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# Symmetry of hypersurfaces and the Hopf Lemma†

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**Abstract:** A classical theorem of A. D. Alexandrov says that a connected compact smooth hypersurface in Euclidean space with constant mean curvature must be a sphere. We give exposition to some results on symmetry properties of hypersurfaces with ordered mean curvature and associated variations of the Hopf Lemma. Some open problems will be discussed.

**Keywords:** symmetry; hypersurfaces; mean curvature; higher order curvature; Hopf Lemma

*Dedicated to Neil Trudinger on his 80th birthday with friendship and admiration.*

## 1. Introduction

H. Hopf established in [3] that an immersion of a topological 2-sphere in $\mathbb{R}^3$ with constant mean curvature must be a standard sphere. He also made the conjecture that the conclusion holds for all immersed connected closed hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature. A. D. Alexandrov proved in [1] that if $M$ is an embedded connected closed hypersurface with constant mean curvature, then $M$ must be a standard sphere. If $M$ is immersed instead of embedded, the conclusion does not hold in general, as shown by W.-Y. Hsiang in [4] for $n \geq 3$ and by Wente in [15] for $n = 2$. A. Ros in [13] gave a different proof for the theorem of Alexandrov making use of the variational properties of the mean curvature.

In this note, we give exposition to some results in [5–9]. It is suggested that the reader read the introductions of [6, 7, 9].

Throughout the paper $M$ is a smooth compact connected embedded hypersurface in $\mathbb{R}^{n+1}$, $k(X) = (k_1(X), \cdots, k_n(X))$ denotes the principal curvatures of $M$ at $X$ with respect to the inner normal, and the

mean curvature of $M$ is

$$H(X) := \frac{1}{n}\left[k_1(X) + \cdots + k_n(X)\right].$$

We use $G$ to denote the open bounded set bounded by $M$.

Li proved in [5] the following result saying that if the mean curvature $H : M \to \mathbb{R}$ has a Lipschitz extension $K : \mathbb{R}^{n+1} \to \mathbb{R}$ which is monotone in the $X_{n+1}$ direction, then $M$ is symmetric about a hyperplane $X_{n+1} = c$.

**Theorem 1.1.** *( [5]) Let $M$ be a smooth compact connected embeded hypersurface without boundary embedded in $\mathbb{R}^{n+1}$, and let $K$ be a Lipschitz function in $\mathbb{R}^{n+1}$ satisfying*

$$K(X', B) \le K(X', A), \quad \forall\, X' \in \mathbb{R}^n,\ A \le B. \tag{1.1}$$

*Suppose that at each point $X$ of $M$ the mean curvature $H(X)$ equals $K(X)$. Then $M$ is symmetric about a hyperplane $X_{n+1} = c$.*

In [5], $K$ was assumed to be $C^1$ for the above result, but the proof there only needs $K$ being Lipschitz.

Li and Nirenberg then considered in [6] and [7] the more general question in which the condition $H(X) = K(X)$ with $K$ satisfying (1.1) is replaced by the weaker, more natural, condition:

**Main Assumption.** For any two points $(X', A), (X', B) \in M$ satisfying $A \le B$ and that $\{(X', \theta A + (1 - \theta)B) : 0 \le \theta \le 1\}$ lies in $\overline{G}$, we have

$$H(X', B) \le H(X', A). \tag{1.2}$$

They showed in [6] that this assumption alone is not enough to guarentee the symmetry of $M$ about some hyperplane $X_{n+1} = c$. The mean curvature $H : M \to \mathbb{R}$ of the counterexample constructed in [6, Figure 4] has a monotone extension $K : \mathbb{R}^{n+1} \to \mathbb{R}$ which is $C^\alpha$ for every $0 < \alpha < 1$, but fails to be Lipschitz. The counterexample actually satisfies (1.2) with an equality. They also constructed a counterexample [6, Section 6] showing that the inequality (1.2) does not imply a pairwise equality.

