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Marcelo Aguiar
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Abstract

Generalized permutahedra are polytopes that arise in combinatorics, algebraic geometry, representation theory, topology, and optimization. They possess a rich combinatorial structure. Out of this structure we build a Hopf monoid in the category of species.

Species provide a unifying framework for organizing families of combinatorial objects. Many species carry a Hopf monoid structure and are related to generalized permutahedra by means of morphisms of Hopf monoids. This includes the species of graphs, matroids, posets, set partitions, linear graphs, hypergraphs, simplicial complexes, and building sets, among others. We employ this algebraic structure to define and study polynomial invariants of the various combinatorial structures.

We pay special attention to the antipode of each Hopf monoid. This map is central to the structure of a Hopf monoid, and it interacts well with its characters and polynomial invariants. It also carries information on the values of the invariants on negative integers. For our Hopf monoid of generalized permutahedra, we show that the antipode maps each polytope to the alternating sum of its faces. This fact has numerous combinatorial consequences.

We highlight some main applications:

- We obtain uniform proofs of numerous old and new results about the Hopf algebraic and combinatorial structures of these families. In particular, we give optimal formulas for the antipode of graphs, posets, matroids, hypergraphs, and building sets. They are optimal in the sense that they provide explicit descriptions for the integers entering in the expansion of the antipode, after all coefficients have been collected and all cancellations have been taken into account.
- We show that reciprocity theorems of Stanley and Billera–Jia–Reiner (BJR) on chromatic polynomials of graphs, order polynomials of posets, and BJR-polynomials of matroids are instances of one such result for generalized permutahedra.

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The first author is affiliated with Cornell University.
The second author is affiliated with San Francisco State University and Universidad de Los Andes.

- We explain why the formulas for the multiplicative and compositional inverses of power series are governed by the face structure of permutahedra and associahedra, respectively, providing an answer to a question of Loday.
- We answer a question of Humpert and Martin on certain invariants of graphs and another of Rota on a certain class of submodular functions.

We hope our work serves as a quick introduction to the theory of Hopf monoids in species, particularly to the reader interested in combinatorial applications. It may be supplemented with Marcelo Aguiar and Swapneel Mahajan's 2010 and 2013 works, which provide longer accounts with a more algebraic focus.

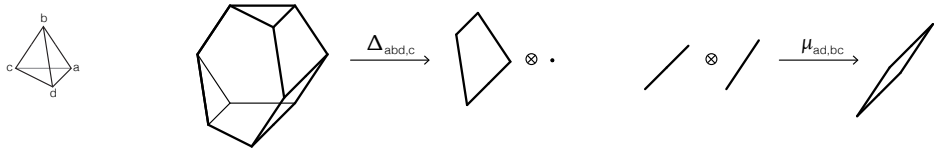
Introduction

Hopf monoids and generalized permutahedra

Joyal [**16**], Joni and Rota [**60**], Schmitt [**84**], Stanley [**93**], and others, taught us that to study combinatorial objects, it is often useful to endow them with algebraic structures. Aguiar and Mahajan's *Hopf monoids in species* [**2**] provide an algebraic framework that supports many familiar combinatorial structures.

Edmonds [**36**], Lovász [**67**], Postnikov [**77**], Stanley [**89**], and others, taught us that to study combinatorial objects, it is often useful to build a polyhedral model for them. *Generalized permutahedra* constitute an ubiquitous family of polytopes which models many combinatorial structures. Generalized permutahedra arose in the theory of combinatorial optimization as polymatroids. Each such polytope is defined by a unique submodular function.

Our work brings together these two points of view. We endow the family of generalized permutahedra with the structure of a Hopf monoid **GP** and show that many other Hopf monoids built out of combinatorial structures find natural models therein, in the sense that they map into **GP** (or certain quotients of **GP**) by means of morphisms of Hopf monoids.



We deal with Hopf monoid structures on (the species of) graphs, matroids, posets, set partitions, simplicial complexes, building sets, and (an additional structure on) simple graphs, to name a few. On these and many other families of combinatorial objects, it is possible to carry out constructions of *merging* and *breaking*: procedures for building a new object out of two, or for decomposing a given object into two. When these procedures obey certain simple rules, the structure can be organized into that of a Hopf monoid in the category of species. The combinatorial objects constitute the elements of the species, merging gives rise to the product, and breaking to the coproduct of the Hopf monoid. A Hopf monoid is a structure akin to that of a Hopf algebra, but better suited to handle these examples rooted in combinatorics.

We use this framework to unify known results, obtain new ones, and answer questions of a combinatorial nature. We discuss some of these applications next.

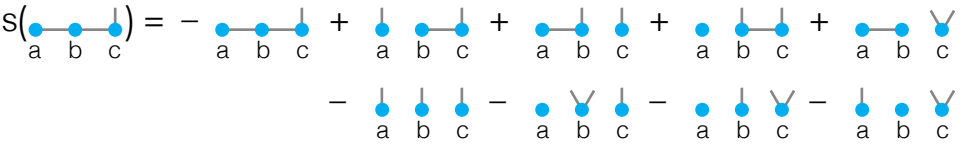
Application A. Antipode formulas

Hopf monoids in species, Hopf algebras, and groups, may all be seen as instances of the general notion of Hopf monoid in a braided (or symmetric) monoidal category.

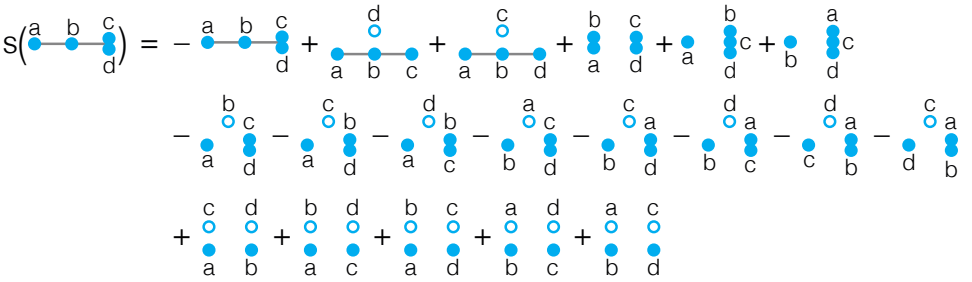
A Hopf monoid $\mathbf{H}$ carries an *antipode* $\mathrm{s} : \mathbf{H} \to \mathbf{H}$, a map which is analogous to inversion in a group. For the Hopf monoids in species we consider, the existence of the antipode is guaranteed, much as is the existence of the reciprocal of a formal power series of the form $1 + xF(x)$. The antipode maps a combinatorial structure to a formal sum of structures of the same kind. It is given by a large alternating sum, usually involving lots of cancellation. A fundamental task is to obtain a cancellation-free formula for the antipode.

Figure 1 gathers a few examples showing the final result of the calculation, after all cancellations have been taken into account. The combinatorial structures are represented by pictures whose meaning is explained in later sections. The formulas arise from alternating sums of 13, 75, and 541 terms, depending on whether the cardinality of the ground set is 3, 4, or 5. One of our main goals is to provide a uniform explanation for these formulas. It turns out that in each case the result is dictated by a polyhedron that models the given combinatorial structure. We explain this in more detail.

