OPTIMIZATION OF A PERTURBED SWEEPING PROCESS BY CONSTRAINED DISCONTINUOUS CONTROLS

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Abstract. This paper deals with optimal control problems described by a controlled version of Moreau's sweeping process governed by convex polyhedra, where measurable control actions enter additive perturbations. This class of problems, which addresses unbounded discontinuous differential inclusions with intrinsic state constraints, is truly challenging and underinvestigated in control theory while being highly important for various applications. To attack such problems with constrained measurable controls, we develop a refined method of discrete approximations with establishing its well-posedness and strong convergence. This approach, married to advanced tools of first-order and second-order variational analysis and generalized differentiation, allows us to derive adequate collections of necessary optimality conditions for local minimizers, first in discrete-time problems and then in the original continuous-time controlled sweeping process by passing to the limit. The new results include an appropriate maximum condition and significantly extend the previous ones obtained under essentially more restrictive assumptions. We compare them with other versions of the maximum principle for controlled sweeping processes that have been recently established for global minimizers in problems with smooth sweeping sets by using different techniques. The obtained necessary optimality conditions are illustrated by several examples.

Key words. Optimal control, sweeping process, variational analysis, discrete approximations, generalized differentiation, necessary optimality conditions.

AMS Subject Classifications. 49M25, 49J53, 90C30.

1 Introduction and Problem Formulation

This paper addresses the following optimal control problem labeled as (P):

Minimize the Mayer-type cost functional

$$J[x,u] := \varphi\big(x(T)\big) \tag{1.1}$$

over the corresponding (described below) pairs $(x(\cdot),u(\cdot))$ satisfying

$$\begin{cases} \dot{x}(t) \in -N\big(x(t);C\big) + g\big(x(t),u(t)\big) \ \text{ a.e. } \ t\in[0,T],\ x(0)=x_0\in C\subset\mathbb{R}^n, \\ u(t)\in U\subset\mathbb{R}^d \ \text{ a.e. } \ t\in[0,T], \end{cases} \tag{1.2}$$

where the set C is a convex *polyhedron* given by

$$C := \bigcap_{j=1}^{s} C^j \ \text{ with } C^j := \big\{x\in\mathbb{R}^n \,\big|\, \langle x_*^j, x\rangle \le c_j\big\}, \tag{1.3}$$

with $\|x_*^j\|=1$, $j=1,\ldots,s$, and where $N(x;C)$ stands for the normal cone of convex analysis defined by

$$N(x;C) := \big\{v\in\mathbb{R}^n \ \big|\ \langle v, y-x\rangle \le 0,\ y\in C\big\} \text{ if } x\in C \text{ and } N(x;C):=\emptyset \text{ if } x\notin C. \tag{1.4}$$

Observe that the second part of definition (1.4) mandatorily yields the presence of the hidden *pointwise state constraints* on the trajectories of (1.2):

$$x(t)\in C,\ \text{ i.e. } \langle x_*^j, x(t)\rangle \le c_j \ \text{ for all } \ t\in[0,T] \ \text{ and } \ j=1,\ldots,s. \tag{1.5}$$

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Considering the differential inclusion in (1.2) without the additive perturbation term $g(x,u)$, we arrive at the framework of the *sweeping process* introduced by Jean-Jacques Moreau who was motivated by applications to problems of elastoplasticity; see [24]. It has been well recognized that the (uncontrolled) Moreau's sweeping process has a *unique* absolutely continuous (or even Lipschitz continuous) solution for convex and mildly nonconvex sets C; see, e.g., [14] and the references therein. Thus there is no room for optimization of the sweeping process unless some additional functions or parameters of choice are inserted into its description. It is very different from control theory for Lipschitzian differential inclusions

$$\dot{x}(t) \in F\big(x(t)\big) \text{ a.e. } t \in [0,T],\ x(0) = x_0 \in \mathbb{R}^n, \tag{1.6}$$

which have multiple solutions. The latter type of dynamics extends the classical ODE control setting with $F(x) := f(x,U)$ in (1.6), where the choice of measurable controls $u(t) \in U \subset \mathbb{R}^d$ a.e. $t \in [0,T]$ creates the possibility to find an optimal one with respect to a prescribed performance. The main issue here is that the normal cone mapping $N(\cdot;C)$ in the sweeping process is *highly non-Lipschitzian* (even discontinuous) while being *maximal monotone*. On the other hand, the well-developed optimal control theory for differential inclusions (1.6) strongly depends on Lipschitzian behavior of $F(\cdot)$; see, e.g., [22, 27] with the references therein as well as more recent publications.

Introducing controls into the perturbation term of (1.2) allows us to have multiple solutions $x(\cdot)$ of this system by the choice of feasible control functions $u(\cdot)$ and thus to minimize the cost functional (1.1) over feasible control-trajectory pairs. Problems of this type were considered in the literature from the viewpoint of the existence of optimal solutions and relaxation; see [1, 10, 16, 26] among other publications.

More recently, *necessary optimality conditions* for local minimizers were derived in [7, 8] by the *method of discrete approximations* for problems of type (P) with smooth (in fact $W^{2,\infty}$) control functions without any constraints. Later on these results were further extended in [9] to nonconvex (and hence nonpolyhedral) problems with prox-regular sets C in the same control setting. Note that both C and g in (1.2) may be time-dependent; we discuss the autonomous case just for simplicity. The discrete approximation approach implemented in [7]–[9] was based on the scheme from [12] developed for the unperturbed sweeping process with controls in the moving set. The later was in turn a sweeping control version of the original discrete approximations method to derive necessary optimality conditions for Lipschitzian differential inclusions (1.6) suggested and implemented in [20]; see also [22].

Quite recently, other approximation procedures were developed to derive necessary optimality conditions for global minimizers of (P) in the class of measurable controls while under rather strong assumptions. The first paper [3] assumes, among other requirements, that the boundary of the sweeping set C in (1.2) is $\mathcal{C}^3$-smooth, the control set U is compact and convex, and its image $g(x,U)$ under g is convex as well. The $\mathcal{C}^3$-smoothness assumption on C was relaxed in [15], by employing a smooth approximation procedure not relying on the distance function as in [3], for the case of $C := \{x \in \mathbb{R}^n \mid \psi(x) \le 0\}$ with ψ being a $\mathcal{C}^2$-smooth convex function. The necessary optimality conditions obtained in both papers [3, 15] can be treated as somewhat different counterparts of the celebrated Pontryagin Maximum Principle (PMP) for state-constrained controlled differential equations $\dot{x} = f(x,u)$.

Note that necessary optimality conditions in some other classes of optimal control problems governed by various controlled versions of the sweeping process were developed in [2, 6, 7, 8, 9, 11, 12, 19].

The main goal of this paper is to derive necessary optimality conditions for local minimizers (in the senses specified below) of the formulated problem (P), with the constraint set U in (1.2) given by an *arbitrary compact* and with the (nonsmooth) *polyhedral* set C from (1.3), by significantly reducing regularity assumptions on the reference control. Although problem (1.2) is stated in the class of *measurable* feasible control actions, we assume that the local *optimal* control under consideration is of *bounded variation*, hence allowing to be discontinuous.

Our approach is based on developing the *method of discrete approximations*, which is certainly of its own interest and has never been implemented before in control theory for sweeping processes with discontinuous controls. The novel results in this direction establish a *strong* approximation of *every feasible* control-state pair for (P) in the sense of the L^2-norm convergence of discretized controls and the

$W^{1,2}$-norm convergence of the corresponding piecewise linear trajectories. Furthermore, we justify such a strong convergence of *optimal* solutions for discrete problems to the given local minimizer of (P).

Dealing further with *intrinsically nonsmooth* and *nonconvex* discrete-time approximation problems, we derive for them necessary optimality conditions of the discrete Euler-Lagrange type by using appropriate unconvexified tools of first-order and second-order variational analysis and generalized differentiation. Employing these tools and passing to the limit from discrete approximations lead us to new nondegenerate necessary optimality conditions for local optimal solutions of the sweeping control problem (P). The obtained results significantly extend those recently established in [8] for unconstrained $W^{2,\infty}$ optimal controls in (P), contain a maximum condition, while being essentially different from the necessary optimality conditions derived in [3, 15] for problems of type (P) with smooth sets C in addition to other assumptions. We present nontrivial examples that illustrate the efficiency of the new results. Further applications to some practical models are considered in our subsequent paper [13].

The rest of the paper is organized as follows. In Section 2 we formulate the standing assumptions, discuss the types of local minimizers under consideration, and present some preliminary results.

Section 3 is devoted to the construction of discrete approximations of the controlled constrained sweeping dynamics (1.2) that allows us to deal with measurable controls (in fact of bounded variation) and to strongly approximate any feasible solutions of (P) as mentioned above. This result plays a major role in the justification of the developed version the method of discrete approximations for problem (P).

In Section 4 we construct a sequence of discrete approximation of a given "intermediate" local minimizer for (P) that occupies an intermediate position between weak and strong minimizers in variational and control problems. The major result of this section justifies the strong $W^{1,2}\times L^2$ approximation of the given local minimum pair $(\bar{x}(\cdot),\bar{u}(\cdot))$ by extended optimal solutions to the discretized problems. It makes a bridge between the continuous-time sweeping control problem (P) and its discrete-time counterparts.

It occurs that the discrete-time approximating problems are unavoidably nonsmooth and nonconvex, even when the initial data are differentiable. It is due to the presence of increasingly many geometric constraints generated by the normal cone graph. To deal with them, we need adequate tools of variational analysis involving not only first-order but also second-order generalized differentiation. The latter is because of the normal cone description of the sweeping process. In Section 5 we present the corresponding definitions of the first-order and second-order generalized differential constructions taken from [21] together with the results of their computations entirely in terms of the given data of (1.2).

Section 6 provides the derivation of necessary optimality conditions for discrete-time problems by reducing them to problems of nondifferentiable programming with many geometric constraints, using necessary optimality conditions for them obtained via variational/extremal principles, and then expressing the latter in terms of the given data of (P) by employing calculus rules of generalized differentiation.

Section 7 is the culmination. We pass to the limit from the necessary optimality conditions for discrete-time problems by using stability of discrete approximations, robustness of generalized differential constructions, and establishing an appropriate convergence of adjoint functions, which is the most difficult part. In this way we arrive at new necessary conditions for local minimizers of (P) expressed in terms of the given data of the original problem. Since signed (not just nonnegative) measures naturally appear in the resulting optimality conditions, dealing with them creates significant difficulties, which have been overcome in our device. The usefulness of the nondegenerate optimality conditions obtained in the main theorem is illustrated in Section 8 by nontrivial examples.

Throughout the paper we use standard notations of variational analysis and optimal control; see, e.g., [21, 22]. Recall that $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$) and that $\mathbb{N}:=\{1,2,\ldots\}$.

2 Standing Assumptions and Basic Notions

Dealing with the polyhedron C from (1.3) and having $\bar{x}\in C$, consider the set of *active constraint indices*

$$I(\bar{x}):=\big\{j\in\{1,\ldots,s\}\;\big|\;\langle x_*^j,\bar{x}\rangle=c_j\big\}. \tag{2.1}$$

Recall that the positive linear independence constraint qualification (PLICQ) holds at $\bar{x}$ if

$$\Big[\sum_{j\in I(\bar{x})}\alpha_j x_*^j=0,\ \alpha_j\in\mathbb{R}_+\Big]\Longrightarrow\big[\alpha_j=0\text{ for all }j\in I(\bar{x})\big],\tag{2.2}$$

and the linear independence constraint qualification (LICQ) holds if the restriction $\alpha_j \in \mathbb{R}_+$ in (2.2) is dropped. Our *standing assumptions* in this paper are as follows:

(H1) The control region $U \neq \emptyset$ is a closed and bounded set in $\mathbb{R}^d$.
(H2) The perturbation mapping $g\colon \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous in (x, u) while being also Lipschitz continuous with respect to x uniformly on U whenever x belongs to a bounded subset of $\mathbb{R}^n$ and satisfies there the sublinear growth condition

$$\|g(x,u)\|\le\beta\big(1+\|x\|\big)\ \text{ for all }\ u\in U$$

with some positive constant β.
(H3) The PLICQ condition (2.2) holds along the reference trajectory $\bar{x}(t)$ of (1.2) for all $t \in [0, T]$.

It follows from [16, Theorem 1] that for each measurable control $u(\cdot)$ there is a unique solution $x(\cdot) \in W^{1,2}([0, T], \mathbb{R}^n)$ to the Cauchy problem in (1.2). Thus by a *feasible process* for (P) we understand a pair $(x(\cdot), u(\cdot))$ such that $u(\cdot)$ is measurable, $x(\cdot) \in W^{1,2}([0, T], \mathbb{R}^n)$, and all the constraints in (1.2) are satisfied. The above discussion tells us that the set of feasible pairs for (P) is nonempty.

Furthermore, it follows from [16, Theorem 2] that under the assumptions above the sweeping control problem (P) admits an *optimal solution* provided that the image set

$$g(x,U):=\big\{y\in\mathbb{R}^n\ \big|\ y=g(x,u)\text{ for some }u\in U\big\}$$

is convex. Since in this paper we are interested in deriving necessary optimality conditions for a given local minimizer of (P), we do not impose the aforementioned convexity assumption.

Let us now specify what we mean by a local minimizer of (P).

Definition 2.1 *We say that a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (P) is a $W^{1,2} \times L^2$*-LOCAL MINIMIZER *in this problem if there exists $\epsilon > 0$ such that $J[\bar{x}, \bar{u}] \le J[x, u]$ for all feasible pairs $(x(\cdot), u(\cdot))$ satisfying*

$$\int_0^T\Big(\|\dot{x}(t)-\dot{\bar{x}}(t)\|^2+\|u(t)-\bar{u}(t)\|^2\Big)\,dt<\epsilon.$$

For the case of differential inclusions of type (1.6) with no explicit controls, this notion corresponds to *intermediate local minimizers* of rank two introduced in [20] and then studied there and in other publications; see, e.g., [22, 27] and the references therein. Quite recently, such minimizers have been revisited in [19] for controlled sweeping processes different from (1.2); namely, for those where continuous control actions enter the moving set $C(t) = C(u(t))$. It is easy to see that *strong* $\mathcal{C} \times L^2$- local minimizers of (P) with $\bar{x}(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ fall into the category of Definition 2.1, but not vice versa.

In the general setting of $W^{1,2} \times L^2$-local minimizers we need to use a certain relaxation procedure in the line of Bogolyubov and Young that has been well understood in the calculus of variations and optimal control; see, e.g., [17, 16, 22, 26, 27] for more recent publications in the case of differential inclusions. Taking into account the convexity and closedness of the normal cone $N(x; C)$ and the compactness of the set $g(x, U)$, the *relaxed* version (R) of problem (P) consists of minimizing the cost functional (1.1) on absolutely continuous trajectories of the convexified differential inclusion

$$\dot{x}(t)\in-N\big(x(t);C\big)+\mathrm{co}\,g\big(x(t),U\big)\ \text{ a.e. }\ t\in[0,T],\ x(0)=x_0\in C\subset\mathbb{R}^n,\tag{2.3}$$

where 'co' signifies the convex hull of the set. Then we come up with the following notion.