A conjecture was made in [7] after the introduction of

**Condition S.** $M$ stays on one side of any hyperplane parallel to the $X_{n+1}$ axis that is tangent to $M$.

**Conjecture 1.** *( [7]) Any smooth compact connected embedded hypersurface $M$ in $\mathbb{R}^{n+1}$ satisfying the Main Assumption and Condition S must be symmetric about a hyperplace $X_{n+1} = c$.*

The conjecture for $n = 1$ was proved in [6]. For $n \ge 2$, they introduced the following condition:

**Condition T.** Every line parallel to the $X_{n+1}$-axis that is tangent to $M$ has contact of finite order.

Note that if $M$ is real analytic then Condition T is automatically satisfied.

They proved in [7, Theorem 1] that $M$ is symmetric about a hyperplane $X_{n+1} = c$ under the Main Assumption, Condition S and T, and a local convexity condition near points where the tangent planes are parallel to the $X_{n+1}$-axis. For convex $M$, their result is

**Theorem 1.2.** *( [7]) Let $M$ be a smooth compact convex hypersurface in $\mathbb{R}^{n+1}$ satisfying the Main Assumption and Condition T. Then $M$ must be symmetric about a hyperplane $X_{n+1} = c$.*

The theorem of Alexandrov is more general in that one can replace the mean curvature by a wide class of symmetric functions of the principal curvatures. Similarly, Theorems 1.1 and 1.2 (as well as the more general [7, Theorem 1]) still hold when the mean curvature function is replaced by more general curvature functions.

Consider a triple $(M, \Gamma, g)$: Let $M$ be a compact connected $C^2$ hypersurface without boundary embedded in $\mathbb{R}^{n+1}$, and let $g(k_1, \cdots, k_n)$ be a $C^3$ function, symmetric in $(k_1, \cdots, k_n)$, defined in an open convex neighborhood $\Gamma$ of $\{(k_1(X), \cdots, k_n(X)) \mid X \in M\}$, and satisfy

$$\frac{\partial g}{\partial k_i}(k) > 0, \quad 1 \le i \le n \quad \text{and} \quad \frac{\partial^2 g}{\partial k_i \partial k_j}(k)\eta^i \eta^j \le 0, \qquad \forall\, k \in \Gamma \text{ and } \eta \in \mathbb{R}^n. \tag{1.3}$$

For convex $M$, their result ( [7, Theorem 2]) is as follows.

**Theorem 1.3.** *( [7]) Let the triple $(M, \Gamma, g)$ satisfy (1.3). In addition, we assume that $M$ is convex and satisfies Condition T and the Main Assumption with inequality (1.2) replaced by*

$$g(k(X', B)) \le g(k(X', A)). \tag{1.4}$$

*Then $M$ must be symmetric about a hyperplane $X_{n+1} = c$.*

For $1 \le m \le n$, let

$$\sigma_m(k_1, \cdots, k_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} k_{i_1} \cdots k_{i_m}$$

be the $m$-th elementary symmetric function, and let

$$g_m := (\sigma_m)^{\frac{1}{m}}.$$

It is known that $g = g_m$ satisfies the above properties in

$$\Gamma_m := \{(k_1, \cdots, k_n) \in \mathbb{R}^n \mid \sigma_j(k_1, \cdots, k_n) > 0 \text{ for } 1 \le j \le m\}.$$

It is known that $\Gamma_1 = \{k \in \mathbb{R}^n \mid k_1 + \cdots + k_n > 0\}$, $\Gamma_n = \{k \in \mathbb{R}^n \mid k_1, \cdots, k_n > 0\}$, $\Gamma_{m+1} \subset \Gamma_m$, and $\Gamma_m$ is the connected component of $\{k \in \mathbb{R}^n \mid \sigma_m(k) > 0\}$ containing $\Gamma_n$.

