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CONTINUOUS FEEDBACK STABILIZATION OF NONLINEAR CONTROL SYSTEMS BY COMPOSITION OPERATORS *

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Abstract. The ability to asymptotically stabilize control systems through the use of continuous feedbacks is an important topic of control theory and applications. In this paper, we provide a complete characterization of continuous feedback stabilizability using a new approach that does not involve control Lyapunov functions. To do so, we first develop a slight generalization of feedback stabilization using composition operators and characterize continuous stabilizability in this expanded setting. Employing the obtained characterizations in the more general context, we establish relationships between continuous stabilizability in the conventional sense and in the generalized composition operator sense. This connection allows us to show that the continuous *stabilizability* of a control system is equivalent to the *stability* of an associated system formed from a local section of the vector field inducing the control system. That is, we reduce the question of continuous *stabilizability* to that of *stability*. Moreover, we provide a universal formula describing all possible continuous stabilizing feedbacks for a given system.

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1. Introduction

In this paper, we consider autonomous control systems of the form

$$\dot{x} = f(x,u),\ t \geq 0. \tag{1}$$

More specifically, we take a neighborhood of the origin $\mathcal{X} \times \mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and, unless otherwise stated, we will always assume that our function f on the right-hand side of (1) satisfies both conditions $f(0,0) = 0$ and $f \in \mathrm{C}^0(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$, where $\mathrm{C}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ denotes the n-times continuously differentiable functions $\mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ (adopting as well the convention that taking $k = -1$ denotes the set of all functions $\mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ without any assumption of continuity).

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Mostly, we are interested in *stabilizing* these systems. That is (as in, e.g., [9, Definition 10.11]), we are looking at the following property.

Definition 1.1 (Local Asymptotic Stabilizability - Feedback Laws)**.** Given a control system of type (1), we say the system is *locally asymptotically stabilizable by means of feedback laws* if there exists a neighborhood of the origin $\mathcal{O} \subseteq \mathcal{X}$ and a feedback $u\colon \mathcal{O} \to \mathcal{U}$ satisfying $u(0) = 0$ that renders the origin a locally asymptotically stable equilibrium [5, Definition 2.1] of the closed-loop system $\dot{x} = f(x, u(x))$.

For some commonly-used additional jargon, if the stability of the system in Definition 1.1 can be made exponential by such a feedback control, we say that the system is *locally exponentially stabilizable by means of feedback laws*. For clarity, note as well that the requirement that $u(0) = 0$ is a mostly-harmless simplification and is no more of an imposition than the requirement that $f(0, 0) = 0$. That is, if we are in some situation where we stabilize our system $\dot{x} = f(x, u)$ with a feedback $u(x)$ such that $u(0) = u_0 \neq 0$, then notice that we must have $f(0, u_0) = 0$. So, we could just consider the system $\tilde{f}(x, u) := f(x, u + u_0)$ instead, which certainly has $\tilde{f}(0, 0) = 0$ and is stabilized by the feedback law $\tilde{u}(x) := u(x) - u_0$ that satisfies $\tilde{u}(0) = 0$.

For brevity (and to avoid the idiosyncrasies of dealing with various non-classical notions of solutions to differential equations), when we reference local asymptotic stability throughout the paper as defined in Definition 1.1, we will *always* assume that the feedback $u(x)$ satisfies the following additional criterion:

Criterion 1.2 (Uniqueness Criterion - Feedback Stabilization)**.** *The state-feedback controller $u(x)$ is such that $f(x, u(x))$ is continuous and $\dot{x} = f(x, u(x))$ has a unique solution $x(t)$ for all $t \geq 0$ and all x_0 in a neighborhood of the origin.*

There is some variation in the literature as to whether Criterion 1.2 ought to be included as part of Definition 1.1 as, e.g., Coron [8] and Zabczyk [35] do. There seems to be no consensus here: On one hand, a number of results may be stated more cleanly when adopting this additional restriction. On the other, this assumption is somewhat unattractive in due to this restrictiveness, since it is also possible to define various other notions of solutions (e.g. Filippov and Krasovskii) which allow for non-unique trajectories and still exhibit convergence in the desired fashion. We will take extra care regarding this point. In particular, when we reference known results in Section 2 while discussing some background materials, we will note which ones would be false without this imposition. Additionally, it is important to note that Criterion 1.2 *does not* require that $u(x)$ must be continuous. That is, it is possible for the composition of a continuous function and a discontinuous function (or even two discontinuous functions) to be continuous. For example, let $g(x) = x^2$ and let $D : \mathbb{R} \to \mathbb{R}$ be such that $D(x) = 1$ if x is rational and $D(x) = -1$ if x is irrational. Then, while D is discontinuous everywhere, both $g \circ D$ and $D \circ D$ are continuous everywhere.

A variety of conditions describing whether the system (1) is locally asymptotically stabilizable by means of continuous feedback laws have been derived. As a nowhere-near-comprehensive listing, see, e.g., the control Lyapunov techniques described in [1, 3, 13, 20], the topological conditions in [4–6, 8, 9, 12], the homological conditions in [8], a standard Lyapunov-theoretic techniques derived from solutions to a Hamilton–Jacobi–Bellman equation in [15], polynomial stabilization in [16], and compilations of many others in [9, 23, 24, 31, 32]. However, fully characterizing whether or not an arbitrary system has this property has proven to be a formidable task. At present, we are not familiar with any conditions, which are simultaneously necessary and sufficient, that allow us to determine the existence or nonexistence of such a controller satisfying Criterion 1.2.

To its credit, the theory of control Lyapunov functions presents some sizable inroads to address the problem–usually in various relaxed settings compared to the traditional notions of solutions and the uniqueness requirement of Criterion 1.2, such as when one allows discontinuous feedbacks, admits various other notions of solutions (e.g. Filippov and Krasovskii), and perhaps enters the broader context of differential inclusions instead of systems (1). However, while these other questions may admit clean characterizations in their own respective contexts (e.g., [1, 7, 13, 20, 28]), the question of stabilizability by continuous feedback in the classical sense remains unsolved by control-Lyapunovian methods. The deficiencies of the control Lyapunov approach in this setting are primarily due to the substantial departure control Lyapunov functions take from the characterization of stability that classical Lyapunov functions enable for non-control systems. As, e.g., Clarke observes in [7], while the existence of a continuous feedback $u(x)$ rendering (1) locally asymptotically stable *does* imply the existence of a smooth control Lyapunov function, the existence of a smooth control Lyapunov function *does not* necessarily imply the existence of such a continuous stabilizing feedback for general systems (1) (though this does hold for certain restricted classes, such as input-affine systems [1]). For example, the system (1) with $f(x,u) = \begin{bmatrix} u_2u_3 & u_1u_3 & u_1u_2 \end{bmatrix}^T$ has $V(x) = \|x\|^2$ as a smooth control Lyapunov function, but is not stabilizable by continuous feedback laws [7, 21]. Accordingly, some refinement of this approach–or some other approach entirely–is needed to resolve the classically-posed form of the stabilizability problem.

In this paper, we develop such an alternate approach and finally resolve this long-standing problem, fully characterizing the classical feedback stabilizability property.

1.1. **Entering the New Approach**

As discussed above, stabilization via feedback satisfying Criterion 1.2 has been recognized as a difficult problem in control theory. One reason for why it might be is that the topic itself could be difficult. That is, information regarding stabilization could, in a colloquial sense, be 'buried' too deep or 'tangled up' too well in the structure of a system to be extracted with ease. Another option is that we've just been phrasing the question in an unfortunate manner and have made things *seem* much more difficult than necessary. One way the latter option might occur, as argued here, is that we opted to ask one particularly messy question instead of asking two much simpler questions that, when combined, produce the original messy one.

More particularly, when we apply a feedback law to a control system of form (1), all we're doing is sending $f(x,u)$ to $f(x,u(x))$. At the end of the day, this amounts to applying a *composition operator* T_h to the vector field f inducing the dynamics of the system, i.e., $T_hf = f \circ h$ for some mapping $h\colon \mathcal{X} \to \mathcal{X} \times \mathcal{U}$. A fairly large amount is known about composition operators, so it seems like the real sticking point in the case of stability is that we require the generating mapping h of our composition operator to be of a *very specific form.* Namely, $h(x) = (x, u(x))$ for some state-feedback control $u(x)$.

Of course, relaxing this condition would change the problem entirely, and answering a different problem instead of the one originally asked doesn't constitute a solution by any means. But, all the same, suppose one *did* decide to do away with the restriction that the mapping h in our composition must be of the form $h(x) = (x, u(x))$. Then, if one were to solve the problem of stability with the very general class of $h(x)$, certainly every necessary condition for the broader case would also be necessary for the particular case of the original question. Indeed, if we knew that a system was stabilizable by some composition operators T_h, all that would then remain is a question about the nature of those stabilizing composition operators–i.e., when can we choose a stabilizing composition operator T_h to have a symbol of the form $h(x) = (x, u(x))$?

Broadly speaking, this is what we will attempt to do here. That is, our approach to stabilizability by feedback will consist of asking the following two questions:

Problem 1.3. Under what conditions does there exist a composition operator T_h, with a symbol h of some desired smoothness, such that the origin of the "closed-loop" system $\dot{x} = T_hf(x)$ is locally asymptotically stable?

Problem 1.4. Given the existence of *some* stabilizing composition operators T_h with a symbol h of the required smoothness so as to resolve Problem 1.3, does there exist at least one that is a control?

As we will show (Theorem 3.2 and Theorem 5.7), the continuous and continuously differentiable formulations of Problem 1.3 are answered in the affirmative when f has a local section α of the desired smoothness at the equilibrium (see definition 1.8). Likewise, we will show (Theorem 6.4) that the corresponding continuous and continuously differentiable formulations of Problem 1.4 are answered in the affirmative when the projection onto its first factor of this local section, i.e. $\mathrm{proj}_1 \circ \alpha$ where $\mathrm{proj}_1(x,u) = x$, induces a diffeo/homeomorphism whose inverse, itself, yields a stable system $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$. Moreover, the answer to these versions of Problem 1.4 will tell us how to produce such a stabilizing control (indeed, all possible continuous or continuously differentiable stabilizing controls) from a local section satisfying the required criteria.

1.2. **Composition Stabilizability**

For precision, we require some preliminary definitions before continuing.

Definition 1.5 (Local Asymptotic Stabilizabilty - Composition Operators)**.** Considering the control system (1) with $f \in \mathrm{C}(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ satisfying $f(0,0) = 0$, we say the system is *locally asymptotically stabilizable by means of a composition operator* if there exists a neighborhood of the origin $\mathcal{O} \subseteq \mathcal{X}$ and a composition operator T_h with symbol $h\colon \mathcal{O} \to \mathcal{X} \times \mathcal{U}$ satisfying $h(0) = (0,0)$ that renders the origin a locally asymptotically stable equilibrium of the system $\dot{x} = T_h f(x)$.