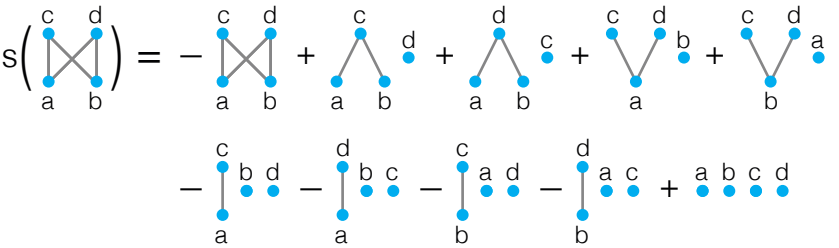
Graphs **G** (Section 3.2):



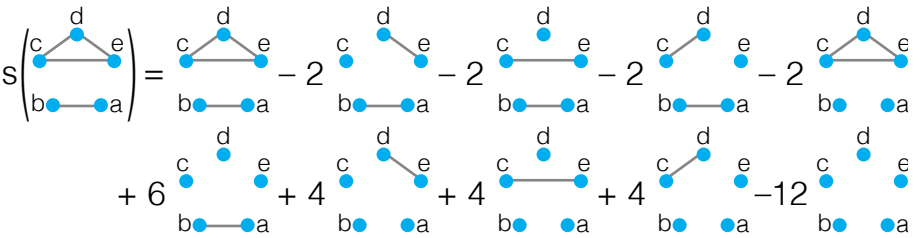
Matroids **M** (Section 3.3):



Posets **P** (Section 3.4):



Partitions **Π** (Section 5.6):



Partitions into paths **F** (Section 5.7):

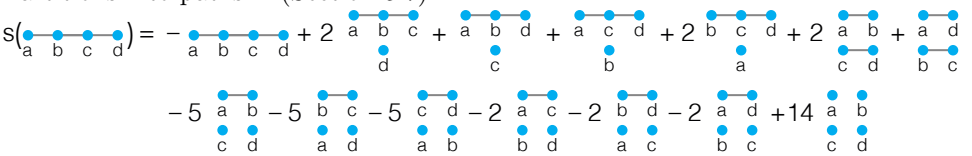


FIGURE 1. Antipode calculations in Hopf monoids

First, we address the antipode problem at the level of generalized permutahedra, where the inherent geometry and topology enable us to understand the cancellation completely. The following is one of our main results.

THEOREM (Theorem 1.6.1). *The antipode for the Hopf monoid of generalized permutahedra* $\mathbf{GP}$ *is given by*

$$\mathrm{s}_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\mathfrak{q} \ face \ of \ \mathfrak{p}} (-1)^{\dim \mathfrak{q}}\, \mathfrak{q}$$

for each generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^I$.

Then, we relate the Hopf monoids $\mathbf{G}, \mathbf{M}, \mathbf{P}, \mathbf{\Pi}, \mathbf{F}$ to the Hopf monoid $\mathbf{GP}$ of generalized permutahedra by means of morphisms of Hopf monoids. Such morphisms preserve antipodes. For example, the graphic zonotope Z_g of a graph g is a generalized permutahedron, and the map $g \mapsto Z_g$ is a morphism of Hopf monoids $\mathbf{G} \to \mathbf{GP}$. To calculate $\mathrm{s}(g)$ in $\mathbf{G}$, we calculate $\mathrm{s}(Z_g)$ in $\mathbf{GP}$ using Theorem 1.6.1. The faces of Z_g are themselves graphic zonotopes associated to certain quotients of g (Lemma 3.2.4). From here, an explicit formula for $\mathrm{s}(g)$ emerges (Corollary 3.2.7). In general, combining Theorem 1.6.1 with an understanding of the combinatorial structure of a given generalized permutahedron, yields a formula generalizing those in Figure 1. The coefficients in the formula of Theorem 1.6.1 are ± 1 (or 0). The larger coefficients in some of the formulas in Figure 1 occur when the morphism to $\mathbf{GP}$ is not injective.

We have gathered the main combinatorial structures that we deal with in the table below. Each one gives rise to a Hopf monoid in species. The Hopf monoids are interrelated by means of morphisms; the table is loosely organized and does not reflect the various connections. The table shows the corresponding class of generalized permutahedra in each case. It is worth mentioning at this point that we deal with possibly unbounded generalized permutahedra. These polyhedra include certain cones associated to posets. The remaining classes of generalized permutahedra in the table are bounded polytopes.

combinatorial structure	polyhedral model	Hopf monoid
(partitions into disjoint) sets	(products of) permutahedra	$\mathbf{\Pi} \cong \overline{\mathbf{\Pi}}$
(partitions into disjoint) paths	(products of) associahedra	$\mathbf{F} \cong \overline{\mathbf{A}}$
graphs	graphic zonotopes	$\mathbf{G}$, $\mathbf{SG}$
hypergraphs	hypergraphic polytopes	$\mathbf{HG}$
simplicial complexes	simplicial complex polytopes	$\mathbf{SC}$
matroids	matroid polytopes	$\mathbf{M}$
graphic matroids	graphic matroid polytopes	$\mathbf{\Gamma}$
building sets	nestohedra	$\mathbf{BS}$
simple graphs	graph associahedra	$\mathbf{W}$
submodular function	generalized permutahedra	$\mathbf{SF} \cong \mathbf{GP}$, $\overline{\mathbf{GP}}$
extended submodular function	extended generalized permutahedra	$\mathbf{SF}^+ \cong \mathbf{GP}^+$, $\overline{\mathbf{GP}^+}$
partial orders (posets)	top-cones in the braid arrangement	$\mathbf{P}$
preorders (preposets)	cones in the braid arrangement	$\mathbf{Q} \cong \mathbf{SF}_{0,\infty} \cong \overline{\mathbf{GP}}_{\mathrm{cone}}$

Some of these Hopf monoids are defined for the first time here, others come from [**2**]. In many cases, the constructions have roots in earlier literature going back to

Joni and Rota [**60**], Schmitt [**84**–**86**], and others. The paper by Haiman and Schmitt [**53**] stands out as the first one to derive an antipode formula of combinatorial significance. Another important landmark is the paper of Billera, Jia, and Reiner [**17**] where Hopf algebraic and polyhedral considerations on matroids were analyzed together for the first time.

These earlier sources deal with a Hopf algebra which we now regard as derived from the Hopf monoid by means of the constructions of [**2**, Chapter 15]. (See Section 1.1.10 for more on this point.) For example, the Hopf monoid of paths gives rise to the *Faà di Bruno* Hopf algebra, an object introduced in [**60**] and with roots in classical work on the composition of two power series. On a few additional occasions, these associated Hopf algebras have been considered in the recent literature, without consideration of Hopf monoids and independently of our work. These include the cases of building sets [**49**], simplicial complexes [**14**], and polymatroids [**32**].

Our results on the antipode encompass new and existing results in a unified manner. The original result of Haiman and Schmitt is on the antipode of the Faà di Bruno Hopf algebra [**53**, Theorem 4]. More recent results are by Humpert and Martin [**58**, Theorem 3.1] (on the antipode for graphs), by Benedetti, Hallam, and Machacek [**14**, Theorem 4] (on the antipode for simplicial complexes), and by Bucher, Eppolito, Jun, and Matherne [**24**, **25**] (on the antipode for matroids).

Application B. Character theory and reciprocity theorems

Consider the following combinatorial invariants: Whitney's *chromatic polynomial* χ_g of a graph g, Stanley's *strict order polynomial* χ_p of a poset p, and the *Billera–Jia–Reiner polynomial* χ_m of a matroid m. These polynomials are determined by the following properties which hold for $n \in \mathbb{N}$:

- $\chi_g(n)$ = number of proper vertex n-colorings of g.
- $\chi_p(n)$ = number of strictly order preserving n-labelings of p.
- $\chi_m(n)$ = number of n-weightings of m under which m has a unique maximum basis.

