2022 American Control Conference (ACC)
Atlanta, USA, June 8-10, 2022

# Prescribed-Time Extremum Seeking with Chirpy Probing for PDEs—Part I: Delay

Cemal Tugrul Yilmaz and Miroslav Krstic

Abstract— We introduce a scalar prescribed–time extremum-seeking control design for static nonlinear maps subject to known input delays. Even in the delay-free case, unlike in the recent efforts by Poveda and Krstic, and by Guay and Benosman, where a user-assigned (prescribed) settling time is enabled by the use of a Newton-based scheme (and by non-smooth feedback of the gradient estimate), our approach employs "chirped" probing and a time varying gain, which ensure convergence to the optimum in prescribed time, independent of the initial distance from the optimizer, even for a gradient scheme. Conventional analysis techniques for extremum seeking, based on infinite-time stability, averaging theory, and PDE backstepping, are leveraged by means of a time dilation (a conversion of time between finite and infinite horizons). This Part I paper is companion to a Part II paper which introduces PT-ES for a three-piece cascade of a heat PDE, an integrator, and a static map, which are inspired by optimization of phase change problems in the form of the Stefan PDE-ODE model.

## I. Introduction

### A. Extremum seeking and prescribed-time stabilization

Extremum seeking (ES) has been an effective optimization technique to find the optimum of static and dynamic systems in real time due to its non-model based nature. Since its proof of stability for ES was developed [9], this technique has been actively studied in terms of theory and has been validated practically in numerous applications (see [17] and references therein). In recent years, ES has also been extended to static maps with time delays observed in the actuation and measurement path. This problem setting can be observed in semiconductor manufacturing, or in many chemical and biochemical processes as discussed in [13]. In addition, the developed delay compensated ES algorithms have been applied to the game theory [14] and the traffic bottleneck congestion [19] problem to name a few.

More recently, the concept of prescribed-time (PT) stabilization convergence has been attracting an extensive interest because the convergence time is prescribed a priori by the user independently of the initial condition and system parameters, which represents an advantage with respect to the other stability concepts such as asymptotic, finite-time and fixed-time stability. Although there has been results on prescribed-time stability of finite-dimensional [6], [16] and infinite-dimensional systems [1], [2], ES with finite-time [4] and fixed-time or prescribed-time convergence [15] are relatively new in the literature.

C. T. Yilmaz and M. Krstic are with Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA, USA. cyilmaz@ucsd.edu, krstic@ucsd.edu

### B. Prescribed-time extremum seeking with chirpy sinusoids

We present the first result in the literature that achieves prescribed-time extremum seeking (PT-ES), first for a static map and then for a map preceded by a delay. In a second, companion paper, we extend our results to static maps with heat PDE actuator dynamics [18].

Two novel ingredients of the PT-ES are (1) the employment of "chirpy" perturbation and demodulation signals (signals whose frequency grows unbounded) and (2) the use of tuning gains that grow unbounded (at least in theory; in implementation they are "clipped" to some moderately large values, sufficient to get very close to the extremum by the prescribed terminal time $T$).

Chirpy signals are common in signal processing and radar technology. In our PT-ES we replace probing sinusoids with a constant frequency $\omega$ by sinusoids with argument $\omega t T/(t_0 + T - t)$, which goes to infinity as the time $t$ advances from the initial time $t_0$ to the terminal time $t_0 + T$, where $T$ is the prescribed horizon of convergence to the extremum. The first derivative of the sinusoid's argument is called the 'instantaneous frequency' and in our case it is given by $\omega(t_0 + T)T/(t_0 + T - t)^2$. The second derivative is called 'chirpiness,' which in our approach is

$$\text{chirpiness}(t) = 2\omega \frac{T(t_0 + T)}{(t_0 + T - t)^3}. \quad (1)$$

We first show a PT-ES design for a static scalar map, to provide the basic understanding for how the existing theory of infinite-horizon averaging can be employed for a stability study in a PT setting. This movement between finite and infinite time is done using the time-dilation transformation $\tau(t) = Tt/(t_0 + T - t)$, where $\tau$ begins at $t_0$ and becomes infinite when $t \to t_0 + T$.

In the absence of delay, it is of interest to compare the result of our Section II with the result in [15]. The fixed-time result in [15] yield settling times that are user-assignable only with Newton-based algorithms, which make the settling times independent of the unknown Hessian. In our paper, thanks to the use of chirped probing and a time-varying gain, we achieve a desired settling time $T$ without the Hessian estimation and without the Newton approach.

Then, in Sections III, IV, and VI, we advance to a static map with an input delay, which is inspired by maximizing flow through traffic bottlenecks [19]. We first represent the delay using a transport PDE and obtain a ODE-PDE cascade. Second, we design a time-varying backstepping transformation, which maps the cascade into a prescribed-time stabilized target system, and obtain a controller. The

978-1-6654-5196-3/$31.00 ©2022 AACC 

perturbation-based estimates of the gradient and Hessian are incorporated into the resulting controller. Then, we prove the PT convergence of the average closed-loop system. We also discuss that the proof of the PT convergence of the output to a small neighborhood of the extremum would be shown if a suitable averaging theorem was developed.