If such a stabilizing composition operator T_h can be chosen with a symbol $h \in \mathrm{C}^k(\mathcal{O}, \mathcal{X} \times \mathcal{U})$ as $-1 \leq k \leq \infty$, we say that the system (1) is *locally asymptotically stabilizable by means of a composition operator with a* C^k *symbol*. As before with controls, note that our assumption that $h(0) = (0,0)$ is again a harmless simplification. Additionally, as with the case of stabilizing feedbacks as well, we make an analogous uniqueness assumption to that of Criterion 1.2.

Criterion 1.6 (Uniqueness Criterion - Composition Stabilization)**.** *The composition operator T_h is such that $T_h f$ is continuous and $\dot{x} = T_h f(x)$ has a unique solution $x(t)$ for all $t \geq 0$ and all x_0 in a neighborhood of the origin.*

The stabilizability of a system by composition operators is properly a weaker property than the stabilizability of a system by feedback laws, as it allows for compositions in the *state space domain* as well as the control domain. More succinctly, we are just treating *everything* as a control in this generalization. To highlight it, let's look at the following straightforward example.

Example 1.7. Let $f(x,u) := x$. Since the application of any control has no effect on the dynamics, the corresponding system (1) is not locally asymptotically stabilizable by means of feedback laws. However, (1) *is* locally asymptotically stabilizable by means of a composition operator (in fact, by one with a continuous symbol). Indeed, take, e.g., $h(x) := (-x, u(x))$ for any continuous feedback law $u(x)$. Then the system

$$\dot{x} = (T_h f)(x) = f(h(x)) = -x$$

has a locally asymptotically stable equilibrium at the origin. In fact, it is easy to see that the origin is a globally exponentially stable equilibrium.

It is also reasonable to consider stabilizability by composition operators as a mere subcase of stabilizability by feedback laws, since treating *everything* as a control is really the same thing as considering local asymptotic stabilizability for the restrictive subclass of systems which *only* depend on controls (i.e., systems of form (1) that depend trivially on the state variable).

In this paper, we will show that this natural (albeit, much weaker) generalization of stabilizability (equivalently, conventional stabilizability for the restrictive subclass of systems) has some interesting connections to the problem of stabilizability by feedback laws given in Definition 1.1. To do so, we will build up this approach and detail how many of the classical theorems on stabilizability have much cleaner forms in the composition operator context. We will show that extracting properties about the composition operators which stabilize a given system of form (1) tends to be, in general, relatively easy. Additionally, we will make a case for the position that this is the correct approach to view the problem of feedback stabilizability. In support of this stance, we will completely resolve the continuous and continuously differentiable formulations of Problem 1.4 and, to some extent, the continuous and continuously differentiable formulations of Problem 1.3.

1.3. **Main Results of the Paper**

The contribution of this paper is to reduce the question of *stabilizability* to that of *stability*. That is, we will take the broader class of control systems of the form (1) and reduce the question of their *stabilizability* to the question of the *stability* of an associated system. This departs substantially from the existing control-Lyapunov-theoretic approaches to the problem of stabilizability for general nonlinear systems (1)–wherein one must determine the existence or nonexistence of a so-called control Lyapunov function, though the existence of smooth control Lyapunov function does not necessarily guarantee the existence of a stabilizing feedback in the setting of classical solutions to (1) and continuous feedbacks satisfying Criterion 1.2. Instead of this incomplete approach, we present a complete characterization of stabilizability that requires no such control-Lyapunov-theoretic tools and can be stated entirely in terms of the stability of a system derived from a local section of the vector field f in (1). Further, this does not only provide a characterization of local exponential and asymptotic stabilizability by feedback laws, but also produces a characterization of all possible stabilizing feedback laws. More interestingly perhaps, this approach provides a somewhat unexpected link between the topic of stabilization and the existence of local sections.

To allow for concise wording, we first need a definition:

Definition 1.8 (Sections, Local Sections)**.** Given topological spaces X, Y and a function $F\colon X \to Y$, a *global section* of F is a continuous right inverse of F. That is, a mapping $\sigma\colon Y \to X$ is a section of F if σ is continuous and $F \circ \sigma = \mathrm{id}_Y$. A *local section* of F is a continuous mapping $\sigma\colon \mathcal{O} \to X$ for some open set $\mathcal{O} \subseteq Y$ such that $F \circ \sigma = \iota \circ \mathrm{id}_{\mathcal{O}}$, where ι denotes the inclusion map $\iota\colon \mathcal{O} \to Y$. If $x_0 \in X$, we say that F has a *local section at* x_0 if there exists a local section $\sigma : \mathcal{O} \to X$ of F such that $\mathcal{O}$ is an open neighborhood of $F(x_0) = y_0$ and $\sigma(y_0) = x_0$.

For the remainder of this subsection, we will *always* assume that Criterion 1.2 and Criterion 1.6 hold whenever we reference stabilizability by feedback laws or stabilizability by composition operators. Our results are as follows:

First, we address the case of local asymptotic stabilizability by composition operators. We begin by addressing the case of local *exponential* stabilizability of smooth systems by composition operators with smooth symbols.

Theorem 3.2 *The system* (1) *with* $f \in \mathrm{C}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ *for* $k \geq 1$ *is locally exponentially stabilizable by means of a composition operator* T_h *with a* C^k *symbol* h *if and only if* f *has a* C^k *local section at the origin.*

For stabilization by composition operators with merely continuous symbols, the case is more interesting.

Theorem 5.7 *The system* (1) *is locally asymptotically stabilizable by means of a composition operator* T_h *with a continuous symbol* h *if and only if there exists a local section of* f *at the origin. Moreover, such a composition operator* T_h *can always be chosen so that this stability is exponential.*

This produces an interesting gap, which is not resolved here. Namely, exponential and asymptotic local stabilization by composition operators with continuous symbols coincide, while they may not in the case where $k > 0$. In particular, for local asymptotic stabilizability by a composition operator with an C^k symbol to be possible when local exponential stabilizability by the same is not, this requires that all of the local sections at the origin of the vector field f in (1) must fail to be differentiable. However, while this is necessary, it is unclear what conditions are sufficient to provide the existence of such a stabilizing composition operator.

We make a connection to Brockett's theorem as well, providing a variety of composition-operator stabilization characterizations of this well-known property. In particular, while Brockett's condition is necessary but not sufficient for stabilization by composition operators with continuous symbols, it turns out (somewhat shockingly) to be *sufficient* in the discontinuous case:

Theorem 4.5 *If the vector field* f *in* (1) *satisfies Brockett's condition, then* (1) *is locally exponentially stabilizable by means of a composition operator (potentially, with a discontinuous symbol).*

We do not know whether Brockett's condition is also necessary for stabilization by composition operators with discontinuous symbols, though we conjecture that it is. In support of this, we show that Brockett's condition is necessary and sufficient for a certain restricted class of systems.

Corollary 4.12 *Suppose the origin is an isolated zero of* f *in* (1)*. Then,* (1) *is locally asymptotically stabilizable by means of a composition operator* T_h *with a symbol which is continuous at the origin if and only if the vector field* f *in* (1) *satisfies Brockett's condition.*

Using the tools developed for the case of stabilization by composition operators, we are able to give a necessary and sufficient condition for the existence of a stabilizing feedback laws as well. Unlike the case of composition operators, this result leaves no remaining gaps.

Theorem 6.4 *For $k \geq 0$, the system* (1) *is locally asymptotically (resp. exponentially) stabilizable by means of a* C^k *feedback law u if and only if there exists a* C^k *local section α of f at the origin such that $proj_1 \circ \alpha$ is a homeomorphism (resp. diffeomorphism) and $\dot{x} = (proj_1 \circ \alpha)^{-1}(x)$ has the origin as a locally asymptotically (resp. exponentially) stable equilibrium. Moreover, every such control u is of the form $u = proj_2 \circ \alpha \circ (proj_1 \circ \alpha)^{-1}$ for some α satisfying the above.*

2. Preliminaries

In this section, we begin by briefly recalling some known results regarding stabilization by feedback laws and detailing a short taxonomy of the obstacles remaining between what is known and a complete characterization of local asymptotic stabilizability.

2.1. Hautus Lemma and Related Results

As mentioned previously, when we reference local asymptotic stability throughout the paper, we will *always* assume that the feedback $u(x)$ satisfies Criterion 1.2. In this section, however, we will be careful to also note which ones would be false without this imposition.

As mentioned previously, a large number of partial characterizations of stabilizability do exist, and some of the more useful elementary ones are worth mentioning quickly. First, let us briefly recall some conventional notation. Given a mapping $f \in \mathrm{C}^1(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ as in (1), we denote the Jacobian of f by J_f and the partial Jacobian matrices of f at $(0,0)$ by

$$A_f := \left.\frac{\partial f}{\partial x}\right|_{(0,0)} \text{ and } B_f := \left.\frac{\partial f}{\partial u}\right|_{(0,0)}. \tag{2}$$

e.g. so that, as a block matrix, $J_f|_{(0,0)} = \begin{bmatrix} A_f & B_f \end{bmatrix}$. Given a linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$, denote its *spectrum* by $\Lambda(T)$ and the subset of $\Lambda(T)$ consisting of the eigenvalues with *nonnegative real part* by

$$\Lambda_+(T) := \{\lambda \in \Lambda(T) \mid \mathrm{Re}(\lambda) \geq 0\}.$$

Next we recount the celebrated Hautus lemma needed below.

Lemma 2.1 (Hautus)**.** *Given an $n \times n$ matrix A and an $n \times m$ matrix B, the linear system $\dot{x} = Ax + Bu$ is locally exponentially stabilizable if and only if for all $\lambda \in \Lambda_+(A)$ it holds that*

$$rank \left[\begin{array}{c|c} \lambda I - A & B \end{array} \right] = n.$$

There is a similar result to the Hautus lemma, which applies to the linearization of a system like that given in (1). That is, using the notation of (2), it applies to the linearized system

$$\dot{x} = A_f x + B_f u, \ \ t \geq 0. \tag{3}$$

This nonlinear analogue of the Hautus lemma via linearization was first proved by Zabczyk in [35], though he mentioned that it was likely known as folklore for some time before a published proof was available.

Theorem 2.2 (Zabczyk)**.** *The control system* (1) *with $f \in \mathrm{C}^1(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ is locally exponentially stabilizable by means of* C^1 *feedback laws if and only if the linearized system* (3) *is locally exponentially stabilizable. Moreover, if the linearized system* (3) *is locally exponentially stabilizable, then the linear feedback law stabilizing* (3) *is a locally exponentially stabilizing feedback law for the original system* (1).

Combining Hautus' lemma and Zabczyk's theorem yields the following consequence.