Billera, Jia, and Reiner were the first to consider the matroid assignment $m \mapsto \chi_m$, and to understand it as a Hopf morphism. They compared it with the graph assignment $g \mapsto \chi_g(n)$, which also arises Hopf algebraically, writing:

"As far as we know, this [Hopf] morphism is of a different nature." [1]

We show that in fact these two morphisms – as well as the poset morphism $p \mapsto \chi_p(n)$ – are of exactly the same nature. To see this, one needs to view them inside the Hopf monoid of extended generalized permutahedra; something that their work helped us foresee.

One may wonder about a combinatorial description for the quantities obtained by plugging in negative integer values into these polynomials. The answer is provided by the following *combinatorial reciprocity theorems*. For $n \in \mathbb{N}$:

- $|\chi_g(-n)|$ = number of compatible pairs of an n-coloring and an acyclic orientation of g.

[1] They wrote this about the assignment of a quasisymmetric function to a matroid m and a graph g, but there is no essential difference with the assignment of a polynomial. As they explain in [**17**], a character on a Hopf algebra or monoid gives rise to a polynomial and a quasisymmetric function. For graphs and matroids, this gives rise to the polynomials we discuss and the quasisymmetric functions they discuss.

- $|\chi_p(-n)|$ = number of weakly order preserving n-labelings of p.
- $|\chi_m(-n)|$ = number of pairs of an n-weighting w of m and a w-maximum basis.

The first two are due to Stanley [**90**, Theorem 1.2], [**88**, Theorem 3] and the third to Billera, Jia, and Reiner [**17**, Theorem 6.3]. In Chapter 4, we cast these results in a unified setting, showing that they are all instances of the same general fact that holds for extended generalized permutahedra. We explain this general fact employing the notion of a *character* on a Hopf monoid.

The choice of a character on a Hopf monoid gives rise to a polynomial $\chi_x(n)$ for each element x of the monoid. This is an invariant of the structure x, in the sense that isomorphic structures yield the same polynomial. Furthermore, this polynomial satisfies a reciprocity rule as follows: up to a sign, $\chi_x(-n)$ equals $\chi_{\mathrm{s}(x)}(n)$. This is Proposition 4.1.5. It relates values of the invariant on negative integers to values on positive integers, with the antipode bridging between the two. A combinatorial understanding of the antipode may thus be exploited to answer the question at hand.

As before, we first construct and analyze the invariant at the level of generalized permutahedra. The starting point is a character which sends points to 1 and all other generalized permutahedra to 0. We then specialize by employing the morphisms from **G**, **M**, and **P** to **GP**. This gives the three combinatorial reciprocity theorems above.

In Chapter 4 we employ the heavier but more precise notation $\chi_I(x)(n)$, where I is the ground set, $x \in \mathbf{H}[I]$ is the given combinatorial structure, and n is the polynomial variable.

Application C. Inversion of formal power series

Figure 2 shows the first few (standard) *permutahedra* π_n and (Loday) *associahedra* $\mathfrak{a}_n$. Both π_1 and $\mathfrak{a}_1$ are points, π_2 and $\mathfrak{a}_2$ are segments. While π_3 is a hexagon, $\mathfrak{a}_3$ is a pentagon. Next come π_4, a truncated octahedron, and $\mathfrak{a}_4$, the three-dimensional associahedron. There is one permutahedron in each dimension, and every face of a permutahedron is a product of permutahedra. There is one associahedron in each dimension, and every face of an associahedron is a product of associahedra.

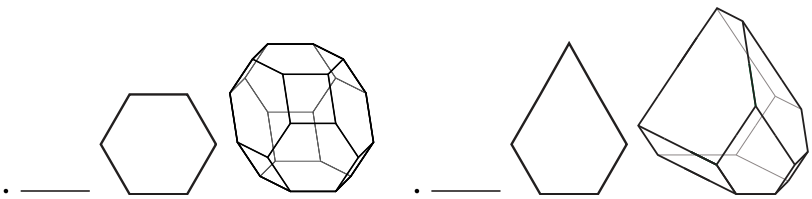


FIGURE 2. The permutahedra $\pi_1, \pi_2, \pi_3, \pi_4$ (left) and associahedra $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4$ (right).

Multiplicative Inversion. Consider formal power series

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!} \quad \text{and} \quad B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!} \quad \text{such that} \quad A(x)B(x) = 1, \tag{1}$$

assuming for simplicity $a_0 = 1$. The first few coefficients of $B(x) = 1/A(x)$ are:

$$\begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 + 2a_1^2 \\ b_3 &= -a_3 + 6a_2a_1 - 6a_1^3 \\ b_4 &= -a_4 + 8a_3a_1 + 6a_2^2 - 36a_2a_1^2 + 24a_1^4 \end{aligned}$$

What do these numbers count? The face structure of permutahedra tells the full story. For example, the formula for b_4 accounts for the faces of the permutahedron π_4: 1 truncated octahedron π_4, 8 hexagons $\pi_3 \times \pi_1$, 6 squares $\pi_2 \times \pi_2$, 36 segments $\pi_2 \times \pi_1 \times \pi_1$, and 24 points $\pi_1 \times \pi_1 \times \pi_1 \times \pi_1$. The signs in the formula are determined by the parity of the face dimensions.

Compositional Inversion. The problem of inverting power series with respect to composition is classical and falls under the heading of *Lagrange inversion*. There exists a variety of approaches to the subject and vast work on variants and generalizations. See [**92**, Chapter 5] for an introduction, and [**46**] for a recent survey.

Consider formal power series

$$(2) \quad C(x) = \sum_{n \geq 1} c_{n-1}x^n \quad \text{and} \quad D(x) = \sum_{n \geq 1} d_{n-1}x^n \text{ such that } \quad C(D(x)) = x,$$

assuming for simplicity $c_0 = 1$. The first few coefficients of $D(x) = C(x)^{\langle -1 \rangle}$ are:

$$\begin{aligned} d_1 &= -c_1 \\ d_2 &= -c_2 + 2c_1^2 \\ d_3 &= -c_3 + 5c_2c_1 - 5c_1^3 \\ d_4 &= -c_4 + 6c_3c_1 + 3c_2^2 - 21c_2c_1^2 + 14c_1^4 \end{aligned}$$

Now it is the face structure of associahedra that tells us what these numbers count. For example, the formula for d_4 accounts for the faces of the associahedron $\mathfrak{a}_4$: 1 three-dimensional associahedron $\mathfrak{a}_4$, 6 pentagons $\mathfrak{a}_3 \times \mathfrak{a}_1$ and 3 squares $\mathfrak{a}_2 \times \mathfrak{a}_2$, 21 segments $\mathfrak{a}_2 \times \mathfrak{a}_1 \times \mathfrak{a}_1$, and 14 points $\mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1 \times \mathfrak{a}_1$. The signs are again determined by the parity of the face dimensions. This description is a form of *combinatorial Lagrange inversion*.

