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Non-Hermitian skin effect in two dimensional continuous systems

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Abstract

An extensive number of the eigenstates can become exponentially localized at one boundary of nonreciprocal non-Hermitian systems. This effect is known as the non-Hermitian skin effect and has been studied mostly in tight-binding lattices. To extend the skin effect to continues systems beyond 1D, we introduce a quadratic imaginary vector potential in the continuous two dimensional Schrödinger equation. We find that inseparable eigenfunctions for separable nonreciprocal Hamiltonians appear under infinite boundary conditions. Introducing boundaries destroy them and hence they can only be used as quasi-stationary states in practice. We show that all eigenstates can be clustered at the point where the imaginary vector potential is minimum in a confined system.

1. Introduction

Models describing one-dimensional tight-binding lattices with gain and loss or asymmetrical hopping amplitudes are frequently used to explore non-Hermitian effects in discrete systems. Nonreciprocal lattices, i.e. lattices with asymmetrical hopping amplitudes that can be induced by an imaginary vector potential on a periodic potential have some remarkable effects which have no analog in Hermitian systems such as single-particle topological phase transition and non-Hermitian skin effect [1–3]. Non-Hermitian skin effect implies that spectrum for the nonreciprocal non-Hermitian lattices changes non-locally as a result of distant boundaries and hence open boundary conditions (OBC) and periodic boundary conditions (PBC) can lead to drastically different spectra [4–28]. Besides, the eigenstates under OBC are exponentially localized at the edge while those under PBC are extended along the lattice as opposed to the Hermitian lattices where it is commonly believed that bulk eigenstates in a long lattice are insensitive to boundary conditions. Because of the boundary condition sensitivity of the spectrum, the standard bulk-boundary correspondence of topological insulators fails in such systems and the non-Bloch band theory defined in the generalized Brillouin zone was proposed to analyze the topological edge modes from the topological bulk invariants [29, 30]. These theoretical predictions have been confirmed experimentally in various platforms [31–39].

The non-Hermitian skin effect, as well as its extension to nonlinear domains, [40, 41] have been explored for discrete systems in previous studies while the literature on its continuous analog is limited [42–44]. Boundary condition sensitivity of the spectrum was shown to exist in the continuous systems governed by a Hamiltonian containing a periodic potential and a constant imaginary vector potential [42] and non-Bloch band theory in continuous systems were also developed [43].

In this work, we go beyond these recent studies on continuous systems and introduce also the effect of the infinite boundary conditions (IBC) which is missing in the previous studies. To this end, we consider two dimensional Schrödinger equation, in which non-reciprocity is introduced by an imaginary vector potential and examine the localization of eigenstates for various forms of imaginary vector potentials. We show that the energy spectrum under IBC, OBC, and PBC can be drastically different from each other. We discuss the condition for the imaginary vector potential to obtain a real spectrum under PBC. It is well known that a two-dimensional Hermitian Hamiltonian that is separable in Cartesian coordinate has separable eigenfunctions. Here, we show

that this is not always the case for non-Hermitian Hamiltonians. Contrary to OBC and PBC, we show that inseparable eigenstates for separable Hamiltonians (IESH) appear for the systems under IBC.

2. Model

Consider a parent Hermitian Hamiltonian $\mathcal{H}'(\mathbf{r},\ \mathbf{p})$, from which we generate a nonreciprocal non-Hermitian Hamiltonian $\mathcal{H}(\mathbf{r},\ \mathbf{p})$ by introducing an imaginary vector potential through $\mathbf{p} \to \mathbf{p} - i\nabla\Lambda$, i. e., $\mathcal{H}(\mathbf{r},\ \mathbf{p}) = \mathcal{H}'(\mathbf{r},\ \mathbf{p} - i\nabla\Lambda)$, where $\Lambda = \Lambda(\mathbf{r})$ is a real-valued function and ∇ is the gradient operator. Let us start with a Hermitian parent Hamiltonian in two dimensions $\mathcal{H}' = \mathbf{p}^2 + V(x,\ y)$, from which we obtain ($\hbar = 1$)

$$\mathcal{H} = (\mathbf{p} - i\nabla\Lambda)^2 + V(x,\ y) \tag{1}$$

There exists non-Hermitian eigenstates ψ that acquire an additional factor with respect to the Hermitian eigenstates ψ'

$$\psi = e^{-\Lambda}\psi' \tag{2}$$

Under this non-unitary transformation, the corresponding non-Hermitian and Hermitian eigenvalues remain the same $E = E'$. However, this doesn't necessarily mean that the Hermitian and non-Hermitian spectra are the same since ψ in (2) are not generally complete and one may find other non-Hermitian eigenstates that are not given by the equation (2).

