



Marginal dynamics of interacting diffusions on unimodular Galton–Watson trees

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Abstract

Consider a system of homogeneous interacting diffusive particles labeled by the nodes of a unimodular Galton–Watson tree, where the state of each node evolves infinitesimally like a d -dimensional diffusion whose drift coefficient depends on (the histories of) its own state and the states of neighboring nodes, and whose diffusion coefficient depends only on (the history of) its own state. Under suitable regularity assumptions on the coefficients, an autonomous characterization is obtained for the marginal distribution of the dynamics of the neighborhood of a typical node in terms of a certain local equation, which is a new kind of stochastic differential equation that is nonlinear in the sense of McKean. This equation describes a finite-dimensional non-Markovian stochastic process whose infinitesimal evolution at any time depends not only on the structure and current state of the neighborhood, but also on the conditional law of the current state given the past of the states of neighboring nodes until that time. Such marginal distributions are of interest because they arise as weak limits of both marginal distributions and empirical measures of interacting diffusions on many sequences of sparse random graphs, including the configuration model and Erdős–Rényi graphs whose average degrees converge to a finite non-zero limit. The results obtained com-

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plement classical results in the mean-field regime, which characterize the limiting dynamics of homogeneous interacting diffusions on complete graphs, as the number of nodes goes to infinity, in terms of a corresponding nonlinear Markov process. However, in the sparse graph setting, the topology of the graph strongly influences the dynamics, and the analysis requires a completely different approach. The proofs of existence and uniqueness of the local equation rely on delicate new conditional independence and symmetry properties of particle trajectories on unimodular Galton–Watson trees, as well as judicious use of changes of measure.

Keywords Interacting diffusions · Sparse graphs · Random graphs · Local weak convergence · Mean-field limits · Nonlinear Markov processes · Erdős–Rényi graphs · Configuration model · Unimodularity · Markov random fields

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1 Introduction

1.1 Background and motivation

Given a (possibly random) simple, (almost surely) locally finite rooted graph $G = (V, E)$, consider interacting diffusions of the form

$$dX_v^G(t) = b(X_v^G(t), \mu_v^G(t))dt + \sigma(X_v^G(t))dW_v(t), \quad v \in V, \quad t \geq 0, \quad (1.1)$$

with initial condition $x(0) \in (\mathbb{R}^d)^V$. Here, $(W_v)_{v \in V}$ are independent d -dimensional standard Brownian motions, b and σ are suitably regular drift and diffusion coefficients, and $\mu_v^G(t)$ is the local (random) empirical measure of the states of the neighbors of v at time $t \geq 0$:

$$\mu_v^G(t) = \frac{1}{|N_v(G)|} \sum_{u \in N_v(G)} \delta_{X_u^G(t)},$$

with $N_v(G) = \{u \in V : (u, v) \in E\}$ denoting the neighborhood of the vertex v in the graph G . (By convention, set $\mu_v^G(t) = \delta_0$ when $N_v(G)$ is empty, that is, when the vertex v is isolated.) Important quantities of interest include the dynamics of the state of a “typical” vertex and, when the graph G is finite, the (global) empirical measure process defined by

$$\bar{\mu}^G(t) = \frac{1}{|V|} \sum_{v \in V} \delta_{X_v^G(t)}, \quad t \geq 0. \quad (1.2)$$

However, these systems are typically too large and complex to be analytically or numerically tractable. Therefore, it is natural to seek approximations that are provably accurate in a suitable asymptotic regime.

Classical works of McKean, Vlasov and others (see [23, 26, 35, 44] and references therein) focused on such particle systems when $G = K_n$, the complete graph on n vertices. They showed that, under suitable conditions, the limit of $X_\emptyset^{K_n}$ in (1.1), where \emptyset is a root vertex, chosen uniformly at random from V , is described by the following nonlinear Markov process:

$$dX(t) = b(X(t), \bar{\mu}(t))dt + \sigma(X(t))dW(t), \quad \bar{\mu}(t) = \mathcal{L}(X(t)), \quad t \geq 0, \quad (1.3)$$

where $\bar{\mu}(t)$ is the (deterministic) weak limit, as $n \rightarrow \infty$, of the global empirical measure $\bar{\mu}^{K_n}(t)$, and $\mathcal{L}(Z)$ denotes the law of a random variable Z . The measure-valued function $\bar{\mu}(\cdot)$ can also be characterized as the unique solution to a nonlinear partial differential equation (namely, the forward Kolmogorov equation associated with this process), whence the name *nonlinear* Markov process. The key property that leads to such a characterization is the observation that particles interact only *weakly*, with the influence of any single particle on any other particle being of order $1/n$. This leads to asymptotic independence of any finite collection of particles and convergence of the random (global) empirical measure process of the finite particle systems to a deterministic limit (see [35, 44] for further discussion of this phenomenon, known as propagation of chaos). An alternative two-step perspective to mean-field limits, taken in [24], is to first use exchangeability to show that X^{K_n} converges to a limit, which is the unique solution of a countably infinite coupled system of diffusions, and then show that the marginal of any vertex in this infinite system can be autonomously described as the nonlinear Markov process in (1.3).

Given the above intuition, it is natural to expect that asymptotic independence and the same mean-field characterization (1.3) for the limiting dynamics of a typical node may continue to hold for suitably “dense” graph sequences $\{G_n\}_{n \in \mathbb{N}}$, where each graph is not necessarily complete, but the (minimum or average) degree of the graphs grows to infinity. Indeed, several recent works [4, 8, 10, 33, 36, 39] have shown that either asymptotic independence or a mean-field characterization like (1.3) continues to hold under different sets of assumptions on the precise nature of denseness of the graph sequence. As in the complete graph case, these works exploit the fact that the local interaction strength at a vertex is inversely proportional to the degree at that vertex, and thus vanishes in dense regimes. The only difference from the complete graph setting is that the proofs are more involved due to a lack of full exchangeability.

In contrast, very few works have studied the limiting behavior of $\bar{\mu}^{G_n}$ or $X_\phi^{G_n}$ in the complementary sparse graph regime, that is, when the average degrees of possibly random graphs in the sequence $\{G_n\}_{n \in \mathbb{N}}$ remain uniformly bounded as the size of the graph goes to infinity. Nonetheless, sparse complex networks and dynamical processes thereon are ubiquitous in applied sciences [3], and the limitations of traditional mean field approximations are increasingly well understood [17]. A particularly interesting diffusive model of the form (1.1) is the Kuramoto model, corresponding to constant σ and $b(x, m) = c \int \sin(y - x)m(dy)$ for $c > 0$, in dimension $d = 1$. The Kuramoto model on complex networks has been studied extensively in recent years, as a paradigm for self-organization and synchronization phenomena [2]. Sparse regimes appear to exhibit a particularly rich range of synchronization behaviors, analyzed in terms of the multiplicity and stability of equilibrium points; see [38, 42] for representative work.

In the sparse regime, neighboring particles are strongly interacting and do not become asymptotically independent as the graph size goes to infinity, and so the limiting dynamics of any finite set of particles is no longer described in terms of the mean-field limit. In Theorem 3.3 of a companion paper [29] (which extends

results of a previous version [27]), we consider a more general class of (possibly non-Markovian) dynamics than (1.1), and show under broad assumptions that if the sequence $\{(G_n, X^{G_n}(0))\}_{n \in \mathbb{N}}$ of (possibly random) rooted graphs, marked with initial conditions, converges in distribution (in the local weak sense) to a limit marked (random) graph $(G, X^G(0))$, then $\{(G_n, X^{G_n})\}_{n \in \mathbb{N}}$ also converges in distribution (again in the local weak sense) to (G, X^G) . This, in particular, implies that the marginal dynamics at the root $X_\emptyset^{G_n}$ converges in law to the corresponding marginal dynamics X_\emptyset^G on the limit graph G , where \emptyset denotes an appropriate root vertex in G . Furthermore, it was also shown in Theorem 3.7 of [29] that if a sequence of finite marked graphs $\{(G_n, X^{G_n}(0))\}_{n \in \mathbb{N}}$ converges in a slightly stronger sense (namely in *probability* in the local weak sense) to a limit graph $(G, X^G(0))$, then the corresponding sequence of global empirical measures $\{\bar{\mu}^{G_n}\}_{n \in \mathbb{N}}$ converges weakly to the law of the root marginal X_\emptyset^G in the infinite graph.

In many cases of interest, the limit graph G is a so-called unimodular Galton–Watson (UGW) tree (see Definition 3.8). This is the case, for example, when G_n is the Erdős–Rényi graph on n vertices with parameter $p_n \in (0, 1)$, with $np_n \rightarrow \theta \in (0, \infty)$, or the graphs G_n are sampled from a configuration model with a converging empirical degree sequence with finite non-zero first moment, and the root is chosen uniformly at random (see, e.g., [11, 46] or Section 2.2.3 of [29]). Furthermore, for suitable initial conditions $(X_v^{G_n}(0))_{v \in V_n}$, for example, when they are independent and identically distributed (i.i.d.) with a distribution independent of n (or, more generally, distributed according to a Gibbs measure with a pairwise interaction potential independent of n), the corresponding sequence of marked graphs $(G_n, X_v^{G_n}(0))_{v \in V_n}$ also converges to the limit graph G marked with initial conditions coming from the same i.i.d. or Gibbs distribution, respectively (see, e.g., Propositions 2.14 or 2.15 of [29]). Thus the above marginal dynamics and global empirical measure convergence results apply to such sequences.

The only other work we know of that considers asymptotic limits for interacting diffusions in the sparse graph regime is [38], which considers a slightly different model of Markovian interacting diffusions with identity diffusion coefficient, weighted pairwise interactions, an i.i.d. random environment and i.i.d. initial conditions, when the largest vertex degree in G_n is additionally assumed to be of order $|G_n|^{o(1)}$. They prove a local convergence result for the interacting processes, but only state, without proof, an empirical measure convergence result. However, as shown in [29] (see Theorem 3.9 and Theorem 6.4 therein), the empirical measure convergence to a deterministic limit can fail under local convergence in distribution, rather than in probability, of the graph sequences, thus demonstrating that the proof of empirical measure convergence is more subtle in the sparse graph regime than in the complete or dense graph regimes.

1.2 Our contributions

1.2.1 Discussion of our results

¹ Both works [29, 38] can be viewed as implementing, for sparse graph sequences, the first step of the two-step approach of [24] mentioned above for complete graphs, namely showing that the weak limits of both $\{X_{\phi_n}^{G_n}\}_{n \in \mathbb{N}}$ and $\{\bar{\mu}^{G_n}\}$ exist and coincide with the law of X_ϕ^G , the root marginal of X^G , which is the unique solution to a countably infinite coupled system of SDEs. However, both these works leave open the important question of providing an autonomous characterization of the law of the root marginal X_ϕ^G . The main contribution of this article is a resolution of this issue in the case when $G = \mathcal{T}$ is a UGW tree. Specifically, we consider the interacting particle system on \mathcal{T} defined in (1.1) (or rather a possibly non-Markovian generalization of the dynamics described in Sect. 3), and show that the marginal dynamics of the root particle and its neighbors can be characterized by an autonomous system of equations that we call the *local equation*. The choice of a UGW tree is in some sense canonical in view of the results in [27, 29, 38] mentioned above that show that the law of the root particle dynamics on a UGW tree \mathcal{T} arises as the limit of both marginal dynamics and the global empirical measure processes of diffusive particle systems on many sequences of sparse random graphs of growing size. Thus, the local equation can be viewed as the analogue, in the sparse graph setting, of the equation (1.3) that characterizes the nonlinear Markov process describing the limiting evolution of a typical particle for suitably dense graph sequences.

To the best of our knowledge, prior to this work, there did not exist even a conjecture regarding the form of the limiting marginal dynamics of a typical particle or global empirical measure process in the sparse graph regime. In this regime the graph structure clearly plays a role, and new ideas are required. In the particular case when \mathcal{T} is the (deterministic) κ -regular tree \mathbb{T}_κ the local equation may be viewed as a path-dependent McKean–Vlasov stochastic differential equation with a non-standard measure-dependence; it characterizes an $(\mathbb{R}^d)^{1+\kappa}$ -valued process whose infinitesimal evolution at any time t depends not only on its current state at time t but also on the conditional law of the current state given the histories of the states of part of the neighborhood up to time t (see Definition 3.5). Notably, even when the original interacting process on \mathbb{T}_κ is Markovian, the solution to the local equation is a non-Markovian process and it is also nonlinear in the sense that at any time, its evolution depends on the law of the process up to that time, although in a non-standard way via a conditional distribution associated with the law. However, while in general these are complicated non-Markovian processes, recent work that establishes analogous results for jump processes [13–15] shows that for certain classes of models, the local equations in fact

¹ This paper, along with [28, 29], supersedes the earlier arXiv version [27], after reorganizing and expanding upon several aspects of the material. Notably, this paper removes the assumption of bounded drift in the derivation of the local equation, whereas [29] sharpens and strengthens the results on local weak convergence of particle systems, and [28] elaborates further on related yet rather separate conditional independence properties. These three papers treat very different, complementary aspects of the same class of particle systems and may be read independently.

reduce to a very tractable system of (nonlinear) Markovian stochastic equations (see, e.g., [13] or Section 4.3.2 of [40]).

To provide insight into the form of the local equation, in Sect. 1.2.2 below we first derive the local equation for a particle system in the simplest case when the UGW tree is \mathbb{T}_2 , or equivalently, \mathbb{Z} . Then, in Sect. 1.2.3 we discuss the significant additional complications that arise in the general case of a random UGW tree \mathcal{T} . The only other result that we are aware of that provides an autonomous characterization of marginals of an infinite system of interacting diffusions on a sparse graph was obtained recently in [12], which treats a Markovian interacting diffusive particle system with identity diffusion coefficient in the special case where the interaction graph is a directed line, without any feedback of the interactions. In this specific setting, a coupling argument is used to obtain an autonomous characterization of the law of the trajectories of any contiguous set of particles in terms of a non-linear diffusion process. As we show in Sect. 1.2.2, even on a line derivation of such an autonomous characterization is more complicated when the graph is no longer directed.

1.2.2 The local equation for a particle system on the line graph

Consider the particular diffusive particle system

$$dX_v(t) = [\beta(X_v(t), X_{v+1}(t)) + \beta(X_v(t), X_{v-1}(t))]dt + \sigma(X_v(t))dW_v(t), \quad v \in \mathbb{Z}, \quad (1.4)$$

where $(X_v(0))_{v \in \mathbb{Z}}$ are i.i.d., and $\beta : (\mathbb{R}^d)^2 \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ are assumed to be sufficiently regular (see Assumption A). Note that this is a particular case of the dynamics (1.1) when G is equal to the (deterministic) 2-regular tree \mathbb{T}_2 , which can be identified with \mathbb{Z} , and the drift b is linear in the measure variable: $b(x, \nu) = 2 \int_{\mathbb{R}^d} \beta(x, y) \nu(dy)$ for $x \in \mathbb{R}^d$ and ν is a probability measure on \mathbb{R}^d . We will also assume that the dependence of β on the second variable is non-trivial so that we have a system of diffusions that are truly interacting. For any $t > 0$, let

$$X_v[t] := (X_v(s))_{s \in [0, t]}$$

represent the trajectory of X_v in the interval $[0, t]$, and for any subset $A \subset \mathbb{Z}$, let $X_A[t] := (X_v[t])_{v \in A}$. Identifying the root node with $0 \in \mathbb{Z}$, we would like to understand the law of the dynamics of the root marginal X_0 , but it turns out that to obtain an autonomous description, one should instead consider the marginal dynamics $X_{\{-1, 0, 1\}} = (X_{-1}, X_0, X_1)$ of the root *and its neighborhood*, rather than just the root. Furthermore, this also allows one to understand correlations between neighboring particles. The characterization via the local equation entails three key ingredients.

(i) *Markov random field structure* First, note that the dynamics of X_0 is completely endogenous in that it only depends on the states of X_{-1} and X_1 , which are part of the neighborhood. On the other hand, the evolution of X_{-1} depends on X_{-2} , the state of node -2 , which lies outside the set $\{-1, 0, 1\}$. Therefore, in order to obtain an autonomous description of the law of the dynamics of $X_{\{-1, 0, 1\}}$, we need to be able to express the conditional law of $X_{-2}(t)$ given $X_{\{-1, 0, 1\}}$ in terms of the (joint) law of

$X_{\{-1,0,1\}}$. As a key first step towards achieving this goal, we establish the following conditional independence property of the particle system (1.4): for each $t > 0$,

$$(X_j[t])_{j < i} \perp\!\!\!\perp (X_j[t])_{j > i+1} \mid (X_i[t], X_{i+1}[t]), \quad \forall i \in \mathbb{Z} \quad (1.5)$$

where for random elements Z_1, Z_2, Z_3 , we use $Z_1 \perp\!\!\!\perp Z_2 \mid Z_3$ to denote that Z_1 and Z_2 are conditionally independent given Z_3 . In other words, we show that for each $t > 0$, $(X_i[t])_{i \in \mathbb{Z}}$ is a second-order Markov chain on \mathbb{Z} .

At first glance, one might conjecture that $(X_j(t))_{j < i} \perp\!\!\!\perp (X_j(t))_{j > i} \mid X_i(t)$ for each $i \in \mathbb{Z}$, that is, for every fixed $t > 0$, the *states* $(X_i(t))_{i \in \mathbb{Z}}$ form a *first-order* Markov chain (on \mathbb{Z}). However, this conjecture is incorrect, because conditioning on $X_i(t)$ provides information about the trajectory $X_i[t]$, which in turn influences the entire trajectories $X_{i-1}[t]$ and $X_{i+1}[t]$. This observation may then prompt the modified conjecture that the *trajectories* $(X_i[t])_{i \in \mathbb{Z}}$ form a first-order Markov chain, i.e., $(X_j[t])_{j < i} \perp\!\!\!\perp (X_j[t])_{j > i} \mid X_i[t]$ for each $i \in \mathbb{Z}$. This too turns out to be incorrect, because conditioning on $X_i[t]$ induces correlations between the Brownian motions $(W_j)_{j < i}$ and $(W_j)_{j > i}$, in a way that the conditioning in (1.5) remarkably does not.

In fact, Theorem 2.7 of [28] shows that for any locally finite graph G and X^G as in (1.1), for every $t > 0$, the trajectories $(X_v^G[t])_{v \in V}$ form a *local second-order Markov random field (MRF)* (assuming the initial conditions do), in the sense that

$$X_A^G[t] \perp\!\!\!\perp X_B^G[t] \mid X_{\partial^2 A}^G[t], \quad \forall A \subset V \text{ finite}, B \subset V \setminus (A \cup \partial^2 A), \quad (1.6)$$

where $\partial^2 A$ is the set of nodes at distance one or two from A (see Sect. 2.1 for graph-theoretic terminology and Sect. 5 for a discussion of MRFs). We would like to emphasize, however, that the conditional independence property (1.5) required here is not implied by the local MRF property established in [28]. Indeed, to obtain the first conditional independence statement in (1.5) one would need to apply (1.6) with $A = \{j \in \mathbb{N} : j < i\}$ and $B = \{j \in \mathbb{N} : j > i+1\}$ in (1.6). In particular, we need (1.6) to also hold for certain infinite sets A . This is analogous to the distinction between (tree-indexed) first-order Markov chains and first-order local MRFs; the former often form a proper subset of the latter, as explained in [16, Chapters 10–12]. More generally, an extension from a *local* MRF property to a *global* MRF property, in which $A \subset V$ in (1.6) is allowed to be infinite, is highly non-trivial and can fail in general; see [20, 22, 47] for works in other contexts that illustrate the underlying subtleties. Nevertheless, we show that the *global* MRF property does hold in our setting; see Propositions 3.17 and 3.18 for a proof in the more general context of random UGW trees. For further intuition into this second-order MRF property and explicit examples that illustrate why the first-order counterparts fail, we refer the reader to Section 3.3 of [28].

(ii) *A projection theorem and symmetry considerations* We now discuss the second ingredient of the proof, recalling that we are interested in an autonomous characterization of $X_{\{-1,0,1\}}$, where $(X_v)_{v \in \mathbb{Z}}$ are as in (1.4). Using an optional projection argument known from filtering theory (see Appendix A), we can conclude that (extending the probability space if necessary) there exist independent Brownian motions

$(\tilde{W}_{-1}, \tilde{W}_0, \tilde{W}_1)$ such that $X = (X_{-1}, X_0, X_1)$ satisfies

$$X_v(t) = \tilde{b}_v(t, X)dt + \sigma(X_v(t))d\tilde{W}_v(t), \quad v \in \{-1, 0, 1\},$$

where, with \mathcal{C} denoting the space of \mathbb{R}^d -valued continuous functions on $[0, \infty)$, $\tilde{b}_v : [0, \infty) \times \mathcal{C}^{(-1,0,1)} \mapsto \mathbb{R}^d$ is a progressively measurable version of the conditional expectation:

$$\tilde{b}_v(t, x) := \mathbb{E}\left[\beta(X_v(t), X_{v+1}(t)) + \beta(X_v(t), X_{v-1}(t)) \mid X_{\{-1,0,1\}}[t] = x[t]\right], \quad v \in \{-1, 0, 1\},$$

where we recall $x[t] = (x(s))_{s \in [0,t]}$. Clearly, the drift coefficient for the root or zero particle remains the same as in the original system described in (1.4):

$$\tilde{b}_0(t, x) = \beta(x_0(t), x_1(t)) + \beta(x_0(t), x_{-1}(t)).$$

On the other hand, \tilde{b}_1 and \tilde{b}_{-1} do not coincide with the corresponding drifts in the original system, but we can simplify the expressions for them using the conditional independence relation of (1.5) along with symmetries of the particle system. Precisely, as justified below, we have

$$\begin{aligned} \tilde{b}_{-1}(t, x) &= \mathbb{E}[\beta(X_{-1}(t), X_0(t)) + \beta(X_{-1}(t), X_{-2}(t)) \mid X_{\{-1,0,1\}}[t] = x[t]] \\ &= \beta(x_{-1}(t), x_0(t)) + \mathbb{E}[\beta(X_{-1}(t), X_{-2}(t)) \mid (X_{-1}, X_0)[t] = (x_{-1}, x_0)[t]] \\ &= \beta(x_{-1}(t), x_0(t)) + \mathbb{E}[\beta(X_0(t), X_{-1}(t)) \mid (X_0, X_1)[t] = (x_{-1}, x_0)[t]]. \end{aligned}$$

Indeed, the crucial steps are the second line, which follows from the conditional independence of $X_{-2}[t]$ and $X_1[t]$ given $X_{\{-1,0\}}[t]$, and the third line, which follows from the shift-invariance of the particle system on \mathbb{Z} , which gives equality in law of (X_{-2}, X_{-1}, X_0) and (X_{-1}, X_0, X_1) . We can derive an analogous expression for $\tilde{b}_1(t, x)$ by using the conditional independence of $X_2[t]$ and $X_{-1}[t]$ given $X_{\{0,1\}}[t]$, and the equality in law between (X_2, X_1, X_0) and (X_{-1}, X_0, X_1) which now follows from both the shift-invariance and reflection-invariance (around $0 \in \mathbb{Z}$) of X :

$$\begin{aligned} \tilde{b}_1(t, x) &= \mathbb{E}[\beta(X_1(t), X_2(t)) \mid (X_1, X_0)[t] = (x_1, x_0)[t]] + \beta(x_1(t), x_0(t)) \\ &= \mathbb{E}[\beta(X_0(t), X_{-1}(t)) \mid (X_0, X_1)[t] = (x_1, x_0)[t]] + \beta(x_1(t), x_0(t)), \end{aligned}$$

for $t > 0$ and $x \in \mathcal{C}^{(-1,0,1)}$. In summary, if we define

$$\tilde{\gamma}_t(x, y) := \mathbb{E}[\beta(X_0(t), X_{-1}(t)) \mid (X_0, X_1)[t] = (x, y)[t]], \quad (x, y) \in \mathcal{C}^2, \quad (1.7)$$

then we find that $X = X_{\{-1,0,1\}}$ solves the coupled system

$$\begin{aligned} dX_{-1}(t) &= [\beta(X_{-1}(t), X_0(t)) + \tilde{\gamma}_t(X_{-1}, X_0)]dt + \sigma(X_{-1}(t))d\tilde{W}_{-1}(t), \\ dX_0(t) &= [\beta(X_0(t), X_1(t)) + \beta(X_0(t), X_{-1}(t))]dt + \sigma(X_0(t))d\tilde{W}_0(t), \\ dX_1(t) &= [\tilde{\gamma}_t(X_1, X_0) + \beta(X_1(t), X_0(t))]dt + \sigma(X_1(t))d\tilde{W}_1(t), \end{aligned} \quad (1.8)$$

where \tilde{W}_{-1} , \tilde{W}_0 and \tilde{W}_1 are independent d -dimensional Brownian motions. Modulo some additional technical conditions, this is precisely the \mathbb{T}_2 local equation associated with the particle system (1.4); see Definition 3.5 with $\kappa = 2$. Observe that even though the original system (1.4) describes a (linear) Markov process, its marginal $X_{\{-1,0,1\}}$, as described by the system (1.8), is a nonlinear, non-Markovian process since $\tilde{\gamma}_t$ is a functional of the law of $X_{\{-1,0,1\}}[t]$ of the process and it takes as arguments the past of coordinates of the process (up to time t). However, also note that this dependence ensures that the coupled system (1.8) is autonomously defined.

(iii) *Proof of well-posedness* The final step of the proof, carried out in full detail in Sect. 4.2, is to show that the law of $X_{\{-1,0,1\}}$ is the unique (weak) solution to the local equation (1.8). Coupling methods combined with Banach fixed point arguments, which are commonly used in the analysis of more standard nonlinear Markov processes that arise as mean-field limits, are rendered unsuitable by the complicated appearance of conditional laws in the local equation. Instead, we once more exploit the conditional independence and symmetry properties described above to essentially rebuild the law of $X_{\mathbb{Z}}$ using just the joint law of the root neighborhood. Specifically, given a solution (Y_{-1}, Y_0, Y_1) to the local equation, let $\mu(dy_{-1}, dy_0, dy_1)$ denote the joint law of the root neighborhood (Y_{-1}, Y_0, Y_1) , and let $\Gamma(dy_1; Y_0, Y_{-1})$ denote the conditional law of Y_{-1} given (Y_0, Y_1) . By (reflection) symmetry, the conditional law of Y_1 given (Y_0, Y_{-1}) is precisely $\Gamma(dy_1; Y_0, Y_{-1})$. We then consider the unique probability measure on $\mathcal{C}^{\mathbb{Z}}$ with (consistent) finite-dimensional distribution on $\mathcal{C}^{\mathbb{Z} \cap [-n,n]}$ given by

$$\mu(dy_{-1}, dy_0, dy_1) \prod_{i=1}^{n-1} \Gamma(dy_{i+1}; y_i, y_{i-1}) \Gamma(dy_{-(i+1)}; y_{-i}, y_{-(i-1)}) \quad (1.9)$$

for each $n \in \mathbb{N}$, where the product of the kernels reflects the conditional independence property of $X_{\mathbb{Z}}$ stated in (1.5). The crux of the argument is to show that this probability measure on $\mathcal{C}^{\mathbb{Z}}$ is the law of a solution of the infinite SDE system (1.4); uniqueness for the local equation then follows from uniqueness for the infinite particle system. The full justification is much more involved but ultimately rests upon conditional independence and symmetry arguments like those used above, as well as judicious use of Girsanov's theorem to characterize Γ and the measures in (1.9). It is worth emphasizing that, by purely measure-theoretic arguments, the law of *any* random sequence $X_{\mathbb{Z}} = (X_i)_{i \in \mathbb{Z}}$ that is invariant under shifts and reflections, and also satisfies the conditional independence property (1.5), is uniquely determined by its root neighborhood marginal via the construction in (1.9). However, the difficulty lies in transferring additional properties (such as the property that the collection $X_{\mathbb{Z}}$ satisfies a certain SDE) from the marginal to the full configuration, and vice versa.

The proof of uniqueness outlined here notably relies on the uniqueness of the infinite particle system $X_{\mathbb{Z}}$ defined by (1.4). Under the more restrictive assumption of bounded drift, it is possible to avoid any reference to the infinite system and instead, provide a direct proof using relative entropy estimates. See the extended arXiv version for details (Sections 1.2.2(iii) and 4.3 of arXiv:2009.11667).

1.2.3 Additional challenges on random trees

As we have seen above, three main ingredients of the proof of characterization of the law of marginal dynamics in terms of the local equation include a certain conditional independence property that is similar in spirit to the second-order MRF property, a stochastic analytic result on projections of Itô processes, and symmetry considerations. These arguments can be extended to more general dynamics and \mathbb{T}_κ for general $\kappa > 2$ in an analogous manner, although the proofs are more involved, with the main change being that one now exploits the class of symmetries arising from the automorphism group on \mathbb{T}_κ , which can be visualized as translation and rotation symmetries (see Sect. 3.2.1 for the form of the local equation in this case). However, the intuition described above is somewhat limited to deterministic trees.

On random UGW trees, the proof of the characterization of marginal dynamics via the local equation (described in Definition 3.9 and Sect. 3.3), is an order of magnitude harder, and requires new ingredients. Firstly, the conditional independence property must now be established in an annealed sense, looking jointly at the particle system and the structure of the underlying tree, and the statement and proof are significantly more involved (see Proposition 3.17). As for the second step, while the projection argument is similar, the symmetry considerations must be significantly altered, as they are not so useful in the quenched form used for deterministic regular trees. Instead, the appropriate notion of symmetry here turns out to be *unimodularity*, which is defined by a certain *mass-transport principle* (elucidated in Sect. 7.2). This can be viewed as a sort of stationarity property, which is often loosely described as the property that *the root is equally likely to be any vertex* [1], although the precise formulation is more subtle. In the course of the proof of our main result, we show in Proposition 7.3 that this unimodularity property is preserved by dynamics of the form (1.1), which may be of interest in its own right.

The unimodularity property is applied to establish a key identity (see Proposition 3.18) that relates certain conditional expectations related to the histories of the process at the root and its neighbors to a suitably reweighted version of corresponding conditional expectations related to the histories of the process at a child of the root and its neighborhood, leading to a more complicated form of the analogue of $\tilde{\gamma}_t$ (as discussed in Remark 3.11). Section 3.4 contains precise statements of these key properties, which are applied in Sect. 4.1 to show that the marginal distributions satisfy the UGW local equation. Finally, the more complicated form of the local equation on the UGW tree also leads to additional subtleties in the last step of establishing uniqueness (see Sect. 4.2). In particular, the proof now entails certain non-trivial change of measure arguments that were not necessary in the case of the deterministic regular tree; see Sect. 4.2.1 for an outline and Sect. 4.2.2 for the details.

Precise statements of our main results are given in Sect. 3. In the next section, we first develop some notation.