Combinatorial formulas for the coefficients of b_n and d_n above and combinatorial formulas for the face enumeration of permutahedra and associahedra have been known for a long time; and these formulas do coincide. However, our treatment seems to be the first to truly explain the geometric connection. We derive these inversion formulas in a unified fashion, exploiting the fact that both permutahedra and associahedra are particular generalized permutahedra. In the case of compositional inversion and associahedra, this answers a 2005 question of Loday [**66**].

Our approach is again Hopf algebraic. The set of characters on a Hopf monoid is endowed with a group structure. The product is convolution and the inversion is precomposition with the antipode. For the Hopf monoid $\mathbf{\Pi}$, characters may be identified with power series $A(x)$ as in (1), with convolution corresponding to multiplication. It follows that to understand the coefficients of $B(x)$, it suffices to understand the antipode of $\mathbf{\Pi}$. The Hopf monoid $\mathbf{F}$ and its antipode may be similarly employed to deal with compositional inversion. We carry this work out in Sections 2.2, 2.4, 5.6 and 5.7.

Outline

The material is organized into five chapters. Chapter 1 sets the foundations and must be read first. The remaining chapters, while interconnected in various ways, may be approached independently of each other. Section 4.3 depends on Chapter 3, Sections 5.6 and 5.7 depend on Chapter 2.

Chapter 1: The Hopf monoid GP and its antipode. This chapter contains the essential background on Hopf monoids in species (Section 1.1), introduces a number of examples (Section 1.2) and goes on to discuss the central object in this work, the Hopf monoid of generalized permutahedra (Sections 1.3–1.5). *Normal equivalence* is a relation among polytopes. The quotient $\overline{\mathbf{GP}}$ of the Hopf monoid $\mathbf{GP}$ under this relation is defined in Section 1.4.3. In Section 1.6 we prove Theorem 1.6.1: this main result establishes that the antipode of $\mathbf{GP}$ maps a polytope to the alternating sum of its faces. The Hopf monoids defined in Section 1.2 are those of graphs $\mathbf{G}$, matroids $\mathbf{M}$, posets $\mathbf{P}$, set partitions $\mathbf{\Pi}$, and partitions into paths $\mathbf{F}$. The chapter may serve as a quick introduction to Hopf monoids in species and to illustrate their ubiquity in combinatorics.

Chapter 2: Permutahedra, associahedra, and inversion. This chapter connects two particular Hopf monoids to operations on power series. This necessitates one more ingredient from the general theory of Hopf monoids: the notion of character and the group structure on the set of characters on a Hopf monoid. This is discussed in the opening Section 2.1. The connection to power series is covered in Sections 2.2 and 2.3. Restricting generalized permutahedra to products of standard permutahedra yields a Hopf submonoid $\overline{\mathbf{\Pi}}$ of $\overline{\mathbf{GP}}$. The group of characters on $\overline{\mathbf{\Pi}}$ is isomorphic to the group of invertible power series under multiplication (normalized by $a_0 = 1$). Interestingly, a parallel story unfolds replacing standard permutahedra by associahedra. This results in a Hopf submonoid $\overline{\mathbf{A}}$ and a group of characters isomorphic to the group of invertible power series under composition (normalized by $c_0 = 1$). Section 2.4 then uses these results and the antipode of $\mathbf{GP}$ to derive Application C and obtain a unified explanation for the formulas computing the inverse of a power series with respect to either multiplication or composition. This material is complemented later in Sections 5.6 and 5.7, where it is shown that $\overline{\mathbf{\Pi}}$ is isomorphic to $\mathbf{\Pi}$, and $\overline{\mathbf{A}}$ to $\mathbf{F}$.

Chapter 3: Submodular functions arising from combinatorial structures. This chapter centers around the notion of submodular function. Each generalized permutahedron in $\mathbb{R}^I$ is determined by a unique submodular function on the Boolean poset 2^I with values in $\mathbb{R}$. Generalized permutahedra and submodular functions thus constitute cryptomorphic notions. We review this fact in Section 3.1. Several combinatorial structures give rise to submodular functions (and hence to generalized permutahedra). The notion of *diminishing returns* offers a useful alternative characterization for these functions. Submodular functions associated to graphs, matroids, and posets are discussed in Sections 3.2, 3.3, and 3.4. These are the cut function of a graph, the rank function of a matroid, and the order ideal indicator function of a poset. To cover the latter case, we consider *extended submodular functions*, which take values in $\mathbb{R} \cup \{\infty\}$. They correspond to certain unbounded polyhedra which we call *extended generalized permutahedra*. They give rise to the Hopf monoid $\mathbf{GP}^+$.

In this manner these combinatorial structures are modeled by particular classes of (extended) generalized permutahedra: graphic zonotopes, matroid polytopes, and poset cones. From our perspective, this allows us to view **G** and **M** as Hopf submonoids of **GP** (and **P** as a Hopf submonoid of $\mathbf{GP}^+$) and then to obtain antipode formulas for each of these Hopf monoids as corollaries to Theorem 1.6.1. This accomplishes Application A. In the case of graphs, a closely related result was obtained independently by Humpert and Martin [**58**, Theorem 3.1]. These authors obtained the corresponding result for the Hopf algebra associated to **G**. In Section 3.2 we also answer some questions from [**58**, Section 5] on characters of complete graphs. In the case of matroids, a formula for the antipode of the associated Hopf algebra was obtained by Bucher, Eppolito, Jun, and Matherne in [**24**, Theorem 4.7].

Chapter 4: Characters, polynomial invariants, and reciprocity. This chapters turns to Application B. The opening Section 4.1 discusses the construction of polynomial invariants of combinatorial structures out of the choice of a character on the corresponding Hopf monoid. We derive general properties of these invariants in Propositions 4.1.1–4.1.3, and obtain a general reciprocity theorem in Proposition 4.1.5. Section 4.2 carries out this construction for a particular character of **GP**. In Section 4.3 we derive the reciprocity theorems of Stanley on graphs and posets and of Billera-Jia-Reiner on matroids as consequences.

Chapter 5: Hypergraphs, building sets, and related combinatorial structures. The final chapter focuses on a particular family of generalized permutahedra, the *hypergraphic polytopes*. In Section 5.1, we provide a characterization of these objects answering a question of Rota. These polytopes give rise to a Hopf submonoid **HGP** of **GP**. Section 5.2 introduces a Hopf monoid **HG** of hypergraphs. It is isomorphic to **HGP** and contains **G**. We study other interesting Hopf submonoids of **HG**. One of them is the Hopf monoid **SC** of simplicial complexes. We employ once again Theorem 1.6.1 to derive an antipode formula for **SC**. This offers a geometric explanation for the antipode formula for the associated Hopf algebra, which was obtained earlier by Benedetti, Hallam, and Machacek [**14**]. Another family of combinatorial objects (*building sets*) and their associated polytopes (*nestohedra*) are considered in Section 5.4, together with the Hopf monoids they give rise to. A third Hopf monoid of graphs, with operations of *ripping* and *sewing*, is introduced in Section 5.5. It is denoted **W** and it maps to the Hopf monoid of building sets. Sections 5.6 and 5.7 discuss how **W** contains both the Hopf monoid **Π** of set partitions, and the Hopf monoid **F** of paths, giving rise to some interesting enumerative consequences.

Conventions. We work over a field $\Bbbk$ of characteristic 0. We use H for Hopf monoids in set species, **H** for Hopf monoids in vector species, and H for Hopf algebras.

