac{n-2}{2}, n-2)$, $\{n-2\}$ or $(n-2, n)$. To keep things simple and without losing depth, we focus our discussion in this paragraph to the case $a_1, a_2 < 0$ and $\beta_1 = \beta_2 = \beta$. When $n - 2 < \beta < n$, the solution set of (1.1) is compact, and the total Leray–Schauder degree of all solutions is -1. When $\frac{n-2}{2} \leq \beta < n-2$, the solution set of (1.1) is compact, and the total Leray–Schauder degree of all solutions is 0. When $\beta = n - 2$, the solution set of (1.1) is compact provided $c \neq 1$ for certain explicit positive number c depending only on $a_1, a_2, K(0)$ and $K(\pi)$, and the total Leray–Schauder degree is -1 when $c > 1$ and 0 when $c < 1$. When $\beta < \frac{n-2}{2}$, there exist functions K for which (1.1) has a blow-up sequence of solutions with unbounded energy.

Table 1
A summary of results for $\max\{\frac{n-2k}{2}, 2\} \le \beta_1, \beta_2 < n$.

Table 1(a): $a_1, a_2 > 0$	
Thm. 1.1, 1.2	
Compactness	T
Degree	-1
Existence	T

Table 1(b): $a_1 > 0 > a_2$	
Thm. 1.1, 1.2, 1.3	
Compactness	T
Degree	0
Existence	T/F

Table 1(c): $a_1, a_2 < 0$

$\frac{1}{\beta_1} + \frac{1}{\beta_2} < \frac{2}{n-2k}$ Thm. 1.1, 1.2		$\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$ Thm. 1.1, 1.2, 1.3, 1.4		$\frac{1}{\beta_1} + \frac{1}{\beta_2} > \frac{2}{n-2k}$ Thm. 1.1, 1.3, 1.4	
Compactness	T	Compactness	*	Compactness	T
Degree	-1	Degree	$-1/0/?$	Degree	0
Existence	T	Existence	T/F	Existence	T/F

T = True. F = False.
T/F = Sometimes True and Sometimes False.
* = Sometimes True. ? = Unknown.

Our present work extends the above results to the case $k \ge 2$. When $2 \le \beta_1, \beta_2 < \frac{n-2k}{2}$ there exist functions K for which (1.1) has a blow-up sequence of solutions with unbounded energy; see Theorem 1.5. For $\max\{\frac{n-2k}{2}, 2\} \le \beta_1, \beta_2 < n$, our results are summarized in Table 1. In Table 1(a), when $a_1, a_2 > 0$, we have that the solution set is compact, the total degree for second order nonlinear elliptic operators is equal to -1 and (1.1) has a positive solution. In Table 1(b), when a_1 and a_2 are of different signs, the solution set is compact, but the total degree is equal to 0. In this case sometimes (1.1) does not have a solution and sometimes it has a solution. If K is strictly monotone, (1.1) has no solution in view of the Kazdan–Warner-type identity. We also give examples of K's for which (1.1) has positive solutions. Let us describe Table 1(c) which concerns the case $a_1, a_2 < 0$ in more details. The analysis is split according to how $\frac{1}{\beta_1} + \frac{1}{\beta_2}$ compares to $\frac{2}{n-2k}$.

- When $\frac{1}{\beta_1} + \frac{1}{\beta_2} < \frac{2}{n-2k}$, the solution set is compact, the total degree is equal to -1 and (1.1) has a positive solution.
- When $\frac{1}{\beta_1} + \frac{1}{\beta_2} > \frac{2}{n-2k}$, the solution set is compact, the total degree is zero, and the existence of positive solution to (1.1) depends on the particular K at hand: there are examples of K's which give existence as well as those which give non-existence for (1.1).
- When $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$, there exist functions K for which the solution set is compact where the total degree can be -1 or 0. Clearly, when the degree is -1, (1.1) has a positive solution. There are examples of K's for which (1.1) has no solution. It is not known if the compactness of the solution set holds for every K.

For any integer $m \geq 0$ and $0 < \alpha \leq 1$, let $C_r^m(\mathbb{S}^n)$ and $C_r^{m,\alpha}(\mathbb{S}^n)$ denote the spaces of C^m and $C^{m,\alpha}$ axisymmetric functions on $\mathbb{S}^n$, respectively.

In the statement of the next two theorems, let $C_{(1)} = C_{n,k}(\beta_1, a_1, K(0))$, $C_{(2)} = C_{n,k}(\beta_2, a_2, K(\pi))$ when $a_1, a_2 < 0$, where

$$C_{n,k}(\beta, a, s) := \frac{1}{2}\left[\frac{2\Gamma(n)s^{\frac{n-\beta}{2k}}}{|a|\beta\Gamma(\frac{n-\beta}{2})\Gamma(\frac{n+\beta}{2})}\right]^{\frac{1}{\beta}} \text{ for } \frac{n(n-2k)}{n+2k} < \beta < n, a < 0, s > 0.$$

Theorem 1.1. *Let $n \geq 5$, $2 \leq k < n/2$, $0 < \alpha \leq 1$, $K \in C_r^{2,\alpha}(\mathbb{S}^n)$ be positive and satisfy (1.2) for some $a_1, a_2 \neq 0$ and $2 \leq \beta_1, \beta_2 < n$. Assume that*

(i) *if $\beta_i < \frac{n-2k}{2}$ for some $i \in \{1, 2\}$, then $a_i > 0$, and*
(ii) *if $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$, $\frac{n(n-2k)}{n+2k} < \beta_1, \beta_2 < n$, and $a_1, a_2 < 0$, then*

$$C_{(1)}C_{(2)} \neq 1. \tag{1.3}$$

Then there exists a constant $C > 0$ such that all positive solutions of (1.1) in $C_r^2(\mathbb{S}^n)$ satisfy

$$\|\ln v\|_{C^{4,\alpha}(\mathbb{S}^n)} < C.$$

See Remark 4.2 for detailed statement on how C depends on the function K. See Subsection 4.2 for further compactness results involving a family of K's.

We make a comment on condition (1.3). In the case $k = 1$ and $\beta_1 = \beta_2 = n - 2$, a similar condition was given in [18]. The relevance of this condition in the study of compactness issues is shown more clearly when one considers a family of K's in (1.1). More precisely, for any positive $K \in C_r^{2,\alpha}(\mathbb{S}^n)$ satisfying (1.2) with $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$, $\frac{n(n-2k)}{n+2k} < \beta_1, \beta_2 < n$, $a_1, a_2 < 0$ for which $C_{(1)}C_{(2)} = 1$, there exists a sequence of positive functions $\{K_i\} \subset C_r^{2,\alpha}(\mathbb{S}^n)$ which satisfies (1.2) with $\beta_{1,i} = \beta_1$, $\beta_{2,i} = \beta_2$, $C_{(1,i)} \to C_{(1)}$, $C_{(2,i)} \to C_{(2)}$ and which converges in $C^{2,\alpha}(\mathbb{S}^n)$ to K such that there exists a blow-up sequence of positive solutions to (1.1) with K replaced by K_i. This is a consequence of the homotopy invariance property of the degree and the degree counting formula in Theorem 1.2 below. The proof is similar to that of [18, Corollary 0.24] in the case $k = 1$. Our analysis also shows that such sequence of solutions blow up at both the north and south poles; see the proof of Theorem 4.4 or Lemma 4.5.

For $k = 1$, analogous compactness results were proved by Li [17,18] and by Chen and Lin [9]. Roughly speaking, compactness of the solution set was obtained in [18, Theorem 0.19] when $n - 2 < \beta_1, \beta_2 < n$, in [18, Theorem 0.20] for $\beta_1 = \beta_2 = n - 2$, and in [9, Theorem 1.2] for $\frac{n-2}{2} < \beta_1, \beta_2 < n$ satisfying $\frac{1}{\beta_1} + \frac{1}{\beta_2} \neq \frac{2}{n-2}$. We remark that when $k = 1$ and $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2}$ the corresponding compactness result also holds but is not available in the literature except for the case $\beta_1 = \beta_2 = n - 2$ mentioned above. We will publish this result elsewhere.

As a direct application of the above compactness result and available degree computation (see [6,17,22]), we have:

Theorem 1.2. *Assume that $n, k, \alpha, K, a_1, a_2, \beta_1, \beta_2$ and C be as in Theorem 1.1. Then*

$$\deg\Big(\sigma_k(\lambda(A_{g_v})) - K, \Big\{v \in C_r^{4,\alpha}(\mathbb{S}^n) : v > 0, \lambda(A_{g_v}) \in \Gamma_k, \|\ln v\|_{C^{4,\alpha}(\mathbb{S}^n)} < C\Big\}, 0\Big)$$

$$= \begin{cases} -1 & \text{if } a_1, a_2 > 0, \\ 0 & \text{if } a_1 < 0 < a_2 \text{ or } a_2 < 0 < a_1, \\ -1 & \text{if } a_1, a_2 < 0 \text{ and } \frac{1}{\beta_1} + \frac{1}{\beta_2} < \frac{2}{n-2k}, \\ -1 & \text{if } a_1, a_2 < 0, \frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}, \text{ and } C_{(1)}C_{(2)} > 1, \\ 0 & \text{if } a_1, a_2 < 0, \frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}, \text{ and } C_{(1)}C_{(2)} < 1, \\ 0 & \text{if } a_1, a_2 < 0 \text{ and } \frac{1}{\beta_1} + \frac{1}{\beta_2} > \frac{2}{n-2k}. \end{cases}$$

Here deg *is the degree for second order nonlinear elliptic operators as defined in [16]. In particular, in the cases where the resulting degree is non-zero,* (1.1) *has at least one positive solution in $C_r^{4,\alpha}(\mathbb{S}^n)$.*

Our next two results concern the case the total degree is zero, i.e. when a_1 and a_2 are of opposite signs, or $a_1, a_2 < 0$ but $\frac{1}{\beta_1} + \frac{1}{\beta_2} > \frac{2}{n-2k}$ or $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$ and $C_{(1)}C_{(2)} < 1$. In these situations, the existence of solutions depends on the particular K at hand. Our next result shows that for any given signs of a_1 and a_2 and any given values of $\beta_1, \beta_2 \geq 2$, there exists K for which (1.1) has a solution.

Theorem 1.3. *Assume that $n \geq 5$ and $2 \leq k < n/2$. For any given signs $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and constants $\beta_1, \beta_2 \geq 2$, there exist some non-zero constants a_1, a_2 with $sign(a_i) = \varepsilon_i$ and a positive function $K \in C_r^2(\mathbb{S}^n) \cap C_{loc}^\infty(\mathbb{S}^n \setminus \{\theta = 0, \pi\})$ satisfying* (1.2) *with the above a_i's and β_i's such that* (1.1) *has at least one positive solution in $C_r^4(\mathbb{S}^n)$.*

On the other hand, as mentioned earlier, if K is monotone in one direction (which implies that a_1 and a_2 are of opposite signs), (1.1) has no solution in view of the Kazdan–Warner-type identity (see [12,26], and also Section 2). Our next result asserts that for any $2 \leq \beta_1, \beta_2 < n$ satisfying $\frac{1}{\beta_1} + \frac{1}{\beta_2} \geq \frac{2}{n-2k}$, there exists a function K with $a_1, a_2 < 0$ for which (1.1) has no solution.

Theorem 1.4. *Let $n \geq 5$ and $2 \leq k < \frac{n}{2}$. For any given $2 \leq \beta_1,\ \beta_2 < n$ with $\frac{1}{\beta_1} + \frac{1}{\beta_2} \geq \frac{2}{n-2k}$, there exists a positive function $K \in C_r^{[\beta], \beta - [\beta]}(\mathbb{S}^n) \cap C_{loc}^\infty(\mathbb{S}^n \setminus \{\theta = 0, \pi\})$, $\beta = \min\{\beta_1, \beta_2\}$ satisfying (1.2) with the above β_1, β_2 and some $a_1,\ a_2 < 0$ such that (1.1) admits no positive solution in $C^2(\mathbb{S}^n)$, with or without axisymmetry.*

When $k = 1$, a similar non-existence result of axisymmetric solutions was proved by Bianchi and Egnell [2, Theorem 0.3] under the assumption $\frac{n(n-2)}{n+2} < \beta_1,\ \beta_2 < n$ and $\frac{1}{\beta_1} + \frac{1}{\beta_2} \geq \frac{2}{n-2}$, and by Chen and Lin [9, Theorem 1.3] under the assumption $\beta_1,\ \beta_2 > 1$

and $\frac{1}{\min\{\beta_1,n\}} + \frac{1}{\min\{\beta_2,n\}} > \frac{2}{n-2}$. Under certain monotonicity of K, it was shown in Bianchi [1] that the axisymmetry of K implies that of solutions for the prescribed scalar curvature equation. These results together give the counterpart of Theorem 1.4 for $k = 1$.

Our next theorem is a non-compactness result when $2 \leq \beta_1 = \beta_2 < \frac{n-2k}{2}$.

Theorem 1.5. *Let $n \geq 5$ and $2 \leq k < \frac{n}{2}$. For any given $2 \leq \beta_1 = \beta_2 < \frac{n-2k}{2}$, there exists a positive function $K \in C^{[\beta_1],\beta_1-[\beta_1]}(\mathbb{S}^n) \cap C^\infty_{loc}(\mathbb{S}^n \setminus \{\theta = 0, \pi\})$ satisfying (1.2) with $a_1 = a_2 < 0$ and a sequence of positive solutions $\{v_i\} \subset C^2_r(\mathbb{S}^n)$ of (1.1) such that, for some constant $C > 0$ depending only on n and β,*

$$C \ln\ln \max_{\mathbb{S}^n} v_i \geq \int_{\mathbb{S}^n} v_i^{\frac{2n}{n-2}}\, dv_{\mathring{g}} \geq \frac{1}{C} \ln\ln \max_{\mathbb{S}^n} v_i \to \infty.$$

For $k = 1$, the existence of blow-up sequences of solutions was proved by Chen and Lin [8, Theorem 1.1], though without an estimate on the rate of blow-up for $\int_{\mathbb{S}^n} v_i^{\frac{2n}{n-2}}\, dv_{\mathring{g}}$ as in our result above.

An ingredient in the proof of Theorems 1.1–1.5 is a fine analysis near a blow-up point in rotational symmetry. Consider in $B_2 \subset \mathbb{R}^n$ the equation

$$\sigma_k(\lambda(A_{u_i^{\frac{4}{n-2}} g_{\text{flat}}})) = K_{Euc}, \quad \lambda(A_{u_i^{\frac{4}{n-2}} g_{\text{flat}}}) \in \Gamma_k \text{ in } B_2, \tag{1.4}$$

where $u_i(0) \to \infty$ and $K_{Euc} \in C^{2,\alpha}(B_2)$ satisfies for some $2 \leq \beta < n$ the condition

$$K_{Euc}(r) = K_{Euc}(0) + ar^\beta + R(r) \tag{1.5}$$

with $|R(r)| + r|R'(r)| = o(r^\beta)$ as $r \to 0$. We give in Theorem 3.1 a description of u_i as a 'sum of bubbles' as $i \to \infty$. To keep things simple in this introduction, let us state here a consequence of it instead of the full result.

Theorem 1.6. *Let $n \geq 5$ and $2 \leq k < \frac{n}{2}$. Suppose that $K_{Euc} \in C^{2,\alpha}(B_2)$, $0 < \alpha \leq 1$, is positive, rotationally symmetric and satisfies (1.5) for some $a \neq 0$ and $\beta \geq 2$. Suppose that $u_i \in C^2(B_2)$ are positive, rotationally symmetric and satisfy (1.4) and that $u_i(0) \to \infty$. Then:*

(i) *When $\frac{n-2k}{2} \leq \beta < n$, the integral $\int_{B_1} u_i^{\frac{2n}{n-2}}\, dx$ is bounded as $i \to \infty$.*
(ii) *When $2 \leq \beta < \frac{n-2k}{2}$,*

$$\lim_{i\to\infty} \frac{1}{\ln\ln u_i(0)} \int_{B_1} u_i^{\frac{2n}{n-2}}\, dx = \frac{C(n,k)}{|\ln(1 - \frac{2\beta}{n-2k})|} K_{Euc}(0)^{-\frac{n}{2k}}$$

for some constant $C(n,k) > 0$ depending only on n and k.

We note that, when $\frac{n-2k}{2} < \beta < n$, the sequence $\{u_i\}$ contains exactly one bubble, i.e. $\int_{B_1} u_i^{\frac{2n}{n-2}}\,dx$ converges to $C(n,k)K_{Euc}(0)^{-\frac{n}{2k}}$ for some positive constant $C(n,k)$ depending only on n and k. When $\beta = \frac{n-2k}{2}$, we know that $\{u_i\}$ contains at least one bubble. (See Theorem 3.1.) It is interesting to understand whether $\{u_i\}$ can contain two or more bubbles.

When $k = 1$, statement (i) in Theorem 1.6 was proved by Li [17] for $\beta \geq n-2$ and by Chen and Lin [7] for $\frac{n-2}{2} \leq \beta < n-2$.

The rest of the paper is structured as follows. In Section 2 we derive some useful integral identities for the σ_k-Nirenberg problem in axisymmetry. These integral identities contain the well-known Pohozaev identity as well as some other identities which we refer to as mass-type identities (see subsection 2.2). In Section 3, we give a fine analysis of near a blow-up point for the σ_k-Yamabe problem on Euclidean balls. In Sections 4–8, we use the local analysis above to prove Theorems 1.1–1.5. We include also an appendix where certain integrals used in the body of the paper are computed in terms of the gamma function.

2. Preliminaries

In this section we give some equivalent forms of (1.1) for positive $v \in C^2_r(\mathbb{S}^n)$ and derive some useful integral identities, among which is the Pohozaev identity.

We let $r = \cot\frac{\theta}{2}$, $t = \ln\cot\frac{\theta}{2}$ and express g_v as a metric conformal to the Euclidean metric or the round cylinder metric:

$$g_v = v(\theta)^{\frac{4}{n-2}}(d\theta^2 + \sin^2\theta\,\mathring{g}_{\mathbb{S}^{n-1}}) = u(r)^{\frac{4}{n-2}}(dr^2 + r^2\mathring{g}_{\mathbb{S}^{n-1}}) = e^{-2\xi(t)}(dt^2 + \mathring{g}_{\mathbb{S}^{n-1}}).$$

Define $K_{Euc}(r) := K(\theta) =: K_{cyl}(t)$.

We will use a prime and a dot to mean differentiation with respect to r and t respectively.

One can explicitly express u in terms of v as follows. Let $\Phi : \mathbb{R}^n \to \mathbb{S}^n$ be the inverse of the stereographic projection:

$$x^i = \frac{2y^i}{1+|y|^2} \text{ for } i = 1,\ldots,n, \text{ and } x^{n+1} = \frac{|y|^2-1}{|y|^2+1}.$$

Then

$$u(y) = \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}} v(x), \qquad K_{Euc} = K\circ\Phi, \tag{2.1}$$

and (1.1) is equivalent to

$$\sigma_k(\lambda(A^u)) = K_{Euc}, \quad \lambda(A^u) \in \Gamma_k \text{ in } \mathbb{R}^n, \tag{2.2}$$

where A^u is the matrix

$$A^u = -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}du\otimes du - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|du|^2\, I.$$

Likewise, with $r = |y|$ and $t = \ln|y|$, we have

$$\xi = -\frac{2}{n-2}\ln u - \ln r, \qquad K_{cyl}(t) = K\circ\Phi(e^t, 0, \ldots, 0), \tag{2.3}$$

and (1.1) gives

$$F_k[\xi] = K_{cyl} \text{ and } |\dot\xi| < 1 \text{ in } (-\infty,\infty), \tag{2.4}$$

where

$$F_k[\xi] := \frac{1}{2^{k-1}}\binom{n-1}{k-1}e^{2k\xi}(1-\dot\xi^2)^{k-1}\Big(\ddot\xi + \frac{n-2k}{2k}(1-\dot\xi^2)\Big). \tag{2.5}$$

The condition (1.2) is equivalent to the condition that

$$\begin{aligned} K_{cyl} &= K_{cyl}(\infty) + 2^{\beta_1}a_1 e^{-\beta_1 t} + o(e^{-\beta_1 t}) \text{ as } t\to\infty,\\ K_{cyl} &= K_{cyl}(-\infty) + 2^{\beta_2}a_2 e^{\beta_2 t} + o(e^{\beta_2 t}) \text{ as } t\to-\infty, \end{aligned}$$

with the error terms being controlled up to and including first order derivatives.

We note a simple property of the equation (2.4) which we will make use of later on: There exists a constant $\bar x$ depending only on n, k and an upper bound for K_{cyl} such that, for ξ satisfying (2.4),

$$\text{if } \xi(t) \geq \bar x \text{ and } \dot\xi(t) = 0, \text{ then } \ddot\xi(t) < 0. \tag{2.6}$$

2.1. Pohozaev-type identities

For $\xi \in C^2(\mathbb{R})$, following [25], define

$$\bar H(\xi,\dot\xi) := \frac{1}{2^k}\binom{n}{k}e^{(2k-n)\xi}(1-\dot\xi^2)^k - e^{-n\xi}.$$

Then $\bar H$ has the property that, for $-\infty < t_1 \leq t_2 < \infty$,

$$\bar H(\xi(t_2),\dot\xi(t_2)) - \bar H(\xi(t_1),\dot\xi(t_1)) = -n\int_{t_1}^{t_2}(F_k[\xi]-1)e^{-n\xi}\dot\xi\,dt. \tag{2.7}$$

We will also consider the quantity

$$H(t,\xi,\dot\xi) := \frac{1}{2^k}\binom{n}{k} e^{(2k-n)\xi}(1-\dot\xi^2)^k - K_{cyl}(t)\, e^{-n\xi}. \tag{2.8}$$

As a consequence of (2.7), we have, for $-\infty < t_1 \le t_2 < \infty$,

$$H(t,\xi(t),\dot\xi(t))\Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \Big[-n(F_k[\xi] - K_{cyl})e^{-n\xi}\dot\xi - \dot K_{cyl}\, e^{-n\xi}\Big]\, dt. \tag{2.9}$$

If ξ satisfies (2.4) and $\xi(t) - |t|$ is bounded in $(-\infty,\infty)$ (e.g. if ξ is related to a solution to (1.1) via (2.1) and (2.3)), we have $H(t,\xi,\dot\xi) \to 0$ as $t \to \pm\infty$ and (2.9) gives

$$H(t,\xi,\dot\xi) = -\int_{-\infty}^{t} \dot K_{cyl}(\tau)\, e^{-n\xi(\tau)}\, d\tau = \int_{t}^{\infty} \dot K_{cyl}(\tau)\, e^{-n\xi(\tau)}\, d\tau. \tag{2.10}$$

Equivalently, if we let u be related to ξ via (2.3) and define

$$H_{Euc}(r,u,u') = \frac{(-1)^k 2^k}{(n-2)^{2k}}\binom{n}{k} r^{n-2k} u^{\frac{2(n-2k)}{n-2}} \Big[\frac{ru'}{u}\Big(\frac{ru'}{u} + n - 2\Big)\Big]^k - K_{Euc}(r) r^n u^{\frac{2n}{n-2}},$$

then

$$H_{Euc}(r,u,u') = -\int_0^r K'_{Euc}(s)\, u(s)^{\frac{2n}{n-2}}\, s^n\, ds = \int_r^\infty K'_{Euc}(s)\, u(s)^{\frac{2n}{n-2}}\, s^n\, ds. \tag{2.11}$$

The identities (2.7), (2.9), (2.10) and (2.11) are known as Pohozaev identities for the σ_k-Yamabe equation. See [3,24] for the case $k = 1$, [12,26] for $k \ge 2$.