In this work, we suppose that Λ is a quadratic polynomial in two variables containing real-valued constants $\alpha_{1,2,3,4,5}$

$$\Lambda = \alpha_1\, x + \alpha_2\, y + \alpha_3\, x^2 + \alpha_4\, y^2 + \alpha_5\, x\, y. \tag{3}$$

Note that the non-Hermitian Hamiltonian is parity-time ($\mathcal{PT}$) symmetric if $V(-x, -y) = V(x, y)$ and $\alpha_1 = \alpha_2 = 0$.

A rather general result of the multiplicative factor $e^{-\Lambda}$ in equation (2) is that with an appropriate choice of the parameters α_j, non-Hermitian eigenstates can be exponentially localized at the point where Λ is minimum. Such a clustered localization of non-Hermitian eigenstates is particularly interesting for a confined system (a system under OBC) when the minimum point of Λ is set at one boundary of the system. This is in fact a continuous analog of the non-Hermitian skin effect. Besides, Λ can be set to have a minimum at an arbitrary point in the confined system to obtain clustered localization at that point. We will discuss this issue below in more detail when we study OBC.

Let us suppose firstly that the system is subject to IBC in two dimensions

$$\psi(x = \mp\infty,\ y) = \psi(x,\ y = \mp\infty) = 0 \tag{4}$$

It is well known that if the two-dimensional potential of the Hermitian Hamiltonian is separable in cartesian coordinates, $V(x, y) = V(x) + V(y)$, then the corresponding eigenfunctions are just products of the one-dimensional eigenfunctions and the two-dimensional problem becomes equivalent to two one-dimensional problems. Therefore, the energies are expressed as the sum of one-dimensional energies (2). However this is not always the case for the non-Hermitian systems under IBC. In fact, the non-Hermitian eigenfunctions ψ are not complete even though the Hermitian eigenfunctions ψ' are complete in equation (2). This means that there may be some other non-Hermitian eigenfunctions that cannot be obtainable from the complete Hermitian eigenfunctions ψ' in equation (2). One strategy to obtain them, if they exist, is to use the general solution of the corresponding Schrödinger equation for the parent Hermitian Hamiltonian instead of using the Hermitian eigenstates. Remarkably, inseparable non-Hermitian eigenstates may come with eigenvalues that are absent in the corresponding Hermitian spectrum. To illustrate our discussion on a toy model, we consider $\alpha_1 = \alpha_2 = \alpha_5 = 0$ and $\alpha_3 > 0$ and $\alpha_4 > 0$ ($\Lambda = \alpha_3 x^2 + \alpha_4 y^2$) and $V(x, y) = 0$ at which the $\mathcal{PT}$ symmetric, nonreciprocal non-Hermitian Hamiltonian becomes

$$\mathcal{H}_{2D} = (p_x - 2i\alpha_3 x)^2 + (p_y - 2i\alpha_4 y)^2 \tag{5}$$

Although this non-Hermitian Hamiltonian is separable in cartesian coordinates, it has some eigenfunctions that cannot be factored into a function of x times a function of y as can be seen from the following exact analytical solutions subject to IBC (4). The complex energy (seperable) and zero energy (mostly inseperable) unnormalized eigenfunctions are given by

$$\psi_{k_x,k_y} = e^{-\alpha_3 x^2 - \alpha_4 y^2}\ e^{i(k_x x + k_y y)} \tag{6}$$

$$\psi_0 = e^{-\alpha_3 x^2 - \alpha_4 y^2}\ f(x \mp iy) \tag{7}$$

where k_x and k_y are complex valued constants, and $f(x \mp iy)$ is an arbitrary function of its arguments to be chosen in such a way that ψ_0 satisfy IBC (figure 1) Therefore, the degree of degeneracy at zero energy level is infinite since