Future directions

The Hopf monoid structure on generalized permutahedra has an interesting connection with McMullen's polytope algebra [**70**]. This structure descends to the quotient Hopf monoid $\mathbb{I}(\mathbf{GP}) = \mathbf{GP}/\mathbf{ie}$, where $\mathbf{ie}$ is generated by the *inclusion-exclusion relations* $P = \sum_{Q \in \mathcal{P}} (-1)^{\dim P - \dim Q} Q$ for any polyhedral subdivision $\mathcal{P}$ of a generalized permutahedron P into generalized permutahedra. In particular, the antipode now takes the elegant form $\mathrm{s}_I(\mathfrak{p}) = (-1)^{|I| - \dim \mathfrak{p}} \mathfrak{p}^\circ$, where $\mathfrak{p}^\circ$ denotes the relative interior of $\mathfrak{p}$. This was shown by Ardila and Sanchez in [**12**, **82**], and by Bastidas in [**13**]. Ardila and Sanchez further show that this Hopf theoretic framework offers a simple, unified explanation for many new and old valuative invariants on matroids, graphs, and posets that have recently arisen in combinatorics and algebraic geometry. They also prove that $\mathbb{I}(\mathbf{GP}^+)$ satisfies a natural universality property, which partially explains the ubiquity of generalized permutahedra and valuative invariants in the theory of Hopf monoids. Bastidas further shows that this polytope algebra is a module over the Tits algebra of the braid arrangement, and describes its composition factors combinatorially in terms of permutation statistics. He also obtains analogous results for the signed braid arrangement and signed permutations.

Another main direction in which this work may be continued consists in extending its constructions to the setting of *deformations of Coxeter permutahedra*. The latter are polytopal deformations of the Coxeter permutahedron π_W corresponding to a finite Coxeter group W. The polyhedral foundations of this theory have been laid out by Ardila, Castllo, Eur, and Postnikov [**8**]. They include connections to Fomin-Zelevinsky and Hohlweg-Lange-Thomas's *Coxeter associahedra* [**40**, **55**], Fujishige's *bisubmodular functions* [**41**], Gelfand and Serganova's *Coxeter matroids* [**21**, **22**], Reiner's *signed posets* [**78**], Stembridge's *Coxeter root cones* [**96**], Zaslavsky's *signed graphs* [**102**], and the *weight polytopes* describing the representations of semisimple Lie algebras [**42**]. An extension of the theory of Hopf monoids that admits a real hyperplane arrangement as an input has been developed in recent years by Aguiar and Mahajan [**4**, **5**]. A theory that incorporates aspects specific to Coxeter arrangements is being developed by those authors and by Rodríguez [**79**].

We list other aspects that appear worth pursuing.

- There is a morphism of Hopf monoids $\mathbf{GP} \to \mathbf{P}$ which maps a generalized permutahedron to the sum over its vertices of the normal cones at those vertices. There is a similar map that sums normal cones over all faces. These and related maps deserve study.
- The primitive part of the Hopf monoid $\mathbf{G}$ is described by Aguiar and Mahajan in [**3**, Section 9.4]. The primitive part of several other Hopf submonoids of $\mathbf{GP}$ are given by Sanchez in [**81**, **82**].

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We do not thank the thieves in Portland, OR who set the project back in 2013 by stealing a backpack with a folder containing five years of work: results, proofs, write-ups, pictures, examples, and counterexamples. We do thank them for not publishing our work in their name.

The main constructions and most of the results in this monograph were obtained in 2008; they were announced in [**3**, Section 9.6] and in numerous talks since January 2009. In the meantime, some of the results presented here were discovered independently by other authors. In particular, some of the results in Sections 1.4, 3.2, 4.3.4, 5.3, and 5.4.1 appear in [**14**, **23**, **32**, **48**, **58**], respectively. We thank the authors of these papers for their patience while we published our work, and for their open communication with us, which has improved our understanding of the material.

MA: I thank Swapneel Mahajan, my long-time close collaborator. This work is an outgrowth of our joint monographs on the subject [**2**, **3**]. His ideas show throughout.

FA: Gian-Carlo Rota assigned a homework problem that led me to Proposition 5.1.4 when I was a first-year graduate student in 1998, and he generously encouraged me to publish it. I said I would, but I did not see the significance of this result at the time and did not publish it then. Over a decade later, I was happily surprised to see the same result reappear in this context, which connects to ideas of his in multiple ways. This project has been a reminder that I still have much to learn from the brief but influential lessons I received from Rota. Also, I am glad to finally keep my word.

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CHAPTER 1

The Hopf monoid of generalized permutahedra

1.1. A brief guide to Hopf monoids in species

The theory of Hopf monoids in species developed in [**2**] constitutes a useful algebraic tool to study many families of combinatorial objects of interest. The structure captures procedures for *merging* two disjoint objects into one, and for *breaking* an object into two disjoint parts. When certain simple axioms are satisfied, these procedures define the product and coproduct in a Hopf monoid. One can then use the general theory to obtain numerous combinatorial consequences. In Sections 1.1, 2.1, and 4.1 we outline the most relevant combinatorial features of this theory. Our exposition is self-contained; the interested reader may find more details on some of these constructions in [**2**].

1.1.1. Set species. We begin by reviewing Joyal's notion of set species [**15**, **61**]. This is a framework, rooted in category theory, used to systematically study combinatorial families and the relationships between them.

DEFINITION 1.1.1. A *set species* P consists of the following data.

- For each finite set I, a set $\mathrm{P}[I]$.
- For each bijection $\sigma : I \to J$, a map $\mathrm{P}[\sigma] : \mathrm{P}[I] \to \mathrm{P}[J]$. These should be such that $\mathrm{P}[\sigma \circ \tau] = \mathrm{P}[\sigma] \circ \mathrm{P}[\tau]$ and $\mathrm{P}[\mathrm{id}] = \mathrm{id}$.

It follows that each map $\mathrm{P}[\sigma]$ is invertible, with inverse $\mathrm{P}[\sigma^{-1}]$. Sometimes we refer to an element $x \in \mathrm{P}[I]$ as a *structure* (of species P) on the set I.

Let $[n] := \{1, \ldots, n\}$. It also follows that, for each for $n \in \mathbb{N}$, the symmetric group S_n acts on the set $\mathrm{P}[n] = \mathrm{P}[\{1, \ldots, n\}]$. The action of $\sigma \in S_n$ is the map $\mathrm{P}[\sigma] : \mathrm{P}[n] \to \mathrm{P}[n]$.

In the examples that interest us, $\mathrm{P}[I]$ is the set of all combinatorial structures of a certain kind that can be constructed on the ground set I. For each bijection $\sigma : I \to J$, the map $\mathrm{P}[\sigma]$ takes each structure on I and relabels its ground set to J according to σ.

EXAMPLE 1.1.2. Define a set species L as follows. For any finite set I, $\mathrm{L}[I]$ is the set of all linear orders on I. If ℓ is a linear order on I and $\sigma : I \to J$ is a bijection, then $\mathrm{L}[\sigma](\ell)$ is the linear order on J for which $j_1 < j_2$ if $\sigma^{-1}(j_1) < \sigma^{-1}(j_2)$ in ℓ. If we regard ℓ as a list of the elements of I, then $\mathrm{L}[\sigma](\ell)$ is the list obtained by replacing each $i \in I$ for $\sigma(i) \in J$.