The method of proof of Theorems 1.2 and 1.3 (as well as the more general [7, Theorems 1 and 2]) begins as in that of the theorem of Alexandrov, using the method of moving planes. Then, as indicated in the introduction of [6], one is led to the need for variations of the classical Hopf Lemma. The Hopf Lemma is a local result. The needed variant of the Hopf Lemma to prove Theorem 1.2 (and Conjecture 1) was raised as an open problem ( [7, Open Problem 2]) which remains open. The proof of Theorems 1.2 and 1.3 (as well as the more general [7, Theorems 1 and 2]) was based on the maximum principle, but also used a global argument.

In a recent paper [9], Li, Yan and Yao proved Conjecture 1 using a method different from that of [6] and [7], exploiting the variational properties of the mean curvature. In fact, they proved the symmetry result under a slightly weaker assumption than Condition S:

**Condition S'.** There exists some constant $r > 0$, such that for every $\overline{X} = (\overline{X}', \overline{X}_{n+1}) \in M$ with a horizontal unit outer normal (denote it by $\bar{\nu} = (\bar{\nu}', 0)$), the vertical cylinder $|X' - (\overline{X}' + r\bar{\nu}')| = r$ has an empty intersection with $G$. ($G$ is the bounded open set in $\mathbb{R}^{n+1}$ bounded by the hypersurface $M$.)

**Theorem 1.4.** *( [9]) Let $M$ be a compact connected $C^2$ hypersurface without boundary embedded in $\mathbb{R}^{n+1}$, which satisfies both the Main Assumption and Condition S'. Then $M$ must be symmetric about a hyperplane $X_{n+1} = c$.*

Here are two conjectures, in increasing strength.

**Conjecture 2.** *For $n \geq 2$ and $2 \leq m \leq n$, let $M$ be a compact connected $C^2$ hypersurface without boundary embedded in $\mathbb{R}^{n+1}$ satisfying Condition S (or the slightly weaker Condition S') and $\{(k_1(X), \cdots, k_n(X)) \mid X \in M\} \subset \Gamma_m$. We assume that $M$ satisfies the Main Assumption with inequality (1.2) replaced by*

$$\sigma_m(k(X', B)) \leq \sigma_m(k(X', A)). \tag{1.5}$$

*Then $M$ must be symmetric about a hyperplane $X_{n+1} = c$.*

The next one is for more general curvature functions.

**Conjecture 3.** *For $n \geq 2$, let the triple $(M, \Gamma, g)$ satisfy (1.3). In addition, we assume that $M$ satisfies Condition S (or the slightly weaker Condition S') and the Main Assumption with inequality (1.2) replaced by (1.4). Then $M$ must be symmetric about a hyperplane $X_{n+1} = c$.*

The above two conjectures are open even for convex $M$.

Conjecture 2 can be approached by two ways. One is by the method of moving planes, and this leads to the study of variations of the Hopf Lemma. Such variations of the Hopf Lemma are of its own interest. A number of open problems and conjectures on such variations of the Hopf Lemma has been discussed in [6–8]. For related works, see [11] and [14]. We will give some discussion on this in Section 1.

Conjecture 2 can also be approached by using the variational properties of the higher order mean curvature (i.e., the $\sigma_m$-curvature). If the answer to Conjecture 2 is affirmative, then the inequality in (1.5) must be an equality. This curvature equality was proved in the following lemma, using the variational properties of the $\sigma_m$-curvature:

**Lemma 1.** *(Y. Y. Li, X. Yan and Y. Yao) For $n \geq 2$ and $2 \leq m \leq n$, let $M$ be a compact connected $C^2$ hypersurface without boundary embedded in $\mathbb{R}^{n+1}$ satisfying Condition S'. We assume that $M$ satisfies the Main Assumption, with inequality (1.2) replaced by (1.5). Then (1.5) must be an equality for every pair of points.*

The proof of Theorem 1.4 and Lemma 1 will be sketched in Section 2.