Definition 2.2 *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a feasible pair for (P). We say that it is a* RELAXED $W^{1,2} \times L^2$-LOCAL MINIMIZER *for (P) if there is $\epsilon > 0$ such that*

$$\varphi\big(\bar{x}(T)\big) \leq \varphi\big(x(T)\big) \quad \textit{whenever} \quad \int_0^T \Big(\|\dot{x}(t) - \dot{\bar{x}}(t)\|^2 + \|u(t) - \bar{u}(t)\|^2\Big)\, dt < \epsilon,$$

where $u(\cdot)$ is a measurable control with $u(t) \in \operatorname{co} U$ a.e. on $[0, T]$, and where $x(\cdot)$ is a trajectory of the convexified inclusion (2.3) that can be strongly approximated in $W^{1,2}([0, T]; \mathbb{R}^n)$ by feasible trajectories to (P) generated by piecewise constant controls $u_m(\cdot)$ on $[0, T]$ the convex combinations of which strongly converges to $u(\cdot)$ in the norm topology $L^2([0, T]; \mathbb{R}^d)$.

Since step functions are dense in the space $L^2([0, T]; \mathbb{R}^d)$, we obviously have that there is no difference between $W^{1,2} \times L^2$-local minimizers for (P) and their relaxed counterparts provided that the sets $g(x, U)$ and U are *convex*, which is not assumed in what follows. Moreover, it is possible to deduce from the proofs of [16, Theorem 2] and [26, Theorem 4.2] that any strong local minimizer for (P) is automatically a relaxed one under the assumptions made, but we are not going to pursue this issue here.

Consider further a set-valued mapping $F\colon \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ defined by

$$F(x, u) := N(x; C) - g(x, u) \tag{2.4}$$

and deduce from the Motzkin's theorem of the alternative the representation

$$F(x, u) := \Big\{ \sum_{j \in I(x)} \lambda^j x_*^j \;\Big|\; \lambda^j \geq 0 \Big\} - g(x, u). \tag{2.5}$$

3 Discrete Approximations of Feasible Solutions

In this section we start developing the *method of discrete approximations* to study the sweeping control problem (P) under our standing assumptions. For simplicity, consider the standard Euler explicit scheme for the replacement of the time derivative in (1.2) by

$$\dot{x}(t) \approx \frac{x(t+h) - x(t)}{h} \text{ as } h \downarrow 0,$$

which we formalize as follows. For any $m \in I\!N$ denote by

$$\Delta_m := \big\{0 = t_m^0 < t_m^1 < \ldots < t_m^{2^m} = T\big\} \text{ with } h_m := t_m^{i+1} - t_m^i = \frac{T}{2^m}$$

the uniform discrete mesh on $[0, T]$ and define the sequence of discrete-time systems

$$x_m^{i+1} \in x_m^i - h_m F(x_m^i, u_m^i),\ i = 0, \ldots, 2^m - 1, \quad x_m^0 := x_0 \in C, \tag{3.1}$$

where we have $u_m^i \in U$ due to the definition of F in (2.4). Let $I_m^i := [t_m^{i-1}, t_m^i)$.

The next result provides a constructive approximation of *any* feasible process for (P) by feasible solutions to a perturbation of (3.1) appropriately extended to the continuous-time interval $[0, T]$. This result plays a major role in the entire subsequent procedure to derive necessary optimality conditions for (P) while certainly being of its independent interest. Recall that a *representative* of a given measurable function on $[0, T]$ is a function that agrees with the given one for a.e. $t \in [0, T]$.

Theorem 3.1 *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a feasible pair for problem (P) such that $\bar{u}(\cdot)$ is of bounded variation (BV) while admitting a right continuous representative on $[0, T]$, which we keep denoting by $\bar{u}(\cdot)$. In addition to* (H1) *and* (H2)*, suppose that the mapping $g(x, u)$ is locally Lipschitzian in both variables around $(\bar{x}(t), \bar{u}(t))$ for all $t \in [0, T]$. Then for each $m \in \mathbb{N}$ there exist state-control pairs $(x_m(t), u_m(t))$ and perturbation terms $r_m(t) \geq 0$ and $\rho_m(t) \in \mathbb{B}$ as $0 \leq t \leq T$ satisfying the following:*
(a) *The sequence of control mappings $u_m\colon [0, T] \to U$, which are constant on each interval I_m^i, converges*

to $\bar{u}(\cdot)$ strongly in $L^2([0,T];\mathbb{R}^d)$ and pointwise on $[0,T]$.
(b) *The sequence of continuous state mappings $x_m\colon [0,T]\to\mathbb{R}^n$, which are affine on each interval I_m^i, converges strongly in $W^{1,2}([0,T];\mathbb{R}^n)$ to $\bar{x}(\cdot)$ while satisfying the state constraints*

$$x_m(t_m^i)=\bar{x}(t_m^i)\in C \text{ for each } i=1,\ldots,2^m \text{ with } x_m(0)=x_0. \tag{3.2}$$

(c) *For all $t\in(t_m^{i-1},t_m^i)$ and $i=1,\ldots,2^m$ we have the differential inclusions*

$$\dot{x}_m(t)\in -N\big(x_m(t_m^i);C\big)+g\big(x_m(t_m^i),u_m(t)\big)+r_m(t)\rho_m(t), \tag{3.3}$$

where the mappings $r_m\colon[0,T]\to[0,\infty)$ and $\rho_m\colon[0,T]\to\mathbb{B}$ are constant on each interval I_m^i with

$$r_m(\cdot)\to 0 \quad \text{in } L^2(0,T) \ \text{ as } \ m\to\infty. \tag{3.4}$$

In the proof of Theorem 3.1 we use the following important lemma, which can be distilled from the book by Brézis [5, Proposition 3.3].

Lemma 3.2 *Given a feasible solution $(\bar{x}(\cdot),\bar{u}(\cdot))$ to (P) under the assumptions of Theorem 3.1, we have:*

- **(i)** *$\bar{x}(\cdot)$ is Lipschitz continuous on $[0,T]$ and right differentiable for every $t\in[0,T)$;*
- **(ii)** *the sweeping differential inclusion*

$$\dot{\bar{x}}(t)\in -N\big(\bar{x}(t);C\big)+g\big(\bar{x}(t),\bar{u}(t)\big),$$

with the right derivative $\dot{\bar{x}}(t)$ taken from (i) *and the right continuous representative of the control $\bar{u}(t)$, is satisfied for each $t\in[0,T)$.*

Now we are ready to proceed with the proof of the major Theorem 3.1.

Proof of Theorem 3.1. Fix $m\in\mathbb{N}$ and for all $t\in[t_m^i,t_m^{i+1})$ and $i=0,\ldots,2^m-1$ define

$$u_m(t):=\bar{u}(t_m^{i+1}),\quad x_m(t):=\bar{x}(t_m^i)+(t-t_m^i)\frac{\bar{x}(t_m^{i+1})-\bar{x}(t_m^i)}{h_m}.$$

Then denote by $\omega_m(\cdot)$ the right derivative of $x_m(\cdot)$, for which we have the representation

$$\omega_m(t)=\omega_m^i:=\frac{\bar{x}(t_m^{i+1})-\bar{x}(t_m^i)}{h_m} \ \text{ whenever } \ t\in[t_m^i,t_m^{i+1}),\ i=0,\ldots,2^m-1.$$

It follows from the right continuity of $\bar{u}(t)$ that $u_m(t)\to\bar{u}(t)$ as $m\to\infty$ for all $t\in[0,T)$. Hence we get that $u_m(\cdot)\to\bar{u}(\cdot)$ strongly in $L^2(0,T)$ by the Lebesgue dominated convergence theorem, which verifies (a). To prove (b) and (c), let $\bar{t}$ be a nodal point of the m-th mesh that by construction remains a nodal point for all m'-mesh with $m'\ge m$. Denote by $i_m(\bar{t})$ the index i such that $\bar{t}=i\frac{T}{2^m}$ and deduce that

$$\lim_{m\to\infty}\omega_m^{i_m(\bar{t})}=\dot{\bar{x}}(\bar{t}) \tag{3.5}$$

from Lemma 3.2(i). We claim now that

$$\lim_{m\to\infty}\|\omega_m-\dot{\bar{x}}\|_{L^2(0,T)}=0. \tag{3.6}$$

Indeed, since $\bar{x}$ is Lipschitz continuous by (i) in Lemma 3.2, by using the dominated convergence theorem it is sufficient to prove that $\omega_m(t)\to\dot{\bar{x}}(t)$ a.e. in $[0,T]$. To proceed, set $\tau_m(t)$ to be the unique nodal point t_m^i such that $t\in[t_m^i,t_m^{i+1})$ and then observe that for a.e. $t\in[0,T]$ we have

$$\begin{aligned}\omega_m(t) &= \frac{1}{h_m}\Big(\frac{\bar{x}(\tau_m(t)+h_m)-\bar{x}(t)}{\tau_m(t)+h_m-t}\big(\tau_m(t)+h_m-t\big)+\frac{\bar{x}(t)-\bar{x}(\tau_m(t))}{t-\tau_m(t)}\big(t-\tau_m(t)\big)\Big)\\ &\quad\text{(since } \bar{x} \text{ is differentiable at } t \text{ for a.e. } t\in[0,T])\\ &= \frac{1}{h_m}\Big(\big(\dot{\bar{x}}(t)+o(1)\big)\big(\tau_m(t)+h_m-t\big)+\big(\dot{\bar{x}}(t)+o(1)\big)\big(t-\tau_m(t)\big)\Big)\to\dot{\bar{x}}(t)\end{aligned}$$

as $m \to \infty$. This verifies therefore the claimed convergence in (3.6).

Recalling again Lemma 3.2, at each nodal point $\bar{t}$ we get

$$\lim_{m\to\infty} \omega_m^{i_m(\bar{t})} = \dot{\bar{x}}(\bar{t}) \in -N\big(\bar{x}(\bar{t});C\big) + g\big(\bar{x}(\bar{t}),\bar{u}(\bar{t})\big).$$

Pick $\bar{\zeta} \in N(\bar{x}(\bar{t});C)$ with $\dot{\bar{x}}(\bar{t}) = -\bar{\zeta} + g(\bar{x}(\bar{t}),\bar{u}(\bar{t}))$. Recall that

$$\omega_m^{i_m(\bar{t})} + \bar{\zeta} - g\big(x_m(\bar{t}),u_m(\bar{t})\big) = \omega_m^{i_m(\bar{t})} + \bar{\zeta} - g\big(x_m(\bar{t}),\bar{u}(\bar{t}+h_m)\big).$$

Remembering that $x_m(\bar{t}) = \bar{x}(\bar{t})$ for each m tells us that the last expression tends to zero as $m \to \infty$ due to the right continuity of $\bar{u}(t)$ and to the (Lipschitz) continuity of g. Thus there exists a sequence $\{r_m(\bar{t})\}$ such that $r_m(\bar{t}) \downarrow 0$ as $m \to \infty$ and

$$\omega_m^{i_m(\bar{t})} \in -N(x_m(\bar{t});C) + g\big(x_m(\bar{t}),u_m(\bar{t})\big) + r_m(\bar{t})\mathbb{B}.$$

By choosing an appropriate vector $\rho_m(\bar{t}) \in \mathbb{B}$ and extending both $r_m(\bar{t})$ and $\rho_m(\bar{t})$ constantly to the interval $I_m^{i_m(\bar{t})}$, we complete the proof of the theorem. $\square$

4 Discrete Approximations of Local Optimal Solutions

As seen above, Theorem 3.1 provides a constructive discrete approximation of *any* feasible solution to problem (P) by feasible solutions to discrete-time problems, with no connections to optimization. The main goal here is to study a *given* local optimal solution to (P) by using discrete approximations as a *vehicle* to derive further necessary optimality conditions for it. To proceed in this direction, we construct a sequence of discrete-time optimization problems such that their optimal solutions always exist and strongly converge in the sense below to the given local minimizer of the original sweeping control problem.

Our main attention in this section is paid to *relaxed* $W^{1,2} \times L^2$*-local minimizers* $(\bar{x}(\cdot),\bar{u}(\cdot))$ for (P) introduced in Definition 2.2 while recalling that the relaxation is not needed if either the set $g(x,U)$ is convex, or $(\bar{x}(\cdot),\bar{u}(\cdot))$ is a *strong* local minimizer for (P); see the discussions in Section 2.

Given a relaxed $W^{1,2} \times L^2$-local minimizer $(\bar{x}(\cdot),\bar{u}(\cdot))$, we construct the following family of discrete-time problems (P_m), $m \in I\!N$, where F is defined in (2.4), $r_m(\cdot)$ and $\rho_m(\cdot)$ are as in the statement of Theorem 3.1, and where ϵ is taken from the definition of local minimizer:

$$\text{minimize } \; J_m[x_m,u_m] := \varphi\big(x_m(T)\big) + \frac{1}{2}\sum_{i=0}^{2^m-1}\int_{t_m^i}^{t_m^{i+1}} \left(\left\|\frac{x_m^{i+1}-x_m^i}{h_m} - \dot{\bar{x}}(t)\right\|^2 + \big\|u_m^i - \bar{u}(t)\big\|^2\right) dt \tag{4.1}$$

over all the discrete functions $(x_m,u_m) = (x_m^0,x_m^1,\ldots,x_m^{2^m},u_m^0,u_m^1,\ldots,u_m^{2^m-1})$ satisfying the constraints

$$x_m^{i+1} \in x_m^i - h_m\big(F(x_m^i,u_m^i) - \rho_m(t_m^i)r_m(t_m^i)\big) =: x_m^i - h_m F_m(t_m^i,x_m^i,u_m^i) \;\text{ for }\; i=0,\ldots,2^m-1, \tag{4.2}$$

where $F_m(t,x,u) := F(x,u) - r_m(t)\rho_m(t)$, and where

$$\langle x_*^j, x_m^{2^m}\rangle \le c_j \;\text{ for all }\; j=1,\ldots,s \;\text{ with }\; x_m^0 := x_0 \in C,\; u_m^0 := \bar{u}(0),$$

$$\sum_{i=0}^{2^m-1}\int_{t_m^i}^{t_m^{i+1}} \left(\left\|\frac{x_m^{i+1}-x_m^i}{h_m} - \dot{\bar{x}}(t)\right\|^2 + \big\|u_m^i - \bar{u}(t)\big\|^2\right) dt \le \frac{\epsilon}{2}, \tag{4.3}$$

$$u_m^i \in U \;\text{ for }\; i=0,\ldots,2^m-1. \tag{4.4}$$

Note that the constraints $\langle x_*^j, x_m^i\rangle \le c_j$ for all $j=1,\ldots,s$ and $i=0,\ldots,2^m-1$ are included in (4.2), since the validity of (4.2) yields the nonemptiness of $N(x_m^i;C)$ for each $i=0,\ldots,2^m-1$, which in turn implies that $x_m^i \in C$ for all $i=0,\ldots,2^m-1$.

To implement the method of discrete approximation, we have to make sure that each problem (P_m) admits an optimal solution. By taking into account Theorem 3.1, we deduce it from the classical Weierstrass existence theorem in finite dimensions due to the construction of (P_m) and the assumptions made.