2.2. Mass-type identities

More generally, one is interested in finding smooth functions $B, P : \mathbb{R}\times\mathbb{R}\times(-1,1) \to \mathbb{R}$ such that

$$B(t,\xi,\dot\xi)\Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} F_k[\xi] P(\tau,\xi,\dot\xi)\, d\tau \text{ for all } -\infty < t_1 \le t_2 < \infty, \tag{2.12}$$

i.e.

$$\frac{d}{dt} B(t,\xi,\dot\xi) = F_k[\xi] P(t,\xi,\dot\xi).$$

Let $A(t,x,y)$ be such that $\partial_y A(t,x,y) = e^{2kx}(1-y^2)^{k-1}P(t,x,y)$ where x and y are dummy variables standing for ξ and $\dot{\xi}$. We compute, using (2.5) and then integrating by parts,

$$\int_{t_1}^{t_2} F_k[\xi]P(\tau,\xi,\dot{\xi})\,d\tau$$

$$= \frac{1}{2^{k-1}}\binom{n-1}{k-1}\int_{t_1}^{t_2}\Big[\partial_y A(\tau,\xi,\dot{\xi})\ddot{\xi} + \frac{n-2k}{2k}\partial_y A(\tau,\xi,\dot{\xi})(1-\dot{\xi}^2)\Big]\,d\tau$$

$$= \frac{1}{2^{k-1}}\binom{n-1}{k-1}A(t,\xi,\dot{\xi})\Big|_{t=t_1}^{t=t_2}$$

$$+ \frac{1}{2^{k-1}}\binom{n-1}{k-1}\int_{t_1}^{t_2}\Big[-\partial_t A(\tau,\xi,\dot{\xi}) - \partial_x A(\tau,\xi,\dot{\xi})\dot{\xi} + \frac{n-2k}{2k}\partial_y A(\tau,\xi,\dot{\xi})(1-\dot{\xi}^2)\Big]\,d\tau.$$

Hence, to obtain (2.12), we impose that A satisfies the first order PDE

$$-\partial_t A(t,x,y) - y\partial_x A(t,x,y) + \frac{n-2k}{2k}(1-y^2)\partial_y A(t,x,y) = 0 \text{ in } \{|y|<1\}. \tag{2.13}$$

The projected characteristic curves of (2.13) are given by $\{t + \frac{k}{n-2k}\ln\frac{1+y}{1-y} = \text{const}, x - \frac{k}{n-2k}\ln(1-y^2) = \text{const}\}$. The general solution to (2.13) thus takes the form

$$A(t,x,y) = G\Big(t + \frac{k}{n-2k}\ln\frac{1+y}{1-y}, x - \frac{k}{n-2k}\ln(1-y^2)\Big)$$

for an arbitrary smooth function $G:\mathbb{R}^2\to\mathbb{R}$. Putting things together we have

$$\frac{n-2k}{2^k n}\binom{n}{k}G\Big(t + \frac{k}{n-2k}\ln\frac{1+\dot{\xi}}{1-\dot{\xi}}, \xi - \frac{k}{n-2k}\ln(1-\dot{\xi}^2)\Big)\Big|_{t=t_1}^{t=t_2}$$

$$= \int_{t_1}^{t_2}\frac{F_k[\xi]}{e^{2k\xi}(1-\dot{\xi}^2)^k}\Big\{\partial_t G\Big(\tau + \frac{k}{n-2k}\ln\frac{1+\dot{\xi}}{1-\dot{\xi}}, \xi - \frac{k}{n-2k}\ln(1-\dot{\xi}^2)\Big) +$$

$$+ \dot{\xi}\partial_x G\Big(\tau + \frac{k}{n-2k}\ln\frac{1+\dot{\xi}}{1-\dot{\xi}}, \xi - \frac{k}{n-2k}\ln(1-\dot{\xi}^2)\Big)\Big\}\,d\tau. \tag{2.14}$$

We have therefore proved that, for any smooth function $G:\mathbb{R}^2\to\mathbb{R}$, the following B and P satisfy (2.12):

$$B(t,x,y) = \frac{n-2k}{2^k n}\binom{n}{k}G\Big(t + \frac{k}{n-2k}\ln\frac{1+y}{1-y}, \xi - \frac{k}{n-2k}\ln(1-y^2)\Big),$$

$$P(t,x,y) = \frac{1}{e^{2kx}(1-y^2)^k}\Big\{\partial_t G\Big(t + \frac{k}{n-2k}\ln\frac{1+y}{1-y}, x - \frac{k}{n-2k}\ln(1-y^2)\Big) + \\ + \dot\xi \partial_x G\Big(t + \frac{k}{n-2k}\ln\frac{1+y}{1-y}, x - \frac{k}{n-2k}\ln(1-y^2)\Big)\Big\}.$$

Example 2.1. It is readily seen that the choice $G(t,x) = \frac{n}{n-2k}e^{-(n-2k)x}$ in (2.14) implies the Pohozaev identities (2.7) and (2.9).

Example 2.2. We will also use in the proof of Theorem 1.1 the choice $G(t,x) = \frac{2}{n-2k}e^{\frac{n-2k}{2}(t-x)}$. This gives the quantity

$$m(t,\xi,\dot\xi) := \frac{1}{2^{k-1}n}\binom{n}{k}(1+\dot\xi(t))^k e^{\frac{n-2k}{2}(-\xi(t)+t)}$$

and the identity, for $-\infty < t_1 \le t_2 < \infty$,

$$m(t,\xi,\dot\xi)\Big|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} F_k[\xi](1-\dot\xi)^{-(k-1)}e^{-\frac{n+2k}{2}\xi}e^{\frac{n-2k}{2}\tau}\,d\tau. \tag{2.15}$$

If ξ satisfies (2.4) and $\xi(t)+|t|$ is bounded as $t \to -\infty$, we have $m(t,\xi,\dot\xi) \to 0$ as $t \to -\infty$ and (2.15) implies

$$m(t,\xi,\dot\xi) = \int_{-\infty}^{t} F_k[\xi](1-\dot\xi)^{-(k-1)}e^{-\frac{n+2k}{2}\xi}e^{\frac{n-2k}{2}\tau}\,d\tau. \tag{2.16}$$

We will refer to identities (2.15) and (2.16) as mass-type identities.

Example 2.3. For further reference, we also note that the separable ansatz $P(t,x,y) = P_1(t)P_2(x)P_3(y)$ leads to the choice $G(t,x) = \frac{2}{n-2k}e^{(n-2k)(bx+ct)}$. This gives the quantity

$$m_{b,c}(t,\xi,\dot\xi) := \frac{1}{2^{k-1}n}\binom{n}{k}(1-\dot\xi)^{-k(b+c)}(1+\dot\xi)^{-k(b-c)}e^{(n-2k)(b\xi+ct)}$$

and the identity, for $-\infty < t_1 \le t_2 < \infty$,

$$m_{b,c}(t,\xi,\dot\xi)\Big|_{t=t_1}^{t=t_2} = 2\int_{t_1}^{t_2} F_k[\xi](1-\dot\xi)^{-k(b+c+1)}(1+\dot\xi)^{-k(b-c+1)}(b\dot\xi+c)\times \\ \times e^{((n-2k)b-2k)\xi+(n-2k)c\tau}\,d\tau.$$

It is readily seen that taking $b=-1$ and $c=0$ gives the Pohozaev identities, and taking $b=-1$ and $c=1$ gives the mass-type identities.

3. Local blow-up analysis

Consider in $B_2 \subset \mathbb{R}^n$ the equation (1.4), i.e.

$$\sigma_k(\lambda(A^u)) = K_{Euc}, \quad \lambda(A^u) \in \Gamma_k \text{ in } B_2$$

where $K_{Euc} \in C^{2,\alpha}(B_2)$ satisfies (1.5) for some $2 \le \beta < n$. In this section, we study the behavior of a sequence of positive rotationally symmetric solutions $\{u_i\}$ of (1.4) with $u_i(0) \to \infty$.

As in the previous section, we work with cylindrical coordinates. Let $t = \ln r$, $\xi(t) = -\frac{2}{n-2} \ln u(r) - \ln r$, and $K_{cyl}(t) = K_{Euc}(r)$. Then $\xi(t) + t$ is bounded as $t \to -\infty$,

$$F_k[\xi] = K_{cyl} \text{ and } |\dot{\xi}| < 1 \text{ in } (-\infty, \ln 2),$$

and

$$K_{cyl} = K_{cyl}(-\infty) + 2^\beta a e^{\beta t} + o(e^{\beta t}) \text{ as } t \to -\infty, \tag{3.1}$$

with the error term being controlled up to and including first order derivatives.

Throughout the section, let

$$\Xi(t) := -\ln \frac{e^t}{1 + e^{2t}} - \ln \Big(2^{\frac{1}{2}} \binom{n}{k}^{\frac{1}{2k}} \Big).$$

Note that solutions to $F_k[E] = 1$ and $|\dot{E}| < 1$ in $(-\infty, \infty)$ satisfying $H(t, E, \dot{E}) \equiv 0$ are given by

$$E(t) = \Xi(t + \ln \lambda) = -\ln \frac{\lambda e^t}{1 + \lambda^2 e^{2t}} - \ln \Big(2^{\frac{1}{2}} \binom{n}{k}^{\frac{1}{2k}} \Big) \text{ for some } \lambda > 0.$$

Theorem 3.1. *Let $n \ge 5$ and $2 \le k < n/2$. Suppose that $K_{Euc} \in C^{2,\alpha}(B_2)$, $0 < \alpha \le 1$, is positive, rotationally symmetric and satisfies (1.5) for some $a \ne 0$ and $\beta \ge 2$. Suppose that $u_i \in C^2(B_2)$ are positive, rotationally symmetric and satisfy (1.4) and that $u_i(0) \to \infty$. Let $t = \ln r$, $\xi_i(t) = -\frac{2}{n-2} \ln u_i(r) - \ln r$ and $\lambda_i = 2^{-\frac{1}{2}} \binom{n}{k}^{-\frac{1}{2k}} K_{Euc}(0)^{\frac{1}{2k}} u_i(0)^{\frac{2}{n-2}}$.*

(a) *One has for some C depending only on n and K_{Euc} that*

$$\xi_i \ge -C \text{ and } |\dot{\xi}_i| + |\ddot{\xi}_i| \le C \text{ in } (-\infty, \ln \frac{3}{2}). \tag{3.2}$$

Furthermore, for every $\varepsilon_i \to 0^+$, $R_i \to \infty$, after passing to a subsequence, one has that $\frac{R_i}{\lambda_i} \to 0$ and, $0 \le \ell \le 2$,

$$\Big| \frac{d^\ell}{dt^\ell} \Big[\xi_i(t) - \Xi(t + \ln \lambda_i) - \frac{1}{2k} \ln K_{Euc}(0) \Big] \Big| \le \varepsilon_i \lambda_i^\ell e^{\ell t} \text{ in } (-\infty, \ln \frac{R_i}{\lambda_i}). \tag{3.3}$$

In particular, $\xi_i(\ln \frac{R_i}{\lambda_i}) = \ln R_i + O(1) \to \infty$ *and there exists* $t_{1,i} = -\ln \lambda_i + o(1)$ *such that* $\dot{\xi}_i < 0$ *in* $(-\infty, t_{1,i})$ *and* $\dot{\xi}_i > 0$ *in* $(t_{1,i}, \ln \frac{R_i}{\lambda_i})$.

(b) *Let*

$$t_{2,i} = \sup \Big\{ t \in [t_{1,i}, 0] : \dot{\xi}_i > 0 \text{ in } (t_{1,i}, t) \Big\}.$$

Then, for large i,

$$t_{2,i} = -\max \Big\{ 1 - \frac{\beta}{n-2k}, 0 \Big\} \ln \lambda_i + O(1) > t_{1,i} \tag{3.4}$$

$\dot{\xi}_i > 0$ *in* $(t_{1,i}, t_{2,i})$ *and*

$$\Big| \xi_i(t) - \Xi(t + \ln \lambda_i) \Big| \leq O(1) \text{ in } (-\infty, t_{2,i}). \tag{3.5}$$

Furthermore, if $\beta \leq n - 2k$, *then* $a < 0$.

(c) *Suppose* $2 \leq \beta < n - 2k$. *Then, for large* i, $t_{2,i} < 0$, $\dot{\xi}_i(t_{2,i}) = 0$, *and* $\ddot{\xi}_i(t_{2,i}) < 0$. *Let*

$$t_{3,i} = \sup \Big\{ t \in [t_{2,i}, 0] : \dot{\xi}_i < 0 \text{ in } (t_{2,i}, t) \Big\}.$$

Then

$$t_{3,i} = -\max \Big\{ 1 - \frac{2\beta}{n-2k}, 0 \Big\} \ln \lambda_i + O(1) > t_{2,i} \tag{3.6}$$

$\dot{\xi}_i < 0$ *in* $(t_{2,i}, t_{3,i})$, *and*

$$\Big| \xi_i(t) - \Xi\Big(t + (1 - \frac{2\beta}{n-2k}) \ln \lambda_i\Big) \Big| \leq O(1) \text{ in } (t_{2,i}, t_{3,i}). \tag{3.7}$$

(d) *If* $2 \leq \beta < \frac{n-2k}{2}$, *then, for large* i, *there exist* $N_i = \left\lfloor \frac{\ln \ln \lambda_i + O(1)}{|\ln(1 - \frac{2\beta}{n-2k})|} \right\rfloor \geq 2$ *and* $2N_i$ *critical points of* ξ_i,

$$t_{1,i} < t_{2,i} < t_{3,i} < t_{4,i} < \ldots < t_{2N_i,i} = O(1)$$

with

$$\begin{aligned} t_{2\ell,i} &= -(1 - \frac{\beta}{n-2k})(1 - \frac{2\beta}{n-2k})^{\ell-1} \ln \lambda_i + O(1), \\ t_{2\ell+1,i} &= -(1 - \frac{2\beta}{n-2k})^{\ell} \ln \lambda_i + O(1), \end{aligned}$$

such that $\dot{\xi}_i < 0$ *in* $(t_{2\ell,i}, t_{2\ell+1,i})$, $\dot{\xi}_i > 0$ *in* $(t_{2\ell+1,i}, t_{2\ell+2,i})$ *and*

$$\Big| \xi_i(t) - \Xi(t - t_{2\ell+1,i}) \Big| \leq O(1) \text{ in } (t_{2\ell,i}, t_{2\ell+2,i}) \text{ for } 1 \leq \ell \leq N_i - 1.$$

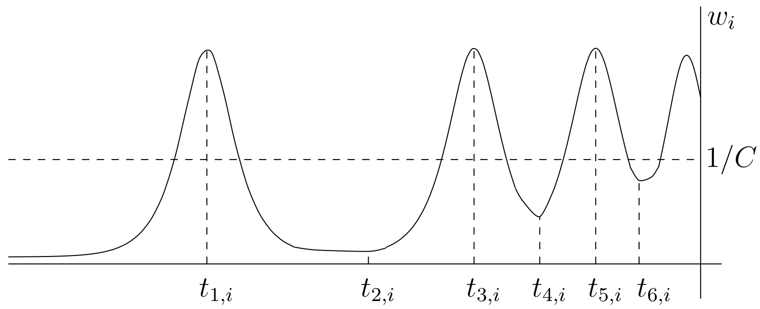

Fig. 1. A profile of $w_i = r^{\frac{n-2}{2}} u_i$ vs. $t = \ln r$ with $N_i = 3$ when $\beta < \frac{n-2k}{2}$. The gap between the peaks decreases exponentially to $O(1)$ in N_i steps. The pieces above the line $w = 1/C$ are close to the standard bubbles.

Furthermore, for every $\varepsilon > 0$, there exists $R_\varepsilon > \frac{1}{\varepsilon}$ independent of i, such that, for any ℓ satisfying $|t_{2\ell+1,i}| \geq R_\varepsilon$, we have

$$\|\xi_i(t) - \Xi(t - t_{2\ell+1,i}) - \frac{1}{2k} \ln K_{Euc}(0)\|_{C^2[t_{2\ell+1,i}-1/\varepsilon, t_{2\ell+1,i}+1/\varepsilon]} \leq \varepsilon.$$

Here $|O(1)| \leq C$, independent of i and ℓ and ε, and $o(1)$ denotes a term which goes to 0 as $i \to \infty$.

A schematic sketch for the conclusion in (d) is given in Fig. 1.

Corollary 3.2. *Under the hypotheses of Theorem 3.1, when $\beta < \frac{n-2k}{2}$ we have*

$$\frac{1}{\ln \ln \lambda_i} \int\limits_{B_1} u_i^{\frac{2n}{n-2}}\, dx \to \frac{C(n,k)}{|\ln(1 - \frac{2\beta}{n-2k})|} K_{Euc}(0)^{-\frac{n}{2k}} \text{ as } i \to \infty,$$

where $C(n,k) > 0$ depends only on n and k. Furthermore, along a subsequence, ξ_i converges in $C^2_{loc}(-\infty, \ln 2)$ to some $\xi_\infty \in C^2(-\infty, \ln 2)$ satisfying $F_k[\xi_\infty] = K_{cyl}$ and $|\dot{\xi}_\infty| < 1$ in $(-\infty, \ln 2)$ and there exist critical points of ξ_∞,

$$0 > t_{0,\infty} > t_{1,\infty} > \ldots \to -\infty$$

with

$$\Big| t_{2j,\infty} - (1 - \frac{2\beta}{n-2k})^{-j} t_{0,\infty} \Big| \leq C,$$
$$\Big| t_{2j+1,\infty} - (1 - \frac{\beta}{n-2k})^{-1} (1 - \frac{2\beta}{n-2k})^{-j} t_{0,\infty} \Big| \leq C,$$

such that $\dot{\xi}_\infty \leq 0$ in $(t_{2j+2,\infty}, t_{2j,\infty})$, $\dot{\xi}_\infty \geq 0$ in $(t_{2j+1,\infty}, t_{2j,\infty})$ and

$$\Big| \xi_\infty(t) - \Xi(t - t_{2j+1,\infty}) \Big| \leq C \text{ in } (t_{2j+2,\infty}, t_{2j,\infty}) \text{ for } j \geq 0$$

for some constant $C > 0$. Finally, for every $\varepsilon > 0$, there exists $R_\varepsilon > \frac{1}{\varepsilon}$, such that, for any j satisfying $|t_{2j+1,\infty}| \geq R_\varepsilon$, we have

$$\|\xi_\infty(t) - \Xi(t - t_{2j+1,\infty}) - \frac{1}{2k} \ln K_{Euc}(0)\|_{C^2[t_{2j+1,\infty}-1/\varepsilon, t_{2j+1,\infty}+1/\varepsilon]} \leq \varepsilon.$$

3.1. An oscillation estimate

We will make use of the following oscillation estimate for sub-solutions to the σ_k-equation.

Lemma 3.3. *Assume that $n \geq 5$ and $2 \leq k < n/2$. There exist large constants $\xi_0 > 0$ and $C_0 > 0$ depending only on n such that if ξ is C^2 and monotone in some interval $[t_1, t_2] \subset (0, \infty)$ and satisfies*

$$0 \leq F_k[\xi] \leq 1, |\dot{\xi}| \leq 1, \text{ and } \xi \geq \xi_0 \text{ in } [t_1, t_2],$$

then

$$t_2 - t_1 \geq |\xi(t_2) - \xi(t_1)| \geq t_2 - t_1 - C_0. \tag{3.8}$$

For future reference, we state also here an equivalent version in Euclidean setting.

Lemma 3.3'. *Assume that $n \geq 5$ and $2 \leq k < n/2$. There exist a small constant $\varepsilon_0 > 0$ and a large constant C depending only on n such that if $u \in C^2(\bar{B}_{r_2} \setminus B_{r_1})$ is positive, rotationally symmetric,*

$$\sigma_k(\lambda(A^u)) \leq 1, \quad \lambda(A^u) \in \Gamma_k \text{ in } B_{r_2} \setminus \bar{B}_{r_1},$$

$r^{\frac{n-2}{2}} u(r)$ is non-increasing (or non-decreasing, resp.), and $r_1^{\frac{n-2}{2}} u(r_1) \leq \varepsilon_0$, then

$$1 \leq \frac{r_2^{n-2} u(r_2)}{r_1^{n-2} u(r_1)} \leq C \qquad \Big(\text{or } \frac{1}{C} \leq \frac{u(r_2)}{u(r_1)} \leq 1, \text{ resp.}\Big).$$

Proof. By considering $\tilde{\xi}(t) = \xi(-t)$ instead of ξ if necessary, it suffices to consider the case ξ is non-decreasing.

The first inequality in (3.8) holds due to the fact that $\dot{\xi} \leq 1$, so we only need to prove the second inequality.

By (2.6) and the fact that $F_k[\xi] \leq 1$, there exists a constant $\bar{x}$ depending only on n and k such that, whenever $\xi(t) > \bar{x}$ and $\dot{\xi}(t) = 0$, it holds that $\ddot{\xi}(t) < 0$. Without loss of generality, we assume that $\xi > \bar{x}$ in $[t_1, t_2]$. Since $\dot{\xi} \geq 0$, this implies that

$$\dot{\xi}(t) > 0 \text{ and } \xi(t) < \xi(t_2) \text{ for } t \in [t_1, t_2). \tag{3.9}$$

For $x, y \in \mathbb{R}$, let

$$\bar{H}(x,y) = c\,e^{(2k-n)x}(1-y^2)^k - e^{-nx} \text{ where } c = \frac{1}{2^k}\binom{n}{k}.$$

By (2.7) and the fact that $F_k[\xi] \le 1$ and $\dot{\xi} \ge 0$ in $[t_1, t_2]$, we have

$$\frac{d}{dt}\bar{H}(\xi,\dot{\xi}) = -ne^{-n\xi}(F_k[\xi]-1)\dot{\xi} \ge 0 \text{ in } [t_1,t_2].$$

Therefore

$$\bar{H}(\xi(t),\dot{\xi}(t)) \le \bar{H}(\xi(t_2),\dot{\xi}(t_2)) \le \bar{H}(\xi(t_2),0) \text{ for } t \in [t_1,t_2]. \tag{3.10}$$

By the explicit expression of $\bar{H}$, by increasing $\bar{x}$ if necessary, we may assume that $\bar{H}(\cdot, 0)$ is decreasing and positive in $(\bar{x}, \infty)$. For $\bar{x} \le x \le a$, define

$$g_a(x) = 1 - c^{-\frac{1}{k}} e^{\frac{n-2k}{k}x}(\bar{H}(a,0) + e^{-nx})^{\frac{1}{k}}.$$

Then $g_a(a) = 0$, and by the monotonicity of $\bar{H}(x,0)$,

$$g_a(x) > 1 - c^{-\frac{1}{k}} e^{\frac{n-2k}{k}x}(\bar{H}(x,0) + e^{-nx})^{\frac{1}{k}} = 0 \text{ for } x \in [\bar{x}, a).$$

Using the explicit expression of $\bar{H}$ and the fact that $0 \le \dot{\xi} \le 1$, we can rewrite (3.10) as

$$\dot{\xi} \ge \sqrt{g_{\xi(t_2)}(\xi)} \text{ in } [t_1,t_2] \text{ provided } \xi(t_1) > \bar{x}. \tag{3.11}$$

Claim. *There exist constants $C > 0$ and $\xi_0 > \bar{x}$ depending only on n and k such that*

$$\int_x^a \frac{d\mu}{\sqrt{g_a(\mu)}} \le a - x + C \textit{ for } \xi_0 \le x \le a. \tag{3.12}$$

Clearly (3.9), (3.11) and (3.12) imply, for $\xi \ge \xi_0$ in $[t_1, t_2]$, that

$$t_2 - t_1 \le \int_{\xi(t_1)}^{\xi(t_2)} \frac{d\mu}{\sqrt{g_{\xi(t_2)}(\mu)}} \le \xi(t_2) - \xi(t_1) + C,$$

which gives the right half of (3.8).

Using the inequality $(1+z)^{\frac{1}{k}} \le 1 + z^{\frac{1}{k}}$ for $z \ge 0$, we have for all $\bar{x} \le x \le a$ that

$$1 - g_a(x) = e^{\frac{n-2k}{k}(x-a)}\Big(1 - c^{-1}e^{-2ka} + c^{-1}e^{-nx+(n-2k)a}\Big)^{\frac{1}{k}}$$

$$
\begin{aligned}
&\le e^{\frac{n-2k}{k}(x-a)}\Big(1 + c^{-1}e^{-nx+(n-2k)a}\Big)^{\frac{1}{k}} \le e^{\frac{n-2k}{k}(x-a)}\Big(1 + c^{-\frac{1}{k}}e^{\frac{-nx+(n-2k)a}{k}}\Big)\\
&= e^{\frac{n-2k}{k}(x-a)} + c^{-\frac{1}{k}}e^{-2x}.
\end{aligned}
$$

In particular, we can choose $\xi_0 = \xi_0(n,k)$ such that, for $\xi_0 \le x \le a-1$,

$$
1 - g_a(x) \le e^{\frac{n-2k}{k}(x-a)} + c^{-\frac{1}{k}}e^{-2x} \le e^{-\frac{n-2k}{k}} + c^{-\frac{1}{k}}e^{-2\xi_0} < 1.
$$

This implies that there exists a constant $C = C(n,k,\xi_0)$ such that

$$
\frac{1}{\sqrt{g_a(x)}} = \frac{1}{\sqrt{1-(1-g_a(x))}} \le 1 + C(e^{\frac{n-2k}{k}(x-a)} + e^{-2x}) \text{ for } \xi_0 \le x \le a-1.
$$

On the other hand, by enlarging ξ_0 and C if necessary, we have for $a-1 \le x \le a$ that

$$
\begin{aligned}
g_a(x) &= 1 - e^{\frac{n-2k}{k}(x-a)}\Big(1 + c^{-1}e^{-2ka}(e^{n(a-x)} - 1)\Big)^{\frac{1}{k}}\\
&\ge 1 - e^{\frac{n-2k}{k}(x-a)}\Big(1 + c^{-1}e^{-2k\xi_0}(e^{n(a-x)} - 1)\Big)^{\frac{1}{k}} \ge \frac{1}{C}(a-x).
\end{aligned}
$$

Combining the above estimates, we have for $\xi_0 \le x \le a$ that

$$
\begin{aligned}
\int_x^a \frac{d\mu}{\sqrt{g_a(\mu)}} &\le \int_x^{a-1}\Big[1 + C(e^{\frac{n-2k}{k}(\mu-a)} + e^{-2\mu})\Big]d\mu + \int_{a-1}^a \frac{C\,d\mu}{\sqrt{a-\mu}}\\
&\le a - x + C(n,k),
\end{aligned}
$$

which gives the claim, and hence completes the proof. $\square$

3.2. Proof of Statement (a) of Theorem 3.1

By first and second derivative estimates for the σ_k-Yamabe equation (see e.g. [11, Theorem 1.1], [20, Theorem 1.10]), to prove (3.2), we only need to show

$$
u_i(r)r^{\frac{n-2}{2}} \le C, \tag{3.13}
$$

where here and below C denotes a constant depending only on n,k and K_{Euc}.

The proof of (3.13) is a standard argument using the Liouville-type theorem [15, Theorem 1.3] and the symmetry of u_i.

Suppose by contradiction that (3.13) does not hold. Then, we can find $y_i \in B_{3/2}$ such that $|y_i|^{\frac{n-2}{2}}u_i(y_i) \to \infty$.