### C. Notation

The partial derivatives of a function $u(x,t)$ are denoted by $\partial_x u(x,t) = \partial u(x,t)/\partial x$, $\partial_t u(x,t) = \partial u(x,t)/\partial t$. The spatial $L_2(0,D)$ norm of $u(x,t)$ is denoted by $||u(\cdot,t)||^2_{L_2(0,D)} = \int_0^D u^2(x,t)dx$. The spatial $L_2(X)$ norm of a function $p(x,y,t)$, where $X := \{0 \le y \le x \le D\}$ for $D \ge 0$, is denoted by $||p(\cdot,\cdot,t)||^2_{L_2(X)} = \int_0^D \int_0^x p^2(x,y,t)dydx$.

## II. Basic PT-ES Free of Delays: Design and Analysis

The main objective of scalar PT-ES is to find the optimum of the unknown static map $Q(\theta)$ in a prescribed-time by employing harmonic excitation to the input $\theta \in \mathbb{R}$, and extracting the gradient information $\hat{G}$ from the output response $y \in \mathbb{R}$. Regarding the structure of the unknown static map, we assume the following:

*Assumption 1:* The unknown nonlinear static map has the following quadratic form,

$$Q(\theta) = y^* + \frac{H}{2}(\theta - \theta^*)^2, \tag{2}$$

where $y^* \in \mathbb{R}$ and $\theta^* \in \mathbb{R}$ are the unknown optimum output and input value, respectively, $H < 0$ is the unknown Hessian of the static map $Q(\theta)$.

We illustrate the basic procedure of the PT-ES scheme in Figure 1. It is clear from (2) and Figure 1 that the output signal $y(t)$ is written as follows

$$y(t) = y^* + \frac{H}{2}(\theta(t) - \theta^*)^2. \tag{3}$$

We define the following dilation and contraction transformations

$$\tau = \frac{t}{\nu(t - t_0)}, \tag{4}$$

$$t = (T + t_0)\frac{\tau}{T + \tau}, \tag{5}$$

with the following smooth functions

$$\nu(t - t_0) = 1 - \frac{t - t_0}{T}, \tag{6}$$

$$\mu_m(t - t_0) = \frac{1}{\nu^m(t - t_0)}, \tag{7}$$

for $t \in [t_0, t_0 + T)$, $\tau \in [t_0, \infty)$ and $m \in \mathbb{N}$, where $t_0$ is the initial time and $T$ is the prescribed-time. In general, the perturbation signals $S(t)$ and $M(t)$ are chosen as $a\sin(\omega t)$ and $\frac{2}{a}\sin(\omega t)$ in order to ensure the exponential stability of the averaged error-dynamics. To achieve PT convergence to the extremum, we replace the sinusoids with "chirpy" perturbation and demodulation signals whose frequency grows rather than being constant:

$$S(t) = a\sin\left(\frac{\omega t}{\nu(t - t_0)}\right), \tag{8}$$

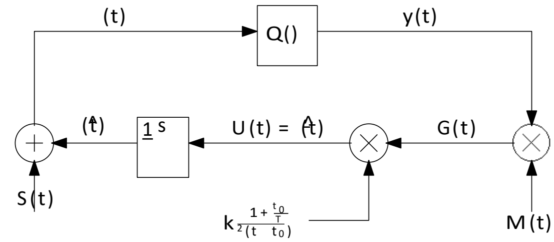


Fig. 1. Basic PT-ES scheme

$$M(t) = \frac{2}{a}\sin\left(\frac{\omega t}{\nu(t - t_0)}\right), \tag{9}$$

Letting $\tilde{\theta} = \hat{\theta} - \theta^*$ and considering Fig. 1, we can write

$$\dot{\tilde{\theta}}(t) = k\frac{1 + \frac{t_0}{T}}{\nu^2(t - t_0)}\frac{2}{a}\sin\left(\frac{\omega t}{\nu(t - t_0)}\right)y(t), \tag{10}$$

for $k > 0$. Considering (5), we define $\tilde{\theta}^\infty(\tau) = \tilde{\theta}\left(\frac{\tau(T + t_0)}{T + \tau}\right)$ and note that $\theta(t) = \hat{\theta}(t) + a\sin\left(\frac{\omega t}{\nu(t - t_0)}\right)$, which yields $\theta(t) - \theta^* = \tilde{\theta}(t) + a\sin\left(\frac{\omega t}{\nu(t - t_0)}\right)$. Then, by recalling (3), we can rewrite (10) in $\tau$-domain as follows

$$\frac{d\tilde{\theta}^\infty(\tau)}{d\tau} = \frac{2k}{a}\sin(\omega\tau)\left[y^* + \frac{H}{2}\left(\tilde{\theta}^\infty(\tau) + a\sin(\omega\tau)\right)^2\right]. \tag{11}$$

The averaging of (11) yields

$$\frac{d\tilde{\theta}^\infty_{av}(\tau)}{d\tau} = kH\tilde{\theta}^\infty_{av}(\tau), \tag{12}$$

which shows the local exponential stability of $\tilde{\theta}^\infty(\tau)$ in $\tau$-domain, $\tau \in [t_0, \infty)$ and implies the local PT stability of $\tilde{\theta}(t)$ in $t$-domain, $t \in [t_0, t_0 + T)$. The following theorem concludes the properties of the basic PT-ES scheme depicted in Figure 1.