For instance, $\mathrm{L}[\{a, b, c\}] = \{abc, bac, acb, bca, cab, cba\}$ and if $\sigma : \{a, b, c\} \to \{1, 2, 3\}$ is given by $\sigma(a) = 1, \sigma(b) = 2, \sigma(c) = 3$, then $\mathrm{L}[\sigma] : \mathrm{L}[\{a, b, c\}] \to \mathrm{L}[\{1, 2, 3\}]$ is given by $\sigma(abc) = 123, \sigma(acb) = 132, \sigma(bac) = 213, \sigma(bca) = 231, \sigma(cab) = 312, \sigma(cba) = 321$.

DEFINITION 1.1.3. A *morphism* $f : \mathrm{P} \to \mathrm{Q}$ between set species P and Q is a collection of maps $f_I : \mathrm{P}[I] \to \mathrm{Q}[I]$ which satisfy the following *naturality* axiom: for each bijection $\sigma : I \to J$, $f_J \circ \mathrm{P}[\sigma] = \mathrm{Q}[\sigma] \circ f_I$.

EXAMPLE 1.1.4. An automorphism of the set species L of linear orders is given by the reversal maps $\mathrm{rev}_I : \mathrm{L}[I] \to \mathrm{L}[I]$ defined by $\mathrm{rev}_I(a_1 a_2 \dots a_i) = a_i \dots a_2 a_1$ for each linear order on I written as a list $a_1 a_2 \dots a_i$.

1.1.2. Hopf monoids in set species. A set species P is *connected* if the set $\mathrm{P}[\emptyset]$ is a singleton. We make the assumption that all species are connected throughout this work; occasionally we do this explicitly. This will let us state the Hopf monoid axioms more briefly. In particular, the existence of an antipode will not be required in the definition, as this is guaranteed by connectedness. The antipode of a Hopf monoid is discussed later in Section 1.1.8.

A *decomposition* of a finite set I is a finite sequence $(S_1, \dots, S_k)$ of pairwise disjoint subsets of I whose union is I. In this situation, we write

$$I = S_1 \sqcup \cdots \sqcup S_k.$$

Note that $I = S \sqcup T$ and $I = T \sqcup S$ are distinct decompositions of I (unless $I = S = T = \emptyset$).

DEFINITION 1.1.5. A *connected Hopf monoid in set species* consists of the following data.

- A connected set species H.
- For each finite set I and each decomposition $I = S \sqcup T$, product and coproduct maps

$$\mathrm{H}[S] \times \mathrm{H}[T] \xrightarrow{\mu_{S,T}} \mathrm{H}[I] \quad \text{and} \quad \mathrm{H}[I] \xrightarrow{\Delta_{S,T}} \mathrm{H}[S] \times \mathrm{H}[T]$$

satisfying the *naturality, unitality, associativity,* and *compatibility* axioms below.

Before stating the axioms of a Hopf monoid, we discuss some terminology and notation. The collection of maps μ (resp. Δ) is called the *product* (resp. the *coproduct*) of the Hopf monoid H. Fix a decomposition $I = S \sqcup T$. For $x \in \mathrm{H}[S]$, $y \in \mathrm{H}[T]$, and $z \in \mathrm{H}[I]$ we write

$$(x, y) \xrightarrow{\mu_{S,T}} x \cdot y \quad \text{and} \quad z \xrightarrow{\Delta_{S,T}} (z|_S, z/_S).$$

We call $x \cdot y \in \mathrm{H}[I]$ the *product* of x and y, $z|_S \in \mathrm{H}[S]$ the *restriction of z to S* and $z/_S \in \mathrm{H}[T]$ the *contraction of S from z*. Finally, we call the element $1 \in \mathrm{H}[\emptyset]$ the *unit* of H.

The product keeps track of how we merge two disjoint structures x on S and y on T into a single structure $x \cdot y$ on I, according to a suitable combinatorial rule. The coproduct keeps track of how we break up a structure z on I into a structure $z|_S$ on S and a structure $z/_S$ on T. Section 1.2 features five important examples.

The axioms are as follows. We note that each axiom can be rephrased in terms of a commutative diagram; we invite the reader to draw those diagrams or to see [**2**, Sections 8.2–8.3] for details.

NATURALITY. For each decomposition $I = S \sqcup T$, each bijection $\sigma : I \to J$, and any choice of $x \in \mathrm{H}[S]$, $y \in \mathrm{H}[T]$, and $z \in \mathrm{H}[I]$, we have

$$\mathrm{H}[\sigma](x \cdot y) = \mathrm{H}[\sigma|_S](x) \cdot \mathrm{H}[\sigma|_T](y),$$
$$\mathrm{H}[\sigma](z)|_S = \mathrm{H}[\sigma|_S](z|_S), \qquad \mathrm{H}[\sigma](z)/_S = \mathrm{H}[\sigma|_T](z/_S).$$

This says that relabeling may be performed either before or after merging and breaking, without altering the result.

UNITALITY. For each I and $x \in \mathrm{H}[I]$, we must have

$$x \cdot 1 = x = 1 \cdot x, \qquad x|_I = x = x/_\emptyset.$$

This says that merging and breaking are trivial when the decomposition of the underlying set I is trivial; here 1 represents the unique structure (of species H) on the empty set.

ASSOCIATIVITY AND COASSOCIATIVITY. For each decomposition $I = R \sqcup S \sqcup T$, and any $x \in \mathrm{H}[R]$, $y \in \mathrm{H}[S]$, $z \in \mathrm{H}[T]$, and $w \in \mathrm{H}[I]$, we must have

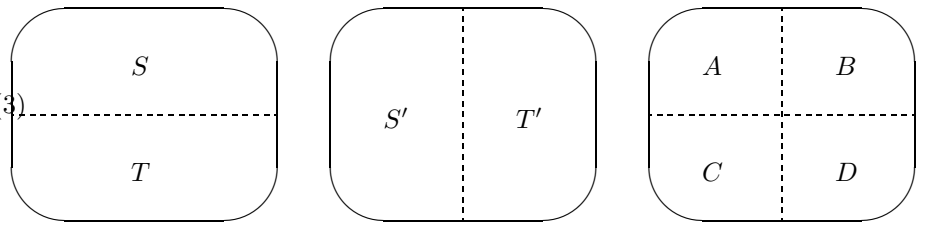
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$
$$(w|_{R\sqcup S})|_R = w|_R, \qquad (w|_{R\sqcup S})/_R = (w/_R)|_S, \qquad w/_{R\sqcup S} = (w/_R)/_S.$$

This says that successively merging three combinatorial structures on R, S, T into one structure on I produces a coherent result (associativity) – namely $\Delta_{R,S,T}(x, y, z) = x \cdot y \cdot z$ – and similarly for breaking a single structure on I into three structures on R, S, T (coassociativity) – namely $\Delta_{R,S,T}(w) = (w|_R, (w/_R)|_S, w/_{R\sqcup S})$. By induction, merging and breaking are then also well-defined for decompositions of I into more than three parts (higher (co)associativity).

COMPATIBILITY. Fix decompositions $S \sqcup T = I = S' \sqcup T'$, and consider the pairwise intersections $A := S \cap S'$, $B := S \cap T'$, $C := T \cap S'$, $D := T \cap T'$ as illustrated below. In this situation, for any $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$, we must have

$$(x \cdot y)|_{S'} = x|_A \cdot y|_C \quad \text{and} \quad (x \cdot y)/_{S'} = x/_A \cdot y/_C.$$

(3)



This says that "merging then breaking" is the same as "breaking then merging". If we start with structures x and y on S and T, we can merge them into a structure $x \cdot y$ on I, and then break the result into structures on S' and T'. We can also break x (resp. y) into two structures on A and B (resp. C and D), and merge the resulting pieces into structures on S' and T'. These two procedures should give the same answer.

