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Letter

Diffraction-free beam propagation at the exceptional point of non-Hermitian Glauber Fock lattices

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Abstract

We construct localized beams in a non-Hermitian Glauber Fock (NGF) lattice of coupled waveguides and show that they can propagate over a long distance withalmost no diffraction. We specifically obtain the diffraction-free beams in a finite NGF lattice at the exceptional point (EP) by using the exact eigenstates of the semi-infinite unidirectional NGF lattice. We provide a numerical approach to finding other lattices that are capable of supporting non-diffracting beams at EPs.

Keywords: diffraction-free beam propagation, exceptional, Hermitian, Glauber, Fock, lattices

(Some figures may appear in colour only in the online journal)

1. Introduction

The physics of photon propagation in discrete lattices is very rich and has been extensively studied, [1] which has resulted in the introduction of synthetic lattices capable of supporting non-diffracting wave propagation [2–8]. This includes lattices supporting Airy beams [9] or flat bands in Hermitian lattices [10, 11] and non-Hermitian parity-time symmetric lattices [12–14]. Specifically, in parity-time symmetric lattices, the flat band can occur at an exceptional point (EP), which allows for non-diffracting beam propagation. EPs are topological singularities in non-Hermitian Hamiltonians [15]. At an nth order EP, n eigenvalues and the corresponding eigenstates of a non-Hermitian Hamiltonian coalesce. EPs are ubiquitous in optics and have interesting transport feature which its properties manifested in manipulating light propagation such as unidirectional invisibility [16, 17], unidirectional lasing [18], lasing and anti-lasing in a cavity [19] and enhanced optical sensitivity [20].

One special system where discrete diffraction can be studied is the semi-infinite and asymmetric Glauber Fock lattice [21]. Hermitian Glauber Fock lattice has recently been implemented and demonstrated in optical lattices [22–24]. Apart from technological application in diffraction management the Glauber Fock photonic lattice allows us to visualize quantum harmonic oscillator which by itself is a strong motivation to study such classical lattices in optics. A Glauber Fock photonic lattice is composed of an array of evanescently coupled waveguides with a square-root distribution of the coupling between adjacent waveguides [22]. The first experimental realization with direct observation of the classical analog of Fock state displacements was reported in [23]. The Glauber–Fock photonic lattice is interesting in the sense that every excited waveguide represents a Fock state and an infinitely long lattice admits an exact analytical solution.

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In a finite unidirectional lattice with open boundary conditions, all eigenstates coalesce as an EP occurs. However, this is not the case in a semi-infinite unidirectional lattice [25, 26]. In this paper, we consider a non-Hermitian Glauber–Fock lattice and obtain a continuous family of eigenstates exactly to analyze a unidirectional semi-infinite lattice from an analytical point of view. In practice, every lattice has a finite number of lattice sites and thus one might think that the continuous family of eigenstates remains a mathematical curiosity. However, we discuss that the continuous family of eigenstates [27] can be used to construct almost non-diffracting waves at the EP in a finite Glauber–Fock lattice. Here an analogy with self-accelerating waves may be constructed. They are non-integrable and physically impossible to be realized but can be used as almost non-diffracting waves up to a large distance in an experiment after truncation [9].

2. Model

A generic 1D waveguide lattice with asymmetric couplings between adjacent sites is described by the following set of equations