2 Preliminaries and notation

In this section, we introduce common notation and definitions used throughout the paper, and which are required to state the main results. Throughout, we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.1 Graphs and the Ulam–Harris–Neveu labeling for trees

2.1.1 General graph terminology

Given a graph $G = (V, E)$, we will often abuse notation by writing $v \in G$ for $v \in V$ to refer to a vertex or node of the graph. In this paper, we will always assume that the graph has a finite or countably infinite vertex set and is simple (no self-edges or multi-edges). Given $u, v \in V$, a path from u to v is a sequence of distinct vertices $u = u_0, u_1, u_2, \dots, u_n = v$ such that $(u_{i-1}, u_i) \in E$ for $i = 1, \dots, n$. The graph G is said to be connected if there exists a path between any two vertices $u, v \in V$. For two vertices $u, v \in V$, the distance between u and v is the length of the shortest path from u to v , or ∞ if no such path exists. The diameter $\text{diam}(A)$ of a set $A \subset V$ is the maximal distance between vertices of A . For $v \in V$, the neighborhood of v in G is defined to be

$$N_v(G) := \{u \in V \setminus \{v\} : (u, v) \in E\}.$$

The degree of a vertex v is $|N_v(G)|$, where as usual $|A|$ denotes the cardinality of a set A . A graph is said to be *locally finite* if each vertex has a finite degree. Given $A \subset V$, its boundary and double boundary are defined to be

$$\partial A := \{u \in V \setminus A : \exists v \in A \text{ such that } (u, v) \in E\}, \quad \partial^2 A := \partial A \cup \partial(A \cup \partial A). \quad (2.1)$$

Note that ∂A (resp. $\partial^2 A$) is the set of vertices that are at a distance 1 (resp. 1 or 2) from A . A *clique* is a complete subgraph, that is, a set $A \subset V$ such that $(u, v) \in E$ for every distinct $u, v \in A$. Equivalently, a clique is a set $A \subset V$ of diameter at most 1. Similarly, we say that a set $A \subset V$ is a *2-clique* if $\text{diam}(A) \leq 2$.

2.1.2 The Ulam–Harris–Neveu labeling for trees

A tree is a (undirected) graph $G = (V, E)$ such that given any two vertices $u, v \in G$, there is a unique path between u and v . It will be convenient to work with the standard labeling scheme for trees, often attributed to Ulam–Harris–Neveu (see, e.g., [19, Section VI.2] or [37]), defined using the vertex set

$$\mathbb{V} := \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \mathbb{N}^k. \quad (2.2)$$

For $u, v \in \mathbb{V}$, let uv denote concatenation, that is, if $u = (u_1, \dots, u_k) \in \mathbb{N}^k$ and $v = (v_1, \dots, v_j) \in \mathbb{N}^j$, then $uv = (u_1, \dots, u_k, v_1, \dots, v_j) \in \mathbb{N}^{k+j}$. The root \emptyset is the identity element, so $\emptyset u = u\emptyset = u$ for all $u \in \mathbb{V}$. For $v \in \mathbb{V} \setminus \{\emptyset\}$, we write π_v for the parent of v ; precisely, π_v is the unique element of \mathbb{V} such that there exists $k \in \mathbb{N}$ satisfying $v = \pi_v k$. We view \mathbb{V} as a graph by declaring two vertices to be adjacent if one is the parent of the other. Thus, the neighborhoods of \mathbb{V} are $N_\emptyset(\mathbb{V}) = \mathbb{N}$ and $N_v(\mathbb{V}) = \{\pi_v\} \cup \{vk : k \in \mathbb{N}\}$ for $v \in \mathbb{V} \setminus \{\emptyset\}$. Note that this graph \mathbb{V} is not locally finite.

There is a natural partial order on \mathbb{V} . We say $u \leq v$ if there exists (a necessarily unique) $w \in \mathbb{V}$ such that $uw = v$, and say $u < v$ when $w \neq \emptyset$. A subset $\mathcal{T} \subset \mathbb{V}$ is defined to be a *tree* if:

- (1) $\emptyset \in \mathcal{T}$;
- (2) If $v \in \mathcal{T}$ and $u \in \mathbb{V}$ with $u \leq v$, then $u \in \mathcal{T}$;
- (3) For each $v \in \mathcal{T}$ there exists an integer $c_v(\mathcal{T}) \geq 0$ such that, for $k \in \mathbb{N}$, we have $vk \in \mathcal{T}$ if and only if $1 \leq k \leq c_v(\mathcal{T})$.

Note that for us a tree, by default, is locally finite. We also use the symbol \mathcal{T} to refer not only to the subset of \mathbb{V} but also to the induced subgraph. Inductively, for $u \in \mathcal{T}$, we think of the elements $(uv)_{v=1}^{c_u(\mathcal{T})}$ as the children of the vertex labeled u . For any $\mathcal{T} \subset \mathbb{V}$ and $v \in \mathbb{V}$, define $N_v(\mathcal{T}) = \mathcal{T} \cap N_v(\mathbb{V})$ to be the set of neighbors of v in \mathcal{T} if $v \in \mathcal{T}$, and set $N_v(\mathcal{T}) = \emptyset$ if $v \notin \mathcal{T}$. It is convenient to define also \mathbb{V}_n to be the labels of the first n generations:

$$\mathbb{V}_n := \{\emptyset\} \cup \bigcup_{k=1}^n \mathbb{N}^k. \quad (2.3)$$

With a minor abuse of notation, we also use \mathbb{V}_n to denote the corresponding induced subgraph.

2.2 Measure spaces

For a Polish space \mathcal{X} , we write $\mathcal{P}(\mathcal{X})$ for the set of Borel probability measures on \mathcal{X} , endowed always with the topology of weak convergence. Note that $\mathcal{P}(\mathcal{X})$ itself becomes a Polish space with this topology, and we equip it with the corresponding Borel σ -field. We write δ_x for the Dirac measure at a point $x \in \mathcal{X}$. For an \mathcal{X} -valued random variable X , we write $\mathcal{L}(X)$ to denote its law, which is an element of $\mathcal{P}(\mathcal{X})$. Given any measure ν on a measurable space and any ν -integrable function f on that space, we use the usual shorthand notation $\langle \nu, f \rangle := \int f d\nu$. Given \mathcal{X} -valued random elements $Y, Y_n, n \in \mathbb{N}$, we write $Y_n \Rightarrow Y$ to mean that the law of Y_n converges weakly to the law of Y .

2.3 Function spaces

For a fixed positive integer d , throughout we write

$$\mathcal{C} := C(\mathbb{R}_+; \mathbb{R}^d)$$

for the path space of continuous functions, endowed with the topology of uniform convergence on compacts. For $t > 0$, we write $\mathcal{C}_t := C([0, t]; \mathbb{R}^d)$, and for $x \in \mathcal{C}$ we write $\|x\|_{*,t} := \sup_{s \in [0,t]} |x(s)|$ and $x[t] := \{x(s), s \in [0, t]\}$ for the truncated path, viewed as an element of \mathcal{C}_t .

2.4 Configuration spaces

For a set \mathcal{X} and a graph $G = (V, E)$, we write \mathcal{X}^V or \mathcal{X}^G for the configuration space $\{(x_v)_{v \in V} : x_v \in \mathcal{X} \text{ for every } v \in V\}$. We make use of a standard notation for configurations on subsets of V : For $x = (x_v)_{v \in V} \in \mathcal{X}^V$ and $A \subset V$, we write x_A for the element $x_A = (x_v)_{v \in A}$ of \mathcal{X}^A .

2.5 Space of unordered terminating sequences

As discussed in the introduction, we will study stochastic differential equations that take values in a sequence of configuration spaces with corresponding underlying interaction graphs that have different numbers of vertices. We want to be able to specify a single “drift function” that takes as input finite sequences of elements of \mathcal{X} of arbitrary length and is insensitive to the order of these elements.

To this end, for a set \mathcal{X} , we define in this paragraph a space $S^{\sqcup}(\mathcal{X})$ of finite unordered \mathcal{X} -valued sequences of arbitrary length (possibly zero). First, for $k \in \mathbb{N}$ we define the symmetric power (or unordered Cartesian product) $S^k(\mathcal{X})$ as the quotient of \mathcal{X}^k by the natural action of the symmetric group on k letters. For convenience, let $S^0(\mathcal{X}) = \{\circ\}$. Define $S^{\sqcup}(\mathcal{X})$ as the disjoint union,

$$S^{\sqcup}(\mathcal{X}) = \bigsqcup_{k=0}^{\infty} S^k(\mathcal{X}).$$

A typical element of $S^{\sqcup}(\mathcal{X})$ will be denoted $(x_v)_{v \in V}$, for a finite (possibly empty) set V ; if the set is empty, then by convention $(x_v)_{v \in V} = \circ \in S^0(\mathcal{X})$. It must be stressed that, of course, the element $(x_v)_{v \in V}$ has no order. The space $S^{\sqcup}(\mathcal{X})$ must not be confused with what is traditionally called the *infinite symmetric product space* in algebraic topology when \mathcal{X} is endowed with a distinguished (base) point e , in which the points (x_1, \dots, x_n, e) and (x_1, \dots, x_n) would be identified; these two points are distinct in $S^{\sqcup}(\mathcal{X})$.

Suppose now that (\mathcal{X}, d) is a metric space, and endow $S^{\sqcup}(\mathcal{X})$, with the usual disjoint union topology, i.e., the finest topology on $S^{\sqcup}(\mathcal{X})$ for which the injection $S^k(\mathcal{X}) \hookrightarrow S^{\sqcup}(\mathcal{X})$ is continuous for each $k \in \mathbb{N}$. A function $F : S^{\sqcup}(\mathcal{X}) \rightarrow \mathcal{Y}$ to a metric space \mathcal{Y} is continuous if and only if there is a sequence $(f_k)_{k=0}^{\infty}$, where $f_0 \in \mathcal{Y}$

and, for each $k \in \mathbb{N}$, $f_k : \mathcal{X}^k \rightarrow \mathcal{Y}$ is a continuous function that is symmetric in its k variables, such that

$$F((x_i)_{i \in \{1, \dots, k\}}) = \begin{cases} f_k(x_1, \dots, x_k) & \text{for } k \in \mathbb{N}, (x_1, \dots, x_k) \in \mathcal{X}^k \\ f_0 & \text{for } k = 0. \end{cases}$$

If \mathcal{X} is separable and completely metrizable, then so is $S^\sqcup(\mathcal{X})$. Note that a sequence $(x_v^n)_{v \in V_n}$ in $S^\sqcup(\mathcal{X})$ converges to $(x_v)_{v \in V}$ if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a bijection $\varphi : V_n \rightarrow V$ such that $\max_{v \in V_n} d(x_v^n, x_{\varphi(v)}) < \epsilon$. (Note that this implicitly requires that $|V_n| = |V|$ for sufficiently large n .) It is worth noting that continuous functions on $S^\sqcup(\mathcal{X})$ are strictly more general than weakly continuous functions on the set of empirical measures, but we refer to [29] for further discussion.

3 Statements of main results

For a tree \mathcal{T} , viewed as a subset of \mathbb{V} as defined in Sect. 2.1.2, we are interested in the SDE system

$$dX_v^T(t) = 1_{\{v \in \mathcal{T}\}} \left(b(t, X_v^T, X_{N_v(\mathcal{T})}^T) dt + \sigma(t, X_v^T) dW_v(t) \right), \quad v \in \mathbb{V}, \quad (3.1)$$

where recall that $N_v(\mathcal{T})$ denotes the set of neighbors of v in \mathcal{T} , and b and σ are suitable progressively measurable coefficients as specified in Assumption A. When the tree \mathcal{T} is random, we always take it to be independent of the initial conditions and Brownian motions. Note that we include even those labels $v \in \mathbb{V} \setminus \mathcal{T}$ that do not belong to the tree, for which the process is constant $X_v^T(t) = X_v^T(0)$; this will be convenient notation and, in the random tree case, will render the tree itself measurable with respect to the initial σ -field (as elaborated in Remark 4.2). Also note that, unlike in the introduction, we allow path-dependent coefficients (b, σ) , both because this arises in applications and because this results in no change in the arguments or in the form of the local equation described in Sect. 3.2, which are inevitably path-dependent regardless of whether b and σ are, as discussed in Sect. 1.2.2(ii).

We state first our standing assumptions in Sect. 3.1. Then we introduce the local equation in Sect. 3.2 and finally state our main results in Sect. 3.3. Throughout, recall the function space $\mathcal{C} = C(\mathbb{R}_+; \mathbb{R}^d)$ and sequence space $S^\sqcup(\mathcal{C})$ defined in Sects. 2.3 and 2.5, respectively.

3.1 Assumptions

Fix a dimension $d \in \mathbb{N}$ and, for $x \in \mathcal{C}$ and $t > 0$, recall from Sect. 2.3 the notation $\|x\|_{*,t} := \sup_{s \in [0,t]} |x(s)|$. We assume the drift coefficient b , diffusion coefficient σ , and an initial distribution λ_0 , satisfy the following:

Assumption A (A.1) The drift coefficient $b : \mathbb{R}_+ \times \mathcal{C} \times S^\sqcup(\mathcal{C}) \rightarrow \mathbb{R}^d$ is continuous and has linear growth, in the sense that for each $T > 0$, there exists $C_T < \infty$ such that, for any $(t, x, (x_v)_{v \in A}) \in [0, T] \times \mathcal{C} \times S^\sqcup(\mathcal{C})$, we have

$$|b(t, x, (x_v)_{v \in A})| \leq C_T \left(1 + \|x\|_{*,t} + \frac{1}{|A|} \sum_{v \in A} \|x_v\|_{*,t} \right),$$

where the average is understood to be zero if $|A| = 0$. Moreover, b is progressively measurable; that is, it is jointly measurable (which is already implied by the above continuity properties) and non-anticipative in the sense that for each $t \geq 0$, $b(t, x, (x_v)_{v \in A}) = b(t, y, (y_v)_{v \in A})$ whenever $x(s) = y(s)$ and $x_v(s) = y_v(s)$ for all $s \leq t$ and $v \in A$.

(A.2) The diffusion matrix $\sigma : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^{d \times d}$ satisfies the following:

(A.2a) σ is bounded and continuous. Moreover, $\sigma(t, x)$ is invertible for each (t, x) , and the inverse is uniformly bounded. Lastly, σ is progressively measurable, which implies that for each $t \geq 0$, $\sigma(t, x) = \sigma(t, y)$ whenever $x(s) = y(s)$ for all $s \leq t$.

(A.2b) The following driftless SDE admits a unique in law weak solution:

$$dX(t) = \sigma(t, X)dW(t), \quad X(0) \sim \lambda_0.$$

(A.3) The initial states $(X_v^T(0))_{v \in \mathbb{V}}$ are i.i.d. with common distribution $\lambda_0 \in \mathcal{P}(\mathbb{R}^d)$, and λ_0 has finite second moment.

(A.4) For each non-random tree $\mathcal{T} \subset \mathbb{V}$, there exists a unique in law weak solution of the SDE system (3.1) with i.i.d. initial positions $(X_v^T(0))_{v \in \mathbb{V}}$ with law λ_0 .

The final condition (A.4) regarding uniqueness in law for (3.1) is not as stringent as it may appear. If the tree \mathcal{T} is finite, it follows automatically from Assumptions (A.1)–(A.2) and Girsanov's theorem (see Lemma B.1). For infinite graphs, Theorem 3.2 below shows that Assumption (A.4) holds if b and σ are suitably Lipschitz. The i.i.d. assumption on the initial conditions in (A.3) can be relaxed, although we do not do so in this article; see Remark 3.16 for further discussion.

Remark 3.1 As an immediate consequence of Assumption (A.4), it follows that the SDE (3.1) is unique in law even when the tree \mathcal{T} is random, since we always take \mathcal{T} to be independent of the initial conditions and the Brownian motions.

Theorem 3.2 Suppose that Assumptions (A.1) and (A.2a) hold. Assume also that the functions b and σ are Lipschitz, in the sense that for each $T > 0$, there exist $K_T, \bar{K}_T < \infty$ such that, for all $t \in [0, T]$, all $x, x' \in \mathcal{C}$, and all $(x_u)_{u \in A}, (x'_u)_{u \in A} \in S^\sqcup(\mathcal{C})$ indexed by the same finite set A , we have

$$|b(t, x, (x_u)_{u \in A}) - b(t, x', (x'_u)_{u \in A})| \leq K_T \left(\|x - x'\|_{*,t} + \frac{1}{|A|} \sum_{u \in A} \|x_u - x'_u\|_{*,t} \right), \quad (3.2)$$

where the average is understood to be zero if $|A| = 0$, and

$$|\sigma(t, x) - \sigma(t, x')| \leq \bar{K}_T \|x - x'\|_{*,t}. \quad (3.3)$$

Then there exists a pathwise unique strong solution for the SDE system (3.1), with any initial conditions $(X_v^T(0))_{v \in \mathcal{T}}$.

Proof This follows from standard arguments; see [29, Theorem 3.1]. \square

Motivated by Theorem 3.2, we will sometimes make the following assumption.

Assumption B Suppose that Assumptions (A.1), (A.2a), and (A.3) hold. Assume also that the functions b and σ are Lipschitz, in the sense that (3.2) and (3.3) hold.

We note that due to Theorem 3.2, Assumption B implies Assumption A.

The main examples of interactions we have in mind for the drift b in Assumption (A.1) take the following forms:

Example 3.3 For a first example, suppose b is of the form

$$b(t, x, (x_v)_{v \in A}) = \begin{cases} \tilde{b}_0(t, x) & \text{if } A = \emptyset, \\ \frac{1}{|A|} \sum_{v \in A} \tilde{b}(t, x, x_v) & \text{if } A \neq \emptyset, \end{cases}$$

for given functions $\tilde{b}_0 : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$ and $\tilde{b} : \mathbb{R}_+ \times \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}^d$. Assumption (A.1) holds if \tilde{b}_0 and \tilde{b} are continuous with linear growth, in the sense that for each $T > 0$ there exists $C_T < \infty$ such that

$$|\tilde{b}_0(t, x)| + |\tilde{b}(t, x, y)| \leq C_T (1 + \|x\|_{*,t} + \|y\|_{*,t}), \quad \text{for all } (t, x, y).$$

Example 3.4 Generalizing Example 3.3, suppose b is of the form

$$b(t, x, (x_v)_{v \in A}) = \begin{cases} \tilde{b}_0(t, x) & \text{if } A = \emptyset, \\ \tilde{b}\left(t, x, \frac{1}{|A|} \sum_{v \in A} \delta_{x_v}\right) & \text{if } A \neq \emptyset, \end{cases}$$

for given functions $\tilde{b}_0 : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$ and $\tilde{b} : \mathbb{R}_+ \times \mathcal{C} \times \mathcal{P}(\mathcal{C}) \rightarrow \mathbb{R}^d$. In fact, \tilde{b} needs only to be defined on the subspace of $\mathcal{P}(\mathcal{C})$ consisting of empirical measures of finitely many points. Assumption (A.1) holds if \tilde{b}_0 and \tilde{b} are continuous (using weak convergence or any Wasserstein metric on $\mathcal{P}(\mathcal{C})$) with linear growth, namely if for each $T > 0$ there exists $C_T < \infty$ such that

$$|\tilde{b}_0(t, x)| + |\tilde{b}(t, x, m)| \leq C_T \left(1 + \|x\|_{*,t} + \int_{\mathcal{C}} \|y\|_{*,t} dm(y) \right), \quad \text{for all } (t, x, m).$$

3.2 The local equation

The local equation describes a novel stochastic dynamical system and is significantly more complicated on the UGW tree than on non-random trees, where its structure is more transparent, especially given the discussion in Sect. 1.2.2. Thus, we first introduce its definition on the infinite regular tree in Sect. 3.2.1 and defer the full formulation for a UGW tree to Sect. 3.2.2. However, the reader may choose to skip directly to Sect. 3.2.2 without loss of continuity.

3.2.1 The local equation for an infinite regular tree

Let \mathbb{T}_κ be the infinite κ -regular tree for some integer $\kappa \geq 2$, and note that it can be identified with the subset $\{\emptyset\} \cup \{1, \dots, \kappa\} \cup \bigcup_{n=2}^{\infty} (\{1, \dots, \kappa\} \times \{1, \dots, \kappa - 1\}^{n-1})$ of the vertex set \mathbb{V} defined in (2.2).

Recall from Sect. 2.3 that for $t > 0$ and $x \in \mathcal{C} = C(\mathbb{R}_+; \mathbb{R}^d)$, we write $x[t] := \{x(s) : s \in [0, t]\}$ for the truncated path, viewed as an element of $\mathcal{C}_t = C([0, t]; \mathbb{R}^d)$. The following generalizes the local equation outlined in Sect. 1.2.2 for a model on \mathbb{T}_2 .

Definition 3.5 Let $\mathbb{T}_{\kappa,1} = \{\emptyset, 1, \dots, \kappa\}$ denote the first generation of the κ -regular tree. A *weak solution of the \mathbb{T}_κ local equation with initial law $\lambda_0 \in \mathcal{P}(\mathbb{R}^d)$* is a tuple $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \gamma, (B_v, Y_v)_{v \in \mathbb{T}_{\kappa,1}})$ such that:

- (1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.
- (2) $(B_v)_{v \in \mathbb{T}_{\kappa,1}}$ are independent d -dimensional \mathbb{F} -Brownian motions.
- (3) $(Y_v)_{v \in \mathbb{T}_{\kappa,1}}$ are continuous d -dimensional \mathbb{F} -adapted processes.
- (4) $(Y_v(0))_{v \in \mathbb{T}_{\kappa,1}}$ are i.i.d. with law λ_0 .
- (5) The function $\mathbb{R}_+ \times \mathcal{C}^2 \ni (t, x_\emptyset, x_1) \mapsto \gamma_t(x_\emptyset, x_1) \in \mathbb{R}^d$ is progressively measurable and satisfies

$$\gamma_t(Y_\emptyset, Y_1) = \mathbb{E} \left[b(t, Y_\emptyset, Y_{\{1, \dots, \kappa\}}) \mid Y_\emptyset[t], Y_1[t] \right], \quad a.s., \text{ for a.e. } t \in [0, T]. \quad (3.4)$$

- (6) The following system of stochastic equations holds:

$$\begin{aligned} dY_\emptyset(t) &= b(t, Y_\emptyset, Y_{\{1, \dots, \kappa\}}) dt + \sigma(t, Y_\emptyset) dB_\emptyset(t), \\ dY_i(t) &= \gamma_t(Y_i, Y_\emptyset) dt + \sigma(t, Y_i) dB_i(t), \quad i = 1, \dots, \kappa. \end{aligned} \quad (3.5)$$

- (7) For each $i = 1, \dots, \kappa$ and $T > 0$, we have

$$\int_0^T \left(|\gamma_t(Y_\emptyset, Y_i)|^2 + |\gamma_t(Y_i, Y_\emptyset)|^2 + |\gamma_t(\widehat{X}_\emptyset, \widehat{X}_i)|^2 + |\gamma_t(\widehat{X}_i, \widehat{X}_\emptyset)|^2 \right) dt < \infty, \quad a.s.,$$

where $(\widehat{X}_v)_{v \in \mathbb{T}_{\kappa,1}}$ is the unique in law (by Assumption (A.2b)) solution to the driftless SDE system

$$d\widehat{X}_v(t) = \sigma(t, \widehat{X}_v)dB_v(t), \quad v \in \mathbb{T}_{\kappa,1},$$

where $(\widehat{X}_v(0))_{v \in \mathbb{T}_{\kappa,1}}$ are i.i.d. with law λ_0 .

Alternatively, we may refer to the law of the $\mathcal{C}^{\kappa+1}$ -valued random variable $(Y_v)_{v \in \mathbb{T}_{\kappa,1}}$ as a weak solution. We say that the \mathbb{T}_κ local equation with initial law λ_0 is *unique in law* if any two weak solutions induce the same law on $\mathcal{C}^{\kappa+1}$.

Remark 3.6 The property (7) in Definition 3.5 will be used to justify certain applications of Girsanov's theorem (as in Lemma B.1). Specifically, it ensures that the joint laws of (Y_\emptyset, Y_i) and $(\widehat{X}_\emptyset, \widehat{X}_i)$ are mutually absolutely continuous.

Remark 3.7 The local equation describes a “nonlinear” process in the sense of a McKean–Vlasov equation because the law of the solution enters the dynamics. However, a crucial yet unusual feature of the local equation (3.5) is that the conditional expectation mapping γ_t appears with different arguments throughout the SDE system. In the related paper [30] (see also [43, 48]), we show that analogous discrete-time local dynamics can be simulated efficiently. In future work, we plan to investigate the analytical and numerical tractability of the local dynamics in the diffusion setting.

It is worth noting how the \mathbb{T}_κ local equation (3.5) simplifies when the drift b takes the form described in Example 3.3 above. Indeed, the law of $(Y_\emptyset, Y_1, \dots, Y_\kappa)$ is clearly invariant under permutations of (Y_1, \dots, Y_κ) , which implies

$$\gamma_t(Y_\emptyset, Y_1) = \frac{1}{\kappa} \widetilde{b}(t, Y_\emptyset, Y_1) + \frac{\kappa-1}{\kappa} \widetilde{\gamma}_t(Y_\emptyset, Y_1),$$

where we define

$$\widetilde{\gamma}_t(Y_\emptyset, Y_1) := \mathbb{E} \left[\widetilde{b}(t, Y_\emptyset, Y_2) \mid Y_\emptyset[t], Y_1[t] \right].$$

We may then write (3.5) as

$$\begin{aligned} dY_\emptyset(t) &= \frac{1}{\kappa} \sum_{i=1}^{\kappa} \widetilde{b}(t, Y_\emptyset, Y_i) dt + \sigma(t, Y_\emptyset) dB_\emptyset(t), \\ dY_i(t) &= \left(\frac{1}{\kappa} \widetilde{b}(t, Y_i, Y_\emptyset) + \frac{\kappa-1}{\kappa} \widetilde{\gamma}_t(Y_i, Y_\emptyset) \right) dt + \sigma(t, Y_i) dB_i(t), \quad i = 1, \dots, \kappa. \end{aligned}$$

The main result on the characterization of the dynamics of the root and its neighborhood $\mathbb{T}_{\kappa,1}$ via the local equation is given in Corollary 3.14. It is a simple consequence of the more general result, Theorem 3.12, for UGW(ρ) trees given in Sect. 3.3. With that in mind, in the next section, we first introduce the general form of the local equation for a UGW tree.

3.2.2 The local equation for unimodular Galton–Watson trees

Fix a distribution $\rho \in \mathcal{P}(\mathbb{N}_0)$ with finite non-zero first moment. We first formally define a UGW(ρ) tree:

Definition 3.8 Given $\rho \in \mathcal{P}(\mathbb{N}_0)$ with a finite nonzero first moment, the random tree UGW(ρ) has a root with offspring distribution ρ , and each vertex of each subsequent generation has a number of offspring according to the distribution $\hat{\rho} \in \mathcal{P}(\mathbb{N}_0)$, where $\hat{\rho}$ is given by

$$\hat{\rho}(k) = \frac{(k+1)\rho(k+1)}{\sum_{n \in \mathbb{N}} n\rho(n)}, \quad k \in \mathbb{N}_0, \quad (3.6)$$

and the numbers of offspring in different generations are all independent of each other. Recalling the Ulam–Harris–Neveu labeling from Sect. 2.1.2, we view a UGW(ρ) tree as a random subset of \mathbb{V} .

As discussed in Sect. 1.1, this kind of random tree arises as the local weak limit of many natural finite random graph models (see Examples 2.2, 2.3, and 2.4 of [29]).

We now give the general form of the local equation for UGW trees. In this case, the structure of the neighborhood of the root is also random. To capture this, it is useful to consider the root neighborhood as a subset of the vertex set $\mathbb{V}_1 = \{\emptyset\} \cup \mathbb{N}$.

Definition 3.9 Given $\rho \in \mathcal{P}(\mathbb{N}_0)$ with finite nonzero first moment and $\lambda_0 \in \mathcal{P}(\mathbb{R}^d)$, a *weak solution of the UGW(ρ) local equation with initial law λ_0* is a tuple $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \mathcal{T}_1, \gamma, (B_v, Y_v)_{v \in \mathbb{V}_1}, \hat{C}_1)$ such that:

- (1) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.
- (2) \mathcal{T}_1 is a random tree with the same law as the first generation of a UGW(ρ) tree. More explicitly, \mathcal{T}_1 has vertex set $\{\emptyset, 1, \dots, \kappa\}$ for some \mathbb{N}_0 -valued \mathcal{F}_0 -measurable random variable κ with law ρ , and the edge set is $\{(\emptyset, k) : k = 1, \dots, \kappa\}$. (If $\kappa = 0$, this means the vertex set is simply $\{\emptyset\}$, there are no edges, and $N_\emptyset(\mathcal{T}_1) = \emptyset$.)
- (3) \hat{C}_1 is an \mathcal{F}_0 -measurable \mathbb{N}_0 -valued random variable with law $\hat{\rho}$, as defined in (3.6).
- (4) $(B_v)_{v \in \mathbb{V}_1}$ are independent d -dimensional \mathbb{F} -Brownian motions.
- (5) $(Y_v)_{v \in \mathbb{V}_1}$ are continuous d -dimensional \mathbb{F} -adapted processes.
- (6) $(Y_v(0))_{v \in \mathbb{V}_1}$ are \mathcal{F}_0 -measurable and i.i.d. with law λ_0 .
- (7) The function $\mathbb{R}_+ \times \mathcal{C}^2 \ni (t, x_\emptyset, x_1) \mapsto \gamma_t(x_\emptyset, x_1) \in \mathbb{R}^d$ is progressively measurable and satisfies

$$\gamma_t(Y_\emptyset, Y_1) = \begin{cases} \frac{\mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T}_1)|}{1+\hat{C}_1} b(t, Y_\emptyset, Y_{N_\emptyset(\mathcal{T}_1)}) \mid Y_\emptyset[t], Y_1[t] \right]}{\mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T}_1)|}{1+\hat{C}_1} \mid Y_\emptyset[t], Y_1[t] \right]} & \text{on } \{N_\emptyset(\mathcal{T}_1) \neq \emptyset\}, \\ b(t, Y_\emptyset, \circ) & \text{on } \{N_\emptyset(\mathcal{T}_1) = \emptyset\}, \end{cases} \quad (3.7)$$

a.s., for a.e. $t \in [0, T]$. Recall our convention that \circ denotes the unique element of the one-point space \mathcal{C}^0 .

(8) \mathcal{T}_1 , $(Y_v(0))_{v \in \mathbb{V}_1}$, \widehat{C}_1 , and $(B_v)_{v \in \mathbb{V}_1}$ are independent.

(9) The following system of stochastic equations is satisfied:

$$\begin{aligned} dY_\emptyset(t) &= b(t, Y_\emptyset, Y_{N_\emptyset(\mathcal{T}_1)}) dt + \sigma(t, Y_\emptyset) dB_\emptyset(t), \\ dY_k(t) &= 1_{\{k \in \mathcal{T}_1\}} \left(\gamma_t(Y_k, Y_\emptyset) dt + \sigma(t, Y_k) dB_k(t) \right), \quad k \in \mathbb{N}. \end{aligned} \quad (3.8)$$

(10) For each $k \in \mathbb{N}$ and $T > 0$, we have

$$\int_0^T \left(|\gamma_t(Y_\emptyset, Y_k)|^2 + |\gamma_t(Y_k, Y_\emptyset)|^2 + |\gamma_t(\widehat{X}_\emptyset, \widehat{X}_k)|^2 + |\gamma_t(\widehat{X}_k, \widehat{X}_\emptyset)|^2 \right) dt < \infty, \quad a.s.,$$

on the event $\{k \in \mathcal{T}_1\}$, where $(\widehat{X}_v)_{v \in \mathbb{V}_1}$ is the unique in law (by Assumption (A.2b)) solution of the driftless SDE system

$$d\widehat{X}_v(t) = 1_{\{v \in \mathcal{T}_1\}} \sigma(t, \widehat{X}_v) dB_v(t), \quad v \in \mathbb{V}_1,$$

where $(\widehat{X}_v(0))_{v \in \mathbb{V}_1}$ are i.i.d. with law λ_0 .