Proposition 4.1 *In addition to the assumptions of Theorem* 3.1*, suppose that the cost function φ is lower semicontinuous (l.s.c.) on $\mathbb{R}^n$. Then each problem (P_m) admits an optimal solution provided that $m \in I\!N$ is sufficiently large.*

Proof. It follows from Theorem 3.1 that the set of feasible solutions (x_m, u_m) to (P_m) is nonempty for any large m. It follows from the constraint structures in (P_m) and the assumptions imposed on U and g that the feasible sets are closed. Furthermore, it easy to deduce from the localization in (4.3) that the feasible sets are bounded as well. Thus the lower semicontinuity assumption on the cost function φ ensures the existence of optimal solutions to (P_m) by the Weierstrass theorem. $\square$

Now we are ready to derive the main result of this section that establishes the strong $W^{1,2}$ convergence of any sequence $(\bar{x}_m(\cdot), \bar{u}_m(\cdot))$ of optimal solutions to (P_m), which are extended to the entire interval $[0,T]$, to the given local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ for the original problem (P).

Theorem 4.2 *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a relaxed $W^{1,2} \times L^2$-local minimizer for the sweeping control problem (P), and let φ be continuous around $\bar{x}(T)$ in addition to the assumptions of Theorem* 3.1*. Consider any sequence of optimal solutions $(\bar{x}_m(\cdot), \bar{u}_m(\cdot))$ to problems (P_m) and extend them to $[0,T]$ piecewise linearly for $\bar{x}_m(\cdot)$ and piecewise constantly for $\bar{u}_m(\cdot)$ without relabeling. Then we have the convergence*

$$\big(\bar{x}_m(\cdot), \bar{u}_m(\cdot)\big) \to \big(\bar{x}(\cdot), \bar{u}(\cdot)\big) \quad as \quad m \to \infty$$

in the strong topology of $W^{1,2}([0,T];\mathbb{R}^n) \times L^2([0,T];\mathbb{R}^d)$.

Proof. It is sufficient to show that

$$\lim_{m\to\infty} \int_0^T \Big(\|\dot{\bar{x}}_m(t) - \dot{\bar{x}}(t)\|^2 + \|\bar{u}_m(t) - \bar{u}(t)\|^2 \Big)\, dt = 0. \tag{4.5}$$

Arguing by contradiction, suppose that there exists a subsequence of the integral values γ_m in (4.5) that converges, without relabeling, to some number $\gamma > 0$. Due to (4.3), the sequence of extended optimal solutions $\{(\dot{\bar{x}}_m(\cdot), \bar{u}_m(\cdot))\}$ to (P_m) is bounded in the reflexive space $L^2([0,T];\mathbb{R}^n) \times L^2([0,T];\mathbb{R}^d)$, and thus it contains a weakly converging subsequence in this product space, again without relabeling. Denote by $(\widetilde{v}(\cdot), \widetilde{u}(\cdot))$ the limit of the latter subsequence and then let

$$\widetilde{x}(t) := x_0 + \int_0^t \widetilde{v}(\tau) d\tau \quad \text{for all} \quad t \in [0,T].$$

Since $\dot{\widetilde{x}}(t) = \widetilde{v}(t)$ for a.e. $t \in [0,T]$, we have that

$$\big(\bar{x}_m(\cdot), \bar{u}_m(\cdot)\big) \to \big(\widetilde{x}(\cdot), \widetilde{u}(\cdot)\big) \quad \text{as} \quad m \to \infty$$

in the topology of $W^{1,2}([0,T];\mathbb{R}^n) \times L^2([0,T];\mathbb{R}^d)$. Invoking the Mazur weak closure theorem tells us that there is a sequence of convex combinations of $(\bar{x}_m(\cdot), \bar{u}_m(\cdot))$, which converges to $(\widetilde{x}(\cdot), \widetilde{u}(\cdot))$ strongly in $W^{1,2}([0,T];\mathbb{R}^n) \times L^2([0,T];\mathbb{R}^d)$, and thus $(\dot{\bar{x}}_m(t), \bar{u}_m(t)) \to (\dot{\widetilde{x}}(t), \widetilde{u}(t))$ for a.e. $t \in [0,T]$ along a subsequence. Furthermore, we can clearly replace above the piecewise linear extensions of the discrete trajectories $\bar{x}_m(\cdot)$ to the interval $[0,T]$ by the trajectories of (1.2) generated by the controls $\bar{u}_m(\cdot)$ piecewise constantly extended to $[0,T]$. The obtained pointwise convergence of convex combinations allows us to conclude that $\widetilde{u}(t) \in \operatorname{co} U$ for a.e. $t \in [0,T]$ and that $\widetilde{x}(\cdot)$ satisfies the convexified differential inclusion (2.3). Passing now to the limit as $m \to \infty$ in the cost functional and constraints (4.1)–(4.4) of problem (P_m) with taking into account the assumed local continuity of φ and the constructions above, we conclude that the pair $(\widetilde{x}(\cdot), \widetilde{u}(\cdot))$ belongs to the prescribed $W^{1,2} \times L^2$-neighborhood of the given local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ and satisfies the inequality

$$J[\widetilde{x}, \widetilde{u}] + \gamma/2 \le J[\bar{x}, \bar{u}] \Longrightarrow J[\widetilde{x}, \widetilde{u}] < J[\bar{x}, \bar{u}] \tag{4.6}$$

due to the strong convergence of $(\bar{x}_m(\cdot), \bar{u}_m(\cdot))$ to $(\widetilde{x}(\cdot), \widetilde{u}(\cdot))$ and the structure of (4.1). Appealing to Definition 2.2 tells us that (4.6) contradicts the very fact that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a relaxed $W^{1,2} \times L^2$-local minimizer of (P). Thus we get (4.5) and complete the proof of the theorem. $\square$

Recalling the discussion after Definition 2.2 leads us to the following consequence of Theorem 4.2, which provides the strong approximation of local minimizers for (P) without an explicit relaxation.

Corollary 4.3 *In addition to the assumptions of Theorem* 4.2*, suppose that the sets* $g(x, U)$ *and* U *are convex. Then the convergence result of Theorem* 4.2 *holds true.*

5 Tools of Variational Analysis

The results of Section 4 make a bridge between the given local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the original problem (P) and (global) optimal solutions for the sequence of discrete approximations (P_m) that exist by Proposition 4.1 and strongly converge to $(\bar{x}(\cdot), \bar{u}(\cdot))$ by Theorem 4.2. This supports our approach to derive necessary optimality conditions for $(\bar{x}(\cdot), \bar{u}(\cdot))$ by establishing firstly necessary conditions for optimal solutions to the discrete-time problems (P_m) and then passing to the limit in them as $m \to \infty$.

Looking at the structures of each problem (P_m) and the equivalent problem of finite-dimensional mathematical programming defined in Section 6, we observe that they are always *nonsmooth* and *nonconvex*, even when the initial data of (P) possess these properties. This is due to the *graphical* set constraints associated with the discrete-time inclusions (4.2) that are generated by the normal cone mapping in (2.4).

To proceed with deriving necessary optimality conditions for (P_m) and then for (P) by passing to the limit, we have to employ appropriate generalized differential constructions of variational analysis. These constructions should be *robust*, enjoy comprehensive *calculus rules*, and such that the corresponding normal cone is *not too large* while being applied to–specifically–graphical sets. It does hold, in particular, for the Clarke normal cone $\overline{N}$, which is always a linear subspace of a maximum dimension for sets that are graphically homeomorphic to graphs of Lipschitzian functions; see [21, 25] for more details and references. For example, we have $\overline{N}((0,0); \operatorname{gph}|x|) = \mathbb{R}^2$ for the graph of the simplest convex function on $\mathbb{R}$.

All the required properties are satisfied for the generalized differential constructions initiated by the second author. Elements of the first-order theory and various applications can be found by now in many books; see, e.g., [21]–[23], [25], [27]. We refer the reader to [22, 23] and the bibliographies therein for second-order constructions used in what follows.

To briefly overview the needed notions, recall first the (Painlevé-Kuratowski) *outer limit* of a set-valued mapping/multifunction $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $\bar{x}$ with $F(\bar{x}) \neq \emptyset$ given by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \big\{ y \in \mathbb{R}^m \;\big|\; \exists \text{ sequences } \; x_k \to \bar{x},\; y_k \to y \; \text{ such that } \; y_k \in F(x_k),\; k \in I\!N \big\} \tag{5.1}$$

Given now a set $\Omega \subset \mathbb{R}^n$ locally closed around $\bar{x} \in \Omega$, we define by using (5.1) the (basic, limiting, Mordukhovich) *normal cone* to Ω at $\bar{x}$ by

$$N(\bar{x}; \Omega) = N_\Omega(\bar{x}) := \operatorname*{Lim\,sup}_{x \to \bar{x}} \big\{ \operatorname{cone}[x - \Pi(x; \Omega)] \big\}. \tag{5.2}$$

where $\Pi(x; \Omega) := \big\{ u \in \Omega \;\big|\; \|x - u\| = \operatorname{dist}(x; \Omega) \big\}$ is the Euclidean projection of x onto Ω, and where 'cone' stands for the (nonconvex) conic hull of the set. When Ω is convex, (5.2) reduces to the normal cone of convex analysis, but it is often nonconvex otherwise.

Given further a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with its domain and graph

$$\operatorname{dom} F := \big\{ x \in \mathbb{R}^n \;\big|\; F(x) \neq \emptyset \big\} \;\text{ and }\; \operatorname{gph} F := \big\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \;\big|\; y \in F(x) \big\}$$

locally closed around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, the *coderivative* of F at $(\bar{x}, \bar{y})$ is generated by (5.2) as

$$D^* F(\bar{x}, \bar{y})(u) := \big\{ v \in \mathbb{R}^n \;\big|\; (v, -u) \in N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) \big\}, \quad u \in \mathbb{R}^m. \tag{5.3}$$

When $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is single-valued and continuously differentiable ($\mathcal{C}^1$-smooth) around $\bar{x}$, we have

$$D^* F(\bar{x})(u) = \big\{ \nabla F(\bar{x})^* u \big\} \;\text{ for all }\; u \in \mathbb{R}^m$$

via the adjoint/transposed Jacobian matrix $\nabla F(\bar{x})^*$, where $\bar{y} = F(\bar{x})$ is omitted.

Let $\phi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ be an extended-real-valued l.s.c. function with

$$\operatorname{dom} \phi := \big\{ x \in \mathbb{R}^n \;\big|\; \varphi(x) < \infty \big\} \;\text{ and }\; \operatorname{epi} \phi := \big\{ (x, \alpha) \in \mathbb{R}^{n+1} \;\big|\; \alpha \geq \phi(x) \big\}$$

standing for its domain and epigraph. The (first-order) *subdifferential* of ϕ at $\bar{x} \in \operatorname{dom}\phi$ is defined geometrically via the normal cone (5.2) by

$$\partial\phi(\bar{x}) := \big\{v \in \mathbb{R}^m \;\big|\; (v,-1) \in N\big((\bar{x},\phi(\bar{x}));\operatorname{epi}\phi\big)\big\} \tag{5.4}$$

while admitting equivalent analytic representations; see, e.g., [21, 25]. Note that $N(\bar{x};\Omega) = \partial\delta(\bar{x};\Omega)$ for any $\bar{x} \in \Omega$, where $\delta(x;\Omega)$ denotes the indicator function of Ω equal to 0 for $x \in \Omega$ and ∞ otherwise. Then given a subgradient $\bar{v} \in \partial\phi(\bar{x})$ and following [21, 23], we define the *second-order subdifferential* (or *generalized Hessian*) of ϕ at $\bar{x}$ relative to $\bar{v}$ by

$$\partial^2\phi(\bar{x},\bar{v})(u) := (D^*\partial\phi)(\bar{x},\bar{v})(u), \quad u \in \mathbb{R}^n,$$

via the coderivative (5.3) of the first-order subdifferential mapping $x \mapsto \partial\phi(x)$ from (5.4). If the function ϕ is $\mathcal{C}^2$-smooth around $\bar{x}$, then we have the representation

$$\partial^2\phi(\bar{x},\bar{v})(u) = \big\{\nabla^2\phi(\bar{x})u\big\} \text{ for all } u \in \mathbb{R}^n,$$

where $\nabla^2\phi(\bar{x})$ stands for the classical (symmetric) Hessian of ϕ at $\bar{x}$ with $\bar{v} = \nabla\phi(\bar{x})$. If $\phi(x) := \delta(x;\Omega)$, then $\partial^2\varphi(\bar{x},\bar{v})(u) = (D^*N_\Omega)(\bar{x},\bar{v})(u)$ for any $\bar{v} \in N(\bar{x};\Omega)$ and $u \in \mathbb{R}^n$. The latter second-order construction is evaluated below in the case of the polyhedral set $\Omega = C$ from (1.3). To proceed, define the index sets corresponding to the generating vectors x_*^j in (1.3) by

$$I_0(w) := \big\{j \in I(x) \;\big|\; \langle x_*^j, w\rangle = c_j\big\} \text{ and } I_>(w) := \big\{j \in I(x) \;\big|\; \langle x_*^j, w\rangle > c_j\big\},\; w \in \mathbb{R}^n. \tag{5.5}$$

where $I(x)$ is taken from (2.1) with $\bar{x} := x \in C$. The next theorem provides an effective upper estimate of the coderivative of F_m from (4.2) with ensuring the equality under an additional assumption on x_*^j.

Theorem 5.1 *Given F_m in (4.2) with C from (1.3), denote $G(x) := N(x;C)$ and suppose in addition to standing assumptions that g is $\mathcal{C}^1$-smooth around the reference points. Then for any $(t,x,u) \in [0,T] \times C \times U$ and $\omega + g(x,u) + r_m(t)\rho_m(t) \in G(x)$ we have the (x,u)-coderivative upper estimate*

$$D^*F_m(t,x,u,\omega)(w) \subset \Big\{z = \Big(-\nabla_x g(x,u)^*w + \sum_{j \in I_0(w)\cup I_>(w)} \gamma^j x_*^j, -\nabla_u g(x,u)^*w\Big)\Big\}, \tag{5.6}$$

where $w \in \operatorname{dom} D^*G(x,\omega+g(x,u)+r_m(t)\rho_m(t))$, where $I_0(w)$ and $I_>(w)$ are taken from (5.5), and where $\gamma^j \in \mathbb{R}$ for $j \in I_0(w)$, while $\gamma^j \ge 0$ for $j \in I_>(w)$. Furthermore, (5.6) holds as an equality and the domain $\operatorname{dom} D^*G(x,\omega + g(x,u) + r_m(t)\rho_m(t))$ can be computed by

$$\operatorname{dom} D^*G\big(x,\omega+g(x,u)+r_m(t)\rho_m(t)\big) = \Big\{w \Big| \exists\lambda^j \ge 0 \text{ with } \omega+g(x,u) = \sum_{j\in I(x)} \lambda^j x_*^j,\; \lambda^j > 0 \Longrightarrow \langle x_*^j, w\rangle = 0\Big\}$$

provided that the generating vectors $\{x_*^j \mid j \in I(x)\}$ of the polyhedron C are linearly independent.