Let $r_i = |y_i|/2$, $\bar{y}_i \in \bar{B}_{r_i}(y_i)$ be a point where $(r_i - |y-y_i|)^{\frac{n-2}{2}}u_i(y)$ attains its maximum in $\bar{B}_{r_i}(y_i)$, and $s_i = (r_i - |\bar{y}_i - y_i|)/2 \in (0, r_i/2]$. It is clear that

$$s_i^{\frac{n-2}{2}} u_i(\bar{y}_i) \geq 2^{-\frac{n-2}{2}} r_i^{\frac{n-2}{2}} u_i(y_i) \to \infty \text{ and } \max_{\bar{B}_{s_i}(\bar{y}_i)} u_i \leq 2^{\frac{n-2}{2}} u_i(\bar{y}_i).$$

Let

$$\hat{u}_i(z) = \frac{1}{u_i(\bar{y}_i)} u_i\Big(\bar{y}_i + u_i(\bar{y}_i)^{-\frac{2}{n-2}} z\Big) \text{ for } |z| \leq s_i\, u_i(\bar{y}_i)^{\frac{2}{n-2}}.$$

Then $\hat{u}_i$ satisfies

$$\sigma_k(\lambda(A^{\hat{u}_i}(z))) = K_{Euc}(\bar{y}_i + u_i(\bar{y}_i)^{-\frac{2}{n-2}} z), \quad \lambda(A^{\hat{u}_i}) \in \Gamma_k \text{ in } \big\{|z| \leq s_i\, u_i(\bar{y}_i)^{\frac{2}{n-2}}\big\}.$$

By first and second derivative estimates for the σ_k-Yamabe equation and the Liouville-type theorem [15, Theorem 1.3], we thus have, after passing to a subsequence, that $\hat{u}_i$ converges in $C^2_{loc}(\mathbb{R}^n)$ to a limit $\hat{u}_*$ of the form

$$\hat{u}_*(z) = b_*(a_* + |z - z_*|^2)^{-\frac{n-2}{2}}$$

for some positive constants a_*, b_* and some $z_* \in \mathbb{R}^n$.

On the other hand, the rotational symmetry of u_i implies that, for every ball $B_r(y) \subset B_{s_i}(\bar{y}_i)$, the level set $\{u_i = u_i(y)\}$ intersects $\partial B_r(y)$ non-trivially. Applying this to balls centered at $\bar{y}_i + u_i(\bar{y}_i)^{-\frac{2}{n-2}} z_*$ and sending $i \to \infty$, we obtain that the level set $\{\hat{u}_* = \hat{u}_*(z_*)\}$ intersects every spheres centered at z_*. This is impossible as z_* is a strict maximum point of u_*. Estimate (3.13) is proved.

Now, define

$$\tilde{u}_i(z) = \frac{1}{u_i(0)} u_i(\lambda_i^{-1} z) \text{ for } z \in \mathbb{R}^n.$$

By first and second derivative estimates for the σ_k-Yamabe equation and the Liouville-type theorem [15, Theorem 1.3], we may assume after passing to a subsequence if necessary that $\tilde{u}_i$ converges in $C^2_{loc}(\mathbb{R}^n)$ to

$$U(r) = (1 + r^2)^{-\frac{n-2}{2}}.$$

Furthermore, for every $\varepsilon_i \to 0^+$ and every $R_i \to \infty$, after passing to a subsequence, we have

$$\|\tilde{u}_i - U\|_{C^2(B_{R_i})} \leq \varepsilon_i.$$

This gives precisely estimate (3.3). The last assertion of (a) also follows.

3.3. Proof of Statement (b) of Theorem 3.1

Note that by (a) with $\varepsilon_i R_i \to 0$, we have $\dot{\xi}_i(t) > 0$ in $(t_{1,i}, \ln \frac{R_i}{\lambda_i})$. Clearly, by the definition of $t_{2,i}$, if $t_{2,i} < 0$ is finite, $\dot{\xi}_i(t_{2,i}) = 0$. Furthermore, we have for $\ln \frac{R_i}{\lambda_i} < t < t_{2,i}$ that

$$\xi_i(t) \geq \xi_i(\ln(R_i/\lambda_i)) \overset{(3.3)}{\geq} \ln R_i - O(1). \tag{3.14}$$

It follows from property (2.6) that $\ddot{\xi}_i < 0$ at every critical point of ξ_i in $[\ln \frac{R_i}{\lambda_i}, t_{2,i}]$ for large i. In particular, for large i, ξ_i is strictly increasing in $[\ln \frac{R_i}{\lambda_i}, t_{2,i})$ and, if $t_{2,i} < 0$ is finite, then as $\dot{\xi}_i(t_{2,i}) = 0$, $\ddot{\xi}_i(t_{2,i}) < 0$.

Estimate (3.5) follows from (3.3), (3.14), the monotonicity of ξ_i and Lemma 3.3.

Let us now prove (3.4) and, when $\beta \leq n - 2k$, the negativity of a. For $t \in (\ln \frac{R_i}{\lambda_i}, t_{2,i})$, we have by the Pohozaev identity (2.10) that

$$\frac{1}{2^k}\binom{n}{k}\left[1 - \dot{\xi}_i(t)^2\right]^k - e^{-2k\xi_i(t)} K_{cyl}(t) = e^{(n-2k)\xi_i(t)} \int_{-\infty}^{t} \dot{K}_{cyl}(\tau) e^{-n\xi_i(\tau)} d\tau.$$

Recalling (1.5), we see that $\dot{K}_{cyl}(\tau) = -a 2^\beta e^{\beta\tau} L(\tau)$ for some bounded function L satisfying $L(\tau) \to 1$ as $\tau \to -\infty$. Hence, using (3.3) with $\ell = 0$ in the interval $(-\infty, \ln \frac{R_i}{\lambda_i})$, (3.5) in the interval $(\ln \frac{R_i}{\lambda_i}, t)$ and noting that $\xi_i(t) = \ln \lambda_i + t + O(1) \geq \ln R_i + O(1) \to \infty$ as $i \to \infty$, we have

$$\begin{aligned} &\left[1 - \dot{\xi}_i(t)^2\right]^k + o(1) \\ &\quad = -(1 + o(1)) e^{(n-2k)\xi_i(t)} \lambda_i^{-\beta_2} a\beta 2^{\beta + \frac{n+2k}{2}} \binom{n}{k}^{\frac{n-2k}{2k}} K_{Euc}(0) \int_0^\infty \frac{r^{n+\beta_2-1}}{(1+r^2)^n} dr. \end{aligned} \tag{3.15}$$

Since the right hand side of (3.15) is $-e^{O(1)} a e^{(n-2k)t} \lambda_i^{n-2k-\beta}$ (by (3.5)) and (3.15) holds for all $t \in (\ln \frac{R_i}{\lambda_i}, t_{2,i})$, we have that $t_{2,i} = O(1)$ when $\beta \geq n - 2k$ and $t_{2,i} = -(1 - \frac{\beta}{n-2k})\lambda_i + O(1)$ for $\beta < n - 2k$, which gives (3.4). In addition, by considering the sign of the left and right sides of (3.15), we have that $a < 0$ when $\beta \leq n - 2k$.

3.4. Proof of Statement (c) of Theorem 3.1

Note that when $\beta < \frac{n-2k}{2}$, Statement (c) is contained in Statement (d). We consider here only the case $\frac{n-2k}{2} \leq \beta < n - 2k$ and leave the case $\beta < \frac{n-2k}{2}$ to the proof of Statement (d).

We have seen that $t_{2,i} < 0$, $\dot{\xi}_i(t_{2,i}) = 0$ and $\ddot{\xi}_i(t_{2,i}) < 0$.

Since $\dot{\xi}_i > -1$, we deduce from (3.5) that

$$\xi_i(t) \geq \xi_i(t_{2,i}) - (t - t_{2,i}) \geq -\Big(1 - \frac{2\beta}{n-2k}\Big) \ln \lambda_i - t - O(1) \text{ for } t \geq t_{2,i}. \tag{3.16}$$

As $\beta \geq \frac{n-2k}{2}$, estimate (3.16) implies that, for every $\xi_0 > 0$, there exists $\tilde{t}_{3,i} = O(1) \gg t_{2,i}$ such that $\xi_i > \xi_0$ in $(t_{2,i}, \tilde{t}_{3,i})$. In view of (2.4) and (2.6), when ξ_0 is sufficiently large, the function $\dot{\xi}_i$ in the interval $(t_{2,i}, \tilde{t}_{3,i})$ has the property that, whenever $\dot{\xi}_i(t) = 0$, it holds that $\ddot{\xi}_i(t) < 0$. Note also that $\dot{\xi}_i(t) < 0$ for $t > t_{2,i}$ and close to $t_{2,i}$ (because $\dot{\xi}_i(t_{2,i}) = 0$ and $\ddot{\xi}_i(t_{2,i}) < 0$). These two properties imply that $\dot{\xi}_i < 0$ in $(t_{2,i}, \tilde{t}_{3,i})$. In particular, $t_{3,i} \geq \tilde{t}_{3,i}$ and so $t_{3,i} = O(1)$, which gives (3.6). Estimate (3.7) follows from Lemma 3.3 applied in the interval $(t_{2,i}, \tilde{t}_{3,i})$ and the fact that $|\dot{\xi}| < 1$ in $[\tilde{t}_{3,i}, t_{3,i}]$.

3.5. Proof of Statement (d) of Theorem 3.1

Suppose $2 \leq \beta < \frac{n-2k}{2}$. Recall from (b) that $a < 0$. For simplicity, we assume instead of (3.1) that $K_{cyl}(t) = 1 - |a|e^{\beta t}$ in $(-\infty, 0)$. The proof in the general case where K_{cyl} satisfies only (3.1) can be done by restricting attention to a small ball B_δ and an easy accommodation for error terms.

Note that $K_{cyl}(t) = 1 - |a|e^{\beta t}$ is decreasing and so by the Pohozaev identity (2.9), the functions $H(t, \xi_i, \dot{\xi}_i)$ are increasing. Noting that $H(t, \xi_i, \dot{\xi}_i) \to 0$ as $t \to -\infty$ (since $\xi_i(t) + t$ is bounded as $t \to -\infty$), we thus have that $H(t, \xi_i, \dot{\xi}_i) > 0$ in $(-\infty, 0]$.

Hence, by estimate (3.2) and the Pohozaev identity (2.10),

$$0 \leq \lim_{t \to -\infty} \sup_i H(t, \xi_i, \dot{\xi}_i) \leq \lim_{t \to -\infty} C \int_{-\infty}^{t} |\dot{K}_{cyl}| \, d\tau = 0, \tag{3.17}$$

where here and below C denotes a constant which remains independent of i.

Recall that all solutions to $F_k[E] = 1$ in $(-\infty, \infty)$ satisfying $H(t, E, \dot{E}) \equiv 0$ are of the form

$$E(t) = \Xi(t + \ln \lambda) \text{ for some } \lambda > 0.$$

This implies:

Lemma 3.4. *Let ξ_i satisfy $F_k[\xi_i] = K_i$ and $|\dot{\xi}_i| < 1$ in $(-\infty, \ln 2)$ where $K_i \in C^2(-\infty, \ln 2)$ satisfies*

$$\sup_i \sup_{t \in (-\infty, \ln 2)} (|\ln K_i(t)| + |\dot{K}_i(t)| + |\ddot{K}_i(t)|) < \infty.$$

For every $c_0 \in \mathbb{R}$, if $s_i \leq 0$, $\xi_i(s_i) \to c_0$, $H(s_i, \xi_i(s_i), \dot{\xi}_i(s_i)) \to 0$ and, for some $0 \leq S_i \leq |s_i|$,

$$\lim_{i\to\infty} K_i(t+s_i) = 1 \textit{ for all } t \le \liminf S_i,$$

then $c_0 \ge \min \Xi = \ln\left(2^{-\frac{1}{2}}\binom{n}{k}^{\frac{1}{2k}}\right)$ *and there exists* $T_i \to \infty$ *such that, after passing to a subsequence,*

$$\|\xi_i(t+s_i) - \Xi(t+\bar{t})\|_{C^2([-T_i,\tilde{T}_i])} \le \delta_i, \qquad \tilde{T}_i = \begin{cases} T_i & \textit{if } S_i \to \infty, \\ S_i & \textit{if } S_i \textit{ is bounded,} \end{cases}$$

where $\bar{t}$ *are one of the two solutions of* $\Xi(\bar{t}) = c_0$ *if* $c_0 > \min \Xi$ *and* $\bar{t} = 0$ *if* $c_0 = \min \Xi$.

In the sequel, we fix some $\bar{\xi}_0 > \min \Xi$ which is larger than the constant ξ_0 in Lemma 3.3 and the constant $\bar{x}$ in (2.6), and has the additional property that

$$\begin{aligned} &\text{For any } C^2 \text{ functions } \xi, \text{ if } t \text{ satisfies } \xi(t) > \bar{\xi}_0 \text{ and } \dot{\xi}(t) = 0, \\ &\text{then } \tfrac{1}{2}e^{-(n-2k)\xi(t)} \le 2^k\binom{n}{k}^{-1} H(t,\xi(t),\dot{\xi}(t)) \le e^{-(n-2k)\xi(t)}. \end{aligned} \tag{3.18}$$

By Lemma 3.4 and in view of (3.17), there exists $m_0 > 0$ depending only on $(n, K, \bar{\xi}_0)$ such that, for each i, the number $\tilde{N}_i$ of points $s < -m_0$ such that $\xi_i(s) = \bar{\xi}_0$ and $\dot{\xi}_i(s) < 0$ is non-zero and finite. We label these points as $s_{1,i} < s_{2,i} < \ldots < s_{\tilde{N}_i,i}$. By the same lemma, if we let $m_0' > 0$ be the solution to $\Xi(-m_0') = \bar{\xi}_0$ with $\dot{\Xi}(-m_0') < 0$, then for every $\varepsilon > 0$, there exists $\tilde{R}_\varepsilon > \frac{2}{\varepsilon}$ independent of i such that for any ℓ satisfying $|s_{\ell,i}| > \tilde{R}_\varepsilon$,

$$\|\xi_i(t+s_{\ell,i}) - \Xi(t-m_0')\|_{C^2[-2/\varepsilon,2/\varepsilon]} \le \varepsilon. \tag{3.19}$$

It is readily seen from (3.19) and (2.6) that $\xi_i^{-1}(\bar{\xi}_0) \cap (-\infty, s_{\tilde{N}_i,i} + 9m_0'/4]$ comprises of $s_{1,i} < s''_{1,i} < s_{2,i} < s''_{2,i} < \ldots < s_{\tilde{N}_i,i} < s''_{\tilde{N}_i,i}$, and $\xi_i\big|_{(-\infty,s''_{\tilde{N}_i,i}]}$ has critical points $t_{1,i},\ldots,t_{2\tilde{N}_i-1,i}$ such that

$$s_{1,i} < t_{1,i} < s''_{1,i} < t_{2,i} < s_{2,i} < \ldots < s''_{\tilde{N}_i,i} < t_{2\tilde{N}_i-2,i} < s_{\tilde{N}_i,i} < t_{2\tilde{N}_i-1,i} < s''_{\tilde{N}_i,i},$$

$\dot{\xi}_i < 0$ in $(-\infty, t_{1,i})$ and $(t_{2\ell,i}, t_{2\ell+1,i})$, and $\dot{\xi}_i > 0$ in $(t_{2\ell-1,i}, t_{2\ell,i})$, for $1 \le \ell \le \tilde{N}_i - 1$. Furthermore, (3.7) holds.

By Statements (a) and (b), we have that $a < 0$, $t_{1,i} = -\ln\lambda_i + o(1)$, $t_{2,i} = -(1 - \frac{\beta}{n-2k})\ln\lambda_i + O(1)$, and $\xi_i(t_{2,i}) = \frac{\beta}{n-2k}\ln\lambda_i + O(1)$.

To conclude, we need to show that there exists $1 \le N_i \le \tilde{N}_i - 1$, $N_i = \lfloor\frac{\ln\ln\lambda_i + O(1)}{|\ln(1-\frac{2\beta}{n-2k})|}\rfloor$ such that

(i) $t_{2\ell,i} = -\alpha_\ell \ln\lambda_i + O(1)$ for $2 \le \ell \le N_i$,
(ii) $t_{2\ell+1,i} = -(\alpha_\ell - \gamma_\ell)\ln\lambda_i + O(1)$ for $1 \le \ell \le N_i - 1$,

where $|O(1)| \le C$, independent of i and ℓ, $\alpha_\ell = (1-\frac{\beta}{n-2k})(1-\frac{2\beta}{n-2k})^{\ell-1}$ and $\gamma_\ell = \frac{\beta}{n-2k}(1-\frac{2\beta}{n-2k})^{\ell-1}$. Note that by applying Lemma 3.3 to the intervals $[t_{2\ell,i}, t_{2\ell+1,i}]$ and $[t_{2\ell+1,i}, t_{2\ell+2,i}]$, we obtain from the above that

$$\xi_i(t) = \xi_i(t_{2\ell+1,i}) - t + t_{2\ell+1,i} + O(1) = -(\alpha_\ell - \gamma_\ell)\ln\lambda_i - t + O(1) \text{ in } [t_{2\ell,i}, t_{2\ell+1,i}],$$
$$\xi_i(t) = \xi_i(t_{2\ell+1,i}) + t - t_{2\ell+1,i} + O(1) = (\alpha_\ell - \gamma_\ell)\ln\lambda_i + t + O(1) \text{ in } [t_{2\ell+1,i}, t_{2\ell+2,i}].$$

In other words

$$\xi_i(t) = \Xi(t - t_{2\ell+1,i}) + O(1) \text{ in } [t_{2\ell,i}, t_{2\ell+2,i}].$$

To prove (i)-(ii), we use the following lemma, which is of independent interest and can be applied in a situation more general that what is described above. (Note that no assumption at $-\infty$ is assumed in the lemma.) Recall that $\bar{\xi}_0$ is a constant larger than the constant ξ_0 in Lemma 3.3 and the constant $\bar{x}$ in (2.6), and has the property (3.18).

Lemma 3.5. *Let $a < 0$ and $\beta \in (0, n-2k)$ and suppose $K_{cyl} \in C^{2,\alpha}(-\infty, \ln 2)$, $0 < \alpha \le 1$, satisfies (3.1). For every given constant $D \ge 0$, there exists some large $M = M(n, K_{cyl}, D, \bar{\xi}_0) > 1$ such that if $\xi \in C^2(-\infty, \ln 2)$ satisfies $F_k[\xi] = K_{cyl}(t)$ and $|\dot{\xi}| < 1$ in $(-\infty, \ln 2)$, and if $t_* < 0$ is a critical point of ξ satisfying*

$$-(n-2k)\xi(t_*) - D \le \beta(t_* + \xi(t_*)) \le -M,$$

then $\xi(t_*) > \bar{\xi}_0$, $\ddot{\xi}(t_*) < 0$, and there exist critical points $t_* < t_{*+1} < t_{*+2} < 0$ of ξ such that $\xi(t_{*+1}) < \ln\left(2^{-\frac{1}{2}}\binom{n}{k}^{\frac{1}{2k}}\right) + \frac{1}{M}$, $\xi(t_{*+2}) > \bar{\xi}_0$, $\ddot{\xi}(t_{*+1}) > 0$, $\ddot{\xi}(t_{*+2}) < 0$, $\dot{\xi} < 0$ in (t_*, t_{*+1}), $\dot{\xi} > 0$ in (t_{*+1}, t_{*+2}), and

$$|t_{*+1} - (t_* + \xi(t_*))| \le M, \tag{3.20}$$
$$\Big|t_{*+2} - \big(1 - \frac{\beta}{n-2k}\big)(t_* + \xi(t_*))\Big| \le M, \tag{3.21}$$
$$|\xi(t) - \Xi(t - t_{*+1})| \le M \text{ in } [t_*, t_{*+2}], \tag{3.22}$$
$$-(n-2k)\xi(t_{*+2}) \le \beta(t_{*+2} + \xi(t_{*+2})), \tag{3.23}$$
$$\Big|(t_{*+2} + \xi(t_{*+2})) - \big(1 - \frac{2\beta}{n-2k}\big)(t_* + \xi(t_*))\Big| \le M. \tag{3.24}$$

Once this lemma is proved, we can obtain the conclusion as follows. Take $D = 0$ and fix M as in the lemma. First, we have for all large i that

$$-(n-2k)\xi_i(t_{2,i}) \le \beta(t_{2,i} + \xi_i(t_{2,i})) \le -M.$$

Let N_i be the largest number in $\{2, \ldots, \tilde{N}_i - 1\}$ such that $\beta(t_{2\ell,i} + \xi_i(t_{2\ell,i}) \le -M$ for $1 \le \ell \le N_i$. Applying the lemma repeatedly with $t_* = t_{2\ell,i} < 0$ for $1 \le \ell \le N_i$, we have

$$\Big|(t_{2\ell+2,i}+\xi_i(t_{2\ell+2,2}))-\Big(1-\frac{2\beta}{n-2k}\Big)(t_{2\ell,i}+\xi_i(t_{2\ell,i}))\Big|\le M. \tag{3.25}$$

(Note that if $N_i=\tilde N_i-1$, the lemma also gives the existence of another local maximum point $t_{2\tilde N_i,i}\in(s''_{\tilde N_i,i},0)$ of ξ_i.) This implies for $2\le\ell\le N_i+1$ that

$$\Big|(t_{2\ell,i}+\xi_i(t_{2\ell,i}))-(1-\frac{2\beta}{n-2k})^{\ell-1}(t_{2,i}+\xi_i(t_{2,i}))\Big|\le M\sum_{j=0}^{\ell-2}(1-\frac{2\beta}{n-2k})^j\le\frac{n-2k}{2\beta}M.$$

Since $t_{2,i}+\xi_i(t_{2,i})+\big(1-\frac{2\beta}{n-2k}\big)\ln\lambda_i$ is bounded as $i\to\infty$, we thus have for $1\le\ell\le N_i+1$ that

$$\Big|t_{2\ell,i}+\xi_i(t_{2\ell,i})+(1-\frac{2\beta}{n-2k})^{\ell}\ln\lambda_i\Big|\le C, \tag{3.26}$$

where C is independent of i and ℓ. Returning to (3.20) and (3.21) (still with $t_*=t_{2\ell,i}$), we see that the declared properties (i) and (ii) hold.

To finish the proof, we show that $N_i\ge\lfloor\frac{\ln\ln\lambda_i+O(1)}{|\ln(1-\frac{2\beta}{n-2k})|}\rfloor=:\hat N_i$. (Note that $t_{2\hat N_i,i}\ge -C$ for some C independent of i and $t_{2\ell+2,i}-t_{2\ell+1,i}\ge m_0'/4$ for all ℓ, this estimate gives $N_i=\hat N_i+O(1)=\lfloor\frac{\ln\ln\lambda_i+O(1)}{|\ln(1-\frac{2\beta}{n-2k})|}\rfloor$.)

In view of (3.26) with $\ell=N_i$ and the fact that $\beta(t_{2N_i,i}+\xi_i(t_{2N_i,i}))\le -M$, we only need to show that $t_{2N_i,i}+\xi_i(t_{2N_i,i})\ge -C$ for some C independent of i. By (3.25), it suffices to show that $t_{2N_i+2,i}+\xi_i(t_{2N_i+2,i})\ge -C$. To this end, we may assume without loss of generality that $\beta(t_{2N_i+2,i}+\xi_i(t_{2N_i+2,i}))\le -M$, as otherwise there is nothing to prove. By the lemma, we can find critical points $t_{2N_i+2,i}<t_{2N_i+3,i}<t_{2N_i+4,i}<0$ of ξ where $\xi(t_{2N_i+2,i})>\bar\xi_0>\xi(t_{2N_i+3,i})$ and $\dot\xi<0$ in $(t_{2N_i+2,i},t_{2N_i+3,i})$. In particular, there exists $s_{N_i+2,i}\in(t_{2N_i+2,i},t_{2N_i+3,i})$ such that $\xi(s_{N_i+2,i})=\bar\xi_0$ and $\dot\xi(s_{N_i+2,i})<0$. By construction of the sequence $\{s_{\ell,i}\}$, we have $s_{N_i+2,i}\ge -m_0$. It follows that $t_{2N_i+3,i}\ge -m_0$. Recalling (3.20) with $t_*=t_{2N_i+2,i}$, we thus have $t_{2N_i+2,i}+\xi_i(t_{2N_i+2,i})\ge -C$ as wanted. Theorem 3.1 follows. □

Proof of Lemma 3.5. In the proof we will frequently use the function H defined in (2.8) and the Pohozaev identity (2.9). For convenience, we write $H(t):=H(t,\xi(t),\dot\xi(t))$.

For simplicity, we consider again only the case $K_{cyl}(t)=1-|a|e^{\beta t}$.

As in (3.2), there exists $C'=C'(n,K_{cyl})$ such that

$$\xi\ge -C' \text{ and } |\dot\xi|+|\ddot\xi|\le C' \text{ in } (-\infty,\ln\frac{3}{2}).$$

By Lemma 3.4, there exist $m_0>10m_0'>0$ depending only on $(n,K,\bar\xi_0)$ such that

$$\begin{aligned}&\text{If } s\le -m_0 \text{ satisfies } \xi(s)=\bar\xi_0,\ \dot\xi(s)<0 \text{ and } |H(s)|\le 1/m_0 \text{ then there}\\ &\text{exist } s''>s'>s \text{ such that } \xi(s'')=\bar\xi_0,\ \dot\xi(s')=0,\ \ddot\xi(s')>0,\ \dot\xi<0 \text{ in } [s,s'),\\ &\dot\xi>0 \text{ in } (s',s''],\ 3m_0'/4\le s'-s\le 5m_0'/4 \text{ and } 7m_0'/4\le s''-s\le 9m_0'/4.\end{aligned} \tag{3.27}$$

In the sequel, M is a large constant which may need to be enlarged at a few instances in the proof but will depend only on n, a, β, D, C' and $\bar{\xi}_0$.

Since $-(n-2k)\xi(t_*) - D \leq -M$, we may take M sufficiently large so that $\xi(t_*) > \bar{\xi}_0$. As $\xi(t_*) > \bar{\xi}_0$ and $\dot{\xi}(t_*) = 0$, we have by (2.6) that $\ddot{\xi}(t_*) < 0$ and t_* is a local maximum point of ξ. We will show the existence of t_{*+1} by showing that ξ will decrease to the value $\bar{\xi}_0$ and appeal to (3.27).