Alternatively, we may refer to the law of the $\mathcal{C}^{\mathbb{V}_1}$ -valued random variable $(Y_v)_{v \in \mathbb{V}_1}$ as a weak solution. We say that the UGW(ρ) local equation with initial law λ_0 is *unique in law* if any two weak solutions induce the same law on $\mathcal{C}^{\mathbb{V}_1}$.

Remark 3.10 It is worth noting how Definition 3.9 reduces to Definition 3.5 when the tree is the deterministic κ -regular tree, i.e., the UGW(ρ) tree with $\rho = \delta_\kappa$ for an integer $\kappa \geq 2$. In this case, we have $\widehat{\rho} = \delta_{\kappa-1}$, $N_\emptyset(\mathcal{T}_1) = \{1, \dots, \kappa\}$, and $\widehat{C}_1 = \kappa - 1$, and the definition of γ_t in (3.7) reduces to (3.4).

Remark 3.11 The more complicated form of γ_t in Definition 3.9 as opposed to Definition 3.5 is due to the subtler symmetries of the UGW tree in comparison with the simpler symmetries of the non-random trees \mathbb{T}_κ . Note that for the UGW tree \mathcal{T} , on the event $\{|N_\emptyset(\mathcal{T})| \neq \emptyset\}$, the random variable \widehat{C}_1 is independent of $|N_\emptyset(\mathcal{T})|$ and represents the number of offspring of vertex 1, which in a UGW(ρ) tree has law $\widehat{\rho}$. The following identity then provides intuition behind the definition of γ_t in (3.7): for bounded functions $h : \mathbb{N} \mapsto \mathbb{R}$,

$$\mathbb{E} \left[h(1 + \widehat{C}_1) 1_{\{N_\emptyset(\mathcal{T}_1) \neq \emptyset\}} \right] = \mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T})|}{1 + \widehat{C}_1} h(|N_\emptyset(\mathcal{T})|) 1_{\{N_\emptyset(\mathcal{T}_1) \neq \emptyset\}} \right],$$

which is easily verified by showing that both sides are equal to $[1 - \rho(0)] \sum_{k=0}^\infty h(k+1) \widehat{\rho}(k)$. This should be interpreted as explaining how to change measure, using the Radon-Nikodym derivative $|N_\emptyset(\mathcal{T})|/(1 + \widehat{C}_1)$, to effectively *re-root* the tree to vertex 1 instead of \emptyset . See Proposition 3.18 for a precise statement. Of course, in the κ -regular tree case discussed in Remark 3.10, no such change of measure is necessary, because the re-rooted tree is isomorphic to the original tree. On a more technical level, it is

worth noting that the presence of the indicators in the SDE system (3.8) ensures that $\{v \in \mathcal{T}\}$ is a.s. $X_v[t]$ -measurable for each $t > 0$ (see Remark 4.2), and thus the conditional expectations appearing in (3.7) implicitly condition on the tree structure in addition to the particle trajectories. Also, our choice of how to define $\gamma_t(Y_\emptyset, Y_1)$ on the event $\{N_\emptyset(\mathcal{T}_1) = \emptyset\}$ is a useful convention but is irrelevant to the form of the local equation.

3.3 Characterization of marginals via the local equation

The following is our main result for particle systems set on UGW trees.

Theorem 3.12 *Suppose Assumption A holds. Let \mathcal{T} denote a $\text{UGW}(\rho)$ tree, where $\rho \in \mathcal{P}(\mathbb{N}_0)$ has finite nonzero first and second moments. Let $X^\mathcal{T} = (X_v^\mathcal{T})_{v \in \mathbb{V}}$ be the solution of the SDE system (3.1). Then the law of the $\mathcal{C}^{\mathbb{V}_1}$ -valued random variable $(X_v^\mathcal{T})_{v \in \mathbb{V}_1}$ is a weak solution of the $\text{UGW}(\rho)$ local equation with initial law λ_0 . Moreover, the $\text{UGW}(\rho)$ local equation with initial law λ_0 is unique in law.*

Remark 3.13 To be absolutely clear about the meaning of Theorem 3.12, we must stress that $(X_v^\mathcal{T})_{v \in \mathbb{V}_1}$ provides a weak solution of the local equation, but $(X_v^{\mathcal{T}_1})_{v \in \mathbb{V}_1}$ does not, where we write $\mathcal{T}_1 := \mathcal{T} \cap \mathbb{V}_1$ for the first generation of \mathcal{T} . The difference is that $X^{\mathcal{T}_1} = (X_v^{\mathcal{T}_1})_{v \in \mathbb{V}}$ denotes the particle system set on the one-generation tree \mathcal{T}_1 , in which the children of the root comprise the leaves of the tree, whereas $(X_v^\mathcal{T})_{v \in \mathbb{V}_1}$ represents the root neighborhood for the particle system set on the potentially infinite $\text{UGW}(\rho)$ tree \mathcal{T} .

Since the κ -regular tree is a special case of the UGW tree defined in Definition 3.8 (see Remark 3.10), the following is an immediate corollary of Theorem 3.12, where we recall that $\mathbb{T}_{\kappa,1} := \{\emptyset, 1, \dots, \kappa\}$ represents one generation of the tree \mathbb{T}_κ .

Corollary 3.14 *Suppose Assumption A holds. Let \mathbb{T}_κ denote the infinite κ -regular tree, for some $\kappa \geq 2$, and let $\mathbb{T}_{\kappa,1}$ denote its first generation. Let $X^{\mathbb{T}_\kappa} = (X_v^{\mathbb{T}_\kappa})_{v \in \mathbb{V}}$ denote the solution of the SDE (3.1) on the tree $\mathcal{T} = \mathbb{T}_\kappa$. Then the law of $(X_v^{\mathbb{T}_\kappa})_{v \in \mathbb{T}_{\kappa,1}}$ is a weak solution of the \mathbb{T}_κ local equation with initial law λ_0 . Moreover, the \mathbb{T}_κ local equation with initial law λ_0 is unique in law.*

As will be discussed in Sect. 3.5, combining Theorem 3.12 with the results of [29] yields a characterization of the limiting marginals and empirical measures of particle systems set on large finite graphs converging locally to UGW trees.

Remark 3.15 The unimodularity condition on the random tree, although convenient and natural in the context of local limits of random graphs, is not entirely necessary for obtaining a form of marginal dynamics. Indeed, in a related paper [30], we obtain analogous results for interacting discrete-time Markov chains (equivalently, stochastic cellular automata), on standard Galton–Watson trees. The marginal dynamics on a general Galton–Watson tree, however, involve the first two generations of the tree instead of just the first generation. The extra symmetry imposed by unimodularity enables the reduction to a single generation, essentially because of the symmetry result of Proposition 3.18 below.

Remark 3.16 We focus in this paper on i.i.d. initial conditions, for the sake of simplicity, but similar results are valid in greater generality. On the regular tree \mathbb{T}_κ , if the SDE system (3.1) starts from a distribution $\lambda \in \mathcal{P}((\mathbb{R}^d)^{\mathbb{T}_\kappa})$ that is automorphism-invariant and a second order MRF (see Definition 5.1), then Corollary 3.14 remains valid with $(Y_v(0))_{v \in \mathbb{T}_{\kappa,1}}$ distributed according to the $\mathbb{T}_{\kappa,1}$ -marginal of λ . A similar result should hold in the UGW case, but the requisite symmetry and conditional independence properties are much more subtle to formulate, and hence, deferred to future work.

3.4 Comments on the proof and two key auxiliary results

The proof of Theorem 3.12, which is given in Sect. 4, relies on two important properties of the interacting particle system (3.1) which we state in this section and which may be of independent interest. The first result, Proposition 3.17, states the form of the conditional independence property that we require. It can be viewed as a (more complicated) analogue of the second-order MRF property for \mathbb{T}_2 -trees discussed in Sect. 1.2.2(i), but the essential message remains the same: by conditioning on the particle trajectories at the root vertex and a child thereof, the particle trajectories in the two disjoint subtrees obtained by removing the edge between these two vertices become independent.

Recall in the following that π_v denotes the parent vertex of any $v \in \mathbb{V} \setminus \{\emptyset\}$ as defined in Sect. 2.1.2.

Proposition 3.17 *Suppose Assumption A holds, and suppose \mathcal{T} is a $\text{UGW}(\rho)$ tree, where $\rho \in \mathcal{P}(\mathbb{N}_0)$ has finite nonzero first moment. Then, for each $t > 0$, the following hold:*

- (i) $(X_{ki}^\mathcal{T}[t])_{i \in \mathbb{N}}$ is conditionally independent of $X_{\pi_1}^\mathcal{T}[t]$ given $X_{\{\emptyset, k\}}^\mathcal{T}[t]$, for any $k \in \mathbb{N}$.
- (ii) For each $t > 0$, the conditional law of $(X_{ki}^\mathcal{T}[t])_{i \in \mathbb{N}}$ given $(X_k^\mathcal{T}[t], X_\emptyset^\mathcal{T}[t])$ does not depend on the choice of $k \in \mathbb{N}$. More precisely, there exists a measurable map $\Lambda_t : \mathcal{C}^2 \rightarrow \mathcal{P}(\mathcal{C}_t^\mathbb{N})$ such that, for every $k \in \mathbb{N}$ and every Borel set $B \subset \mathcal{C}_t^\mathbb{N}$, we have

$$\Lambda_t(X_k^\mathcal{T}, X_\emptyset^\mathcal{T})(B) = \mathbb{P}((X_{ki}^\mathcal{T}[t])_{i \in \mathbb{N}} \in B \mid X_\emptyset^\mathcal{T}[t], X_k^\mathcal{T}[t]) \text{ a.s.}$$

The proof of Proposition 3.17 is given in Sect. 6 and relies on general definitions and properties of MRFs on finite graphs outlined in Sect. 5. We first study finite truncations of the UGW tree in Proposition 6.2, prove a version of this property on the truncated graph, and then carefully take limits.

While Proposition 3.17(ii) captures some of the symmetry of the $\text{UGW}(\rho)$ tree \mathcal{T} , the next result, Proposition 3.18, provides one more crucial symmetry property and is where unimodularity comes into play; this might be contrasted with the simpler symmetry considerations used in the case of \mathbb{T}_2 as outlined in Sect. 1.2.2(ii). Proposition 3.18 below is where the measure change described in Remark 3.11 appears, which explains the form of γ_t in Definition 3.9.

Recall the definition of the space $S^\cup(\mathcal{X})$ from Sect. 2.5. For Proposition 3.18 and its proof, it is helpful to introduce some notation to emphasize when we are working with

unordered vectors (elements of $S^{\sqcup}(\mathcal{X})$) versus ordered vectors. For a finite set A and a (ordered) vector $x_A = (x_v)_{v \in A} \in \mathcal{X}^A$, we write $\langle x_A \rangle$ to denote the corresponding element (equivalence class) of $S^{\sqcup}(\mathcal{X})$. The canonical labeling scheme \mathbb{V} introduced in Sect. 2.1.2 and adopted in this section carries with it a natural order, and we will find it helpful to use this notation $\langle \cdot \rangle$ when it is important to stress that we are dealing with an unordered vector.

Proposition 3.18 *Suppose Assumption A holds, and suppose \mathcal{T} is a UGW(ρ) tree, where $\rho \in \mathcal{P}(\mathbb{N}_0)$ has finite nonzero first moment. Let $t > 0$, and let $h : \mathcal{C}_t^2 \times S^{\sqcup}(\mathcal{C}_t) \rightarrow \mathbb{R}$ be bounded and measurable. Suppose we are given a measurable function $\Xi_t : \mathcal{C}^2 \rightarrow \mathbb{R}$ that satisfies*

$$\Xi_t(X_\emptyset, X_1) = 1_{\{N_\emptyset(\mathcal{T}) \neq \emptyset\}} \frac{\mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} h(X_\emptyset[t], X_1[t], \langle X_{N_\emptyset(\mathcal{T}_1)}[t] \rangle) \mid X_\emptyset[t], X_1[t] \right]}{\mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} \mid X_\emptyset[t], X_1[t] \right]}, \quad a.s.$$

Then, for each $k \in \mathbb{N}$,

$$\Xi_t(X_k, X_\emptyset) = \mathbb{E} \left[h(X_k[t], X_\emptyset[t], \langle X_{N_k(\mathcal{T})}[t] \rangle) \mid X_\emptyset[t], X_k[t] \right], \quad a.s., \text{ on } \{k \in \mathcal{T}\}. \quad (3.9)$$

The proof of Proposition 3.18 is given in Sect. 7. It is worth noting that the statement of Proposition 3.18 would be far less succinct if we did not define the SDE as in (3.1) with the canonical labeling scheme.

3.5 Limits of finite-graph systems

This section presents the natural application of our local equation to characterizing the limiting behavior of finite particle systems, drawing on our recent results in [29]. For a finite and possibly random graph G with non-random vertex set V , we define $X^G = (X_v^G)_{v \in V}$ as the unique in law solution of the SDE

$$dX_v^G(t) = b(t, X_v^G, X_{N_v(G)}^G)dt + \sigma(t, X_v^G)dW_v(t), \quad v \in V, \text{ with } (X_v^G(0))_{v \in V} \text{ i.i.d. } \sim \lambda_0. \quad (3.10)$$

Here $(W_v)_{v \in V}$ are independent Brownian motions, the initial law $\lambda_0 \in \mathcal{P}(\mathbb{R}^d)$ is given, and $N_v(G)$ denotes the set of vertices in G which are adjacent to v . Moreover, we assume as always that the graph G , if random, is independent of $(W_v, X_v^G(0))_{v \in V}$. Note that under Assumption A, as discussed thereafter, existence and uniqueness in law for the SDE (3.10) hold by Girsanov's theorem. We define also the (global) empirical measure

$$\bar{\mu}^G := \frac{1}{|V|} \sum_{v \in V} \delta_{X_v^G}, \quad (3.11)$$

which we view as a random element of $\mathcal{P}(\mathcal{C})$.

Using Theorems 3.3 and 3.7 of [29], we could now state a rather general theorem that applies to any (random) graph sequence that *converges in the local weak sense* to a $\text{UGW}(\rho)$ tree. Indeed, [29, Theorem 3.7] shows that if G_n *converges in probability in the local weak sense* to a limiting graph G , then both μ^{G_n} and $X_{\phi_n}^{G_n}$, where ϕ_n is a uniformly random vertex in G_n , converge, with the limits characterized in terms of the root particle in the SDE system (3.10) set on the limit graph G . When G is a $\text{UGW}(\rho)$ tree, we then characterize this root particle via our local equation. To avoid giving a full definition of local weak convergence of (marked) graphs (which can be found in [29] in Section 2.2.4, including Definitions 2.8 and 2.10 therein, and Appendix A), we prefer not to state the most general result possible here, and instead we focus on three noteworthy random graph models and associated graph sequences:

1. $\text{ER}(\theta)$ The Erdős-Rényi graph $G \sim \mathcal{G}(n, p)$ is defined for $n \in \mathbb{N}$ and $p \in (0, 1)$ by considering a graph with n vertices and independently connecting each pair of distinct vertices with probability p each. Consider the sequence $G_n = \mathcal{G}(n, p_n)$, with $p_n \in (0, 1)$ and $np_n \rightarrow \theta \in (0, \infty)$.
2. $\text{RR}(\kappa)$ The random κ -regular graph $G \sim \text{Reg}(n, \kappa)$ is defined for $n \in \mathbb{N}$ by choosing a κ -regular graph (meaning each vertex has exactly κ neighbors) uniformly at random from among all κ -regular graphs on n vertices. It is well known that a κ -regular graph on n vertices exists as long as $n\kappa$ is even and $n \geq \kappa + 1$. Consider the sequence $G_n = \text{Reg}(n, \kappa)$ if $n\kappa$ is even and $G_n = \text{Reg}(n+1, \kappa)$ if $n\kappa$ is odd.
3. $\text{CM}(d)$ The configuration model $G \sim \text{CM}(n, d^n)$, for any degree sequence $d^n = (d_1^n, \dots, d_n^n) \in \mathbb{N}^n$, is the uniformly random graph from among all graphs on n vertices with degree sequence d^n ; see [45, Chapter 7] for more information. Of course if $d^n = (\kappa, \dots, \kappa)$ then this reduces to the random κ -regular graph. Consider the sequence $G_n = \text{CM}(n, d^n)$, with $\frac{1}{n} \sum_{k=1}^n \delta_{d_k^n}$ converges weakly to some $d \in \mathcal{P}(\mathbb{N}_0)$, $\frac{1}{n} \sum_{k=1}^n d_k^n \rightarrow \sum_{k=0}^{\infty} kd(k)$, and assume d has finite first and second moments, that is, $\sum_{k \in \mathbb{N}_0} k^2 d(k) < \infty$.

The following convergence result is a consequences of Theorems 3.7 and 3.9 and Examples 2.2, 2.3 and 2.4 of [29]. Recall here that $\mathcal{L}(Z)$ denotes the law of a random variable Z , and \Rightarrow denotes convergence in law.

Theorem 3.19 [29] *Suppose Assumption B holds, assume the initial distribution λ_0 has bounded support, and let $\{G_n\}_{n \in \mathbb{N}}$ be an $\text{ER}(\theta)$, $\text{RR}(\kappa)$ or $\text{CM}(d)$ graph sequence, as defined above, and correspondingly, define $\rho \in \mathcal{P}(\mathbb{N}_0)$ by $\rho := \text{Poisson}(\theta)$, $\rho := \delta_\kappa$ or $\rho := d$. Further, let X^{G_n} and $\bar{\mu}^{G_n}$ be as in (3.10) and (3.11), respectively, let ϕ_n denote a uniformly random vertex in G_n , and let $T = \text{UGW}(\rho)$ be the rooted UGW tree with offspring distribution ρ . Then $X_{\phi_n}^{G_n} \Rightarrow X_\phi^T$ and $\bar{\mu}^{G_n} \Rightarrow \mathcal{L}(X_\phi^T)$.*

Note that this result notably points out that convergence (in probability) in the *local weak sense* of each of the above three random graph models is strong enough to imply the convergence of the *global* empirical measure $\bar{\mu}^{G_n}$, in fact to a non-random limit. However, in Theorem 3.19, this limit is characterized as the marginal of an *infinite-dimensional* stochastic differential equation. When combined with Theorem 3.12, this implies the following result, which states that these limits can in fact be characterized as the marginal of an (almost surely) *finite-dimensional* stochastic system, thus achieving a remarkable dimension reduction.

Corollary 3.20 *Suppose Assumption B holds, assume the initial distribution λ_0 has bounded support, and let $\{G_n\}_{n \in \mathbb{N}}$ be an ER(θ), RR(κ) or CM(d) graph sequence, and correspondingly, define $\rho \in \mathcal{P}(\mathbb{N}_0)$ by $\rho := \text{Poisson}(\theta)$, $\rho := \delta_\kappa$ or $\rho := d$. Further, let X^{G_n} and $\bar{\mu}^{G_n}$ be as in (3.10) and (3.11), respectively, let ϕ_n denote a uniformly random vertex in G_n , let $\mathcal{T} = \text{UGW}(\rho)$, and let Y be a solution to the UGW(ρ) local equation with initial law λ_0 , as stated in Definition 3.9 (equivalently, in the case $\rho = \delta_\kappa$, Definition 3.5). Then $X_{\phi_n}^{G_n} \Rightarrow Y_\phi$ and $\bar{\mu}^{G_n} \Rightarrow \mathcal{L}(Y_\phi)$.*

Note that this result asserts that in each case, the sequence of random empirical measures $\{\mu^{G_n}\}_{n \in \mathbb{N}}$ converges in law to the *non-random limit* $\mathcal{L}(Y_\phi)$. By standard propagation of chaos arguments (see [44] or Lemma 2.12 of [29]), it follows that if $k \in \mathbb{N}$ is fixed and if v_n^1, \dots, v_n^k are k independent uniformly random vertices in G_n , then $\mathcal{L}(X_{v_n^1}^{G_n}, \dots, X_{v_n^k}^{G_n})$ converges weakly to the k -fold product measure $\mathcal{L}(Y_\phi) \times \dots \times \mathcal{L}(Y_\phi)$ as $n \rightarrow \infty$. The same is then true if (v_n^1, \dots, v_n^k) is chosen uniformly at random from among the $\binom{n}{k}$ k -tuples of *distinct* vertices. However, it is important to emphasize that unlike mean-field limits, in our setting this convergence does not hold for any arbitrary chosen finite set of vertices. In particular, if $v_n^1 = \phi_n$ and v_n^2 is a neighbor of ϕ_n chosen uniformly at random (assuming one exists, else set v_n^2 to be a uniformly at random vertex from $V \setminus \phi_n$), then the laws of $X_{v_n^1}^{G_n}$ and $X_{v_n^2}^{G_n}$ are not asymptotically independent but remain correlated in the limit, with the limiting correlations captured by the local equation.

4 Proof of Theorem 3.12

This section is devoted to the proof of Theorem 3.12 using the results stated in Propositions 3.17 and 3.18. Throughout, let $(X_v^T)_{v \in \mathbb{V}}$ be a solution to the SDE system (3.1). In Sect. 4.1 we first verify that the marginal $(X_v^T)_{v \in \mathbb{V}_1}$ is a weak solution of the local equation, in particular establishing existence of a solution to the local equation. Then, in Sect. 4.2, we show that the local equation is well-posed in the sense that it has a unique weak solution. In the proofs we will use the notation \mathcal{E} to denote the Doleans exponential, or

$$\mathcal{E}_t(M) := \exp(M_t - \frac{1}{2}[M]_t), \quad t \geq 0, \quad (4.1)$$

for a continuous local martingale M , where $[M]$ denotes the (optional) quadratic variation process of M . We also recall that H denotes the relative entropy functional defined in (4.2).

4.1 Verification result

We prove in this section the first claim of Theorem 3.12, which asserts that the law of the root neighborhood particles $(X_v)_{v \in \mathbb{V}_1}$ provides a weak solution of the local equation of Definition 3.9.

We first state a fairly standard integrability estimate, which explains the need for the average $1/|A|$ in the linear growth Assumption (A.1). We defer the proof to Appendix C, as it is similar to [28, Lemma 5.1]. In the following, let H denote the relative entropy functional: for probability measures $\nu, \tilde{\nu}$ on a common measurable space, let

$$H(\nu|\tilde{\nu}) := \int \log \frac{d\nu}{d\tilde{\nu}} d\nu \quad \text{if } \nu \ll \tilde{\nu}, \quad H(\nu|\tilde{\nu}) = \infty \quad \text{if } \nu \not\ll \tilde{\nu}, \quad (4.2)$$

where $\nu \ll \tilde{\nu}$ signifies ν is absolutely continuous with respect to $\tilde{\nu}$. For any random tree \mathcal{T} , let $(\hat{X}_v^{\mathcal{T}})_{v \in \mathbb{V}}$ denote the unique in law [by Assumption (A.2b)] solution of the driftless SDE system

$$d\hat{X}_v^{\mathcal{T}}(t) = 1_{\{v \in \mathcal{T}\}} \sigma(t, \hat{X}_v^{\mathcal{T}}) dB_v(t), \quad v \in \mathbb{V}, \quad (4.3)$$

where $(\hat{X}_v^{\mathcal{T}}(0))_{v \in \mathbb{V}}$ are i.i.d. with law λ_0 , and as usual the tree, initial conditions, and Brownian motions are independent. Recall in the following that $\mathcal{L}(Z)$ denotes the law of a random variable Z , and $x_A = (x_v)_{v \in A}$ denotes a sub-configuration of $x = (x_v)_{v \in \mathbb{V}}$ for $A \subset \mathbb{V}$.

Lemma 4.1 *Suppose Assumption A holds. For each $T \in (0, \infty)$ there exists a constant $C_T^* < \infty$ such that, for any random tree $\mathcal{T} \subset \mathbb{V}$, letting $X^{\mathcal{T}}$ be the solution of (3.1), we have*

$$\sup_{v \in \mathbb{V}} \mathbb{E}[\|X_v^{\mathcal{T}}\|_{*,T}^2 | \mathcal{T}] \leq C_T^*, \quad a.s., \quad (4.4)$$

and also, for any finite set $A \subset \mathbb{V}$,

$$H(\mathcal{L}(X_A^{\mathcal{T}}[T]) | \mathcal{L}(\hat{X}_A^{\mathcal{T}}[T])) \leq C_T^*(1 + |A|), \quad (4.5)$$

$$H(\mathcal{L}(\hat{X}_A^{\mathcal{T}}[T]) | \mathcal{L}(X_A^{\mathcal{T}}[T])) \leq C_T^*(1 + |A|). \quad (4.6)$$

Now, we work for the rest of Sect. 4.1 on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, supporting a UGW(ρ) tree \mathcal{T} , independent d -dimensional Brownian motions $(W_v)_{v \in \mathbb{V}}$, and continuous d -dimensional processes $(X_v^{\mathcal{T}})_{v \in \mathbb{V}}$ satisfying the SDE system (3.1). As always we assume \mathcal{T} , $(W_v)_{v \in \mathbb{V}}$, and $(X_v^{\mathcal{T}}(0))_{v \in \mathbb{V}}$ are independent, and $(X_v^{\mathcal{T}}(0))_{v \in \mathbb{V}}$ are i.i.d. with common law λ_0 . The offspring distribution $\rho \in \mathcal{P}(\mathbb{N}_0)$ has finite nonzero first moment and finite second moment. For ease of notation, for the rest of Sect. 4.1 we omit the superscript by writing $(X_v)_{v \in \mathbb{V}} = (X_v^{\mathcal{T}})_{v \in \mathbb{V}}$ and $(\hat{X}_v)_{v \in \mathbb{V}} = (\hat{X}_v^{\mathcal{T}})_{v \in \mathbb{V}}$. The driftless process $\hat{X} = (\hat{X}_v^{\mathcal{T}})_{v \in \mathbb{V}}$ defined in (4.3) may live on a different probability space that we do not specify.

Remark 4.2 The dynamics (3.1) include the “fictional” particles $v \notin \mathcal{T}$ in such a way that the random tree \mathcal{T} can be recovered from $(X_v[t])_{v \in \mathbb{V}}$ for any $t > 0$. Indeed, almost surely, $v \notin \mathcal{T}$ if and only if there exists an interval on which $t \mapsto X_v(t)$ is constant. (Note that this holds because the diffusion coefficient is assumed non-degenerate.)

More precisely, \mathcal{T} is measurable with respect to the “just after time zero” σ -field, or

$$\{v \in \mathcal{T}\} \in \bigcap_{t>0} \sigma(X_v(s) : s \leq t), \quad \text{a.s. for each } v \in \mathbb{V}. \quad (4.7)$$

Here “a.s.” means that the event $\{v \in \mathcal{T}\}$ belongs to the completion of the σ -field appearing on the right-hand side. Moreover, there exists a deterministic mapping $\tau : \mathcal{C} \rightarrow \{0, 1\}$, measurable with respect to $\cap_{t>0} \mathcal{F}_t^{\mathcal{C}}$ where $(\mathcal{F}_t^{\mathcal{C}})_{t \geq 0}$ is the canonical filtration on \mathcal{C} , such that

$$1_{\{v \in \mathcal{T}\}} = \tau(X_v), \quad \text{a.s. for each } v \in \mathbb{V}. \quad (4.8)$$

In particular, this function τ does not depend on v . These observations will be exploited several times throughout this section.

The proof is decomposed into several steps.

Step 1 The first step of the proof will be to project onto the root neighborhood \mathbb{V}_1 using the projection theorem (Theorem A.2). It follows from (3.1) that $(X_v)_{v \in \mathbb{V}_1}$ satisfies

$$\begin{aligned} dX_\emptyset(t) &= b(t, X_\emptyset, X_{N_\emptyset(\mathcal{T})}) dt + \sigma(t, X_\emptyset) dW_\emptyset(t), \\ dX_k(t) &= 1_{\{k \in \mathcal{T}_1\}} \left(b(t, X_k, X_{N_k(\mathcal{T})}) dt + \sigma(t, X_k) dW_k(t) \right), \quad k \in \mathbb{V}_1 \setminus \{\emptyset\}, \end{aligned}$$

where we write $\mathcal{T}_1 := \mathcal{T} \cap \mathbb{V}_1$ for the first generation of \mathcal{T} . By Theorem A.2, by extending the probability space if necessary, we may find independent d -dimensional Brownian motions $(B_v)_{v \in \mathbb{V}_1}$ such that

$$dX_v(t) = \tilde{b}_v(t, X_{\mathbb{V}_1}) dt + \tilde{\sigma}_v(t, X_{\mathbb{V}_1}) dB_v(t), \quad v \in \mathbb{V}_1, \quad (4.9)$$

where $\tilde{b}_v : \mathbb{R}_+ \times \mathcal{C}^{\mathbb{V}_1} \mapsto \mathbb{R}^d$ and $\tilde{\sigma}_v : \mathbb{R}_+ \times \mathcal{C}^{\mathbb{V}_1} \mapsto \mathbb{R}^{d \times d}$ are any progressively measurable functions satisfying

$$\tilde{b}_v(t, X_{\mathbb{V}_1}) = \mathbb{E} \left[1_{\{v \in \mathcal{T}_1\}} b(t, X_v, X_{N_v(\mathcal{T})}) \mid X_{\mathbb{V}_1}[t] \right], \quad (4.10)$$

$$\tilde{\sigma}_v \tilde{\sigma}_v^\top(t, X_{\mathbb{V}_1}) = \mathbb{E} \left[1_{\{v \in \mathcal{T}_1\}} \sigma \sigma^\top(t, X_v) \mid X_{\mathbb{V}_1}[t] \right], \quad (4.11)$$

a.s. for a.e. $t > 0$. Such progressively measurable functions always exist by Lemma A.1. Now, the functions \tilde{b}_\emptyset and $(\tilde{\sigma}_v)_{v \in \mathbb{V}_1}$ can be simplified because the corresponding integrands are $X_{\mathbb{V}_1}[t]$ -measurable. Indeed, because $X_{N_\emptyset(\mathcal{T})}$ and \mathcal{T}_1 are $X_{\mathbb{V}_1}[t]$ -measurable for each $t > 0$ (as a consequence of Remark 4.2), we may take

$$\tilde{b}_\emptyset(t, X_{\mathbb{V}_1}) = b(t, X_\emptyset, X_{N_\emptyset(\mathcal{T}_1)}), \quad \tilde{\sigma}_v(t, X_{\mathbb{V}_1}) = 1_{\{v \in \mathcal{T}_1\}} \sigma(t, X_v), \quad v \in \mathbb{V}_1. \quad (4.12)$$

Step 2 Next, we simplify the form of \tilde{b}_v for $v \in \mathbb{V}_1 \setminus \emptyset$, using symmetry and conditional independence results. Noting that $\mathbb{V}_1 \setminus \{\emptyset\}$ can be identified with \mathbb{N} , for a given $k \in \mathbb{N}$,

we first apply the conditional independence result of Proposition 3.17(i) to deduce that all particles except \emptyset and k may be safely omitted from the conditioning in the definition of \tilde{b}_k . That is, recalling also that $\{k \in \mathcal{T}_1\}$ is $X_k[t]$ -measurable (again by Remark 4.2), we have

$$\tilde{b}_k(t, X_{\mathbb{V}_1}) = 1_{\{k \in \mathcal{T}_1\}} \mathbb{E} [b(t, X_k, X_{N_k(\mathcal{T})}) \mid X_{\emptyset}[t], X_k[t]], \quad a.s., \quad a.e. \, t > 0.$$

Now, fix $t > 0$. Since b is progressively measurable, there exists a measurable function $h : \mathcal{C}_t \times S^{\sqcup}(\mathcal{C}_t) \mapsto \mathbb{R}$ such that $b(t, x, \bar{x}) = h(x[t], \bar{x}[t])$ for $x \in \mathcal{C}$, $\bar{x} \in S^{\sqcup}(\mathcal{C})$. Then, on $\{k \in \mathcal{T}_1\}$, $\tilde{b}_k(t, X_{\mathbb{V}_1})$ is equal to the right-hand side of (3.9) with this choice of h . Although h is not bounded as is required in Proposition 3.18, both $h(X_{\emptyset}[t], \langle X_{N_{\emptyset}}[t] \rangle)$ and $h(X_k[t], \langle X_{N_k}[t] \rangle)$ are square-integrable due to the linear growth of b from Assumption (A.1) and Lemma 4.1, and we know also that $|N_{\emptyset}(\mathcal{T})|$ is square-integrable as we assumed ρ has finite second moment. Hence, by truncating h and taking limits, we easily extend the validity of the formula in Proposition 3.18 to cover such an h . Ultimately, we deduce that

$$\tilde{b}_k(t, X_{\mathbb{V}_1}) = \gamma_t(X_k, X_{\emptyset}), \quad \text{on } \{k \in \mathcal{T}_1\}, \quad (4.13)$$

where $\gamma_t : \mathcal{C}^2 \mapsto \mathbb{R}^d$ is a progressively measurable function satisfying

$$\gamma_t(X_{\emptyset}, X_1) = \frac{\mathbb{E} \left[\frac{|N_{\emptyset}(\mathcal{T})|}{|N_1(\mathcal{T})|} b(t, X_{\emptyset}, X_{N_{\emptyset}(\mathcal{T}_1)}) \mid X_{\emptyset}[t], X_1[t] \right]}{\mathbb{E} \left[\frac{|N_{\emptyset}(\mathcal{T})|}{|N_1(\mathcal{T})|} \mid X_{\emptyset}[t], X_1[t] \right]} \quad \text{on } \{N_{\emptyset}(\mathcal{T}_1) \neq \emptyset\}, \quad (4.14)$$

and $\gamma_t(X_{\emptyset}, X_1) = b(t, X_{\emptyset}, \circ)$ on $\{N_{\emptyset}(\mathcal{T}_1) = \emptyset\}$, where we recall that \circ denotes the element of the one-point space \mathcal{C}^0 . Note that $|N_1(\mathcal{T})| \geq 1$ a.s., $\mathbb{E}[|N_{\emptyset}(\mathcal{T})|^2] < \infty$ and $\mathbb{E} \left[\int_0^T |b(t, X_{\emptyset}[t], X_{N_{\emptyset}}[t])|^2 dt \right] < \infty$, which together imply

$$\mathbb{E} \left[1_{\{N_{\emptyset}(\mathcal{T}) \neq \emptyset\}} \frac{|N_{\emptyset}(\mathcal{T})|}{|N_1(\mathcal{T})|} \int_0^T |b(t, X_{\emptyset}[t], X_{N_{\emptyset}}[t])| dt \right] < \infty,$$

for each $T \in (0, \infty)$. Since X is continuous, the existence of a progressively measurable version of $(t, x_{\emptyset}, x_1) \mapsto \gamma_t(x_{\emptyset}, x_1)$ is then guaranteed by Lemma A.1.