Corollary 2.3 (Hautus-Zabczyk)**.** *The control system* (1) *with* $f \in \mathrm{C}^1(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ *is locally exponentially stabilizable by means of* C^1 *feedback laws if and only if we have for all* $\lambda \in \Lambda_+(A_f)$ *that*

$$\mathit{rank} \left[\ \lambda I - A_f \ \middle| \ B_f \ \right] = n.$$

The most famous result regarding asymptotic stabilizability is probably a remarkable and easily formulated necessary condition given by Brockett [4]. We discuss it in the next subsection.

2.2. **Brockett's Theorem**

Brockett gave in [4] a necessary condition for feedback asymptotic stabilizability of nonlinear systems, which has attained a great attention in control theory. This condition constitutes (with the proof based on topological degree theory) that f must satisfy a certain 'local openness' property if such a controller exists.

Definition 2.4 (Locally Open)**.** A mapping $f\colon \mathbb{R}^\ell \to \mathbb{R}^n$ is said to be *open at a point* $\bar{z} \in \mathbb{R}^\ell$ if we have $f(\bar{z}) \in \mathrm{int} f(\mathcal{O})$ for any neighborhood $\mathcal{O}$ of $\bar{z}$.

The origin of this property goes back to the classical Banach-Schauder open mapping theorem, which is one of the central results of functional analysis. We give it in the form of a well-known extension of its classical statement from smooth to continuous system by means of continuous vs. smooth feedback laws is given by Coron [8] with the usage of a result by Zabczyk from [35].

Theorem 2.5 (Brockett's Theorem - Continuous Extension)**.** *If the system* (1) *is locally asymptotically stabilizable by means of continuous feedback laws satisfying Criterion* 1.2*, then it is necessary that the mapping* f *is open at* $(0, 0)$.

Stated a bit differently, Brockett's theorem says that any system (1), which is asymptotically stabilizable by means of continuously differentiable feedback laws, has a solution to $f(x, u) = y$ for all $\|y\|$ sufficiently small. More compactly, Brockett's condition says that such a system must be 'locally surjective' on any neighborhood of the origin, i.e., f is surjective onto a neighborhood of the origin in $\mathbb{R}^n$ when restricted to some appropriate neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}^m$. For intuition, the 'archetypal' example of a mapping that satisfies this condition would be–in the spirit of the inverse function theorem–any such f that has a Jacobian of full row-rank at the origin. To contrast, while having a full row-rank Jacobian at the origin is certainly *sufficient*, it is definitely not *necessary*; take, for example, $f(x) := -x^3$.

As a small note, observe that it is *not* known whether or not Brockett's condition is required for local asymptotic stabilizability in the more general case of merely continuous feedback laws *without* the assumption of Criterion 1.2 regarding the uniqueness of locally defined solutions. That being said, it is worth mentioning (as Sontag notes in [31, Section 5.8, 5.9] and [33, p. 554]) that continuous feedback laws can often be 'smoothed out' away from the origin. Thus, combined with Sontag's proof in [33, p. 560] that stabilization via continuous feedback laws which are locally Lipschitz away from the origin yields Brockett's condition (without making the assumption of uniqueness of trajectories), this does lend some support to the *conjecture* that Brockett's theorem may be necessary in the general case of Definition 1.1 sans the strong requirement of unique trajectories, as evidenced by partial results to this effect such as, e.g., [27].

Strong attempts have been made to extend Brockett's theorem (and results like it) in some or another form and to 'close the gap' between necessity and sufficiency when it comes to characterizing stabilizable systems. There are a large number of results like that, accounting for a similarly large number of relatively-involved distinctions regarding the continuity and differentiability classes of stabilizing controls–for example, by Coron in [9], by Sontag in [31,33], by Zabczyk in [35], and many others. For an easy reference on a fairly large portion of the subject (and a truly excellent treatment of the topic overall), Byrnes [5] provides nice, streamlined proofs of both Coron's and Zabczyk's results among other developments.

To conclude this short overview of known results in the area particularly related to Brockett's theorem, let us mention a new *variational approach* to feedback stabilizability of nonlinear control systems that was initiated in [12] and then further developed in our paper [6]. Though variational analytic connections will not be our focus here, we should note that some results in [6] were reached by arguments involving the use of composition operators. Particularly, [6, Theorem 9] gives a condition (less restrictive than what is required by Corollary 2.3) which ensures the existence of stabilizing controls which are C^1 on only a *punctured* neighborhood of the origin (that is, $\mathcal{N} \setminus \{(0,0)\}$ for some neighborhood of the origin $\mathcal{N}$). This extends results like Corollary 2.3 to cover some instances of the case where locally asymptotically stabilization of (1) is only possible by *continuous* (and not C^1), feedback laws. The techniques employed in [6] seem to suggest that further applications of composition operators–utilized more adroitly–might allow for further progress in this direction. This constitutes, in large part, the motivation for our approach herein.

2.3. **Outline**

The rest of the paper is organized as follows. Section 3 is devoted to developing the composition counterpart of the extended Hautus-Zabczyk result in the novel context. In Section 4, we derive new results for composition operators under Brockett's openness condition to familiarize ourselves with the notion of composition stabilization. In particular, Subsection 4.1 establishes some subtle properties of local quotient maps under the openness condition. In Subsection 4.2 we show that Brockett's theorem is still necessary for the composition generalization of stabilizability and, in some cases, sufficient. In Section 5, we solve the problem of exponential stabilization by means of composition operators. In the process, we will also solve a portion of the problem of asymptotic stabilization by composition operators.

In Section 6, we return to the case of stabilization via feedback laws and leverage what we have learned about stabilization via composition operators in the previous sections. We produce a characterization of local asymptotic and exponential stabilizability by feedbacks, also characterizing what stabilizing controls must be like. The conclusion, Section 7, summarizes the main developments of the paper and discusses some open problems for future research.

3. A Generalized Hautus Lemma for Composition Operators

Let's begin by seeing how a well-known feedback stabilizability result carries over to the composition stabilizability setting. In the spirit of the Hautus-Zabczyk result (i.e., Corollary 2.3) regarding the local exponential stabilizability of a system by C^k feedback laws, we establish an analogous criterion for the local exponential stabilizability of a system by means of composition operators with a C^k symbol.

The next result in this subsection was proven previously by the authors in [6] (see, e.g. [6, Lemma 5]), although the context was not so explicitly stated. We opt to include a modified (and much simpler) version of the proof here.

Corollary 3.1 (Hautus-Zabczyk - Composition Stabilizability)**.** *The system* (1) *with* $f \in \mathrm{C}^1(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ *is locally exponentially stabilizable by a composition operator with a* C^1 *symbol if and only if*

$$\mathit{rank}\left(J_f|_{(0,0)}\right) = n. \tag{4}$$

Proof. Writing $w = (x, u)$, consider the system $\dot{y} = F(y, w) := f(w)$ with trivial dependence on the state variable y. Clearly, stabilization of this system by feedback laws is equivalent to the stabilization of (1) by composition operators. As such, Corollary 2.3 gives the result. □

This result isn't surprising at all–it is just a sub-case of that given in Corollary 2.3. Indeed, the condition (4) must happen in the case of stabilization by feedback laws as well: suppose that (1) is locally exponentially stabilizable by means of continuously differentiable feedback laws. Then there are two possibilities: either A_f is full rank, or it is not. In the first case, if A_f is of full rank, then $J_f|_{(0,0)}$ certainly has full row-rank. In the second case, if A_f is not of full rank, then $\lambda = 0$ is certainly an eigenvalue of A_f with nonnegative real part, and thus the Hautus lemma dictates that we have

$$\operatorname{rank}\left[\begin{array}{c|c} \lambda I - A_f & B_f \end{array}\right] = \operatorname{rank}\left(J_f|_{(0,0)}\right) = n.$$

The result of Corollary 3.1 serves as good motivation for what we will proceed to do in much of the remainder of the paper. Namely, notice that if (4) holds, then $J_f|_{(0,0)}$ has a *continuous right inverse*–namely, the Moore-Penrose pseudoinverse $\left(J_f|_{(0,0)}\right)^+$. In fact, this holds more generally.

In the following theorem, we show that the existence of a C^k local section at zero is equivalent to exponential stabilizability of (1) by means of a C^k composition operator.

Theorem 3.2 (Composition Stabilizability and Local Sections - C^k Version)**.** *The system* (1) *with* $f \in \mathrm{C}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$*, where* $k \geq 1$*, is locally exponentially stabilizable by a composition operator with a* C^k *symbol if and only if* f *has a* C^k *local section at the origin.*

Proof. The reverse direction is obvious: Suppose f has a C^k local section at the origin $\alpha : \mathcal{O} \to \mathcal{X}$. Take any system $\dot{x} = g(x)$ for which the origin is a locally exponentially stable equilibrium and g is C^k. Let $\iota : \mathcal{O} \to \mathbb{R}^n$ denote the inclusion map and let $\tilde{g} : \mathcal{N} \to \mathcal{O}$ be such that $\iota \circ \tilde{g} = g|_{\mathcal{N}}$ for some neighborhood of the origin $\mathcal{N}$ (i.e. so that $g(\mathcal{N}) \subseteq \mathcal{O}$, as is always possible by continuity). Choosing T_h to be the composition operator with symbol $h = \alpha \circ \tilde{g}$ so that

$$T_h f = f \circ \alpha \circ \tilde{g} = \iota \circ \tilde{g} = g|_{\mathcal{N}},$$

it is clear that T_h locally exponentially stabilizes (1).

In the forward direction, assume that the system (1) is locally exponentially stabilizable by means of a composition operator T_h with a C^k symbol. Then, the system $\dot{x} = T_h f(x)$ is locally exponentially stabilizable by means of the composition operator T_x, which certainly has a C^1 symbol. By Corollary 3.1, $J_{T_h f}|_0$ has full row-rank and the function $F(x, y) := T_h f(y) - x$ satisfies the conditions of the Implicit Function Theorem. So, there exists an open neighborhood of the origin $\mathcal{O} \subseteq \mathbb{R}^n$ and a C^k function $g \colon \mathcal{O} \to \mathbb{R}^n$ such that $g(0) = 0$ and $F(x, g(x)) = 0$ for all $x \in \mathcal{O}$. Correspondingly, the function $\alpha \colon \mathcal{O} \to \mathcal{X} \times \mathcal{U}$ defined by $\alpha := h \circ g$ is C^k and satisfies $(f \circ \alpha)(x) = x$ for all $x \in \mathcal{O}$. That is, α is a C^k local section of f at the origin. □

The central thesis of this paper's approach to stabilization by composition operators is largely characterized by the result above. That is, in the remainder of the composition-operator-focused portion of this paper, we will show that Theorem 3.2 generalizes to characterize exponential and asymptotic stabilizability by means of composition operators in the continuous case. To this aim, we will begin by showing that Brockett's theorem holds in the context of stabilization by composition operators in the next section. After some prerequisite details are dealt with following this, we will show that the obvious non-differentiable analog of the result of Theorem 3.2 holds as well.