We have discussed in the above symmetry properties of hypersurfaces in the Euclidean space. It is also interesting to study symmetry properties of hypersurfaces under ordered curvature assumptions in the hyperbolic space, including the study of the counter part of Theorem 1.1, Theorem 1.4, and Conjecture 2 in the hyperbolic space. Extensions of the Alexandrov-Bernstein theorems in the hyperbolic space were given by Do Carmo and Lawson in [2]; see also Nelli [10] for a survey on Alexandrov-Bernstein-Hopf theorems.

## 2. Discussion on Conjecture 2 and the proof of Theorem 1.3

Let

$$\Omega = \{(t, y) \mid y \in \mathbb{R}^{n-1}, |y| < 1, 0 < t < 1\}, \tag{2.1}$$

$$u, v \in C^\infty(\overline{\Omega}),$$

$$u \geq v \geq 0, \qquad \text{in } \Omega,$$

$$u(0,y) = v(0,y), \quad \forall\ |y| < 1; \qquad u(0,0) = v(0,0) = 0,$$
$$u_t(0,0) = 0,$$
$$u_t > 0, \qquad \text{in } \Omega.$$

We use $k^u(t,y) = (k_1^u(t,y), \cdots, k_n(t,y))$ to denote the principal curvatures of the graph of $u$ at $(t,y)$. Similarly, $k^v = (k_1^v, \cdots, k_n^v)$ denotes the principal curvatures of the graph of $v$.

Here are two plausible variations of the Hopf Lemma.

**Open Problem 1.** *For $n \geq 2$ and $1 \leq m \leq n$, let $u$ and $v$ satisfy the above. Assume*

$$\begin{cases} \textit{whenever } u(t,y) = v(s,y), 0 < s < 1, |y| < 1, \textit{ then there} \\ \sigma_m(k^u)(t,y) \leq \sigma_m(k^v)(s,y). \end{cases}$$

*Is it true that either*

$$u \equiv v \ \textit{ near } (0,0) \tag{2.2}$$

*or*

$$v \equiv 0 \ \textit{ near } (0,0)? \tag{2.3}$$

A weaker version is

**Open Problem 2.** *In addition to the assumption in Open Problem 1, we further assume that*

$$w(t,y) := \begin{cases} v(t,y), & t \geq 0, |y| < 1 \\ u(-t,y), & t < 0, |y| < 1 \end{cases} \quad \textit{is } C^\infty \textit{ in } \{(t,y) \mid |t| < 1, |y| < 1\}.$$

*Is it true that either (2.2) or (2.3) holds?*

Open Problems 1 and 2 for $m = 1$ are exactly the same as [7, Open Problems 1 and 2], where it was pointed out that an affirmative answer to Open Problem 2 for $m = 1$ would yield a proof of Conjecture 1 by modification of the arguments in [6, 7]. This applies to $2 \leq m \leq n$ as well: An affirmative answer to Open Problem 2 for some $2 \leq m \leq n$ would yield a proof of Conjecture 2 (with Condition S) for the $m$.

As mentioned earlier, the answer to Open Problem 1 for n=1 is yes, and was proved in [6]. For $n \geq 2$, a number of conjectures and open problems on plausible variations to the Hopf Lemma were given in [6–8]. The study of such variations of the Hopf Lemma can first be made for the Laplace operator instead of the curvature operators. The following was studied in [8].

Let $u \geq v$ be in $C^\infty(\overline{\Omega})$ where $\Omega$ is given by (2.1). Assume that

$$u > 0,\ v > 0,\ u_t > 0 \quad \text{in } \Omega$$

and

$$u(0,y) = 0 \quad \text{for } |y| < 1.$$

We impose a main condition for the Laplace operator:

$$\text{whenever } u(t,y) = v(s,y) \text{ for } 0 < t \leq s < 1, \text{there } \Delta u(t,y) \leq \Delta v(s,y).$$

Under some conditions we wish to conclude that

$$u \equiv v \ \text{ in } \Omega. \tag{2.4}$$

The following two conjectures, in decreasing strength, were given in [8].