*Theorem 1:* Consider the system in Figure 1 and the transformations (4), (5) under Assumption 1. There exists $\omega^* > 0$ such that $\forall\omega > \omega^*$, the error $\tilde{\theta}^\infty(\tau) = \hat{\theta}\left(\frac{\tau(T + t_0)}{T + \tau}\right) - \theta^*$ has a unique prescribed-time stable solution in $t$-domain, denoted by $\tilde{\theta}^\Pi\left(\frac{t}{\nu(t - t_0)}\right)$, where $\tilde{\theta}^\Pi(\tau)$ is the unique exponentially stable periodic solution in $\tau$ of period $\Pi = 2\pi/\omega$ satisfying $\forall\tau \ge t_0$:

$$|\tilde{\theta}^\Pi(\tau)| \le O(1/\omega). \tag{13}$$

Furthermore,

$$\limsup_{t \to t_0 + T} |\theta(t) - \theta^*| = O(a + 1/\omega), \tag{14}$$

$$\limsup_{t \to t_0 + T} |y(t) - y^*| = O(a^2 + 1/\omega^2). \tag{15}$$

*Proof:* The proof of (13) follows from the application of the averaging theorem [7] to (12). Performing the time contraction $\tau \to t$ in (13), we deduce the PT stable solution of $\tilde{\theta}(t)$. In view of the fact (13), we can write

$$\limsup_{\tau \to \infty} |\tilde{\theta}^\infty(\tau)|^2 = \limsup_{\tau \to \infty} \left\{|\tilde{\theta}^\infty(\tau) + \tilde{\theta}^\Pi(\tau) - \tilde{\theta}^\Pi(\tau)|^2\right\}, \tag{16}$$

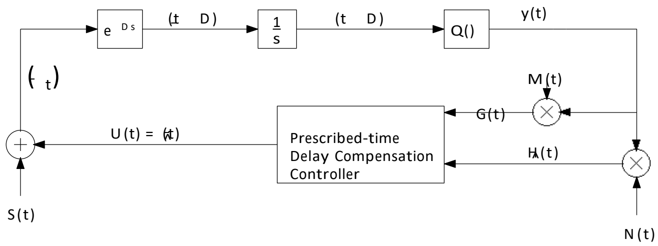


Fig. 2. Block diagram of the developed PT-ES scheme for delay compensation

$$\leq \sqrt{2 \lim_{\tau\to\infty} \sup \left\{ |\tilde{\theta}^{\infty}(\tau) - \tilde{\theta}^{\Pi}(\tau)|^2 + |\tilde{\theta}^{\Pi}(\tau)|^2 \right\}}, \quad (17)$$

$$\leq O(1/\omega). \quad (18)$$

Considering the property of $\theta^{\infty}(\tau) - \theta^* = \tilde{\theta}^{\infty}(\tau) + a\sin(\omega\tau)$ and performing the time contraction $\tau \to t$, we get (14). Considering (3) and (14), we conclude (15). ■

## III. Problem Statement for PT-ES with Delays

In this paper, we consider the case that the output $y(t)$ has a known and constant delay $D \in \mathbb{R}$, i.e.,

$$y(t) = Q(\theta(t - D)). \quad (19)$$

Note that the static map $Q(\theta)$ allows us to write the delay $D$ as $D = D_u + D_y$, where $D_u$ and $D_y$ are the delays in the actuation and measurement path, respectively. Our aim is to design a scalar PT-ES which compensates the delay and stabilizes the output signal $y(t)$ around a small neighborhood of the extremum in the prescribed time $T \geq D$. In addition to Assumption 1, we make the following assumption regarding the bound of the Hessian.

*Assumption 2:* The lower bound of the unknown Hessian of the static map, $\bar{H} < H$, is known.

Considering Assumption 1, we can write the output of the static map as follows

$$y(t) = y^* + \frac{H}{2}(\theta(t - D) - \theta^*)^2. \quad (20)$$

Figure 2 illustrates the closed-loop ES with prescribed-time delay compensation controller, which is to be designed.

## IV. Perturbation Signals and Gradient/Hessian Estimate

In this section, we define the signals introduced in Figure 2. Let us first define the following error state

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*, \quad (21)$$

where $\hat{\theta}(t)$ is the estimate of $\theta^*$ and $\tilde{\theta}(t)$ is the estimation error. In view of (21), we can write the estimation error dynamics from Figure 2 as follows

$$\dot{\tilde{\theta}}(t - D) = U(t - D). \quad (22)$$

Moreover, we get

$$\dot{\theta}(t) = \dot{\hat{\theta}}(t) + S(t), \quad (23)$$

Here, $S(t)$ is the derivative of the future state of (8) in delay-free case since the integrator acts before the map $Q(\cdot)$, whereas the integrator comes after $U(t)$ in Figure 1. $S(t)$ is expressed as follows

$$S(t) = a\omega \cos\left(\frac{\omega(t + D)}{\nu(t + D - t_0)}\right) \frac{1 + \frac{t_0}{T}}{\nu^2(t + D - t_0)}. \quad (24)$$