4. Extended Brockett Theorem and Composition Properties under Openness

This section studies the role of the openness property of vector fields of control systems in stabilizability of such systems by means of composition operators. We begin in Subsection 4.1 by first investigating the openness property of Brockett's theorem, yielding a connection to quotient maps that will be important throughout the remainder of this paper. After this, in Subsection 4.2, we show that Brockett's openness property is not only necessary, but is also sufficient, for stabilization by composition operators with symbols that are continuous at the origin provided the origin is an isolated zero of f in (1). For systems with the origin as non-isolated zero, we conjecture that the same holds as well.

4.1. **Openness Condition and Local Quotient Maps**

The first lemma here showcases a useful fact about continuous surjections between arbitrary topological spaces satisfying the property given in Definition 2.4 for some point x_0 in their domain.

Lemma 4.1. *Let X and Y be topological spaces. Fix $x_0 \in X$ and suppose that $F\colon X \to Y$ is a continuous surjection which is open at x_0. Then, for any topological space Z and any $g\colon Y \to Z$, the mapping g is continuous at $F(x_0)$ if and only if the composition $g \circ F$ is continuous at x_0.*

Proof. In the forward direction, suppose that g is continuous at $F(x_0)$. The continuity of the composition $g \circ F\colon X \to Z$ at x_0 follows immediately, since continuity is preserved under compositions. That is, if $Z_0 \subseteq Z$ is a neighborhood of $g(F(x_0))$, then the continuity of g at $F(x_0)$ ensures that $g^{-1}(Z_0)$ is a neighborhood of $F(x_0)$. Using the continuity of F at x_0 tells us that the inverse image $F^{-1}(g^{-1}(Z_0))$ is a neighborhood of x_0.

In the reverse direction, suppose that $g \circ F$ is continuous at x_0. Then, for any neighborhood Z_0 of $g(F(x_0))$, there exists a neighborhood $\mathcal{O} \subseteq X$ of x_0 such that $\mathcal{O} \subseteq (g \circ F)^{-1}(Z_0)$. Observing that

$$\mathcal{O} \subseteq (g \circ F)^{-1}(Z_0) = F^{-1}(g^{-1}(Z_0))$$

and employing the surjectivity of F imply that $F(F^{-1}(S)) = S$ for all subsets $S \subseteq X$. Therefore, it yields

$$F(\mathcal{O}) \subseteq F(F^{-1}(g^{-1}(Z_0))) = g^{-1}(Z_0).$$

Since F is open at x_0 and $\mathcal{O}$ is a neighborhood of x_0, we get that $F(\mathcal{O})$ is a neighborhood of $F(x_0)$. Remembering that Z_0 was an arbitrarily chosen neighborhood of $g(F(x_0))$, the continuity of g at $F(x_0)$ follows. □

Lemma 4.1 has an interesting interpretation in terms of *quotient maps*.

Definition 4.2. Let X and Y be topological spaces. We say that a continuous function $f : X \to Y$ is a quotient map if f is surjective and $f^{-1}(U) \subseteq X$ is open if and only if $U \subseteq Y$ is open

Recall that quotient maps are characterized among continuous surjections as follows; see, e.g., [34]:

Theorem 4.3. *Let $q\colon X \to Y$ be a continuous surjection. The following are equivalent:*

(i) *q is a quotient map.*
(ii) *For any topological space Z and any mapping $g\colon Y \to Z$, we have that g is continuous if and only if the composition $g \circ q$ is continuous.*

Similarly, to show that a continuous surjection q is a quotient map, recall that it is sufficient (though not necessary) to verify that q is either an *open map* or a *closed map*. That is, if a continuous surjection is to be a quotient map, it is sufficient that it is open at *every point in its domain*. Essentially, it is the *global* analogue to the *local* version given in the assumptions of Lemma 4.1.

In fact, the similarities between this and what we have in the Definition 2.4 are really quite apparent. Namely, instead of a *continuous surjection* which is *open at every point*, we have a continuous surjection that is *open at a selected point*. In this sense, a continuous surjection which satisfies Definition 2.4 must be something like a *local* quotient map, i.e., a quotient map *at a point*. In light of the characteristic *global* property of quotient maps (i.e., Theorem 4.3), it's not surprising that the *local analogue* of a quotient map induces a *local* analogue of this characteristic property (i.e., Lemma 4.1).

As it turns out, we can go further in some situations. Recall that we say a map $F\colon X \to Y$ is *proper* if $F^{-1}(K)$ is compact for every compact set $K \subseteq Y$. Similarly, we say that a topological space X is *locally compact* if every point $x \in X$ has a compact neighborhood. As is well-known, if X is Hausdorff, this is equivalent to the statement that every neighborhood N of a point $x \in X$ contains a compact neighborhood K of x. Finally, recall, as mentioned previously following Theorem 4.3, that if one desires to show that a mapping $F\colon X \to Y$ is a quotient map, then it is sufficient to show that F is a closed, continuous surjection.

Lemma 4.4. *Let X and Y be topological spaces and suppose that $F\colon X \to Y$ is continuous, $F(x_0) = y_0$, and F is open at $x_0 \in X$. Then there exist an open neighborhood $Y_0 \subseteq Y$ of y_0, an open neighborhood $X_0 \subseteq X$ of x_0, and a continuous surjection $\tilde{F}\colon X_0 \to Y_0$ which is open at x_0 and satisfies $\iota \circ \tilde{F} = F|_{X_0}$, where $\iota\colon Y_0 \to Y$ denotes the inclusion map.*

If, in addition, both X and Y are locally compact and Y is Hausdorff, then we can instead take X_0 and Y_0 to both be compact and such that $\tilde{F}$ is a closed quotient map.

Proof. Since F is open at $x_0 \in X$, it follows that $F(x_0) \in \mathrm{int}F(X_0)$ for any neighborhood $X_0 \subseteq X$ of x_0. In particular, by $x_0 \in \mathrm{int}(X)$ we have $F(x_0) \in \mathrm{int}F(X)$. It allows us to choose an open neighborhood Y_0 of $F(x_0) = y_0 \in Y$ such that $Y_0 \subseteq F(X)$. Write $X_0 := F^{-1}(Y_0)$ and take the inclusion map $\iota\colon Y_0 \to Y$. Then X_0 is an open neighborhood of x_0 by the continuity of F, and the mapping $\tilde{F}\colon X_0 \to Y_0$ defined by $\iota \circ \tilde{F} := F|_{X_0}$ is a surjection by construction. Since $\tilde{F}$ agrees with F on X_0, the claimed openness and continuity follow as well.

Now, let us also assume that X is locally compact and that Y is locally compact and Hausdorff. Note that by the local compactness of X, there must exist a compact neighborhood $K \subseteq X$ of x_0. If Y is Hausdorff, then $F|_K : K \to Y$ is a continuous map from a compact space to a Hausdorff space. By the Closed Map Lemma, $F|_K$ is therefore both proper and closed. By the openness of F at x_0, we have that $F(x_0) \in \mathrm{int}F(K)$, and there exists a neighborhood of $F(x_0)$ contained in $F(K)$. As we have assumed that Y is both locally compact and Hausdorff, we may take this to be a compact neighborhood $Y_0 \subseteq F(K)$ of $F(x_0)$. Since $F|_K$ is proper and continuous, $X_0 := F|_K^{-1}(Y_0) = K \cap F^{-1}(Y_0) \subseteq K$ is a compact neighborhood of x_0. Define the mapping $\tilde{F}\colon X_0 \to Y_0$ by $\iota \circ \tilde{F} = (F|_K)|_{X_0} = F|_{X_0}$, where $\iota\colon Y_0 \to Y$ is again the inclusion map. Noting that a subspace of a Hausdorff space is itself a Hausdorff space, Y_0 is Hausdorff and the mapping $\tilde{F}$ is continuous map from a compact space to a Hausdorff space and a surjection by construction. Hence, via another application of the Closed Map Lemma, $\tilde{F}$ is a proper, closed, continuous surjection and therefore, a closed quotient map. □

Every function factors as a surjection (onto its image) and an injection (of the image into the codomain), so a factorization of this form is nothing surprising. What Lemma 4.4 really tells us, in essence, is that systems (1) satisfying Brockett's necessary condition factor in this way, but with *neighborhoods of the origin* as the respective domain and codomain of the corresponding surjection and inclusion. Moreover, systems (1) that satisfy Brockett's condition do not just have a *local analogue* to the characteristic property of quotient maps, but factor as inclusions and quotient maps *themselves* on some suitable restriction of their domain.

Recall that any surjection $F\colon X \to Y$ necessarily has a right inverse $\alpha\colon Y \to X$, i.e. a mapping α such that $F \circ \alpha = \mathrm{id}_Y$. This gives us a nice, albeit obvious, result. For clarity, there is *no assumption* on the continuity of the symbol of the stabilizing composition operator in what follows.

Theorem 4.5 (Discontinuous Composition Stabilizability). *If the vector field f in* (1) *satisfies Brockett's condition, then* (1) *is locally exponentially stabilizable by means of a composition operator.*

Proof. If (1) satisfies the necessary condition of Brockett's theorem, then f is open at $(0, 0)$. As such, Lemma 4.4 implies that there exists a neighborhood of the origin $X_0 \subseteq \mathcal{X} \times \mathcal{U}$ and a neighborhood of the origin $Y_0 \subseteq \mathbb{R}^n$ such that the mapping $\tilde{f}\colon X_0 \to Y_0$, defined by $\iota_1 \circ \tilde{f} = f|_{X_0}$ via the inclusion map $\iota_1 : Y_0 \to \mathbb{R}^n$, is a surjection. Hence, $\tilde{f}$ has a right inverse $\alpha\colon Y_0 \to X_0$, and as $f|_{X_0}(0, 0) = 0$, we may certainly choose this right inverse so that it satisfies $\alpha(0) = (0, 0)$. Take any system $\dot{x} = g(x)$ for which the origin is a locally exponentially stable equilibrium and g is continuous. Let $\tilde{g} : \mathcal{N} \to Y_0$ be such that $\iota_1 \circ \tilde{g} = g|_{\mathcal{N}}$ for some neighborhood of the origin $\mathcal{N}$ (i.e. so that $g(\mathcal{N}) \subseteq Y_0$, as is always possible by continuity). Set $h = \iota_2 \circ \alpha \circ \tilde{g}$, where $\iota_2 : X_0 \to X$ is the inclusion map, and notice that, since $\iota_1 \circ \tilde{f} = f|_{X_0}$ and $f \circ \iota_2 = f|_{X_0}$, we have

$$\begin{aligned} T_h f &= f \circ \iota_2 \circ \alpha \circ \tilde{g} \\ &= f|_{X_0} \circ \alpha \circ \tilde{g} \\ &= \iota_1 \circ \tilde{f} \circ \alpha \circ \tilde{g} = \iota_1 \circ \tilde{g} = g|_{\mathcal{N}}. \end{aligned}$$

Since $\dot{x} = g(x)$ has the origin as a locally exponentially stable equilibrium and satisfies $g(0) = 0$, so does $\dot{x} = g|_{\mathcal{N}}(x)$. It then follows that T_h is a composition operator which locally exponentially stabilizes (1). □

To reiterate, we stress that there is *absolutely no assurance of continuity* for the symbol h of the stabilizing composition operator T_h. Instead, Theorem 4.5 simply highlights the fact that, while stabilization by composition operators is remarkably easy to obtain in the presence of Brockett's condition (on a theoretical level, at least), the real trick is obtaining *continuity* for the stabilizing composition operator's symbol. We will proceed towards this end with some more characterizations of systems satisfying Brockett's condition and the potential continuity of their restrictions' right inverses.