Define

$$s_0 = \sup\Big\{t \in [t_*, 0) : \xi(t) > \bar{\xi}_0 \text{ in } [t_*, t]\Big\}.$$

Since $\dot{\xi}(t) < 0$ for $t > t_*$ and close to t_*, we have by (2.6) that $\dot{\xi} < 0$ in (t_*, s_0). Applying Lemma 3.3, we have

$$\xi(t_*) - (t - t_*) \leq \xi(t) \leq \xi(t_*) - (t - t_*) + C_0 \text{ for } t \in [t_*, s_0], \tag{3.28}$$

where $C_0 > 0$ is the constant in Lemma 3.3. Taking $t = s_0$ in (3.28) and using the fact that $\beta(t_* + \xi(t_*)) \leq -M$, we obtain, after possibly enlarging M, that

$$s_0 \leq t_* + \xi(t_*) - \xi(s_0) + C_0 \leq -\frac{1}{\beta}M - \bar{\xi}_0 + C_0 \leq -m_0 < 0,$$

which implies that $\xi(s_0) = \bar{\xi}_0$ and

$$(t_* + \xi(t_*)) - \bar{\xi}_0 \leq s_0 \leq (t_* + \xi(t_*)) - \bar{\xi}_0 + C_0. \tag{3.29}$$

To use (3.27), we need to estimate $H(s_0)$. On one hand, by (3.18) and the relation $-(n-2k)\xi(t_*) - D \leq \beta(t_* + \xi(t_*))$, we have

$$0 < H(t_*) \leq \frac{1}{2^k}\binom{n}{k} e^{-(n-2k)\xi(t_*)} \leq \frac{1}{2^k}\binom{n}{k} e^D e^{\beta(t_*+\xi(t_*))}.$$

On the other hand, we have

$$0 < -\int_{t_*}^{s_0} \dot{K}_{cyl} e^{-n\xi}\, d\tau \overset{(3.28),(3.29)}{\leq} \frac{|a|\beta e^{(n+\beta)(\bar{\xi}_0+C_0)}}{n+\beta} e^{\beta(t_*+\xi(t_*))}.$$

Thus, by the Pohozaev identity (2.9) and the fact that $\beta(t_* + \xi(t_*)) \leq -M$ and by possibly enlarging M,

$$\begin{aligned} 0 < H(s_0) &= H(t_*) - \int_{t_*}^{s_0} \dot{K}_{cyl} e^{-n\xi}\, d\tau \\ &\leq \Big(\frac{1}{2^k}\binom{n}{k} e^D + \frac{|a|\beta\, e^{(n+\beta)(\bar{\xi}_0+C_0)}}{n+\beta}\Big) e^{\beta(t_*+\xi(t_*))} \leq \frac{1}{m_0}. \end{aligned} \tag{3.30}$$

Therefore, by (3.27), there exist $s_1 > t_{*+1} > s_0$ such that $\xi(s_1) = \bar{\xi}_0$, $\dot{\xi}_i(t_{*+1}) = 0$, $\dot{\xi} < 0$ in $[s_0, t_{*+1})$, $\dot{\xi} > 0$ in $(t_{*+1}, s_1]$, $3m_0'/4 \leq t_{*+1} - s_0 \leq 5m_0'/4$ and $7m_0'/4 \leq s_1 - s_0 \leq 9m_0'/4$.

Clearly, (3.20) follows from (3.29) and the bound $s_0 + 3m_0'/4 < t_{*+1} < s_0 + 5m_0'/4$.

From the above, we know that $\dot{\xi} > 0$ in (t_{*+1}, s_1). Define

$$t_{*+2} = \sup\Big\{t \in [s_1, 0) : \dot{\xi}(t) > 0 \text{ in } [s_1, t]\Big\}.$$

Note that $\xi \geq \bar{\xi}_0$ in $[s_1, t_{*+2}]$, and so by (2.6), $\dot{\xi} > 0$ in $[s_1, t_{*+2})$. We will show that when M is suitably large, $t_{*+2} < 0$ and hence t_{*+2} is a critical point of ξ.

By Lemma 3.3, (3.29) and the fact that $\xi(s_1) = \bar{\xi}_0$ and $s_0 + 7m_0'/4 < s_1 < s_0 + 9m_0'/4$, we have

$$t - (t_* + \xi(t_*)) - C \leq \xi(t) \leq t - (t_* + \xi(t_*)) + C \text{ in } [s_1, t_{*+2}], \tag{3.31}$$

where here and below C denotes a positive constant depending only on $n, a, \beta, D, C', \bar{\xi}_0$, C_0 and m_0. This together with (3.20) and (3.29) gives (3.22) after possibly enlarging M.

Let us now estimate $H(t_{*+2})$ in terms of $t_* + \xi(t_*)$. By the Pohozaev identity (2.9), we have $H(t_{*+2}) = H(s_0) - \int_{s_0}^{t_{*+2}} \dot{K}_{cyl} e^{-n\xi}\, d\tau$. Using (3.29) and the inequalities $-C' \leq \xi \leq \bar{\xi}_0$ in $[s_0, s_1]$ and $7m_0'/4 \leq s_1 - s_0 \leq 9m_0'/4$, we have that

$$\frac{1}{C} e^{\beta(t_* + \xi(t_*))} \leq \int_{s_0}^{s_1} \dot{K}_{cyl} e^{-n\xi}\, d\tau \leq C e^{\beta(t_* + \xi(t_*))}. \tag{3.32}$$

By (3.31), we have

$$\frac{1}{C} e^{(\beta-n)t} e^{-n(t_* + \xi(t_*))} \leq -\dot{K}_{cyl}(t) e^{-n\xi(t)} \leq C e^{(\beta-n)t} e^{-n(t_* + \xi(t_*))} \text{ in } [s_1, t_{*+2}],$$

and so, as $\beta < n$,

$$0 \leq -\int_{s_1}^{t_{*+2}} \dot{K}_{cyl} e^{-n\xi}\, d\tau \leq C e^{\beta(t_* + \xi(t_*))}. \tag{3.33}$$

Putting together (3.30), (3.32) and (3.33), we thus have

$$\frac{1}{C} e^{\beta(t_* + \xi(t_*))} \leq H(t_{*+2}) \leq C e^{\beta(t_* + \xi(t_*))}. \tag{3.34}$$

Recalling the expression of H in (2.8) and using $t = t_{*+2}$ in (3.31), we obtain

$$1 \overset{(2.8)}{\geq} 2^k \binom{n}{k}^{-1} e^{(n-2k)\xi(t_{*+2})} H(t_{*+2}) \overset{(3.31),\ (3.34)}{\geq} \frac{1}{C} e^{(n-2k)t_{*+2}} e^{-(n-2k-\beta)(t_* + \xi(t_*))},$$

which, in view of the fact $\beta(t_* + \xi(t_*)) \le -M$, leads to

$$t_{*+2} \le \frac{n-2k-\beta}{n-2k}(t_* + \xi(t_*)) + C \le -\frac{n-2k-\beta}{\beta(n-2k)}M + C. \tag{3.35}$$

As $\beta < n - 2k$, the right hand side of (3.35) can be made negative by enlarging M. Recalling the definition of t_{*+2}, we thus have $\dot{\xi}(t_{*+2}) = 0$ and, by (2.6), $\ddot{\xi}(t_{*+2}) < 0$.

As $\xi(t_{*+2}) > \xi(s_1) = \bar{\xi}_0$ and $\dot{\xi}(t_{*+2}) = 0$ and in view of (3.18) and (3.34), we have

$$-\frac{\beta}{n-2k}(t_* + \xi(t_*)) - C \le \xi(t_{*+2}) \le -\frac{\beta}{n-2k}(t_* + \xi(t_*)) + C,$$

and, in view of (3.31) with $t = t_{*+2}$,

$$\frac{n-2k-\beta}{n-2k}(t_* + \xi(t_*)) - C \le t_{*+2} \le \frac{n-2k-\beta}{n-2k}(t_* + \xi(t_*)) + C.$$

These give (3.21). They also give

$$(n-2k)\xi(t_{*+2}) + \beta(t_{*+2} + \xi(t_{*+2})) \ge -\frac{2\beta^2}{n-2k}(t_* + \xi(t_*)) - C,$$

$$\Big(1 - \frac{2\beta}{n-2k}\Big)(t_* + \xi(t_*)) - C \le t_{*+2} + \xi(t_{*+2}) \le \Big(1 - \frac{2\beta}{n-2k}\Big)(t_* + \xi(t_*)) + C.$$

In view of the fact that $\beta(t_* + \xi(t_*)) \le -M$, by enlarging M one final time, we obtain (3.23) and (3.24) as desired. □

4. Compactness estimates: proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1 together with some extensions.

4.1. Proof of Theorem 1.1

By first and second derivative estimates for the σ_k-Yamabe equation (see e.g. [11, Theorem 1.1], [20, Theorem 1.10]), it suffices to show that

$$v \le C_1 \text{ for all positive } C^2_r \text{ solutions } v \text{ of (1.1)}$$

where C_1 depends only on n, k and K. Suppose by contradiction that there exist positive functions $v_i \in C^2_r(\mathbb{S}^n)$ satisfying (1.1) such that $\max v_i \to \infty$.

Let $u_i : \mathbb{R}^n \to \mathbb{R}$ be related to v_i as in (2.1). As u_i is super-harmonic and rotationally symmetric, the maximum principle implies that $u_i(0)$ is the maximum of u_i in any closed ball centered at the origin. Recalling (2.1), we have

$$v_i(x) \le v_i(S)\Big(\frac{2}{1 + \cos d_{\mathring{g}}(x, S)}\Big)^{\frac{n-2}{2}} \text{ for all } x \in \mathbb{S}^n \setminus \{N\}, \tag{4.1}$$

where N and S are respectively the north and south poles of $\mathbb{S}^n$. In particular, $v_i \leq 2^{\frac{n-2}{2}} v_i(S)$ in the lower closed hemi-sphere. Likewise $v_i \leq 2^{\frac{n-2}{2}} v_i(N)$ in the upper closed hemi-sphere. As $\max v_i \to \infty$, this implies that

$$\max\{v_i(S), v_i(N)\} \to \infty. \tag{4.2}$$

Throughout the proof, C denotes some generic positive constant which may change from one line to another but will remain independent of i, $O(1)$ denotes a term which is bounded as $i \to \infty$, and $o(1)$ denotes a term which tends to zero as $i \to \infty$.

Step 1: We show that

$$v_i(x) d_{\mathring{g}}(x, \{N, S\})^{\frac{n-2}{2}} \leq C,$$

and

$$|\nabla^\ell \ln v_i(x)| d_{\mathring{g}}(x, \{N, S\})^\ell \leq C \text{ for } \ell = 1, 2. \tag{4.3}$$

These estimates follow from Theorem 3.1(a) and (4.1).

In the next step, let a_2 and β_2 be as given in (1.2), $t = \ln r$, ξ_i be related to u_i as in (2.3) and $\lambda_i := 2^{-\frac{1}{2}} \binom{n}{k}^{-\frac{1}{2k}} K(S)^{\frac{1}{2k}} u_i(0)^{\frac{2}{n-2}} = 2^{\frac{1}{2}} \binom{n}{k}^{-\frac{1}{2k}} K(S)^{\frac{1}{2k}} v_i(S)^{\frac{2}{n-2}}$.

Step 2: Making use of Pohozaev-type and mass-type identities, we show that if $v_i(S) \to \infty$, then $a_2 < 0$, and, for large i, there exist

$$\delta_i = e^{O(1)} \lambda_i^{-(1 - \frac{\beta_2}{n-2k})}, \tag{4.4}$$

$$\nu_i = e^{O(1)} \lambda_i^{-(1 - \frac{2\beta_2}{n-2k})} \tag{4.5}$$

such that ξ_i is strictly increasing in $(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, is strictly decreasing in $(\ln \delta_i, \ln \nu_i)$, has a strict local maximum at $\ln \delta_i$, and

$$\xi_i(\ln \delta_i) = \frac{\beta_2}{n-2k} \ln \lambda_i + p_2 + o(1), \tag{4.6}$$

$$\xi_i(\ln \delta_i) = \ln \lambda_i + \ln \delta_i + q_2 + o(1), \tag{4.7}$$

$$\xi_i(t) = \ln \lambda_i + t + O(1) \text{ in } (\ln \frac{2}{\lambda_i}, \ln \delta_i), \tag{4.8}$$

$$\xi_i(t) = -(1 - \frac{2\beta_2}{n-2k}) \ln \lambda_i - t + O(1) \text{ in } (\ln \delta_i, \ln \nu_i), \tag{4.9}$$

where

$$p_2 := -\frac{1}{n-2k}\ln\Big[2^{\beta_2+\frac{n+2k}{2}}\binom{n}{k}^{\frac{n-2k}{2k}}\frac{\Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2\Gamma(n)}|a_2|\beta_2 K(S)^{-\frac{n}{2k}}\Big],$$

$$q_2 := -\ln\Big[2^{\frac{n+2k}{2(n-2k)}}\binom{n}{k}^{\frac{1}{2k}}K(S)^{-\frac{1}{2k}}\Big].$$

If $\beta < n-2k$, the negativity of a_2, and estimates (4.4), (4.5), (4.8) and (4.9) follow from Theorem 3.1(b) and (c). Using the fact that ξ_i is now defined on all of $\mathbb{R}$ (rather than $(-\infty, \ln 2)$ in Theorem 3.1), the same proof can be used to treat the case $n-2k \le \beta < n$. Estimate (4.6) will be obtained by using $\dot{\xi}_i(\ln\delta_i) = 0$ in the relevant Pohozaev identity. Estimate (4.7) will be proved using a mass-type identity. Let us now give the details.

Proof of (4.8).

By Theorem 3.1(a), for every $\varepsilon_i \to 0^+$ and every $R_i \to \infty$, after passing to a subsequence, we have for $0 \le \ell \le 2$ that

$$\Big|\frac{d^\ell}{dt^\ell}\Big[\xi_i(t) + \ln\frac{\lambda_i e^t}{1+\lambda_i^2 e^{2t}} + \ln\Big(2^{\frac{1}{2}}\binom{n}{k}^{\frac{1}{2k}}K(S)^{-\frac{1}{2k}}\Big)\Big]\Big| \le \varepsilon_i \lambda_i^\ell e^{\ell t} \text{ in } (-\infty, \ln\frac{R_i}{\lambda_i}). \tag{4.10}$$

Note that by (4.10), $\dot{\xi}_i(t) \ge 0$ in $(\ln\frac{2}{\lambda_i}, \ln\frac{R_i}{\lambda_i})$. Let

$$\delta_i = \sup\Big\{s \ge \frac{2}{\lambda_i} : \dot{\xi}_i \ge 0 \text{ in } (\ln\frac{2}{\lambda_i}, \ln s)\Big\} \in [R_i\lambda_i^{-1}, \infty].$$

Clearly, if δ_i is finite, $\dot{\xi}_i(\ln\delta_i) = 0$. Furthermore, we have for $\ln\frac{R_i}{\lambda_i} < t < \ln\delta_i$ that

$$\xi_i(t) \ge \xi_i(\ln(R_i/\lambda_i)) \overset{(4.10)}{\ge} \ln R_i - O(1). \tag{4.11}$$

It follows from property (2.6) that $\ddot{\xi}_i < 0$ at every critical point of ξ_i in $[\ln\frac{R_i}{\lambda_i}, \ln\delta_i]$ for large i. In particular, for large i, ξ_i is strictly increasing in $[\ln\frac{R_i}{\lambda_i}, \ln\delta_i)$ and, if δ_i is finite, then as $\dot{\xi}_i(\ln\delta_i) = 0$, $\ddot{\xi}_i(\ln\delta_i) < 0$, and $\ln\delta_i$ is a strict local maximum of ξ_i.

Estimate (4.8) follows from (4.11), the monotonicity of ξ_i and Lemma 3.3.

Proof of the negativity of a_2 and estimates (4.4) and (4.6).

As in the proof of Theorem 3.1 (see (3.15)), we have for $t \in (\ln\frac{R_i}{\lambda_i}, \ln\delta_i)$ that

$$\Big[1-\dot{\xi}_i(t)^2\Big]^k + o(1)$$
$$= -(1+o(1))e^{(n-2k)\xi_i(t)}\lambda_i^{-\beta_2}a_2\beta_2 2^{\beta_2+\frac{n+2k}{2}}\binom{n}{k}^{\frac{n-2k}{2k}}K(S)^{-\frac{n}{2k}}\int_0^\infty \frac{r^{n+\beta_2-1}}{(1+r^2)^n}\,dr.$$

Using Corollary A.2, we get

$$\Big[1-\dot{\xi}_i(t)^2\Big]^k + o(1) = -(1+o(1))e^{(n-2k)\xi_i(t)}\lambda_i^{-\beta_2}e^{-(n-2k)p_2}. \tag{4.12}$$

Since the right hand side of (4.12) is $-e^{O(1)}a_2e^{(n-2k)t}\lambda_i^{n-2k-\beta_2}$ (by (4.8)) and (4.12) holds for all $t \in (\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, it follows that δ_i is finite and, in view of the definition of δ_i, $\dot{\xi}_i(\ln \delta_i) = 0$. In particular, we can also take $t = \ln \delta_i$ in (4.12), yielding the assertion $a_2 < 0$ and estimates (4.4) and (4.6).

As $a_2 < 0$, item (i) of the hypotheses of the theorem gives $\beta_2 \geq \frac{n-2k}{2}$.

Proof of estimates (4.5) and (4.9).

The proof is similar to the proof of Theorem 3.1(c). We omit the details.

Proof of estimate (4.7).

We start by using the mass-type identity (2.16) and the fact that $\dot{\xi}_i(\ln \delta_i) = 0$ to obtain

$$\begin{aligned} e^{\frac{n-2k}{2}(-\xi_i(\ln \delta_i)+\ln \delta_i)} &= 2^{k-1}n\binom{n}{k}^{-1} m(\ln \delta_i, \xi_i(\ln \delta_i), \dot{\xi}_i(\ln \delta_i)) \\ &= 2^{k-1}n\binom{n}{k}^{-1} \int\limits_{-\infty}^{\ln \delta_i} K_{cyl}(\tau)(1-\dot{\xi}_i)^{-(k-1)}e^{-\frac{n+2k}{2}\xi_i}e^{\frac{n-2k}{2}\tau}\,d\tau. \end{aligned} \tag{4.13}$$

We proceed to estimate the integral on the right hand side of (4.13). The integration over $(-\infty, \ln \frac{R_i}{\lambda_i})$ can be estimated using the continuity of K and (4.10) with $\varepsilon_i \ll R_i^{-3}$ and Corollary A.2:

$$\begin{aligned} &\int\limits_{-\infty}^{\ln \frac{R_i}{\lambda_i}} K_{cyl}(\tau)(1-\dot{\xi}_i)^{-(k-1)}e^{-\frac{n+2k}{2}\xi_i}e^{\frac{n-2k}{2}\tau}\,d\tau \\ &\quad = (1+o(1))2^{\frac{n-2k+4}{4}}\binom{n}{k}^{\frac{n+2k}{4k}} K(S)^{-\frac{n-2k}{4k}}\lambda_i^{-\frac{n-2k}{2}} \int\limits_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{n+2}{2}}}\,dr \\ &\quad = (1+o(1))\frac{1}{n}2^{\frac{n-2k+4}{4}}\binom{n}{k}^{\frac{n+2k}{4k}} K(S)^{-\frac{n-2k}{4k}}\lambda_i^{-\frac{n-2k}{2}}. \end{aligned} \tag{4.14}$$

To estimate the integration over $(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, we need to bound $(1-\dot{\xi}_i)^{-(k-1)}$. Recall from Step 2 that $\dot{\xi}_i > 0$ in $(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$. Let $X_i = e^{2\xi_i}(1-\dot{\xi}_i^2) > 0$, which is, up to a harmless multiplicative constant, the repeated eigenvalue of the Schouten tensor of g_{v_i}. Note that (2.4) can be recast as

$$X_i^{k-1}e^{2\xi_i}\ddot{\xi}_i + \frac{n-2k}{2k}X_i^k = 2^{k-1}\binom{n-1}{k-1}^{-1}K_{cyl}.$$

Thus, there exists a small $\chi_0 > 0$ depending only on n, k and a positive lower bound for K such that $\ddot{\xi}_i(t) \geq 0$ whenever $X_i(t) < \chi_0$. As $\dot{X}_i = -2\dot{\xi}_i(e^{2\xi_i}\ddot{\xi}_i + X_i)$ and $\dot{\xi}_i > 0$ in

$(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, this implies that $\dot{X}_i(t) \le 0$ whenever $X_i(t) < \chi_0$ for $t \in (\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$. On the other hand, since $X_i(\ln \delta_i) = e^{O(1)} \lambda_i^{\frac{2\beta_2}{n-2k}} > \chi_0$ (in view of (4.6) and $\dot{\xi}_i(\ln \delta_i) = 0$), we deduce that X_i is nowhere less than χ_0 in $(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, i.e.

$$X_i \ge \chi_0 \text{ in } (\ln \frac{R_i}{\lambda_i}, \ln \delta_i).$$

It follows that $1 - \dot{\xi}_i = \frac{X_i e^{-2\xi_i}}{1+\dot{\xi}_i} \ge \frac{\chi_0}{2} e^{-2\xi_i}$ in $(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, and so, in view of (4.8),

$$\begin{aligned}
&\int_{\ln \frac{R_i}{\lambda_i}}^{\ln \delta_i} K_{cyl}(\tau)(1 - \dot{\xi}_i)^{-(k-1)} e^{-\frac{n+2k}{2}\xi_i} e^{\frac{n-2k}{2}\tau}\, d\tau \\
&\qquad \le C \int_{\ln \frac{R_i}{\lambda_i}}^{\ln \delta_i} e^{-\frac{n-2k+4}{2}\xi_i} e^{\frac{n-2k}{2}\tau}\, d\tau \le C R_i^{-2} \lambda_i^{-\frac{n-2k}{2}}. \qquad (4.15)
\end{aligned}$$

Putting (4.14) and (4.15) into (4.13) we obtain (4.7), which concludes Step 2.

Step 3: We draw a contradiction.

By (4.2), we may assume without loss of generality that $v_i(S) \to \infty$. By Step 2 and point (i) of the hypotheses, we have that $\beta_2 \ge \frac{n-2k}{2}$. We consider the cases $\beta_2 \ge n - 2k$ and $\frac{n-2k}{2} \le \beta_2 < n - 2k$ separately.

Case (a): $\beta_2 \ge n - 2k$. We will show that $a_1 < 0$, $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$ and that (1.3) is violated, which amounts to a contradiction to our hypotheses.

We first prove that $v_i(N) \to \infty$. Indeed, by (4.9), the oscillation of $\xi_i(t) - t$ in $[0, \infty)$ tends to infinity as $i \to \infty$. This gives $\mathrm{osc}_{\bar{\mathbb{S}}^n_+} \ln v_i \to \infty$. Now, if $v_i(N)$ was bounded, we would have by (4.1) that v_i is uniformly bounded away from the south pole, and hence, by the Harnack estimate, $|\nabla \ln v_i| \le C$ on $\bar{\mathbb{S}}^n_+$, which is a contradiction to the above estimate on the oscillation of $\ln v_i$.

The rough idea of the proof is as follows: Let $\lambda_i = 2^{-\frac{k}{2}+1} \binom{n}{k}^{-1/2} K(S)^{\frac{1}{2}} v_i(S)^{\frac{2}{n-2}} \to \infty$ and $\tilde{\lambda}_i = 2^{-\frac{k}{2}+1} \binom{n}{k}^{-1/2} K(N)^{\frac{1}{2}} v_i(N)^{\frac{2}{n-2}} \to \infty$. We apply Step 2 to both the north and the south poles to obtain that ξ_i has exactly three critical points, is decreasing in $(-\infty, -\ln \lambda_i + o(1))$, increasing in $(-\ln \lambda_i + o(1), \ln \delta_i)$, decreasing in $(\ln \delta_i, \ln \tilde{\lambda}_i + o(1))$ and increasing in $(\ln \tilde{\lambda}_i + o(1), \infty)$, and that $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$. We then show that the 4-vector $V_i = (\ln \lambda_i, \ln \tilde{\lambda}_i, \xi_i(\ln \delta_i), \ln \delta_i)^T$ satisfies a linear equation of the form $MV_i = P + o(1)$ where the 4×4-matrix M and the 4-vector P are independent of i. It follows that P is orthogonal to the kernel of M^T, which gives $C_{(1)} C_{(2)} = 1$ where $C_{(1)} = C_{n,k}(\beta_1, a_1, K(N))$ and $C_{(2)} = C_{n,k}(\beta_2, a_2, K(S))$.