**Conjecture 4.** *Assume, in addition to the above, that*

$$u_t(0,0) = 0. \tag{2.5}$$

*Then (2.4) holds:*

$$u \equiv v \ \text{ in } \Omega.$$

**Conjecture 5.** *In addition to (2.5) assume that*

$$u(t,0) \text{ and } v(t,0) \text{ vanish at } t = 0 \text{ of finite order.}$$

*Then*

$$u \equiv v \ \text{ in } \Omega.$$

Partial results were given in [8] concerning these conjectures. On the other hand, the conjectures remain largely open.

## 3. Discussion on Conjecture 2 and the proof of Theorem 1.4

Theorem 1.4 was proved in [9] by making use of the variational properties of the mean curvature operator. We sketch the proof of Theorem 1.4 below, see [9] for details.

For any smooth, closed hypersurface $M$ embeded in $\mathbb{R}^{n+1}$, let $V : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a smooth vector field. Consider, for $|t| < 1$,

$$M(t) := \{x + tV(x) \mid x \in M\}, \tag{3.1}$$

and

$$S(t) := \int_{M(t)} d\sigma = \text{area of } M(t).$$

It is well known that

$$\frac{d}{dt} S(t)\Big|_{t=0} = -\int_M V(x) \cdot \nu(x) H(x) d\sigma(x), \tag{3.2}$$

where $H(x)$ is the mean curvature of $M$ at $x$ with respect to the inner unit normal $\nu$.

Define the projection map $\pi : (x', x_{n+1}) \to x'$, and set $R := \pi(M)$.

Condition S' assures that $\nu(\bar{x})$, $\bar{x} \in M$, is horizontal iff $\bar{x}' \in \partial R$; $\partial R$ is $C^{1,1}$ (with $C^{1,1}$ normal under control); and

$$M = M_1 \cup M_2 \cup \widehat{M},$$

where $M_1, M_2$ are respectively graphs of functions $f_1, f_2 : R^\circ \to R$, $f_1, f_2 \in C^2(R^\circ)$, $f_1 > f_2$ in $R^\circ$, and $\widehat{M} := \{(x', x_{n+1}) \in M \mid x' \in \partial R\} \equiv M \cap \pi^{-1}(\partial R)$. Note that $f_1, f_2$ are not in $C^0(R)$ in general.

**Lemma 2.**

$$H(x', f_1(x')) = H(x', f_2(x')) \quad \forall\ x' \in R^\circ. \tag{3.3}$$

*Proof.* Take $V(x) = e_{n+1} = (0, ..., 0, 1)$, and let $M(t)$ and $S(t)$ be defined as above with this choice of $V(x)$. Clearly, $S(t)$ is independent of $t$. So we have, using (3.2) and the order assumption on the mean curvature, that

$$\begin{aligned} 0 &= \frac{d}{dt}S(t)\Big|_{t=0} = -\sum_{i=1}^{2}\int_{M_i} e_{n+1}\cdot\nu(x)H(x)d\sigma(x) \\ &= -\int_{R^\circ}\left[H(x', f_1(x')) - H(x', f_2(x'))\right]dx' \geq 0. \end{aligned} \tag{3.4}$$

Using again the order assumption on the mean curvature we obtain The curvature equality (3.3).

For any $v \in C^\infty(R^n)$, let $V(x) := v(x')e_{n+1}$, and let $M(t)$ and $S(t)$ be defined as above with this choice of $V(x)$. We have, using (3.2) and (3.3), that

$$\begin{aligned} 0 &= \frac{d}{dt}S(t)\Big|_{t=0} = -\sum_{i=1}^{2}\int_{M_i} v(x')e_{n+1}\cdot\nu(x)H(x)d\sigma(x) \\ &= -\int_{R^\circ} v(x')\left[H(x', f_1(x')) - H(x', f_2(x'))\right]dx' = 0. \end{aligned} \tag{3.5}$$

Theorem 1.4 is proved by contradition as follows: If $M$ is not symmetric about a hyperplane, then $\nabla(f_1 + f_2)$ is not identically zero. We will find a particular $V(x) = v(x')e_{n+1}$, $v \in C^2_{loc}(R^\circ)$, to make

$$\frac{d}{dt}S(t)\Big|_{t=0} \neq 0,$$

which contradicts to (3.5).