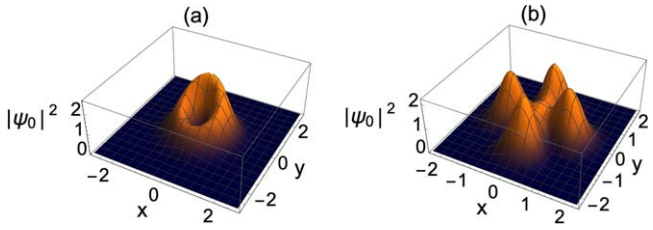

Figure 1. The density for the unnormalized inseparable zero energy eigenfunctions with $\psi_0 = e^{-x^2-y^2}\,(0.5 + 3i(x - iy))$ (a) and $\psi_0 = e^{-x^2-y^2}\,(1 - 3.5i(x^2 - y^2))$ (b).

one can in principle find infinitely many such $f(x \mp iy)$. The corresponding spectrum is continuous and covers the whole complex plane since the eigenvalues $(k_x^2 + k_y^2)$ associated with (6) can take any complex values. Note that the eigenfunctions (6) and (7) are not $\mathcal{PT}$ symmetric when the corresponding eigenvalues are complex and $f(x + iy) \neq \mp f(-x+iy)$, respectively.

The eigenstates of the Hamiltonian (5) are not orthogonal, so finding time evolution of an arbitrary initial wave packet is not straightforward. We obtain some exact solutions by solving the time-dependent Schrödinger equation $\mathcal{H}_{2D}\,\psi = i\,\partial_t\psi$. For example, we consider the initial wave packet $\psi(t=0) = e^{-\Lambda}\;e^{ik_x x + ik_y y}\;f(x \mp iy)$. We observe that its time evolution is given by $\psi(t) = e^{-\Lambda}\;e^{ik_x x + ik_y y}e^{-i(k_x^2+k_y^2)t}\;(f(x \mp iy - 2(k_x \mp ik_y)t)$. We find other solutions such as $\psi = (cx - y^2 - 2it)e^{-\Lambda}$ and $\psi = (x^2 + icy + 2it)e^{-\Lambda}$, and $\psi = (x^2 + 2it)ye^{-\Lambda}$, where c is an arbitrary constant.

To this end, we say that the infinitely degenerate zero-energy solutions don't appear in 1D (there exists only one zero energy eigenfunction $X_0(x) = x\;\;e^{-\alpha_3 x^2}$ subject to IBC for the 1D Hamiltonian $\mathcal{H}_{1D} = (p_x - 2i\alpha_3 x)^2$. On the other hand, 3D systems give rise to richer zero energy solutions as can be seen from the solution of the 3D Hamiltonian $\mathcal{H}_{3D} = \mathcal{H}_{2D} + (p_z - 2iz)^2$: $\Psi_0 = e^{-\Lambda}\;f(a_1 x + a_2 y + ia_3 z)$, where a_1, a_2 and a_3 are complex constants satisfying $a_3^2 = a_1^2 + a_2^2$ and f is an arbitrary function and $\Lambda = \alpha_3 x^2 + \alpha_4 y^2 + z^2$. Using this 3D solution, one can find infinitely many solutions that are inseparable in every directions.

The spectrum of a long Hermitian OBC system can be practically calculated by approximating it as an infinite system. However, this is not the case in non-Hermitian systems with non-Hermitian skin effect [16]. To study sensitivity of the non-Hermitian spectrum to the boundary conditions, we nextconsider the OBC where the system is confined to a rectangular box of length L_x and width L_y with impenetrable walls.