Proof. Picking any $w \in \operatorname{dom} D^*G(x,\omega + g(x,u) + r_m(t)\rho_m(t))$ and $z \in D^*F_m(t,x,u,y)(w)$ and then denoting $\widetilde{G}(x,u) := G(x)$ and $\widetilde{f}(x,u) := -g(x,u)$, we deduce from [22, Theorem 1.62] that

$$z \in \nabla\widetilde{f}(x,u)^*w + D^*\widetilde{G}\big(x,u,\omega + g(x,u) + r_m(t)\rho_m(t)\big)(w).$$

Observe then the obvious composition representation

$$\widetilde{G}(x,u) = G \circ \widetilde{g}(x,u) \text{ with } \widetilde{g}(x,u) := x,$$

where the latter mapping has the surjective derivative. It follows from [22, Theorem 1.66] that

$$z \in \nabla\widetilde{f}(x,u)^*w + \nabla\widetilde{g}(x,u)^*D^*G\big(x,\omega + g(x,u) + r_m(t)\rho_m(t)\big)(w). \tag{5.7}$$

Employing now in (5.7) the coderivative estimate for the normal cone mapping G obtained in [18, Theorem 4.5] with the exact coderivative calculation given in [18, Theorem 4.6] under the linear independence of the generating vectors x_*^j and also taking into account the structure of the mapping $\widetilde{f}$ in (5.7), we arrive at (5.6) and the equality therein under the aforementioned assumption. □

6 Necessary Optimality Conditions for Discrete-Time Problems

Here we derive necessary optimality conditions for solutions to each problem (P_m), $m \in I\!N$, formulated in (4.1)–(4.4). It is done by reducing each (P_m) to a nondynamic problem of nondifferentiable programming with functional and many geometric constraints, then employing necessary optimality conditions for the latter problem obtained in terms of generalized differential constructions of Section 5, and finally expressing the obtained conditions in terms of the given data of (P_m) by using calculus rules of generalized differentiation. In this way we arrive at the following necessary conditions, which will be further specified below by applying the second-order calculations presented in Section 5.

Theorem 6.1 *Let $(\bar{x}_m, \bar{u}_m) = (\bar{x}_m^0, \ldots, \bar{x}_m^{2^m}, \bar{u}_m^0, \ldots, \bar{u}_m^{2^m-1})$ be an optimal solution to problem (P_m) along which the general assumptions of Theorem* 5.1 *are satisfied. Suppose in addition that the cost function φ is Lipschitz continuous around the point $\bar{x}_m(T)$. Then there are elements $\lambda_m \geq 0$, $\psi_m = (\psi_m^0, \ldots, \psi_m^{2^m-1})$ with $\psi_m^i \in N(\bar{u}_m^i; U)$, as $i = 0, \ldots, 2^m - 1$, $\xi_m = (\xi_m^1, \ldots, \xi_m^s) \in \mathbb{R}_+^s$, and $p_m^i \in \mathbb{R}^n$ as $i = 0, \ldots, 2^m$ satisfying the conditions*

$$\lambda_m + \|\xi_m\| + \sum_{i=0}^{2^m-1} \left\|p_m^i\right\| + \|\psi_m\| \neq 0, \tag{6.1}$$

$$\xi_m^j \big(\langle x_*^j, x_m^{2^m}\rangle - c_j\big) = 0, \quad j = 1, \ldots, s, \tag{6.2}$$

$$-p_m^{2^m} = \lambda_m \vartheta_m^{2^m} + \sum_{j=1}^{s} \xi_m^j x_*^j \quad \text{with} \quad \vartheta_m^{2^m} \in \partial\varphi(\bar{x}_m^{2^m}), \tag{6.3}$$

$$\begin{gathered}\Big(\frac{p_m^{i+1} - p_m^i}{h_m}, -\frac{1}{h_m}\lambda_m \theta_m^{iu}, \frac{1}{h_m}\lambda_m \theta_m^{iy} - p_m^{i+1}\Big) \\ \in \Big(0, \frac{1}{h_m}\psi_m^i, 0\Big) + N\Big(\Big(\bar{x}_m^i, \bar{u}_m^i, -\frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m}\Big); \operatorname{gph} F_m\Big)\end{gathered} \tag{6.4}$$

for $i = 0, \ldots, 2^m - 1$, where we use the notation

$$\theta_m^i = \big(\theta_m^{iy}, \theta_m^{iu}\big) := \Big(\int_{t_m^i}^{t_m^{i+1}} \Big(\frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} - \dot{\bar{x}}(t)\Big)\,dt, \int_{t_m^i}^{t_m^{i+1}} \big(\bar{u}_m^i - \bar{u}(t)\big)\,dt\Big). \tag{6.5}$$

Proof. Denote $z := (x_m^0, \ldots, x_m^{2^m}, u_m^0, \ldots, u_m^{2^m-1}, y_m^0, \ldots, y_m^{2^m-1}) \in \mathbb{R}^{(2\cdot 2^m+1)n+2^m\cdot d}$, where the starting point x_m^0 is fixed. Taking $\epsilon > 0$ from (P_m), consider the following problem of mathematical programming (MP) with respect to the variable z:

$$\text{minimize} \;\; \phi_0(z) := \varphi\big(x(T)\big) + \frac{1}{2}\sum_{i=0}^{2^m-1} \int_{t_m^i}^{t_m^{i+1}} \left\|\big(y_m^i - \dot{\bar{x}}(t), u_m^i - \bar{u}(t)\big)\right\|^2 dt$$

subject to finitely many equality, inequality, and geometric constraints given by

$$\phi(z) := \sum_{i=0}^{2^m-1} \int_{t_m^i}^{t_m^{i+1}} \left\|\big(y_m^i, u_m^i\big) - \big(\dot{\bar{x}}(t), \bar{u}(t)\big)\right\|^2 dt - \frac{\epsilon}{2} \leq 0,$$

$$g_i(z) := x_m^{i+1} - x_m^i - h_m y_m^i = 0, \quad i = 0, \ldots, 2^m - 1,$$

$$h_j(z) := \langle x_*^j, x_m^{2^m}\rangle - c_j \leq 0, \quad j = 1, \ldots, s,$$

$$z \in \Xi_i := \Big\{(x_m^0, \ldots, y_m^{2^m-1}) \in \mathbb{R}^{(2\cdot 2^m+1)n+2^m\cdot d} \;\Big|\; -y_m^i \in F_m(t_m^i, x_m^i, u_m^i)\Big\}, \quad i = 0, \ldots, 2^m - 1,$$

$$z \in \Xi_{2^m} := \big\{(x_m^0, \ldots, y_m^{2^m-1}) \in \mathbb{R}^{(2\cdot 2^m+1)n+2^m\cdot d} \;\big|\; x_m^0 \;\text{ is fixed}\big\},$$

$$z \in \Omega_i := \big\{(x_m^0, \ldots, y_m^{2^m-1}) \in \mathbb{R}^{(2\cdot 2^m+1)n+2^m\cdot d} \;\big|\; u_m^i \in U\big\}, \quad i = 0, \ldots, 2^m - 1.$$

Necessary optimality conditions for (MP) in terms of the generalized differential tools reviewed above can be deduced from [23, Proposition 6.4 and Theorem 6.5]. We specify them for the optimal solution

$$\bar{z} := \left(\bar{x}_m^0, \ldots, \bar{x}_m^{2^m}, \bar{u}_m^0, \ldots, \bar{u}_m^{2^m-1}, \bar{y}_m^0, \ldots, \bar{y}_m^{2^m-1}\right)$$

to (MP). It follows from Theorem 4.2 that the inequality constraint in (MP) defined by ϕ is inactive for large m, and so the corresponding multiplier does not appear in the optimality conditions. Thus we can find $\lambda_m \geq 0$, $\xi_m = (\xi_m^1, \ldots, \xi_m^s) \in \mathbb{R}_+^s$, $p_m^i \in \mathbb{R}^n$ as $i = 1, \ldots, 2^m$, and

$$z_i^* = \left(x_{0i}^*, \ldots, x_{2^m i}^*, u_{0i}^*, \ldots, u_{(2^m-1)i}^*, y_{0i}^*, y_{1i}^*, \ldots, y_{(2^m-1)i}^*\right), \quad i = 0, \ldots, 2^m,$$

which are not zero simultaneously while satisfying the conditions

$$z_i^* \in \begin{cases} N(\bar{z}; \Xi_i) + N(\bar{z}; \Omega_i) & \text{if } i \in \{0, \ldots, 2^m - 1\}, \\ N(\bar{z}; \Xi_i) & \text{if } i = 2^m, \end{cases} \tag{6.6}$$

$$-z_0^* - \ldots - z_{2^m}^* \in \lambda_m \partial \phi_0(\bar{z}) + \sum_{j=1}^{s} \xi_m^j \nabla h_j(\bar{z}) + \sum_{i=0}^{2^m-1} \nabla g_i(\bar{z})^* p_m^{i+1},$$

$$\xi_m^j h_j(\bar{z}) = 0, \quad j = 1, \ldots, s. \tag{6.7}$$

Note that the first line in (6.6) comes by applying the normal cone intersection formula from [21, Corollary 3.5] to $\bar{z} \in \Omega_i \cap \Xi_i$ for $i = 0, \ldots, 2^m - 1$, where the required qualification condition

$$N(\bar{z}; \Omega_i) \cap \big(-N(\bar{z}; \Xi_i)\big) = \{0\}, \quad i = 0, \ldots, 2^m - 1,$$

follows directly from the coderivative estimate (5.6) of Theorem 5.1 under the imposed PLICQ. We deduce from the structure of Ω_i and Ξ_i that the inclusions in (6.6) can be equivalently written as

$$\left(x_{ii}^*, u_{ii}^* - \psi_m^i, -y_{ii}^*\right) \in N\left(\left(\bar{x}_m^i, \bar{u}_m^i, -\frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m}\right); \operatorname{gph} F_m(t_m^i, \cdot, \cdot)\right) \text{ for } i = 0, \ldots, 2^m - 1 \tag{6.8}$$

with every other components of z_i^* equal to zero, where $\psi_m^i \in N(\bar{u}_m^i; U)$ for all $i = 0, \ldots, 2^m - 1$. Observe furthermore that x_{0m}^* and u_{0m}^* determined by the normal cone to Ξ_{2^m} are the only nonzero components of $z_{2^m}^*$. This implies by using (6.6) and (6.7) that

$$-z_0^* - \ldots - z_{2^m}^* \in \lambda_m \partial \phi_0(\bar{z}) + \sum_{j=1}^{s} \xi_m^j \nabla h_j(\bar{z}) + \sum_{i=0}^{2^m-1} \nabla g_i(\bar{z})^* p_m^{i+1}$$

with $\xi_m^j\big(\langle z_m^{j2^m}, x_m^{2^m}\rangle - c_m^{j2^m}\big) = 0$, $j = 1, \ldots, s$. Using the expressions for ϕ_0, g_i, and h_j above together with the elementary subdifferential sum rule from [21, Proposition 1.107] gives the calculations

$$\Big(\sum_{j=1}^{s} \xi_m^j \nabla h_j(\bar{z})\Big)_{x_m^{2^m}} = \Big(\sum_{j=1}^{s} \xi_m^j x_*^j\Big),$$

$$\Big(\sum_{i=0}^{2^m-1} \nabla g_i(\bar{z})^* p_m^{i+1}\Big)_{x_m^i} = \begin{cases} -p_m^1 & \text{if } i = 0, \\ p_m^i - p_m^{i+1} & \text{if } i = 1, \ldots, 2^m - 1, \\ p_m^{2^m} & \text{if } i = 2^m, \end{cases}$$

$$\Big(\sum_{i=0}^{2^m-1} \nabla g_i(\bar{z})^* p_m^{i+1}\Big)_{y_m^i} = \left(-h_m p_m^1, -h_m p_m^2, \ldots, -h_m p_m^{2^m}\right),$$

$$\partial \phi_0(\bar{z}) = \partial \varphi(\bar{x}_m^m) + \frac{1}{2} \sum_{i=0}^{2^m-1} \nabla \rho_i(\bar{z}) \text{ with } \rho_i(\bar{z}) := \int_{t_m^i}^{t_m^{i+1}} \left\| \Big(\frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} - \dot{\bar{x}}(t), \bar{u}_m^i - \bar{u}(t)\Big) \right\|^2 dt.$$

The set $\lambda_m\partial\phi_0(\bar z)$ is represented as the collection of

$$\lambda_m\big(0,\ldots,0,\vartheta_m^{2^m},\theta_m^{0u},\ldots,\theta_m^{(2^m-1)u},\theta_m^{0y},\ldots,\theta_m^{(2^m-1)y}\big)\ \text{ with }\ \vartheta_m^{2^m}\in\partial\varphi(\bar x_m^{2^m}),$$

$$(\theta_m^{iu},\theta_m^{iy})=\left(\int_{t_m^i}^{t_m^{i+1}}\big(\bar u_m^i-\bar u(t)\big)\,dt,\ \int_{t_m^i}^{t_m^{i+1}}\left(\frac{\bar x_m^{i+1}-\bar x_m^i}{h_m}-\dot{\bar x}(t)\right)dt\right),\quad i=0,\ldots,2^m-1.$$

Thus we obtain the following relationships

$$-x_{00}^*-x_{02^m}^*=-p_m^1, \tag{6.9}$$

$$-x_{ii}^*=p_m^i-p_m^{i+1},\quad i=1,\ldots,2^m-1, \tag{6.10}$$

$$0=\lambda_m\vartheta_m^{2^m}+p_m^{2^m}+\sum_{j=1}^s\xi_m^jx_*^j\ \text{ with }\ \vartheta_m^{2^m}\in\partial\varphi(\bar x_m^{2^m}), \tag{6.11}$$

$$-u_{00}^*=\lambda_m\theta_m^{0u}\ \text{ and }\ -u_{ii}^*=\lambda_m\theta_m^{iu},\quad i=1,\ldots,2^m-1, \tag{6.12}$$

$$-y_{ii}^*=\lambda_m\theta_m^{iy}-h_mp_m^{i+1},\quad i=0,\ldots,2^m-1, \tag{6.13}$$

which allow us to arrive at all the necessary optimality conditions claimed in the theorem. Indeed, observe first that (6.7) yields (6.2). Extending p_m by $p_m^0:=x_{02^m}^*$ ensures that (6.3) follows from (6.11). Then we deduce from (6.10), (6.12), and (6.13) that

$$\frac{x_{ii}^*}{h_m}=\frac{p_m^{i+1}-p_m^i}{h_m},\quad \frac{u_{ii}^*}{h_m}=-\frac{1}{h_m}\lambda_m\theta_m^{iu},\ \text{ and }\ \frac{y_{ii}^*}{h_m}=-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}.$$

Substituting this into the left-hand side of (6.8) justifies the discrete-time adjoint inclusion (6.4).