Step 3 It remains to check that we have all of the ingredients required by Definition 3.9 for a solution of the local equation. We begin with the integrability condition stated as property (10) in Definition 3.9. Note that Lemma 4.1 and the linear growth of b from Assumption (A.1) ensure that, by Jensen's inequality and (4.14),

$$\mathbb{E} \left[1_{\{k \in \mathcal{T}\}} \int_0^T |\gamma_t(X_k, X_{\emptyset})|^2 dt \right] \leq \mathbb{E} \left[1_{\{k \in \mathcal{T}\}} \int_0^T |b(t, X_k, X_{N_k(\mathcal{T})})|^2 dt \right] < \infty. \quad (4.15)$$

Next, recall from Remark 4.2 that $\{N_\emptyset(T) \neq \emptyset\}$ is $X_\emptyset[t]$ -measurable for $t > 0$. Applying the conditional Jensen's inequality, and invoking (4.14), Assumption (A.1) and Lemma 4.1 yields

$$\begin{aligned} & \mathbb{E} \left[1_{\{N_\emptyset(T) \neq \emptyset\}} \frac{|N_\emptyset(T)|}{|N_1(T)|} \int_0^T |\gamma_t(X_\emptyset, X_k)|^2 dt \right] \\ & \leq \mathbb{E} \left[1_{\{N_\emptyset(T) \neq \emptyset\}} \frac{|N_\emptyset(T)|}{|N_1(T)|} \int_0^T |b(t, X_\emptyset, X_{N_\emptyset(T_1)})|^2 dt \right] \\ & \leq 3C_T^2 T \mathbb{E} \left[1_{\{N_\emptyset(T) \neq \emptyset\}} \frac{|N_\emptyset(T)|}{|N_1(T)|} \left(1 + \|X_\emptyset\|_{*,T}^2 + \frac{1}{|N_\emptyset(T)|} \sum_{k \in N_\emptyset(T)} \|X_k\|_{*,T}^2 \right) \right] \\ & \leq 3C_T^2 T (1 + 2C_T^*) \mathbb{E} \left[1_{\{N_\emptyset(T) \neq \emptyset\}} \frac{|N_\emptyset(T)|}{|N_1(T)|} \right] < \infty, \end{aligned} \quad (4.16)$$

where $C_T < \infty$ and $C_T^* < \infty$ are the constants from Assumption (A.1) and Lemma 4.1, respectively. Recalling that $\gamma_t(X_\emptyset, X_1) = b(t, X_\emptyset, \circ)$ on $\{N_\emptyset(T) = \emptyset\}$ we deduce from (4.15) and (4.16) that the following two integrals are a.s. finite, for each $k \in \mathbb{N}$:

$$\int_0^T |\gamma_t(X_\emptyset, X_k)|^2 dt, \quad \int_0^T |\gamma_t(X_k, X_\emptyset)|^2 dt.$$

The finite entropies of Lemma 4.1 ensure that the laws of (X_\emptyset, X_k) and $(\widehat{X}_\emptyset, \widehat{X}_k)$ are equivalent (i.e., mutually absolutely continuous) for each $k \in \mathbb{N}$, and therefore the following integrals are also a.s. finite:

$$\int_0^T |\gamma_t(\widehat{X}_\emptyset, \widehat{X}_k)|^2 dt, \quad \int_0^T |\gamma_t(\widehat{X}_k, \widehat{X}_\emptyset)|^2 dt.$$

Along with the definition of γ_t (see (4.14) and the subsequent line), this verifies both properties (7) and (10) of Definition 3.9, with $Y = X$.

Finally, by enlarging the probability space if necessary, let $\widehat{C}_1^{\text{ext}}$ be an \mathcal{F}_0 -measurable \mathbb{N}_0 -valued random variable with law $\widehat{\rho}$, independent of $(T, (X_v(0))_{v \in \mathbb{V}})$. Define $\widehat{C}_1 := |N_1(T)| - 1$ on the event $\{N_\emptyset(T_1) \neq \emptyset\}$, and on the complementary event $\{N_\emptyset(T_1) = \emptyset\}$ define $\widehat{C}_1 := \widehat{C}_1^{\text{ext}}$. This way, using the definition of the UGW(ρ) tree T , one may easily check that \widehat{C}_1 has law $\widehat{\rho}$, T_1 is the first generation of a UGW(ρ) tree, $(X_v(0))_{v \in \mathbb{V}}$ are i.i.d. and \mathcal{F}_0 -measurable with law λ_0 , and moreover, \widehat{C}_1 , T_1 , and $(X_v(0))_{v \in \mathbb{V}}$ are independent. This verifies properties (1)–(3), (6), and (8) of Definition 3.9. (The definition of \widehat{C}_1 on $\{N_\emptyset(T_1) = \emptyset\}$ is made in this way for the sole purpose of meeting the independence requirement of Definition 3.9(8), and $\widehat{C}_1^{\text{ext}}$ serves no other purpose.) Combining relations (4.9)–(4.14), we see that the stochastic equations (3.8) are satisfied with $Y_k = X_k$ for all $k \in \mathbb{V}_1$, and thus properties (4), (5), and (9) of Definition 3.9 hold. Putting this together, we see that $(X_v)_{v \in \mathbb{V}_1}$ is a weak solution of the UGW(ρ) local equation with initial law λ_0 , as in Definition 3.9.

4.2 Proof of well-posedness of the UGW(ρ) local equation

Fix $\rho \in \mathcal{P}(\mathbb{N}_0)$ with finite first and second moments. As briefly described in Sect. 1.2.2 in the simplest case of a 2-regular tree, the basic idea behind the proof of uniqueness is to use the weak solution to the local equation to construct a solution to the infinite particle system (3.1) on the UGW(ρ) tree \mathcal{T} , and then invoke uniqueness (in law) of the latter to deduce that of the former. However, the construction is more involved when $\kappa > 2$ and substantially more complicated in the case of the random UGW tree. To make the proof more transparent, we first provide an outline and introduce some common notation in Sect. 4.2.1, then prove the main technical lemmas in Sect. 4.2.2, and finally, in Sect. 4.2.3, show that the uniqueness property in Theorem 3.12 is a consequence of these lemmas.

4.2.1 Outline of proof and some common terminology

Let $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \mathcal{T}_1, \gamma, (B_v, Y_v)_{v \in \mathbb{V}_1}, \widehat{C}_1)$ be any weak solution to the UGW(ρ) local equation with initial law λ_0 , as specified in Definition 3.9. Due to properties (2), (3), (4), and (8) of Definition 3.9, by extending the probability space if needed, we can assume without loss of generality that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ also supports a UGW(ρ) tree \mathcal{T} , independent of the standard d -dimensional \mathbb{F} -Brownian motions $(B_v)_{v \in \mathbb{V}}$, and i.i.d. initial conditions $(Y_v(0))_{v \in \mathbb{V}}$, such that $\mathcal{T}_1 = \mathcal{T} \cap \mathbb{V}_1$ and $\widehat{C}_1 + 1 = |N_1(\mathcal{T})|$ on $\{N_\emptyset(\mathcal{T}) \neq \emptyset\}$. Next, again on the event $\{N_\emptyset(\mathcal{T}) \neq \emptyset\}$, we aim to extend the local solution to \mathbb{V}_2 in such a way that the law of the particle system on the random tree $\mathcal{T}_2 := \mathcal{T} \cap \mathbb{V}_2$ of depth 2 is consistent with the \mathcal{T}_2 -marginal of the interacting particle system (3.1), where recall that, for any $n \in \mathbb{N}$, \mathbb{V}_n was defined in (2.3). For this it suffices to specify the conditional joint law of the states of vertices in $\mathcal{T}_2 \setminus \mathbb{V}_1$ given $Y_{\mathbb{V}_1}$. In view of the second-order MRF property and exchangeability (as encapsulated in Proposition 3.17), this is equal to the product of the conditional joint laws of the states of the offspring of each $i \in \mathcal{T}_1 \setminus \{\emptyset\}$, given the states of the vertices i and \emptyset , and each of these conditional laws is identical in form.

Now, in the case when $\mathcal{T} = \mathbb{T}_\kappa$ for some $\kappa \geq 2$, this conditional law can be identified from the weak solution to the local equation since, by the symmetry of the tree, it has the same form as the conditional law, given the trajectories of vertices \emptyset and 1, of the remaining children $\mathcal{T}_1 \setminus \{\emptyset, 1\} = \{2, \dots, \kappa\}$ of the root \emptyset , except that the roles of \emptyset and 1 are now reversed, since 1 now acts as the new root (see Fig. 1).

In the case of the UGW(ρ) tree, while the conditional joint laws are the same *given* the structure of the tree, re-rooting the tree at 1 changes the distribution of the tree. To account for this, we define a new “tilted” measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ via the relation

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} 1_{\{N_\emptyset(\mathcal{T}_1) \neq \emptyset\}} + 1_{\{N_\emptyset(\mathcal{T}) = \emptyset\}}. \quad (4.17)$$

The fact that this defines a true probability measure $\widetilde{\mathbb{P}}$ is justified in Lemma 4.4 below.

We then characterize the joint law of (Y_\emptyset, Y_1) under this tilted measure $\widetilde{\mathbb{P}}$ in Lemma 4.4, and then use the unimodularity of the tree (in particular, Proposition 3.18) to compute the conditional law on each time interval $[0, t]$ of the trajectories

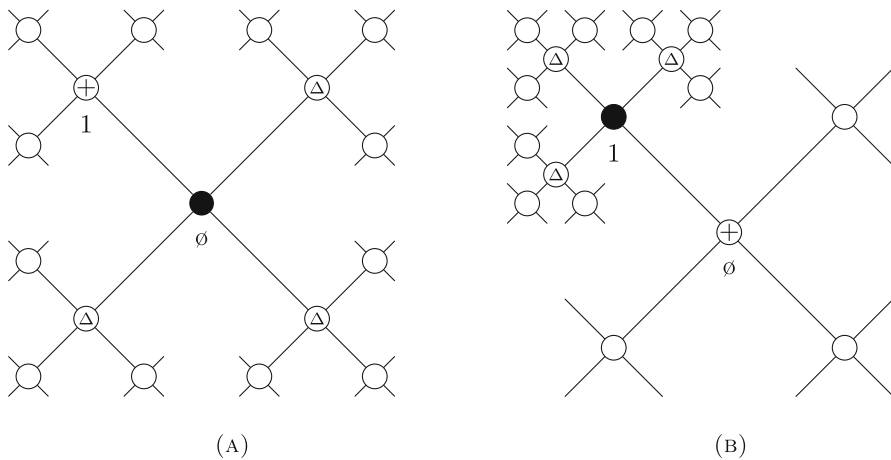


Fig. 1 The case $\mathcal{T} = \mathbb{T}_4$, relating the conditional law $\mathcal{L}(Y_\Delta \in \cdot | Y_\bullet, Y_+)$ when **A** the tree is rooted at ‘ \bullet ’ and **B** the tree is rerooted at ‘ $+$ ’

of the neighborhood $N_\emptyset(\mathcal{T})$ of the root given those of \emptyset and 1 in Lemma 4.6. Using this conditional law, which is denoted by Z_t , we extend the particle system to \mathbb{V}_2 , and recursively to \mathbb{V}_n , and denote the latter law as $Q^n \in \mathcal{P}(\mathcal{C}^{\mathbb{V}_n})$; see (4.30). Finally, in Proposition 4.7 we show that the family $\{Q^n\}$ is consistent, in the sense that the projection of Q^n to $\mathcal{C}^{\mathbb{V}_k}$ coincides with Q^k for $k < n$, and that its unique extension to a law $Q \in \mathcal{P}(\mathcal{C}^{\mathbb{V}})$ coincides with the unique law of a weak solution to the infinite particle system (3.1).

We close this discussion by introducing some additional notation that will be used throughout the proof. Let $\nu := \mathbb{P} \circ Y_{\mathbb{V}_1}^{-1} \in \mathcal{P}(\mathcal{C}^{\mathbb{V}_1})$ denote the law of (the Y -marginal of) the weak solution of the UGW (ρ) local equation, and define the corresponding “tilted” measure $\tilde{\nu} \in \mathcal{P}(\mathcal{C}^{\mathbb{V}_1})$ by $\tilde{\nu} := \tilde{\mathbb{P}} \circ Y_{\mathbb{V}_1}^{-1}$. In other words, letting $\mathbb{E}^{\mathbb{P}}$ denote expectation with respect to \mathbb{P} , $\tilde{\nu}$ is defined by the Radon-Nikodym derivative

$$\frac{d\tilde{\nu}}{d\nu}(Y_{\mathbb{V}_1}) = \mathbb{E}^{\mathbb{P}} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} 1_{\{N_\emptyset(\mathcal{T}) \neq \emptyset\}} + 1_{\{N_\emptyset(\mathcal{T}) = \emptyset\}} \mid Y_{\mathbb{V}_1} \right],$$

though we will make no use of this precise form. Also, throughout, to compute various laws and conditional laws, it will be convenient to introduce some reference measures. For this, we introduce again the solution $(\widehat{X}_v)_{v \in \mathbb{V}}$ to the driftless SDE system (omitting the superscript \mathcal{T})

$$d\widehat{X}_v(t) = 1_{\{v \in \mathcal{T}\}} \sigma(t, \widehat{X}_v) dB_v(t), \quad v \in \mathbb{V}, \quad t > 0. \quad (4.18)$$

Note that this SDE is unique in law due to Assumption (A.2b). We also introduce the canonical probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^{*,n})$ to be used throughout the proof. Here, $\Omega^n = \mathcal{C}^{\mathbb{V}_n}$, \mathcal{F}^n is the Borel σ -field, and $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \geq 0}$ is the natural right-continuous filtration generated by the canonical coordinate processes, which are denoted by $(X_v)_{v \in \mathbb{V}_n}$, and $P^{*,n} := \mathbb{P} \circ \widehat{X}_{\mathbb{V}_n}^{-1}$, and $\tilde{P}^{*,n} := \tilde{\mathbb{P}} \circ \widehat{X}_{\mathbb{V}_n}^{-1}$ serve as references mea-

sures that represent the laws of the first n generations of the processes defined in (4.18) under the probability measures \mathbb{P} and $\tilde{\mathbb{P}}$, respectively. We define $\mathcal{T}_n \subset \mathbb{V}$ as the random tree with vertex set $\{v \in \mathbb{V}_n : \tau(X_v) = 1\}$, where τ is given as in Remark 4.2. In this way, \mathcal{T}_n agrees in law with $\mathcal{T} \cap \mathbb{V}_n$, the height- n truncation of the $\text{UGW}(\rho)$ tree \mathcal{T} . To be clear, $(X_v)_{v \in \mathbb{V}_n}$ and \mathcal{T}_n live on the canonical space Ω^n , whereas the other random variables such as $(\mathcal{T}, \widehat{X}, Y)$ are defined on Ω .

We make special note of the conventions we use to help the reader keep track of the various notations. We use a tilde for measures associated with the measure change, namely $\tilde{\mathbb{P}}$ and its descendants $\tilde{P}^{*,n}$ and $\tilde{\nu}$. The superscripts $*$ and n on $P^{*,n}$ and $\tilde{P}^{*,n}$ indicate that these measures are to be viewed as reference measures on the canonical space associated with n generations Ω^n . Lastly, the letter ν (and its decorated versions) will refer to measures constructed from the given solution $Y_{\mathbb{V}_1}$ of the local equation.

Remark 4.3 It is worth emphasizing again, as in Remark 3.10, how the argument simplifies when the tree is the deterministic κ -regular tree, i.e., the $\text{UGW}(\rho)$ tree with $\rho = \delta_\kappa$ for an integer $\kappa \geq 2$. In this case, we have $\widehat{\rho} = \delta_{\kappa-1}$, and we have deterministically $|N_v(\mathcal{T})| = \kappa$ for all $v \in \mathbb{V}$. In this case, $\tilde{\mathbb{P}} = \mathbb{P}$, $\tilde{\nu} = \nu$, and $\tilde{P}^{*,n} = P^{*,n}$. On a first reading it may help to keep these substitutions in mind.

4.2.2 Details of the proof

Once again, we break down the detailed justification into three steps.

Step 1 Our first goal is to identify the marginal law of (Y_\emptyset, Y_1) under the tilted measure $\tilde{\mathbb{P}}$ defined in (4.17). Specifically, recalling the definitions of $\nu, \tilde{\nu}$, the reference measures and canonical processes introduced in the last section, we define the marginal laws

$$\begin{aligned}
 \nu^{\emptyset,1} &= \nu \circ (X_\emptyset, X_1)^{-1} = \mathbb{P} \circ (Y_\emptyset, Y_1)^{-1}, \\
 \tilde{\nu}^{\emptyset,1} &= \tilde{\nu} \circ (X_\emptyset, X_1)^{-1} = \tilde{\mathbb{P}} \circ (Y_\emptyset, Y_1)^{-1}, \\
 \tilde{P}^{*,\emptyset,1} &= \tilde{P}^{*,1} \circ (X_\emptyset, X_1)^{-1},
 \end{aligned}$$

which are all elements of $\mathcal{P}(\mathcal{C}^{\{\emptyset,1\}})$. We start with a lemma that uses the projection theorem (Theorem A.2) to characterize the law $\tilde{\nu}^{\emptyset,1}$ as the weak solution to an SDE.

Lemma 4.4 *The measure $\tilde{\mathbb{P}}$ specified in (4.17) defines a probability measure on $(\Omega, \mathcal{F}, \mathbb{F})$. Moreover, by extending the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \tilde{\mathbb{P}})$ if necessary, we may find independent d -dimensional standard $\tilde{\mathbb{P}}$ -Brownian motions $(\tilde{W}_v)_{v \in \{\emptyset,1\}}$ such that (Y_\emptyset, Y_1) satisfies the following SDE system:*

$$dY_\emptyset(t) = \gamma_t(Y_\emptyset, Y_1)dt + \sigma(t, Y_\emptyset)d\tilde{W}_\emptyset(t), \quad (4.19)$$

$$dY_1(t) = 1_{\{1 \in \mathcal{T}_1\}}(\gamma_t(Y_1, Y_\emptyset)dt + \sigma(t, Y_1)d\tilde{W}_1(t)), \quad (4.20)$$

where $\gamma_t : \mathbb{R}_+ \times \mathcal{C}^2 \mapsto \mathbb{R}^d$ is the progressively measurable mapping defined in (3.7).

Proof To see that (4.17) indeed defines a probability measure, note that $\mathbb{P}(N_\emptyset(\mathcal{T}) = \emptyset) = \rho(0)$ and

$$\begin{aligned} \mathbb{E} \mathbb{P} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} 1_{\{N_\emptyset(\mathcal{T}_1) \neq \emptyset\}} \right] &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{k}{j+1} \rho(k) \widehat{\rho}(j) \\ &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{k}{j+1} \rho(k) \frac{(j+1)\rho(j+1)}{\sum_{i=1}^{\infty} i\rho(i)} = 1 - \rho(0). \end{aligned}$$

We stress that \mathcal{T} is a UGW(ρ) tree under \mathbb{P} but not under $\widetilde{\mathbb{P}}$, although both measures give rise to the same conditional law of the particles $\widehat{X}_{\mathbb{V}}$ given the tree \mathcal{T} .

We now turn to the proof of the second assertion of the lemma. Observe first that $(Y_v)_{v \in \mathbb{V}_1}$ solves the SDE system (3.8), where γ_t is defined as in (3.7). Note that the change of measure from \mathbb{P} to $\widetilde{\mathbb{P}}$ alters the law of the tree \mathcal{T} but not the Brownian motions or initial states. We can then apply Theorem A.2 to construct, by again extending the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, d -dimensional independent \mathbb{F} -Brownian motions $(\widetilde{W}_v)_{v \in \{\emptyset, 1\}}$ such that (Y_\emptyset, Y_1) satisfy the following SDE system:

$$\begin{aligned} dY_\emptyset(t) &= \widetilde{b}_\emptyset(t, Y_\emptyset, Y_1)dt + \widetilde{\sigma}_\emptyset(t, Y_\emptyset, Y_1)d\widetilde{W}_\emptyset(t), \\ dY_1(t) &= \widetilde{b}_1(t, Y_\emptyset, Y_1)dt + \widetilde{\sigma}_1(t, Y_\emptyset, Y_1)d\widetilde{W}_1(t), \end{aligned}$$

where $\widetilde{b}_v : \mathbb{R}_+ \times \mathcal{C}^2 \rightarrow \mathbb{R}^d$ and $\widetilde{\sigma}_v : \mathbb{R}_+ \times \mathcal{C}^2 \rightarrow \mathbb{R}^{d \times d}$ are any progressively measurable functions satisfying

$$\begin{aligned} \widetilde{b}_\emptyset(t, Y_\emptyset, Y_1) &= \mathbb{E}^{\widetilde{\mathbb{P}}} \left[b(t, Y_\emptyset, Y_{N_\emptyset(\mathcal{T}_1)}) \mid Y_\emptyset[t], Y_1[t] \right], \\ \widetilde{b}_1(t, Y_\emptyset, Y_1) &= \mathbb{E}^{\widetilde{\mathbb{P}}} \left[1_{\{1 \in \mathcal{T}_1\}} \gamma_t(Y_1, Y_\emptyset) \mid Y_\emptyset[t], Y_1[t] \right], \\ \widetilde{\sigma}_\emptyset \widetilde{\sigma}_\emptyset^\top(t, Y_\emptyset, Y_1) &= \mathbb{E}^{\widetilde{\mathbb{P}}} \left[\sigma \sigma^\top(t, Y_\emptyset) \mid Y_\emptyset[t], Y_1[t] \right], \\ \widetilde{\sigma}_1 \widetilde{\sigma}_1^\top(t, Y_\emptyset, Y_1) &= \mathbb{E}^{\widetilde{\mathbb{P}}} \left[1_{\{1 \in \mathcal{T}_1\}} \sigma \sigma^\top(t, Y_1) \mid Y_\emptyset[t], Y_1[t] \right]. \end{aligned}$$

Note again that progressively measurable versions exist by Lemma A.1.

Now, by Remark 4.2 and in particular (4.8), $\{1 \in \mathcal{T}_1\}$ is $Y_1[t]$ -measurable for each $t > 0$. Together with the progressive measurability of $(t, x, x') \mapsto \gamma_t(x, x')$, this shows that

$$\begin{aligned} \widetilde{\sigma}_\emptyset(t, Y_\emptyset, Y_1) &= \sigma(t, Y_\emptyset), \quad \widetilde{\sigma}_1(t, Y_\emptyset, Y_1) = 1_{\{1 \in \mathcal{T}_1\}} \sigma(t, Y_1), \\ \widetilde{b}_1(t, Y_\emptyset, Y_1) &= 1_{\{1 \in \mathcal{T}_1\}} \gamma_t(Y_1, Y_\emptyset). \end{aligned}$$

On the other hand, in terms of the Radon-Nikodym derivative $d\widetilde{\mathbb{P}}/d\mathbb{P}$ we can rewrite

$$\widetilde{b}_\emptyset(t, Y_\emptyset, Y_1) = \mathbb{E}^{\mathbb{P}} \left[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} b(t, Y_\emptyset, Y_{N_\emptyset(\mathcal{T}_1)}) \mid Y_\emptyset[t], Y_1[t] \right] / \mathbb{E}^{\mathbb{P}} \left[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \mid Y_\emptyset[t], Y_1[t] \right].$$

On the $Y_1[t]$ -measurable event $\{N_\emptyset(\mathcal{T}_1) = \emptyset\}$, we have $\tilde{b}_\emptyset(t, Y_\emptyset, Y_1) = b(t, Y_\emptyset, \circ)$, where we recall the convention that \circ denotes the unique element of the one-point space \mathcal{C}^0 . On the other hand, recalling the definitions of $d\tilde{\mathbb{P}}/d\mathbb{P}$ and γ_t from (4.17) and (3.7), respectively, on the complementary event $\{N_\emptyset(\mathcal{T}_1) \neq \emptyset\}$ we have

$$\begin{aligned}\tilde{b}_\emptyset(t, Y_\emptyset, Y_1) &= \mathbb{E}^\mathbb{P} \left[\frac{|N_\emptyset(\mathcal{T}_1)|}{|N_1(\mathcal{T}_1)|} b(t, Y_\emptyset, Y_{N_\emptyset(\mathcal{T}_1)}) \mid Y_\emptyset[t], Y_1[t] \right] / \mathbb{E}^\mathbb{P} \left[\frac{|N_\emptyset(\mathcal{T}_1)|}{|N_1(\mathcal{T}_1)|} \mid Y_\emptyset[t], Y_1[t] \right] \\ &= \gamma_t(Y_\emptyset, Y_1).\end{aligned}$$

Thus, in either case, $\tilde{b}_\emptyset(t, Y_\emptyset, Y_1) = \gamma_t(Y_\emptyset, Y_1)$, and in fact this identity is precisely the purpose of the change of measure $\tilde{\mathbb{P}}$. This concludes the proof. \square

Step 2 We now express (in Lemmas 4.5, 4.6 below) the (conditional) density $d\tilde{v}_t/d\tilde{P}_t^{*,1}$ explicitly in terms of certain local martingales that we now define. We recall the canonical space Ω^n and canonical processes $X = (X_v)_{v \in \mathbb{V}_1}$ introduced in Sect. 4.2.1 and define the processes M_v^n , R_v , and R_\emptyset on Ω^n as follows:

$$\begin{aligned}M_v^n &:= \int_0^\cdot (\sigma \sigma^\top)^{-1}(s, X_v) b(s, X_v, X_{N_v(\mathcal{T}_n)}) \cdot dX_v(s), \quad n \in \mathbb{N}, v \in \mathbb{V}_{n-1}, \\ R_v &:= \int_0^\cdot (\sigma \sigma^\top)^{-1}(s, X_v) \gamma_s(X_v, X_{\pi_v}) \cdot dX_v(s), \quad v \in \mathbb{V} \setminus \{\emptyset\}, \\ R_\emptyset &:= \int_0^\cdot (\sigma \sigma^\top)^{-1}(s, X_\emptyset) \gamma_s(X_\emptyset, X_1) \cdot dX_\emptyset(s),\end{aligned}\tag{4.21}$$

where we have omitted the arguments from M_v^n , R_v , and R_\emptyset for notational conciseness. It will be important later to take note of the following consistency property of M_v^n when we stay away from the leaves of \mathbb{V}_n :

$$M_v^n((x_u)_{u \in \mathbb{V}_n}) = M_v^{n+1}((x_u)_{u \in \mathbb{V}_{n+1}}), \quad \text{for } v \in \mathbb{V}_{n-1}, (x_u)_{u \in \mathbb{V}_{n+1}} \in \mathcal{C}^{\mathbb{V}_{n+1}}. \tag{4.22}$$

Recall the Doleans exponential \mathcal{E}_t defined in (4.1).