Considering (23) and (24), we can write the following relation

$$\theta(t) = \hat{\theta}(t) + a\sin\left(\frac{\omega(t + D)}{\nu(t + D - t_0)}\right). \quad (25)$$

Let us define

$$\vartheta(t) = \tilde{\theta}(t - D). \quad (26)$$

Combining (21) and (25) in view of (26), we get the following relation

$$\vartheta(t) + a\sin\left(\frac{\omega t}{\nu(t - t_0)}\right) = \theta(t - D) - \theta^*. \quad (27)$$

The final step before designing a controller is to generate the estimate of the gradient and Hessian as follows

$$G(t) = M(t)y(t), \quad (28)$$

$$\hat{H}(t) = N(t)y(t), \quad (29)$$

with the following multiplicative excitation signals

$$M(t) = \frac{2}{a}\sin\left(\frac{\omega t}{\nu(t - t_0)}\right), \quad (30)$$

$$N(t) = -\frac{8}{a^2}\cos\left(\frac{2\omega t}{\nu(t - t_0)}\right). \quad (31)$$

Performing time dilation $t \to \tau$ and following [3], the averaged version of the gradient (28) and Hessian estimate (29) along with (20), (27) can be calculated as follows

$$G_{av}(t) = H\vartheta_{av}(t), \quad (32)$$

$$\hat{H}_{av}(t) = H. \quad (33)$$

## V. Controller Design

Recalling (22), (26) and using the technique in [8], the delayed input $U(t - D)$ can be represented as the boundary of a transport partial differential equation (PDE) as follows,

$$\dot{\vartheta}(t) = u(0, t), \quad (34)$$

$$\partial_t u(x, t) = \partial_x u(x, t), \quad (35)$$

$$u(D, t) = U(t), \quad (36)$$

where the solution of (35) and (36) is given by

$$u(x, t) = U(t + x - D). \quad (37)$$

Following the technique discussed in [8], we consider the following backstepping transformation

$$w(x, t) = u(x, t) + \bar{k}\mu_2(t - t_0 + x)\left(\vartheta(t) + \int_0^x u(y, t)dy\right) \quad (38)$$

which maps (34)–(36) into the target system

$$\dot{\vartheta}(t) = -\bar{k}\mu_2(t - t_0)\vartheta(t) + w(0, t), \quad (39)$$

$$\partial_t w(x, t) = \partial_x w(x, t), \quad (40)$$

$$\partial_t w(D, t) = -\mu_0\mu_2(t - t_0 + D)w(D, t), \quad (41)$$

where $\bar{k}, \mu_0 > 0$. Now, in order to design a dynamic stabilizing feedback law, let us take the derivative of (38) with respect to $x$

$$\partial_t w(x,t) = \partial_t u(x,t) + \bar{k}\mu_2(t-t_0+x)u(x,t) + 2\bar{k}/T\,\mu_3(t-t_0+D)\left(\vartheta(t) + \int_0^x u(y,t)dy\right). \tag{42}$$

In (42), let $x = D$ and consider the dynamic boundary condition (41). Then, we obtain the controller as follows

$$\begin{aligned}\dot{U}(t) = &-(\mu_0+\bar{k})\mu_2(t-t_0+D)U(t) \\ &-\bar{k}\mu_3(t-t_0+D)\left(\mu_0\mu_1(t-t_0+D) + 2/T\right. \\ &\left.\times\left(\vartheta(t) + \int_0^D u(y,t)dy\right)\right).\end{aligned} \tag{43}$$

The implementation of the controller (43) is not possible since there is no measurement of $\vartheta(t)$. To achieve this, let us define $\bar{k} = kH$ where $H < 0$ is the unknown Hessian and the gain $k < 0$ is assigned by the user. Then, we can rewrite the controller (43) as follows

$$\dot{U}(t) = -\mu_0\mu_2(t-t_0+D)U(t) + \hat{H}(t)Q(t) + G(t)P(t), \tag{44}$$

where

$$\begin{aligned}Q(t) = &-k\mu_2(t-t_0+D)\left(U(t) + \mu_1(t-t_0+D)\right. \\ &\left.\times\left(\mu_0\mu_1(t-t_0+D) + 2/T\right)\int_0^D u(y,t)dy\right),\end{aligned} \tag{45}$$

$$P(t) = -k\mu_3(t-t_0+D)\left[\mu_0\mu_1(t-t_0+D) + 2/T\right]. \tag{46}$$

Let us note that imposing a dynamic boundary condition on the target systems (41) is inspired by [10]. This enables us to design a dynamic stabilizing feedback law directly. This is a different controller design approach than [11] where a static stabilizing feedback law is designed first and then, a low-pass filter applied to the controller. Substituting (28) and (29) into (44), we obtain the following error-dynamics

$$\dot{\vartheta}(t) = u(0,t), \tag{47}$$

$$\partial_t u(x,t) = \partial_x u(x,t), \tag{48}$$

$$\begin{aligned}\partial_t u(D,t) = &-\mu_0\mu_2(t-t_0+D)u(D,t) + HP(t)\vartheta(t) \\ &+ HQ(t) + \sin\left(\frac{\omega t}{\nu(t-t_0)}\right)f_1(t) \\ &-\cos\left(\frac{2\omega t}{\nu(t-t_0)}\right)f_2(t) - \sin\left(\frac{3\omega t}{\nu(t-t_0)}\right)f_3(t) + \\ &\cos\left(\frac{4\omega t}{\nu(t-t_0)}\right)f_4(t),\end{aligned} \tag{49}$$