$$H\psi_n = \gamma_{n+}\ \psi_{n+1} + \gamma_{n-}\ \psi_{n-1} \tag{1}$$

where γ_{n-} and γ_{n+} describe the site-dependent forward and backward hopping amplitudes, respectively, and ψ_n is the complex field amplitude and H is the corresponding non-Hermitian Hamiltonian. In the case of unidirectional hoppings, i.e., either $\gamma_{n+}=0$ or $\gamma_{n-}=0$ for all n, all eigenstates coalesce to a unique exceptional state with zero eigenvalue. This is true as long as the unidirectional lattice is finite and has open boundary conditions. Here, we are particularly interested in an exactly solvable semi-infinite unidirectional lattice, $N\rightarrow\infty$. The model we consider is a variant of the semi-infinite Glauber–Fock lattice, where the hopping amplitudes increase with the square root of the site number, $\gamma_{n+}=\gamma_+\sqrt{n+1}$ and $\gamma_{n-}=\gamma_-\sqrt{n}$, where γ_+ and γ_- are constants describing site-independent forward and backward hopping amplitudes, respectively. The equation satisfied by the complex field amplitude ψ_n at the nth waveguide in this model is then given by

$$-i\partial_z\psi_n + \gamma_+\sqrt{n+1}\ \psi_{n+1} + \gamma_-\sqrt{n}\ \psi_{n-1} = 0 \tag{2}$$

where z is the normalized propagation distance, $n=0,1,2,\ldots,N-1$. Note that the system is non-Hermitian when $\gamma_+\neq\gamma_-$.

First consider a finite unidirectional lattice. In this case, the exceptional state is well localized at the right edge when $\gamma_+=0$ and at left edge when $\gamma_-=0$. However, this is not the case for the semi-finite unidirectional lattice. Specifically, such a localization is not available when $\gamma_+=0$ since there is no right edge in the semi-infinite lattice. In the case of semi-infinite lattice with $\gamma_-=0$, we get a continuous family of eigenstates instead of a unique exceptional state. To obtain the most general form of the exact analytical solution when $\gamma_+\neq\gamma_-\neq 0$, let us use a trick by writing the state vector as $|\Phi\rangle=\sum_{n=0}^{\infty}\psi_n(z)|n\rangle$, where the Fock state $|n\rangle$ corresponds to a situation when only the waveguide with number n is excited [22, 23] and $\psi_n(z)$ is defined above. Substituting this solution into the equation (2) yields the Schrodinger-like equation $\mathcal{H}\Phi=i\frac{\partial\Phi}{\partial z}$, where time is replaced by the propagation distance. The corresponding Hamiltonian reads $\mathcal{H}=\gamma_-\ \hat{a}^\dagger+\ \gamma_+\ \hat{a}$, where $a^\dagger$ and a are the well-known bosonic creation and annihilation operators satisfying $\hat{a}^\dagger|n>=\sqrt{n+1}|n+1>$ and $\hat{a}|n>=\sqrt{n}|n-1>$, respectively. We can transform this Hamiltonian using $\hat{a}=\frac{q+ip}{\sqrt{2}}$ and $\hat{a}^\dagger=\frac{q-ip}{\sqrt{2}}$, where q and p are the normalized position and momentum operators, respectively. Then the Hamiltonian can be rewritten in the following form $\mathcal{H}=\frac{i(\gamma_+-\gamma_-)}{\sqrt{2}}p+\frac{\gamma_++\gamma_-}{\sqrt{2}}q$. Notice that we assume $\gamma_-\neq\gamma_+$, which is the condition for non-Hermiticity. Let us substitute $p=-i\partial_q$ and solve the corresponding Schrödinger-like equation. The resulting equation is of first order and admits an exact analytic solution. If γ_+ and γ_- are z-dependent, then the most general form of the solution is given by

$$\Phi(q,z)=e^{-\int_0^q S(Z+iq-is)sds}\ F(Z+iq) \tag{3}$$

where $S(Z)=\frac{\gamma_++\gamma_-}{\gamma_+-\gamma_-}$, where $Z=\int_0^z\frac{\gamma_+-\gamma_-}{\sqrt{2}}dz$ and $F(q)$ is an arbitrary continuous function. If γ_+ and γ_- are constants, then it is reduced to

$$\Phi(q,z)=e^{\frac{\gamma_-+\gamma_+}{2(\gamma_--\gamma_+)}q^2}f\left(z-\frac{\sqrt{2}i}{\gamma_+-\gamma_-}q\right). \tag{4}$$