Lemma 4.5 *For each $t > 0$, we have*

$$\frac{d\tilde{v}_t}{d\tilde{P}_t^{*,1}} = \frac{dv_t}{dP_t^{*,1}} = \mathcal{E}_t(M_\emptyset^1) \prod_{v \in \mathcal{T}_1 \setminus \{\emptyset\}} \mathcal{E}_t(R_v). \tag{4.23}$$

Proof The continuity of b and the processes \widehat{X}_v and Y_v for each v ensures that the following integrals are trivially a.s. finite:

$$\int_0^T |b(t, \widehat{X}_\emptyset, \widehat{X}_{N_\emptyset(\mathcal{T}_1)})|^2 dt, \quad \int_0^T |b(t, Y_\emptyset, Y_{N_\emptyset(\mathcal{T}_1)})|^2 dt.$$

We know also from condition (10) of Definition 3.9 that the following integrals are a.s. finite:

$$\int_0^T |\gamma_t(Y_k, Y_\emptyset)|^2 dt, \quad \int_0^T |\gamma_t(\widehat{X}_k, \widehat{X}_\emptyset)|^2 dt.$$

Recalling the form of the SDE systems for $Y = (Y_v)_{v \in \mathbb{V}_1}$ and $(\widehat{X}_v)_{v \in \mathbb{V}_1}$ in (3.8) and (4.18), respectively, and the definitions of ν_t , $\widetilde{\nu}_t$ and $P_t^{*,1}$, $\widetilde{P}_t^{*,1}$ as the laws of $(Y_v)_{v \in \mathbb{V}_1}$ and $(\widehat{X}_v)_{v \in \mathbb{V}_1}$ under \mathbb{P} and $\widetilde{\mathbb{P}}$, respectively, these facts justify an application of Girsanov's theorem in the form of Lemma B.1. By expanding the expression analogous to (B.3) in the above setting, we see that the Radon-Nikodym derivative of ν_t with respect to $P_t^{*,1}$ takes the form announced in the second equality in (4.23). The same logic (noting that $\widetilde{\mathbb{P}}$ and \mathbb{P} are mutually absolutely continuous) also yields the same form for $d\widetilde{\nu}_t/d\widetilde{P}_t^{*,1}$, thus justifying the first equality in (4.23). \square

Our next goal is to calculate the following conditional density process for each $t > 0$:

$$Z_t((\tilde{x}_k)_{k \in \mathbb{N}}; x_\emptyset, x_1) := \frac{d\widetilde{\nu}((X_{1+k}[t])_{k \in \mathbb{N}} \in \cdot \mid X_1[t] = x_1[t], X_\emptyset[t] = x_\emptyset[t])}{d\widetilde{P}^{*,1}((X_{1+k}[t])_{k \in \mathbb{N}} \in \cdot \mid X_1[t] = x_1[t], X_\emptyset[t] = x_\emptyset[t])}((\tilde{x}_k[t])_{k \in \mathbb{N}}), \quad (4.24)$$

for $(x_\emptyset, x_1) \in \mathcal{C}^{\emptyset,1}$ and $(\tilde{x}_k)_{k \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$. Recall the definition of $\widetilde{P}_t^{*,\emptyset,1}$ just prior to Lemma 4.4 as the marginal of $\widetilde{P}_t^{*,1}$ on $\mathcal{C}_t^{\{\emptyset,1\}}$. Since $Z_t(\cdot; X_\emptyset, X_1)$ is a well-defined conditional density by Lemma 4.5, for $\widetilde{P}_t^{*,\emptyset,1}$ -a.e. $(x_\emptyset, x_1) \in \mathcal{C}_t^2$ we have

$$\begin{aligned} 1 &= \mathbb{E}^{\widetilde{P}_t^{*,1}} [Z_t((X_{1+k})_{k \in \mathbb{N}}; X_\emptyset, X_1) \mid X_\emptyset[t] = x_\emptyset, X_1[t] = x_1] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}} [Z_t((\widehat{X}_{1+k})_{k \in \mathbb{N}}; \widehat{X}_\emptyset, \widehat{X}_1) \mid \widehat{X}_\emptyset[t] = x_\emptyset, \widehat{X}_1[t] = x_1]. \end{aligned} \quad (4.25)$$

In particular, on the $\widehat{X}_1[t]$ -measurable event $\{1 \notin \mathcal{T}\}$, note that $Z_t((\widehat{X}_{1+k})_{k \in \mathbb{N}}; \widehat{X}_\emptyset, \widehat{X}_1)$ is $(\widehat{X}_\emptyset[t], \widehat{X}_1[t])$ -measurable and must therefore equal 1.

Lemma 4.6 *For each $t > 0$, we have*

$$Z_t((X_{1+k})_{k \in \mathbb{N}}; X_\emptyset, X_1) = \frac{\mathcal{E}_t(M_\emptyset^1)}{\mathcal{E}_t(R_\emptyset)} \prod_{v \in N_\emptyset(T) \setminus \{1\}} \mathcal{E}_t(R_v), \quad \widetilde{P}^{*,1} - a.s. \quad (4.26)$$

Moreover, for each $n \in \mathbb{N}$ and $v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}$, we have a.s.

$$1 = \mathbb{E}^{\mathbb{P}} [Z_t(\widehat{X}_{C_v(T)}; \widehat{X}_v, \widehat{X}_{\pi_v}) \mid \widehat{X}_v[t], \widehat{X}_{\pi_v}[t]] = \mathbb{E}^{\mathbb{P}} [Z_t(\widehat{X}_{C_v(T)}; \widehat{X}_v, \widehat{X}_{\pi_v}) \mid \widehat{X}_{\mathbb{V}_n}[t]], \quad (4.27)$$

where we write $C_v(T) := N_v(T) \setminus \{\pi_v\}$ for the children of the vertex v .

Proof We first compute the density $d\tilde{\nu}_t^{\emptyset,1}/d\tilde{P}_t^{*,\emptyset,1}$. By Lemma 4.4, $\tilde{\nu}^{\emptyset,1}$ is the law of the solution (Y_\emptyset, Y_1) to the SDE system defined by (4.19) and (4.20). Hence, condition (10) of Definition 3.9 justifies an application of Girsanov's theorem, in the form of Lemma B.1, which yields

$$\frac{d\tilde{\nu}_t^{\emptyset,1}}{d\tilde{P}_t^{*,\emptyset,1}}(X_\emptyset, X_1) = \begin{cases} \mathcal{E}_t(R_\emptyset)\mathcal{E}_t(R_1) & \text{if } 1 \in \mathcal{T}_1, \\ \mathcal{E}_t(R_\emptyset) & \text{if } 1 \notin \mathcal{T}_1. \end{cases} \quad (4.28)$$

Moreover, using Bayes' rule we obtain

$$Z_t((X_{1+k})_{k \in \mathbb{N}}; X_\emptyset, X_1) = \frac{d\tilde{\nu}_t}{d\tilde{P}_t^{*,1}}(X_{\mathbb{V}_1}) \Big/ \frac{d\tilde{\nu}_t^{\emptyset,1}}{d\tilde{P}_t^{*,\emptyset,1}}(X_\emptyset, X_1).$$

Appealing to (4.28) and (4.23), we then obtain (4.26). Alternatively, recalling the definitions of the martingales R_v and M_\emptyset^1 , shows that Z_t is really a function of $(\langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, X_\emptyset[t], X_1[t])$; that is, the dependence on the coordinates $(X_{1+k}[t])_{k \in \mathbb{N}}$ is only through the equivalence class $\langle X_{N_\emptyset(\mathcal{T})}[t] \rangle$ (which is a random element of $S^\sqcup(\mathcal{C}_t)$). Thus, we can write

$$Z_t((X_{1+k})_{k \in \mathbb{N}}; X_\emptyset, X_1) = \widehat{Z}_t(\langle X_{N_\emptyset(\mathcal{T})} \rangle; X_\emptyset, X_1), \quad (4.29)$$

where $\widehat{Z}_t : S^\sqcup(\mathcal{C}) \times \mathcal{C}^2 \mapsto \mathbb{R}_+$ is defined by

$$\widehat{Z}_t(\langle X_{N_\emptyset(\mathcal{T})} \rangle; X_\emptyset, X_1) := \frac{\mathcal{E}_t(M_\emptyset^1)}{\mathcal{E}_t(R_\emptyset)\mathcal{E}_t(R_1)} \prod_{v \in N_\emptyset(\mathcal{T})} \mathcal{E}_t(R_v).$$

For the proof of the second (and last) assertion of the lemma, we take advantage of some symmetries of the driftless particle system $\widehat{X}_{\mathbb{V}}$ defined in (4.18). First note that, by inspecting (4.18), and recalling the conditional independence properties of the UGW tree \mathcal{T} itself, it is clear that $\widehat{X}_{C_v(\mathcal{T})}$ is conditionally independent of $\widehat{X}_{\mathbb{V}_n}$ given $\{v \in \mathcal{T}\}$ under \mathbb{P} , for each $n \in \mathbb{N}$ and $v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}$. This immediately implies the second identity in (4.27). Second, we claim that in order to prove the first identity in (4.27) it suffices to prove it only for the case $v = 1$. This is because each non-root vertex in the UGW(ρ) tree \mathcal{T} has the same offspring distribution $\widehat{\rho}$ under \mathbb{P} , and thus the conditional law of $\widehat{X}_{C_v(\mathcal{T})}$ given $\{v \in \mathcal{T}\}$ does not depend on the choice of $v \in \mathbb{V} \setminus \{\emptyset\}$.

To prove the first identity in (4.27) for the case $v = 1$, first recall that, as noted just after (4.25), on the event $\{1 \notin \mathcal{T}\}$ it holds that $Z_t(\widehat{X}_{C_1(\mathcal{T})}; \widehat{X}_1, \widehat{X}_\emptyset) = 1$. Hence, we focus on the complementary event. Recall the notation of (4.29), which gives

$$\mathbb{E}^{\mathbb{P}} [Z_t(\widehat{X}_{C_1(\mathcal{T})}; \widehat{X}_1, \widehat{X}_\emptyset) \mid \widehat{X}_1[t], \widehat{X}_\emptyset[t]] = \mathbb{E}^{\mathbb{P}} [\widehat{Z}_t(\langle \widehat{X}_{N_1(\mathcal{T})} \rangle; \widehat{X}_1, \widehat{X}_\emptyset) \mid \widehat{X}_1[t], \widehat{X}_\emptyset[t]].$$

We are now in a position to apply Proposition 3.18. Indeed, Proposition 3.18 applies not just to the original SDE system $X_{\mathbb{V}}$ of (3.1) but also to the system $\widehat{X}_{\mathbb{V}}$ defined

in (4.18), simply because the latter is the special case of the former corresponding to $b \equiv 0$. We deduce that, on the event $\{1 \in \mathcal{T}\}$, we have

$$\mathbb{E}^{\mathbb{P}} \left[\widehat{Z}_t((\widehat{X}_{N_1(\mathcal{T})}); \widehat{X}_1, \widehat{X}_\emptyset) \mid \widehat{X}_1[t], \widehat{X}_\emptyset[t] \right] = \Xi_t(\widehat{X}_1, \widehat{X}_\emptyset),$$

where we define $\Xi_t : \mathcal{C}_t^2 \rightarrow \mathbb{R}$ by

$$\Xi_t(\widehat{X}_\emptyset, \widehat{X}_1) := 1_{\{1 \in \mathcal{T}\}} \frac{\mathbb{E}^{\mathbb{P}} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} \widehat{Z}_t((\widehat{X}_{N_1(\mathcal{T})}); \widehat{X}_\emptyset, \widehat{X}_1) \mid \widehat{X}_\emptyset[t], \widehat{X}_1[t] \right]}{\mathbb{E}^{\mathbb{P}} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} \mid \widehat{X}_\emptyset[t], \widehat{X}_1[t] \right]}.$$

Recalling from (4.17) that $d\widetilde{\mathbb{P}}/d\mathbb{P} = |N_\emptyset(\mathcal{T})|/|N_1(\mathcal{T})|$ on $\{1 \in \mathcal{T}\}$, it follows from Bayes' rule that

$$\Xi_t(\widehat{X}_\emptyset, \widehat{X}_1) = \mathbb{E}^{\widetilde{\mathbb{P}}} \left[\widehat{Z}_t((\widehat{X}_{N_1(\mathcal{T})}); \widehat{X}_\emptyset, \widehat{X}_1) \mid \widehat{X}_\emptyset[t], \widehat{X}_1[t] \right], \quad \text{on } \{1 \in \mathcal{T}\}.$$

Reverting back from the \widehat{Z} to Z notation as in (4.29), this can be rewritten as

$$\Xi_t(\widehat{X}_\emptyset, \widehat{X}_1) = \mathbb{E}^{\widetilde{\mathbb{P}}} \left[Z_t((\widehat{X}_{1+k})_{k \in \mathbb{N}}; \widehat{X}_\emptyset, \widehat{X}_1) \mid \widehat{X}_\emptyset[t], \widehat{X}_1[t] \right], \quad \text{on } \{1 \in \mathcal{T}\}.$$

It follows from (4.25) that $\Xi_t(\widehat{X}_\emptyset, \widehat{X}_1) = 1$ on $\{1 \in \mathcal{T}\}$, which completes the proof of (4.27). \square

Step 3 We finally present the main construction of the argument, which involves establishing a one-to-one correspondence between solutions of the local equation and solutions of the infinite SDE system (3.1) via a recursive construction and an extension. Recall the definition of the law $P^{*,n} \in \mathcal{P}(\mathcal{C}^n)$ of the driftless process introduced in Sect. 4.2.1, and as usual, let $P_t^{*,n}$ denote its projection onto $\mathcal{P}(\mathcal{C}_t^n)$. For each $t > 0$ and $n \geq 1$, define a probability measure $Q_t^n \in \mathcal{P}(\mathcal{C}_t^{\mathbb{V}_n})$ via the density

$$\frac{dQ_t^n}{dP_t^{*,n}}((x_v)_{v \in \mathbb{V}_n}) = \frac{dv_t}{dP_t^{*,1}}((x_v)_{v \in \mathbb{V}_1}) \prod_{v \in \mathcal{T}_{n-1} \setminus \{\emptyset\}} Z_t((x_{vk})_{k \in \mathbb{N}}; x_v, x_{\pi_v}), \quad (4.30)$$

with Z_t as defined in (4.24). We now establish the following.

Proposition 4.7 *We have $Q_t^1 = v_t$ for each $t > 0$. Moreover, $\{Q_t^n : t > 0, n \in \mathbb{N}\}$ is a well defined and consistent family of probability measures in the sense that for $t > s \geq 0$ and $n \geq k$ the projection of Q_t^n from $\mathcal{C}_t^{\mathbb{V}_n}$ to $\mathcal{C}_s^{\mathbb{V}_k}$ is precisely Q_s^k . Furthermore, the unique extension $Q \in \mathcal{P}(\mathcal{C}^{\mathbb{V}})$ of $\{Q^n\}$ to $\mathcal{P}(\mathcal{C}^{\mathbb{V}})$ coincides with the (unique) law of a weak solution of the SDE system (3.1) with \mathcal{T} given as a UGW(ρ) tree.*

Proof Note that $Q_t^1 = v_t$ for $t > 0$ follows immediately from the definition (4.30). For the next assertion, note that (as justified below) for each $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{E}^{P^{*,n+1}} \left[\frac{dQ_t^{n+1}}{dP_t^{*,n+1}} (X_{\mathbb{V}_{n+1}}[t]) \mid X_{\mathbb{V}_n}[t] \right] \\ &= \mathbb{E}^{P^{*,n+1}} \left[\frac{dQ_t^n}{dP_t^{*,n}} (X_{\mathbb{V}_n}[t]) \prod_{v \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}} Z_t(X_{C_v}(\mathcal{T}); X_v, X_{\pi_v}) \mid X_{\mathbb{V}_n}[t] \right] \\ &= \frac{dQ_t^n}{dP_t^{*,n}} (X_{\mathbb{V}_n}[t]) \prod_{v \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}} \mathbb{E}^{P^{*,n+1}} \left[Z_t(X_{C_v}(\mathcal{T}); X_v, X_{\pi_v}) \mid X_{\mathbb{V}_n}[t] \right] \\ &= \frac{dQ_t^n}{dP_t^{*,n}} (X_{\mathbb{V}_n}[t]). \end{aligned}$$

Indeed, the last line uses the relation (4.27), and the penultimate line uses the fact that for $n \in \mathbb{N}$, $(\widehat{X}_{C_v}(\mathcal{T}))_{v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}}$ are conditionally independent given $\widehat{X}_{\mathbb{V}_n}$, which follows from the conditional independence structure of the tree itself; $(C_v)_{v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}}$ are conditionally independent of each other given $(1_{\{v \in \mathcal{T}\}})_{v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}}$. Iterating this, we find for each $t > 0$ and $n \geq k$ with $n, k \in \mathbb{N}$ that

$$\mathbb{E}^{P^{*,n}} \left[\frac{dQ_t^n}{dP_t^{*,n}} (X_{\mathbb{V}_n}[t]) \mid X_{\mathbb{V}_k}[t] \right] = \frac{dQ_t^k}{dP_t^{*,k}} (X_{\mathbb{V}_k}[t]), \quad a.s. \quad (4.31)$$

In particular,

$$\mathbb{E}^{P^{*,n}} \left[\frac{dQ_t^n}{dP_t^{*,n}} (X_{\mathbb{V}_n}[t]) \right] = \mathbb{E}^{P^{*,1}} \left[\frac{dQ_t^1}{dP_t^{*,1}} (X_{\mathbb{V}_1}[t]) \right] = 1 \quad (4.32)$$

and Q_t^n is a well-defined probability measure.

Next, we rewrite the Radon-Nikodym derivative (4.30) in a more useful form. Recalling the definitions of the martingales M_v^n and R_v given in (4.21), the consistency equations (4.22) and the relation (4.26), it is straightforward to check that for each $v \in \mathcal{T}_{n-1} \setminus \{\emptyset\}$,

$$Z_t((X_{vk})_{k \in \mathbb{N}}; X_v, X_{\pi_v}) = \frac{\mathcal{E}_t(M_v^n)}{\mathcal{E}_t(R_v)} \prod_{u \in N_v(\mathcal{T}) \setminus \{\pi_v\}} \mathcal{E}_t(R_u) = \frac{\mathcal{E}_t(M_v^n)}{\mathcal{E}_t(R_v)} \prod_{u \in C_v(\mathcal{T})} \mathcal{E}_t(R_u),$$

where we again abbreviate $C_v(T) = N_v(T) \setminus \{\pi_v\}$. Combining this relation with (4.30) and the form of $d\nu_t/dP_t^{*,1}$ given in (4.23), we obtain

$$\begin{aligned} \frac{dQ_t^n}{dP_t^{*,n}} &= \mathcal{E}_t(M_\emptyset^n) \prod_{v \in \mathcal{T}_1 \setminus \{\emptyset\}} \mathcal{E}_t(R_v) \prod_{v \in \mathcal{T}_{n-1} \setminus \{\emptyset\}} Z_t((X_{vk})_{k \in \mathbb{N}}; X_v, X_{\pi_v}) \\ &= \mathcal{E}_t(M_\emptyset^n) \prod_{v \in \mathcal{T}_1 \setminus \{\emptyset\}} \mathcal{E}_t(R_v) \prod_{v \in \mathcal{T}_{n-1} \setminus \{\emptyset\}} \left(\frac{\mathcal{E}_t(M_v^n)}{\mathcal{E}_t(R_v)} \prod_{u \in C_v(T)} \mathcal{E}_t(R_u) \right). \end{aligned}$$

For each $v \in \mathcal{T}_{n-1} \setminus \{\emptyset\}$, the factor $\mathcal{E}_t(R_v)$ appears exactly once in the numerator and once in the denominator. Hence, the above reduces to

$$\frac{dQ_t^n}{dP_t^{*,n}} = \prod_{v \in \mathcal{T}_{n-1}} \mathcal{E}_t(M_v^n) \prod_{v \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}} \mathcal{E}_t(R_v) = \mathcal{E}_t \left(\sum_{v \in \mathcal{T}_{n-1}} M_v^n + \sum_{v \in \mathcal{T}_n \setminus \mathcal{T}_{n-1}} R_v \right), \quad (4.33)$$

where the second equality follows from the fact that the local martingales $\{M_v^n : v \in \mathbb{V}_{n-1}\} \cup \{R_v : v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}\}$ are orthogonal. Combining this with (4.32) gives the martingale property

$$\mathbb{E}^{P^{*,n}} \left[\frac{dQ_t^n}{dP_t^{*,n}} (X_{\mathbb{V}_n}[t]) \mid X_{\mathbb{V}_n}[s] \right] = \frac{dQ_s^n}{dP_s^{*,n}} (X_{\mathbb{V}_n}[s]), \quad a.s., \quad (4.34)$$

for $t > s > 0$.

Together, Eqs. (4.31) and (4.34) prove the stated consistency property of the family $\{Q^n\}$. Due to the Daniell-Kolmogorov theorem, we deduce from this that there is a unique $Q \in \mathcal{P}(\mathcal{C}^{\mathbb{V}})$ whose restriction to $\mathcal{C}_t^{\mathbb{V}_n}$ is Q_t^n for each $n \in \mathbb{N}$ and $t > 0$.

We now turn to the proof of the last statement of the proposition, which asserts that Q is the unique law of a weak solution to the SDE system (3.1). To this end, for each $n \geq 1$ and $t > 0$, we identify Q_t^n as the law of an SDE solution as follows. Recalling the definition $P^{*,n} = \mathbb{P} \circ \widehat{X}_{\mathbb{V}_n}^{-1}$, where \widehat{X}_v satisfies (4.18), and the definitions of M_v^n and R_v , we deduce from (4.32), (4.33), and Girsanov's theorem that Q^n is precisely the law of a weak solution $(Y_v)_{v \in \mathbb{V}_n}$ of the SDE system

$$\begin{aligned} dY_v(t) &= 1_{\{v \in \widetilde{\mathcal{T}}\}} \left(b(t, Y_v, Y_{N_v}(\widetilde{\mathcal{T}})) dt + \sigma(t, Y_v) dB_v(t) \right), \quad v \in \mathbb{V}_{n-1} \\ dY_v(t) &= 1_{\{v \in \widetilde{\mathcal{T}}\}} \left(\gamma_t(Y_v, Y_{\pi_v}) dt + \sigma(t, Y_v) dB_v(t) \right), \quad v \in \mathbb{V}_n \setminus \mathbb{V}_{n-1}, \end{aligned} \quad (4.35)$$

where $(B_v)_{v \in \mathbb{V}_n}$ are independent Brownian motions, $(Y_v(0))_{v \in \mathbb{V}_n}$ are i.i.d. with law λ_0 , and $\widetilde{\mathcal{T}}$ is an independent UGW(ρ) tree.

Now, define $\widehat{Q}^n \in \mathcal{P}(\mathcal{C}^{\mathbb{V}})$ so that the projection onto $\mathcal{C}^{\mathbb{V}_n}$ is precisely Q^n and the coordinates on $\mathbb{V} \setminus \mathbb{V}_n$ are (arbitrarily) chosen to be identically zero, with probability 1. It is immediate that \widehat{Q}^n converges weakly to Q , due to the consistency property of $\{Q_t^n : t \geq 0, n \in \mathbb{N}\}$ established above. On the other hand, we argue that if $\{\widehat{Q}^n\}$

converges to some limit, then this limit must be the law of a weak solution of the infinite SDE system (3.1). Indeed, if $\{\widehat{Q}^n\}$ converges to $Q = \mathcal{L}((\widehat{Y}_v)_{v \in \mathbb{V}})$, then we may pass to the limit in (4.35) (using again the weak continuity of stochastic integrals provided by Kurtz and Protter [25, Theorem 2.2] and the continuity of b and σ in Assumption A) to find that for each n , the \mathbb{V}_{n-1} -coordinates $(\widehat{Y}_v)_{v \in \mathbb{V}_{n-1}}$ satisfy the same SDE system as in (4.35). As this holds for each fixed n , we conclude that $(\widehat{Y}_v)_{v \in \mathbb{V}_{n-1}}$ satisfies the infinite SDE system (3.1). In light of the uniqueness in law of solutions of (3.1) (see Assumption (A.1) and Remark 3.1), we conclude that $Q = \mathcal{L}((X_v)_{v \in \mathbb{V}})$, where $(X_v)_{v \in \mathbb{V}}$ was the unique in law solution of (3.1) on the UGW(ρ) tree \mathcal{T} . This completes the proof, as we know from the beginning of the proof that the \mathbb{V}_1 -marginal of Q is precisely $Q^1 = \nu$. \square

4.2.3 Completing the proof of uniqueness in Theorem 3.12

The lemmas of the previous section contain the proof of the uniqueness assertion in Theorem 3.12. Indeed, we began in Sect. 4.2.1 with an arbitrary weak solution $((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}), \mathcal{T}_1, \gamma, (B_v, Y_v)_{v \in \mathbb{V}_1})$ to the UGW(ρ) local equation with initial law λ_0 . In Proposition 4.7, recalling the notation $\nu = \mathcal{L}((Y_v)_{v \in \mathbb{V}_1})$, we deduced that necessarily $\nu = \mathcal{L}((X_v)_{v \in \mathbb{V}_1})$, where $(X_v)_{v \in \mathbb{V}}$ solves the SDE system (3.1). We know from Assumption (A.1) (and Remark 3.1) that the SDE system (3.1) is unique in law. Hence, the law of $(Y_v)_{v \in \mathbb{V}_1}$ does not depend on the choice of weak solution to the UGW(ρ) local equation.

5 Second-order Markov random fields

The rest of the paper is devoted to justifying the two key Propositions 3.17 and 3.18. We begin by summarizing some general properties of Markov random fields (MRFs) which will play a key role in the former proposition. Throughout this section, we work with a fixed Polish space \mathcal{X} and a fixed (non-random) graph $G = (V, E)$, assumed to have finite or countable vertex set. We assume that G is simple (no self-loops or multi-edges), but it need not be locally finite (so that we may use $G = \mathbb{V}$). We fix a reference measure $\lambda \in \mathcal{P}(\mathcal{X})$. The goal of this section is to summarize how conditional independence properties of a measure $\mu \in \mathcal{P}(\mathcal{X}^V)$ can be deduced from factorization properties of its density with respect to the product measure λ^V .

We recall the basic graph-theoretic definitions given in Sect. 2.1.1, in particular the notion of boundary and double boundary of a set A of vertices in a graph $G = (V, E)$ defined in (2.1). In what follows, for any random elements $Y_i, i = 1, 2, 3$, we write $Y_1 \perp\!\!\!\perp Y_2 \mid Y_3$ to denote that Y_1 is conditionally independent of Y_2 given Y_3 .

Definition 5.1 (*Second-order MRF*) A collection of \mathcal{X} -valued random elements $(Y_v)_{v \in G}$ is said to form a (global) second-order MRF with respect to G if for any sets $A \subset V$, $B \subset V \setminus (A \cup \partial^2 A)$, we have the following conditional independence structure:

$$Y_A \perp\!\!\!\perp Y_B \mid Y_{\partial^2 A}.$$

Note that a first-order MRF (with respect to G), sometimes also referred to as a Gibbs measure, would require the same to hold but with ∂A in place of $\partial^2 A$.

We state here a variant of a well known theorem, which can be found in various forms in [16, Theorem 2.30] and [31, Proposition 3.8, Theorem 3.9], for first-order MRFs on finite graphs. We do not state the more difficult converse, often attributed to Hammersley-Clifford, as we will not need it. Recall that a 2-clique of a graph is a set of vertices of diameter at most 2.

Theorem 5.2 *Assume the graph G is finite. Assume $\mu \in \mathcal{P}(\mathcal{X}^V)$ is absolutely continuous with respect to λ^V . Suppose there exists a set \mathcal{K} of 2-cliques of G such that the density of μ with respect to λ^V factorizes in the form*

$$\frac{d\mu}{d\lambda^V}(x_V) = \prod_{K \in \mathcal{K}} f_K(x_K), \quad (5.1)$$

for some measurable functions $f_K : \mathcal{X}^K \rightarrow \mathbb{R}_+$, for $K \in \mathcal{K}$. Then μ is a second-order MRF.

Proof Let $A \subset V$, and let $\varphi : \mathcal{X}^{\partial^2 A} \rightarrow \mathbb{R}_+$ denote the marginal density of $X_{\partial^2 A}$. Let $B = (A \cup \partial^2 A)^c$. Then the conditional density of (X_A, X_B) given $X_{\partial^2 A}$ is precisely

$$\frac{1}{\varphi(x_{\partial^2 A})} \prod_{K \in \mathcal{K}} f_K(x_K).$$

No 2-clique of G that intersects A can also intersect B , and vice versa, because any pair of vertices $u \in A$ and $v \in B$ have distance at least 3. Thus, for $x_{\partial^2 A}$ frozen, the above conditional density as a function of (x_A, x_B) factorizes into a function of x_A times a function of x_B . This implies X_A and X_B are conditionally independent given $X_{\partial^2 A}$. \square

The second-order MRF property is more intuitive, but the factorization property of Theorem 5.2 will be quite useful in our analysis. Hence, we give it a name:

Definition 5.3 We say that $\mu \in \mathcal{P}(\mathcal{X}^V)$ admits a 2-clique factorization with respect to λ^V if the density $d\mu/d\lambda^V$ exists and takes the form (5.1), for some set \mathcal{K} of 2-cliques of G .

It is clear that Theorem 5.2 admits a generalization to m -order MRFs, defined in the obvious way for $m \in \mathbb{N}$, where one must assume the density factorizes over m -cliques, but we have no use for such a generalization.

6 Proof of the conditional independence property

We now turn to the proof of the conditional independence property stated in Proposition 3.17, which played a crucial role in the proof of existence for Theorem 3.12. The strategy is to first establish the property on certain finite truncations of the tree,

and then use an approximation argument. Specifically, in Sect. 6.1 we first establish the desired conditional independence property on a truncation of the infinite tree \mathbb{V} to one of finite depth and width by explicitly identifying the joint density with respect to a product measure and then invoking Theorem 5.2. In Sect. 6.2 we then implement a rather delicate limiting argument to show that the conditional independence property is preserved when the infinite tree is approximated by trees of finite depth and width.

6.1 Truncated systems

We begin by studying the particle system set on the truncated (finite) tree $\mathcal{T}_n := \mathcal{T} \cap \mathbb{V}_{n,n}$, where \mathcal{T} is a UGW(ρ) tree, and

$$\mathbb{V}_{m,n} := \{\emptyset\} \cup \bigcup_{k=1}^m \{1, \dots, n\}^k, \quad \text{for } n, m \in \mathbb{N}.$$

That is, $\mathbb{V}_{m,n}$ is the set of labels of trees of height m with at most n offspring per generation. Let $(X_v^n)_{v \in \mathbb{V}} := (X_v^{T_n})_{v \in \mathbb{V}}$ be a solution to the SDE system

$$dX_v^n(t) = 1_{\{v \in \mathcal{T}_n\}} \left(b(t, X_v^n, X_{N_v(\mathcal{T}_n)}^n) dt + \sigma(t, X_v^n) dW_v(t) \right), \quad v \in \mathbb{V}, \quad (6.1)$$

where $(X_v^n(0))_{v \in \mathbb{V}}$ are i.i.d. with law λ_0 , and as usual the tree \mathcal{T} , the initial conditions $(X_v^n(0))_{v \in \mathbb{V}}$, and the driving Brownian motions $(W_v)_{v \in \mathbb{V}}$ are independent. Also, for $v \in \mathbb{V} \setminus \mathcal{T}_n$, note as usual that the particles are constant over time, with $X_v^n(t) = X_v^n(0)$ for all $t > 0$. Let $P^n \in \mathcal{P}(\{0, 1\} \times \mathcal{C})^{\mathbb{V}}$ denote the law of

$$\left(1_{\{v \in \mathcal{T}_n\}}, X_v^n \right)_{v \in \mathbb{V}}. \quad (6.2)$$

We will identify P^n by way of its Radon–Nikodym derivative with respect to a certain reference measure (in the process showing that the SDE (6.1) is unique in law). In this case, as a reference measure we use $\mathcal{W} \in \mathcal{P}(\{0, 1\} \times \mathcal{C})^{\mathbb{V}}$, defined as the law of $(\xi_v, \widehat{X}_v)_{v \in \mathbb{V}}$, where $(\xi_v)_{v \in \mathbb{V}}$ are independent Bernoulli(1/2) random variables, and where \widehat{X} solves the driftless SDE system

$$d\widehat{X}_v(t) = \xi_v \sigma(t, \widehat{X}_v) dB_v(t), \quad \widehat{X}_v(0) \sim \lambda_0, \quad v \in \mathbb{V}, \quad (6.3)$$

with $(B_v)_{v \in \mathbb{V}}$ as independent standard d -dimensional Brownian motions, and with $(B_v)_{v \in \mathbb{V}}, (\xi_v)_{v \in \mathbb{V}}$, and with $(\widehat{X}_v(0))_{v \in \mathbb{V}}$ independent. Note that the SDE (6.3) is well-posed due to Assumption (A.4). Note in particular that \mathcal{W} is an i.i.d. product measure.