Let us now give the details. Applying Step 2 to S, we have $a_2 < 0$ and there exist δ_i and ν_i satisfying (4.4) and (4.5) such that ξ_i is strictly increasing in $(\ln \frac{R_i}{\lambda_i}, \ln \delta_i)$, is strictly decreasing in $(\ln\delta_i, \ln\nu_i)$, has a strict local maximum at $\ln\delta_i$, and (4.6)–(4.9) hold. Applying Step 2 to N, we have that $a_1 < 0$, $\beta_1 \geq \frac{n-2k}{2}$, and there exist

$$\tilde{\delta}_i = e^{O(1)}\tilde{\lambda}_i^{-(1-\frac{\beta_1}{n-2k})}, \tag{4.16}$$

$$\tilde{\nu}_i = e^{O(1)}\tilde{\lambda}_i^{-(1-\frac{2\beta_1}{n-2k})} \tag{4.17}$$

such that ξ_i is strictly decreasing in $(-\ln\tilde{\delta}_i, -\ln\frac{R_i}{\tilde{\lambda}_i})$, strictly increasing in $(-\ln\tilde{\nu}_i, -\ln\tilde{\delta}_i)$, has a strict local maximum at $-\ln\tilde{\delta}_i$,

$$\xi_i(-\ln\tilde{\delta}_i) = \frac{\beta_1}{n-2k}\ln\tilde{\lambda}_i + p_1 + o(1), \tag{4.18}$$

$$\xi_i(-\ln\tilde{\delta}_i) = \ln\tilde{\lambda}_i + \ln\tilde{\delta}_i + q_1 + o(1), \tag{4.19}$$

$$\xi_i(t) = \ln\tilde{\lambda}_i - t + O(1) \text{ in } (-\ln\tilde{\delta}_i, \ln\frac{\tilde{\lambda}_i}{2}), \tag{4.20}$$

$$\xi_i(t) = -(1-\frac{2\beta_1}{n-2k})\ln\tilde{\lambda}_i + t + O(1) \text{ in } (-\ln\tilde{\nu}_i, -\ln\tilde{\delta}_i), \tag{4.21}$$

where

$$p_1 := -\frac{1}{n-2k}\ln\Big[2^{\beta_1+\frac{n+2k}{2}}\binom{n}{k}^{\frac{n-2k}{2k}}\frac{\Gamma(\frac{n-\beta_1}{2})\Gamma(\frac{n+\beta_1}{2})}{2\Gamma(n)}|a_1|\beta_1 K(N)^{-\frac{n}{2k}}\Big],$$

$$q_1 := -\ln\Big[2^{\frac{n+2k}{2(n-2k)}}\binom{n}{k}^{\frac{1}{2k}}K(N)^{-\frac{1}{2k}}\Big].$$

Comparing the value of $\xi_i(0)$ from (4.8) and (4.21), we have

$$\lambda_i = e^{O(1)}\tilde{\lambda}_i^{-1+\frac{2\beta_1}{n-2k}}. \tag{4.22}$$

This implies that $\beta_1 > \frac{n-2k}{2}$.

Note that, by (4.10) and the definition of δ_i and ν_i, ξ_i is strictly decreasing in $(-\infty, \ln\frac{1+o(1)}{\lambda_i})$, strictly increasing in $(\ln\frac{1+o(1)}{\lambda_i}, \ln\delta_i)$, strictly decreasing in $(\ln\delta_i, \ln\nu_i)$, and has exactly two critical points in $(-\infty, \ln\nu_i)$ at $\ln\frac{1+o(1)}{\lambda_i}$ and $\ln\delta_i$. Now, since $\beta_2 \geq n-2k$ and $\beta_1 > \frac{n-2k}{2}$, we have by (4.4) and (4.17) that

$$-\ln\tilde{\nu}_i = -(1-\frac{2\beta_1}{n-2k})\ln\tilde{\lambda}_i \ll O(1) \leq \ln\delta_i = -(1-\frac{\beta_2}{n-2k})\ln\lambda_i + O(1).$$

Since ξ_i is strictly decreasing in $(-\ln\tilde{\nu}_i, -\ln\tilde{\delta}_i)$ and $\dot{\xi}_i(-\ln\tilde{\delta}_i) = \dot{\xi}_i(\ln\delta_i) = 0$, we have that $\ln\delta_i = -\ln\tilde{\delta}_i$, which implies (in view of (4.4) and (4.16))

$$\lambda_i^{-(1-\frac{\beta_2}{n-2k})} = e^{O(1)}\tilde{\lambda}_i^{(1-\frac{\beta_1}{n-2k})}. \tag{4.23}$$

Substituting (4.22) into (4.23), we obtain that

$$\frac{1}{\beta_1}+\frac{1}{\beta_2}=\frac{2}{n-2k}. \tag{4.24}$$

Now, let $V_i=(\ln\lambda_i,\ln\tilde{\lambda}_i,\xi_i(\ln\delta_i),\ln\delta_i)^T$ and observe that (4.6), (4.7), (4.18) and (4.19) give a linear system of the form

$$MV_i=P+o(1),\ \text{where } M=\begin{pmatrix}-\frac{\beta_2}{n-2k} & 0 & 1 & 0\\ -1 & 0 & 1 & -1\\ 0 & -\frac{\beta_1}{n-2k} & 1 & 0\\ 0 & -1 & 1 & 1\end{pmatrix}\ \text{and } P=\begin{pmatrix}p_2\\ q_2\\ p_1\\ q_1\end{pmatrix}.$$

A straightforward computation gives that $\det M=\frac{\beta_1\beta_2}{n-2k}\Big(-\frac{2}{n-2k}+\frac{1}{\beta_1}+\frac{1}{\beta_2}\Big)=0$, and the kernel of M^T is generated by $W_0:=(\frac{n-2k}{\beta_2},-1,\frac{n-2k}{\beta_1},-1)^T$. The fact that $MV_i=P+o(1)$ implies that $P\cdot W_0=0$, i.e.

$$\frac{n-2k}{\beta_2}p_2-q_2+\frac{n-2k}{\beta_1}p_1-q_1=0.$$

Recalling the expression of p_1,p_2,q_1,q_2, we see that this is equivalent to $C_{(1)}C_{(2)}=1$. However, since (4.24) holds and $a_1,a_2<0$, we have by our hypotheses that (1.3) holds, which is contradiction to the above identity. This finishes the proof when $\beta_2\ge n-2k$.

Case (b): $\frac{n-2k}{2}\le\beta_2<n-2k$.

Take a point p on the equator of $\mathbb{S}^n$. Recall that $v_i(S)\to\infty$. By Step 2, we know that $v_i(p)\to 0$. Let $\check{v}_i=\frac{1}{v_i(p)}v_i$. By the first and second derivatives estimates (4.3), after passing to a subsequence if necessary, we may assume that $\check{v}_i$ converges in $C^{1,\alpha}_{loc}(\mathbb{S}^n\setminus\{S,N\})$ to some positive function $\check{v}_\infty\in C^{1,1}_{loc}(\mathbb{S}^n\setminus\{S,N\})$ which satisfies

$$\lambda(A_{g_{\check{v}_\infty}})\in\partial\Gamma_k\ \text{in } \mathbb{S}^n\setminus\{S,N\} \tag{4.25}$$

in the viscosity sense. Note that as $\frac{n-2k}{2}\le\beta_2<n-2k$, we have in Step 2 that $\delta_i\to 0$ and $\nu_i\ge\frac{1}{C}$. Hence, by estimate (4.9) in Step 2, there exists $r_i=O(\delta_i)\to 0$ such that $\frac{1}{C}\le\check{v}_i\le C$ in $\{x: r_i\le d_{\mathring{g}}(x,S)\le\pi/2\}$. It follows that

$$\frac{1}{C}\le\check{v}_\infty\le C\ \text{near } S. \tag{4.26}$$

We proceed according to whether $v_i(N)$ is bounded or not. Suppose first that $v_i(N)$ is bounded. Then $\sup_{\mathbb{S}^n_+}v_i$ is also bounded (see (4.1) and the sentence following it). The estimates in Step 1 are thus improved to

$$v_i(x)d_{\mathring{g}}(x,\{S\})^{\frac{n-2}{2}}\le C\ \text{and } |\nabla^\ell\ln v_i(x)|d_{\mathring{g}}(x,\{S\})^\ell\le C\ \text{for } \ell=1,2.$$

It follows that the function $\check{v}_\infty$ satisfies

$$\lambda(A_{g_{\check{v}_\infty}}) \in \partial\Gamma_k \text{ in } \mathbb{S}^n \setminus \{S\}.$$

In view of the Liouville-type theorem [19, Theorem 1.3], this is impossible: No such $\check{v}_\infty$ can satisfy (4.26).

Finally, consider the case that N is a blow-up point. In view of Case (a) above, by exchanging the role of the north pole and the south pole, we may assume that $\frac{n-2k}{2} \le \beta_1 < n - 2k$. The proof of (4.26) also applies near N giving that

$$\frac{1}{C} \le \check{v}_\infty \le C \text{ in } \mathbb{S}^n \setminus \{S, N\}.$$

By the classification result [21, Theorem 1.6], no axisymmetric solution $\check{v}_\infty$ to (4.25) satisfies the above inequality. This finishes the proof of Theorem 1.1. $\square$

The following remark is easily seen from the above proof:

Remark 4.1. If $\max(a_1, a_2) > 0$, the constant C_1 in Theorem 1.1 depends only on an upper bound of $|a_1|, |a_2|$, $|a_1|^{-1}, |a_2|^{-1}$, $(n-\beta_1)^{-1}, (n-\beta_2)^{-1}$, $\|\ln K\|_{C_r^{2,\alpha}(\mathbb{S}^n)}$, and a non-negative function $\phi : [0, \pi/2) \to [0, \infty)$ such that $\phi(\theta) \to 0$ as $\theta \to 0$ and

$$\frac{|R_1(\theta)| + |\theta||R_1'(\theta)|}{|\theta|^{\beta_1}} \le \phi(\theta) \text{ and } \frac{|R_2(\theta)| + |\pi - \theta||R_2'(\theta)|}{|\pi - \theta|^{\beta_2}} \le \phi(\pi - \theta).$$

If $\frac{1}{\beta_1} + \frac{1}{\beta_2} \neq \frac{2}{n-2k}$ and $a_1, a_2 < 0$, the constant C_1 depends only on an upper bound of $|a_1|, |a_2|, |a_1|^{-1}, |a_2|^{-1}$, $(n-\beta_1)^{-1}, (n-\beta_2)^{-1}$, $\|\ln K\|_{C_r^{2,\alpha}(\mathbb{S}^n)}$, $|\frac{1}{\beta_1} + \frac{1}{\beta_2} - \frac{2}{n-2k}|^{-1}$, and a function ϕ as above.

If $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$ and $a_1, a_2 < 0$, the constant C_1 depends only on an upper bound of $|a_1|, |a_2|$, $|a_1|^{-1}, |a_2|^{-1}$, $(n-\beta_1)^{-1}, (n-\beta_2)^{-1}$, $\|\ln K\|_{C_r^{2,\alpha}(\mathbb{S}^n)}$, $|C_{(1)}C_{(2)} - 1|^{-1}$, and a function ϕ as above.

4.2. Some extensions of Theorem 1.1

In many situations, we will often consider (1.1) in a family of equations of the form

$$\sigma_k(\lambda(A_{g_v})) = K_\mu \text{ and } \lambda(A_{g_v}) \in \Gamma_k \text{ on } \mathbb{S}^n \tag{4.27}$$

where K_μ depends on a certain parameter μ in some index set I. Analogous to (1.2), we will assume that there exist $a_{1,\mu}, a_{2\mu} \neq 0$ and $2 \le \beta_{1,\mu}, \beta_{2,\mu} < n$ such that if we write

$$K_\mu(\theta) = K_\mu(0) + a_{1,\mu}\theta^{\beta_{1,\mu}} + R_{1,\mu}(\theta) = K_\mu(\pi) + a_{2,\mu}(\pi - \theta)^{\beta_{2,\mu}} + R_{2,\mu}(\theta)$$

then

$$\limsup_{\theta\to 0}\sup_{\mu\in I}\frac{|R_{1,\mu}(\theta)|+|\theta||R'_{1,\mu}(\theta)|}{|\theta|^{\beta_{1,\mu}}}=\limsup_{\theta\to \pi}\sup_{\mu\in I}\frac{|R_{2,\mu}(\theta)|+|\pi-\theta||R'_{2,\mu}(\theta)|}{|\pi-\theta|^{\beta_{2,\mu}}}=0. \tag{4.28}$$

Remark 4.2. It is not hard to see from the proof of Theorem 1.1 that if each K_μ satisfies the hypotheses of Theorem 1.1, $\frac{1}{C}\le |a_{1,\mu}|,|a_{2,\mu}|\le C$, $|n-\beta_{1,\mu}|,|n-\beta_{2,\mu}|\ge \frac{1}{C}$, $|\frac{1}{\beta_{1,\mu}}+\frac{1}{\beta_{2,\mu}}-\frac{2}{n-2k}|\ge\frac{1}{C}$, and $\|K_\mu\|_{C^{2,\alpha}_r(\mathbb{S}^n)}\le C$ for some constant C, then there exists a constant $C_1>0$ such that all positive solutions to (4.27) with $\mu\in I$ satisfy

$$\|\ln v\|_{C^{4,\alpha}(\mathbb{S}^n)}<C_1.$$

Furthermore, if $\max(a_{1,\mu},a_{2,\mu})\ge\frac{1}{C}$, the assumption that $|\frac{1}{\beta_{1,\mu}}+\frac{1}{\beta_{2,\mu}}-\frac{2}{n-2k}|\ge\frac{1}{C}$ can be dropped.

To prove Theorem 1.2 later on, we embed K in a family $\{K_\mu\}$ in two specific ways for which Remarks 4.1 and 4.2 do not apply. Let us now show how the proof of Theorem 1.1 can be adapted to cover those situations.

Theorem 4.3. *Assume that $n\ge 5$, $2\le k<n/2$, $0<\alpha<1$, $K\in C^{2,\alpha}_r(\mathbb{S}^n)$ is positive and satisfies (1.2) for some $a_1,a_2\ne 0$ and $2\le\beta_1,\beta_2<n$. Assume further that $a_i>0$ if $\beta_i<\frac{n-2k}{2}$ for some i, and $\max(a_1,a_2)>0$ if $\frac{1}{\beta_1}+\frac{1}{\beta_2}\ge\frac{2}{n-2k}$. For $\mu\in(0,1]$, let $K_\mu=\mu K+(1-\mu)2^{-k}\binom{n}{k}$. Then there exists some positive constant C_1 such that all $C^2_r(\mathbb{S}^n)$ positive solutions to (4.27) with $0<\mu\le 1$ satisfy*

$$\|\ln v\|_{C^{4,\alpha}(\mathbb{S}^n)}<C_1.$$

Proof. The proof is almost identical to that of Theorem 1.1. We will only indicate the necessary changes. We suppose by contradiction that there exists $\mu_i\in(0,1]$ and positive functions $v_i\in C^2_r(\mathbb{S}^n)$ satisfying (4.27) with $\mu=\mu_i$ such that

$$\max\{v_i(N),v_i(S)\}\to\infty.$$

There is no change to Step 1.

Step 2 is modified as follows: One shows that if $v_i(S)\to\infty$, then $a_2<0$, $\beta_2\ge\frac{n-2k}{2}$ and, for large i, there exist $\delta_i=e^{O(1)}\mu_i^{-\frac{1}{n-2k}}\lambda_i^{-(1-\frac{\beta_2}{n-2k})}$ and $\nu_i=e^{O(1)}\mu_i^{-\frac{2}{n-2k}}\lambda_i^{-(1-\frac{2\beta_2}{n-2k})}$ such that ξ_i is strictly increasing in $(\ln\frac{R_i}{\lambda_i},\ln\delta_i)$, is strictly decreasing in $(\ln\delta_i,\ln\nu_i)$, has a strict local maximum at $\ln\delta_i$, and

$$\xi_i(\ln\delta_i)=\frac{\beta_2}{n-2k}\ln\lambda_i-\frac{1}{n-2k}\ln\mu_i+p_2+o(1),$$

$$\xi_i(t)=\ln\lambda_i+t+O(1)\text{ in }(\ln\frac{2}{\lambda_i},\ln\delta_i),$$

$$\xi_i(t)=-\frac{2}{n-2k}\ln\mu_i-(1-\frac{2\beta_2}{n-2k})\ln\lambda_i-t+O(1)\text{ in }(\ln\delta_i,\ln\nu_i).$$

The appearance of μ_i in the above is due to the fact that, in the present case, one needs to include a multiplicative factor of μ_i on the right hand side of (4.12).

Step 3 is modified as follows. If $\frac{n-2k}{2} \leq \beta_2 < n-2k$, the proof remains unchanged. If $\beta_2 \geq n-2k$, one still has that $v_i(N) \to \infty$ and $v_i(S) \to 0$, $a_1, a_2 < 0$, $\beta_1 \geq \frac{n-2k}{2}$, $\delta_i = \tilde{\delta}_i^{-1}$ and $\lambda_i = e^{O(1)}\tilde{\lambda}_i^{-(1-\frac{2\beta_1}{n-2k})}$. Recalling that $\delta_i = e^{O(1)}\mu_i^{-\frac{1}{n-2k}}\lambda_i^{-(1-\frac{\beta_2}{n-2k})}$ and $\tilde{\delta}_i = e^{O(1)}\mu_i^{-\frac{1}{n-2k}}\tilde{\lambda}_i^{-(1-\frac{\beta_1}{n-2k})}$, one obtains that

$$\tilde{\lambda}_i^{-\frac{\beta_1+\beta_2}{n-2k}+\frac{2\beta_1\beta_2}{n-2k}} = e^{O(1)}\mu_i^{\frac{2}{n-2}} \leq C.$$

This implies $\frac{1}{\beta_1} + \frac{1}{\beta_2} \geq \frac{2}{n-2k}$. By our hypotheses on the signs of a_1 and a_2, we thus have $\max(a_1, a_2) > 0$, contradicting the earlier conclusion that a_1 and a_2 are both negative. $\square$

Theorem 4.4. *Assume that $n \geq 5$, $2 \leq k < n/2$, $0 < \alpha < 1$, $0 < \varepsilon_0 < 1$, $\{K_\mu\}$ is a bounded sequence of positive functions in $C_r^{2,\alpha}(\mathbb{S}^n)$ which satisfies (4.28) for some $-\varepsilon_0^{-1} < a_{1,\mu}, a_{2,\mu} < -\varepsilon_0 < 0$ and $\frac{n-2k}{2} \leq \beta_{1,\mu}, \beta_{2,\mu} \leq n - \varepsilon_0$, $\frac{1}{\beta_{1,\mu}} + \frac{1}{\beta_{2,\mu}} \to \frac{2}{n-2k}$. Let $C_{(1),\mu} = C_{n,k}(\beta_{1,\mu}, a_{1,\mu}, K_\mu(0))$ and $C_{(2),\mu} = C_{n,k}(\beta_{2,\mu}, a_{2,\mu}, K_\mu(\pi))$ and assume further that either*

(i) $\frac{1}{\beta_{1,\mu}} + \frac{1}{\beta_{2,\mu}} \geq \frac{2}{n-2k}$ *and* $C_{(1),\mu}C_{(2),\mu} < 1 - \varepsilon_0$,

or

(ii) $\frac{1}{\beta_{1,\mu}} + \frac{1}{\beta_{2,\mu}} \leq \frac{2}{n-2k}$ *and* $C_{(1),\mu}C_{(2),\mu} > 1 + \varepsilon_0$.

Then there exists a constant $C_1 > 0$ such that all $C_r^2(\mathbb{S}^n)$ positive solutions to (1.1) satisfy

$$\|\ln v\|_{C^{4,\alpha}(\mathbb{S}^n)} < C_1.$$

Proof. We amend the proof of Theorem 1.1, and we will indicate only the necessary changes. We suppose by contradiction that the conclusion fails. By passing to a subsequence, we may assume that there exist $\mu_i \to \infty$ and positive functions $v_i \in C_r^2(\mathbb{S}^n)$ satisfying (4.27) with $\mu = \mu_i$ such that

$$\max\{v_i(N), v_i(S)\} \to \infty.$$

Let $\gamma_i = \frac{n-2k}{\beta_{1,\mu_i}} + \frac{n-2k}{\beta_{2,\mu_i}} - 2$. Passing again to a subsequence, we may further assume that $\gamma_i \geq 0$ for all i or $\gamma_i \leq 0$ for all i.

Steps 1 and 2 remain unchanged. In Step 3, we again have that both the north and the south poles are blow-up points, $\delta_i = \tilde{\delta}_i^{-1}$, and

$$\xi_i(\ln\delta_i) = \frac{\beta_{2,\mu_i}}{n-2k}\ln\lambda_i + p_{2,\mu_i} + o(1), \tag{4.29}$$

$$\xi_i(\ln\delta_i) = \ln\lambda_i + \ln\delta_i + q_{2,\mu} + o(1), \tag{4.30}$$

$$\xi_i(-\ln\tilde{\delta}_i) = \frac{\beta_{1,\mu_i}}{n-2k}\ln\tilde{\lambda}_i + p_{1,\mu_i} + o(1), \tag{4.31}$$

$$\xi_i(-\ln\tilde{\delta}_i) = \ln\tilde{\lambda}_i + \ln\tilde{\delta}_i + q_{1,\mu_i} + o(1), \tag{4.32}$$

where

$$p_{2,\mu} := -\frac{1}{n-2k}\ln\Big[2^{\beta_{2,\mu}+\frac{n+2k}{2}}\binom{n}{k}^{\frac{n-2k}{2k}}\frac{\Gamma(\frac{n-\beta_{2,\mu}}{2})\Gamma(\frac{n+\beta_{2,\mu}}{2})}{2\Gamma(n)}|a_{2,\mu}|\beta_{2,\mu}K_\mu(S)^{-\frac{n}{2k}}\Big],$$

$$q_{2,\mu} := -\ln\Big[2^{\frac{n+2k}{2(n-2k)}}\binom{n}{k}^{\frac{1}{2k}}K_\mu(S)^{-\frac{1}{2k}}\Big],$$

$$p_{1,\mu} := -\frac{1}{n-2k}\ln\Big[2^{\beta_{1,\mu}+\frac{n+2k}{2}}\binom{n}{k}^{\frac{n-2k}{2k}}\frac{\Gamma(\frac{n-\beta_{1,\mu}}{2})\Gamma(\frac{n+\beta_{1,\mu}}{2})}{2\Gamma(n)}|a_{1,\mu}|\beta_{1,\mu}K_\mu(N)^{-\frac{n}{2k}}\Big],$$

$$q_{1,\mu} := -\ln\Big[2^{\frac{n+2k}{2(n-2k)}}\binom{n}{k}^{\frac{1}{2k}}K_\mu(N)^{-\frac{1}{2k}}\Big].$$

Now, adding (4.30) and (4.32) gives

$$2\xi_i(\ln\delta_i) - \ln\lambda_i - \ln\tilde{\lambda}_i = q_{1,\mu_i} + q_{2,\mu_i} + o(1).$$

Multiplying (4.29) by $\frac{n-2k}{\beta_{2,\mu_i}}$ and (4.31) by $\frac{n-2k}{\beta_{1,\mu_i}}$ and adding the resulting identities together give

$$(2+\gamma_i)\xi_i(\ln\delta_i) - \ln\lambda_i - \ln\tilde{\lambda}_i = \frac{n-2k}{\beta_{1,\mu_i}}p_{1,\mu_i} + \frac{n-2k}{\beta_{2,\mu_i}}p_{2,\mu_i} + o(1).$$

Recalling that $\xi_i(\ln\delta_i) = \frac{\beta_{2,\mu_i}}{n-2k}\ln\lambda_i \to \infty$, we thus have in the case of non-negative γ_i's that

$$0 \le \liminf_{i\to\infty}\Big[\frac{n-2k}{\beta_{1,\mu_i}}p_{1,\mu_i} + \frac{n-2k}{\beta_{2,\mu_i}}p_{2,\mu_i} - q_{1,\mu_i} - q_{2,\mu_i}\Big] = \liminf_{i\to\infty}\ln[C_{(1),\mu_i}C_{(2),\mu_i}],$$

and in the case of non-positive γ_i's that

$$0 \ge \limsup_{i\to\infty}\Big[\frac{n-2k}{\beta_{1,\mu_i}}p_{1,\mu_i} + \frac{n-2k}{\beta_{2,\mu_i}}p_{2,\mu_i} - q_{1,\mu_i} - q_{2,\mu_i}\Big] = \limsup_{i\to\infty}\ln[C_{(1),\mu_i}C_{(2),\mu_i}].$$

These contradict our hypotheses. $\square$

4.3. Second proof of Theorem 1.1 in the case $\frac{1}{\beta_1}+\frac{1}{\beta_2}=\frac{2}{n-2k}$

In this subsection, we give an alternative proof of Theorem 1.1 in the case $1/\beta_1 + 1/\beta_2 = 2/(n-2k)$ and a_1, $a_2 < 0$.

By the assumptions on β_1 and β_2,

$$(n(n-2k))/(n+2k) < \min\{\beta_1, \beta_2\} \le n-2k \le \max\{\beta_1, \beta_2\} < n.$$

By the first and second derivative estimates for the σ_k-Yamabe equation, it suffices to show that

$$v \le C_1 \text{ for all positive } C_r^2 \text{ solutions } v \text{ of (1.1).}$$

Suppose by contradiction that there exist positive functions $v_i \in C_r^2(\mathbb{S}^n)$ satisfying (1.1) such that $\max v_i \to \infty$. Let N and S denote respectively the north and south poles of $\mathbb{S}^n$.

Throughout the proof, C denotes some generic positive constant which may change from one line to another but will remain independent of i,

Step 1: Making the same argument as in the beginning of the proof of Theorem 1.1, we can conclude that (4.2)-(4.3) still hold.

Step 2: We show that

$$\min\{v_i(S), v_i(N)\} \to \infty. \tag{4.33}$$

This follows from (4.2) and the following lemma (which does not use $1/\beta_1 + 1/\beta_2 = 2/(n-2k)$).

Lemma 4.5. *Assume that $n \ge 5$, $2 \le k < n/2$, $0 < \alpha < 1$, $K \in C_r^{2,\alpha}(\mathbb{S}^n)$ is positive and satisfies (1.2) for some $a_1, a_2 < 0$ and $(n(n-2k))/(n+2k) < \beta_1, \beta_2 < n$. Assume that $\{v_i\} \subset C_r^2(\mathbb{S}^n)$ is a sequence of positive solutions of (1.1) satisfying (4.2). Then we have (4.33).*

Proof. Assume $v_i(S) \to \infty$. Let u_i be related to v_i as in (2.1). By Theorem 3.1(a), for every $\varepsilon_i \to 0^+$ and every $R_i \to \infty$,

$$|u_i(0)^{-1}u_i(r) - (1+\lambda_i^2 r^2)^{\frac{2-n}{2}}| \le \varepsilon_i \quad \text{in} \quad \{0 \le r \le r_i := \lambda_i^{-1}R_i\}, \tag{4.34}$$

where $\lambda_i = 2^{-\frac{1}{2}}\binom{n}{k}^{-\frac{1}{2k}}K(S)^{\frac{1}{2k}}u_i(0)^{\frac{2}{n-2}}$. In particular, we can choose R_i such that $R_i u_i(0)^{-\frac{\beta_2}{n(n-2)}} \to 0^+$ and $\varepsilon_i R_i^{n-2} \to 0^+$. By Theorem 3.1 (a)-(c), we have that

$$u_i(r) = e^{O(1)}u_i(0)^{-1}r^{2-n} \quad \text{in} \quad \{r_i \le r \le \bar{r}_i\}, \tag{4.35}$$

and

$$u_i(r) = e^{O(1)} u_i(0)^{-\min\{\frac{2\beta_2}{n-2k}-1,1\}} \quad \text{in } \{\bar{r}_i \le r \le 1\}, \tag{4.36}$$

where $\bar{r}_i = e^{O(1)} u_i(0)^{-\frac{2}{n-2}\max\{1-\frac{\beta_2}{n-2k},0\}}$.