where

$$f_1(t) = P(t)\left(\frac{2y^*}{a} + \frac{H}{a}\vartheta^2(t) + \frac{3Ha}{4}\right) + Q(t)\left(\frac{4H}{a}\vartheta(t)\right), \tag{50}$$

$$f_2(t) = P(t)[H\vartheta(t)] + Q(t)\left(2H + \frac{8y^*}{a^2} + \frac{4H}{a^2}\vartheta^2(t)\right), \tag{51}$$

$$f_3(t) = P(t)\left(\frac{Ha}{4}\right) + Q(t)\left(\frac{4H}{a}\vartheta(t)\right), \tag{52}$$

$$f_4(t) = Q(t)H. \tag{53}$$

## VI. Stability Analysis

Let us define $\vartheta^\infty(\tau) = \vartheta(\tau(T+t_0)/(T+\tau))$, $u^\infty(x,\tau) = u(x,\tau(T+t_0)/(T+\tau))$, $U^\infty(\tau) = U(\tau(T+t_0)/(T+\tau))$ and write the system (47)–(49) in $\tau$-domain as follows

$$\frac{d}{d\tau}\vartheta^\infty(\tau) = \frac{T(T+t_0)}{(T+\tau)^2}u^\infty(0,\tau), \tag{54}$$

$$\partial_\tau u^\infty(x,\tau) = \frac{T(T+t_0)}{(T+\tau)^2}\partial_x u^\infty(x,\tau), \tag{55}$$

$$\frac{d}{d\tau}u^\infty(D,\tau) = \frac{d}{d\tau}U^\infty(\tau), \tag{56}$$

for $\tau \in [t_0,\infty)$ and $x \in [0,D]$. The average version of the error-dynamics (54)–(56) in $\tau$-domain can be directly written as follows

$$\frac{d}{d\tau}\vartheta^\infty_{av}(\tau) = \frac{T(T+t_0)}{(T+\tau)}u^\infty_{av}(0,\tau), \tag{57}$$

$$\partial_\tau u^\infty_{av}(x,\tau) = \frac{T(T+t_0)}{(T+\tau)^2}\partial_x u^\infty_{av}(x,\tau), \tag{58}$$

$$\frac{d}{d\tau}u^\infty_{av}(D,\tau) = \frac{d}{d\tau}U^\infty_{av}(\tau). \tag{59}$$

Let us write (57)–(59) in t-domain again as follows

$$\dot{\vartheta}_{av}(t) = u_{av}(0,t), \tag{60}$$

$$\partial_t u_{av}(x,t) = \partial_x u_{av}(x,t), \tag{61}$$

$$\begin{aligned}\frac{d}{dt}u_{av}(D,t) = &-(\mu_0+kH)\mu_2(t-t_0+D)u_{av}(D,t) \\ &-kH\left(\mu_0\mu_4(t-t_0+D) + \frac{2}{T}\mu_3(t-t_0+D)\right) \\ &\times\left(\vartheta_{av}(t) + \int_0^D u_{av}(y,t)dy\right).\end{aligned} \tag{62}$$

for $(x,t) \in T_2$. The main theorem is stated as follows.

*Theorem 2:* The average system (60)–(62) is prescribed-time stable and the solution satisfies

$$\lim_{t\to t_0+T}\left[\vartheta^2_{av}(t) + \|u_{av}(\cdot,t)\|^2_{L_2(0,D)}\right] = 0, \tag{63}$$

$$\lim_{t\to t_0+T-D} U^2_{av}(t) = 0 \tag{64}$$

*Proof:*

Step 1: (Time dilation $t \to \tau$) This step is done in (54)–(56) by writing the system (47)–(49) in $\tau$-domain.

Step 2: (Averaging operation) We perform this step in (57)–(59) by taking the average of (54)–(56).

Step 3: (Time contraction $\tau \to t$) We convert (57)–(59) to (60)–(62).

Step 4: (Backstepping transformation) The backstepping transformation

$$\begin{aligned}w(x,t) = &\ u_{av}(x,t) + kH\mu_2(t-t_0+x) \\ &\times\left(\vartheta_{av}(t) + \int_0^x u_{av}(y,t)dy\right)\end{aligned} \tag{65}$$

transforms (60)–(62) into the target system

$$\dot{\vartheta}_{av}(t) = -kH\mu_2(t-t_0)\vartheta_{av}(t) + w(0,t), \tag{66}$$