To show that P^n of (6.2) is a second-order MRF, we will study how its density with respect to \mathcal{W} factorizes, and then apply Theorem 5.2. As a first step, we identify the density of the $\{0, 1\}^{\mathbb{V}_{n,n}}$ -marginal:

Lemma 6.1 *Suppose ρ has a finite nonzero first moment. The law of $(1_{\{v \in \mathcal{T}\}})_{v \in \mathbb{V}_{n,n}}$ on $\{0, 1\}^{\mathbb{V}_{n,n}}$ is absolutely continuous with respect to that of $(\xi_v)_{v \in \mathbb{V}_{n,n}}$. Moreover, the*

Radon-Nikodym derivative is of the form

$$F_n((a_v)_{v \in \mathbb{V}_{n,n}}) = f_\emptyset(a_\emptyset, (a_k)_{k=1}^n) \prod_{v \in \mathbb{V}_{n-1,n} \setminus \{\emptyset\}} f_1(a_v, (a_{vk})_{k=1}^n), \quad (6.4)$$

for measurable functions $f_\emptyset, f_1 : \{0, 1\}^{n+1} \rightarrow \mathbb{R}_+$.

Proof This is an easy consequence of the conditional independence structure of the tree \mathcal{T} and the fact that, aside from the root, every vertex has an identical offspring distribution. \square

Next, we establish the desired second-order MRF property for P^n . We make use of the following notation. For $t > 0$, a set $A \subset \mathbb{V}$, and a probability measure Q on $(\{0, 1\} \times \mathcal{C})^\mathbb{V}$, we write Q_t and $Q_t[A]$ for the projections onto $(\{0, 1\} \times \mathcal{C}_t)^\mathbb{V}$ and $(\{0, 1\} \times \mathcal{C}_t)^A$, respectively. For example, $Q_t[A]$ is the image of Q through the map $(a_v, x_v)_{v \in \mathbb{V}} \mapsto (a_v, x_v[t])_{v \in A}$.

Proposition 6.2 *Suppose Assumption A holds, and assume the offspring distribution ρ has a finite nonzero first moment. Then, for each $t > 0$ and $n \geq 3$, the following hold:*

- (i) $(1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in \mathbb{V}}$ is a global second-order MRF.
- (ii) $(X_v^n[t])_{v \in \mathbb{V}}$ is a global second-order MRF.

Proof The property (ii) easily follows from (i), after noting as in Remark 4.2 that $1_{\{v \in \mathcal{T}_n\}}$ is measurable with respect to $X_v^n[t]$. Hence, we only prove (i).

Fix $t > 0$ and $n \geq 3$. Because the coordinates of $\mathbb{V} \setminus \mathbb{V}_{n,n}$ are all independent of those in $\mathbb{V}_{n,n}$, it clearly suffices to show that $(1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in \mathbb{V}_{n,n}}$ is a global second-order MRF. By Definition 5.1, we must show that

$$(1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in A} \perp\!\!\!\perp (1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in B} \mid (1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in \partial^2 A}, \quad (6.5)$$

for any sets $A, B \subset \mathbb{V}_{n,n}$ with $B \cap (A \cup \partial^2 A) = \emptyset$, where ∂^2 denotes the double boundary operation in the tree $\mathbb{V}_{n,n}$. Recall that $P_t^n[\mathbb{V}_{n,n}]$ is the restriction of the law P^n of the random process $(1_{\{v \in \mathcal{T}_n\}}, X_v^n)_{v \in \mathbb{V}}$ in (6.2) to $(\{0, 1\} \times \mathcal{C}_t)^\mathbb{V}$, and similarly for $\mathcal{W}_t[\mathbb{V}_{n,n}]$, where $\mathcal{W} \in \mathcal{P}((\{0, 1\} \times \mathcal{C})^\mathbb{V})$ is the law of the process $(\xi_v, \widehat{X}_v)_{v \in \mathbb{V}}$ defined just prior to (6.3). To prove (i), we show that the density $dP_t^n[\mathbb{V}_{n,n}]/d\mathcal{W}_t[\mathbb{V}_{n,n}]$ admits a 2-clique factorization in the sense of Definition 5.3. To show this, we will use Girsanov's theorem to identify a conditional density *given the realization of the tree*, and then note that $dP_t^n[\mathbb{V}_{n,n}]/d\mathcal{W}_t[\mathbb{V}_{n,n}]$ is nothing but the product of this conditional density with the density of the law of $(1_{\{v \in \mathcal{T}\}})_{v \in \mathbb{V}_{n,n}}$ with respect to the law of $(\xi_v)_{v \in \mathbb{V}_{n,n}}$, the form of which was identified in Lemma 6.1.

To identify this conditional density, we need a bit more notation. Define

$$\mathcal{D}_{n,n} := \{(1_{\{v \in \mathcal{T}\}})_{v \in \mathbb{V}_{n,n}} \in \{0, 1\}^{\mathbb{V}_{n,n}} : T \subset \mathbb{V}_{n,n} \text{ is a tree}\}.$$

Define $\widehat{\mathcal{T}}_n : \mathcal{D}_{n,n} \rightarrow 2^{\mathbb{V}_{n,n}}$ by setting $\widehat{\mathcal{T}}_n((1_{\{v \in \mathcal{T}\}})_{v \in \mathbb{V}_{n,n}}) = T$ for each tree $T \subset \mathbb{V}_{n,n}$, and extend $\widehat{\mathcal{T}}_n$ to all of $\{0, 1\}^{\mathbb{V}_{n,n}}$ by (arbitrarily) setting $\widehat{\mathcal{T}}_n(a) := \{\emptyset\}$ for $a \notin \mathcal{D}_{n,n}$.

Note that $(1_{\{v \in \mathcal{T}_n\}})_{v \in \mathbb{V}_{n,n}}$ belongs a.s. to $\mathcal{D}_{n,n}$ and that $(\xi_v)_{v \in \mathbb{V}_{n,n}}$ is measurable with respect to $\widehat{\mathcal{T}}_n((\xi_v)_{v \in \mathbb{V}_{n,n}})$ on the event $\{(\xi_v)_{v \in \mathbb{V}_{n,n}} \in \mathcal{D}_{n,n}\}$. We may additionally extend the domain $\widehat{\mathcal{T}}_n$ to all of $(\{0, 1\} \times \mathcal{C})^{\mathbb{V}_{n,n}}$ by the identification $\widehat{\mathcal{T}}_n((a_v, x_v)_{v \in \mathbb{V}_{n,n}}) = \widehat{\mathcal{T}}_n((a_v)_{v \in \mathbb{V}_{n,n}})$. Intuitively, under the measure $P_t^n[\mathbb{V}_{n,n}]$, $\widehat{\mathcal{T}}_n$ will represent the truncated random UGW(ρ) tree \mathcal{T}_n , with the advantage that $\widehat{\mathcal{T}}_n$ is defined on the canonical space $(\{0, 1\} \times \mathcal{C})^{\mathbb{V}_{n,n}}$.

Given these definitions, we may now identify the density of $P_t^n[\mathbb{V}_{n,n}]$ with respect to $\mathcal{W}_t[\mathbb{V}_{n,n}]$, conditionally on $\widehat{\mathcal{T}}_n$. Since $\mathbb{V}_{n,n}$ is a finite set, we may apply Girsanov's theorem in the form of Lemma B.1 (which is applicable since (B.2) is satisfied due to Assumption (A.1) and Remark B.2): recalling the definition of X^n in (6.1), the conditional density of $P_t^n[\mathbb{V}_{n,n}]$ with respect to $\mathcal{W}_t[\mathbb{V}_{n,n}]$ given $\widehat{\mathcal{T}}_n$ is

$$\frac{dP_t^n[\mathbb{V}_{n,n}](\cdot | \widehat{\mathcal{T}}_n)}{d\mathcal{W}_t[\mathbb{V}_{n,n}](\cdot | \widehat{\mathcal{T}}_n)} = \prod_{v \in \mathbb{V}_{n,n}} \mathcal{E}_t(M_v^n), \quad (6.6)$$

where \mathcal{E}_t is the Doleans exponential defined in (4.1), and $M_v^n = M_v^n((a_v, x_v)_{v \in \mathbb{V}_{n,n}})$ is given by

$$M_v^n(t)((a_v, x_v)_{v \in \mathbb{V}_{n,n}}) := 1_{\{v \in \widehat{\mathcal{T}}_n\}} \int_0^t (\sigma \sigma^\top)^{-1} b(s, x_v, x_{N_v(\widehat{\mathcal{T}}_n)}) \cdot dx_v(s),$$

where we suppressed the arguments $(a_v)_{v \in \mathbb{V}_{n,n}}$ of $\widehat{\mathcal{T}}_n$. Observe that for each $v_0 \in \mathbb{V}_{n,n}$, $M_{v_0}^n$ depends on $(a_v, x_v)_{v \in \mathbb{V}_{n,n}}$ only through a_{v_0}, x_{v_0} and $(a_v, x_v)_{v \in N_{v_0}(\mathbb{V}_{n,n})}$, recalling that $N_v(\mathbb{V}_{n,n})$ denotes the set of neighbors of v within the tree $\mathbb{V}_{n,n}$.

Letting F_n be as in Lemma 6.1, the entire (joint) density takes the form

$$\frac{dP_t^n[\mathbb{V}_{n,n}]}{d\mathcal{W}_t[\mathbb{V}_{n,n}]}((a_v, x_v)_{v \in \mathbb{V}_{n,n}}) = F_n((a_v)_{v \in \mathbb{V}_{n,n}}) \frac{dP_t^n[\mathbb{V}_{n,n}](\cdot | \widehat{\mathcal{T}}_n)}{d\mathcal{W}_t[\mathbb{V}_{n,n}](\cdot | \widehat{\mathcal{T}}_n)}((a_v, x_v)_{v \in \mathbb{V}_{n,n}}).$$

Together, (6.6) and Lemma 6.1 imply that this can be rewritten as

$$\frac{dP_t^n[\mathbb{V}_{n,n}]}{d\mathcal{W}_t[\mathbb{V}_{n,n}]}((a_v, x_v)_{v \in \mathbb{V}_{n,n}}) = \prod_{v \in \mathbb{V}_{n,n}} g_v^n((a_v, x_v), (a_u, x_u)_{u \in N_v(\mathbb{V}_{n,n})}),$$

for appropriate functions $(g_v^n)_{v \in \mathbb{V}_{n,n}}$. More precisely, with f_\emptyset and f_1 as in Lemma 6.1, we have

$$g_v^n((a_v, x_v), (a_u, x_u)_{u \in N_v(\mathbb{V}_{n,n})}) = \begin{cases} f_\emptyset(a_\emptyset, (a_k)_{k=1}^n) \mathcal{E}_t(M_\emptyset^n) & \text{if } v = \emptyset, \\ f_1(a_v, (a_{vk})_{k=1}^n) \mathcal{E}_t(M_v^n) & \text{if } v \in \mathbb{V}_{n-1,n} \setminus \{\emptyset\}, \\ \mathcal{E}_t(M_v^n) & \text{if } v \in \mathbb{V}_{n,n} \setminus \mathbb{V}_{n-1,n}. \end{cases}$$

Observing that for each $v \in \mathbb{V}_{n,n}$, the set $\{v\} \cup N_v(\mathbb{V}_{n,n})$ is a 2-clique in $\mathbb{V}_{n,n}$, property (i) now follows from Theorem 5.2. \square

6.2 Convergence to the infinite system

With the second-order MRF property now established for the truncated systems P^n , we wish to pass to the limit $n \rightarrow \infty$ to deduce a similar property for the infinite system. We begin by checking that the law P^n of $(1_{\{v \in \mathcal{T}_n\}}, X_v^n)_{v \in \mathbb{V}}$ converges to the law P of $(1_{\{v \in \mathcal{T}\}}, X_v)_{v \in \mathbb{V}}$ and also that conditional laws converge in a suitable sense, where we recall that X^n and X , respectively, denote the solutions of the SDE systems (6.1) and (3.1).

Lemma 6.3 *Suppose Assumption A holds. Assume also that ρ has a finite nonzero first moment. Then $P^n \rightarrow P$ weakly on $(\{0, 1\} \times \mathcal{C})^{\mathbb{V}}$. Moreover, for any $k \in \mathbb{N}$, any $t > 0$, and any bounded continuous function $\varphi : \mathcal{C}_t^{\mathbb{V}} \rightarrow \mathbb{R}$, we have*

$$\mathbb{E}[\varphi(X_{\mathbb{V} \setminus \{\emptyset, k\}}^n[t], X_{\{\emptyset, k\}}^n[t]) \mid X_{\{\emptyset, k\}}^n[t]] \Rightarrow \mathbb{E}[\varphi(X_{\mathbb{V} \setminus \{\emptyset, k\}}[t], X_{\{\emptyset, k\}}[t]) \mid X_{\{\emptyset, k\}}[t]], \quad (6.7)$$

where we recall that \Rightarrow denotes convergence in law.

Proof Recall $\mathcal{T}_n = \mathcal{T} \cap \mathbb{V}_{n,n}$, where \mathcal{T} is a UGW(ρ) tree, and $\mathbb{V}_{n,n}$ is as defined in (2.3), and note that $(1_{\{v \in \mathcal{T}_n\}})_{v \in \mathbb{V}}$ therefore converges in law to $(1_{\{v \in \mathcal{T}\}})_{v \in \mathbb{V}}$ in $\{0, 1\}^{\mathbb{V}}$. It is straightforward to check that the family of \mathcal{C} -valued random variables $\{X_v^n : v \in \mathbb{V}, n \in \mathbb{N}\}$ is tight, by standard arguments or by using the relative entropy estimates of Lemma 4.1. Hence, $\{(1_{\{v \in \mathcal{T}_n\}}, X_v^n)_{v \in \mathbb{V}} : n \in \mathbb{N}\}$ is a tight family of $(\{0, 1\} \times \mathcal{C})^{\mathbb{V}}$ -valued random variables. Let $(1_{\{v \in \mathcal{T}\}}, X_v^\infty)_{v \in \mathbb{V}}$ denote any weak limit point, and assume by Skorokhod representation that it is in fact an a.s. limit. For $m \in \mathbb{N}$, we have $N_v(\mathcal{T}_n) = N_v(\mathcal{T})$ for all $n > m + 1$ and $v \in \mathbb{V}_m$, and using weak convergence of stochastic integrals (see [25, Theorem 2.2]) we deduce that $(X_v^\infty)_{v \in \mathbb{V}_m}$ satisfies

$$dX_v^\infty(t) = 1_{\{v \in \mathcal{T}\}} \left(b(t, X_v^\infty, X_{N_v(\mathcal{T})}^\infty) dt + \sigma(t, X_v^\infty) dW_v^\infty(t) \right), \quad v \in \mathbb{V}_m,$$

for some independent Brownian motions $(W_v^\infty)_{v \in \mathbb{V}_m}$. As this is true for each m , we deduce that $(X_v^\infty)_{v \in \mathbb{V}}$ and $(X_v)_{v \in \mathbb{V}}$ solve the same SDE system (3.1). The SDE (3.1) is unique in law by Assumption (A.1) (and Remark 3.1), and so the law of $(1_{\{v \in \mathcal{T}\}}, X_v^\infty)_{v \in \mathbb{V}}$ must be $P := \mathcal{L}((1_{\{v \in \mathcal{T}\}}, X_v)_{v \in \mathbb{V}})$, which shows that $P^n \rightarrow P$.

The second claim requires more care, and we will ultimately appeal to [9, Theorem 2.1], which gives a criterion for the weak convergence of conditional expectations. We introduce the following systems that are parallel to X^n and X but are driftless for nodes in \mathbb{V}_2 . Let $Q^n \in \mathcal{P}((\{0, 1\} \times \mathcal{C})^{\mathbb{V}})$ denote the law of

$$\left(1_{\{v \in \mathcal{T}\}}, Y_v^n \right)_{v \in \mathbb{V}}, \quad (6.8)$$

where $(Y_v^n(0))_{v \in \mathbb{V}} = (X_v(0))_{v \in \mathbb{V}}$ and $(Y_v^n)_{v \in \mathbb{V}}$ solves the SDE system

$$\begin{aligned} dY_v^n(t) &= 1_{\{v \in \mathcal{T}_n\}} \left(b(t, Y_v^n, Y_{N_v(\mathcal{T}_n)}^n) dt + \sigma(t, Y_v^n) dW_v(t) \right), & v \in \mathbb{V} \setminus \mathbb{V}_2, \\ dY_v^n(t) &= 1_{\{v \in \mathcal{T}_n\}} \sigma(t, Y_v^n) dW_v(t), & v \in \mathbb{V}_2. \end{aligned} \quad (6.9)$$

Recall that the tree \mathcal{T} , the initial conditions $(X_v(0))_{v \in \mathbb{V}}$, and the driving Brownian motions $(W_v)_{v \in \mathbb{V}}$ are independent. To see that the SDE (6.9) is unique in law (and hence, Q^n is well-defined), condition on the (finite) tree \mathcal{T}_n , use the independence properties just stated, the fact that the driftless SDE is unique in law by Assumption (A.4) and Lemma B.1 (along with Remark B.2 and Assumption (A.1)).

Similarly, let $Q \in \mathcal{P}(\{0, 1\} \times \mathcal{C})^{\mathbb{V}}$ denote the law of

$$\left(1_{\{v \in \mathcal{T}\}}, Y_v\right)_{v \in \mathbb{V}}, \quad (6.10)$$

where $(Y_v(0))_{v \in \mathbb{V}} = (X_v(0))_{v \in \mathbb{V}}$ and $(Y_v)_{v \in \mathbb{V}}$ solves the SDE system

$$\begin{aligned} dY_v(t) &= 1_{\{v \in \mathcal{T}\}} \left(b(t, Y_v, Y_{N_v(\mathcal{T})}) dt + \sigma(t, Y_v) dW_v(t) \right), \quad v \in \mathbb{V} \setminus \mathbb{V}_2, \\ dY_v(t) &= 1_{\{v \in \mathcal{T}\}} \sigma(t, Y_v) dW_v(t), \quad v \in \mathbb{V}_2. \end{aligned} \quad (6.11)$$

That the SDE (6.11) is unique in law (and thus Q is well-defined) can be deduced by applying Lemma B.4 with $X^2 = (X_v^2)_{v \in \mathbb{V}}$ equal to the solution to the SDE (3.1), which is unique in law by Remark 3.1, and $X^1 = (Y_v)_{v \in \mathbb{V}}$ as above, noting that the two differ only for v in the finite set \mathbb{V}_2 , and that condition (B.5) of Lemma B.4 holds by Remark B.2 and Assumption (A.1).

It is easily checked that $Q^n \rightarrow Q$ weakly, using the same argument which showed that $P^n \rightarrow P$ above. Fix $t > 0$. We may now apply Girsanov's theorem, in the precise infinite-dimensional form developed in Lemma B.4, whose application is justified by the uniqueness in law of the SDEs in (6.9) and (6.11) and the fact that the condition (B.5) holds on account of Remark B.2 and Assumption (A.1), to obtain

$$\begin{aligned} \frac{dP_t^n}{dQ_t^n}((1_{\{v \in \mathcal{T}_n\}}, Y_v^n)_{v \in \mathbb{V}}) &= \mathcal{E}_t \left(\sum_{v \in \mathbb{V}_2} \int_0^t 1_{\{v \in \mathcal{T}_n\}} \sigma(s, Y_v^n)^{-1} b(s, Y_v^n, Y_{N_v(\mathcal{T}_n)}^n) \cdot dW_v(s) \right), \\ \frac{dP_t}{dQ_t}((1_{\{v \in \mathcal{T}\}}, Y_v)_{v \in \mathbb{V}}) &= \mathcal{E}_t \left(\sum_{v \in \mathbb{V}_2} \int_0^t 1_{\{v \in \mathcal{T}\}} \sigma(s, Y_v)^{-1} b(s, Y_v, Y_{N_v(\mathcal{T})}) \cdot dW_v(s) \right). \end{aligned}$$

Note that the summations are a.s. finite, since all but finitely many of the indicators $1_{\{v \in \mathcal{T}_n\}}$ and $1_{\{v \in \mathcal{T}\}}$ are zero for $v \in \mathbb{V}_2$.

From the weak convergence $Q^n \rightarrow Q$ (of the laws of $(1_{\{v \in \mathcal{T}_n\}}, Y_v^n)_{v \in \mathbb{V}}$ to that of $(1_{\{v \in \mathcal{T}\}}, Y_v)_{v \in \mathbb{V}}$) and using weak convergence of stochastic integrals (see [25, Theorem 2.2]), we easily deduce the following weak convergence in $(\{0, 1\} \times \mathcal{C}_t)^{\mathbb{V}} \times \mathbb{R}$:

$$\begin{aligned} &\left(\left(1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t]\right)_{v \in \mathbb{V}}, \frac{dP_t^n}{dQ_t^n}((1_{\{v \in \mathcal{T}_n\}}, Y_v^n)_{v \in \mathbb{V}}) \right) \\ &\Rightarrow \left(\left(1_{\{v \in \mathcal{T}\}}, Y_v[t]\right)_{v \in \mathbb{V}}, \frac{dP_t}{dQ_t}((1_{\{v \in \mathcal{T}\}}, Y_v)_{v \in \mathbb{V}}) \right). \end{aligned} \quad (6.12)$$

To use this to deduce the desired convergence of related conditional distributions, we now verify an additional condition in [9, Theorem 2.1]. Fix $k \in \mathbb{N}$ and a bounded

continuous function g on $(\{0, 1\} \times \mathcal{C}_t)^{\mathbb{V} \setminus \{\emptyset, k\}}$. It is clear from the form of (6.9) that $(1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}}$ and $(Y_v^n[t])_{v \in \{\emptyset, k\}}$ are conditionally independent given $\{k \in \mathcal{T}\}$, and similarly with (\mathcal{T}_n, Y^n) replaced by (\mathcal{T}, Y) , when $n > k$. For $n > k$, we have $\{k \in \mathcal{T}_n\} = \{k \in \mathcal{T}\}$, and thus

$$\begin{aligned} & \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) \middle| (1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t])_{v \in \{\emptyset, k\}} \right] \\ &= \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) \middle| 1_{\{k \in \mathcal{T}\}} \right] \\ &= \frac{1_{\{k \in \mathcal{T}\}}}{\mathbb{P}(k \in \mathcal{T})} \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) 1_{\{k \in \mathcal{T}\}} \right] \\ &\quad + \frac{1_{\{k \notin \mathcal{T}\}}}{\mathbb{P}(k \notin \mathcal{T})} \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}_n\}}, Y_v^n[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) 1_{\{k \notin \mathcal{T}\}} \right] \\ &\Rightarrow \frac{1_{\{k \in \mathcal{T}\}}}{\mathbb{P}(k \in \mathcal{T})} \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}\}}, Y_v[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) 1_{\{k \in \mathcal{T}\}} \right] \\ &\quad + \frac{1_{\{k \notin \mathcal{T}\}}}{\mathbb{P}(k \notin \mathcal{T})} \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}\}}, Y_v[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) 1_{\{k \notin \mathcal{T}\}} \right] \\ &= \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}\}}, Y_v[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) \middle| 1_{\{k \in \mathcal{T}\}} \right] \\ &= \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}\}}, Y_v[t])_{v \in \mathbb{V} \setminus \{\emptyset, k\}} \right) \middle| (1_{\{v \in \mathcal{T}\}}, Y_v[t])_{v \in \{\emptyset, k\}} \right]. \end{aligned}$$

This and (6.12) are precisely the two conditions assumed in [9, Theorem 2.1], which we may now apply to deduce that

$$\begin{aligned} & \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in \mathbb{V}} \right) \middle| (1_{\{v \in \mathcal{T}_n\}}, X_v^n[t])_{v \in \{\emptyset, k\}} \right] \\ &\Rightarrow \mathbb{E} \left[g \left((1_{\{v \in \mathcal{T}\}}, X_v[t])_{v \in \mathbb{V}} \right) \middle| (1_{\{v \in \mathcal{T}\}}, X_v[t])_{v \in \{\emptyset, k\}} \right], \end{aligned}$$

for each bounded continuous function g on $(\{0, 1\} \times \mathcal{C})^{\mathbb{V}}$. Specializing to functions on $\mathcal{C}^{\mathbb{V}}$ yields the claim (6.7). \square

6.3 Proof of Proposition 3.17

We finally prove Proposition 3.17, starting with claim (i). Fix $k \in \mathbb{N}$ and let $C_k = \{ki : i \in \mathbb{N}\}$. Fix two bounded continuous functions f and g on $\mathcal{C}_t^{C_k}$ and $\mathcal{C}_t^{\mathbb{V}_1}$. From Lemma 6.3 we have that

$$\begin{aligned} & s_1 \mathbb{E}[f(X_{C_k}^n[t]) \mid X_{\{\emptyset, k\}}^n[t]] + s_2 \mathbb{E}[g(X_{\mathbb{V}_1}^n[t]) \mid X_{\{\emptyset, k\}}^n[t]] \\ & quad \Rightarrow s_1 \mathbb{E}[f(X_{C_k}[t]) \mid X_{\{\emptyset, k\}}[t]] + s_2 \mathbb{E}[g(X_{\mathbb{V}_1}[t]) \mid X_{\{\emptyset, k\}}[t]] \end{aligned}$$

for every $s_1, s_2 \in \mathbb{R}$. Therefore, by the Cramér-Wold theorem,

$$\begin{aligned} & \left(\mathbb{E}[f(X_{C_k}^n[t]) | X_{\{\emptyset, k\}}^n[t]], \mathbb{E}[g(X_{V_1}^n[t]) | X_{\{\emptyset, k\}}^n[t]] \right) \\ & \Rightarrow \left(\mathbb{E}[f(X_{C_k}[t]) | X_{\{\emptyset, k\}}[t]], \mathbb{E}[g(X_{V_1}[t]) | X_{\{\emptyset, k\}}[t]] \right). \end{aligned}$$

By Proposition 6.2(i), $X_{C_k}^n[t]$ and $X_{V_1}^n[t]$ are conditionally independent given $X_{\{\emptyset, k\}}^n[t]$ for each n ; indeed, apply Definition 5.1 of a second-order MRF with the set A given as the set of all descendants of k , so that $\partial^2 A = \{\emptyset, k\}$. Thus, we have

$$\begin{aligned} \mathbb{E}[f(X_{C_k}[t])g(X_{V_1}[t])] &= \lim_{n \rightarrow \infty} \mathbb{E}[f(X_{C_k}^n[t])g(X_{V_1}^n[t])] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbb{E}[f(X_{C_k}^n[t]) | X_{\{\emptyset, k\}}^n[t]] \mathbb{E}[g(X_{V_1}^n[t]) | X_{\{\emptyset, k\}}^n[t]] \right] \\ &= \mathbb{E} \left[\mathbb{E}[f(X_{C_k}[t]) | X_{\{\emptyset, k\}}[t]] \mathbb{E}[g(X_{V_1}[t]) | X_{\{\emptyset, k\}}[t]] \right]. \end{aligned}$$

As this holds for any pair of bounded continuous functions (f, g) , we conclude as desired that $X_{C_k}[t]$ and $X_{V_1}[t]$ are conditionally independent given $X_{\{\emptyset, k\}}[t]$.

To prove part (ii) of Proposition 3.17, we use a symmetry argument. Fix $k \in \mathbb{N}$, and let $\varphi : \mathbb{V} \rightarrow \mathbb{V}$ denote the transposition of the subtrees rooted at 1 and k , defined by setting $\varphi(1u) = ku$ and $\varphi(ku) = 1u$ for all $u \in \mathbb{V}$ as well as $\varphi(v) = v$ for all $v \in \mathbb{V}$ which satisfy neither $v \geq 1$ nor $v \geq k$ (i.e., for all $v \in \mathbb{V}$ that are not descendants of 1 or k). Due to the recursive structure of the tree $\mathcal{T} \sim \text{UGW}(\rho)$, we have $\mathcal{L}(\mathcal{T} | k \in \mathcal{T}) = \mathcal{L}(\varphi(\mathcal{T}) | k \in \mathcal{T})$. Using uniqueness of the SDE system in Assumption (A.4), we deduce that $\mathcal{L}(X_\emptyset, X_1, (X_{1j})_{j \in \mathbb{N}} | k \in \mathcal{T}) = \mathcal{L}(X_\emptyset, X_k, (X_{kj})_{j \in \mathbb{N}} | k \in \mathcal{T})$. Now, fix $t > 0$ and let $\Lambda_t : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C}_t^{\mathbb{N}})$ denote a version of the conditional law of $(X_{1j}[t])_{j \in \mathbb{N}}$ given $(X_1[t], X_\emptyset[t])$. Then, for bounded measurable functions f, g, h , we combine this symmetry property with the conditional independence of Proposition 3.17(i) proven above to obtain

$$\begin{aligned} & \mathbb{E} \left[f(X_\emptyset[t])g(X_k[t])h((X_{kj}[t])_{j \in \mathbb{N}})1_{\{k \in \mathcal{T}\}} \right] \\ &= \mathbb{E} \left[f(X_\emptyset[t])g(X_1[t])h((X_{1j}[t])_{j \in \mathbb{N}})1_{\{k \in \mathcal{T}\}} \right] \\ &= \mathbb{E} \left[f(X_\emptyset[t])g(X_1[t])\langle \Lambda_t(X_1, X_\emptyset), h \rangle 1_{\{k \in \mathcal{T}\}} \right] \\ &= \mathbb{E} \left[f(X_\emptyset[t])g(X_k[t])\langle \Lambda_t(X_k, X_\emptyset), h \rangle 1_{\{k \in \mathcal{T}\}} \right]. \end{aligned}$$

Indeed, the second step followed from the conditional independence of $(X_{1j}[t])_{j \in \mathbb{N}}$ and $\{k \in \mathcal{T}\}$ (which is $X_k[t]$ -measurable by Remark 4.2) given $(X_\emptyset[t], X_1[t])$. This shows that

$$\langle \Lambda_t(X_k, X_\emptyset), h \rangle = \mathbb{E} \left[h((X_{kj}[t])_{j \in \mathbb{N}}) | X_k[t], X_\emptyset[t] \right], \quad \text{a.s. on } \{k \in \mathcal{T}\}. \quad (6.13)$$

Recalling how Λ_t was defined above, the proof would now be complete if not for the qualification “on $\{k \in \mathcal{T}\}$,” so we lastly take care of the complementary set. Let $Y_v(t) = X_v(0)$ for all $t \geq 0$ and $v \in \mathbb{V}$, and note that $Y_v = X_v$ a.s. on $\{v \notin \mathcal{T}\}$

by construction. Note also that $(Y_v)_{v \in \mathbb{V}}$ are i.i.d. On the event $\{k \notin \mathcal{T}\}$, we know $X_{kj} \equiv Y_{kj}$ for all $j \in \mathbb{N}$, and so

$$\mathbb{E} [h((X_{kj}[t])_{j \in \mathbb{N}}) \mid X_k[t], X_\emptyset[t]] = \mathbb{E} [h((Y_{kj}[t])_{j \in \mathbb{N}})] = \mathbb{E} [h((Y_{1j}[t])_{j \in \mathbb{N}})], \quad a.s. \quad (6.14)$$

Repeating this independence argument with $k = 1$ and using the definition of Λ_t , we find

$$\langle \Lambda_t(X_1, X_\emptyset), h \rangle = \mathbb{E} [h((Y_{1j}[t])_{j \in \mathbb{N}})], \quad a.s. \text{ on } \{1 \notin \mathcal{T}\}. \quad (6.15)$$

Recalling from Remark 4.2 that there is a measurable function τ such that $1_{\{v \in \mathcal{T}\}} = \tau(X_v)$ a.s. for each v , it is straightforward to deduce from (6.14) and (6.15) that the same identity (6.13) holds also on the event $\{k \notin \mathcal{T}\}$. \square

7 Proof of the symmetry property

The last remaining point is to prove Proposition 3.18, which was the second key ingredient in the first (verification) part of Theorem 3.12. As a first step, in Sect. 7.1 we show that the children of the root are exchangeable, in a suitable conditional sense. Then, in Sect. 7.2, we use unimodularity to prove Proposition 3.18. Recall here that for a finite set A and for $x_A \in \mathcal{X}^A$ we write $\langle x_A \rangle$ for the corresponding element (equivalence class) in $S^\sqcup(\mathcal{X})$.