Note that we made the replacement $F(\frac{\gamma_+-\gamma_-}{\sqrt{2}}z+iq)\rightarrow f(z+\frac{\sqrt{2}i}{\gamma_+-\gamma_-}q)$ in the last step. We stress that we impose the boundary condition on ψ_n not on $\Phi(q)$, that is $\psi_{n\rightarrow\infty}(z)=0$. The arbitrary function $f(q)$ can be determined from the initial condition at $z=0$. Equation (3) allows us to find the time evolution of any initial state. In q-space, the wave packet translates with a constant speed. In the original space, this may imply growing or decaying diffracting solutions depending on the form of $f(q)$. We finally find the form of the wave packet in the original space using $\psi_n(z)=\langle n|\Phi\rangle$

$$\psi_n(z)=\int_{-\infty}^{\infty}\frac{\pi^{-1/4}}{\sqrt{2^n n!}}e^{-q^2/2}H_n(q)\ \Phi(q,z)\ dq \tag{5}$$

where $H_n(q)$ is the nth order Hermite polynomials. This is the most general solution of our original system and one can analytically study not only the eigenstates but also time evolution of any given initial wave packet for the semi-infinite non-Hermitian lattice.

As a special case, consider $\gamma_-=-\gamma_+$. In this case, the transformed Hamiltonian is reduced to $\mathcal{H}=\sqrt{2}\,i\,\gamma_-p$, which is an anti-Hermitian Hamiltonian, $\mathcal{H}=-\mathcal{H}^\dagger$. In this case the solution (3) becomes $\Phi=f(z+\frac{i}{\sqrt{2}\gamma_-}q)$. We get a specific zero energy stationary solution if we choose $f=1$. One can choose

many different forms of the function f such as $f(q) = e^{-q^2/4}$ and $f(q) = e^{q^2/4}$, which are not stationary solutions (they are decaying and growing solutions).

Next, let us obtain the eigenstates (stationary solutions) of the semi-infinite Glauber Fock lattice from the solution (5). We choose the arbitrary function $f = f_E$ in (3) in such a way that Φ becomes z-independent

$$f_E = \exp\left(-iE\left(z + \frac{\sqrt{2}i}{\gamma_+ - \gamma_-}q\right)\right) \tag{6}$$

where the constant E are eigenvalues. Below, we show that E does not take discrete values but continuous values. One can directly substitute them into the integral (5) to get the continuous family of eigenstates with energy E. One can see that as the contrast between γ_+ and γ_- increases, the localization length of the eigenstates get increased.

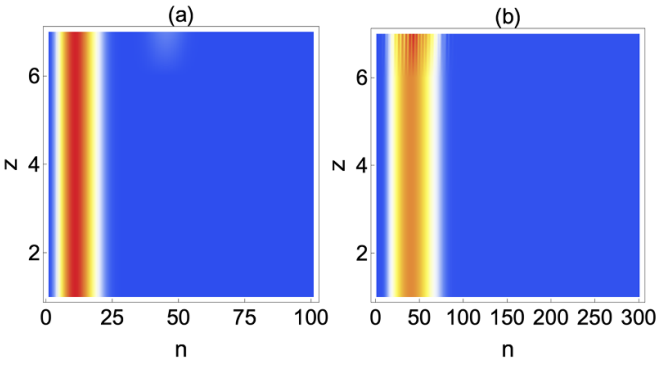

Figure 1. The density plots for $N = 100$ (a) and $N = 300$ (b). α is chosen a larger value in (b) to shift the center of the wave packet to the right. At the EP $\gamma_- = 0$, we can construct non-diffracting wave packet. Note that the true exceptional eigenstate is the one where only the $n = 0$th lattice site is excited.