Finally, to verify (6.1) we argue by contradiction and suppose that $\lambda_m=0,\xi_m=0,\psi_m=0$, and $p_m^i=0$ as $i=0,\ldots,2^m-1$, which yield $x_{02^m}^*=p_m^0=0$. Then it follows from (6.11) that $p_m^{2^m}=0$, and so $p_m^i=0$ whenever $i=0,\ldots,2^m$. By (6.9) and (6.10) we get $x_{ii}^*=0$ for all $i=0,\ldots,2^m-1$. Using (6.12) tells us that $u_{ii}^*=0$ as $i=1,\ldots,2^m-1$. Since the first condition in (6.12) yields also $u_{00}^*=0$, it follows that $u_{ii}^*=0$ for $i=0,\ldots,2^m-1$. In addition we have by (6.13) that $y_{ii}^*=0$ for all $i=0,\ldots,2^m-1$. Remembering that the components of z_i^* different from $(x_{ii}^*,u_{ii}^*,y_{ii}^*)$ are zero for $i=0,\ldots,2^m-1$ ensures that $z_i^*=0$ for $i=0,\ldots,2^m-1$ and similarly $z_{2^m}^*=0$. Therefore $z_i^*=0$ for all $i=0,\ldots,2^m$, which violates the nontriviality condition for (MP) and thus completes the proof. $\square$

The next theorem applies to (6.4) the calculation result of Theorem 4.2 and provides in this way necessary optimality conditions for problem (P_m) expressed entirely via its initial data.

Theorem 6.2 *Let $(\bar x_m,\bar u_m)$ be an optimal solution to problem (P_m) formulated in (4.1)–(4.4), where the cost function φ is locally Lipschitzian around $\bar x_m(T)$, and where the sweeping mapping F is defined in (2.4). Using the notation and assumptions of Theorem 5.1, take $(\theta_m^{iu},\theta_m^{iy})$ from (6.5). Then for all $m\in\mathbb{N}$ there exist dual elements (λ_m,ψ_m,p_m) as in Theorem 6.1 together with vectors $\eta_m^i\in\mathbb{R}_+^s$ for $i=0,\ldots,2^m$ and $\gamma_m^i\in\mathbb{R}^s$ for $i=0,\ldots,2^m-1$ satisfying the nontriviality conditions*

$$\lambda_m+\left\|\eta_m^{2^m}\right\|+\sum_{i=0}^{2^m-1}\left\|p_m^i\right\|+\|\psi_m\|\neq 0, \tag{6.14}$$

the primal-dual relationships given for all $i=0,\ldots,2^m-1$ and $j=1,\ldots,s$ by

$$r_m(t_m^i)\rho_m(t_m^i)-\frac{\bar x_m^{i+1}-\bar x_m^i}{h_m}+g(\bar x_m^i,\bar u_m^i)=\sum_{j\in I(\bar x_m^i)}\eta_m^{ij}x_*^j, \tag{6.15}$$

$$\begin{aligned}\frac{p_m^{i+1}-p_m^i}{h_m}\quad&=-\nabla_xg(\bar x_m^i,\bar u_m^i)^*\Big(-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}\Big)\\&\quad+\sum_{j\in I_0(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})\cup I_>(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})}\gamma_m^{ij}x_*^j,\end{aligned} \tag{6.16}$$

$$-\frac{1}{h_m}\lambda_m\theta_m^{iu}-\frac{1}{h_m}\psi_m^i=-\nabla_u g(\bar{x}_m^i,\bar{u}_m^i)^*\Big(-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}\Big) \tag{6.17}$$

with $\psi_m^i\in N(\bar{u}_m^i;U)$ *as* $i=0,\ldots,2^m-1$ *taken from Theorem* 6.1, *the transversality condition*

$$-p_m^{2^m}=\lambda_m\vartheta_m^{2^m}+\sum_{j=1}^s\eta_m^{2^m j}x_*^j\ \ \textit{with}\ \ \vartheta_m^{2^m}\in\partial\varphi(\bar{x}_m^{2^m}) \tag{6.18}$$

and such that the following implications hold for $i=0,\ldots,2^m-1$ *and* $j=1,\ldots,s$*:*

$$\Big[\langle x_*^j,\bar{x}_m^i\rangle<c_j\Big]\Longrightarrow\eta_m^{ij}=0, \tag{6.19}$$

$$\begin{cases}\Big[j\in I_>(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})\Big]\Longrightarrow\gamma_m^{ij}\ge 0,\\ \Big[j\notin I_0(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})\cup I_>(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})\Big]\Longrightarrow\gamma_m^{ij}=0.\end{cases} \tag{6.20}$$

We also have the complementary slackness condition (6.2) *together with*

$$\big[\langle x_*^j,\bar{x}_m^i\rangle<c_j\big]\Longrightarrow\gamma_m^{ij}=0\ \ \textit{for}\ \ i=0,\ldots,2^m-1\ \ \textit{and}\ \ j=1,\ldots,s, \tag{6.21}$$

$$\big[\langle x_*^j,\bar{x}_m^{2^m}\rangle<c_j\big]\Longrightarrow\eta_m^{2^m j}=0\ \ \textit{for}\ \ j=1,\ldots,s, \tag{6.22}$$

Furthermore, the linear independence of the vectors $\{x_*^j|\ j\in I(\bar{x}_m^i)\}$ *ensures the implication*

$$\eta_m^{ij}>0\Longrightarrow\Big[\Big\langle x_*^j,p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy}\Big\rangle=0\Big] \tag{6.23}$$

Assuming in addition that the matrices $\nabla_u g(\bar{x}_m^i,\bar{u}_m^i)$ *are of full rank for all* $i=0,\ldots,2^m-1$ *and* $m\in\mathbb{N}$ *sufficiently large, we get the enhanced nontriviality condition*

$$\lambda_m+\|p_m^0\|+\|\psi_m\|\neq 0. \tag{6.24}$$

Proof. Using the necessary optimality conditions of Theorem 6.1, we can rewrite (6.4) as

$$\Big(\frac{p_m^{i+1}-p_m^i}{h_m},-\frac{1}{h_m}\lambda_m\theta_m^{iu}-\frac{1}{h_m}\psi_m^i\Big)\in D^*F_m\Big(\bar{x}_m^i,\bar{u}_m^i,-\frac{\bar{x}_m^{i+1}-\bar{x}_m^i}{h_m}\Big)\Big(-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}\Big) \tag{6.25}$$

for all $i=0,\ldots,2^m-1$ by the coderivative definition (5.3). Taking into account that

$$r_m(t_m^i)\rho_m(t_m^i)-\frac{\bar{x}_m^{i+1}-\bar{x}_m^i}{h_m}+g(\bar{x}_m^i,\bar{u}_m^i)\in G(\bar{x}_m^i)\ \ \text{for}\ \ i=0,\ldots,2^m-1 \tag{6.26}$$

with $G(x)=N(x;C)$, we find vectors $\eta_m^i\in\mathbb{R}_+^s$ as $i=0,\ldots,2^m-1$ such that conditions (6.15) and (6.19) hold. Employing now the coderivative evaluation (5.6) from Theorem 5.1 with $x:=\bar{x}_m^i$, $u:=\bar{u}_m^i$, $\omega:=-\frac{\bar{x}_m^{i+1}-\bar{x}_m^i}{h_m}$, and $w:=-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}$ for $i=0,\ldots,2^m-1$ gives us $\gamma_m^i\in\mathbb{R}^s$ and the relationships

$$\Big(\frac{p_m^{i+1}-p_m^i}{h_m},-\frac{1}{h_m}\lambda_m\theta_m^{iu}-\frac{\psi_m^{iu}}{h_m}\Big)$$

$$=\begin{pmatrix}-\nabla_x g(\bar{x}_m^i,\bar{u}_m^i)^*\Big(-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}\Big)+\displaystyle\sum_{j\in I_0(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})\cup I_>(p_m^{i+1}-\frac{1}{h_m}\lambda_m\theta_m^{iy})}\gamma_m^{ij}x_*^j,\\ -\nabla_u g(\bar{x}_m^i,\bar{u}_m^i)^*\Big(-\frac{1}{h_m}\lambda_m\theta_m^{iy}+p_m^{i+1}\Big)\end{pmatrix},$$

$$\psi_m^i\in N(\bar{u}_m^i;U)\ \ \text{as}\ \ i=0,\ldots,2^m-1.$$

This ensures the validity of all the conditions in (6.16), (6.17), (6.20), and (6.21). Denoting $\eta_m^{2^m} := \xi_m$ with ξ_m taken from Theorem 6.1, we get $\eta_m^i \in \mathbb{R}_+^s$ for all $i = 0, \ldots, 2^m$ and deduce (6.14) and (6.18) from those in (6.1) and (6.3). Implications (6.22) follow directly from (6.2) and the definition of $\eta_m^{2^m}$.

Assume finally that the generating vectors $\{x_*^j \mid j \in I(\bar{x}_m^i)\}$ are linearly independent. In this case we deduce from (6.15) and (6.25) and the domain formula in Theorem 5.1 that condition (6.23) is satisfied. It remains to verify the enhanced nontriviality (6.24) under the additional assumption on the full rank of the matrices $\nabla_u g(\bar{x}_m^i, \bar{u}_m^i)$. Suppose on the contrary that $\lambda_m = 0$, $p_m^0 = 0$, and $\psi_m = 0$. Then $p_m^{i+1} = 0$ as $i = 0, \ldots, 2^m - 1$ by (6.17). Then it follows from (6.16) the equality

$$\sum_{j \in I_0(p_m^{i+1} - \frac{1}{h_m}\lambda_m \theta_m^{iy}) \cup I_>(p_m^{i+1} - \frac{1}{h_m}\lambda_m \theta_m^{iy})} \gamma_m^{ij} x_*^j = 0.$$

Invoking now (6.18) and $p_m^{2^m} = 0$ tells us that $\sum_{j=1}^s \eta_m^{2^m j} x_*^j = 0$. This implies by definition (2.1) of the active constraint indices and the imposed linear independence of x_*^j over this index set that $\eta_m^{2^m} = 0$. Thus (6.14) is violated, which verifies (6.24) and completes the proof of the theorem. $\square$

7 Optimality Conditions for the Controlled Sweeping Process

In this section we derive necessary optimality conditions for the local minimizer under consideration in the original problem (P) by passing to the limit as $m \to \infty$ in the necessary optimality conditions of Theorem 6.1 for the discrete-time problems (P_m). Furnishing the limiting procedure requires the usage of Theorem 4.2 and the tools of generalized differentiation reviewed in Section 5.

Theorem 7.1 *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a relaxed $W^{1,2} \times L^2$-local minimizer of problem (P) such that $\bar{u}(\cdot)$ is of bounded variation and admits a right continuous representative on $[0,T]$. In addition to* (H1)–(H3), *suppose that $g(\cdot,\cdot)$ is $\mathcal{C}^1$-smooth around $(\bar{x}(t), \bar{u}(t))$ with the full rank of the matrices $\nabla_u g(\bar{x}(t), \bar{u}(t))$ on $[0,T]$, and that φ is locally Lipschitzian around $\bar{x}(T)$. Then there exist a multiplier $\lambda \geq 0$, a signed vector measure $\gamma = (\gamma^1, \ldots, \gamma^s) \in C^*([0,T];\mathbb{R}^s)$ as well as adjoint arcs $p(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ and $q(\cdot) \in BV([0,T];\mathbb{R}^n)$ such that the following conditions are fulfilled:*

(i) *The* PRIMAL-DUAL DYNAMIC RELATIONSHIPS *consisting of:*

- *The* PRIMAL ARC REPRESENTATION

$$-\dot{\bar{x}}(t) = \sum_{j=1}^s \eta^j(t) x_*^j - g\big(\bar{x}(t), \bar{u}(t)\big) \quad \textit{for a.e. } \ t \in [0,T), \tag{7.1}$$

where the functions $\eta^j(\cdot) \in L^2([0,T);\mathbb{R}_+)$ are uniquely determined for a.e. $t \in [0,T)$ by representation (7.1). *In fact,* (7.1) *holds at all $t \in [0,T)$ provided that $\dot{\bar{x}}(t)$ denotes the right derivative.*

- *The* ADJOINT DYNAMIC SYSTEM

$$\dot{p}(t) = -\nabla_x g\big(\bar{x}(t), \bar{u}(t)\big)^* q(t) \quad \textit{for a.e. } \ t \in [0,T], \tag{7.2}$$

where the right continuous representative of $q(\cdot)$, with the same notation, satisfies

$$q(t) = p(t) - \int_{(t,T]} \sum_{j=1}^s d\gamma^j(\tau) x_*^j, \tag{7.3}$$

for all $t \in [0,T]$ except at most a countable subset, and moreover $p(T) = q(T)$.

- *The* LOCAL MAXIMUM PRINCIPLE

$$\psi(t) := \nabla_u g\big(\bar{x}(t), \bar{u}(t)\big)^* q(t) \in \operatorname{co} N\big(\bar{u}(t); U\big) \quad \textit{for a.e. } \ t \in [0,T], \tag{7.4}$$

which gives us the GLOBAL MAXIMIZATION CONDITION

$$\big\langle \psi(t), \bar{u}(t) \big\rangle = \max_{u \in U} \big\langle \psi(t), u \big\rangle \quad \textit{for a.e. } \ t \in [0,T] \tag{7.5}$$

provided that the control set U is convex.

• *The* DYNAMIC COMPLEMENTARY SLACKNESS CONDITIONS

$$\big\langle x_*^j, \bar{x}(t)\big\rangle < c_j \Longrightarrow \eta^j(t) = 0 \;\; \text{and} \;\; \eta^j(t) > 0 \Longrightarrow \big\langle x_*^j, q(t)\big\rangle = 0 \tag{7.6}$$

for a.e. $t \in [0,T)$ and all $j = 1, \ldots, s$ provided that LICQ at $\bar{x}(t)$ is additionally imposed.

(ii) *The* ENDPOINT RELATIONSHIPS *consisting of:*

• *The* TRANSVERSALITY CONDITIONS*: there exist numbers $\eta^j(T) \geq 0$ for $j \in I(\bar{x}(T))$ such that*

$$-p(T) - \sum_{j \in I(\bar{x}(T))} \eta^j(T) x_*^j \in \lambda \partial \varphi\big(\bar{x}(T)\big) \;\; \text{and} \;\; \sum_{j \in I(\bar{x}(T))} \eta^j(T) x_*^j \in N\big(\bar{x}(T); C\big). \tag{7.7}$$

• *The* ENDPOINT COMPLEMENTARY SLACKNESS CONDITIONS

$$\big\langle x_*^j, \bar{x}(T)\big\rangle < c_j \Longrightarrow \eta^j(T) = 0, \tag{7.8}$$

with the numbers $\eta^j(T)$ are from (7.7).

(iii) *The* MEASURE NONATOMICITY CONDITION: *If $t \in [0,T)$ and $\langle x_*^j, \bar{x}(t)\rangle < c_j$ for all $j = 1, \ldots, s$, then there exists a neighborhood V_t of t in $[0,T)$ such that $\gamma(V) = 0$ for all the Borel subsets V of V_t.*

(iv) *The* NONTRIVIALITY RELATIONSHIPS *consisting of:*

• *The* GENERAL NONTRIVIALITY CONDITIONS*: we always have*

$$(\lambda, p, \|\gamma\|_{TV}) \neq 0, \tag{7.9}$$

which is equivalent to $(\lambda, p, q) \neq 0$ provided that LICQ holds at $\bar{x}(t)$ on $[0,T]$.