7.1 Conditional exchangeability at the generation level

We first show how to use Proposition 3.17 to derive a useful conditional exchangeability property.

Lemma 7.1 *Suppose Assumption A holds, and assume that $\rho \in \mathcal{P}(\mathbb{N}_0)$ has a finite nonzero first moment. For each $t > 0$ and each bounded measurable function $h : \mathcal{C}_t^2 \times S^\sqcup(C_t)^2 \rightarrow \mathbb{R}$, it holds almost surely on the event $\{1 \in \mathcal{T}\}$ that*

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} h(X_\emptyset[t], X_k[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, \langle X_{N_k(\mathcal{T})}[t] \rangle) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle \right] \\ &= \mathbb{E} [h(X_\emptyset[t], X_1[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, \langle X_{N_1(\mathcal{T})}[t] \rangle) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle]. \end{aligned} \quad (7.1)$$

Proof We first prove (7.1) assuming that h has the following form: there exists a bounded measurable mapping $f : \mathcal{C}_t^2 \rightarrow \mathbb{R}$ such that

$$h(x, y, \tilde{x}, \tilde{y}) = f(x, y), \quad x, y \in \mathcal{C}_t, \quad \tilde{x}, \tilde{y} \in S^\sqcup(C_t). \quad (7.2)$$

Fix $k, n \in \mathbb{N}$ with $k \leq n$, and let $\varphi : \mathbb{V} \rightarrow \mathbb{V}$ denote the transposition of the subtrees rooted at 1 and k , defined by setting $\varphi(1u) = ku$ and $\varphi(ku) = 1u$ for all

$u \in \mathbb{V}$ as well as $\varphi(v) = v$ for all $v \in \mathbb{V}$ which satisfy neither $v \geq 1$ nor $v \geq k$ with respect to the Ulam–Harris–Neveu labeling (i.e., for all $v \in \mathbb{V}$ that are neither descendants of 1 nor k). Due to the recursive structure of the tree $\mathcal{T} \sim \text{UGW}(\rho)$, we have $\mathcal{L}(\mathcal{T} \mid |N_\emptyset(\mathcal{T})| = n) = \mathcal{L}(\varphi(\mathcal{T}) \mid |N_\emptyset(\mathcal{T})| = n)$. Using uniqueness of the SDE system in Assumption (A.4), we deduce that

$$\begin{aligned}
 & \mathcal{L}(X_\emptyset[t], X_1[t] \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, |N_\emptyset(\mathcal{T})| = n) \\
 &= \mathcal{L}(X_\emptyset[t], X_k[t] \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, |N_\emptyset(\mathcal{T})| = n).
 \end{aligned}$$

From this we have

$$\begin{aligned}
 \frac{1}{n} \sum_{k=1}^n f(X_\emptyset[t], X_k[t]) &= \mathbb{E} \left[\frac{1}{n} \sum_{k=1}^n f(X_\emptyset[t], X_k[t]) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, |N_\emptyset(\mathcal{T})| = n \right] \\
 &= \mathbb{E} [f(X_\emptyset[t], X_1[t]) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, |N_\emptyset(\mathcal{T})| = n]. \quad (7.3)
 \end{aligned}$$

In other words, it holds a.s. on $\{1 \in \mathcal{T}\} = \{N_\emptyset(\mathcal{T}) \neq \emptyset\}$ that

$$\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} f(X_\emptyset[t], X_k[t]) = \mathbb{E} [f(X_\emptyset[t], X_1[t]) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle, |N_\emptyset(\mathcal{T})|].$$

Because $|N_\emptyset(\mathcal{T})|$ is a.s. $\langle X_{N_\emptyset(\mathcal{T})}[t] \rangle$ -measurable for each $t > 0$, this implies

$$\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} f(X_\emptyset[t], X_k[t]) = \mathbb{E} [f(X_\emptyset[t], X_1[t]) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle],$$

again on the event $\{1 \in \mathcal{T}\}$. Thus, the proof is complete for h of the form (7.2).

We now prove (7.1) for general h . Since both sides of (7.1) are conditional on $X_\emptyset[t]$ and $\langle X_{N_\emptyset(\mathcal{T})}[t] \rangle$, by general measure-theoretic considerations, it suffices to prove the relation (7.1) for $h(x, y, \tilde{x}, \tilde{y}) = g(y, \tilde{y})$ depending only on the variables that are not being conditioned upon. That is, it suffices to show that for all bounded measurable functions $g : \mathcal{C}_t \times S^\cup(\mathcal{C}_t) \rightarrow \mathbb{R}$ we have

$$\begin{aligned}
 & \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} g(X_k[t], \langle X_{N_k(\mathcal{T})}[t] \rangle) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle \right] \\
 &= \mathbb{E} [g(X_1[t], \langle X_{N_1(\mathcal{T})}[t] \rangle) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle], \quad \text{a.s., on } \{1 \in \mathcal{T}\}. \quad (7.4)
 \end{aligned}$$

To prove this, recall first from Proposition 3.17(ii) that there is a measurable function $\Lambda_t : \mathcal{C}_t^2 \rightarrow \mathcal{P}(\mathcal{C}_t^\mathbb{N})$ such that

$$\Lambda_t(X_k[t], X_\emptyset[t]) = \mathcal{L}((X_{ki}[t])_{i \in \mathbb{N}} \mid X_k[t], X_\emptyset[t]), \quad \text{a.s., on } \{k \in \mathcal{T}\}.$$

Using the conditional independence of Proposition 3.17(i), we have also

$$\Lambda_t(X_k[t], X_\emptyset[t]) = \mathcal{L}((X_{ki}[t])_{i \in \mathbb{N}} \mid X_{\mathbb{V}_1}[t]), \quad \text{a.s., on } \{k \in \mathcal{T}\}. \quad (7.5)$$

Noting again that $|N_\emptyset(\mathcal{T})|$ is $\langle X_{N_\emptyset(\mathcal{T})}[t] \rangle$ -measurable, we may use the tower property of conditional expectation (and other relations specified below) to obtain, on $\{1 \in \mathcal{T}\}$,

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} g(X_k[t], \langle X_{N_k(\mathcal{T})}[t] \rangle) \middle| X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle \right] \\ &= \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} \mathbb{E}[g(X_k[t], \langle X_{N_k(\mathcal{T})}[t] \rangle) \mid X_{\mathbb{V}_1}[t]] \middle| X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle \right]. \\ &= \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} \langle \Lambda_t(X_k[t], X_\emptyset[t]), g(X_k[t], \langle \cdot \rangle) \rangle \middle| X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle \right] \\ &= \mathbb{E} [\langle \Lambda_t(X_1[t], X_\emptyset[t]), g(X_1[t], \langle \cdot \rangle) \rangle \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle], \end{aligned}$$

where the second equality used (7.5) and our short-hand notation $\langle v, f \rangle = \int f d\nu$ for any measure ν and ν -integrable function f , and the last equality used the relation (7.3) with $f(x_\emptyset, x_k) = \langle \Lambda_t(x_k, x_\emptyset), g(x_k, \langle \cdot \rangle) \rangle$ for $x_\emptyset, x_k \in \mathcal{C}_t^2$. Now, apply (7.5) once again to rewrite the right-hand side as

$$\begin{aligned} & \mathbb{E} [\mathbb{E}[g(X_1[t], \langle X_{N_1(\mathcal{T})}[t] \rangle) \mid X_{\mathbb{V}_1}[t]] \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle] \\ &= \mathbb{E} [g(X_1[t], \langle X_{N_1(\mathcal{T})}[t] \rangle) \mid X_\emptyset[t], \langle X_{N_\emptyset(\mathcal{T})}[t] \rangle], \quad \text{on } \{1 \in \mathcal{T}\}. \end{aligned}$$

This shows (7.4), thus completing the proof of the lemma. \square

7.2 Unimodular random graphs

So far we only needed the notion of a unimodular Galton–Watson tree, which could be defined simply as in Definition 3.8. However, the final step of the proof of Proposition 3.18 uses crucially the notion of *unimodularity* on general graphs, which we now briefly define; referring to [1] for a more thorough discussion. For this, we will need to introduce the notation for (doubly) rooted (marked) graphs. We recall the general graph terminology introduced in Sect. 2.1.

A *rooted graph* (G, o) is a connected graph equipped with a distinguished vertex o , where we assume G has finite or countable vertex set and is locally finite, meaning each vertex has finitely many neighbors. An *isomorphism* from one rooted graph (G_1, o_1) to another (G_2, o_2) is a bijection φ from the vertex set of G_1 to that of G_2 such that $\varphi(o_1) = o_2$ and such that (u, v) is an edge in G_1 if and only if $(\varphi(u), \varphi(v))$ is an edge in G_2 . We say two rooted graphs are *isomorphic* if there exists an isomorphism between them, and we let \mathcal{G}_* denote the set of isomorphism classes of rooted graphs. Similarly, a *doubly rooted graph* (G, o, o') is a rooted graph (G, o) with an additional distinguished vertex o' (which may equal o). Two doubly rooted graphs (G_i, o_i, o'_i) are isomorphic if there is an isomorphism from (G_1, o_1) to (G_2, o_2) which also maps o'_1 to o'_2 . We write \mathcal{G}_{**} for the set of isomorphism classes of doubly rooted graphs.

There are analogous definitions for *marked rooted graphs*. An \mathcal{X} -marked rooted graph is a tuple (G, x, o) , where (G, o) is a rooted graph and $x = (x_v)_{v \in G} \in \mathcal{X}^G$ is a vector of marks, indexed by vertices of G . We say that two marked rooted graphs (G_1, x^1, o_1) and (G_2, x^2, o_2) are *isomorphic* if there exists an isomorphism φ between the rooted graphs (G_1, o_1) and (G_2, o_2) that maps the marks of one to the marks of the other (i.e., for which $x_v^1 = x_{\varphi(v)}^2$ for all $v \in G_1$). Let $\mathcal{G}_*[\mathcal{X}]$ denote the set of isomorphism classes of \mathcal{X} -marked rooted graphs. A doubly rooted marked graph is defined in the obvious way, and $\mathcal{G}_{**}[\mathcal{X}]$ denotes the set of isomorphism classes of doubly rooted marked graphs.

These spaces of graphs come with natural topologies. For $r \in \mathbb{N}$ and $(G, o) \in \mathcal{G}_*$, let $B_r(G, o)$ denote the induced subgraph of G (rooted at o) containing only those vertices with (graph) distance at most r from the root o . The distance between (G_1, o_1) and (G_2, o_2) is defined as the value $1/(1 + \bar{r})$, where \bar{r} is the supremum over $r \in \mathbb{N}_0$ such that $B_r(G_1, o_1)$ and $B_r(G_2, o_2)$ are isomorphic, where we interpret $B_0(G_i, o_i) = \{o_i\}$. The distance between two marked graphs (G_i, x^i, o_i) , $i = 1, 2$, is likewise defined as the value $1/(1 + \bar{r})$, where \bar{r} is the supremum over $r \in \mathbb{N}_0$ such that there exists an isomorphism φ from $B_r(G_1, o_1)$ to $B_r(G_2, o_2)$ such that $d(x_v^1, x_{\varphi(v)}^2) \leq 1/r$ for all $v \in B_r(G_1, o_1)$. We equip \mathcal{G}_{**} and $\mathcal{G}_{**}[\mathcal{X}]$ with similar metrics, just using the union of the balls at the two roots in place of the ball around a single root. Metrized in this manner, the spaces \mathcal{G}_* and \mathcal{G}_{**} are Polish spaces, as are $\mathcal{G}_*[\mathcal{X}]$ and $\mathcal{G}_{**}[\mathcal{X}]$ if \mathcal{X} is itself a Polish space. See [6, Lemma 3.4] (or [29, Appendix A]) for a proof that $\mathcal{G}_*[\mathcal{X}]$ is a Polish space. Each space $\mathcal{G}_*[\mathcal{X}]$ and $\mathcal{G}_{**}[\mathcal{X}]$ is equipped with its Borel σ -algebra.

We are now ready to introduce the definition of unimodularity for general graphs.

Definition 7.2 For a metric space \mathcal{X} , we say that a $\mathcal{G}_*[\mathcal{X}]$ -valued random element (G, X, o) is *unimodular* if the following *mass-transport principle* holds: for every (non-negative) bounded Borel measurable function $F : \mathcal{G}_{**}[\mathcal{X}] \rightarrow \mathbb{R}_+$,

$$\mathbb{E} \left[\sum_{o' \in G} F(G, X, o, o') \right] = \mathbb{E} \left[\sum_{o' \in G} F(G, X, o', o) \right]. \quad (7.6)$$

A \mathcal{G}_* -valued random variable (G, o) is said to be *unimodular* if the same identity holds, but with X removed, that is, if for every bounded Borel measurable function $F : \mathcal{G}_{**} \rightarrow \mathbb{R}_+$,

$$\mathbb{E} \left[\sum_{o' \in G} F(G, o, o') \right] = \mathbb{E} \left[\sum_{o' \in G} F(G, o', o) \right].$$

Recalling the canonical Ulam–Harris–Neveu labeling introduced in Sect. 2.1.2, as described therein, a (countable, locally finite) tree may always be viewed as a subset of \mathbb{V} satisfying the appropriate properties. Recall that $\emptyset \in \mathbb{V}$ denotes the root of any tree in this canonical labeling, and let \mathbb{T}_* denote the collection of subsets of \mathbb{V} described in Sect. 2.1.2 that define a rooted tree. A tree $\mathcal{T} \in \mathbb{T}_*$ induces an element (\mathcal{T}, \emptyset) of \mathcal{G}_* , and we say a random $(\mathbb{T}_*$ -valued) tree \mathcal{T} is unimodular if (\mathcal{T}, \emptyset) is a unimodular random graph in the sense of Definition 7.2.

Recall from Assumption (A.4) and Remark 3.1 that there is a unique solution $X^{\mathcal{T}} = (X_v^{\mathcal{T}})_{v \in \mathbb{V}}$ to the system (3.1) for any tree $\mathcal{T} \in \mathbb{T}_*$. We may then view $(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, \emptyset)$ as a rooted graph marked by the trajectories of the process $X^{\mathcal{T}}$, i.e., as a $\mathcal{G}_*[C]$ -valued random element.

Proposition 7.3 *Suppose Assumption A holds. Let \mathcal{T} be any unimodular (\mathbb{T}_* -valued) random tree, and let $X^{\mathcal{T}} = (X_v^{\mathcal{T}})_{v \in \mathbb{V}}$ be the unique solution of the SDE system (3.1). Then the $\mathcal{G}_*[C]$ -valued random variable $(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, \emptyset)$ is unimodular.*

Proof It will help to temporarily free ourselves from the canonical labels of \mathbb{V} . For any (countable, locally finite) tree \mathcal{T} (labeled in any manner), consider the SDE system

$$dX_v^{\mathcal{T}}(t) = b(t, X_v^{\mathcal{T}}, X_{N_v(\mathcal{T})}^{\mathcal{T}})dt + \sigma(t, X_v^{\mathcal{T}})dW_v(t), \quad v \in \mathcal{T}, \quad (7.7)$$

where $N_v(\mathcal{T})$ denotes the neighbors of v in \mathcal{T} , $(W_v)_{v \in \mathcal{T}}$ are independent Brownian motions, and $(X_v(0))_{v \in \mathcal{T}}$ are i.i.d. with law λ_0 . Note that this SDE system is unique in law by Assumption (A.4), as the tree \mathcal{T} can always be viewed up to isomorphism as a subset of \mathbb{V} . For any non-random doubly rooted tree (\mathcal{T}, o_1, o_2) , the unique solution of (7.7) gives rise to a $\mathcal{C}^{\mathcal{T}}$ -valued random variable $X^{\mathcal{T}} = (X_v^{\mathcal{T}})_{v \in \mathcal{T}}$, which in turn induces a $\mathcal{G}_{**}[C]$ -valued random variable $(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, o_1, o_2)$, whose law we denote by $Q[\mathcal{T}, o_1, o_2]$.

We claim first that $Q[\mathcal{T}, o_1, o_2] = Q[\mathcal{T}', o'_1, o'_2]$ whenever (\mathcal{T}, o_1, o_2) and $(\mathcal{T}', o'_1, o'_2)$ are isomorphic as doubly rooted graphs. To see this, let $\varphi : \mathcal{T} \rightarrow \mathcal{T}'$ denote any isomorphism. It is clear from the structure of the SDE (7.7) that the $\mathcal{C}^{\mathcal{T}}$ -valued random elements $(X_{\varphi(v)}^{\mathcal{T}'})_{v \in \mathcal{T}}$ and $(X_v^{\mathcal{T}})_{v \in \mathcal{T}}$ solve the same SDE and thus have the same law, due to the aforementioned uniqueness in law. In particular, the $\mathcal{G}_{**}[C]$ -valued random variables $(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, o_1, o_2)$ and $(\mathcal{T}', (X_v^{\mathcal{T}'})_{v \in \mathcal{T}'}, o'_1, o'_2)$ have the same law.

This shows that $Q[\mathcal{T}, o_1, o_2]$ depends on (\mathcal{T}, o_1, o_2) only through its isomorphism class. We may thus view Q as a (measurable) map from the set $\mathbb{T}_{**} \subset \mathcal{G}_{**}$ of doubly rooted trees to $\mathcal{P}(\mathcal{G}_{**}[C])$. (For a justification of the measurability of Q , see Remark 7.4 below.) For a bounded measurable function $F : \mathcal{G}_{**} \rightarrow \mathbb{R}_+$, the function $\mathbb{T}_{**} \ni (\mathcal{T}, o_1, o_2) \mapsto \langle Q[(\mathcal{T}, o_1, o_2)], F \rangle \in \mathbb{R}_+$ is also bounded and measurable, and we extend it to be zero on $\mathcal{G}_{**} \setminus \mathbb{T}_{**}$. Then, for a given unimodular (\mathbb{T}_* -valued) random tree \mathcal{T} , we have (as justified subsequently)

$$\begin{aligned} \mathbb{E} \left[\sum_{o \in \mathcal{T}} F(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, \emptyset, o) \right] &= \mathbb{E} \left[\sum_{o \in \mathcal{T}} \mathbb{E}[F(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, \emptyset, o) \mid \mathcal{T}] \right] \\ &= \mathbb{E} \left[\sum_{o \in \mathcal{T}} \langle Q[(\mathcal{T}, \emptyset, o)], F \rangle \right] \\ &= \mathbb{E} \left[\sum_{o \in \mathcal{T}} \langle Q[(\mathcal{T}, o, \emptyset)], F \rangle \right] = \mathbb{E} \left[\sum_{o \in \mathcal{T}} \mathbb{E}[F(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, o, \emptyset) \mid \mathcal{T}] \right] \\ &= \mathbb{E} \left[\sum_{o \in \mathcal{T}} F(\mathcal{T}, (X_v^{\mathcal{T}})_{v \in \mathcal{T}}, o, \emptyset) \right]. \end{aligned}$$

Indeed, the second and fourth steps used the fact that a random tree $\mathcal{T} \subset \mathbb{V}$ in the SDE system (3.1) is always assumed to be independent of the Brownian motions and initial conditions, which ensures that the conditional law of $(\mathcal{T}, (X_v^T)_{v \in \mathcal{T}}, \emptyset, o)$ given \mathcal{T} is precisely $Q[\mathcal{T}, \emptyset, o]$. \square

Remark 7.4 For completeness, we sketch here a proof of the measurability of Q introduced in the last proof. For $r \in \mathbb{N}$ and (G, o_1, o_2) for which the graph distance $d_G(o_1, o_2)$ is at most r , let $B_r(G, o_1, o_2) \in \mathcal{G}_{**}$ denote the union of the balls of radius r around o_1 and o_2 . The topology of the subspace $\{(\mathcal{T}, o_1, o_2) \in \mathbb{T}_{**} : (\mathcal{T}, o_1, o_2) = B_r(\mathcal{T}, o_1, o_2)\}$ is discrete for each r , so the map $(\mathcal{T}, o_1, o_2) \mapsto Q[B_r(\mathcal{T}, o_1, o_2)]$ is trivially measurable for each r . To complete the proof, it suffices to argue that $\lim_{r \rightarrow \infty} Q[B_r(\mathcal{T}, o_1, o_2)] = Q[(\mathcal{T}, o_1, o_2)]$ for each $(\mathcal{T}, o_1, o_2) \in \mathbb{T}_{**}$. If we fix a doubly rooted tree (\mathcal{T}, o_1, o_2) (with labels, i.e., not an element of \mathcal{G}_{**} but rather a representative from an equivalence class therein), then straightforward weak convergence arguments show that, for each $k \in \mathbb{N}$, $(X_v^{B_r(\mathcal{T}, o_1, o_2)})_{v \in B_k(\mathcal{T}, o_1, o_2)}$ converges in law to $(X_v^{(\mathcal{T}, o_1, o_2)})_{v \in B_k(\mathcal{T}, o_1, o_2)}$ as $r \rightarrow \infty$, which proves the claim.

Remark 7.5 It is well known that a UGW(ρ) tree (\mathcal{T}, \emptyset) is unimodular (hence the name), for $\rho \in \mathcal{P}(\mathbb{N}_0)$ with finite nonzero first moment, and from Proposition 7.3 we then deduce that $(\mathcal{T}, (X_v^T)_{v \in \mathcal{T}}, \emptyset)$ is unimodular. A direct proof of the mass-transport principle for (\mathcal{T}, \emptyset) is attributed to [34], but one can argue instead by approximation by finite uniformly rooted graphs; see [1, Example 10.2] or [11, Proposition 2.5].

7.3 Proof of Proposition 3.18

As in the statement of Proposition 3.18, let $h : \mathcal{C}_t^2 \times S^\sqcup(\mathcal{C}_t) \mapsto \mathbb{R}$ be bounded and measurable. To prove the proposition, we may assume without loss of generality that in addition $h \geq 0$. Fix $t > 0$, and let $g : \mathcal{C}_t^2 \rightarrow \mathbb{R}_+$ be any bounded measurable function. Because t is fixed, throughout this proof we will omit the argument $[t]$ for the sake of readability, with the understanding that every appearance of X_v below should be written more precisely as $X_v[t]$. Recall once more that for a finite set A and for $x_A \in \mathcal{X}^A$ we write $\langle x_A \rangle$ for the corresponding element (equivalence class) in $S^\sqcup(\mathcal{X})$. We will take advantage of the unimodularity of $(\mathcal{T}, X, \emptyset)$ shown in Proposition 7.3, by applying the mass-transport principle with

$$F(G, x, \emptyset, o) := g(x_\emptyset, x_o)h(x_o, x_\emptyset, \langle x_{N_o(G)} \rangle)1_{\{o \in N_\emptyset(G)\}}/|N_\emptyset(G)|.$$

Note that F is well defined on $\mathcal{G}_{**}[\mathcal{C}_t]$ because it is invariant under isomorphisms of (G, x, \emptyset, o) . We recall also that $\{v \in \mathcal{T}\}$ is measurable with respect to X_v for each $v \in \mathbb{V}$, as explained in Remark 4.2, which in particular implies that $\{1 \in \mathcal{T}\}$ and $|N_\emptyset(\mathcal{T})|$ are $\langle X_{N_\emptyset(\mathcal{T})} \rangle$ -measurable, and $|N_k(\mathcal{T})|$ is $\langle X_{N_k(\mathcal{T})} \rangle$ -measurable. The following calculation will use Lemma 7.1 and the aforementioned measurability properties in the first and last equality, unimodularity as in (7.6) with F as above in the third equality, and the fact that $\emptyset \in N_v(\mathcal{T})$ if and only if $v \in N_\emptyset(\mathcal{T})$ in the fourth equality

(recalling also our convention that $\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} = 0$ when $N_\emptyset(\mathcal{T}) = \emptyset$):

$$\begin{aligned}
 & \mathbb{E} \left[g(X_\emptyset, X_1) h(X_1, X_\emptyset, \langle X_{N_1(\mathcal{T})} \rangle) 1_{\{1 \in \mathcal{T}\}} \right] \\
 &= \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} g(X_\emptyset, X_k) h(X_k, X_\emptyset, \langle X_{N_k(\mathcal{T})} \rangle) \right] \\
 &= \mathbb{E} \left[\sum_{v \in \mathcal{T}} g(X_\emptyset, X_v) h(X_v, X_\emptyset, \langle X_{N_v(\mathcal{T})} \rangle) 1_{\{v \in N_\emptyset(\mathcal{T})\}} \frac{1}{|N_\emptyset(\mathcal{T})|} \right] \\
 &= \mathbb{E} \left[\sum_{v \in \mathcal{T}} g(X_v, X_\emptyset) h(X_\emptyset, X_v, \langle X_{N_\emptyset(\mathcal{T})} \rangle) 1_{\{\emptyset \in N_v(\mathcal{T})\}} \frac{1}{|N_v(\mathcal{T})|} \right] \\
 &= \mathbb{E} \left[\frac{1}{|N_\emptyset(\mathcal{T})|} \sum_{k \in N_\emptyset(\mathcal{T})} g(X_k, X_\emptyset) h(X_\emptyset, X_k, \langle X_{N_\emptyset(\mathcal{T})} \rangle) \frac{|N_\emptyset(\mathcal{T})|}{|N_k(\mathcal{T})|} \right] \\
 &= \mathbb{E} \left[g(X_1, X_\emptyset) h(X_\emptyset, X_1, \langle X_{N_\emptyset(\mathcal{T})} \rangle) \frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} 1_{\{1 \in \mathcal{T}\}} \right]. \tag{7.8}
 \end{aligned}$$

If $\varphi_h : \mathcal{C}_t^2 \rightarrow \mathbb{R}$ is defined by

$$\varphi_h(X_\emptyset, X_1) = 1_{\{1 \in \mathcal{T}\}} \mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} h(X_\emptyset, X_1, \langle X_{N_\emptyset(\mathcal{T})} \rangle) \mid X_\emptyset, X_1 \right],$$

then (7.8) can be rewritten as

$$\mathbb{E} \left[g(X_\emptyset, X_1) h(X_1, X_\emptyset, \langle X_{N_1(\mathcal{T})} \rangle) 1_{\{1 \in \mathcal{T}\}} \right] = \mathbb{E} \left[g(X_1, X_\emptyset) \varphi_h(X_\emptyset, X_1) 1_{\{1 \in \mathcal{T}\}} \right]. \tag{7.9}$$

Similarly, define $\varphi_1 : \mathcal{C}_t^2 \rightarrow \mathbb{R}$ by

$$\varphi_1(X_\emptyset, X_1) = 1_{\{1 \in \mathcal{T}\}} \mathbb{E} \left[\frac{|N_\emptyset(\mathcal{T})|}{|N_1(\mathcal{T})|} \mid X_\emptyset, X_1 \right].$$

Apply the identity (7.9), with h replaced by the constant function 1 and with $g(x_\emptyset, x_1)$ replaced by $g(x_1, x_\emptyset) \varphi_h(x_\emptyset, x_1)$, to obtain

$$\mathbb{E} \left[g(X_1, X_\emptyset) \varphi_h(X_\emptyset, X_1) 1_{\{1 \in \mathcal{T}\}} \right] = \mathbb{E} \left[g(X_\emptyset, X_1) \varphi_h(X_1, X_\emptyset) \varphi_1(X_\emptyset, X_1) 1_{\{1 \in \mathcal{T}\}} \right].$$

Substitution of this identity into the right-hand side of (7.9) yields

$$\begin{aligned}
 & \mathbb{E} \left[g(X_\emptyset, X_1) h(X_1, X_\emptyset, \langle X_{N_1(\mathcal{T})} \rangle) 1_{\{1 \in \mathcal{T}\}} \right] \\
 &= \mathbb{E} \left[g(X_\emptyset, X_1) \varphi_h(X_1, X_\emptyset) \varphi_1(X_\emptyset, X_1) 1_{\{1 \in \mathcal{T}\}} \right]. \tag{7.10}
 \end{aligned}$$

The fact that this holds for any g implies that, a.s. on $\{1 \in \mathcal{T}\}$,

$$\mathbb{E} \left[h(X_1, X_\emptyset, \langle X_{N_1(\mathcal{T})} \rangle) \mid X_\emptyset, X_1 \right] = \varphi_h(X_1, X_\emptyset) \varphi_1(X_\emptyset, X_1).$$

On the other hand, applying (7.10) with h replaced by the constant function 1, we deduce that $\varphi_1(X_1, X_\emptyset) \varphi_1(X_\emptyset, X_1) = 1$ a.s. on $\{1 \in \mathcal{T}\}$, and so

$$\mathbb{E} \left[h(X_1, X_\emptyset, \langle X_{N_1(\mathcal{T})} \rangle) \mid X_\emptyset, X_1 \right] = \frac{\varphi_h(X_1, X_\emptyset)}{\varphi_1(X_1, X_\emptyset)}.$$

Now recalling the definition of Ξ_t given in the statement of Proposition 3.18, (still omitting $[t]$ from the notation), it follows that

$$\Xi_t(X_\emptyset, X_1) = 1_{\{1 \in \mathcal{T}\}} \frac{\varphi_h(X_\emptyset, X_1)}{\varphi_1(X_\emptyset, X_1)}.$$

Thus, the last two displays establish (3.9) with $k = 1$. In light of the symmetry provided by Proposition 3.17(ii), this is enough to complete the proof. \square

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Appendix A: A projection theorem

Here we state and prove a result, used crucially in deriving the local equation, which can be seen as a *projection* or *mimicking* theorem for Itô processes. Theorem A.2 below seems to be reasonably well known, particularly in filtering theory, appearing (in various different forms) for instance in [32, Theorem 7.17], [7, Corollary 3.11], and [41, Section VI.8] but we give a short and mostly self-contained proof. Theorem A.2 can be seen also as a path-dependent counterpart of the famous mimicking theorem of Gyöngy [18].