Recall now that, given a mapping $F\colon X \to Y$, we say a subset $S \subseteq X$ is *F-saturated* if $S = F^{-1}(F(S))$. As illustrated by the following lemma (the proof of which can be found in, e.g., [29, Lemma 2.3]), F-saturated subsets of closed mappings behave nicely.

Lemma 4.6. *Let X and Y be topological spaces, let $F\colon X \to Y$ be a closed map, and assume $\mathcal{M} \subseteq X$ is closed and F-saturated. Then, for every open set $\mathcal{O} \subseteq X$ such that $\mathcal{M} \subseteq \mathcal{O}$, there exists an F-saturated open set $\mathcal{N} \subseteq X$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{O}$.*

This immediately leads to the following result:

Lemma 4.7. *Let X and Y be locally compact and let Y be Hausdorff. Suppose that $F\colon X \to Y$ is continuous, $F(x_0) = y_0$, and F is open at $x_0 \in X$. Then, if $F(x) = y_0$ has only the trivial solution $x = x_0$ for all x in some sufficiently small neighborhood of x_0, any right inverse of the closed quotient map $\tilde{F}\colon X_0 \to Y_0$ (guaranteed to exist by Lemma 4.4) is continuous at y_0 whenever X_0 and Y_0 are chosen to be sufficiently small.*

Proof. By Lemma 4.4, the existence of the closed quotient map follows immediately from our assumptions. Noting that if $F(x) = y_0$ has only the trivial solution $x = x_0$ for all x in some sufficiently small neighborhood of x_0, suppose we have chosen X_0 and Y_0 such that $\tilde{F}(x) = y_0$ has only the solution $x = x_0$ (as is always possible, viz. the proof of Lemma 4.4). Then, the subset $\mathcal{M} = \{x_0\} \subseteq X_0$ satisfies

$$\tilde{F}^{-1}(\tilde{F}(\mathcal{M})) = \tilde{F}^{-1}(\tilde{F}(x_0)) = \tilde{F}^{-1}(y_0) = x_0$$

and is therefore $\tilde{F}$-saturated. Since $\tilde{F}$ is a surjection, there exists a (potentially non-unique) right inverse $\alpha\colon Y_0 \to X_0$ of $\tilde{F}$. Taking $\mathcal{O}$ to be any neighborhood of x_0, notice that $\mathcal{O}$ must necessarily contain an open set containing $\mathcal{M}$. Hence, by Lemma 4.6, there exists an open neighborhood $\mathcal{N}$ of x_0 such that $\mathcal{N}$ is $\tilde{F}$-saturated and $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{O}$. So, since $\mathcal{O} \supseteq \mathcal{N}$ and $\tilde{F}^{-1}(\tilde{F}(\mathcal{N})) = \mathcal{N}$, we have

$$\alpha^{-1}(\mathcal{O}) \supseteq \alpha^{-1}(\mathcal{N}) = \alpha^{-1}(\tilde{F}^{-1}(\tilde{F}(\mathcal{N}))) = (\tilde{F} \circ \alpha)^{-1}(\tilde{F}(\mathcal{N})) = (\mathrm{id}_{Y_0})^{-1}(\tilde{F}(\mathcal{N})) = \tilde{F}(\mathcal{N}).$$

By openness, $\tilde{F}(\mathcal{N})$ contains a neighborhood of y_0, and therefore $\alpha^{-1}(\mathcal{O})$ must as well for any neighborhood $\mathcal{O}$ of $\alpha(y_0) = x_0$. That is, any right inverse α of $\tilde{F}$ is continuous at y_0. □

Essentially, Lemma 4.7 just tells us that systems (1), which satisfy Brockett's necessary condition and have the origin as an isolated zero of f in (1), *automatically* have (viz. Theorem 4.5) locally asymptotically (and exponentially) stabilizing composition operators with symbols that are continuous at the origin. This will lead nicely into the results of the following subsection.

4.2. Brockett's Theorem in the Class of Composition Operators

The next result provides a natural extension of Brockett's theorem to the class of stabilizing composition operators with continuous symbols.

Theorem 4.8 (Brockett's Theorem for Composition Stabilizability)**.** *If the system* (1) *is locally asymptotically stabilizable by means of a composition operator with a continuous symbol, then it is necessary that the vector field $f(x, u)$ is open at* $(0, 0)$*.*

Proof. Let $y \in \mathbb{R}^n$ and write $w := (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. Set $\mathcal{W} := \mathcal{X} \times \mathcal{U}$ and choose a neighborhood of the origin $\mathcal{Y} \subseteq \mathbb{R}^n$. Let $F\colon \mathcal{Y} \times \mathcal{W} \to \mathbb{R}^n$ be given by $F(y, w) := f(w)$; so F is independent of y. If the system (1) is locally asymptotically stabilizable by means of a composition operator T_h with a continuous symbol $h(\cdot)$, then the one in $\dot{y} = F(y, w)$ is locally asymptotically stabilizable by means of continuous feedback laws, namely

$w(y) := h(y)$. Applying Theorem 2.5 tells us it must be the case that $0 \in \text{int}F(\mathcal{Y}_0 \times \mathcal{W}_0) = \text{int}f(\mathcal{W}_0)$ for any neighborhood of the origin $\mathcal{Y}_0 \times \mathcal{W}_0 \subseteq \mathcal{Y} \times \mathcal{W}$. Since $\mathcal{W}_0$ is an arbitrary neighborhood of the origin in $\mathcal{X} \times \mathcal{U}$, we justify the result. □

The fact that Brockett's theorem holds when we treat *everything*–that is, both the state variable x and the control variable u in the standard setting–as a control gives us a good perspective to understand why Brockett's condition is necessary for stabilizability by feedback laws, but far from sufficient. Specifically, straightforward cases like that of Example 1.7 demonstrate that stabilizability by a composition operator is a much weaker property than stabilizability by a feedback law. Yet Brockett's condition is necessary in either case.

As it turns out, Brockett's condition remains necessary under even weaker conditions. The following theorem presents one such situation.

Theorem 4.9 (Brockett's Theorem for Composition Stabilizability - Continuity at Zero)**.** *If the system* (1) *is locally asymptotically stabilizable by means of a composition operator T_h with a symbol which is continuous at the origin, then it is necessary that the vector field f in* (1) *satisfies Brockett's condition.*

Proof. If (1) is locally asymptotically stabilizable by means of a composition operator T_h with a symbol h which is continuous at the origin and is such that $T_h f$ is continuous, then the system $\dot{x} = T_h f(x)$ is locally asymptotically stabilizable by the composition operator T_h with h given by the identity, which is certainly continuous. Hence, by Theorem 4.8, $T_h f$ must be open. Since h is assumed to be continuous at zero and satisfy $h(0) = (0, 0)$, it follows that for every neighborhood $\mathcal{V}$ of $h(0) = (0, 0)$, there exists a neighborhood $\mathcal{O}$ of 0 such that $h(\mathcal{O}) \subseteq \mathcal{V}$. Correspondingly, by the openness of $T_h f = f \circ h$, there exists a neighborhood $\mathcal{N}$ of 0 such that $\mathcal{N} \subseteq T_h f(\mathcal{O}) = f(h(\mathcal{O})) \subseteq f(\mathcal{V})$. Therefore, f is open at the origin. □

Example 4.10. Consider Brockett's famous integrator, the system (1) with $f \in \mathrm{C}^1(\mathbb{R}^3 \times \mathbb{R}^2, \mathbb{R}^3)$ given by

$$f(x, u) = \begin{bmatrix} u_1 \\ u_2 \\ x_1 u_2 - x_2 u_1 \end{bmatrix}.$$

It is easy to see that f fails to satisfy Brockett's condition as stated in Theorem 2.5, as the f-image of any neighborhood of the origin fails to contain points of the form $(0, 0, \epsilon)$ for $\epsilon \neq 0$. Consequently, there can be no continuous feedback law stabilizing this system. In light of Theorem 4.9, this deficiency can be extended and is, in fact, even stronger–not only can there be no continuous stabilizing feedback, this continuity is impossible to achieve even just at the equilibrium. Moreover, neither of these deficiencies can be remedied by moving to the larger context of stabilization by composition operators.

After some consideration, Theorem 4.9 is relatively natural, as openness at the origin is a local property and continuity at the origin is the local analogue to continuity. In light of Theorem 4.9, Lemma 4.7, and Theorem 4.5, we pose the following conjecture:

Conjecture 4.11. If the vector field f in (1) satisfies Brockett's condition, then (1) is locally exponentially stabilizable by means of a composition operator T_h with a symbol which is continuous at the origin.

If Conjecture 4.11 is true, then Brockett's condition is (by Theorem 4.5) necessary and sufficient for stabilizability by a composition operator with a symbol that is continuous at the origin. We suspect that this could possibly be the case, but can also imagine that there may be pathologies preventing such a neat characterization. In support of this conjecture, however, we do have the following immediate result via Theorem 4.9 and Theorem 4.7:

Corollary 4.12 (Brockett's Theorem for Composition Stabilizability - Continuity at Zero)**.** *Suppose the origin is an isolated zero of f in* (1)*. Then,* (1) *is locally asymptotically stabilizable by means of a composition operator T_h with a symbol which is continuous at the origin if and only if the vector field f in* (1) *satisfies Brockett's condition.*

Having now touched on Brockett's condition in the context of composition stabilization, we are now ready to proceed to composition stabilization for non-differentiable systems.

5. Composition Stabilizability without Differentiability

To proceed, we must develop a method to characterize stabilizability by a composition operator with a continuous, but not continuously differentiable, symbol. To do this, it will be helpful first to state a much more general version of the implicit function theorem that allows us to deal with this non-differentiability a bit better. Such a version of the (generalized) implicit function theorem was obtained by Kumagai in [19], which is a refinement of the origin result given by Jittorntrum in [17].

The major goal of this section is to answer this remaining question and solve the problem of exponential stabilization by means of composition operators. In the process, we solve a portion of the problem of asymptotic stabilization by composition operators as well. To do so, we will begin in Subsection 5.1 by applying the Kumagai-Jittorntrum implicit function theorem judiciously.

5.1. Composition Stabilizability via Implicit Functions

We start this section with formulating the aforementioned Kumagai-Jittorntrum theorem .