Write

$$S(t) = \sum_{i=1}^{2}\int_{R^\circ}\sqrt{1 + |\nabla[f_i(x') + v(x')]t|^2}dx' + \widehat{S},$$

where $\widehat{S}$, the area of the vertical part of $M$, is independent of $t$ (since $v$ is zero near $\partial R$, so the vertical part of $M$ is not moved).

A calculation gives

$$\frac{d}{dt}S(t)\Big|_{t=0} = \int_{R^\circ}\sum_{i=1}^{2}\left[\nabla A(\nabla f_1(x')) - \nabla A(-\nabla f_2(x'))\right]\cdot\nabla v(x')dx',$$

where

$$A(q) := \sqrt{1 + |q|^2}, \quad q \in R^n.$$

We know that

$$\nabla A(q) = \frac{q}{\sqrt{1 + |q|^2}} \quad \text{and} \quad \nabla^2 A(q) \geq (1 + |q|^2)^{-3/2}I > 0 \ \ \forall q.$$

So $[\nabla A(q_1) - \nabla A(q_2)]\cdot(q_1 - q_2) > 0$ for any $q_1 \neq q_2$.

If $\nabla(f_1 + f_2) \in C^2_{loc}(R^\circ)u \setminus \{0\}$, we would take $v = \nabla(f_1 + f_2)$ and obtain

$$\begin{aligned}\frac{d}{dt}S(t)\Big|_{t=0} &= \int_{R^\circ} [\nabla A(\nabla f_1(x')) - \nabla A(-\nabla f_2(x'))] \cdot \nabla v(x')dx' \\ &= \int_{R^\circ} [\nabla A(\nabla f_1(x')) - \nabla A(-\nabla f_2(x'))] \cdot [\nabla f_1(x') + \nabla f_2(x')]dx' \\ &> 0.\end{aligned}$$

In general, $\nabla(f_1 + f_2)$ would not be in $C^2_{loc}(R^\circ)$. It turns out that Condition S' allows us to do a smooth cutoff near $\partial R$, and conclude the proof. We skip the crucial details, which can be found in the last few pages of [9].

Now we give the

**Proof of Lemma 1.** The proof is similar to that of Lemma 2, see also the proof of [9, Proposition 3] for more details. We still take $V(X) = e_{n+1}$ and let $M(t)$ be as in (3.1). Consider

$$S_{m-1}(t) := \int_{M(t)} \sigma_{m-1}(x)d\sigma.$$

Clearly, $S_{m-1}(t)$ is independent of $t$.

The variational properties of higher order curvature [12, Theorem B] gives

$$\frac{d}{dt}S_{m-1}(t)\Big|_{t=0} = -m\int_M V(x) \cdot \nu(x)\sigma_m(x)d\sigma(x),$$

thus the same argument as (3.4) yields

$$\begin{aligned}0 &= \frac{d}{dt}S_{m-1}(t)\Big|_{t=0} = -m\int_M V(x) \cdot \nu(x)\sigma_m(x)d\sigma(x) \\ &= -\int_{R^\circ} [\sigma_m(x', f_1(x')) - \sigma_m(x', f_2(x'))]\, dx' \geq 0.\end{aligned}$$

We deduce from the above, using the curvature inequality (1.5), that the equality in (1.5) must hold for every pair of points. Lemma 1 is proved.

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## Conflict of interest

The author declares no conflict of interest.

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