$$\partial_t w(x,t) = \partial_x w(x,t), \tag{67}$$

$$\frac{d}{dt}w(D,t) = -\mu_0\mu_2(t-t_0+D)w(D,t). \tag{68}$$

The explicit solution of the PDE system (67), (68) is given by

$$w(x,t) = e^{-\mu_0 T(\mu_1(t-t_0+x)-\mu_1(D))} w(D,t_0). \tag{69}$$

Considering (69), we obtain the solution of the ODE system (66) as follows

$$\begin{aligned}\vartheta_{av}(t) &= e^{\int_{t_0}^{t} -kH\mu_2(\eta-t_0)d\eta}\vartheta_{av}(t_0) + w(D,t_0)\\ &\times \int_{t_0}^{t} e^{\int_{\eta}^{t} -kH\mu_2(\tau-t_0)d\tau} e^{-\mu_0 T(\mu_1(\eta-t_0)-\mu_1(D))}d\eta\\ &= e^{-kHT(\mu_1(t-t_0)-1)}\vartheta_{av}(t_0) + w(D,t_0)e^{\mu_0 T\mu_1(D)}\\ &\times e^{-kHT\mu_1(t-t_0)}\int_{t_0}^{t} e^{-T\mu_1(\eta-t_0)(\mu_0-kH)}d\eta\\ &\leq C_1 e^{-kHT\mu_1(t-t_0)}\vartheta_{av}(t_0),\end{aligned} \tag{70}$$

for $C_1 > 0$. Choosing $\mu_0 > k\bar{H}$, where $\bar{H}$ is the known lower bound of $H$, we conclude that $\vartheta_{av}(t) \to 0$ as $t \to t_0 + T$. Moreover, we can also prove the prescribed-time stability of the norm $W(t) := ||w(\cdot,t)||^2_{L^2(0,D)}$. To do this, we take the derivative of $W(t)$ as follows

$$\begin{aligned}\dot{W}(t) &= \int_0^D -2\mu_0\mu_2(t-t_0+x)w^2(x,t)dx\\ &\leq -2\mu_0\mu_2(t-t_0)W(t).\end{aligned} \tag{71}$$

It follows from (71) that

$$||w(\cdot,t)||_{L^2(0,D)} \leq e^{-2\mu_0 T(\mu_1(t-t_0)-1)}||w(\cdot,t_0)||_{L^2(0,D)}, \tag{72}$$

from which we can conclude that $||w(\cdot,t)||_{L^2(0,D)} \to 0$ as $t \to t_0 + T$. In order to show the prescribed-time stability the original system (60)–(62) as well, the backstepping transformation (65) needs to be invertible. The inverse transformation is given by

$$u_{av}(x,t) = w(x,t) + \int_0^x p(x,y,t)w(y,t)dy + \Gamma(x,t)\vartheta_{av}(t) \tag{73}$$

where the kernels $p(x,y,t)$ satisfies

$$\partial_t p(x,y,t) = \partial_x p(x,y,t) + \partial_y p(x,y,t), \tag{74}$$

$$p(x,0,t) = \Gamma(x,t), \tag{75}$$

for $(x,y,t) \in T_1$ and the kernel $\Gamma(x,t)$ satisfies

$$\partial_t \Gamma(x,t) = \partial_x \Gamma(x,t) + kH\mu_2(t-t_0)\Gamma(x,t), \tag{76}$$

$$\Gamma(0,t) = -kH\mu_2(t-t_0), \tag{77}$$

for $(x,t) \in T_2$. The well-posed solutions of the PDE system (74)–(77) are given as follows

$$p(x,y,t) = -kH\mu_2(t-t_0+x)e^{kHT\mu_1(t-t_0+y)-kHT\mu_1(t-t_0+x)} \tag{78}$$

$$\Gamma(x,t) = -kH\mu_2(t-t_0+x)e^{kHT\mu_1(t-t_0)-kHT\mu_1(t-t_0+x)}. \tag{79}$$

The inverse backstepping transformation (73) is bounded by

$$\begin{aligned}||u_{av}(\cdot,t)||^2_{L^2(0,D)} &\leq 3||w(\cdot,t)||^2_{L^2(0,D)} +\\ 3||\Gamma(\cdot,t)||^2_{L^2(0,D)}\vartheta^2_{av}(t) &+ 3D||p(\cdot,\cdot,t)w(\cdot,t)||^2_{L^2(X)},\end{aligned} \tag{80}$$

where $X := \{0 \leq y \leq x \leq D\}$. It is easy to recover the following bounds

$$\Gamma^2(D,t)\vartheta^2_{av}(t) \leq C_2\mu_4(t-t_0+D)e^{-2kHT\mu_1(t-t_0+D)}, \tag{81}$$

$$\begin{aligned}&||p(D,\cdot,t)w(\cdot,t)||^2_{L^2(0,D)}\\ &\quad\leq C_3\mu_4(t-t_0+D)e^{-2kHT\mu_1(t-t_0+D)},\end{aligned} \tag{82}$$

$$||\Gamma(\cdot,t)||^2_{L^2(0,D)}\vartheta^2_{av}(t) \leq C_4\mu_2(t-t_0)e^{-2kHT\mu_1(t-t_0)}, \tag{83}$$

$$||p(\cdot,\cdot,t)w(\cdot,t)||^2_{L^2(X)} \leq C_5\mu_2(t-t_0)e^{-2kHT\mu_1(t-t_0)}, \tag{84}$$

provided that $\mu_0 > k\bar{H}$ for some $C_2, C_3, C_4, C_5 > 0$ by recalling (70), (72), (78) and (79). The stability bound for the original system (60)–(62) can be written from (80) as follows

$$\begin{aligned}&\vartheta^2_{av}(t) + ||u_{av}(\cdot,t)||^2_{L^2(0,D)}\\ &\quad\leq 3||w(\cdot,t)||^2_{L^2(0,D)} + 3D||p(\cdot,\cdot,t)w(\cdot,t)||^2_{L^2(X)}\\ &\quad+ \left(1 + 3||\Gamma(\cdot,t)||^2_{L^2(0,D)}\right)\vartheta^2_{av}(t).\end{aligned} \tag{85}$$