We begin with a technical lemma to clear up any concerns about the existence of suitable versions of conditional expectations, of the sort that appear in the definitions of γ_t in (3.4) and (3.7). As usual, write $\mathcal{C} = C(\mathbb{R}_+; \mathbb{R}^d)$ and $\mathcal{C}_t = C([0, t]; \mathbb{R}^d)$ for the spaces of \mathbb{R}^d -valued paths, for $t > 0$, and $x[t]$ for the path up to time t of any $x \in \mathcal{C}$. Recall that we call a function f from $\mathbb{R}_+ \times \mathcal{C}$ to a measurable space S *progressively measurable* if it is jointly measurable and satisfies $f(t, x) = f(t, y)$ whenever $t \geq 0$ and $x, y \in \mathcal{C}$ satisfy $x[t] = y[t]$.

Lemma A.1 *Suppose $\Gamma = (\Gamma(t))_{t \geq 0}$ and $Y = (Y(t))_{t \geq 0}$ are stochastic processes with values in \mathbb{R}^k and \mathbb{R}^d , respectively. Suppose Y is continuous, and $\mathbb{E}[\int_0^T |\Gamma(t)| dt] < \infty$ for each $T > 0$. Then there exists a progressively measurable function $\gamma : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^k$ such that*

$$\gamma(t, Y) = \mathbb{E}[\Gamma(t) \mid Y[t]], \quad \text{a.s., for a.e. } t \geq 0.$$

Proof Apply [7, Proposition 5.1], taking the Polish-space-valued process Z_t therein to be the \mathcal{C} -valued process $Y[t]$, to find a Borel measurable function $\widehat{\gamma} : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^k$ such that

$$\widehat{\gamma}(t, Y[t]) = \mathbb{E}[\Gamma(t) \mid Y[t]], \quad \text{a.s., for a.e. } t \geq 0.$$

Then set $\gamma(t, x) = \widehat{\gamma}(t, x[t])$ for $(t, x) \in \mathbb{R}_+ \times \mathcal{C}$. \square

Theorem A.2 Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space supporting an \mathbb{F} -Brownian motion W of dimension m as well as a continuous \mathbb{F} -adapted process X of dimension d such that X admits the differential

$$dX(t) = b(t)dt + \sigma(t)dW(t),$$

where b and σ are \mathbb{F} -progressively processes taking values in \mathbb{R}^d , and $\mathbb{R}^{d \times m}$, respectively, with

$$\mathbb{E} \left[\int_0^t \left(|b(s)| + \text{Tr}[\sigma \sigma^\top(s)] \right) ds \right] < \infty, \quad \text{for } t > 0. \quad (\text{A.1})$$

Let $\widetilde{b} : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^d$ and $\widetilde{\sigma} : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}^{d \times d}$ be any progressively measurable functions satisfying

$$\widetilde{b}(t, X[t]) = \mathbb{E}[b(t) \mid X[t]], \quad \widetilde{\sigma} \widetilde{\sigma}^\top(t, X[t]) = \mathbb{E}[\sigma \sigma^\top(t) \mid X[t]], \quad \text{a.s., for a.e. } t \geq 0.$$

Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$ denote the filtration generated by X , defined by $\mathcal{F}_t^X = \sigma(X[t])$. Then there exists an extension $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}}, \check{\mathbb{P}})$ of the probability space $(\Omega, \mathcal{F}, \mathbb{F}^X, \mathbb{P})$ supporting a standard d -dimensional $\check{\mathbb{F}}$ -Brownian motion \widetilde{W} such that

$$dX(t) = \widetilde{b}(t, X)dt + \widetilde{\sigma}(t, X)d\widetilde{W}(t), \quad t \geq 0.$$

Proof Let $C_c^\infty(\mathbb{R}^d)$ denote the set of smooth functions on \mathbb{R}^d with compact support. Write ∇ and ∇^2 for the gradient and Hessian operators, respectively. By Itô's formula and the condition (A.1), for each $\varphi \in C_c^\infty(\mathbb{R}^d)$ the process

$$\varphi(X(t)) - \int_0^t \left(b(u) \cdot \nabla \varphi(X(u)) + \frac{1}{2} \text{Tr}[\sigma \sigma^\top(u) \nabla^2 \varphi(X(u))] \right) du$$

is a \mathbb{F} -martingale. In particular, if $t > s$, and if Z is any bounded \mathcal{F}_s -measurable random variable then

$$0 = \mathbb{E} \left[Z \left(\varphi(X(t)) - \varphi(X(s)) - \int_s^t \left(b(u) \cdot \nabla \varphi(X(u)) + \frac{1}{2} \text{Tr}[\sigma \sigma^\top(u) \nabla^2 \varphi(X(u))] \right) du \right) \right].$$

Now, If Z is measurable with respect to $\mathcal{F}_s^X \subset \mathcal{F}_s$, then we may use Fubini's theorem and the tower property of conditional expectations to obtain

$$0 = \mathbb{E} \left[Z \left(\varphi(X(t)) - \varphi(X(s)) - \int_s^t \left(\tilde{b}(u, X) \cdot \nabla \varphi(X(u)) + \frac{1}{2} \text{Tr}[\tilde{\sigma} \tilde{\sigma}^\top(u, X(u)) \nabla^2 \varphi(X(u))] \right) du \right) \right].$$

This shows that the process

$$\varphi(X(t)) - \int_0^t \left(\tilde{b}(u, X) \cdot \nabla \varphi(X(u)) + \frac{1}{2} \text{Tr}[\tilde{\sigma} \tilde{\sigma}^\top(u, X) \nabla^2 \varphi(X(u))] \right) du$$

is a \mathbb{R}^X -martingale, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$.

The claim now follows from the usual construction of weak solutions from solutions to martingale problems (e.g., using the arguments in Proposition 5.4.6 and Theorem 3.4.2 of [21] or [41, Theorem (20.1), p. 160]). \square

Appendix B: Forms of Girsanov's theorem

We develop here two simple forms of Girsanov's theorem tailored to the needs of proofs of results in this paper. No aspects of these results should come as a surprise to specialists, but we were unable to locate a reference that covered our precise requirements, which fall beyond the scope of the standard Novikov condition. Our drift b in Assumption A has linear growth, and thus, at least for the first lemma below, fairly standard results could cover some of our needs, such as [21, Corollary 3.5.16] or [32, Theorem 7.7]. But those results, strictly speaking, do not allow a general diffusion coefficient σ . The result [32, Theorem 7.7] is extended in [32, Section 7.6] but still requires Lipschitz coefficients, which is not good enough for us because of the γ_t term in the local equation (3.5), which need not be Lipschitz even when b is. Our second result below, Lemma B.4, is not directly covered by the aforementioned results either, because it involves an infinite-dimensional SDE system, though we only consider a change in drift for a finite number of coordinates. In any case, we give simple proofs of our two results using an elegant recent criterion of [5].

Lemma B.1 *Let $d \in \mathbb{N}$ and $\lambda_0 \in \mathcal{P}(\mathbb{R}^d)$. For $T \in (0, \infty)$, suppose $b : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}^{d \times d}$ are progressively measurable. Assume $\sigma(t, x)$ is invertible for each (t, x) and that σ and σ^{-1} are uniformly bounded. For $i = 1, 2$, suppose $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i = \{\mathcal{F}_t^i\}_{t \geq 0}, \mathbb{P}^i)$ is a filtered probability space supporting a d -dimensional \mathbb{P}^i -Brownian motion W^i and continuous d -dimensional \mathbb{P}^i -adapted process X^i , which satisfy for $t \in [0, T]$,*

$$\begin{aligned} dX^1(t) &= b(t, X^1)dt + \sigma(t, X^1)dW^1(t), & X^1(0) &\sim \lambda_0, \\ dX^2(t) &= \sigma(t, X^2)dW^2(t), & X^2(0) &\sim \lambda_0. \end{aligned} \quad (\text{B.1})$$

Assume the latter SDE is unique in law, and that

$$\mathbb{P}^i \left(\int_0^T |b(t, X^i)|^2 dt < \infty \right) = 1, \quad i = 1, 2. \quad (\text{B.2})$$

Then $\mathcal{L}(X^1[T])$ and $\mathcal{L}(X^2[T])$ are equivalent, and for $x \in \mathcal{C}_T$,

$$\begin{aligned} \frac{d\mathcal{L}(X^1[T])}{d\mathcal{L}(X^2[T])}(x) &= \exp \left(\int_0^T (\sigma \sigma^\top)^{-1} b(t, x) \cdot dx(t) \right. \\ &\quad \left. - \frac{1}{2} \int_0^T |\sigma^{-1} b(t, x)|^2 dt \right). \end{aligned} \quad (\text{B.3})$$

Remark B.2 If $t \mapsto b(t, x)$ is continuous for each x , then $\int_0^T |b(t, x)|^2 dt \leq T \sup_{t \in [0, T]} |b(t, x)|^2 < \infty$ for each x , and the key assumption (B.2) in Lemma B.1 holds automatically.

Proof of Lemma B.1 If b is uniformly bounded, then uniqueness in law of the SDE for X^1 and (B.3) are completely standard, following from Girsanov's theorem. Now, fix $T \in (0, \infty)$ and assume more generally that $\mathbb{P}(\int_0^T |b(t, X^1)|^2 dt < \infty) = \mathbb{P}(\int_0^T |b(t, X^2)|^2 dt < \infty) = 1$. Define $\tau_n : \mathcal{C} \rightarrow [0, T] \cup \{\infty\}$ and $b_n : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}^d$ by

$$b_n(t, x) := 1_{\{t \leq \tau_n(x)\}} b(t, x), \quad \tau_n(x) := \inf \left\{ t \in [0, T] : \int_0^t |b(s, x)|^2 ds \geq n \right\}.$$

Abbreviate $P^2 = \mathcal{L}(X^2[T])$. Now, define $R : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}_+$ by

$$R(t, x) := \exp \left(\int_0^t (\sigma \sigma^\top)^{-1} b(s, x) \cdot dx(s) - \frac{1}{2} \int_0^t |\sigma^{-1} b(s, x)|^2 ds \right).$$

Note that the uniform boundedness of σ^{-1} and the bound (B.2) ensure that $(R(t, \cdot))_{t \in [0, T]}$ is well defined P^2 -a.e. Moreover, the uniform boundedness of σ^{-1} and the definition of b_n guarantee that $\int_0^T |\sigma^{-1} b_n(t, x)|^2 dt = \int_0^{T \wedge \tau_n(x)} |\sigma^{-1} b(t, x)|^2 ds \leq n$ for all $x \in \mathcal{C}$, and thus Novikov's condition is satisfied. Hence, $(R(t \wedge \tau_n(X^2), X^2), \mathcal{F}_t^2)_{t \in [0, T]}$ is a \mathbb{P}^2 -martingale for each n [21, Corollary 3.5.13]. Thus, by Girsanov's theorem (see, e.g., [21, Theorem 3.5.1]), the SDE

$$dX^{1,n}(t) = b_n(t, X^{1,n})dt + \sigma(t, X^{1,n})dW(t), \quad X^{1,n}(0) \sim \lambda_0,$$

is unique in law, with its law $P^{1,n}$ satisfying $P^{1,n} \ll P^2$, where

$$\begin{aligned} \frac{dP^{1,n}}{dP^2}(x) &:= R(T \wedge \tau_n(x), x) \\ &= \exp \left(\int_0^T (\sigma \sigma^\top)^{-1} b_n(t, x) \cdot dx(t) - \frac{1}{2} \int_0^T |\sigma^{-1} b_n(t, x)|^2 dt \right), \end{aligned}$$

for P^2 -almost every $x \in \mathcal{C}$. Assume $X^{1,n}$ is constructed on a probability space $(\Omega^{1,n}, \mathcal{F}^{1,n}, \mathbb{P}^{1,n}, \mathbb{P}^{1,n})$.

We will now apply the criterion of [5, Corollary 2.1] to prove that under P^2 , the process $(R(t, \cdot), \mathcal{F}_t^2)_{t \in [0, T]}$ is not only a local martingale but is in fact a true martingale. To this end, note that the assumption $\mathbb{P}^2(\int_0^T |b(t, X^2)|^2 dt < \infty) = 1$ from (B.2) and the uniform boundedness of σ and σ^{-1} ensure that $\tau_n(X^2) \rightarrow \infty$ and $R(t \wedge \tau_n(X^2), X^2) \rightarrow R(t, X^2)$ a.s. as $n \rightarrow \infty$. Now, for each $n \in \mathbb{N}$ and $t \in [0, T]$, define $Q_n^t \ll P^2$ by

$$\frac{dQ_n^t}{dP^2}(x) = R(t \wedge \tau_n(x), x), \quad x \in \mathcal{C}.$$

Then [5, Corollary 2.1] states that $(R(t, \cdot))_{t \in [0, T]}$ is a P^2 -martingale if and only if

$$\lim_{n \rightarrow \infty} Q_n^t(\tau_n \leq t) = 0, \quad \text{for each } t \in [0, T]. \quad (\text{B.4})$$

But the latter follows from the assumption $\mathbb{P}^1(\int_0^T |b(t, X^1)|^2 dt < \infty) = 1$ imposed in (B.2), since recalling $P^2 = \mathbb{P}^2 \circ (X^2)^{-1}$ and $P^{1,n} = \mathbb{P}^{1,n} \circ (X^{1,n})^{-1}$ and letting \mathbb{E}^2 and $\mathbb{E}^{1,n}$ denote expectation under \mathbb{P}^2 and $\mathbb{P}^{1,n}$, respectively, we have

$$\begin{aligned} Q_n^t(\tau_n \leq t) &= \mathbb{E}^2[R(t \wedge \tau_n(X^2), X^2) 1_{\{\tau_n(X^2) \leq t\}}] = \mathbb{E}^2[R(T \wedge \tau_n(X^2), X^2) 1_{\{\tau_n(X^2) \leq t\}}] \\ &= \mathbb{P}^{1,n}(\tau_n(X^{1,n}) \leq t) = \mathbb{P}^1(\tau_n(X^1) \leq t) = \mathbb{P}^1\left(\int_0^t |b(s, X^1)|^2 ds \geq n\right), \end{aligned}$$

where the penultimate step used the fact that $(X_{t \wedge \tau_n(X^1)}^1)_{t \in [0, T]}$ satisfies the SDE (B.1) with b replaced by b_n and thus, by uniqueness in law of the latter SDE, the law of $(X_{t \wedge \tau_n(X^1)}^1)_{t \in [0, T]}$ under \mathbb{P}^1 coincides with that of $(X_{t \wedge \tau_n(X^{1,n})}^{1,n})_{t \in [0, T]}$ under $\mathbb{P}^{1,n}$. Since the right-hand side of the last display vanishes as $n \rightarrow \infty$ due to (B.2), this proves (B.4).

Hence, under P^2 , we have shown that R is a martingale on a finite time horizon, and thus a uniformly integrable martingale on that time horizon. Since $dP^{1,n}/dP^2 = R(T \wedge \tau_n(\cdot), \cdot)$ for each n , we deduce easily that $dP^1/dP^2 = R(T, \cdot)$. Since $R(T, \cdot) > 0$, we deduce that P^1 and P^2 are equivalent. \square

Recalling the definition of relative entropy functional H from (4.2), we record the following well-known relative entropy identity as a corollary:

Corollary B.3 *Let $d \in \mathbb{N}$ and $\lambda_0 \in \mathcal{P}(\mathbb{R}^d)$. Suppose $b^1, b^2 : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathcal{C} \rightarrow \mathbb{R}^{d \times d}$ are progressively measurable and bounded. Assume $\sigma(t, x)$ is invertible for each (t, x) and that σ^{-1} is uniformly bounded. For $i = 1, 2$, suppose $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i)$ is a filtered probability space supporting a d -dimensional \mathbb{F}^i -Brownian motion W^i and continuous d -dimensional \mathbb{F}^i -adapted process X^i satisfying*

$$dX^i(t) = b^i(t, X^i)dt + \sigma(t, X^i)dW^i(t), \quad X^i(0) \sim \lambda_0.$$

Assume the driftless SDE

$$dX(t) = \sigma(t, X)dW(t), \quad X(0) \sim \lambda_0$$

is unique in law. Then the following relative entropy identity holds:

$$H(\mathcal{L}(X^1[T]) | \mathcal{L}(X^2[T])) = \frac{1}{2} \mathbb{E}^{\mathbb{P}^1} \left[\int_0^T |\sigma^{-1}b^1(t, X^1) - \sigma^{-1}b^2(t, X^1)|^2 dt \right].$$

Proof Abbreviate $P^i = \mathcal{L}(X^i[T])$ for $i = 1, 2$. The boundedness of b^i ensures that (B.2) holds trivially. We may therefore apply Lemma B.1 twice to get

$$\begin{aligned} \frac{dP^1}{dP^2}(x) &= \exp \left(\int_0^T (\sigma \sigma^\top)^{-1} (b^1 - b^2)(t, x) \cdot dx(t) \right. \\ &\quad \left. + \frac{1}{2} \int_0^T (|\sigma^{-1}b^2(t, x)|^2 - |\sigma^{-1}b^1(t, x)|^2) dt \right). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} H(P^1 | P^2) &= \mathbb{E}^{\mathbb{P}^1} \left[\int_0^T (\sigma \sigma^\top)^{-1} (b^1 - b^2)(t, X^1) \cdot dX^1(t) \right. \\ &\quad \left. + \frac{1}{2} \int_0^T (|\sigma^{-1}b^2(t, X^1)|^2 - |\sigma^{-1}b^1(t, X^1)|^2) dt \right] \\ &= \frac{1}{2} \mathbb{E}^{\mathbb{P}^1} \left[\int_0^T |\sigma^{-1}b^1(t, X^1) - \sigma^{-1}b^2(t, X^1)|^2 dt \right]. \end{aligned}$$

This completes the proof. \square

Lastly, we prove an infinite-dimensional result similar to Lemma B.1, tailor-made for its use in the proof of Lemma 6.3.

Lemma B.4 *Let $d \in \mathbb{N}$, and let V be a countable set. Let $\lambda_0 \in \mathcal{P}((\mathbb{R}^d)^V)$. Suppose $b_v^1, b_v^2 : [0, T] \times C^V \rightarrow \mathbb{R}^d$ for $v \in V$ and $\sigma : [0, T] \times C \rightarrow \mathbb{R}^{d \times d}$ are progressively measurable. Assume $\sigma(t, x)$ is invertible for each (t, x) and that σ and σ^{-1} are uniformly bounded. For $i = 1, 2$, suppose $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i = \{\mathcal{F}_t^i\}_{t \geq 0}, \mathbb{P}^i)$ is a filtered probability space supporting independent d -dimensional \mathbb{F}^i -Brownian motions $(W_v^i)_{v \in V}$ as well as continuous d -dimensional \mathbb{F}^i -adapted processes $(X_v^i)_{v \in V}$ satisfying*

$$dX_v^i(t) = b_v^i(t, X)dt + \sigma(t, X_v^i)dW_v^i(t), \quad v \in V, \quad X^i(0) = (X_v^i(0))_{v \in V} \sim \lambda_0,$$

where the SDE system for X^2 is assumed to be unique in law. Assume that $b_v^1 \equiv b_v^2$ except for at most finitely many $v \in V$, and that for $i = 1, 2$,

$$\mathbb{P}^i \left(\int_0^T |b_v^1(t, X^i) - b_v^2(t, X^i)|^2 dt < \infty \right) = 1, \quad \text{for each } v \in V. \quad (\text{B.5})$$

Then, if $P^i \in \mathcal{P}(\mathcal{C}^V)$ denotes the law of $X^i = (X_v^i)_{v \in V}$ under \mathbb{P}^i for $i = 1, 2$, then P^1 and P^2 are equivalent, and

$$\frac{dP^1}{dP^2}(X^2) = \exp \left\{ \sum_{v \in V} \left(\int_0^T \sigma^{-1}(b_v^1 - b_v^2)(t, X^2) \cdot dW_v(t) - \frac{1}{2} \int_0^T |\sigma^{-1}(b_v^1 - b_v^2)(t, X^2)|^2 dt \right) \right\}$$

almost surely, where $\sigma^{-1}(b_v^1 - b_v^2)$ denotes the function $[0, T] \times \mathcal{C}^V \ni (t, x) \mapsto \sigma^{-1}(t, x_v)(b_v^1(t, x) - b_v^2(t, x))$ for $v \in V$.

Proof Let $V_0 := V \setminus \{v \in V : b_v^1 \equiv b_v^2\}$, and note that V_0 is finite by assumption. If $\sum_{v \in V_0} |b_v^1 - b_v^2|^2$ is uniformly bounded, then the claim is a standard application of Girsanov's theorem. For the general case, define $\tau_n : \mathcal{C}^V \rightarrow [0, T] \cup \{\infty\}$ and $b_v^{1,n} : [0, T] \times \mathcal{C}^V \rightarrow \mathbb{R}^d$ for $v \in V$ by

$$\tau_n(x) := \inf \left\{ t \in [0, T] : \sum_{v \in V_0} \int_0^t |b_v^1(s, x) - b_v^2(s, x)|^2 ds \geq n \right\},$$

$$b_v^{1,n}(t, x) := 1_{\{t \leq \tau_n(x)\}} b_v^1(t, x) + 1_{\{t > \tau_n(x)\}} b_v^2(t, x).$$

With these definitions, the remainder of the proof follows that of Lemma B.1 very closely, so we give fewer details. Define $R : [0, T] \times \mathcal{C}^V \rightarrow \mathbb{R}_+$ by

$$R(t, x) := \exp \sum_{v \in V} \left(\int_0^t (\sigma \sigma^\top)^{-1}(b_v^1 - b_v^2)(s, x) \cdot (dx_v(s) - b_v^2(s, x) ds) - \frac{1}{2} \int_0^t |\sigma^{-1}(b_v^1 - b_v^2)(s, x)|^2 ds \right),$$

which is well-defined for P^2 -a.e. $x = (x_v)_{v \in V} \in \mathcal{C}^V$. Note that $b_v^{1,n} \equiv b_v^1 \equiv b_v^2$ for $v \in V \setminus V_0$, so that the summation in the definition of R is actually over the finite set V_0 . Since also $\sum_{v \in V_0} \int_0^{\tau_n(x)} |\sigma^{-1}(b_v^1 - b_v^2)(t, x)|^2 dt \leq n$ for all $x \in \mathcal{C}^V$ by construction, Novikov's condition ensures that $(R(t \wedge \tau_n(X^2), X^2), \mathcal{F}_t^2)_{t \in [0, T]}$ is a \mathbb{P}^2 -martingale, for each n . Hence, by Girsanov's theorem and uniqueness in law of the X^2 equation, the SDE system

$$dX_v^{1,n}(t) = b_v^{1,n}(t, X^{1,n}) dt + \sigma(t, X_v^{1,n}) dW_v(t), \quad v \in V, \quad X^{1,n}(0) \sim \lambda_0,$$

is unique in law, and its law $P^{1,n}$ satisfies $P^{1,n} \ll P^2$ and, a.s.,

$$\frac{dP^{1,n}}{dP^2}(X^2) = \exp \left\{ \sum_{v \in V} \left(\int_0^T \sigma^{-1}(b_v^{1,n} - b_v^2)(t, X^2) \cdot dW_v^2(t) - \frac{1}{2} \int_0^T |\sigma^{-1}(b_v^{1,n} - b_v^2)(t, X^2)|^2 dt \right) \right\}.$$

Assume $X^{1,n}$ is defined on a filtered probability space $(\Omega^{1,n}, \mathcal{F}^{1,n}, \mathbb{P}^{1,n}, \mathbb{P}^{1,n})$.

To complete the proof, as in Lemma B.1, it suffices to show that the local martingale R is a true martingale. To this end, note that the assumption (B.5) and boundedness of σ and σ^{-1} ensure $\tau_n(X^2) \rightarrow \infty$ and $R(t \wedge \tau_n(X^2), X^2) \rightarrow R(t, X^2)$ a.s. as $n \rightarrow \infty$. For each $t \in [0, T]$ and $n \in \mathbb{N}$, we define $Q_n^t \ll P^2$ by $dQ_n^t/dP^2(x) = R(t \wedge \tau_n(x), x)$, $x \in C^V$. Then, by [5, Corollary 2.1], R is a P^2 -martingale if and only if $\lim_{n \rightarrow \infty} Q_n^t(\tau_n \leq t) = 0$ for each $t \in [0, T]$. The latter follows from assumption (B.5) by means of a calculation similar to that used in Lemma B.1: Since the laws of $(X^{1,n}(t \wedge \tau_n(X^{1,n})))_{t \in [0, T]}$ under $\mathbb{P}^{1,n}$ and $(X^1(t \wedge \tau_n(X^1)))_{t \in [0, T]}$ under \mathbb{P}^1 coincide, we have

$$\begin{aligned} Q_n^t(\tau_n \leq t) &= \mathbb{P}^{1,n}(\tau_n(X^{1,n}) \leq t) = \mathbb{P}^1(\tau_n(X^1) \leq t) \\ &= \mathbb{P}^1 \left(\int_0^t |b_v^1(s, X^1) - b_v^2(s, X^1)|^2 ds \geq n \right), \end{aligned}$$

which converges to zero as $n \rightarrow \infty$ due to (B.5). \square

Appendix C: Proof of Lemma 4.1

Recall that $(X_v(0))_{v \in \mathbb{V}}$ are independent of \mathcal{T} and are i.i.d. and square-integrable by Assumption (A.3), $X = (X_v^T)_{v \in \mathcal{T}}$ satisfies the SDE system (3.1). Using the linear growth of Assumption (A.1) and the boundedness of σ of Assumption (A.2), we thus find, for all $t \in [0, T]$,

$$\mathbb{E}[\|X_v\|_{*,t}^2 | \mathcal{T}] \leq C \left(1 + \int_0^t \left(\mathbb{E}[\|X_v\|_{*,s}^2 | \mathcal{T}] + \frac{1}{|N_v(\mathcal{T})|} \sum_{u \in N_v(\mathcal{T})} \mathbb{E}[\|X_u\|_{*,s}^2 | \mathcal{T}] \right) ds \right),$$

where $C < \infty$ is a constant depending only on T, λ_0 , and the constants of Assumptions (A.1) and (A.2). (As usual, the average over $N_v(\mathcal{T})$ is understood to be zero when $N_v(\mathcal{T}) = \emptyset$ or $v \notin \mathcal{T}$.) This implies

$$\sup_{v \in \mathbb{V}} \mathbb{E}[\|X_v\|_{*,t}^2 | \mathcal{T}] \leq 2C \left(1 + \int_0^t \sup_{v \in \mathbb{V}} \mathbb{E}[\|X_v\|_{*,s}^2 | \mathcal{T}] ds \right).$$

The proof of (4.4) can be completed using Gronwall's inequality.

To derive the entropy bounds, fix a finite set $A \subset \mathbb{V}$ and a time horizon $T \in (0, \infty)$. Suppose first that the tree \mathcal{T} is a.s. finite. Define a change of probability measure $\tilde{\mathbb{P}}^A$ by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}^A}{d\mathbb{P}} = \mathcal{E}_T \left(- \sum_{v \in A} \int_0^T \sigma^{-1} b(t, X_v, X_{N_v(\mathcal{T})}) \cdot dW_v(t) \right).$$

Working conditionally on the (finite) tree, we may apply Girsanov's theorem in the form of Lemma B.1, due to Assumption (A.1) and Remark B.2, to deduce that this change of measure is well defined (i.e., $d\tilde{\mathbb{P}}^A/d\mathbb{P}$ has mean 1), and the processes

$$W_v^A(t) := W_v(t) + \int_0^t \sigma^{-1} b(s, X_v, X_{N_v(\mathcal{T})}) ds, \quad v \in \mathbb{V}, \quad t \in [0, T],$$

are independent Brownian motions under $\tilde{\mathbb{P}}^A$ by Girsanov's theorem. Thus, under $\tilde{\mathbb{P}}^A$, we find that $(X_v)_{v \in A}$ satisfy the driftless SDE

$$dX_v(t) = 1_{\{v \in \mathcal{T}\}} \sigma(t, X_v) dW_v(t), \quad v \in A.$$

As this SDE is unique in law by Assumption (A.2b), we deduce that

$$\tilde{\mathbb{P}}^A \circ X_A^{-1} = \mathbb{P} \circ \hat{X}_A^{-1},$$

where $\hat{X} = (\hat{X}_v^T)_{v \in \mathbb{V}}$, is the solution to the SDE system (4.3) and we have assumed (for notational simplicity) that X and \hat{X} are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By the data processing inequality of relative entropy, we have

$$\begin{aligned} H(\mathcal{L}(X_A[T]) | \mathcal{L}(\hat{X}_A[T])) &= H(\mathbb{P} \circ X_A[T]^{-1} | \mathbb{P} \circ \hat{X}_A[T]^{-1}) \\ &= H(\mathbb{P} \circ X_A[T]^{-1} | \tilde{\mathbb{P}}^A \circ X_A[T]^{-1}) \\ &\leq H(\mathbb{P} | \tilde{\mathbb{P}}^A) = \frac{1}{2} \mathbb{E}^{\mathbb{P}} \left[\sum_{v \in A} \int_0^T |\sigma^{-1} b(t, X_v, X_{N_v(\mathcal{T})})|^2 dt \right]. \end{aligned}$$

The proof of (4.5) can be completed by using the boundedness of σ^{-1} , the linear growth of b , and the result (4.4) of the first part (possibly changing the constant). Similarly, to prove (4.6), still in the case of an a.s. finite tree \mathcal{T} , we compute

$$\begin{aligned} H(\mathcal{L}(\hat{X}_A[T]) | \mathcal{L}(X_A[T])) &= H(\mathbb{P} \circ \hat{X}_A[T]^{-1} | \mathbb{P} \circ X_A[T]^{-1}) \\ &= H(\tilde{\mathbb{P}}^A \circ X_A[T]^{-1} | \mathbb{P} \circ X_A[T]^{-1}) \\ &\leq H(\tilde{\mathbb{P}}^A | \mathbb{P}) = \frac{1}{2} \mathbb{E}^{\tilde{\mathbb{P}}^A} \left[\sum_{v \in A} \int_0^T |\sigma^{-1} b(t, X_v, X_{N_v(\mathcal{T})})|^2 dt \right]. \end{aligned}$$

The SDE system (3.1) under $\tilde{\mathbb{P}}^A$ takes the form

$$\begin{aligned} dX_v(t) &= 1_{\{v \in \mathcal{T}\}} \left(b(t, X_v, X_{N_v(\mathcal{T})}) dt + \sigma(t, X_v) dW_v^A(t) \right), \quad v \in \mathbb{V} \setminus A, \\ dX_v(t) &= 1_{\{v \in \mathcal{T}\}} \sigma(t, X_v) dW_v^A(t), \quad v \in A, \end{aligned}$$

and it is straightforward to argue that the SDE system under $\tilde{\mathbb{P}}^A$ enjoys an identical second moment bound as in (4.4). This completes the proof under the additional assumption that \mathcal{T} is a.s. finite. We prove the case of a general random tree \mathcal{T} by truncating the tree to the first n generations, $\mathcal{T}_n := \mathcal{T} \cap \mathbb{V}_n$, and deducing from above that the bounds (4.5) and (4.6) hold when \mathcal{T} is replaced with \mathcal{T}_n . The particle system $(X_v^{\mathcal{T}_n})_{v \in \mathbb{V}}$ clearly converges to $(X_v^{\mathcal{T}})_{v \in \mathbb{V}} = (X_v)_{v \in \mathbb{V}}$ in law, and the lower semicontinuity of relative entropy lets us take limits as $n \rightarrow \infty$ on both sides of (4.5) and (4.6) to show that these bounds hold for \mathcal{T} . \square

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