$$\psi(0, y) = \psi(L_x, y) = \psi(x, 0) = \psi(x, L_y) = 0 \tag{8}$$

We stress that the zero energy states (7) disappear when we introduce boundaries to the system and all eigenstates become separable under OBC. In fact, ψ satisfy OBC only if ψ' satisfy OBC according to equation (2). Therefore the non-Hermitian and parent Hermitian Hamiltonians have the same spectrum under OBC $(E = \pi^2(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2})$, where n_x, n_y are positive integers), revealing that the spectra for IBC and OBC are drastically different from each other as the former one is continuous and covers the whole complex plane while the latter one is discrete and real. This implies that non-Hermitian skin effect occurs. Let us now study the eigenstates under OBC. The multiplicative factor $e^{-\Lambda}$ in (2) leads to exponential localization of the whole non-Hermitian eigenstates at around the point where Λ is minimum in the box. Therefore, we conclude that all eigenstates are exponentially localized at the corner (0, 0). This is a continuous system analogue of the non-Hermitian skin effect. In fact, skin localization is just a special case of eigenstate clustering since we have full control of where localization occurs with appropriate choice of α_j. For example, all localized eigenstates are clustered around the point $(x, y) = (\frac{-\alpha_1}{2\alpha_3}, \frac{-\alpha_2}{2\alpha_4})$ when $\alpha_{1,2} < 0$ and $\alpha_{3,4} > 0$, and $\alpha_5 = 0$. As another example, one can choose $\Lambda = -\alpha_5 xy$ (a uniform magnetic field with imaginary strength). In this case, localization occurs around the corner $(x, y) = (L_x, L_y)$ when $\alpha_5 > 0$. To this end, we remark that Λ can also be chosen to be a periodic function with multiple minimum points instead of the polynomial (3). This allow us to design a clustering of eigenstates at multiple localization points.

Under OBC, time-dependent solutions $\psi(t)$ for the non-Hermitian Hamiltonian $\mathcal{H}$ can be constructed by using a linear combination of the parent Hermitian eigenstates ψ'_{n_x,n_y}

$$\psi(t) = e^{-\Lambda}\sum_{n_x,n_y} c_{n_x,n_y}\,e^{-iE\,t}\;\;\psi'_{n_x,n_y} \tag{9}$$

where $c_{n_x,n_y} = <\psi'_{n_x,n_y}|\;e^{\Lambda}\;\psi(t=0)>$ are arbitrary complex coefficients to be determined from the initial condition and $E = E_{n_x,n_y}$ are the corresponding eigenvalues for the Hermitian eigenfunctions ψ'_{n_x,n_y}. From this solution, time evolution of an initial wave packet for $\mathcal{H}$ under OBC can be derived. The exponential factor $e^{-\Lambda}$

breaks the orthogonality condition and hence the total power of such a wave packet $\psi(t)$ is either constant or oscillates in time.

Having examined OBC, we can now study PBC and compare the spectra under OBC and PBC. Consider that $\mathcal{H}$ and $\mathcal{H}'$ are spatially periodic with the same periodicity and the system has PBC. We emphasize that $\mathcal{H}$ includes the potential $\nabla\Lambda$ (1) while equation (2) includes Λ. The key point here is the fact that the derivative of a periodic function is also a periodic function, but the reverse is not always true. This has interesting implications. If both Λ and $\nabla\Lambda$ are periodic, then the non-Hermitian spectrum under PBC is real valued and strong sensitivity to the boundary conditions is lost. On the other hand, if Λ is not periodic but $\mathcal{H}$ is, then PBC is no longer maintained under the transformation (2). In this case, the energy spectrum is complex valued and strongly sensitive to the boundary conditions (OBC and PBC spectra are dramatically different from each other). Furthermore, the eigenstates under PBC are extended as opposed to the eigenstates under OBC that are exponentially localized. For example, suppose that $V(x, y)$ is a periodic potential and $\Lambda = \alpha_1 x + \alpha_2 y$. Then the corresponding non-Hermitian Hamiltonian $\mathcal{H}$ is periodic, but the solution (2) can no longer be used for PBC. Instead, we start with the solution expressed as an expansion over plain waves with all permitted values of wave vector $\mathbf{k}$ by the boundary condition, $\psi_k = \sum_{\mathbf{k}} c_{\mathbf{k}}\, e^{i\mathbf{k}.\mathbf{r}}$. By substituting this solution into the corresponding eigenvalue equation, one finds a set of algebraic equations for $c_{\mathbf{k}}$ and numerically solve them to find the spectrum for a specific periodic potential V. In this case, the imaginary vector potential in $\mathcal{H}$ introduces an asymmetry between the imaginary parts of $E_{\mathbf{k}}$ and $E_{-\mathbf{k}}$.