2.1. Continuous family of stationary solutions

Let us now explore the stationary solutions specifically for the semi-infinite unidirectional lattice. Suppose first that $\gamma_+ = 0$ and $\gamma_- \neq 0$. In this case, the integral (5) diverges since $\Phi = e^{\frac{q^2}{2}} f_E$. This implies that no stationary solution is available. Note that the exceptional state is well localized at the right edge of a finite lattice, but such an exceptional state does not appear in the semi-infinite lattice because of the absence of the right edge. Instead, one can construct non-stationary solutions by choosing various form of $f(q)$. Suppose next that $\gamma_- = 0$ and $\gamma_+ \neq 0$. For a finite lattice, the zero energy exceptional eigenstate is well localized at the left edge. That zero energy eigenstate is also an eigenstate for the semi-infinite lattice. There are, in fact, infinitely many eigenstates in the semi-infinite lattice, whereas there is just one eigenstate in a finite lattice. To find them, we evaluate the integral (5) with (6). We then get the well known coherent states, which satisfy the completeness condition but not the orthogonality condition

$$\psi_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{7}$$

where $\alpha = \frac{E}{\gamma_+}$. We stress that E can take complex values, from which we can construct either decaying or growing solutions. The solution (7) allows us to obtain a continuous family of eigenstates by varying the parameter E. At $E = 0$, $\psi_n = \delta_{n,0}$, which implies that the zero energy eigenstate exactly matches the eigenstate well localized at the left edge and increasing E shifts the center of the wave packet to the right.

Next, let us discuss the practical application of the above solution since a lattice has practically a finite number of sites. Before discussing this point, we remember the self-accelerating waves, which is a non-integrable mathematical solution and hence physically impossible to be realized. However, it was shown that such solutions can still be used as non-diffracting waves up to a large distance if they were truncated [28, 29]. In a similar fashion, we state that the solutions for the semi-infinite lattice can be used to construct non-diffracting beams in the finite lattices. In other words, truncating our solution allows us to construct almost stationary wave packets (or non-diffracting waves). To get practical non-diffracting waves at the EP, let us truncate the solution (7). This truncation works if N is large enough such that $\frac{\alpha^N}{\sqrt{N!}} \ll 1$. In this case, the non-truncated terms are negligibly small and only contribute to the system in a very large time. Note that this truncated solution is not an exact analytical solution for a finite lattice with N lattice sites as all eigenstates coalesce at the EP.

Let us perform a numerical approach to support our idea. Suppose $\gamma_- = 0$, $\gamma_+ = 1$ and a N is sufficiently large. In figure 1, one can see the density plots for $N = 100$ (a) and $N = 300$ (b). In figure 1(b), the constant α is chosen to have a larger value to shift the center of the wave packet to the right. The system has only one exceptional eigenstate which is perfectly localized at the $n = 0$ lattice site. However, one can see from figure 1 that a non-diffracting wave packet can still be obtained up to a large propagation distance at the EP, $\gamma_- = 0$. The longer the lattice is, the more lifetime the non-diffracting waves can have. At large values of z, the contribution from the right edge comes into play as the lattice is not infinitely extended. Therefore the non-diffracting character is eventually lost.

We have so far studied stationary eigenstates. In the unidirectional lattice, one can also obtain a monotonically growing or decaying solutions and power oscillating solutions up to a large distance. For example, choosing $f(q) = e^{q^2/2}$ leads to growing solution while choosing $f(q) = e^{-q^2/2}$ leads to decaying solution at the EP. If we choose $f(q) = \sin q$, then power oscillation occurs at the EP.