Theorem 5.1 (Kumagai-Jittorntrum Implicit Function Theorem)**.** *Let $F\colon \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^k$ be a continuous mapping, and let $F(a_0, b_0) = 0$. Then the following statements are equivalent:*

- (i) *There exists open neighborhoods $A_0 \subseteq \mathbb{R}^k$ and $B_0 \subseteq \mathbb{R}^\ell$ of a_0 and b_0, respectively, such that for all $b \in B_0$ the equation $F(a, b) = 0$ has a unique solution $a = w(b)$ for some continuous mapping $w\colon B_0 \to A_0$.*
- (ii) *There exist open neighborhoods $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^\ell$ of a_0 and b_0, respectively, such that for all $b \in B$, the mappings $a \mapsto F(a, b)$ defined on A are injective.*

For the sake of completeness, it's worth noting that the Kumagai-Jittorntrum implicit function theorem can be applied to control systems of form (1) whose domain are neighborhoods of points in $\mathbb{R}^k \times \mathbb{R}^\ell$. This is due to the fact that a neighborhood of a point in $\mathbb{R}^p$ contains an open set homeomorphic to $\mathbb{R}^p$ itself.

Simple examples, such as $\dot{x} = x + u^3$, show that local asymptotic stabilizability by a composition operator with a smooth symbol is still possible in certain cases where local asymptotic stabilizability by a continuously differentiable feedback law is not. Let's use the Kumagai-Jittorntrum implicit function theorem again to investigate this phenomenon. We will need a criterion suitably formulated for compositions to do this.

Criterion 5.2. *There exists an open neighborhood of the origin $\mathcal{N} \subseteq \mathcal{X}$ and an auxiliary continuous function $\beta\colon \mathcal{N} \to \mathcal{X} \times \mathcal{U}$ satisfying $\beta(0) = (0, 0)$ such that the composition $f \circ \beta\colon \mathcal{N} \to \mathbb{R}^n$ is injective.*

The connection between Criterion 5.2 and the Kumagai-Jittorntrum implicit function theorem is pretty straightforward: Criterion 5.2 simply asserts the existence of some function β that serves as an appropriate 'adapter' so that we can apply the second item of the equivalence given in Theorem 5.1. The first theorem here shows how this may be applied to stabilization by composition operators.

Theorem 5.3. *If Criterion* 5.2 *holds for the system* (1)*, then* (1) *is locally exponentially stabilizable by means of a composition operator with a continuous symbol.*

Proof. Let $g \in \mathrm{C}(\mathcal{X}, \mathbb{R}^n)$ be any vector field such that $\dot{x} = g(x)$ is locally exponentially stable around the origin. Observe that if there exists an open neighborhood of the origin $\mathcal{N} \subseteq \mathcal{X}$ and a continuous function $\beta\colon \mathcal{N} \to \mathcal{X} \times \mathcal{U}$ such that the mappings $y \mapsto f(\beta(y))$ defined on $\mathcal{N}$ are injective, then given any open neighborhood of the origin $\mathcal{O} \subseteq \mathcal{X}$, the mappings $y \mapsto g(x) - f(\beta(y))$ defined on $\mathcal{N}$ are also injective for each fixed $x \in \mathcal{O}$. Theorem 5.1 tells us that there exist open neighborhoods of the origin $\mathcal{O}_0, \mathcal{N}_0 \subseteq \mathcal{X}$ and a unique continuous mapping $w\colon \mathcal{O}_0 \to \mathcal{N}_0$ such that $g(x) - f(\beta(w(x))) = 0$. Since the system $\dot{x} = g(x)$ is locally exponentially stable, and since taking $h = \beta \circ w$ yields $g(x) = (T_h f)(x)$, the claimed local exponential stabilizability follows. $\square$

While Theorem 5.3 yields the existence of a stabilizing composition operator whenever Criterion 5.2 is satisfied, the question remains: *when does such an auxiliary mapping satisfying Criterion* 5.2 *exist?* The answer is: *when there exists a stabilizing composition operator with a continuous symbol.* That is, we will show that the two are equivalent. Before doing so, the reader should recall that the continuous extension of Brockett's theorem (i.e. Theorem 2.5) requires the uniqueness condition given in Criterion 1.2. As it should be, the following result requires the composition stabilizability uniqueness condition given in Criterion 1.6 in the same fashion.

Lemma 5.4. *Let T_h be a composition operator with a continuous symbol locally asymptotically stabilizing* (1)*. Then, $T_h f|_{\mathcal{N}}$ is injective for some open neighborhood of the origin $\mathcal{N}$ and $h|_{\mathcal{N}}$ is a continuous function satisfying Criterion* 5.2*.*

Proof. If $\dot{x} = T_h f(x)$ has the origin as a locally asymptotically stable equilibrium, then by our assumption in Criterion 1.6, the trajectories of the system are unique when initialized in some open neighborhood $\mathcal{O}$ of the origin. Fix y in $\mathcal{O}$, choose some $T > 0$, and note that $\mathcal{O} \times (\mathcal{O} \times (-T, \infty))$ is an open neighborhood of the origin in $\mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R})$ such that the function $F\colon \mathcal{O} \times (\mathcal{O} \times (-T, \infty)) \to \mathbb{R}^n$ given by $F(x, (y, t)) = T_h f(x) - \dot{\Phi}_y(t + T)$ satisfies $F(0, (0, 0)) = 0$ and is such that for all $(y, t) \in \mathcal{O} \times (-T, \infty)$, the equation $F(x, (y, t)) = 0$ has a unique solution $x = w(y, t)$ given by the continuous mapping $w(y, t) = \Phi_y(t + T)$. By the Kumagai-Jittorntrum Implicit Function Theorem, this then implies that for all (y, t) in a neighborhood of $(0, 0)$ contained in $\mathcal{O} \times (-T, \infty)$, the mappings $x \mapsto F(x, (y, t))$ defined on some open neighborhood of the origin are injective. That is, for all sufficiently small (y, t), the mappings $x \mapsto T_h f(x) - T_h f(\Phi_y(t))$ defined on an open neighborhood of the origin $\mathcal{N} \subseteq \mathcal{O}$ are injective. Correspondingly, taking $(y, t) = (0, 0)$, it follows that the mapping $T_h f(x) - \dot{\Phi}_0(T) = T_h f(x)$ defined on $\mathcal{N}$ is injective, and $h|_{\mathcal{N}}$ is a continuous function satisfying Criterion 5.2. □

Since exponential stabilizability implies asymptotic stabilizability, the results of Lemma 5.3 and Lemma 5.4 produce the following immediate corollary:

Corollary 5.5. *There exists a composition operator with a continuous symbol locally asymptotically stabilizing* (1) *if and only if Criterion* 5.2 *holds for the system* (1)*. Moreover, if* (1) *is locally asymptotically stabilizable by means of a composition operator with a continuous symbol, then it is locally exponentially stabilizable by means of a composition operator with a continuous symbol.*

The equivalence of stabilizability by a composition operator with a continuous symbol and Criterion 5.2 might seem strange at first, but is not so unusual on further inspection. As it turns out, Criterion 5.2 is equivalent to a more familiar condition: the existence of a local section at the origin.

Lemma 5.6. *Criterion* 5.2 *is satisfied for the system* (1) *if and only if the vector field f has a local section at the origin. Moreover, if Criterion* 5.2 *is satisfied, then $f(\beta(\mathcal{N}))$ is a neighborhood of the origin and the local section of f at the origin is given by $\alpha = \beta \circ g^{-1}$, where $g\colon \mathcal{N} \to f(\beta(\mathcal{N}))$ is a homeomorphism.*

Proof. One direction is clear: If α is a local section of f at the origin, then $f \circ \alpha$ is the identity on the domain of α and is certainly injective, therefore satisfying Criterion 5.2. In the other direction, suppose that Criterion 5.2 holds for the system (1). Then by the invariance of domain, the image set $f(\beta(\mathcal{N}))$ is an open neighborhood of the origin and the composition $f \circ \beta$ induces a homeomorphism $g\colon \mathcal{N} \to f(\beta(\mathcal{N}))$ such that $\iota \circ g = f \circ \beta$, where ι denotes the inclusion map $\iota\colon f(\beta(\mathcal{N})) \to \mathbb{R}^n$. Since g is a homeomorphism, the inverse mapping $g^{-1}\colon f(\beta(\mathcal{N})) \to \mathcal{N}$ is continuous, as is $\beta \circ g^{-1}$. Observing that $g(0) = 0$ and that $f \circ \beta \circ g^{-1} = \iota \circ g \circ g^{-1} = \iota$, it follows that f has a local section at the origin given by $\alpha := \beta \circ g^{-1}$. □

Therefore, we have that stabilizability by means of a composition operator with a continuous symbol is *equivalent* to Criterion 5.2, which is *equivalent* to the existence of a local section of f at the origin. Hence, the theme of the Theorem 3.2 continues to hold in the continuous case:

Theorem 5.7. *The system* (1) *is locally asymptotically stabilizable by means of a composition operator T_h with a continuous symbol h if and only if there exists a local section of f at the origin. Moreover, such a composition operator T_h can always be chosen so that this stability is exponential.*

Having now resolved a large part of the continuous formulation of Problem 1.3 posed in the introduction via the results of this section and Section 3.1, we move on to the case of feedback laws.

6. Stabilizability by Feedback Laws

In this section, we will apply the composition stabilizability results of the previous sections to resolve the problem of asymptotic and exponential stabilizability by continuous and continuously differentiable feedback laws. In doing so, we solve the continuous and continuously differentiable formulation of Problem 1.4 as posed in the introduction. To do this, we will reduce the problem of *stabilizability* to the problem of *stability*.

6.1. Stabilizability and Stability of Local Sections

Before delving into the results of this subsection, it behooves us to take brief diversion to stress a point. Namely, there are different conventions as to whether or not it is appropriate to write $F = G$ given two functions $F : X \to Y$ and $G : X \to Z$ with $Z \subseteq Y$ such that $F(x) = G(x)$ for all $x \in X$. Though it is technically more correct to write $F = \iota \circ G$ for the inclusion $\iota : Z \to Y$, it is common practice to abuse notation and omit this detail when the codomain is either not particularly important or is sufficiently important so as to make the necessity of ι clearly apparent. Accordingly, most theorems in this section admit two formulations–a longer and messier-looking one with explicitly stated inclusions and a more concise, readable one in which they are implied. We opt for the latter option in the statements of the results themselves, but use the former more verbose, more technically correct convention in the proofs for greater clarity.