In view of the bounds (83), (84) and noting the fact that the decaying exponential terms dominate the the right hand side of (83), (84) as $t \to t_0 + T$, we conclude that $\vartheta^2_{av}(t) + ||u_{av}(\cdot,t)||^2_{L^2(0,D)} \to 0$ as $t \to t_0 + T$. This proves (63).

The next step is to recover the bound of the average controller input $U_{av}(t) = u_{av}(D,t)$. Setting $x = D$ in (65), we obtain the following bound

$$\begin{aligned}U^2_{av}(t) &\leq 3w^2(D,t) + 3D||p(D,\cdot,t)w(\cdot,t)||^2_{L^2(0,D)}\\ &+ 3\Gamma^2(D,t)\vartheta^2_{av}(t).\end{aligned} \tag{86}$$

Recalling the bounds (69), (81), (82) and noting the domination of the exponential component at the right hand side as $t \to t_0+T-D$, we ensure that $U_{av}(t) \to 0$ as $t \to t_0+T-D$. This proves (64) and completes the proof of Theorem 2. ■

In Theorem 2, we prove that the average closed-loop system (60)–(62) is PT stable. Note that the right hand side of (54) and (55) has a decaying function of time. If these functions were a constant, the averaging technique introduced in [5] would be used. However, for the system (54)–(56), there is no suitable averaging theorem and remains as an open problem in the literature. If this theorem existed, then we would apply it to (54)–(56), then perform the time contraction $\tau \to t$ and prove the following conjecture.

Conjecture 1: Consider the transformations (4), (5), the error-dynamics (47)–(49) in t-domain and the error-dynamics (54)–(56) in $\tau$-domain under Assumption 1 and 2. There exists $\omega^* > 0$ such that $\forall\omega > \omega^*$, the error dynamical system (47)–(49) has a unique prescribed-time stable solution in t-domain, denoted by $\vartheta^\Pi\left(\frac{t}{\nu(t-t_0)}\right)$, $u^\Pi\left(x, \frac{t}{\nu(t-t_0)}\right)$, where $\vartheta^\Pi(\tau), u^\Pi(x,\tau)$ is the unique exponentially stable periodic solution of (54)–(56) in $\tau$ of period $\Pi = 2\pi/\omega$ satisfying $\forall\tau \geq t_0$:

$$\left((\vartheta^\Pi(\tau))^2 + ||u^\Pi(\cdot,\tau)||^2_{L^2(0,D)} + (u^\Pi(D,\tau))^2\right)^{1/2} \leq O(1/\omega). \tag{87}$$

Furthermore,

$$\lim_{t\to t_0+T-D}\sup|\theta(t) - \theta^*| = O(a + 1/\omega), \tag{88}$$

$$\lim_{t\to t_0+T}\sup|y(t) - y^*| = O(a^2 + 1/\omega^2). \tag{89}$$

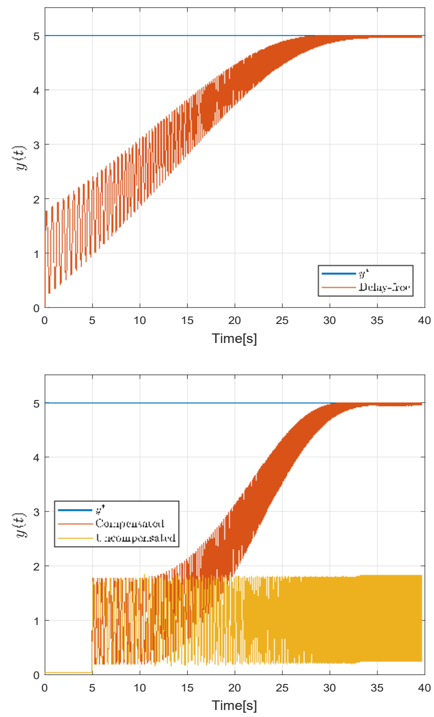


Fig. 3. The upper plot shows the prescribed-time stability of the output signal $y(t)$ under the PT-ES scheme (10) in the presence of no delay. In the lower plot, this scheme fails to achieve the convergence to the optimum value $y^\star$ in the presence of delay, while, the developed PT-ES algorithm (44) compensates the delay and achieves the controller objective in the prescribed time $T = 40$s.