This completes the definition of connected Hopf monoid in set species. In the cases that interest us, naturality and unitality are immediate and associativity is very easy; usually the only non-trivial condition to be checked is compatibility.

We remark that in more general contexts, the definition of a Hopf monoid also requires the existence of an *antipode map*, which we introduce in Section 1.1.8. In the connected case, which is the one that interests us, this map always exists; see also Remark 1.1.12.

Note also that the unitality axiom determines the maps $\mu_{S,T}$ and $\Delta_{S,T}$ uniquely when one of the subsets S or T is empty. Thus, when specifying a Hopf monoid structure, one may restrict attention to the case when both are proper and nonempty.

DEFINITION 1.1.6. A *morphism* $f : \mathrm{H} \to \mathrm{K}$ between Hopf monoids H and K is a morphism of species which preserves products, restrictions and contractions; that is, we have

$$\begin{aligned} f_J(\mathrm{H}[\sigma](x)) &= \mathrm{K}[\sigma](f_I(x)) && \text{for all bijections } \sigma : I \to J \text{ and all } x \in \mathrm{H}[I], \\ f_I(x \cdot y) &= f_S(x) \cdot f_T(y) && \text{for all } I = S \sqcup T \text{ and all } x \in \mathrm{H}[S],\ y \in \mathrm{H}[T], \\ f_S(z|_S) &= f_I(z)|_S, \quad f_T(z/_S) = f_I(z)/_S && \text{for all } I = S \sqcup T \text{ and all } z \in \mathrm{H}[I]. \end{aligned}$$

Units are preserved by connectedness.

Suppose H is a Hopf monoid. Note that if $I = S \sqcup T$ is a decomposition, then $I = T \sqcup S$ is another. Therefore, any $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$ give rise to two structures $x \cdot y$ and $y \cdot x$ on I. Similarly, any $z \in \mathrm{H}[I]$ gives rise to two pairs of structures $(z|_S, z/_S)$ and $(z/_T, z|_T)$ on (S, T).

DEFINITION 1.1.7. A Hopf monoid H is *commutative* if $x \cdot y = y \cdot x$ for any $I = S \sqcup T$, $x \in \mathrm{H}[S]$ and $y \in \mathrm{H}[T]$. It is *cocommutative* if $(z|_S, z/_S) = (z/_T, z|_T)$ for any $I = S \sqcup T$ and $z \in \mathrm{H}[I]$; it is enough to check that $z/_S = z|_T$ for any $I = S \sqcup T$ and $z \in \mathrm{H}[I]$.

EXAMPLE 1.1.8. We now define a Hopf monoid structure on the species L of linear orders of Example 1.1.2. To this end, we define the operations of *concatenation* and *restriction*. Let $I = S \sqcup T$. If $\ell_1 = s_1 \dots s_i$ is a linear order on S and $\ell_2 = t_1 \dots t_j$ is a linear order on T, their concatenation is the following linear order on I:

$$\ell_1 \cdot \ell_2 := s_1 \dots s_i \, t_1 \dots t_j.$$

Given a linear order ℓ on I, the restriction $\ell\,|_S$ is the list consisting of the elements of S written in the order in which they appear in ℓ.

The product (merging) and coproduct (breaking) of the Hopf monoid L are defined by

$$\begin{aligned} \mathrm{L}[S] \times \mathrm{L}[T] &\xrightarrow{\mu_{S,T}} \mathrm{L}[I] & \qquad \mathrm{L}[I] &\xrightarrow{\Delta_{S,T}} \mathrm{L}[S] \times \mathrm{L}[T] \\ (\ell_1, \ell_2) &\longrightarrow \ell_1 \cdot \ell_2 & \qquad \ell &\longrightarrow (\ell|_S, \ell|_T). \end{aligned}$$

Given linear orders ℓ_1 on S and ℓ_2 on T, the compatibility axiom in Definition 1.1.5 boils down to the fact that the concatenation of $\ell_1|_A$ and $\ell_2|_C$ agrees with the restriction to S' of $\ell_1 \cdot \ell_2$. The verification of the remaining axioms is similar.

By definition, $\ell/_S = \ell\,|_T$, so L is cocommutative.

Many Hopf monoids are presented in this monograph, revolving around the main example of the Hopf monoid of generalized permutahedra (Section 1.4). Additional examples are given in [**2**, Chapters 11–13].

1.1.3. Opposite and co-opposite. Given a Hopf monoid H, the *opposite* Hopf monoid H^{op} has the same coproduct as H, while the product is reversed: $\mu_{S,T}(x, y)$ in H^{op} is $\mu_{T,S}(y, x)$ in H. For example, for ℓ_1 and ℓ_2 as in Example 1.1.8, the product in L^{op} is

$$\ell_1 \cdot \ell_2 = t_1 \dots t_j \, s_1 \dots s_i.$$

The *co-opposite* Hopf monoid $\mathrm{H}^{\mathrm{cop}}$ is defined by keeping the product and reversing the coproduct: if $\Delta_{S,T}(z) = (z|_S, z/_S)$ in H, then $\Delta_{S,T}(z) = (z/_T, z|_T)$ in $\mathrm{H}^{\mathrm{cop}}$.

One easily verifies that H^{op} and $\mathrm{H}^{\mathrm{cop}}$ are Hopf monoids. H is (co)commutative if and only if $\mathrm{H} = \mathrm{H}^{\mathrm{op}}$ ($\mathrm{H} = \mathrm{H}^{\mathrm{cop}}$).

The reader may verify that the automorphism of the species L of Example 1.1.4 is an isomorphism of Hopf monoids $\mathrm{L} \to \mathrm{L}^{\mathrm{op}}$.

1.1.4. Vector species. All vector spaces and tensor products below are over a fixed field $\Bbbk$.

A *vector species* $\mathbf{P}$ consists of the following data.

- For each finite set I, a vector space $\mathbf{P}[I]$.
- For each bijection $\sigma : I \to J$, a linear map $\mathbf{P}[\sigma] : \mathbf{P}[I] \to \mathbf{P}[J]$.

These are subject to the same axioms as in Definition 1.1.1. Again, these axioms imply that every such map $\mathbf{P}[\sigma]$ is invertible. A morphism of vector species $f : \mathbf{P} \to \mathbf{Q}$ is a collection of linear maps $f_I : \mathbf{P}[I] \to \mathbf{Q}[I]$ satisfying the naturality axiom of Definition 1.1.3.

1.1.5. Hopf monoids in vector species.

Definition 1.1.9. A *connected Hopf monoid in vector species* is a vector species $\mathbf{H}$ with $\mathbf{H}[\emptyset] = \Bbbk$ that is equipped with linear maps

$$\mathbf{H}[S] \otimes \mathbf{H}[T] \xrightarrow{\mu_{S,T}} \mathbf{H}[I] \quad \text{and} \quad \mathbf{H}[I] \xrightarrow{\Delta_{S,T}} \mathbf{H}[S] \otimes \mathbf{H}[T]$$

for each decomposition $I = S \sqcup T$, subject to axioms that naturally generalize those in Definition 1.1.5. The axioms require the commutativity of certain diagrams; see [**2**, Sections 8.2–8.3] for details.