We have so far considered systems in cartesian coordinates. Let us now briefly discuss systems in cylindrical coordinates and the effect of the imaginary field on vortex solutions. Here we are mainly interested in non-diffracting (shape-preserving during propagation) vortex beams, among which Bessel beams are probably the most widely known [45]. Consider the Helmholtz equation for monochromatic light (the Schrödinger equation can be rearranged to look like the Helmholtz equation) $\nabla^2\psi' = k^2\psi'$, where ∇^2 is the Laplacian, k is the wavenumber. It admits the Bessel beam solution $\psi' = Ae^{ik_z z} J_n(k_r r) e^{\mp in\theta}$, where J_n represents the nth-order Bessel function of the first kind, (r, θ, z) are cylindrical coordinates and $k^2 = k_z^2 + k_r^2$ [45]. The zero-order beam has a maximum intensity on the axis while the densities are zero when $n > 0$. It is impossible to realize a perfect Bessel beam in practice because it has infinite power. Instead, other methods were proposed in the literature to obtain approximate Bessel beams to observe some of their characteristic features up to some distances. For example, a finite power carrying Bessel Gauss beams [46] was proposed for experimental realization, though its propagation is nearly diffraction-free up to some distances. We show here that such non-diffracting vortex beams can be realizable in non-Hermitian systems. Let us introduce the imaginary transformation $\nabla \to \nabla - \nabla\Lambda$, where $\Lambda = (r - r_0)^2$ and r_0 is a constant. Then we get a perfectly diffraction-free Bessel Gauss beam in a non-Hermitian regime $|\psi|^2 = e^{-2(r-r_0)^2} J_n(k_r r)$. By varying the constant r_0, we can adjust the position of the main peak and the location of the density dip. Interestingly, we can generate a density dip at the center even for non-vortex Bessel beam with $n = 0$ when $r_0 > 2$.

One can also study IESH in cylindrically symmetric separable potentials. This can be seen on the toy non-Hermitian Hamiltonian in cylindrical coordinates $\mathcal{H} = (\mathbf{p} - i\nabla\Lambda)^2$, where $\Lambda = r^2 + z^2$. We consider IBC in both radial and z directions. Obtaining exact analytical solutions is not easy, so we write specific zero energy inseparable localized eigenstates with zero vorticity to prove the existence of IESH. It is given by $\psi = e^{-\Lambda}(r^2 - 2z^2 + c_1 z + c_2)$, where c_1, c_2 are constants. Notice the density dip at $r = z = 0$ and $c_2 = 0$ even though the solution has no vorticity.

3. Conclusion

To conclude, we introduce nonreciprocity to Schrödinger equation through an imaginary vector potential to study the non-Hermitian skin effect in continuous systems. Localization of bulk eigenstates under OBC at the edge is a well-known result of non-reciprocity in tight binding lattices. Here, we show that localization is also possible for continuous systems under OBC and further discuss how to control the localization center. We study how and why PBC and OBC spectra become different from each other. We consider IBC and show that IESH can arise due to nonreciprocity in two dimensions. We find infinitely many zero energy eigenstates for the toy model (5). Strictly speaking, IESH can only be available for infinite systems, whereas an experiment has a finite length . Fortunately, IESH can still be used in some practical applications as quasi-stationary solutions . For example, in the non-Hermitian optics community, the zero-energy localized IESH may be utilized to realize numerous almost non-diffracting 2D wave packets in systems where paraxial wave approximation is used.

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Data availability statement

No new data were created or analysed in this study.