## VII. Numerical Simulation

In order to test the effectiveness of the developed extremum seeking controller, we consider the following static quadratic map with the delayed measurement

$$Q(\theta) = 5 - (\theta - 2)^2, \quad (90)$$

$$y(t) = Q(\theta(t - 5)), \quad (91)$$

which is borrowed from [12]. It is clear from (90) and (91) that the extremum point is $(\theta^\star, y^\star) = (2, 5)$, the Hessian of the map is $H = -2$ and the input-output delay is $D = 5$s. We choose the following parameters to perform our simulation: $k = -0.01$, $a = 0.2$, $w = 10$, $\mu_0 = 0.03$, $U(t_0) = 0, \hat{\theta}(t_0) = 0$. The initial time is $t_0 = 0$ and the prescribed-time is selected as $T = 40$s. Figure 3 shows the evolution of the output signal $y(t)$ in three situations: (i) the PT-ES (10) without any input-output delay, (ii) the same controller in the presence of the input-output delay and (iii) the developed delay-compensated PT-ES (44) in the presence of the input-output delay.

## VIII. Conclusion

In this paper, we introduce a scalar gradient-based PT-ES algorithm in the presence of known input-output delay. Although the problem of PT stability of finite-dimensional and infinite-dimensional system as well as the exponential stability of ES algorithms have been actively researched in the recent years, this is the first attempt which combines two concepts, namely PT stability and delay compensation for ES algorithms. We design an average-based controller through a backstepping transformation which maps the system into a desired PT stable target. We prove that the average closed-loop system converges to zero in the prescribed time. We discuss that if there was a generalized averaging theorem for infinite-dimensional systems, we would prove the existence of the PT stable solution of the closed-loop system. The numerical simulations illustrate that the developed algorithm achieves the convergence of the system output to a small neighborhood of the extremum point in the prescribed time.

## References


[1] N. Espitia and W. Perruquetti. Predictor-feedback prescribed-time stabilization of lti systems with input delay. IEEE Transactions on Automatic Control, 2021.

[2] N. Espitia, A. Polyakov, D. Efimov, and W. Perruquetti. Bound-ary time-varying feedbacks for fixed-time stabilization of constant-parameter reaction–diffusion systems. Automatica, 103:398–407, 2019.

[3] J. Feiling, S. Koga, M. Krstić, and T. R. Oliveira. Gradient extremum seeking for static maps with actuation dynamics governed by diffusion pdes. Automatica, 95:197–206, 2018.

[4] M. Guay and M. Benosman. Finite-time extremum seeking control for a class of unknown static maps. International Journal of Adaptive Control and Signal Processing, 35(7):1188–1201, 2021.

[5] J. K. Hale and S. M. V. Lunel. Averaging in infinite dimensions. The Journal of Integral Equations and Applications, pages 463–494, 1990.

[6] J. Holloway and M. Krstic. Prescribed-time observers for linear systems in observer canonical form. IEEE Transactions on Automatic Control, 64(9):3905–3912, 2019.

[7] H. K. Khalil. Nonlinear Systems. New Jersey:Prenctice-Hall, 1996.

[8] M. Krstic. Delay Compensation for Nonlinear, Adaptive, and PDE Systems. Springer, 2009.

[9] M. Krstic and H.-H. Wang. Stability of extremum seeking feed-back for general nonlinear dynamic systems. Automatica-Kidlington, 36(4):595–602, 2000.

[10] P. Nikdel, F. Sheikholeslam, M. Zekri, and M. Ghadiri-Modarres. Newton-based extremum seeking for a static map coupled with a diffusion pde at an arbitrary interior point. In 2021 7th International Conference on Control, Instrumentation and Automation (ICCIA), pages 1–5. IEEE, 2021.

[11] T. R. Oliveira, J. Feiling, and M. Krstic. Extremum seeking for maximizing higher derivatives of unknown maps in cascade with reaction-advection-diffusion pdes. IFAC-PapersOnLine, 52(29):210–215, 2019.

[12] T. R. Oliveira and M. Krstić. Gradient extremum seeking with delays. IFAC-PapersOnLine, 48(12):227–232, 2015.

[13] T. R. Oliveira, M. Krstić, and D. Tsubakino. Extremum seeking for static maps with delays. IEEE Transactions on Automatic Control, 62(4):1911–1926, 2016.

[14] T. R. Oliveira, V. H. P. Rodrigues, M. Krstić, and T. Başar. Nash equilibrium seeking in quadratic noncooperative games under two delayed information-sharing schemes. Journal of Optimization Theory and Applications, pages 1–36, 2020.

[15] J. I. Poveda and M. Krstić. Nonsmooth extremum seeking control with user-prescribed fixed-time convergence. IEEE Transactions on Automatic Control, 66(12):6156–6163, 2021.

[16] Y. Song, Y. Wang, J. Holloway, and M. Krstic. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. Automatica, 83:243–251, 2017.

[17] Y. Tan, W. H. Moase, C. Manzie, D. Nešić, and I. M. Y. Mareels. Extremum seeking from 1922 to 2010. In Proceedings of the 29th Chinese control conference, pages 14–26. IEEE, 2010.

[18] C. T. Yilmaz and M. Krstic. Prescribed-time extremum seeking for pdes—part 2: Heat pde dynamics. In 2022 American Control Conference (ACC). IEEE, 2022.

[19] H. Yu, S. Koga, T. R. Oliveira, and M. Krstic. Extremum seeking for traffic congestion control with a downstream bottleneck. Journal of Dynamic Systems, Measurement, and Control, 143(3):031007, 2021.