We will prove by contradiction that $v_i(N) \to \infty$. Suppose the contrary, then by (4.3) and (4.36), for any $0 < \varepsilon < 1$, we have for large i that

$$u_i(r) \le C u_i(0)^{-\min\{\frac{2\beta_2}{n-2k}-1,1\}} r^{2-n}, \quad \forall\, r \ge \varepsilon. \tag{4.37}$$

On one hand, the Kazdan–Warner-type identity (see (2.11)) gives

$$\int_0^\infty r^n K'_{Euc}(r) u_i^{\frac{2n}{n-2}}\, dr = 0. \tag{4.38}$$

On the other hand, by (4.37), we have for large i that

$$\Big|\int_\varepsilon^\infty r^n K'_{Euc}(r) u_i^{\frac{2n}{n-2}}\, dr\Big| \le C \int_\varepsilon^\infty r^n u_i^{\frac{2n}{n-2}}\, dr \le C(\varepsilon) u_i(0)^{-\frac{2n}{n-2}\min\{\frac{2\beta_2}{n-2k}-1,1\}}.$$

For some $\varepsilon > 0$ sufficiently small so that $K'_{Euc} < 0$ in $(0,\varepsilon)$ (see (1.2)), we deduce from (4.34) that

$$\begin{aligned} -\int_0^\varepsilon r^n K'_{Euc}(r) u_i^{\frac{2n}{n-2}}\, dr &\ge -\int_0^{r_i} r^n K'_{Euc}(r) u_i^{\frac{2n}{n-2}}\, dr \\ &\ge C \int_0^{r_i} r^{n+\beta_2-1} u_i^{\frac{2n}{n-2}}\, dr = C u_i(0)^{-\frac{2\beta_2}{n-2}}. \end{aligned}$$

Multiplying the above two inequalities by $u_i(0)^{\frac{2\beta_2}{n-2}}$, letting $i \to \infty$ and using the fact $n(n-2k)/(n+2k) < \beta_2 < n$, we have

$$\liminf_{i\to\infty} u_i(0)^{\frac{2\beta_2}{n-2}} \Big(-\int_0^\infty r^n K'_{Euc}(r) u_i^{\frac{2n}{n-2}}\, dr \Big) \ge C > 0,$$

which is a contradiction with (4.38). $\square$

Step 3: We show that, for a fixed $x_0 \in \mathbb{S}^n$ and $d_{\mathring{g}}(x_0, S) = \pi/2$,

$$v_i(x_0) v_i(S)^{\min\{\beta_1,\beta_2\}/\beta_1} = e^{O(1)} \tag{4.39}$$

and

$$v_i(x_0)v_i(N)^{\min\{\beta_1,\beta_2\}/\beta_2} = e^{O(1)}. \tag{4.40}$$

The above two estimates follow from Theorem 3.1 (a)-(c) and the facts $1/\beta_1 + 1/\beta_2 = 2/(n-2k)$, a_1, $a_2 < 0$.

Step 4: By (4.33) and (4.39), we have that, $v_i(x_0) = e^{O(1)} v_i(S)^{-\frac{\min\{\beta_1,\beta_2\}}{\beta_1}} \to 0$. By (4.3), after passing to a subsequence if necessary, $v_i(x_0)^{-1}v_i(x)$ converges in $C^{1,\alpha}_{loc}(\mathbb{S}^n\backslash\{N,S\})$ to some positive axisymmetric function $G \in C^{1,1}_{loc}(\mathbb{S}^n\backslash\{N,S\})$ which satisfies

$$\lambda(A_{g_G}) \in \partial\Gamma_k \quad \text{in } \mathbb{S}^n\backslash\{N,S\}.$$

By the classification result [21, Theorem 1.6], we have that $c_1 := \lim_{x\to S} d_{\mathring{g}}(x,S)^{n-2}G(x) \in [0,\infty)$, $c_2 := \lim_{x\to N} d_{\mathring{g}}(x,N)^{n-2}G(x) \in [0,\infty)$, $\max\{c_1,c_2\} > 0$, and in the stereographic projection coordinates as at the beginning of Section 2,

$$G(x) = 2^{2-n}(1+r^2)^{\frac{n-2}{2}}\Big(c_1^{\frac{n-2k}{k(n-2)}} r^{-\frac{n-2k}{k}} + c_2^{\frac{n-2k}{k(n-2)}}\Big)^{\frac{k(n-2)}{n-2k}}. \tag{4.41}$$

By (4.39) and (4.40) and after passing to a subsequence if necessary, we have that $v_i(x_0)v_i(S)^{\min\{\beta_1,\beta_2\}/\beta_1}$ and $v_i(x_0)v_i(N)^{\min\{\beta_1,\beta_2\}/\beta_2}$ converge respectively to two positive constants c_3 and c_4. Therefore,

$$v_i(S)^{\min\{\beta_1,\beta_2\}/\beta_1}v_i(x) \to c_3G(x), \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{S}^n\backslash\{N,S\}), \tag{4.42}$$

and

$$v_i(N)^{\min\{\beta_1,\beta_2\}/\beta_2}v_i(x) \to c_4G(x), \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{S}^n\backslash\{N,S\}).$$

Next we show that

$$c_1c_3 = \begin{cases} \Big(\frac{K(S)}{2^k\binom{n}{k}}\Big)^{-\frac{n-2}{2k}} & \text{if } \beta_2 \geq \beta_1, \\ 0 & \text{if } \beta_2 < \beta_1, \end{cases} \tag{4.43}$$

$$c_2c_3 = \begin{cases} 2^{n-2}\Big(-a_2\beta_2\frac{\Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2\Gamma(n)}\Big)^{\frac{n-2}{n-2k}}\Big(\frac{2^{k\beta_2}\binom{n}{k}^{\beta_2}}{K(S)^{2k+\beta_2}}\Big)^{\frac{n-2}{2k(n-2k)}} & \text{if } \beta_2 \leq \beta_1 \\ 0 & \text{if } \beta_2 > \beta_1, \end{cases} \tag{4.44}$$

$$c_2c_4 = \begin{cases} \Big(\frac{K(N)}{2^k\binom{n}{k}}\Big)^{-\frac{n-2}{2k}} & \text{if } \beta_2 \leq \beta_1, \\ 0 & \text{if } \beta_2 > \beta_1, \end{cases} \tag{4.45}$$

and

$$c_1c_4 = \begin{cases} 2^{n-2}\Big(-a_1\beta_1\frac{\Gamma(\frac{n-\beta_1}{2})\Gamma(\frac{n+\beta_1}{2})}{2\Gamma(n)}\Big)^{\frac{n-2}{n-2k}}\Big(\frac{2^{k\beta_1}\binom{n}{k}^{\beta_1}}{K(N)^{2k+\beta_1}}\Big)^{\frac{n-2}{2k(n-2k)}} & \text{if } \beta_2 \geq \beta_1, \\ 0 & \text{if } \beta_2 < \beta_1. \end{cases} \tag{4.46}$$

We will only need to prove (4.43) and (4.44), since (4.45) and (4.46) follow by switching the roles of S and N.

Let u_i be related to v_i as in (2.1) and let $w_i = u_i^{(n-2k)/k(n-2)}$. Fix a small $\sigma > 0$ such that $K'_{Euc}(r) < 0$ on $[0,\sigma]$. To prove (4.43) and (4.44), we first establish the following two identities.

$$\sigma^{\frac{n-k}{k}} w_i'(\sigma) = -\frac{n(n-2k)}{2k^2\binom{n}{k}^{1/k}}\int_0^\sigma K_{Euc}(r)^{1/k} r^{(n-k)/k} w_i^{\frac{n+2k}{n-2k}}(1-\rho_i(r))^{\frac{1-k}{k}}\,dr, \tag{4.47}$$

and

$$\frac{E(\sigma, w_i, w_i')}{\sigma^{(n-k)/k}w_i'(\sigma)} = -\frac{2k^2}{n(n-2k)}\frac{(K_{Euc}(\sigma)(1-\rho_i(\sigma)))^{1/k}}{\int_0^\sigma K_{Euc}(r)^{1/k}r^{(n-k)/k}w_i^{\frac{n+2k}{n-2k}}(1-\rho_i(r))^{\frac{1-k}{k}}\,dr}, \tag{4.48}$$

where $\rho_i(r) = K_{Euc}(r)^{-1}r^{-n}w_i(r)^{-\frac{2nk}{n-2k}}\int_0^r K'_{Euc}(s)s^n w_i(s)^{\frac{2nk}{n-2k}}\,ds$ and

$$E(r, w_i, w_i') = \frac{2k}{n-2k}w_i^{-\frac{2n}{n-2k}}\Big[-\frac{w_iw_i'}{r} - \frac{k}{n-2k}(w_i')^2\Big].$$

Equations (2.2) and (2.11) can be rewritten in terms of w_i as

$$\begin{cases} w_i'' + \frac{n-k}{k}\frac{w_i'}{r} = -\frac{n(n-2k)}{2k^2\binom{n}{k}}K_{Euc}(r)w_i^{\frac{n+2k}{n-2k}}E(r,w_i,w_i')^{1-k} & \text{in } [0,\infty), \\ E(r,w_i,w_i') > 0 \quad \text{in } [0,\infty), \end{cases} \tag{4.49}$$

and

$$E(r,w_i,w_i')^k = \frac{K_{Euc}(r)}{\binom{n}{k}}(1-\rho_i(r)), \tag{4.50}$$

respectively. Raising (4.50) to the power of $\frac{1-k}{k}$ and then inserting it into (4.49), we have

$$w_i'' + \frac{n-k}{k}\frac{w_i'}{r} = -\frac{n(n-2k)}{2k^2\binom{n}{k}^{1/k}}K_{Euc}(r)^{1/k}w_i^{\frac{n+2k}{n-2k}}(1-\rho_i(r))^{\frac{1-k}{k}}.$$

Multiplying the above identity by $r^{\frac{n-k}{k}}$ and then integrating it on $[0,\sigma]$ give (4.47). Raising (4.50) to the power of $1/k$, evaluating it at $r=\sigma$, and then dividing it by (4.47), we obtain (4.48).

Now we use identities (4.47) and (4.48) to obtain (4.43) and (4.44). By (4.42) and (4.41), we have

$$u_i(0)^{\min\{\beta_1,\beta_2\}/\beta_1} u_i(r) = (2^{\frac{n-2}{2}} v_i(S))^{\min\{\beta_1,\beta_2\}/\beta_1} ((\frac{2}{1+r^2})^{\frac{n-2}{2}} v_i(x))$$
$$= 2^{\frac{n-2}{2}(\frac{\min\{\beta_1,\beta_2\}}{\beta_1}-1)} c_3 (c_1^{\frac{n-2k}{k(n-2)}} r^{-\frac{n-2k}{k}} + c_2^{\frac{n-2k}{k(n-2)}})^{\frac{k(n-2)}{n-2k}} + o(1)$$

in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$. It follows that

$$w_i(0)^{\min\{\beta_1,\beta_2\}/\beta_1} \times \text{LHS of } (4.47) = -\frac{n-2k}{k} 2^{\frac{n-2k}{2k}(\frac{\min\{\beta_1,\beta_2\}}{\beta_1}-1)} (c_1 c_3)^{\frac{n-2k}{k(n-2)}} + o(1), \tag{4.51}$$

and

$$\frac{(n-2k)^2}{2k^2} w_i(0)^{\min\{\beta_1,\beta_2\}/\beta_1} w_i(\sigma)^{\frac{2n}{n-2k}} \sigma^{\frac{n}{k}} \times \text{LHS of } (4.48)$$
$$= -\frac{n-2k}{k} 2^{\frac{n-2k}{2k}(\frac{\min\{\beta_1,\beta_2\}}{\beta_1}-1)} (c_2 c_3)^{\frac{n-2k}{k(n-2)}} + o(1). \tag{4.52}$$

Before estimating the left hand sides of the above identities, we will first give the following estimates:

$$I_1 := \int_0^\sigma K_{Euc}(r)^{1/k} r^{(n-k)/k} w_i^{\frac{n+2k}{n-2k}} (1-\rho_i(r))^{\frac{1-k}{k}}\, dr$$
$$= (1+o(1)) 2^{\frac{n}{2k}} \binom{n}{k}^{\frac{n}{2k^2}} \frac{k}{n} K(S)^{-\frac{n-2k}{2k^2}} w_i(0)^{-1}, \tag{4.53}$$

and

$$I_2 := K_{Euc}(\sigma)\sigma^n w_i(\sigma)^{\frac{2nk}{n-2k}} - \int_0^\sigma K'_{Euc}(s) s^n w_i^{\frac{2nk}{n-2k}}\, ds$$
$$= (1+o(1)) a_2 \beta_2 2^{\frac{n+3\beta_2}{2}} \binom{n}{k}^{\frac{n+\beta_2}{2k}} \frac{\Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2\Gamma(n)} K(S)^{-\frac{n+\beta_2}{2k}} w_i(0)^{-\frac{2k\beta_2}{n-2k}}. \tag{4.54}$$

Recall $r_i = \lambda_i^{-1} R_i$ as in (4.34) and write $I_1 = I_{1,1} + I_{1,2}$ where $I_{1,1}$ and $I_{1,2}$ correspond to the integrals over $[0, r_i]$ and $[r_i, \sigma]$ respectively. By (4.34), we have

$$I_{1,1} = (1+o(1)) \int_0^{r_i} K_{Euc}^{1/k}(r) r^{(n-k)/k} w_i^{\frac{n+2k}{n-2k}}\, dr$$
$$= (1+o(1)) K_{Euc}^{1/k}(0) \int_0^{r_i} r^{(n-k)/k} w_i^{\frac{n+2k}{n-2k}}\, dr$$

$$= (1+o(1))K_{Euc}^{1/k}(0)\lambda_i^{-n/k} w_i(0)^{\frac{n+2k}{n-2k}} \int_0^\infty s^{(n-k)/k}(1+s^2)^{-\frac{n+2k}{2k}}\, ds$$

$$= (1+o(1))\Big(2^k \binom{n}{k}\Big)^{\frac{n}{2k^2}} K(S)^{-\frac{n-2k}{2k^2}} w_i(0)^{-1} \int_0^\infty s^{(n-k)/k}(1+s^2)^{-\frac{n+2k}{2k}}\, ds,$$

where, in the first equality, we have used the fact that for any $0 < r \le r_i$,

$$|\rho_i(r)| \overset{(1.2)}{\le} C w_i(r)^{\frac{-2nk}{n-2k}} \int_0^r s^{\beta_2 - 1} w_i(s)^{\frac{2nk}{n-2k}}\, ds$$

$$\overset{w_i(r)\ge w_i(r_i)}{\le} C w_i(r_i)^{\frac{-2nk}{n-2k}} \int_0^{r_i} s^{\beta_2 - 1} w_i(s)^{\frac{2nk}{n-2k}}\, ds$$

$$\overset{(4.34)}{\le} C w_i^{-\frac{2nk}{n-2k}}(0)(1+\lambda_i^2 r_i^2)^n \int_0^{r_i} s^{\beta_2-1} w_i^{\frac{2nk}{n-2k}}\, ds \overset{(4.34)}{\le} C w_i(0)^{-\frac{2k\beta_2}{n-2k}} R_i^{2n} = o(1).$$

Using the fact that $K' < 0$ on $[r_i, \sigma]$, and estimate (4.35) in the interval $[r_i, \bar{r}_i]$ and estimate (4.36) in the interval $[\bar{r}_i, \sigma]$, we have

$$I_{1,2} \overset{K'<0}{\le} \int_{r_i}^{\sigma} r^{(n-k)/k} w_i^{\frac{n+2k}{n-2k}}\, dr \overset{(4.35),(4.36)}{=} o(1) w_i(0)^{-1}.$$

Combining the above estimates of $I_{1,1}$ and $I_{1,2}$ and using Corollary A.2 give (4.53).

Now we estimate (4.54). We write I_2 as

$$I_2 = K_{Euc}(\sigma)\sigma^n w_i(\sigma)^{\frac{2nk}{n-2k}} - \Big(\int_0^{r_i} + \int_{r_i}^{\sigma}\Big)\Big(K'_{Euc}(s) s^n w_i^{\frac{2nk}{n-2k}}\, ds\Big) =: I_{2,1} + I_{2,2} + I_{2,3}.$$

By (4.36) and the fact $1/\beta_1 + 1/\beta_2 = 2/(n-2k)$, $|I_{2,1}| \le C w_i(0)^{-\frac{2nk}{n-2k}\frac{\min\{\beta_1,\beta_2\}}{\beta_1}}$. By (1.2) and (4.34),

$$I_{2,2} = (-a_2\beta_2 2^{\beta_2} + o(1)) \int_0^{r_i} s^{n+\beta_2-1} w_i^{\frac{2nk}{n-2k}}\, ds$$

$$= (-a_2\beta_2 2^{\beta_2} + o(1))\lambda_i^{-n-\beta_2} w_i(0)^{\frac{2nk}{n-2k}} \int_0^\infty r^{n+\beta_2-1}(1+r^2)^{-n}\, dr$$

$$= (-a_2\beta_2 2^{\beta_2} + o(1))\Big(2^k \binom{n}{k}\Big)^{\frac{n+\beta_2}{2k}} K(S)^{-\frac{n+\beta_2}{2k}} w_i(0)^{-\frac{2k\beta_2}{n-2k}} \int_0^\infty r^{n+\beta_2-1}(1+r^2)^{-n}.$$

By (1.2), (4.35) and (4.36),

$$
\begin{aligned}
|I_{2,3}| &\le C\Big(\int_{r_i}^{\bar r_i} + \int_{\bar r_i}^{\sigma}\Big) s^{n+\beta_2-1} w_i^{\frac{2nk}{n-2k}} \\
&= o(1) w_i(0)^{-\frac{2k\beta_2}{n-2k}} + O(1) w_i(0)^{-\frac{2nk}{n-2k}\frac{\min\{\beta_1,\beta_2\}}{\beta_1}} = o(1) w_i(0)^{-\frac{2k\beta_2}{n-2k}}.
\end{aligned}
$$

Combining the above estimates of $I_{2,1}$, $I_{2,2}$ and $I_{2,3}$ together and using Corollary A.2 give (4.54).

Now, by (4.53)

$$
\begin{aligned}
w_i(0)^{\min\{\frac{2\beta_2}{n-2k}-1,1\}} \times \text{RHS of (4.47)} &= -\frac{n(n-2k)}{2k^2\binom{n}{k}^{1/k}} w_i(0)^{\frac{\min\{\beta_1,\beta_2\}}{\beta_1}} I_1 \\
&= -(1+o(1))\frac{n-2k}{k}\Big(\frac{2^k\binom{n}{k}}{K(S)}\Big)^{\frac{n-2k}{2k^2}} w_i(0)^{\frac{\min\{\beta_1,\beta_2\}}{\beta_1}-1}.
\end{aligned} \tag{4.55}
$$

By (4.53) and (4.54),

$$
\begin{aligned}
&\frac{(n-2k)^2}{2k^2} w_i(0)^{\min\{\frac{2\beta_2}{n-2k}-1,1\}} w_i(\sigma)^{\frac{2n}{n-2k}} \sigma^{\frac{n}{k}} \times \text{RHS of (4.48)} \\
&\quad = -\frac{n-2k}{n} w_i(0)^{\frac{\min\{\beta_1,\beta_2\}}{\beta_1}} (I_2^{1/k}/I_1) \\
&\quad = -(1+o(1))\frac{n-2k}{k} w_i(0)^{\frac{\min\{\beta_1,\beta_2\}-\beta_2}{\beta_1}} \frac{\Big(-a_2\beta_2 2^{\beta_2}\frac{\Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2\Gamma(n)}\Big)^{\frac{1}{k}}\big(2^k\binom{n}{k}\big)^{\frac{\beta_2}{2k^2}}}{K(S)^{\frac{2k+\beta_2}{2k^2}}}.
\end{aligned} \tag{4.56}
$$

Inserting (4.51) and (4.55) into (4.47), passing to limit, and raising to the power of $\frac{k(n-2)}{n-2k}$ give (4.43). Inserting (4.52) and (4.56) into (4.48), passing to limit, and raising to the power of $\frac{k(n-2)}{n-2k}$ give (4.44).

Step 5: We make use of the Kazdan–Warner-type identity to show that

$$
\Big(\frac{K(S)^{k+\beta_2}}{K(N)^{k+\beta_1}}\Big)^{\frac{n-2}{4k}} \Big(\lim_{i\to\infty}\frac{v_i(S)^{\beta_2}}{v_i(N)^{\beta_1}}\Big) = \left(\frac{2^{\frac{\beta_2}{2}} a_2\beta_2\binom{n}{k}^{\frac{\beta_2}{2k}}\Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2^{\frac{\beta_1}{2}} a_1\beta_1\binom{n}{k}^{\frac{\beta_1}{2k}}\Gamma(\frac{n-\beta_1}{2})\Gamma(\frac{n+\beta_1}{2})}\right)^{\frac{n-2}{2}}. \tag{4.57}
$$

Indeed, the Kazdan–Warner-type identity (see (2.11)) gives

$$
\int_{\mathbb{S}^n} \langle \nabla K(x), \nabla x_{n+1}\rangle v_i^{\frac{2n}{n-2}} = 0,
$$

or equivalently,

$$\int\limits_{d_{\hat{g}}(x,S)\le\pi/2} \langle\nabla K(x), \nabla x_{n+1}\rangle v_i^{\frac{2n}{n-2}} = \int\limits_{d_{\hat{g}}(x,N)\le\pi/2} \langle\nabla K(x), \nabla(-x_{n+1})\rangle v_i^{\frac{2n}{n-2}}. \tag{4.58}$$

We will show that

$$\text{LHS of (4.58)} = (1+o(1))2^{\frac{n+\beta_2}{2}} a_2\beta_2 \binom{n}{k}^{\frac{n+\beta_2}{2k}} \frac{\Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2\Gamma(n)} K(S)^{-\frac{k+\beta_2}{2k}} v_i(S)^{-\frac{2\beta_2}{n-2}}, \tag{4.59}$$

and

$$\text{RHS of (4.58)} = (1+o(1))2^{\frac{n+\beta_1}{2}} a_1\beta_1 \binom{n}{k}^{\frac{n+\beta_1}{2k}} \frac{\Gamma(\frac{n-\beta_1}{2})\Gamma(\frac{n+\beta_1}{2})}{2\Gamma(n)} K(N)^{-\frac{k+\beta_1}{2k}} v_i(N)^{-\frac{2\beta_1}{n-2}}. \tag{4.60}$$

Inserting (4.59) and (4.60) into (4.58) gives

$$\frac{K(S)^{\frac{k+\beta_2}{2k}} v_i(S)^{\frac{2\beta_2}{n-2}}}{K(N)^{\frac{k+\beta_1}{2k}} v_i(N)^{\frac{2\beta_1}{n-2}}} = \frac{2^{\frac{\beta_2}{2}} a_2\beta_2 \binom{n}{k}^{\frac{\beta_2}{2k}} \Gamma(\frac{n-\beta_2}{2})\Gamma(\frac{n+\beta_2}{2})}{2^{\frac{\beta_1}{2}} a_1\beta_1 \binom{n}{k}^{\frac{\beta_1}{2k}} \Gamma(\frac{n-\beta_1}{2})\Gamma(\frac{n+\beta_1}{2})} + o(1).$$

Raising to the power of $(n-2)/2$ and letting $i \to \infty$ give (4.57).

We will only need to prove (4.59), since (4.60) follows by switching the roles of S and N.

Now we prove (4.59). Let u_i be related to v_i as in (2.1), then LHS of (4.58) $=$ $\int_0^1 K'_{Euc}(r) r^n u_i^{\frac{2n}{n-2}}$. In order to estimate this integral, we divide the integral into two parts: I_4, the integral on $[0, r_i]$, and I_5, the integral on $[r_i, 1]$. By (1.2) and (4.34),

$$\begin{aligned}
I_4 &\overset{(1.2)}{=} 2^{\beta_2} a_2\beta_2(1+o(1)) \int\limits_0^{r_i} r^{n+\beta_2-1} u_i^{\frac{2n}{n-2}}\, dr \\
&\overset{(4.34)}{=} (1+o(1))2^{\beta_2} a_2\beta_2 2^{\frac{n+\beta_2}{2}} \binom{n}{k}^{\frac{n+\beta_2}{2k}} K(S)^{-\frac{n+\beta_2}{2k}} u_i(0)^{-\frac{2\beta_2}{n-2}} \int\limits_0^{\infty} s^{n+\beta_2-1}(1+s^2)^{-n}\, ds \\
&= (1+o(1)) a_2\beta_2 2^{\frac{n+\beta_2}{2}} \binom{n}{k}^{\frac{n+\beta_2}{2k}} K(S)^{-\frac{n+\beta_2}{2k}} v_i(S)^{-\frac{2\beta_2}{n-2}} \int\limits_0^{\infty} s^{n+\beta_2-1}(1+s^2)^{-n}\, ds.
\end{aligned}$$

By (4.35) and (4.36),

$$\begin{aligned}
|I_5| &\le C \int\limits_{r_i}^{1} r^n u_i^{\frac{2n}{n-2}} \le C\Big[\int\limits_{r_i}^{\bar{r}_i} r^n (u_i(0)(1+\lambda_i^2 r^2))^{\frac{2n}{n-2}}\, dr + u_i(0)^{-\frac{2n}{n-2}\min\{\frac{2\beta_2}{n-2k}-1,1\}}\Big] \\
&= o(u_i(0)^{-\frac{2\beta_2}{n-2}}).
\end{aligned}$$

Combining the above two estimates together and using Corollary A.2 give (4.59).