• *The* ENHANCED NONTRIVIALITY CONDITION

$$(\lambda, p) \neq 0 \tag{7.10}$$

holds provided that $\langle x_*^j, \bar{x}(t)\rangle < c_j$ for all $t \in [0,T)$ and all indices $j = 1, \ldots, s$.

Proof. Given the local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (P), construct the discrete-time problems (P_m) for which optimal solutions $(\bar{x}_m(\cdot), \bar{u}_m(\cdot))$ exist by Proposition 3.1 and converge to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the sense of Theorem 4.2. We derive each of the claimed necessary conditions in (P) by passing to the limit from those in Theorem 6.1. Let us split the derivation into several steps.

Step 1: *Verifying the primal equation and the dynamic complementary slackness conditions.* First we prove (7.1) together with the first complementarity condition in (7.6). Based on (6.5), define the functions

$$\theta_m(t) := \frac{\theta_m^i}{h_m} \;\; \text{for} \;\; t \in [t_m^i, t_m^{i+1}) \;\text{and}\; i = 0, \ldots, 2^m - 1$$

on $[0,T]$ whenever $m \in I\!N$. It is easy to see that

$$\begin{aligned}
\int_0^T \|\theta_m^y(t)\|^2 \, dt &= \sum_{i=0}^{2^m-1} \frac{\big\|\theta_m^{iy}\big\|^2}{h_m} \leq \frac{1}{h_m} \sum_{i=0}^{2^m-1} \Big(\int_{t_m^i}^{t_m^{i+1}} \Big\| \dot{\bar{x}}(t) - \frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} \Big\| \, dt \Big)^2 \\
&\leq \sum_{i=0}^{2^m-1} \int_{t_m^i}^{t_m^{i+1}} \Big\| \dot{\bar{x}}(t) - \frac{\bar{x}_m^{i+1} - \bar{x}_m^i}{h_m} \Big\|^2 dt = \int_0^T \|\dot{\bar{x}}(t) - \dot{\bar{x}}_m(t)\|^2 \, dt.
\end{aligned}$$

Using the strong convergence $(\bar{x}_m(\cdot), \bar{u}_m(\cdot)) \to (\bar{x}(\cdot), \bar{u}(\cdot))$ in Theorem 4.2 ensures that

$$\int_0^T \|\theta_m^y(t)\|^2 \, dt \leq \int_0^T \|\dot{\bar{x}}(t) - \dot{\bar{x}}_m(t)\|^2 \, dt \to 0 \;\; \text{as} \;\; m \to \infty. \tag{7.11}$$

This implies that a subsequence of $\{\theta^y_m(t)\}$ converges, without relabeling, to zero a.e. on $[0,T]$. Likewise

$$\begin{aligned}\int_0^T \Big\|\theta^u_m(t)\Big\|^2 dt &= \sum_{i=0}^{2^m-1}\frac{\Big\|\theta^{iu}_m\Big\|^2}{h_m}\le\frac{1}{h_m}\sum_{i=0}^{2^m-1}\Big(\int_{t^i_m}^{t^{i+1}_m}\|\bar u^i_m-\bar u(t)\|\,dt\Big)^2\\ &\le \sum_{i=0}^{2^m-1}\int_{t^i_m}^{t^{i+1}_m}\Big\|\bar u^i_m-\bar u(t)\Big\|^2 dt=\int_0^T\|\bar u_m(t)-\bar u(t)\|^2\,dt,\end{aligned}$$

which tells us, again by using Theorem 4.2, that

$$\int_0^T \|\theta^u_m(t)\|^2\,dt\le\int_0^T\|\bar u_m(t)-\bar u(t)\|^2\,dt\to 0\ \text{ as }\ m\to\infty, \tag{7.12}$$

and so $\theta^u_m(t)\to 0$ for a.e. $t\in[0,T]$ along a subsequence. The assumed PLICQ along $\bar x(\cdot)$ and the robustness of this condition yields by the choice of x^j_* and the convergence in Theorem 4.2 that the vectors $\{x^j_*\mid j\in I(\bar x^i_m)\}$ are positively linearly independent for each $i=1,\ldots,2^m$ and $m\in I\!N$ sufficiently large.

Taking $\eta^i_m\in\mathbb{R}^s_+$ from Theorem 6.2, we construct the piecewise constant functions $\eta_m(\cdot)$ on $[0,T)$ by $\eta_m(t):=\eta^i_m$ for $t\in[t^i_m,t^{i+1}_m)$ as $i=0,\ldots,2^m-1$. It follows from (6.15) that

$$-\dot{\bar x}_m(t)=\sum_{j=1}^s\eta^j_m(t)x^j_*-g\big(\bar x_m(t^i_m),\bar u_m(t^i_m)\big)-r_m(t^i_m)\rho_m(t^i_m)\ \text{ whenever }\ t\in(t^i_m,t^{i+1}_m),\quad m\in I\!N. \tag{7.13}$$

Furthermore, we get $-\dot{\bar x}(t)\in G(\bar x(t))-g(\bar x(t),\bar u(t))$ for a.e. $t\in[0,T)$ with the mapping $G(\cdot)=N(\cdot;C)$, which is measurable by [25, Theorem 4.26]. The well-known measurable selection result (see, e.g., [25, Corollary 4.6]) allows us to find nonnegative measurable functions $\eta^j(\cdot)$ on $[0,T)$ for $j=1,\ldots,s$ such that equation (7.1) holds. Combining (7.13) and (7.1) ensures that

$$\dot{\bar x}(t)-\dot{\bar x}_m(t)=\sum_{j=1}^s\big[\eta^j_m(t)-\eta^j(t)\big]x^j_*+g\big(\bar x(t),\bar u(t)\big)-g\big(\bar x_m(t^i_m),\bar u_m(t^i_m)\big)-r_m(t^i_m)\rho_m(t^i_m)$$

for $t\in(t^i_m,t^{i+1}_m)$ and $i=0,\ldots,2^m-1$. It follows from the imposed PLICQ that the functions $\eta^j_m(t)$ and $\eta^j(t)$ are uniquely defined for a.e. $t\in[0,T)$ and belong to $L^2([0,T);\mathbb{R}_+)$. The constructions and arguments presented above readily imply the estimate

$$\Big\|\sum_{j=1}^s\big[\eta^j(t)-\eta^j_m(t)\big]x^j_*\Big\|_{L^2}\le\|\dot{\bar x}_m(t)-\dot{\bar x}(t)\|_{L^2}+\Big\|g\big(\bar x(t),\bar u(t)\big)-g\big(\bar x_m(t),\bar u_m(t)\big)\Big\|_{L^2}+r_m(t^i_m)$$

whenever $t\in(t^i_m,t^{i+1}_m)$. Passing to the limit therein with the usage of Theorem 4.2 gives us

$$\sum_{j\in I(\bar x(t))}\big[\eta^j(t)-\eta^j_m(t)\big]x^j_*\to 0\ \text{ as }\ m\to\infty\ \text{ for a.e. }\ t\in[0,T)$$

and yields the a.e. convergence $\eta_m(t)\to\eta(t)$ on $[0,T)$ by the imposed (robust) LICQ in this case. Then the first complementary slackness condition in (7.6) follows from (6.19).

Step 2: *Continuous-time extensions of approximating dual elements.* In the notation of Theorem 6.2, define $q_m(t)$ by extending p^i_m piecewise linearly on $[0,T]$ with $q_m(t^i_m):=p^i_m$ for $i=0,\ldots,2^m$. Construct further $\gamma_m(t)$ and $\psi_m(t)$ on $[0,T]$ by

$$\gamma_m(t):=\gamma^i_m,\quad \psi_m(t):=\frac{1}{h_m}\psi^i_m\ \text{ for }\ t\in[t^i_m,t^{i+1}_m)\ \text{ and }\ i=0,\ldots,2^m-1 \tag{7.14}$$

with $\gamma_m(T):=0$ and $\psi_m(T):=0$. Define now the functions

$$\nu_m(t):=\max\big\{t^i_m\big|\ t^i_m\le t,\ 0\le i\le 2^m-1\big\}\ \text{ for all }\ t\in[0,T],\quad m\in I\!N,$$

and deduce respectively from (6.16) and (6.17) that

$$\begin{aligned}\dot q_m(t) = \quad & -\nabla_x g\big(\bar x_m(\nu_m(t)),\bar u_m(\nu_m(t))\big)^*\big(-\lambda_m\theta^y_m(t)+q_m(\nu_m(t)+h_m)\big)\\ & + \sum_{j\in I_0(-\lambda_m\theta^y_m(t)+q_m(\nu_m(t)+h_m))\cup I_>(-\lambda_m\theta^y_m(t)+q_m(\nu_m(t)+h_m))}\gamma^j_m(t)x^j_*,\qquad\text{and}\end{aligned}\tag{7.15}$$

$$-\lambda_m\theta^u_m(t)-\psi_m(t)=-\nabla_u g\big(\bar x_m(\nu_m(t)),\bar u_m(\nu_m(t))\big)^*\big(-\lambda_m\theta^y_m(t)+q_m(\nu_m(t)+h_m)\big)\tag{7.16}$$

for every $t\in(t^i_m,t^{i+1}_m)$ and $i=0,\ldots,2^m-1$. Next we define the adjoint arcs $p_m(\cdot)$ to $[0,T]$ by

$$p_m(t):=q_m(t)+\int_t^T\Big(\sum_{j=1}^s\gamma^j_m(\tau)x^j_*\Big)d\tau\ \text{ for every }\ t\in[0,T].\tag{7.17}$$

This shows that $p_m(T)=q_m(T)$ and that

$$\dot p_m(t)=\dot q_m(t)-\sum_{j=1}^s\gamma^j_m(t)x^j_*\ \text{ a.e. }\ t\in[0,T].\tag{7.18}$$

The latter implies due to (7.15), (6.20), and the index definitions in (5.5) that

$$\dot p_m(t)=-\nabla_x g\big(\bar x_m(\nu_m(t)),\bar u_m(\nu_m(t))\big)^*\big(-\lambda_m\theta^y_m(t)+q_m(\nu_m(t)+h_m)\big)\tag{7.19}$$

for every $t\in(t^i_m,t^{i+1}_m)$ and $i=0,\ldots,2^m-1$. Define now the vector measures γ^{mes}_m on $[0,T]$ by

$$\int\limits_B d\gamma^{mes}_m:=\int\limits_B\sum_{i=0}^{2^m-1}\frac{1}{h_m}\gamma_m(t)\mathbb{1}_{I^i_m}(t)dt\tag{7.20}$$

for every Borel subset $B\subset[0,T]$, where $\mathbb{1}_\Omega$ signifies the characteristic function of the set Ω that equals to 1 on Ω and to 0 otherwise. We drop for simplicity the index "mes" in what follows if no confusion arises. Since all the expressions in the statement of Theorem 6.1 are positively homogeneous of degree one with respect to $(\lambda_m,p_m,\gamma_m,\psi_m)$, the enhanced nontriviality condition (6.24) and the constructions above allow us to normalize them by imposing the sequential equality

$$\lambda_m+\|p_m(T)\|+\|q_m(0)\|+\sum_{j=1}^s\sum_{i=0}^{2^m-1}\big|\gamma^{ij}_m\big|+\int_0^T\|\psi_m(t)\|\,dt=1,\quad m\in I\!N,\tag{7.21}$$

which tells us, in particular, that all the terms in (7.21) are uniformly bounded.

Step 3: *Verifying the dual dynamic relationships and the maximization conditions.* By (7.21), suppose without loss of generality that $\lambda_m\to\lambda$ as $m\to\infty$ for some $\lambda\ge 0$. To prove the uniform boundedness of the sequence $\{p^0_m,\ldots,p^{2^m}_m\}_{m\in I\!N}$ for all $i=0,\ldots,2^m-1$, $m\in\mathbb{N}$, observe first from (6.16) that

$$p^{i+1}_m=p^i_m-h_m\nabla_x g(\bar x^i_m,\bar u^i_m)^*\Big(-\frac{1}{h_m}\lambda_m\theta^{iy}_m+p^{i+1}_m\Big)+h_m\sum_{j=1}^s\gamma^{ij}_m x^j_*$$

for all $i=0,\ldots,2^m-1$. This implies that

$$\begin{aligned}\|p^i_m\|&\le\|p^{i+1}_m\|+h_m\|\nabla_x g(\bar x^i_m,\bar u^i_m)^*\|\cdot\Big\|\Big(-\frac{1}{h_m}\lambda_m\theta^{iy}_m+p^{i+1}_m\Big)\Big\|+h_m\Big\|\sum_{j=1}^s\gamma^{ij}_m x^j_*\Big\|\\ &=\big(1+h_m\|\nabla_x g(\bar x^i_m,\bar u^i_m)^*\|\big)\|p^{i+1}_m\|+h_m\lambda_m\|\theta^y_m(t^i_m)\|\cdot\|\nabla_x g(\bar x^i_m,\bar u^i_m)^*\|+h_m\Big\|\sum_{j=1}^s\gamma^{ij}_m x^j_*\Big\|\end{aligned}$$

whenever $i = 0, \ldots, 2^m - 1$. It follows from (7.11), (7.21), and (H2) that the quantities $\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)$, $\lambda_m \theta_m^{iy}$ are uniformly bounded for $i = 0, \ldots, 2^m - 1$. Thus we find a constant $M_1 > 0$ such that

$$h_m \lambda_m \left\|\theta_m^y(t_m^i)\right\| \cdot \left\|\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)^*\right\| \le M_1 h_m \left\|\theta_m^y(t_m^i)\right\| = M_1 \sqrt{h_m \int_{t_m^i}^{t_m^{i+1}} \|\theta_m^y(t)\|^2\, dt}$$

for all $i = 0, \ldots, 2^m - 1$ and $m \in I\!N$. It implies that

$$\sum_{i=0}^{2^m-1} h_m \lambda_m \left\|\theta_m^y(t_m^i)\right\| \cdot \left\|\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)^*\right\| \le M_1 \sqrt{\int_0^T \|\theta_m^y(t)\|^2\, dt} \to 0 \;\text{ as }\; m \to \infty.$$

On the other hand, we get due to (7.21) that

$$h_m \sum_{i=0}^{2^m-1} \Big\| \sum_{j=1}^s \gamma_m^{ij} x_*^j \Big\| = \int_0^T \Big\| \sum_{j=1}^s \gamma_m^j(t) x_*^j \Big\| dt \le 1. \tag{7.22}$$

Considering now the numbers

$$A_m^i := h_m \lambda_m \left\|\theta_m^y(t_m^i)\right\| \cdot \left\|\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)^*\right\| + h_m \Big\| \sum_{j=1}^s \gamma_m^{ij} x_*^j \Big\|$$

for $i = 0, \ldots, 2^m - 1$ and using the aforementioned uniform boundedness, find a constant $M_2 > 0$ such that $\sum_{i=0}^{2^m-1} A_m^i \le M_2$. Combining the latter with the estimates above tells us that

$$\|p_m^i\| \le \big(1 + M_1 h_m\big)\|p_m^{i+1}\| + A_m^i, \quad i = 0, \ldots, 2^m - 1. \tag{7.23}$$

Proceeding further step by step, we get the inequalities

$$\begin{aligned} \|p_m^i\| &\le \big(1 + M_1 h_m\big)^{2^m - i}\|p_m^{2^m}\| + \sum_{j=i}^{2^m-1} A_m^j (1 + M_1 h_m)^{j-i} \\ &\le e^{M_1 T} + e^{M_1 T} \sum_{i=0}^{2^m-1} A_m^i \le e^{M_1}(1 + M_2) \;\text{ for }\; i = 2, \ldots, 2^m - 1, \end{aligned}$$

which imply in turn the estimate

$$\|p_m^i\| \le M_3 \;\text{ for some }\; M_3 > 0 \;\text{ and all }\; i = 2, \ldots, 2^m - 1.$$

Hence the boundedness of $\{p_m^0\}$ and $\{p_m^1\}$ follows from (7.23) and the boundedness of $\{p_m^i\}_{2 \le i \le 2^m}$, which thus justifies the boundedness of the entire bundle $\{(p_m^0, \ldots, p_m^{2^m})\}_{m \in I\!N}$.