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References

[1] Hatano N and Nelson D R 1996 *Phys. Rev. Lett.* **77** 570
[2] Yao S and Wang Z 2018 *Phys. Rev. Lett.* **121** 086803
[3] Yao S, Song F and Wang Z 2018 *Phys. Rev. Lett.* **121** 136802
[4] Martinez Alvarez V M, Barrios Vargas J E and Foa Torres L E F 2018 *Phys. Rev.* B **97** 121401(R)
[5] Yuce C 2020 *Phys. Lett.* A **384** 126094
[6] Song F, Yao S and Wang Z 2019 *Phys. Rev. Lett.* **123** 170401
[7] Zhang Z-X, Huang R, Qi L, Xing Y, Zhang Z-J and Wang H-F 2020 *Ann. Phys. (Berlin)* **533** 2000272
[8] Wang P, Jin L and Song Z 2019 *Phys. Rev.* A **99** 062112
[9] Jin L and Song Z 2019 *Phys. Rev.* B **99** 081103
[10] Yuce C 2021 *Phys. Lett.* A **403** 127384
[11] Cao Y, Li Y and Yang X 2021 *Phys. Rev.* B **103** 075126
[12] Ezawa M 2019 *Phys. Rev.* B **99** 121411(R)
[13] Zhong J, Wang K, Park Y, Asadchy V, Wojcik C C, Dutt A and Fan S 2021 *Phys. Rev.* B **104** 125416
[14] Imura K-I and Takane Y 2019 *Phys. Rev.* B **100** 165430
[15] Jiang H, Lü R and Chen S 2020 *Eur. Phys. J.* B **93** 125
[16] Okuma N, Kawabata K, Shiozaki K and Sato M 2020 *Phys. Rev. Lett.* **124** 086801
[17] Fu N, Fu Z, Zhang H, Liao Q, Zhao D and Ke S 2020 *Opt. Quantum Electron.* **52** 61
[18] Yuce C 2020 *Ann. Phys.* **415** 168098
[19] Franca S, Konye V, Hassler F, van den Brink J and Fulga I C 2022 *Phys. Rev. Lett.* **129** 086601
[20] Li L, Lee C H, Mu S and Gong J 2020 *Nat. Commun.* **12** 5491
[21] Lee C H, Li L and Gong J 2019 *Phys. Rev. Lett.* **123** 016805
[22] Longhi S 2019 *Phys. Rev. Research* **1** 023013
[23] Jiang H, Lang L-J, Yang C, Zhu S-L and Chen S 2019 *Phys. Rev.* B **100** 054301
[24] Lee J Y, Ahn J, Zhou H and Vishwanath A 2019 *Phys. Rev. Lett.* **123** 206404
[25] Kunst F K, van Miert G and Bergholtz E J 2019 *Phys. Rev.* B **99** 085427
[26] Yuce C 2020 *Phys. Rev.* A **102** 032203
[27] Shao K, Cai Z-T, Geng H, Chen W and Xing D Y 2022 *Phys. Rev.*B **106** L081402
[28] Lee C H and Thomale R 2019 *Phys. Rev.* B **99** 201103
[29] Yokomizo K and Murakami S 2019 *Phys. Rev. Lett.* **123** 066404
[30] Longhi S 2020 *Phys. Rev. Lett.* **124** 066602
[31] Helbig T *et al* 2020 *Nat. Phys.* **16** 47
[32] Liu S, Shao R, Ma S, Zhang L, You O, Wu H, Cui T and Zhang S 2021 *Research* **2021** 5608038
[33] Xiao L, Deng T, Wang K, Zhu G, Wang Z, Yi W and Xue P 2020 *Nat. Phys.* **16** 761
[34] Ghatak A, Brandenbourger M, Van Wezel J and Coulais C 2020 *Proc. Natl. Ac. Sc* **117** 29561
[35] Weidemann S *et al* 2020 *Science* **368** 311
[36] Zhu X, Wang H, Gupta S K, Zhang H, Xie B, Lu M and Chen Y 2020 *Phys. Rev. Research* **2** 013280
[37] Zhang X, Tian Y, Jiang J-H, Lu M-H and Chen Y-F 2021 *Nat. Comm.* **13** 5377
[38] Zhang W, Di F, Yuan H, Wang H, Zhang X, He1 L, Sun H and Zhang X 2021 arXiv:2109.08334
[39] Zhang L *et al* 2021 *Nat. Commun.* **12** 6297
[40] Yuce C 2021 *Phys. Lett.* A **408** 127484
[41] Ezawa M 2022 *Phys. Rev. B* **105** 125421
[42] Longhi S 2021 *Phys. Rev.* B **104** 125109
[43] Yokomizo K, Yoda T and Murakami S 2022 *Phys. Rev. Research* **4** 023089
[44] Lu M, Zhang X-X and Franz M 2021 *Phys. Rev. Lett.* **127** 256402
[45] Durnin J 1987 *J. Opt Soc. Am.*A **4** 651–4
[46] Gori F, Guattari G and Padovani C 1987 *Opt. Commun.* **64** 491