Step 6: We reach a contradiction. Let $c_5 := \lim_{i\to\infty}(v_i(S)^{\beta_2}/v_i(N)^{\beta_1})$. Then it is easy to see that

$$c_5 = (c_3/c_4)^{\frac{\beta_1\beta_2}{\min\{\beta_1,\beta_2\}}} \tag{4.61}$$

If $\beta_2 \geq \beta_1$, by (4.43), $c_1 > 0$. Dividing (4.43) by (4.46), inserting it into the right hand side of (4.61) and inserting (4.57) into the left hand side of (4.61), we have

$$C_{n,k}(a_1, K(N))C_{n,k}(a_2, K(S)) = 1, \tag{4.62}$$

which is a contradiction to (1.3). If $\beta_1 \geq \beta_2$, by (4.44), $c_2 > 0$. Dividing (4.44) by (4.45), inserting it into the right hand side of (4.61) and inserting (4.57) into the left hand side of (4.61), we have (4.62), which is again a contradiction to (1.3). □

5. The total degree: proof of Theorem 1.2

The computation of the degree is a direct adaptation of the computation in [17,22] to the case of axisymmetry. For completeness, we present a sketch.

Fix some $0 < \alpha' \leq \alpha < 1$. By Theorem 1.1 and first and second derivative estimates for the σ_k-Yamabe equation (see [10,11], [20, Theorem 1.10], [14, Theorem 1.20], [27]), we can select C_* sufficiently large such that all axisymmetric positive solutions to (1.1) belong to the set

$$\mathscr{O} = \Big\{\tilde v \in C_r^{4,\alpha'}(\mathbb{S}^n) : \|\ln \tilde v\|_{C^{4,\alpha'}(\mathbb{S}^n)} < C_*, \lambda(A_{g_{\tilde v}}) \in \Gamma_k\Big\}.$$

Consider the nonlinear operator $F : \mathscr{O} \to C_r^{2,\alpha'}(\mathbb{S}^n)$ defined by

$$F[v] := \sigma_k(\lambda(A_{g_v})) - K, \quad \forall\, v \in \mathscr{O}.$$

By [16], the degree $\deg(F, \mathscr{O}, 0)$ is well-defined and is independent of $\alpha' \in (0, \alpha]$ (see [17, Theorem B.1]).

If $a_1, a_2 < 0$ and $\frac{1}{\beta_1} + \frac{1}{\beta_2} > \frac{2}{n-2k}$, we have in view of the homotopy invariance property of the degree, the non-existence result Theorem 1.4 and Remark 4.2 that $\deg(F, \mathscr{O}, 0) = 0$.

If $a_1, a_2 < 0$, $\frac{1}{\beta_1} + \frac{1}{\beta_2} = \frac{2}{n-2k}$ and $C_{n,k}(\beta_1, a_1, K(0))C_{n,k}(\beta_2, a_2, K(\pi)) < 1$, it follows the compactness estimate Theorem 4.4 and the above statement that $\deg(F, \mathscr{O}, 0) = 0$.

In all remaining cases, in view of the compactness estimate Theorem 4.4 and Remark 4.2, we may assume without loss of generality that $\beta_1, \beta_2 > n - 2k$.

We continue by deforming K to a constant. For $\mu \in (0, 1]$, we let $K_\mu = \mu K + (1 - \mu)2^{-k}\binom{n}{k}$ and consider the equation (4.27). By Theorem 4.3, we may assume that all axisymmetric positive solutions to (4.27) for $\mu \in (0, 1]$ belong to the set $\mathscr{O}$.

Let $F_\mu : \mathscr{O} \to C_r^{2,\alpha'}(\mathbb{S}^n)$ be defined by

$$F_\mu[v] := \sigma_k(\lambda(A_{g_v})) - K_\mu, \quad \forall\, v \in \mathscr{O}. \tag{5.1}$$

Then the degree $\deg(F_\mu, \mathscr{O}, 0)$ is well-defined and is independent of $\mu \in (0,1]$ and of $\alpha' \in (0,\alpha]$. We would now like to compute this degree for small μ and some $\alpha' \in (0,\alpha)$, using the Lyapunov-Schmidt reduction.

We parametrize $C_r^{4,\alpha'}(\mathbb{S}^n)$ as $\mathscr{S}_0 \times \mathbb{R}$ where the $\mathbb{R}$-factor takes into account the action of the Möbius group on $\mathbb{S}^n$ on axisymmetric functions and where the element $1 \in \mathscr{S}_0$ corresponds to the so-called axisymmetric standard bubbles on $\mathbb{S}^n$. To this end, for $t \in \mathbb{R}$, let φ_t be the Möbius transformation on $\mathbb{S}^n$ which, under stereographic projection with respect to the north pole, sends y to ty. For function v defined on $\mathbb{S}^n$, we let

$$T_t v := v \circ \varphi_t |\det d\varphi_t|^{\frac{n-2}{2n}}$$

where $d\varphi_t$ denotes the Jacobian of φ_t. In particular, the pull-back metric of $g_v = v^{\frac{4}{n-2}}\mathring{g}$ under φ_t is given by $\varphi_t^*(g_v) = g_{T_t v}$.

Let

$$\mathscr{S}_0 = \Big\{ v \in C_r^{4,\alpha'}(\mathbb{S}^n) : \int_{\mathbb{S}^n} x^{n+1} |v(x)|^{\frac{2n}{n-2}}\, dv_{\mathring{g}}(x) = 0 \Big\}.$$

For $w \in \mathscr{S}_0$ and $t \in \mathbb{R}$, let $\pi(w,t)$ be defined by $\pi(w,0) = w$ and

$$\pi(w,t) = T_t^{-1}(w).$$

It can be checked that the map $\pi : \mathscr{S}_0 \times \mathbb{R} \mapsto C_r^{4,\alpha'}(\mathbb{S}^n)$ is a C^2 diffeomorphism.

As in [22], Theorem 4.3 and the Liouville-type theorem give

Lemma 5.1. *Let $n \geq 5$, $2 \leq k < n/2$, and $0 < \alpha' < \alpha < 1$. Suppose that $K \in C_r^{2,\alpha}(\mathbb{S}^n)$ is as in Theorem 1.2 with $\beta_1, \beta_2 > n - 2k$. If $v_{\mu_j} = \pi(w_{\mu_j}, t_{\mu_j})$ solves (4.27) for some sequence $\mu_j \to 0^+$, then t_{μ_j} stays in a compact interval of $\mathbb{R}$ and*

$$\lim_{j\to\infty} \|w_{\mu_j} - 1\|_{C^{4,\alpha'}(\mathbb{S}^n)} = 0.$$

The linearized operator of $F_\mu[\pi(\cdot,t)]$ at $\bar{w} \equiv 1$ is readily found to be

$$\mathscr{L} := D_w(F_\mu \circ \pi)(w,\xi)]\Big|_{w=\bar{w}} = -d_{n,k}(\Delta + n) \quad \text{with} \quad d_{n,k} := \frac{2^{2-k}}{n-2}\binom{n}{k}$$

and with domain $D(\mathscr{L})$ being the tangent plane to $\mathscr{S}_0$ at $w = \bar{w}$:

$$D(\mathscr{L}) := T_1(\mathscr{S}_0) = \Big\{ \eta \in C_r^{4,\alpha'}(\mathbb{S}^n) : \int_{\mathbb{S}^n} x\eta(x)\, dv_{\mathring{g}}(x) = 0 \Big\}.$$

It is well-known that $\mathscr{L}$ is an isomorphism from $D(\mathscr{L})$ to

$$R(\mathscr{L}) := \Big\{ f \in C_r^{2,\alpha'}(\mathbb{S}^n) : \int_{\mathbb{S}^n} x^{n+1} f(x)\, dv_{\mathring{g}}(x) = 0 \Big\}.$$

Let Π be a projection from $C_r^{2,\alpha'}(\mathbb{S}^n)$ onto $R(\mathscr{L})$ defined by

$$\Pi f(x) = f(x) - \frac{n+1}{|\mathbb{S}^n|} x^{n+1} \int_{\mathbb{S}^n} y^{n+1} f(y)\, dv_{\mathring{g}}(y).$$

As in [22], we have:

Proposition 5.2. *Let $n \geq 5$, $2 \leq k < n/2$, and $0 < \alpha' < \alpha < 1$. Suppose that $K \in C_r^{2,\alpha}(\mathbb{S}^n)$ is positive and let F_μ be defined by (5.1). Then for every $s_0 \geq 1$, there exists a constant $\mu_0 \in (0, 1]$ and a neighborhood $\mathscr{N}$ of 1 in $\mathscr{S}_0$ such that, for every $\mu \in (0, \mu_0]$ and $\frac{1}{s_0} \leq t \leq s_0$, there exists a unique $w_{t,\mu} \in \mathscr{N}$, depending smoothly on (t, μ), such that*

$$\Pi(F_\mu[\pi(w_{t,\mu}, \xi)]) = 0. \tag{5.2}$$

Furthermore, there exists some $C > 0$ such that, for $\mu \in (0, \mu_0]$ and $\frac{1}{s_0} \leq t, t' \leq s_0$,

$$\|w_{t,\mu} - 1\|_{C^{4,\alpha'}(\mathbb{S}^n)} \leq C\mu \Big\| K - 2^{-k}\binom{n}{k} \Big\|_{C^{2,\alpha}(\mathbb{S}^n)},$$
$$\|w_{t,\mu} - w_{t',\mu}\|_{C^{4,\alpha'}(\mathbb{S}^n)} \leq C\mu |t - t'| \Big\| K - 2^{-k}\binom{n}{k} \Big\|_{C^{2,\alpha}(\mathbb{S}^n)}.$$

Note that equation (5.2) can be equivalently rewritten as

$$\sigma_k(\lambda(A_{g_{w_{t,\mu}}})) = K_\mu \circ \varphi_t(x) - \Lambda_{t,\mu} x^{n+1} \quad \text{on } \mathbb{S}^n,$$

where $\Lambda_{t,\mu} \in \mathbb{R}$ is given by

$$\Lambda_{t,\mu} = -\frac{n+1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} F_\mu[\pi(w_{t,\mu}, \xi)](x) x^{n+1}\, dv_{\mathring{g}}(x). \tag{5.3}$$

Furthermore, for μ sufficiently close to 0, v_μ solves (4.27) if and only if $v_\mu = \pi(w_{t_\mu,\mu}, t_\mu)$ and $\Lambda_{t_\mu,\mu} = 0$ for some t_μ.

Note that, in view of the Kazdan–Warner-type identity (2.11), $\Lambda_{t,\mu}$ can be expressed more directly in terms of K as

$$\frac{1}{\mu}\Lambda_{t,\mu} \int_{\mathbb{S}^n} |\nabla x^{n+1}|^2 w_{t,\mu}^{\frac{2n}{n-2}}\, dv_{\mathring{g}}(x) = \int_{\mathbb{S}^n} \langle \nabla (K \circ \varphi_t), \nabla x^{n+1} \rangle w_{t,\mu}^{\frac{2n}{n-2}}\, dv_{\mathring{g}}(x). \tag{5.4}$$

The degree of the function $t \mapsto \Lambda_{t,\mu}$ can be computed in the same way as in [22]:

Lemma 5.3. *Let $n \geq 5$, $2 \leq k < n/2$, $\alpha \in (0,1)$ and $K \in C_+^{2,\alpha}(\mathbb{S}^n)$ be as in Theorem 1.2 with $\beta_1, \beta_2 > n-2k$. Let $\Lambda_{t,\mu}$ be defined as in (5.3). Then there exist $\mu_0 \in (0,1]$ and $s_0 \in (1,\infty)$ such that, for all $\mu \in (0,\mu_0]$ and $s \in (1,s_0]$, the Brouwer degrees $\deg(\Lambda_{t,\mu},[s^{-1},s],0)$ are well-defined and*

$$\deg(\Lambda_{t,\mu},[s^{-1},s],0) = -\frac{1}{2}(-1)^n[\operatorname{sign}(a_1) + \operatorname{sign}(a_2)].$$

Proof of Theorem 1.2. As explained at the beginning of the section, we only need to consider the case $\beta_1, \beta_2 > n-2k$. In this case, as in [22], there exist $\mu_0 \in (0,1]$ and $s_0 > 1$ such that

$$\deg(F_\mu, \mathscr{O}, 0) = (-1)^n \deg(\Lambda_{t,\mu},[s^{-1},s],0) \text{ for all } \mu \in (0,\mu_0], s \in (1,s_0).$$

The conclusion follows from Lemma 5.3. $\square$

6. Perturbation method: proof of Theorem 1.3

Proof of Theorem 1.3. After a renaming of K to K_μ, it suffices to exhibit a function K satisfying (1.2) such that $\operatorname{sign}(a_i) = \varepsilon_i$ and that the equation (4.27) has a solution for some sufficiently small μ.

Fix some $s_0 > 1$ for the moment, and let $\Lambda_{t,\mu}$ be as in Proposition 5.2. Then (4.27) has a positive solution if the map $t \mapsto \Lambda_{t,\mu}$ has a zero in $[s_0^{-1}, s_0]$.

Prompted by formula (5.4) for $\Lambda_{t,\mu}$ and the fact that $w_{t,\mu} \approx 1$ for small μ, we consider the function

$$H_K(t) = \int_{\mathbb{S}^n} \langle \nabla(K \circ \varphi_t), \nabla x^{n+1} \rangle dv_{\mathring{g}}(x) = n \int_{\mathbb{S}^n} K \circ \varphi_t \, x^{n+1} dv_{\mathring{g}}(x).$$

Clearly, if K and s_0 are such that $H_K(1)$ and $H_K(s_0)$ are of opposite signs, then for all sufficiently small μ, $\Lambda_{1,\mu}$ and $\Lambda_{s_0,\mu}$ are also of opposite signs and the conclusion will follow.

We now proceed to construct K and s_0. Let $K_\#(x) = (x^{n+1})^{2m}$ for some large $m > \beta_1, \beta_2$. Then $H_{K_\#}(1) = 0$ and $H'_{K_\#}(1) > 0$. In particular, there exists $s_0 > 1$ such that $H_{K_\#}(s_0) > 0$.

Take a function K_* satisfying (1.2) with $\operatorname{sign}(a_i) = \varepsilon_i$. By considering the behavior of $H_{K_*}(t)$ as $t \to 0$, we have that $H_{K_*} \not\equiv 0$. Replacing K_* with $K_* \circ \varphi_t$ for some suitable t, we may assume also that $H_{K_*}(1) \neq 0$.

Since $H_{K_\#}(1) = 0$ and $H_{K_\#}(s_0) > 0$, there exists some $\gamma \in \mathbb{R}$ such that $H_{K_* + \gamma K_\#}(1)$ and $H_{K_* + \gamma K_\#}(s_0)$ are of opposite signs.

The desired function K then takes the form $C + K_* + \gamma K_\#$ for some sufficiently large C such that K is positive. $\square$

7. Non-existence: proof of Theorem 1.4

Proof of Theorem 1.4. Let us first prove the non-existence of positive axisymmetric solutions for some suitable K with the declared properties. The fact that this implies the theorem will be dealt with at the last stage.

It is more convenient to work in cylindrical coordinates. Fix $2 \leq \beta_1, \beta_2 < n$ such that $\frac{1}{\beta_1} + \frac{1}{\beta_2} \geq \frac{2}{n-2k}$. For small $0 < \varepsilon \ll 1$ and large $T \geq 1$, fix a positive function $\hat{K}_{\varepsilon,T} \in C^\infty(\mathbb{R})$ such that

$$\hat{K}_{\varepsilon,T}(t) = 1 - \frac{1}{2}e^{\beta_2(t+T+1)} \text{ for } t \leq -T-1, \tag{7.1}$$

$$-2 \leq \frac{d}{dt}\hat{K}_{\varepsilon,T}(t) \leq 0 \text{ for } -T-1 < t < -T, \tag{7.2}$$

$$\hat{K}_{\varepsilon,T}(t) = 1 - \frac{1}{2}e^{-\beta_1(t-T-1)} \text{ for } t \geq T+1, \tag{7.3}$$

$$2 \geq \frac{d}{dt}\hat{K}_{\varepsilon,T}(t) \geq 0 \text{ for } T < t < T+1, \tag{7.4}$$

$$\hat{K}_{\varepsilon,T}(t) = \varepsilon \text{ for } -T \leq t \leq T. \tag{7.5}$$

Let $t = \ln\cot\frac{\theta}{2}$ and $K_{\varepsilon,T}(\theta) = \hat{K}_{\varepsilon,T}(t)$. We will show that there exists $N \gg 1$ such that, whenever $T \geq N$ and $\varepsilon e^{(n+2k)T} \leq \frac{1}{N}$, there is no positive axisymmetric solution of (1.1) with $K = K_{\varepsilon,T}$.

Suppose by contradiction that there exist T_i and ε_i with $T_i \geq i$ and $\varepsilon_i e^{(n+2k)T_i} \leq \frac{1}{i}$ such that the problem (1.1) with $K = K_{\varepsilon_i,T_i}$ has a solution v_i.

Let $r = e^t$ and let $u_i : \mathbb{R}^n \to \mathbb{R}$ and $\xi_i : \mathbb{R} \to \mathbb{R}$ be related to v_i as in (2.1) and (2.3). In particular, ξ_i satisfies

$$F_k[\xi_i] = \hat{K}_{\varepsilon_i,T_i} \text{ and } |\dot{\xi}_i| < 1 \text{ in } (-\infty,\infty), \tag{7.6}$$

and

$$\xi_i(t) - |t| \text{ is bounded as } |t| \to \infty.$$

In the sequel, we use C to denote some positive generic constant which is always independent of i, $O(1)$ to denote a term which is bounded as $i \to \infty$, and $o(1)$ to denote a term which tends to 0 as $i \to \infty$.

Observe that the arguments in the proof of Theorem 3.1(a) give $\xi_i(t) \geq -C$ for $|t| > T_i + 2$. Since $|\dot{\xi}| < 1$, this implies that, for every $m \geq 0$

$$\xi_i(t) \geq -C(m) \text{ for } |t| \geq T_i - m. \tag{7.7}$$

Applying first and second derivative estimates for the σ_k-Yamabe equation to (7.6), we have, for every $m > 0$,

$$|\dot{\xi}_i(t)| + |\ddot{\xi}_i(t)| \le C(m) \text{ for } |t| \ge T_i - m. \tag{7.8}$$

Step 1: Let $Y_i = e^{-\frac{n-2k}{2k}\xi_i}$. We show that

$$Y_i(-T_i) + \frac{2k}{n-2k}\dot{Y}_i(-T_i) = o(1)Y_i(T_i), \tag{7.9}$$

$$Y_i(T_i) - \frac{2k}{n-2k}\dot{Y}_i(T_i) = o(1)Y_i(-T_i). \tag{7.10}$$

We start by rewriting the equation $F_k[\xi_i] = \hat{K}_{\varepsilon_i,T_i}$ in the form

$$e^{2\xi_i}\Big(\ddot{\xi}_i + \frac{n-2k}{2k}(1-\dot{\xi}_i^2)\Big) = 2^{k-1}\binom{n-1}{k-1}^{-1}\frac{\hat{K}_{\varepsilon_i,T_i}}{e^{2(k-1)\xi_i}(1-\dot{\xi}_i^2)^{k-1}}.$$

We proceed by estimating the term on the denominator on the right hand side. Recall the function H defined in (2.8), and note that, by the Pohozaev identity (2.9) and the monotonicity of $\hat{K}_{\varepsilon_i,T_i}$ in $(-\infty, T_i)$, $H(t,\xi_i,\dot{\xi}_i)$ is non-decreasing in $(-\infty,T_i)$. Also, since $|\dot{\xi}_i| < 1$, $\xi_i(t) + t$ is bounded as $t \to -\infty$ and $k < \frac{n}{2}$, $H(t,\xi_i,\dot{\xi}_i) \to 0$ as $t \to -\infty$. Therefore $H(t,\xi_i,\dot{\xi}_i) \ge 0$, i.e.

$$\frac{1}{2^k}\binom{n}{k}e^{2k\xi_i}(1-\dot{\xi}_i^2)^k \ge \hat{K}_{\varepsilon_i,T_i}(t) > 0 \text{ in } (-\infty,T_i).$$

Inserting this into the previous equation, we obtain

$$0 \le e^{2\xi}\Big(\ddot{\xi}_i + \frac{n-2k}{2k}(1-\dot{\xi}_i^2)\Big) \le a_{n,k}\hat{K}_{\varepsilon_i,T_i}^{1/k}, \qquad a_{n,k} = \binom{n}{k}^{\frac{k-1}{k}}\binom{n-1}{k-1}^{-1}.$$

Multiplying this equation by $e^{\pm\frac{n-2k}{2k}t - \frac{n+2k}{2k}\xi}$, we get

$$0 \le \pm\frac{d}{dt}\Big[e^{\frac{n-2k}{2k}(\pm t-\xi_i)}(1\pm\dot{\xi}_i)\Big] \le C\varepsilon_i^{\frac{1}{k}}e^{\pm\frac{n-2k}{2k}t - \frac{n+2k}{2k}\xi_i}. \tag{7.11}$$

Using the fact that $\hat{K}_{\varepsilon_i,T_i} = \varepsilon_i$ in $(-T_i,T_i)$, $\xi_i(t) \ge \xi_i(\pm T_i) - (t \mp T_i)$ in $(-T_i,T_i)$ (since $|\xi_i| < 1$), $T_i \to \infty$ and $\varepsilon_i e^{(n+2k)T_i} \to 0$, we can integrate (7.11) to obtain

$$e^{-\frac{n-2k}{2k}\xi_i(T_i)}\big(1+\dot{\xi}_i(T_i)\big) = o(1)e^{-\frac{n-2k}{2k}\xi_i(-T_i)} + o(1)e^{-\frac{n+2k}{2k}\xi_i(-T_i)},$$
$$e^{-\frac{n-2k}{2k}\xi_i(-T_i)}\big(1-\dot{\xi}_i(-T_i)\big) = o(1)e^{-\frac{n-2k}{2k}\xi_i(T_i)} + o(1)e^{-\frac{n+2k}{2k}\xi_i(T_i)}.$$

In view of (7.7) and the expression of Y_i, (7.9) and (7.10) follows. Step 1 is finished.

Step 2: We show that[3]

$$\alpha_- := \limsup_{i\to\infty}\dot{\xi}_i(-T_i) < 1 \text{ and } \alpha_+ = \liminf_{i\to 0}\dot{\xi}_i(T_i) > -1.$$

[3] In fact, it can be seen from the proof that, when $\beta_1 \ne n-2k$, $\alpha_+ = 1$, and when $\beta_2 \ne n-2k$, $\alpha_- = -1$.

Once this is done, after passing to a subsequence, we have

$$\dot{Y}_i(-T_i) = -\frac{n-2k}{2k}(\alpha_- + o(1))Y_i(-T_i) \text{ and } \dot{Y}_i(T_i) = -\frac{n-2k}{2k}(\alpha_+ + o(1))Y_i(T_i),$$

which together with (7.9) and (7.10) gives

$$0 < Y_i(T_i) = o(1)Y_i(-T_i) = o(1)Y_i(T_i),$$

which yields a contradiction and finishes the proof of Theorem 1.4.

We will only prove that $\alpha_- < 1$. The proof of $\alpha_+ > -1$ is similar.

Let us first show that $\xi_i(-T_i) \to \infty$ as $i \to \infty$. Indeed, by (7.9), $Y_i(-T_i)(1+\dot{\xi}_i(-T_i)) = o(1)$. Using (7.7) and the expression of H (see (2.8)), we have that $H(t, \xi_i, \dot{\xi}_i)\Big|_{t=-T_i} = o(1)$. Recalling the Pohozaev identity (2.10), the fact that $\frac{d}{dt}\dot{\hat{K}}_{\varepsilon_i,T_i} \le -\frac{1}{C}e^{-\beta_2(t+T_i)}$ in $(-\infty, -T_i)$ (see (7.1) and (7.2)), and $\xi_i(t) \le \xi_i(-T_i) - (t + T_i)$ in $(-\infty, -T_i)$ (since $|\dot{\xi}_i| < 1$), we thus have

$$o(1) = \int_{-\infty}^{-T_i} |\frac{d}{dt}\hat{K}_{\varepsilon_i,T_i}|e^{-n\xi_i}\,dt \ge \frac{1}{C}e^{-n\xi_i(-T_i)},$$

which gives $\xi_i(-T_i) \to \infty$ as $i \to \infty$ as wanted.