To verify the uniform boundedness properties of $q_m(\cdot)$, derive from their constructions and (6.16) that

$$\begin{aligned} \sum_{i=0}^{2^m-1} \left\|q_m(t_m^{i+1}) - q_m(t_m^i)\right\| &\le h_m \sum_{i=0}^{2^m-1} \|\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)^*(-\lambda_m \theta_m^y(t^i) + p_m^{i+1})\| \\ &\quad + \int_0^T \Big\| \sum_{j=1}^s \gamma_m^j(t) x_*^j \Big\| dt \end{aligned} \tag{7.24}$$

and observe furthermore that

$$h_m \sum_{i=0}^{2^m-1} \|\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)^*(-\lambda_m \theta_m^y(t^i) + p_m^{i+1})\| \le T \max_{0 \le i \le 2^m-1} \big\{\|\nabla_x g(\bar{x}_m^i, \bar{u}_m^i)^*(-\lambda_m \theta_m^y(t^i) + p_m^{i+1})\|\big\}.$$

The latter ensures the boundedness of the first term on the right-hand side of (7.24) due to the boundedness of $\{p_m^i\}_{m\in I\!N}$, while the boundedness of the second term therein follows from (7.22). Thus we get from (7.24) that the functions $q_m(\cdot)$ on $[0,T]$ are of uniform bounded variation on $[0,T]$ and that

$$2\,\|q_m(t)\| - \|q_m(0)\| - \|q_m(T)\| \le \|q_m(t) - q_m(0)\| + \|q_m(T) - q_m(t)\| \le \mathrm{var}(q_m;[0,T])$$

for all $t \in [0,T]$. Thus the sequence $\{q_m(\cdot)\}$ is bounded on $[0,T]$ since the boundedness of $\{q_m(0)\}$ and $\{q_m(T)\}$ follows from (7.21). Applying now Helly's selection theorem gives us a function of bounded variation $q(\cdot)$ such that $q_m(t) \to q(t)$ as $m \to \infty$ pointwise on $[0,T]$.

We see from (6.16), (7.20) and (7.21) that the measure sequence $\{\gamma_m\}$ is bounded in $C^*([0,T];\mathbb{R}^s)$. Thus the weak* sequential compactness of bounded sets in this space allows us to find a measure $\gamma \in C^*([0,T];\mathbb{R}^s)$ such that $\{\gamma_m\}$ weak* converges to γ in $C^*([0,T];\mathbb{R}^s)$ along a subsequence. It follows from (7.19), (7.21), and the uniform boundedness of $q_m(\cdot)$ on $[0,T]$ that the sequence $\{p_m(\cdot)\}$ is bounded in $W^{1,2}([0,T];\mathbb{R}^n)$ and thus weakly compact in this space. By Mazur's theorem we conclude that a sequence of convex combinations of $\dot p_m(\cdot)$ converges to some $\dot p(\cdot) \in L^2([0,T];\mathbb{R}^n)$ a.e. pointwise on $[0,T]$. This gives us (7.2) by passing to the limit along (7.19) as $m \to \infty$ with the usage of (7.11) up to choosing the right continuous representation of q. Note also that

$$\Big\|\int_t^T \sum_{j=1}^s \gamma_m^j(\tau)x_*^j d\tau - \int_{(t,T]} \sum_{j=1}^s d\gamma^j(\tau)x_*^j\Big\| \to 0 \ \text{ as } \ m \to \infty$$

for all $t \in [0,T]$ except a countable subset of $[0,T]$ by the weak* convergence of the measures γ_m to γ in $C^*([0,T];\mathbb{R}^n)$; cf. [27, p. 325] for similar arguments. Hence we get the convergence

$$\int_t^T \sum_{j=1}^s \gamma_m^j(\tau)x_*^j d\tau \to \int_{(t,T]} \sum_{j=1}^s d\gamma^j(\tau)x_*^j \ \text{ on } \ [0,T] \ \text{ as } \ m \to \infty$$

and thus arrive at (7.3) by passing to the limit in (7.17). The claimed condition $p(T) = q(T)$ in (i) follows directly by passing to the limit in the equalities $p_m(T) = q_m(T)$, $m \in I\!N$. The second complementary slackness condition in (7.6) follows from (6.23) under LICQ while arguing by contradiction with the usage of the established a.e. pointwise convergence of the functions involved therein.

To finish the proof of (i), it remains to verify the validity of the local maximum principle in (7.4) and the global maximization condition (7.5) with referring the reader to Remark 7.2 for more discussions about the terminology. We get (7.4) by passing to the strong L^2-limit as $k \to \infty$ in the relationships (6.17) and in the inclusions $\psi_m^i \in N(\bar u_m^i;U)$, $i = 0,\ldots,2^m-1$, of Theorem 6.2 as $k \to \infty$. This is achieved by employing the strong convergence of the discrete optimal solutions from Theorem 4.2, the convergence of $(\theta_m^y(t),\theta_m^u(t)) \to (0,0)$ for a.e. $t \in [0,T]$ obtained above as well as the robustness of the normal cone (5.2). If U is convex, the maximization condition (7.5) follows directly from (7.4) due to the structure (1.4) of the normal cone to convex sets.

Step 4: *Verifying the endpoint relationships.* Relying on the discrete necessary optimality conditions of Theorem 6.2, define $\eta_m(T) := \eta_m^{2^m}$ and deduce from the normalization of the nontriviality conditions in (6.14) that the sequence $\{\eta_m^{2^m}\}$ converges, along a subsequence, to some vector $(\eta^1(T),\ldots,\eta^{2^m}(T))$. It follows from (6.18) and representation (2.5) that

$$-p_m^{2^m} - \lambda_m \vartheta_m^{2^m} = \sum_{j=1}^s \eta_m^{2^m j} x_*^{2^m} = \sum_{j\in I(\bar x_m^{2^m})} \eta_m^{2^m j} x_*^{2^m} \in N(\bar x_m^{2^m};C), \tag{7.25}$$

where $\eta_m^{2^m j} = 0$ for $j \in \{1,\ldots,s\} \setminus I(\bar x_m^{2^m})$. Denoting $\zeta_m := \sum_{j\in I(\bar x_m^{2^m})} \eta_m^{2^m j} x_*^{2^m}$, observe that a subsequence $\{\zeta_m\}$ converges to some $\zeta \in \mathbb{R}^n$ due to the boundedness of λ_m by (7.21) and the convergence of $\{p_m^{2^m}\}$ and $\{\bar x_m^{2^m}\}$ with taking into account the robustness of the subdifferential. It follows from the robustness of the normal cone in (7.25), the convergence of $\bar x_m^{2^m} \to \bar x(T)$, and the inclusion $I(\bar x_m^{2^m}) \subset I(\bar x(T))$ for all m sufficiently large, that $\zeta \in N(\bar x(T);C)$. Thus we get from (6.18) that

$$-p_m^{2^m} - \zeta_m \in \lambda_m \partial\varphi(\bar x_m^{2^m}) \ \text{ for all } \ m \in I\!N.$$

Passing now to the limit therein as $m \to \infty$ verifies both transversality inclusions in (7.7). The fulfillment of the claimed endpoint complementary slackness conditions in (7.8) follows from the above proof by passing to the limit as $m \to \infty$ in their discrete counterparts established in (6.22) of Theorem 6.2.

Step 5: *Verifying measure nonatomicity.* Take $t \in [0, T)$ with $\langle x_*^j, \bar{x}(t)\rangle < c_j$ for all $j = 1, \ldots, s$ and by continuity of $\bar{x}(\cdot)$ find a neighborhood V_t of t such that $\langle x_*^j, \bar{x}(\tau)\rangle < c_j$ whenever $\tau \in V_t$ and $j = 1, \ldots, s$. Invoking Theorem 4.2 tells us that $\langle x_*^j, \bar{x}_m(t_m^i)\rangle < c_j$ if $t_m^i \in V_t$ for all $j = 1, \ldots, s$ and $m \in I\!N$ sufficiently large. Then we deduce from (6.21) that $\gamma_m(t) = 0$ on any Borel subset V of V_t. Hence

$$\|\gamma_m\|(V) = \int_V d\|\gamma_m\| = \int_V \|\gamma_m(t)\| dt = 0 \tag{7.26}$$

by the construction of γ_m in (7.20). Passing now to the limit therein and taking into account the measure convergence obtained above, we get $\|\gamma\|(V) = 0$, which justifies the claimed measure nonatomicity.

Step 6: *Verifying nontriviality conditions.* We begin with the proof of the nontriviality condition (7.9) under the general assumptions of the theorem. Arguing by contradiction, suppose that $\lambda = 0$, $p(t) = 0$ for all $t \in [0, T]$, and $\|\gamma\|_{TV} = 0$. This implies by (7.3) that $q(t) = 0$ for the right continuous representative of $q(\cdot)$. The assumed negation of nontriviality tells us that $\lambda_m \to 0$ and $p_m(t) \to 0$ for all $t \in [0, T]$. Furthermore, with the usage of (7.17) and the convergence result from [27, p. 325], we get that

$$\begin{aligned}
\lim_{m\to\infty} q_m(t) &= \lim_{m\to\infty} \left(p_m(t) - \int_t^T \sum_{j=1}^s \gamma_m^j(\tau) x_*^j d\tau \right) \\
&= \lim_{m\to\infty} p_m(t) - \int_t^T \sum_{j=1}^s \gamma(\tau) x_*^j d\tau = 0.
\end{aligned}$$

Combining this with (7.16), we deduce that $\psi_m(t) \to 0$ a.e. $t \in [0, T]$, which implies that $\sum_{j=1}^s \sum_{i=0}^{2^m-1} \left|\gamma_m^{ij}\right| \to 1$ as $m \to \infty$ due to (7.21). Define now the sequence of measurable mappings $\alpha_m : [0, T] \to \mathbb{R}^s$ as follows:

$$\alpha_m^i(t) := \frac{\gamma_m^i(t)}{|\gamma_m^i(t)|} \ \text{ if } \ \gamma_m^i(t) \neq 0 \ \text{ and } \ \alpha_m^i(t) := 0 \ \text{ if } \ \gamma_m^i(t) = 0, \ i = 1, \ldots, s, \ \text{ for all } \ t \in [0, T].$$

Taking into account the Jordan measure decompositions $\gamma_m = (\gamma_m)^+ - (\gamma_m)^-$ and $\gamma = \gamma^+ - \gamma^-$ as well as the separability of $C^*([0, T]; \mathbb{R}^s)$, we find a subsequence of measures $\{\gamma_m\}$ with the weak* convergence

$$\{(\gamma_m)^+\} \xrightarrow{w^*} \gamma^+ \ \text{ and } \ \{(\gamma_m)^-\} \xrightarrow{w^*} \gamma^- \ \text{ in } \ C^*\left([0, T]; \mathbb{R}^s\right).$$

Since the sequence $\{\alpha_m(\cdot)\}$ is bounded on $[0, T]$, a straightforward application of [27, Proposition 9.2.1] (where our index m corresponds to the index i in that result) with $A = A_m := [-1, 1]^s$ for all $m \in I\!N$ therein yields the existence of Borel measurable vector functions $\alpha^+, \alpha^- : [0, T] \to \mathbb{R}^s$ satisfying

$$\{\alpha_m^i(\gamma_m^i)^+\} \xrightarrow{w^*} (\alpha^+)^i(\gamma^+)^i \ \text{ and } \ \{\alpha_m^i(\gamma_m^i)^-\} \xrightarrow{w^*} (\alpha^-)^i(\gamma^-)^i, \ i = 1, \ldots, s.$$

With the understanding that, in the sequel of this proof, for s-dimensional vectors α and measures γ, we mean $\alpha d\gamma = (\alpha^i d\gamma^i, \ldots, \alpha^s d\gamma^s)$, the following relationships hold:

$$\begin{aligned}
\left\| \int_{[0,T]\setminus S} \alpha^+(t) d\gamma^+(t) - \int_{[0,T]\setminus S} \alpha^-(t) d\gamma^-(t) \right\| &= \lim_{m\to\infty} \left\| \int_{[0,T]\setminus S} \alpha_m(t) d\left(\gamma_m\right)^+(t) - \int_{[0,T]\setminus S} \alpha_m(t) d\left(\gamma_m\right)^-(t) \right\| \\
&= \lim_{m\to\infty} \left\| \int_{[0,T]\setminus S} \alpha_m(t) d\gamma_m(t) \right\| \\
&= \lim_{m\to\infty} \left\| \int_{[0,T]\setminus S} \left(\alpha_m^1(t) d\gamma_m^1(t), \ldots, \alpha_m^s(t) d\gamma_m^s(t) \right) \right\| \\
&= \lim_{m\to\infty} \left\| \left(\sum_{i=0}^{2^m-1} \left|\gamma_m^{i1}\right|, \ldots, \sum_{i=0}^{2^m-1} \left|\gamma_m^{is}\right| \right) \right\| \\
&= \lim_{m\to\infty} \sqrt{ \sum_{j=1}^{s} \left[\sum_{i=0}^{2^m-1} \left|\gamma_m^{ij}\right| \right]^2 } \\
&\geq \lim_{m\to\infty} \frac{1}{\sqrt{s}} \sum_{j=1}^{s} \sum_{i=0}^{2^m-1} \left|\gamma_m^{ij}\right| = \frac{1}{\sqrt{s}} > 0,
\end{aligned}$$

where $S \subset [0,T]$ is a countable set. On the other hand, we have

$$\begin{aligned}
\left\| \int_{[0,T]\setminus S} \alpha^+(t) d\gamma^+(t) - \int_{[0,T]\setminus S} \alpha^-(t) d\gamma^-(t) \right\| &\leq \left\| \int_{[0,T]\setminus S} \alpha^+(t) d\gamma^+(t) \right\| + \left\| \int_{[0,T]\setminus S} \alpha^-(t) d\gamma^-(t) \right\| \\
&\leq \int_{[0,T]\setminus S} d\|\gamma^+(t)\| + \int_{[0,T]\setminus S} d\|\gamma^-(t)\| \\
&\leq \left\|\gamma^+\right\|_{TV} + \left\|\gamma^-\right\|_{TV} = \|\gamma\|_{TV}.
\end{aligned}$$

Combining the above inequalities gives us $\|\gamma\|_{TV} > 0$, which contradicts the assumed fact that $\|\gamma\|_{TV} = 0$. Hence we justify the fulfillment of the general nontriviality condition (7.9).