We have explored almost non-diffracting waves at EPs for the specific Glauber–Fock lattice. One may ask if we can find such waves for other unidirectional lattice. One example of such other systems is provided in [27] and there are many other such systems. We consider a disordered unidirectional lattice, i.e. a lattice with random unidirectional hopping amplitudes. The corresponding Hamiltonian reads

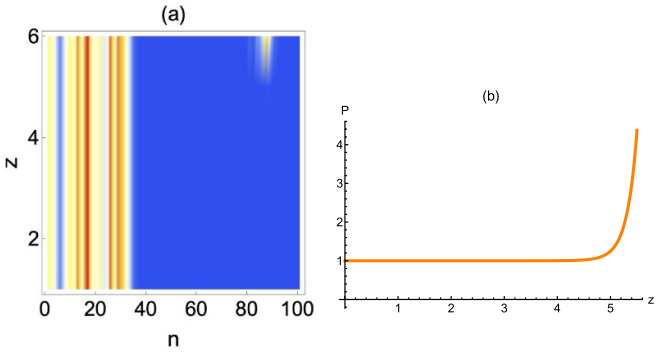

Figure 2. The density plot for $N = 100$ (a) and the ratio of total power to the initial one as a function of z in the unit of coupling (b). The forward hopping amplitudes are set to to zero to obtain that the only eigenstate is the exceptional state well localized at the left edge. The backward hopping amplitudes are chosen randomly in the interval $\gamma_n \in [1,6]$ and we take $\psi_{101}(z=0) = 10^{-8}$. At around $z = 5$, the non-diffracting character is lost ant the total power starts to increase exponentially.

$$H = \sum_{n=1}^{N} \gamma_n \, |\psi_{n+1}><\psi_n| \tag{8}$$

where γ_n take random values and the system has open boundary conditions. One can simply solve the corresponding eigenvalue equation to obtain the zero energy exceptional eigenstate. Here, our aim is to get almost non-diffracting localized waves. Our approach is as follows: instead of setting $\psi_{N+1} = 0$ (due to the open boundary condition at the right edge), here we assume that ψ_{N+1} is a very small number (for example it can be chosen to be equal to 10^{-8}). Then, we can get the field amplitude ψ_n recursively and get the almost non-diffracting waves. By varying E in the corresponding eigenvalue equation, we can obtain a continuous family of such solutions. The resulting solution is not exact, but can be used as almost non-diffracting waves up to a large distance as long as ψ_{N+1} is a very small number and N is large.

Let us apply our above strategy in a disordered system. We consider a disordered lattice, where γ_n take random values in the interval $\gamma_n \in [1,6]$. There are $N = 100$ lattice sites in the system and assume that $\psi_{101}(z=0) = 10^{-8}$. Then we get ψ_n for $n = 1,2,\ldots,N$ and find its time evolution numerically to check whether it shows non-diffracting behavior. In figure 2(a), one can see the density plot as a function of propagation length z at $E = 2.8$. The non-diffracting wavepacket up to $z \sim 5$ can be seen from the figure, which shows that our strategy is good enough to obtain such waves even in a disordered lattice up to a large distance. In figure 2(b), we see that the total power $P = \sum_n \psi(z)$ does not change up to $z \sim 5$. After that point, edge effects come into play and non-diffracting behavior is lost. This is already expected as our solution is not an exact solution. To this end, we say that systems with second order and third order EPs are not good candidates to see almost non-diffracting waves. They can only be seen in systems with higher order EP as our strategy is based upon the fact that N is large.

3. Conclusion

In summary, we provide an exact analytical solution for the non-Hermitian Glauber–Fock lattice with asymmetric hopping amplitudes. In the unidirectional finite lattice, an EP occurs and all eigenstates coalesce. In the semi-infinite unidirectional lattice, a continuous family of eigenstates with complex eigenvalues appear. The latter one is not physical, nevertheless its solutions can be used as almost non-diffracting waves for the finite lattice at EP. We discuss that our method is generic and we provide a numerical approach to construct such non-diffracting waves in a unidirectional lattice with randomized couplings. We stress that the non-diffracting solutions which are stationary solutions of the infinitely long lattice are different than the delta-function excitation at the left edge of the lattice. Such delta-function excitation is a special case of the stationary states that we studied here. Our proposed idea can be experimentally realized in a variety of systems including microwave where commercially available optical isolators can be used to break reciprocity, mechanical metamaterials [30], and optical domain [31], to name a few.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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