Let $\hat{\xi}_i(t) := \xi_i(t-T_i) - \xi_i(-T_i)$. Using (7.7)–(7.8) and the fact that $\hat{\xi}_i(0) = 0$, we have, after passing to a subsequence, that $\hat{\xi}_i$ converges in $C^{1,\alpha}_{loc}(\mathbb{R})$ to a function $\hat{\xi}_\infty \in C^{1,1}_{loc}(\mathbb{R})$. Also, in view of (7.6), $F_k[\hat{\xi}_i](t) = e^{-2k\xi_i(-T_i)}\hat{K}_{\varepsilon_i,T_i}(t-T_i)$. Hence, since $\xi_i(-T_i) \to \infty$ as $i \to \infty$, $\hat{\xi}_\infty$ satisfies in the viscosity sense the equation

$$F_k[\hat{\xi}_\infty] = 0 \text{ and } |\dot{\hat{\xi}}_\infty| \le 1 \text{ in } (-\infty, \infty).$$

By the classification result [21, Theorem 1.6], $\tilde{\xi}_\infty$ takes the form

$$\hat{\xi}_\infty(t) = -\frac{2k}{n-2k}\ln(ae^{-\frac{n-2k}{2k}t} + be^{\frac{n-2k}{2k}t}) \text{ for some } a, b \ge 0 \text{ with } a + b > 0. \tag{7.12}$$

Now, as $\dot{\xi}_i(-T_i) \to \dot{\hat{\xi}}_\infty(0) = \frac{a-b}{a+b}$, in order to conclude Step 2 (and therefore the proof of the theorem), it suffices to show that $b > 0$.[4]

Claim. *The following statements hold.*

(i) *Either* $\{e^{-\frac{n-2}{2}T_i}v_i(S)\}$ *is bounded, or* $e^{-\frac{n-2}{2}T_i}v_i(S) \to \infty$ *and* $\beta_2 \le n - 2k$.
(ii) *Either* $\{e^{-\frac{n-2}{2}T_i}v_i(N)\}$ *is bounded, or* $e^{-\frac{n-2}{2}T_i}v_i(N) \to \infty$ *and* $\beta_1 \le n - 2k$.

[4] In fact, it will be seen from the proof below that, when $\beta_2 \ne n - 2k$, we also have $a = 0$.

Before proving the claim, let us remark that statement (i) implies that $b > 0$ (and hence $\alpha_- < 1$) as follows. (Likewise, (ii) implies that $\alpha_+ > -1$.) Let

$$\check{u}_i(r) = e^{-\frac{n-2}{2}(t+\xi_i(t-T_i))} = e^{-\frac{n-2}{2}T_i} u_i(e^{-T_i} r).$$

Note that $\check{u}_i$ satisfies $\sigma_k(A^{\check{u}_i}(r)) = \hat{K}_{\varepsilon_i,T_i}(\ln r - T_i)$ on $\mathbb{R}^n$, and, by the claim, either $\{\check{u}_i(0)\}$ is bounded, or $\check{u}_i(0) \to \infty$ and $\beta_2 \le n - 2k$. In the case that $\{\check{u}_i(0)\}$ is bounded, as $\check{u}_i(0)$ is the maximum of $\check{u}_i$ on $\mathbb{R}^n$ (by the super-harmonicity of $\check{u}_i$), the first derivative estimates for the σ_k-Yamabe equation give that $\frac{1}{C}\check{u}_i(1) \le \check{u}_i(r) \le C\check{u}_i(1)$ for $r \le 1$, i.e. $|\hat{\xi}_i(t) + t - \hat{\xi}_i(0)| \le C$ in $(-\infty, 0)$. In particular, $\hat{\xi}_\infty(t) + t$ is bounded as $t \to -\infty$. Clearly this is true in (7.12) if and only if $a = 0$ and $b > 0$. In the case that $\check{u}_i(0) \to \infty$ and $\beta_2 < n - 2k$, we have by Theorem 3.1(c) and (d) that there exists an exponent $\varkappa = \varkappa(\beta_2) > 0$ such that

$$\frac{1}{C}\check{u}_i(1) \le \check{u}_i(r) \le C\check{u}_i(1) \text{ in } (\check{u}_i(0)^{-\varkappa}, 1).$$

As $\check{u}_i(0)^{-\varkappa} \to 0$, this again implies that $\hat{\xi}_\infty(t) + t$ is bounded as $t \to -\infty$ and so $a = 0$ and $b > 0$. In the case that $\check{u}_i(0) \to \infty$ and $\beta_2 = n - 2k$, we can apply Step 2 of the proof of Theorem 1.1 to $\check{\xi}_i(t) := \xi_i(t - T_i)$ to obtain that $\hat{\xi}_i(t) = \check{\xi}_i(t) - \xi_i(-T_i)$ has a critical point at some $\ln \check{\delta}_i = O(1)$. It follows that $\hat{\xi}_\infty$ has at least one critical point, which implies that $a, b > 0$.

It remains to prove the claim. Note that the claim clearly holds if $\beta_1, \beta_2 \le n - 2k$. Therefore, we may assume without loss of generality that $\beta_2 = \max(\beta_1, \beta_2) > n - 2k$. As $\frac{1}{\beta_1} + \frac{1}{\beta_2} \ge \frac{2}{n-2k}$, we have that $\beta_1 < n - 2k$, and so (ii) clearly holds. In particular, $\alpha_+ > -1$. It remains to prove (i).

Assume by contradiction that (i) does not hold. Then, since $\beta_2 > n - 2k$, $e^{-\frac{n-2}{2}T_i} v_i(S) \to \infty$ and $\check{u}_i(0) \to \infty$. The proof builds upon the identity

$$H(-T_i, \xi_i(-T_i), \dot{\xi}_i(-T_i)) = H(T_i, \xi_i(T_i), \dot{\xi}_i(T_i)), \tag{7.13}$$

which holds in view of the Pohozaev identity (2.9) and (7.5).

As $\alpha_+ > -1$, we have $\dot{\xi}_i(T_i) = \alpha_+ + o(1) > -1 + \frac{1}{C}$. In particular, by (7.10)

$$e^{-\xi_i(T_i)} = o(1) e^{-\xi_i(-T_i)}. \tag{7.14}$$

Let $\lambda_i := 2^{-\frac{1}{2}} \binom{n}{k}^{-\frac{1}{2k}} \check{u}_i(0)^{\frac{2}{n-2}} = 2^{-\frac{1}{2}} \binom{n}{k}^{-\frac{1}{2k}} e^{-\frac{n-2}{2}T_i} u_i(0)^{\frac{2}{n-2}} \to \infty$. Applying Step 2 of the proof of Theorem 1.1 (see (4.8) and (4.12)) to $\check{\xi}_i(t) := \xi_i(t - T_i)$, we obtain

$$H(-T_i, \xi_i(-T_i), \dot{\xi}_i(-T_i)) \overset{(4.12)}{=} e^{O(1)} \lambda_i^{-\beta_2} \overset{(4.8)}{=} e^{O(1)} e^{-\beta_2 \xi_i(-T_i)}. \tag{7.15}$$

To estimate $H(T_i, \xi_i(T_i), \dot{\xi}_i(T_i))$, consider

$$\tilde{u}_i(r) = e^{-\frac{n-2}{2}(t+\xi_i(-t+T_i))} = \Big(\frac{e^{T_i/2}}{r}\Big)^{n-2} u_i\big(\frac{e^{T_i}}{r}\big)$$

and let us treat separately the case $\{\tilde{u}_i(0)\}$ is bounded and the case $\tilde{u}_i(0) \to \infty$.

Let us start with the case that $\{\tilde{u}_i(0)\}$ is bounded. As seen earlier, this implies that $\frac{1}{C}\tilde{u}_i(1) \le \tilde{u}_i(r) \le C\tilde{u}_i(1)$ for $r \le 1$, and so

$$\xi_i(t+T_i) = \xi_i(T_i) + t + O(1) \text{ for } t \ge 0.$$

By the Pohozaev identity (2.10), (7.3) and (7.4),

$$H(T_i, \xi_i(T_i), \dot{\xi}_i(T_i)) = \int_{T_i}^{\infty} \dot{\hat{K}}_{\varepsilon_i,T_i}(t) e^{-n\xi_i}\, dt = e^{-n\xi_i(T_i)+O(1)}.$$

Using (7.14) in the above gives

$$H(T_i, \xi_i(T_i), \dot{\xi}_i(T_i)) = o(1) e^{-n\xi_i(-T_i)},$$

which gives a contradiction to (7.13) and (7.15), since $\beta_2 < n$.

Let us now turn to the case $\tilde{u}_i(0) \to \infty$. Note that by the same argument that gives $\xi_i(-T_i) \to \infty$, we also have that $\xi_i(T_i) \to \infty$. This implies that $\tilde{u}_i(1) = o(1)$. By Theorem 3.1(c) and (d), we thus have $\beta_1 > \frac{n-2k}{2}$. Apply Step 2 of the proof of Theorem 1.1 (see (4.9) and (4.12)) to $\tilde{\xi}_i(t) := \xi_i(-t+T_i)$, we can find $\tilde{\delta}_i = e^{O(1)} \check{\lambda}_i^{-(1-\frac{\beta_1}{n-2k})}$ such that

$$\xi_i(t) = \xi_i(T_i) - T_i + t + O(1) \text{ in } (T_i, T_i - \ln \tilde{\delta}_i), \tag{7.16}$$

$$H(t, \xi_i, \dot{\xi}_i)\Big|_{t=T_i-\ln\tilde{\delta}_i} = e^{O(1)} \check{\lambda}_i^{-\beta_1} = e^{O(1)} e^{-\beta_1(\frac{2\beta_1}{n-2k}-1)^{-1}\xi_i(T_i)}. \tag{7.17}$$

Using (7.3), (7.4) and (7.16) in the Pohozaev identity (2.9), we have

$$H(t, \xi_i, \dot{\xi}_i)\Big|_{t=T_i}^{t=T_i-\ln\tilde{\delta}_i} = \int_{T_i}^{T_i-\ln\tilde{\delta}_i} \dot{\hat{K}}_{\varepsilon_i,T_i}(t) e^{-n\xi_i}\, dt = e^{O(1)} e^{-n\xi_i(T_i)}.$$

Putting this and (7.17) together and then using (7.14), we get

$$\begin{aligned} H(T_i, \xi_i(T_i), \dot{\xi}_i(T_i)) &= e^{O(1)} e^{-n\xi_i(T_i)} + e^{O(1)} e^{-\beta_1(\frac{2\beta_1}{n-2k}-1)^{-1}\xi_i(T_i)} \\ &= o(1) e^{-n\xi_i(-T_i)} + o(1) e^{-\beta_1(\frac{2\beta_1}{n-2k}-1)^{-1}\xi_i(-T_i)}. \end{aligned}$$

As $\beta_2 < n$, this together with (7.15) and (7.13) implies that $\beta_2 > \beta_1(\frac{2\beta_1}{n-2k} - 1)^{-1}$, which contradicts the hypothesis that $\frac{1}{\beta_1} + \frac{1}{\beta_2} \ge \frac{2}{n-2k}$.

Finally, to conclude, we show that with $T \geq N$ and $\varepsilon e^{(n+2k)T} \leq \frac{1}{N}$ as above, (1.1) with $K = K_{\varepsilon,T}$ has no positive solution, with or without axisymmetry. This follows from Proposition 7.1 below. $\square$

Proposition 7.1. *Suppose $K \in C^1_r(\mathbb{S}^n)$ is positive, non-constant and satisfies $\frac{d}{d\theta}K(\theta) \leq 0$ in $(0, \pi/2)$ and $\frac{d}{d\theta}K(\theta) \geq 0$ in $(\pi/2, \pi)$. Then every positive solution $v \in C^2(\mathbb{S}^n \setminus \{\theta = 0, \pi\})$ to*

$$\sigma_k(\lambda(A_{g_v})) = K \text{ and } \lambda(A_{g_v}) \in \Gamma_k \text{ on } \mathbb{S}^n \setminus \{\theta = 0, \pi\}$$

is axisymmetric.

Remark 7.2. The conclusion remains valid if $K(x)$ is replaced by $K(x)u^{-a}$ for any constant $a \geq 0$, and/or if (σ_k, Γ_k) is replaced by more general operators (f, Γ) as in [13].

Proof. Let $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be related to v by (2.1). Then u is super-harmonic and positive in $\mathbb{R}^n \setminus \{0\}$. It follows that $\liminf_{y\to 0} u(y) > 0$ and so

$$\liminf_{d(x,S)\to 0} v(x) > 0.$$

Likewise,

$$\liminf_{d(x,N)\to 0} v(x) > 0.$$

Note that by [4, Theorem 1.1], it holds in the viscosity sense that

$$\sigma_k(\lambda(A_{g_v})) \geq K \text{ and } \lambda(A_{g_v}) \in \Gamma_k \text{ on } \mathbb{S}^n. \tag{7.18}$$

We can now use the method of moving spheres as in the proof of [13, Theorem 1.5] to reach the conclusion. For readers' convenience, we give here a sketch: For any point on p on the equator of $\mathbb{S}^n$ and any $\lambda \in (0, \pi)$, let $\varphi_{p,\lambda} : \mathbb{S}^n \to \mathbb{S}^n$ be the Möbius transformation that reflects about the sphere $\partial B_\lambda(p)$ centered at p and of radius λ and let $v_{p,\lambda} = |Jac(\varphi_{p,\lambda})|^{\frac{n-2}{2n}} v \circ \varphi_{p,\lambda}$. By the conformal invariance of the equation (1.1) and the monotonicity property of K with respect to θ,

$$\sigma_k(\lambda(A_{g_{v_{p,\lambda}}})) = K \circ \varphi_{p,\lambda} \leq K \text{ in } \mathbb{S}^n \setminus B_\lambda(p). \tag{7.19}$$

Using [13, Lemmas 3.5 and 3.6], the number

$$\bar{\lambda}_p := \sup\left\{\lambda \in (0, \pi) : v_{p,\mu} \leq v \text{ in } \mathbb{S}^n \setminus B_\lambda(p)\right\}$$

is well-defined and belongs to $(0, \pi]$. One can then imitate the proof of [13, Lemma 3.3] using (7.18), (7.19) and the strong maximum principle [4, Theorem 3.1] to show that $\bar{\lambda}_p \geq \pi/2$. Since this holds for every p on the equator, we have that v is axisymmetric. $\square$

8. Non-compactness: proof of Theorem 1.5

Proof of Theorem 1.5. We will work in cylindrical coordinates. Fix $2 \leq \beta < \frac{n-2k}{2}$. Consider $\hat{K}_\varepsilon = 2^{-k}\binom{n}{k} + \varepsilon J$ where ε is sufficiently small and $J \in C^\infty(\mathbb{R})$ is a fixed even function satisfying

$$J(t) = -e^{\beta t} \text{ for } t \leq -1,$$
$$\dot{J}(t) \leq 0 \text{ for } t \leq 0.$$

For $j \geq 0$, let X_j denote the Banach space of functions $\eta \in C^j((-\infty, 0])$ such that

$$\|\eta\|_j := \sup_{t\in(-\infty,0]} e^{-(2+\beta)t} \sum_{\ell=0}^{j} |\frac{d^\ell}{dt^\ell}\eta| < \infty.$$

We will show that, for a suitably small but fixed $\varepsilon > 0$, the equation

$$F_k[\xi] = \hat{K}_\varepsilon, \text{ and } |\dot{\xi}| < 1 \text{ in } (-\infty, \infty) \tag{8.1}$$

has a sequence of even solutions $\xi_i \in C^2(\mathbb{R})$ such that

$$\Big(\xi_i - \log\cosh(t + T_i)\Big)\Big|_{(-\infty,0]} \in X_2 \tag{8.2}$$

where $T_i \to \infty$ as $i \to \infty$. Once this is done, the conclusion of the theorem follows from Corollary 3.2.

<u>Step 1:</u> We prove that there exists some small $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and $T \geq 1$ there exists $\xi = \xi(\cdot; \varepsilon, T) \in C^2((-\infty, 0])$ which satisfies

$$F_k[\xi] = \hat{K}_\varepsilon, \text{ and } |\dot{\xi}| < 1 \text{ in } (-\infty, 0), \tag{8.3}$$
$$\xi - \log\cosh(t + T) \in X_2 \tag{8.4}$$

and the family $\xi(\cdot; \varepsilon, T)$ depends continuously on (ε, T) in the sense that $(\varepsilon, T) \mapsto \xi(\cdot; \varepsilon, T) - \log\cosh(\cdot + T)$ belongs to $C^1((0, \varepsilon_0) \times [1, \infty); X_2)$.

We claim that it is enough to find $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and $T \geq 1$ there exists $\xi = \xi(\cdot; \varepsilon, T) \in C^2((-\infty, -T])$ such that $F_k[\xi] = \hat{K}_\varepsilon$ in $(-\infty, -T)$ and the function $\eta(t; \varepsilon, T) := \xi(t - T; \varepsilon, T) - \log\cosh t$ belongs to X_2 and that $(\varepsilon, T) \mapsto \eta(\cdot; \varepsilon, T)$ belongs to $C^1((0, \varepsilon_0) \times [1, \infty); X_2)$. Indeed, let $(-\infty, T_{\max}) \subset (-\infty, 0)$ be the maximal such that ξ satisfies the equation $F_k[\xi] = \hat{K}_\varepsilon$ in $(-\infty, T_{\max})$, then by the Pohozaev identity (2.10) and the monotonicity of $\hat{K}_\varepsilon$, we have

$$
\begin{aligned}
H(t,\xi,\dot\xi) = \frac{1}{2^k}\binom{n}{k} e^{(2k-n)\xi}(1-\dot\xi^2)^k - \hat K_\varepsilon(t)e^{-n\xi} \\
= -\int_{-\infty}^{t} \dot K_\varepsilon(\tau) e^{-n\xi(\tau)}\,d\tau > 0 \text{ for } t \in (-\infty, T_{\max}). \quad (8.5)
\end{aligned}
$$

This implies that, $1-\dot\xi^2 > 0$ in $(-\infty, T_{\max})$ and $\limsup_{t\to T_{\max}} |\xi(t)| < \infty$. Standard results on local existence, uniqueness and continuous dependence for ODEs imply that $T_{\max} = 0$ and the claim follows.

By considering $\tilde\xi = \xi(\cdot - T)$ and using the claim, to finish Step 1, we need to show the existence of some $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$ and $T \geq 1$, there is a solution $\tilde\xi = \tilde\xi(\cdot;\varepsilon,T)$ to

$$
F_k[\tilde\xi] = \hat K_{\varepsilon e^{-\beta T}} \text{ in } (-\infty,0), \tag{8.6}
$$

$$
\tilde\xi - \log\cosh t \in X_2 \tag{8.7}
$$

and that the map $T \mapsto \eta(\cdot;\varepsilon,T) = \tilde\xi(\cdot;\varepsilon,T) - \log\cosh t$ belongs to $C^1((0,\varepsilon_0)\times[1,\infty);X_2)$.

Using η, we recast (8.6)–(8.7) as

$$
\mathscr{A}[\eta] = -2^{k-1}\binom{n-1}{k-1}^{-1} \varepsilon\, e^{-\beta T}\operatorname{sech}^2 t\, e^{\beta t}
$$

where $\mathscr{A} : X_2 \to X_0$ is given by

$$
\begin{aligned}
\mathscr{A}[\eta] &:= 2^{k-1}\binom{n-1}{k-1}^{-1}\operatorname{sech}^2 t\left\{F_k[\log\cosh t + \eta] - \frac{1}{2^k}\binom{n}{k}\right\}\\
&= \operatorname{sech}^2 t\Big\{e^{2k\eta}\Big(1 - 2\cosh t\,\sinh t\,\dot\eta - \cosh^2 t\,\dot\eta^2\Big)^{k-1}\times\\
&\qquad \times\Big(\frac{n}{2k} + \cosh^2 t\,\ddot\eta - \frac{n-2k}{k}\cosh t\,\sinh t\,\dot\eta - \frac{n-2k}{2k}\cosh^2 t\,\dot\eta^2\Big) - \frac{n}{2k}\Big\}\\
&=: P(t,\eta,\dot\eta,\ddot\eta).
\end{aligned}
$$

Note that for every $\eta \in X_2$, $P(t,\eta,\dot\eta,\ddot\eta)$, $P_\eta(t,\eta,\dot\eta,\ddot\eta)$, $P_{\dot\eta}(t,\eta,\dot\eta,\ddot\eta)$ and $P_{\ddot\eta}(t,\eta,\dot\eta,\ddot\eta)$ are continuous and bounded in $(-\infty,0)$. It follows that $\mathscr{A}$ is C^1 with derivative

$$
D\mathscr{A}[\eta][\varphi] = P_{\ddot\eta}(t,\eta,\dot\eta,\ddot\eta)\ddot\varphi + P_{\dot\eta}(t,\eta,\dot\eta,\ddot\eta)\dot\varphi + P_\eta(t,\eta,\dot\eta,\ddot\eta)\varphi.
$$

Since $\mathscr{A}[0] = 0$, by the implicit function theorem (see e.g. [23, Theorem 2.7.2]), it suffices to check that $\mathscr{L} := D\mathscr{A}[0]$ is invertible. A direct computation gives

$$
\mathscr{L}[\varphi] = \ddot\varphi - (n-2)\tanh t\,\dot\varphi + n\operatorname{sech}^2 t\,\varphi.
$$

The homogeneous equation $\mathscr{L}[\varphi] = 0$ has two linearly independent solutions $\varphi_1(t) = \tanh t$ and $\varphi_2(t) = e^{(n-2)|t|}(1 + O(e^t))$ as $t \to -\infty$. (For example, we can choose $\varphi_2(t) =$

$\tanh t \int_c^t \frac{\cosh^n \tau}{\sinh^2 \tau}\, d\tau$ for some $c < 0$.) In particular, the only solution to $\mathscr{L}[\varphi] = 0$ in X_2 is the trivial element. Furthermore, for every $\zeta \in X_0$, the solution to $\mathscr{L}[\varphi] = \zeta$ in X_2 is given by

$$\varphi(t) = -\varphi_1(t) \int\limits_{-\infty}^{t} \frac{\zeta(\tau)\varphi_2(\tau)}{\cosh^{n-2}\tau}\, d\tau + \varphi_2(t) \int\limits_{-\infty}^{t} \frac{\zeta(\tau)\varphi_1(\tau)}{\cosh^{n-2}\tau} d\tau \text{ for } t \in (-\infty, 0].$$

We thus have that $\mathscr{L}$ is a bijection from X_2 onto X_0. This completes Step 1.

Step 2: Since $\hat{K}_\varepsilon$ is even, to show the existence of even solutions to (8.1)–(8.2), it suffices to show that, after possibly shrinking ε_0, for every $\varepsilon \in (0, \varepsilon_0)$ there exists a sequence $T_i \to \infty$ such that the solution $\xi(\cdot; \varepsilon, T_i)$ obtained in Step 1 satisfies in addition that $\dot{\xi}(0; \varepsilon, T_i) = 0$.

Claim. *By shrinking ε_0 if necessary, we have that if $\dot{\xi}(t; \varepsilon, T) = 0$ for some $t \in (-\infty, 0]$, $\varepsilon \in (0, \varepsilon_0)$ and $T \geq 1$, then $|\ddot{\xi}(t; \varepsilon, T)| \neq 0$.*

Arguing by contradiction, we assume that there exist $\varepsilon_i \to 0$, $\xi_i = \xi(\cdot; \varepsilon_i, T_i)$ and $s_i \in (-\infty, 0]$ such that $\dot{\xi}_i(s_i) = 0$ and $\ddot{\xi}_i(s_i) \to 0$. From the expression of $F_k[\xi_i]$ and (8.3), we have that $\{\xi_i(s_i)\}$ is bounded. Furthermore, the argument in Section 3.2 (see (3.13)), we have $\xi_i \geq -C$ in $(-\infty, s_i]$ for some C independent of i. Recalling (8.5), we have

$$\lim_{i \to \infty} H(s_i, \xi_i(s_i), \dot{\xi}_i(s_i)) = \lim_{i \to \infty} \varepsilon_i \beta \int\limits_{-\infty}^{s_i} e^{\beta\tau} e^{-n\xi_i(\tau)}\, d\tau = 0.$$

By Lemma 3.4, we then have $\lim_{i \to \infty} \ddot{\xi}_i(s_i) = \ddot{\Xi}(0) > 0$, which is a contradiction.

We now fix an arbitrary $0 < \varepsilon < \varepsilon_0$. Let $m(T)$ be the number of solutions to $\dot{\xi}(\cdot; \varepsilon, T) = 0$ in $(-\infty, 0]$. Note that by (8.4), $\dot{\xi}(t; \varepsilon, T) \neq 0$ for large negative t. Thus, by the claim, $m(T)$ is finite for every $T \geq 1$. Since $T \mapsto \xi(\cdot; \varepsilon, T) - \log\cosh(\cdot + T)$ belongs to $C^0([1, \infty); X_2)$, we deduce again from the claim that if an interval $(c, d) \subset [1, \infty)$ is such that $\dot{\xi}(0; \varepsilon, T) \neq 0$ for $T \in (c, d)$, then $m(T)$ is constant for $T \in (c, d)$. On the other hand, by Theorem 3.1(d), $m(T) \to \infty$ as $T \to \infty$. The conclusion is readily seen. $\square$

Appendix A. The values of certain integrals

Lemma A.1. *For $0 < b < 2a$, it holds that*

$$\int\limits_0^\infty (1 + r^2)^{-a}\, r^{b-1}\, dr = \frac{\Gamma(a - \frac{b}{2})\Gamma(\frac{b}{2})}{2\Gamma(a)}.$$

Corollary A.2. *Suppose $n > 0$. We have*

$$\int_0^\infty \frac{r^{n+\beta-1}}{(1+r^2)^n}\,dr = \frac{\Gamma(\frac{n-\beta}{2})\Gamma(\frac{n+\beta}{2})}{2\Gamma(n)} \text{ for } -n < \beta < n,$$

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{n+2}{2}}}\,dr = \frac{1}{n}.$$

Proof. We perform the change of variable $x = \frac{1}{1+r^2}$. Noting that $r^2 = \frac{1-x}{x}$ and $2rdr = -\frac{dx}{x^2}$, we have

$$\int_0^\infty (1+r^2)^{-a}\, r^{b-1}\,dr = \frac{1}{2}\int_0^1 x^{a-\frac{b}{2}-1}(1-x)^{\frac{b}{2}-1}\,dx = \frac{1}{2}B(a-\frac{b}{2},\frac{b}{2}),$$

where B is the beta function. The conclusion follows from a well-known relation between beta and Gamma functions. $\square$

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