To compare (7.9) with $(\lambda, p, q) \neq 0$, we immediately deduce from (7.3) that $[(\lambda, p, q) \neq 0] \Longrightarrow$(7.9). The converse implication is also clear under the additional LICQ assumption.

It remains to verify the validity of the *enhanced nontriviality condition* (7.10) under the interiority assumption made therein. Suppose on the contrary that $\lambda = 0$ and $p(t) = 0$ for all $t \in [0,T]$ while $\langle x_*^j, \bar{x}(t)\rangle < c_j$ for all $t \in [0,T)$ and $j = 1, \ldots, s$. It follows from the discrete endpoint complementary slackness condition (6.21), the arguments in Step 5 together with (7.3) and (7.26) that

$$q(t) = -\int_{(t,T]} \sum_{j=1}^{s} d\gamma^j(\tau) x_*^j = 0 \ \text{ for all } \ t \in [0,T] \setminus A, \tag{7.27}$$

where $A \subset [0,T]$ is a countable set. Since $q(\cdot)$ is right continuous, we always have $q(t) = 0$ in (7.27) and thus show in this way that the failure of (7.10) contradicts the validity of the general nontriviality condition (7.9). This completes the proof of the theorem. □

Remark 7.2 Note that we use the terminology of the *local maximum principle* for (7.4), since it plays a role similar to the conventional maximum principle around the optimal control $\bar{u}(t)$ and reduces to the global maximization condition (7.5) if the set U is convex. In the broad case of the duality correspondence

$$N(\bar{u}(t); U) = T^*(\bar{u}(t); U) := \left\{ v \in \mathbb{R}^n \,\middle|\, \langle v, u\rangle \leq 0 \ \text{ for all } \ u \in T\big(\bar{u}(t); U\big) \right\}$$

between the normal cone in question and some tangent cone $T(\bar{u}(t);U)$ to U at $\bar{u}(t)$, the local condition (7.4) reads as the (global) maximization

$$\big\langle\psi(t),\bar{u}(t)\big\rangle=\max_{u\in T(\bar{u}(t);U)}\big\langle\psi(t),u\big\rangle\ \text{ for a.e. }\ t\in[0,T]$$

of the *linearized Hamilton-Pontryagin function* $\langle\psi(t),u\rangle$ over the tangent cone $T(\bar{u}(t);U)$ without assuming the convexity of either the control set U or the cone $T(\bar{u}(t);U)$.

8 Numerical Examples

In this section we consider two examples illustrating some characteristic features and strength of the necessary optimality conditions for the sweeping control problem (P) obtained in Theorem 7.1.

Prior to dealing with specific examples, let us present the following useful assertion, which is a consequence of the measure nonatomicity condition.

Proposition 8.1 *Assume that $\langle x^*,\bar{x}(\tau)\rangle<c_j$ for all $\tau\in[t_1,t_2]$ with $t_1,t_2\in[0,T)$ and some vector $x^*\in\{x^j_*\mid j=1,\ldots,s\}$, and that the measure nonatomicity condition of Theorem* 7.1 *is satisfied with the measure γ. Then we have $\gamma([t_1,t_2])=0$ and $\gamma(\{\tau\})=0$ whenever $\tau\in[t_1,t_2]$, and so $\gamma((t_1,t_2))=\gamma([t_1,t_2))=\gamma((t_1,t_2])=0$.*

Proof. Pick any $\tau\in[t_1,t_2]$ with $\langle x^*_1,\bar{x}(t)\rangle<c_j$ and find by the measure nonatomicity condition a neighborhood V_τ of τ in $[0,T]$ such that $\gamma(V)=0$ for all the Borel subsets V of V_τ; in particular, $\gamma(\{\tau\})=0$. By $[t_1,t_2]\subset\bigcup_{\tau\in[t_1,t_2]}V_\tau$ and the compactness of $[t_1,t_2]$ we find $\tau_1,\ldots,\tau_l\in[t_1,t_2]$ with $[t_1,t_2]\subset\bigcup_{i=1}^l V_{\tau_i}$. Fix $i=1,\ldots,l-1$ and take $\widetilde{\tau}_i\in V_{\tau_i}\cap V_{\tau_{i+1}}$ with $[\tau_i,\widetilde{\tau}_i]\subset V_{\tau_i}$ and $[\widetilde{\tau}_i,\tau_{i+1}]\subset V_{\tau_{i+1}}$, where $\tau_1:=t_1$ and $\tau_l:=t_2$. Then we arrive at the equalities

$$\gamma([t_1,t_2])=\gamma\Big(\bigcup_{i=1}^{p-1}[\tau_i,\widetilde{\tau}_i)\cup[\widetilde{\tau}_i,\tau_{i+1})\Big)=\sum_{i=1}^{p-1}\Big(\gamma([\tau_i,\widetilde{\tau}_i))+\gamma([\widetilde{\tau}_i,\tau_{i+1}))\Big)=0,$$

which verify the claimed properties of the measure. $\square$

Our first example is two-dimensional with respect to both state and control variables.

Example 8.2 Consider the sweeping control problem of minimizing the cost functional

$$x_1(1)+x_2(1)\ \text{ subject to}$$

$$\begin{cases}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=-N\left(\begin{pmatrix}x_1\\ x_2\end{pmatrix};C\right)+\begin{pmatrix}u_1\\ u_2\end{pmatrix}\\ \qquad\text{with }\begin{pmatrix}x_1\\ x_2\end{pmatrix}(0)=\begin{pmatrix}0\\ x_2^0\end{pmatrix}\end{cases}\tag{8.1}$$

where $C:=\{(x_1,x_2)\in\mathbb{R}^2\mid x_2\geq 0\}$ and $(u_1,u_2)\in U:=[-1,1]\times[-1,1]$. We rewrite the dynamics as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}(t)=\begin{pmatrix}u_1\\ u_2\end{pmatrix}(t)+\eta(t)\begin{pmatrix}0\\ 1\end{pmatrix},\quad \eta(t)\geq 0\ \text{ a.e. }\ t\in[0,1].$$

A direct checking shows that if $x_2^0\geq 1$ then the constraint is irrelevant and the optimal control is constant being equal to $(-1,-1)$. If instead $0\leq x_2^0<1$, then the optimal couple is $\bar{u}_1(t)\equiv-1$ together with any measurable component $\bar{u}_2(t)$ such that $\bar{x}_2(1)=0$.

The conditions of Theorem 7.1 tell us that:

(1) $p=\begin{pmatrix}p_1\\ p_2\end{pmatrix}$ is constant on $[0,1]$ (by (7.2));

(2) $\begin{pmatrix}-p_1\\ -p_2\end{pmatrix}-\begin{pmatrix}0\\ -\eta(1)\end{pmatrix}=\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$, $\lambda\geq 0$ (by (7.7));

(3) $(\lambda, p, \|\gamma\|_{TV}) \neq 0$ (by (7.9));

(4) $q(t) = p - \int_{(t,1]} d\gamma(\tau) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi(t) \in N_{[-1,1]^2} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}$ (by (7.4) and (7.3));

(5) $x_2(t) > 0 \; \forall t \in [0, T) \Longrightarrow \lambda + \|p\| > 0$ (by (7.10));

(6) $\eta(t) = 0$ for a.e. $t \in [0,1]$ with $\bar{x}_2(t) > 0$ and $\left[\eta(t) > 0 \Longrightarrow q(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0\right]$ a.e. $t \in [0,1]$ (by (7.6));

(7) $d\gamma\big|_{\{t \mid \bar{x}_2(t) > 0\}} = 0$ (by the measure nonatomicity condition).

To apply these conditions, consider first the case where $x_2^0 > 1$ in which the constraint is automatically satisfied for all the trajectories. Since $\bar{x}_2(1) > 0$, we get $\eta(1) = 0$ from (6). If $\lambda = 0$, then $p \equiv 0$ and the nontriviality condition (3) is violated. Thus we can suppose that $\lambda = 1$, and so $p = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Condition (7) implies that $d\gamma = 0$ on the set in question; hence $q \equiv p = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \equiv \psi$. This shows that $\psi = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Since $\psi \in N_{[-1,1]^2} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}$, the optimal control is $\bar{u}(t) \equiv \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. It confirms that in this case we do not loose information with respect to the classical PMP.

Consider now the case where $0 \leq x_2^0 \leq 1$. Assuming that $x_2(1) > 0$ yields $\eta(1) = 0$ by (7.8). Repeating the above arguments with the usage of (4) gives us the control $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ on $[0,1]$ while implying that $x_2(1) = 0$, a contradiction. Thus we get $x_2(1) = 0$, and actually all controls u_2 satisfying this property are optimal. In particular, we obtain that $\bar{u}_2 \equiv -1$ in the case where $x_2^0 = 1$. Let us now deal with the first component u_1, which instead reveals a kind of degeneracy in the necessary conditions. Indeed, the following two cases may occur. First the reference trajectory touches the boundary only at the final time. In this case the enhanced nontriviality condition (4) holds, and the analysis goes along the same lines as for $x_2^0 = 1$. Instead, when the reference trajectory remains on the boundary on a set of positive measure, the case $(p, \lambda) = (0, 0)$ is possible (with $\gamma \neq 0$), but then the first components of both (2) and (4) provide no information on u_1. This difficulty can be overcome in this case thanks to the fact that the two variables x_1 and x_2 can be made uncoupled. Indeed the problem is equivalent to minimizing x_1 and x_2 separately, each variable being subject to the dynamics given by the correspondent component of (8.1). Then the problem involving x_1 is classical, and the optimal control $u_1 \equiv -1$ is easily obtained. On the contrary, the problem involving x_2 is of the sweeping type, and its analysis can be performed according to the previous arguments. This verifies the optimality of any control u_2 such that $x_2(1) = 0$.

Note that system (8.1) was also treated in [3], and the given discussion allows us to compare the two sets of necessary conditions: those obtained in [3] and in this paper. The conditions in [3] are generally different from the ones we establish here. Let us mention to this end that those presented here deal only with reference trajectories where the control has bounded variation, but are more detailed in comparison with the conditions in [3] and are more effective for the control u_2 while being more difficult to use for u_1. This difference can be explained by the methods that are used to obtain the necessary conditions. Actually the arguments presented here take into account the constraint at all the steps of the procedure. On the contrary, the method used in [3] is based on penalization, and so it does not see the hard constraint in the approximation steps. This explains why it behaves well with respect to u_1, which is not influenced by the constraint (indeed, the multiplier λ corresponding to the terminal cost is nondegenerate), while obtaining some information on u_2 is more difficult. Observe finally that the method developed here allows us to treat also nonconvex control sets. For example, we can consider the minimization of the same terminal cost subject to (8.1) but with the control set given by

$$U_1 = \{-1, 1\} \times \{-1, 1\}.$$

This problem enjoys relaxation stability (because the value of the nonconvex problem is the same as the convex one), and the above analysis can be performed in the same way with U_1 in place of U.

The next example is also two-dimensional while addressing a more complicated polyhedral set C in comparison with the halfspace in Example 8.2.

Example 8.3 Consider problem (P) with the following initial data:

$$n=m=2,\ T=1,\ x_0:=\Big(-\frac{1}{2},-\frac{1}{2}\Big),\ x_*^1:=(1,0),\ x_*^2:=(0,1),\ c_1=c_2=0,\ \varphi(x):=\frac{\|x\|^2}{2},\ g(u)=u,$$

where feasible controls $u(t)=(u^1(t),u^2(t))\in U$ a.e. $t\in[0,1]$ take values in the unit square $U:=[-1,1]\times[-1,1]\subset\mathbb{R}^2$. Applying necessary optimality conditions of Theorem 7.1, we seek for solutions to (P) satisfying the properties

$$\langle x_*^j,\bar{x}(t)\rangle<c_j=0\ \text{ for all }\ t\in[0,1),\ j=1,2, \tag{8.2}$$

and show that (8.2) holds for $\bar{x}(\cdot)$ that is determined below. In the case of (P) under consideration these conditions say that there exist $\lambda\ge 0$ together with adjoint vectors p and q and $\eta(\cdot)=\big(\eta^1(\cdot),\eta^2(\cdot)\big)\in L^2([0,1];\mathbb{R}^2_+)$ well defined at $t=1$ such that:

(1) $\langle x_j^*,\bar{x}(t)\rangle<c_j\Longrightarrow\eta^j(t)=0$ for $j=1,2$ and a.e. $t\in[0,1]$ including $t=1$;

(2) $-\dot{\bar{x}}(t)=\big(-\dot{\bar{x}}^1(t),-\dot{\bar{x}}^2(t)\big)=(\eta^1(t),\eta^2(t))-\big(\bar{u}^1(t),\bar{u}^2(t)\big)$ for a.e. $t\in[0,1]$;

(3) $\big(\dot{p}^1(t),\dot{p}^2(t)\big)=\big(0,0\big)$ for a.e. $t\in[0,1]$;

(4) $\big(\psi^1(t),\psi^2(t)\big)=(q^1(t),q^2(t))\in N\left(\bar{u}(t);U\right)$ for a.e. $t\in[0,1]$;

(5) $q(t)=p(t)-\gamma((t,1])$ for a.e. $t\in[0,1]$;

(6) $-(p^1(1),p^2(1))=\lambda\big(\bar{x}^1(1),\bar{x}^2(1)\big)+\big(\eta^1(1),\eta^2(1)\big)$ with $\big(\eta^1(1),\eta^2(1)\big)\in N\big(\bar{x}(1);C\big)$;

(7) $(\lambda,p)\neq 0$ due to (8.2) and (7.10).

Employing the first condition in (8.2) together with (1) and (2), gives us $\dot{\bar{x}}(t)=\bar{u}(t)$ for a.e. $t\in[0,1]$. It also follows from (4) and (5) that q can be written in the maximization form (7.5). It follows from (3) that $p(\cdot)$ is constant on $[0,1]$, i.e., $p(t)\equiv p(1)$. This allows us to deduce that

$$q(t)=p(1)-\gamma((t,1])\equiv\ p(1)-\gamma(\{1\})\ \text{ for a.e. }\ t\in[0,1]$$

by using the measure nonatomicity condition of Theorem 7.1 and Proposition 8.1. Several cases may occur. If $\lambda>0$ and $p=\eta(1)=(0,0)$, then (6) implies that $x^1(1)=x^2(1)=0$. In this case the terminal cost is zero. Thus each measurable control pair (u_1,u_2) that steers the initial point to $(0,0)$ exactly in time $t=1$ is optimal, as it is expected. A similar argument shows that if $\lambda=0$, then at least one component must vanish at the final time. In the case where both λ and p do not vanish, and is easy to see that any final point satisfies necessary conditions.

Summarizing, necessary conditions may exhibit degeneracy. Finding sufficient conditions to avoid this behavior is therefore the next major challenge that must be addressed. All the results that are presented in the literature (see, e.g., [27] and the very recent survey [4]) dealing with classical control problems with state constraints do not apply to our setting, because they are designed for more regular dynamics.

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