Theorem 6.1. *For $k \geq 0$ and $\ell \geq 1$, suppose T_h is a composition operator with C^k symbol h such that $T_h f$ is C^ℓ and has the origin as a locally exponentially stable equilibrium. Then, there exists some open connected neighborhood of the origin $\mathcal{V}$ contained in the domain of h such that $T_h f|_\mathcal{V}$ is a diffeomorphism.*

Proof. Let T_h be a composition operator with C^k symbol h such that $T_h f$ is C^ℓ and has the origin as a locally exponentially stable equilibrium. Treating $\dot{x} = T_h f(x)$ as a control system with trivial dependence on the control variable, Corollary 2.3 implies that $J_{T_h f}|_0 \in \mathrm{GL}_n(\mathbb{R})$, and the constant rank theorem yields there is a neighborhood of the origin such that $J_{T_h f}|_x \in \mathrm{GL}_n(\mathbb{R})$ for each x in this neighborhood. By Lemma 5.4, there exists another neighborhood of the origin such that the restriction of $T_h f$ to this neighborhood is injective. Take $\mathcal{V}$ to be a connected open neighborhood of the origin (any sufficiently small open ball about the origin will suffice) contained in the intersection of these two regions region. Then, since $T_h f$ is C^ℓ, $T_h f|_\mathcal{V}$ is C^ℓ. By the invariance of domain, $T_h f|_\mathcal{V} = \iota \circ g$ where $g : \mathcal{V} \to T_h f(\mathcal{V})$ is a homeomorphism and $\iota : T_h f(\mathcal{V}) \to \mathbb{R}^n$. So, $T_h f(\mathcal{V})$ is simply connected as well, g is C^ℓ, and $J_g|_x = J_{T_h f}|_x$ for each $x \in \mathcal{V}$. Since the fibers of g are singletons, they are certainly compact, and since g is a homeomorphism, it is certainly closed. As $\mathcal{V}$ is locally compact and Hausdorff, g is correspondingly proper. Therefore, g has a bijective derivative for each $x \in \mathcal{V}$ and is a proper differentiable map between an open subset $\mathcal{V} \subseteq \mathbb{R}^n$ and the open simply connected subset $T_h f(\mathcal{V})$. By the Hadamard-Caccioppoli theorem, it then follows that g is a C^ℓ diffeomorphism. □

We can use this to great effect in the situations where we seek an exponentially stabilizing feedback law $u(x)$ such that $f(\cdot, u(\cdot))$ is, at least, of class C^1.

Theorem 6.2. *For $k \geq 0$ and $\ell \geq \max\{1, k\}$, the system* (1) *is locally exponentially stabilizable by means of a C^k feedback law u such that $f(\cdot, u(\cdot))$ is C^ℓ if and only if $u = proj_2 \circ \alpha \circ (proj_1 \circ \alpha)^{-1}$ for some C^k local section of f at the origin α such that $proj_1 \circ \alpha$ is a C^ℓ diffeomorphism and $\Lambda_+(J_{\alpha_1}|_0^{-1}) = \emptyset$.*

Proof. In the forward direction, suppose (1) is locally exponentially stabilizable by the C^k feedback law u inducing $h(x) = (x, u(x))$ such that $T_h f$ is C^ℓ for $\ell \geq \max\{1, k\}$. By Theorem 6.1, $T_h f|_\mathcal{V} = \iota \circ g$ for some

C^ℓ diffeomorphism $g : \mathcal{V} \to T_h f(\mathcal{V})$ on some open connected neighborhood of the origin $\mathcal{V}$. So, the function $\alpha : T_h f(\mathcal{V}) \to \mathcal{X} \times \mathcal{U}$ given by $\alpha := h|_{\mathcal{V}} \circ g^{-1}$ is a C^k local section of f at the origin and

$$\mathrm{proj}_1 \circ \alpha = \mathrm{proj}_1 \circ h|_{\mathcal{V}} \circ g^{-1} = \iota \circ g^{-1}$$

where g^{-1} is a C^ℓ diffeomorphism. Observe as well that $\mathrm{proj}_2 \circ \alpha \circ g = u|_{\mathcal{N}}$ is certainly a C^k stabilizing control for (1). Treating $\dot{x} = T_w f|_{\mathcal{N}}(x)$ as a control system with trivial dependence on the control variable, Corollary 2.3 implies that $\Lambda_+ \left(J_{T_w f|_{\mathcal{V}}} \Big|_0 \right) = \emptyset$. Since $T_w f(0) = 0$, the inverse function theorem tells us that

$$J_{\mathrm{proj}_1 \circ \alpha}\big|_0^{-1} = J_{g^{-1}}\big|_{g(0)}^{-1} = J_g\big|_0 = J_{T_w f|_{\mathcal{N}}} \Big|_0 .$$

Hence, we conclude that $\Lambda_+ \left(J_{\mathrm{proj}_1 \circ \alpha}\big|_0^{-1} \right) = \emptyset$.

In the reverse direction, suppose f has a C^k local section at the origin α such that $\mathrm{proj}_1 \circ \alpha = \iota_1 \circ g$ where g is an C^ℓ diffeomorphism for $\ell \geq \max\{1, k\}$ and ι_1 is the appropriate inclusion, and that $\Lambda_+ \left(J_{\mathrm{proj}_1 \circ \alpha}\big|_0^{-1} \right) = \emptyset$. Since $f \circ \alpha = \iota_2$ for ι_2 the corresponding inclusion, it follows that $f \circ \alpha \circ g^{-1} = \iota_2 \circ g^{-1}$. Since $J_g^{-1}\big|_{g(0)} = J_g\big|_0^{-1} = J_{\mathrm{proj}_1 \circ \alpha}\big|_0^{-1}$ by the inverse function theorem, the local exponential stability of $\dot{x} = (f \circ \alpha \circ g^{-1})(x)$ at the origin follows from $\Lambda_+ \left(J_{\alpha_1}\big|_0^{-1} \right) = \emptyset$. Clearly, this yields that $u = \mathrm{proj}_2 \circ \alpha \circ (\mathrm{proj}_1 \circ \alpha)^{-1}$ is a C^k feedback law locally exponentially stabilizing (1). □

The case for asymptotic and exponential stabilizability when $f(\cdot, u(\cdot))$ is only C^0 follows similarly.

Theorem 6.3. *The system* (1) *is locally asymptotically (resp. exponentially) stabilizable by means of continuous feedback laws if and only if there exists a local section of f at the origin, α, such that $\mathrm{proj}_1 \circ \alpha$ is a homeomorphism and $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$ has the origin as a locally asymptotically (resp. exponentially) stable equilibrium point.*

Proof. In the forward direction, suppose (1) is locally asymptotically (resp. exponentially) stabilizable by a continuous feedback law $u : \mathcal{O} \to \mathcal{U}$ defined on some neighborhood of the origin $\mathcal{O} \subseteq \mathcal{X}$. This induces the stabilizing composition operator T_h with $h : \mathcal{O} \to \mathcal{X} \times \mathcal{U}$ given by $h(x) = (x, u(x))$. By the equivalences established in Corollary 5.5 and Lemma 5.4, there exists an open neighborhood of the origin $\mathcal{N} \subseteq \mathcal{O}$ and a homeomorphism $g : \mathcal{N} \to T_h f(\mathcal{N})$ such that the function $\alpha : T_h f(\mathcal{N}) \to \mathcal{X} \times \mathcal{U}$ given by $\alpha := h|_{\mathcal{N}} \circ g^{-1}$ is a local section of f at the origin. Correspondingly,

$$\begin{aligned} T_h f|_{\mathcal{N}} &= f \circ h|_{\mathcal{N}} \circ g^{-1} \circ g \\ &= f \circ \alpha \circ g \\ &= \iota_1 \circ g \end{aligned}$$

for the inclusion $\iota_1 : T_h f(\mathcal{N}) \to \mathbb{R}^n$. Hence, $\mathrm{proj}_1 \circ \alpha = \iota_2 \circ g^{-1}$ for the inclusion $\iota_2 : \mathcal{N} \to \mathcal{X}$ and $\dot{x} = (\iota_1 \circ g)(x)$ has the origin as a locally asymptotically (resp. exponentially) stable equilibrium. Observe as well that $\mathrm{proj}_2 \circ \alpha \circ g = u|_{\mathcal{N}}$ is certainly a continuous feedback law stabilizing (1).

In the reverse direction, suppose f has a local section at the origin α such that $\mathrm{proj}_1 \circ \alpha = \iota_2 \circ h$ for some homeomorphism h and $\dot{x} = (\iota_1 \circ h^{-1})(x)$ has the origin as a locally asymptotically (resp. exponentially) stable equilibrium, where ι_1 and ι_2 are the appropriate inclusions. Since $f \circ \alpha = \iota_1$, we have $f \circ \alpha \circ h^{-1} = \iota_1 \circ h^{-1}$ and the local asymptotic (resp. exponential) stability of $\dot{x} = (f \circ \alpha \circ h^{-1})(x)$ at the origin follows. As $(f \circ \alpha \circ (\mathrm{proj}_1 \circ \alpha)^{-1})(x) = f(x, \mathrm{proj}_2 \circ \alpha \circ h^{-1}(x))$, this yields that $u = \mathrm{proj}_2 \circ \alpha \circ h^{-1}$ is a continuous feedback law locally asymptotically (resp. exponentially) stabilizing (1). □

With the results of Theorem 6.2 and Theorem 6.3 in hand, we can answer the question of stabilizability by feedback laws quite nicely.

Theorem 6.4. *For $k \geq 0$, the system* (1) *is locally asymptotically (resp. exponentially) stabilizable by means of a* C^k *feedback law u if and only if there exists a* C^k *local section α of f at the origin such that $proj_1 \circ \alpha$ is a homeomorphism (resp. diffeomorphism) and $\dot{x} = (proj_1 \circ \alpha)^{-1}(x)$ has the origin as a locally asymptotically (resp. exponentially) stable equilibrium. Moreover, every such control u is of the form $u = proj_2 \circ \alpha \circ (proj_1 \circ \alpha)^{-1}$ for some α satisfying the above.*

As a minor aside, it is interesting to note that Theorem 6.4 allows us to characterize continuous feedback stabilizability in terms of classical Lyapunov functions, as opposed to the more cumbersome control-Lyapunov functions that have been used previously to provide necessary conditions. That is, the existence of a Lyapunov function for $(\mathrm{proj}_1 \circ \alpha)^{-1}$ is equivalent to the local asymptotic stability of the origin for $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$, which is equivalent to the local asymptotic/exponential stabilizability of (1) by the result above.

Broadly speaking, Theorem 6.4 ties a nice bow on the question of stabilizability in a very easy to understand way. Speaking colloquially and in clear terms, here is a summary of what we have learned:

- Exponential stabilizability by a composition operator with C^k symbols is possible if and only if we locally have a C^k local section at the origin. More simply, when we can change *every argument* of the dynamics, stabilization at an equilibrium is possible if and only if we have a tool to *change the dynamics at will.*
- Stabilizability by C^k feedback laws is possible if and only if we locally have a C^k local section at the origin which is invertible in the non-control variables and whose inverse induces a stable system. More simply, when we impose that we may change *only some arguments* of the dynamics, stabilization at an equilibrium is possible if and only if we have two things: (1) a tool to *change the dynamics at will* and (2) a second tool to undo the effects of the first tool on the arguments we cannot change that, itself, *stabilizes those unchanged arguments.*

Let us conclude with some examples to drive home the theme of Theorem 6.4.

Example 6.5. In this example, we consider Artstein's circles; i.e., the system (1) where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is given as

$$f(x,u) := \begin{bmatrix} u(x_2^2 - x_1^2) \\ -2ux_1x_2 \end{bmatrix}.$$

It is well known that this system is not feedback stabilizable by C^1 feedback laws, since the conditions of Corollary 2.3 are not satisfied. Since taking $z = x_2 + ix_1$ in $\mathbb{C}$ renders (1) as the system $\dot{z} = uz^2$, it is straightforward to see that Brockett's condition is satisfied. So, continuous stabilizing feedbacks are not ruled out by this criteria, though it is a well-known fact that they are indeed impossible. We will confirm this through the use of Theorem 6.4. To wit, suppose that f is stabilizable by a continuous feedback. Accordingly, f must then have a local section at the origin α such that $\mathrm{proj}_1 \circ \alpha$ is a homeomorphism. Since $(f \circ \alpha)(z) = (\mathrm{proj}_2 \circ \alpha)(z)((\mathrm{proj}_1 \circ \alpha)(z))^2 = z$ must hold, and as $\mathrm{proj}_2 \circ \alpha$ must then be a nonzero real function away from the origin, it follows that the inverse function of $\mathrm{proj}_1 \circ \alpha$ must satisfy $(\mathrm{proj}_1 \circ \alpha)^{-1}(z) = (\mathrm{proj}_2 \circ \alpha \circ (\mathrm{proj}_1 \circ \alpha)^{-1})(z)z^2$. That is, $(\mathrm{proj}_1 \circ \alpha)^{-1}$ is a continuous rescaling of z^2, which has a ramification point at the origin, thereby contradicting our supposition that it is a homeomorphism. Correspondingly, Artstein's circles are not asymptotically stabilizable by means of continuous feedback laws.

Example 6.6. Consider the system (1) where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is defined by:

$$f(x,u) := \begin{bmatrix} x_1^2 + x_2^2 + x_2 \\ x_1x_2 + x_2^2 + u^3 \end{bmatrix}$$

We begin by noting that Corollary 2.3 yields, by routine computation, that this system fails to be exponentially stabilizable by C^1 feedback laws. It does, however, turn out to be exponentially stabilizable by C^0 feedback laws. We will employ Theorem 6.4 to this end.

We will construct a local section for f at the origin. To do so, let $\alpha_1, \alpha_2, \alpha_3 : \mathbb{R}^2 \to \mathbb{R}$ be the (as-of-yet, unspecified) component functions of this sought right inverse $\alpha : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}$. When constructing α, we must be careful to do so in such a way so as to satisfy the constraints of Theorem 6.4–namely, $\mathrm{proj}_1 \circ \alpha$ must be a diffeomorphism defined on some neighborhood of the origin and $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$ must be locally exponentially stable.

To this end, we start in a straightforward fashion and observe that the first component of f yields that α_1 and α_2 must satisfy $\alpha_1^2 + \alpha_2^2 + \alpha_2 = x_1$. Using the quadratic formula, it is easy to see that this equation will be satisfied if

$$\alpha_2 = \frac{1}{2}\left(-1 + \sqrt{1 - 4(\alpha_1^2 - x_1)}\right).$$

Hence, if $\mathrm{proj}_1 \circ \alpha$ is to be invertible, then this inverse function must satisfy $\alpha_1((\mathrm{proj}_1 \circ \alpha)^{-1}) = x_1$, as well as:

$$\begin{aligned} \alpha_2((\mathrm{proj}_1 \circ \alpha)^{-1}) &= x_2 \\ \frac{1}{2}\left(-1 + \sqrt{1 - 4(x_1^2 - (\mathrm{proj}_1 \circ \alpha)_1^{-1})}\right) &= x_2 \\ (\mathrm{proj}_1 \circ \alpha)_1^{-1} &= x_1^2 + x_2^2 + x_2. \end{aligned}$$

Applying this information, note that the linearization of $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$ would then be of the form

$$J_{(\mathrm{proj}_1 \circ \alpha)^{-1}}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \frac{\partial(\mathrm{proj}_1 \circ \alpha)_2^{-1}}{\partial x_1} & \frac{\partial(\mathrm{proj}_1 \circ \alpha)_2^{-1}}{\partial x_2} \end{bmatrix}.$$

So, let us take $(\mathrm{proj}_1 \circ \alpha)_2^{-1} = -2x_2 - \frac{1}{2}x_1$, producing eigenvalues of $\lambda = 1/2(-2 \pm \sqrt{2}) < 0$. Correspondingly, $(\mathrm{proj}_1 \circ \alpha)^{-1}$ is now well-defined and the system $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$ is locally exponentially stable. Via an application of the inverse function theorem, we reach $\mathrm{proj}_1 \circ \alpha$ as the inverse of this function at the origin, thus specifying α_1 and α_2.

To conclude, we complete our construction of α by solving $\alpha_1\alpha_2 + \alpha_2^2 + \alpha_3^3 = x_2$ for α_3, producing $\alpha_3 = \sqrt[3]{x_2 - \alpha_1\alpha_2 - \alpha_2^2}$. By Theorem 6.4, the C^0 feedback law

$$u(x) = \sqrt[3]{(\mathrm{proj}_1 \circ \alpha)_2^{-1} - x_1x_2 - x_2^2} = \sqrt[3]{-2x_2 - \frac{1}{2}x_1 - x_1x_2 - x_2^2}$$

is a stabilizing control. To confirm this, observe that $f(x, u(x)) = \begin{bmatrix} x_1^2 + x_2^2 + x_2 \\ -2x_2 - \frac{1}{2}x_1 \end{bmatrix}$. Linearizing the right-hand side, we produce $J_{f\circ(\mathrm{id},u)}\big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -2 \end{bmatrix}$, the eigenvalues of which are given by $\lambda = 1/2(-2 \pm \sqrt{2}) < 0$ and are, as they should be, the same as the eigenvalues of the linearization of $\dot{x} = (\mathrm{proj}_1 \circ \alpha)^{-1}(x)$.

Before finishing, it is worth providing some words here regarding the use of Theorem 6.4 in practice. In particular, not all examples will work out as nicely as the one above. Indeed, even with the characterization given by Theorem 6.4, explicit constructions of stabilizing controls is still, quite often, very challenging. In large part, this is due to the fact that the local sections for the vector field f in (1), should they exist, might not be possible to express in a closed form.

In some situations where this difficulty arises, it does not necessarily present a problem. For example, as illustrated by the technique employed in Example 6.6 above, one does not always need to actually compute $\mathrm{proj}_1 \circ \alpha$ directly to obtain our stabilizing feedback law. That is, the actual computation involved in constructing a stabilizing feedback law can, in some particular circumstances, be accomplished without the explicit construction of a local section (correspondingly, allowing us to occasionally overcome the aforementioned difficulty of obtaining a closed form expression for the section). Unfortunately, such tricks do not always resolve similar issues that may arise. For example, even if a local section of f *can* be expressed in a closed form, this does not guarantee that $(\mathrm{proj}_1 \circ \alpha)^{-1}$ enjoys the same property. Likewise, it may be the case that it is possible to express $(\mathrm{proj}_1 \circ \alpha)^{-1}$ in a closed form while the same is not possible for $\mathrm{proj}_2 \circ \alpha$. To add to one's potential confusion, it can *also* be the case that a stabilizing control $u(x)$ exists, is expressible in a closed form, and satisfies $u = u = \mathrm{proj}_2 \circ \alpha \circ (\mathrm{proj}_1 \circ \alpha)^{-1}$ for a local section α of f where both $(\mathrm{proj}_1 \circ \alpha)^{-1}$ and $\mathrm{proj}_2 \circ \alpha$ cannot be expressed in a closed form (since composing two functions that cannot be expressed in closed form does not guarantee that the same holds for the result).

Finally (and, perhaps, frustratingly), it is important to note that we cannot simply employ some approximation technique and ignore the challenge of finding an exact closed-form expression for our control, as Theorem 6.4 tells us that *every possible stabilizing feedback* must be of the form $u = \mathrm{proj}_2 \circ \alpha \circ (\mathrm{proj}_1 \circ \alpha)^{-1}$ for some local section α satisfying the conditions of the theorem. As such, one should expect to occasionally encounter systems which are stabilizable, but whose stabilizing controls are beyond our practical reach.

7. Conclusion and Future Work

In this paper, we have demonstrated that many of the most well-known classical theorems regarding stabilizability have somewhat cleaner forms in the composition operator context. We also, hopefully, were able to convince the reader that extracting properties about the composition operators which stabilize a given system (1) tends to be, in general, relatively easy (at least, in comparison with extracting information about stabilizing feedback control laws). By employing these composition-stabilizability results in the classical feedback-stabilizability context, we have shown that the *stabilizability* of a control systems is equivalent to the *stability* of an associated system. That is, we reduce the question of *stabilizability* to that of *stability*.

There is a lot of potential for further work characterizing stabilizability via composition operators. In particular, completing the distinction between asymptotic and exponential stabilization via composition operators–that is, closing the gap between Theorem 3.2 and Theorem 5.7–could be a worthwhile endeavor. Additionally, conjecture 4.11 would be interesting to investigate further.

From the characterization of stabilizing controls and stabilizable systems, there seem to be a variety of potential avenues for further study. For example, Coron's result from [8] give us a stronger necessary condition than Brockett's theorem by establishing a relationship between the existence of continuous feedback laws and the existence of an isomorphism between certain singular homology (or, equivalently, stable homotopy) groups associated to the system. Refining arguments along homological lines could prove fruitful, in light of the characterizations provided by Theorem 6.4. Additionally, since stability can be completely characterized via Lyapunov functions (as Zubov's theorem shows; see [36]), investigations of Lyapunov-theoretic connections in the context of the results presented here could provide a bridge between the existence of local sections and the existence of Lyapunov functions.

It also would be worth pursuing variational-analytic connections. Namely, extending the established characterizations of local asymptotic and exponential stabilizability to the case of *set-valued* differential inclusions could shed some light on the topic of stabilizability in a more general sense (particularly, by removing the uniqueness criteria imposed throughout this paper). As mentioned and exploited in [6, 12], variational analysis achieves complete characterizations of linear openness (see, e.g., [22] and the references therein), a notion which might potentially serve as the counterpart to Brockett's openness property from Definition 2.4 for general nonsmooth mappings and multifunctions (indeed, Brockett's property has already been investigated in this context to some degree [11]). So, much of the groundwork for further developments has already been completed. The real challenge would be implementing the classical, known results regarding feedback stabilizability (or, better, stabilizability via composition operators) in this much wilder context.

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