We employ similar notations as for Hopf monoids in set species; namely,

$$\mu_{S,T}(x \otimes y) = x \cdot y \quad \text{and} \quad \Delta_{S,T}(z) = \sum z|_S \otimes z/_S,$$

the latter being a variant of *Sweedler's notation* for Hopf algebras. In general, $\sum z|_S \otimes z/_S$ stands for a tensor in $\mathbf{H}[S] \otimes \mathbf{H}[T]$; individual elements $z|_S$ and $z/_S$ may not be defined.

A *morphism of Hopf monoids in vector species* is a morphism of vector species which preserves products, coproducts, and the unit, as in Definition 1.1.6.

1.1.6. Linearization. Consider the *linearization functor*

$$\mathsf{Set} \longrightarrow \mathsf{Vec},$$

which sends a set to the vector space with basis the given set. Composing a set species P with the linearization functor gives a vector species, which we denote $\mathbf{P}$. If H is a Hopf monoid in set species, then its linearization $\mathbf{H}$ is a Hopf monoid in vector species. In this situation, the coproduct of $\mathbf{H}$ is of the form $\Delta_{S,T}(z) = z|_S \otimes z/_S$, where $z \in \mathrm{H}[I]$ is a basis element of $\mathbf{H}[I]$, and the right-hand side is a pure tensor.

Most, but not all, of the Hopf monoids considered in this monograph are in set species. The linearization functor allows us to regard them as Hopf monoids in vector species also.

Remark 1.1.10. The category of vector species carries a symmetric monoidal structure. In any symmetric monoidal category one may consider the notion of *Hopf monoid*. A Hopf monoid in the category of sets under cartesian product is precisely a group. A Hopf monoid in vector species is a Hopf monoid in this categorical sense. For more details about this point of view, and a discussion of set species versus vector species, see [**2**, Chapter 8].

1.1.7. Higher products and coproducts. Let $\mathbf{H}$ be a Hopf monoid in vector species. The following is a consequence of the associativity axiom. For any decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ with $k \geq 2$, there are unique maps

$$\mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \xrightarrow{\mu_{S_1,\ldots,S_k}} \mathbf{H}[I], \qquad \mathbf{H}[I] \xrightarrow{\Delta_{S_1,\ldots,S_k}} \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \tag{4}$$

obtained by respectively iterating the product maps μ or the coproduct maps Δ in any meaningful way. As mentioned when discussing the associativity axiom in Section 1.1.2, these maps are well-defined; we refer to them as the *higher products and coproducts* of $\mathbf{H}$.

For $k = 1$, we define μ_I and Δ_I to be the identity map $\mathrm{id} : \mathbf{H}[I] \to \mathbf{H}[I]$. For $k = 0$, the only set with a decomposition into 0 parts is the empty set, and in that case we let $\mu_{(\,)} : \Bbbk \to \mathbf{H}[\emptyset]$ and $\Delta_{(\,)} : \mathbf{H}[\emptyset] \to \Bbbk$ be the linear maps that send 1 to 1.

When $\mathbf{H}$ is the linearization of a Hopf monoid H in set species, we have higher (co)products

$$\mu_{S_1,\ldots,S_k}(x_1, \ldots, x_k) = x_1 \cdot \ldots \cdot x_k \in \mathrm{H}[I], \qquad \Delta_{S_1,\ldots,S_k}(z) = (z_1, \ldots, z_k)$$

whenever $x_i \in \mathrm{H}[S_i]$ for $i = 1, \ldots, k$, and $z \in \mathrm{H}[I]$, respectively. We refer to $z_i \in \mathrm{H}[S_i]$ as the *i-th minor* of z corresponding to the decomposition $I = S_1 \sqcup \cdots \sqcup S_k$; it is obtained from z by combining restrictions and contractions in any meaningful way.

1.1.8. The antipode and the antipode problem. A *composition* of a finite set I is a decomposition $I = S_1 \sqcup \cdots \sqcup S_k$ in which each subset S_i is nonempty; we write

$$(S_1, \ldots, S_k) \vDash I.$$

There is a unique composition of the empty set. It has no parts ($k = 0$).

If $F = (S_1, \ldots, S_k)$, we write

$$\mu_F = \mu_{S_1,\ldots,S_k} \quad \text{and} \quad \Delta_F = \Delta_{S_1,\ldots,S_k}$$

for the higher (co)products (4). Each composite $\mu_F \Delta_F$ maps $\mathbf{H}[I]$ to itself.

We let $\ell(F) = k$ denote the number of parts of F.

Definition 1.1.11. Let $\mathbf{H}$ be a (connected) Hopf monoid in vector species. The *antipode* of $\mathbf{H}$ is the collection of maps

$$\mathrm{s}_I : \mathbf{H}[I] \to \mathbf{H}[I],$$

one for each finite set I, given by

$$\mathrm{s}_I = \sum_{F \vDash I} (-1)^{\ell(F)} \mu_F \Delta_F = \sum_{\substack{(S_1,\ldots,S_k) \vDash I \\ k \geq 0}} (-1)^k \mu_{S_1,\ldots,S_k} \Delta_{S_1,\ldots,S_k} \tag{5}$$

Note that $\mathrm{s}_\emptyset = \mathrm{id}$ and when I is nonempty, the sum effectively starts at $k = 1$. The right hand side of (5) involves the higher (co)products of (4). Since a composition of I can have at most $|I|$ parts, the sum is finite. We refer to (5) as *Takeuchi's formula*. For alternate formulas and axioms defining the antipode of a Hopf monoid, see [**2**, Section 8.4].

Remark 1.1.12. In the general context of Remark 1.1.10, the antipode is part of the definition of a Hopf monoid in a symmetric monoidal category. A *Hopf monoid* is a bimonoid H for which the identity map is invertible in the convolution monoid $\mathrm{Hom}(H, H)$; its inverse is called the *antipode*. For example, when a group

is regarded as a Hopf monoid (in the category of sets), the antipode is the function that sends each element of the group to its inverse. In the connected situation, Takeuchi's formula (5) automatically guarantees the existence of the antipode.

The antipode is a central part of the structure of a Hopf monoid, and the following is a fundamental problem.

ANTIPODE PROBLEM 1.1.13 ([**2**, Section 8.4.2]). *Find an explicit, cancellation-free formula for the antipode of a given Hopf monoid.*

If $\mathbf{H}$ is the linearization of a Hopf monoid in set species H, the sum in (5) takes place in the vector space $\mathbf{H}[I]$ with basis $\mathrm{H}[I]$. The antipode problem asks for an understanding of the coefficients of $\mathrm{s}_I(h)$ for each basis element h in $\mathrm{H}[I]$.

REMARK 1.1.14. The number of terms in Takeuchi's formula (5) is the *ordered Bell number* $\omega(n) \approx n!/2(\log 2)^{n+1}$; the first few terms in this sequence are $1, 1, 3, 13, 75, 541, 4683, 47293, 545835$ [**47**]. Their rapid growth makes this equation impractical, even for moderate values of n. To solve the Antipode Problem 1.1.13, one needs further insight into the Hopf monoid in question.

1.1.9. Properties of the antipode. The following properties of the antipode follow from general results for Hopf monoids in monoidal categories. The first result states that the antipode reverses products and coproducts.

PROPOSITION 1.1.15. *Let* $\mathbf{H}$ *be a Hopf monoid and* $I = S \sqcup T$ *a decomposition. Then*

$$\mathrm{s}_I(x \cdot y) = \mathrm{s}_T(y) \cdot \mathrm{s}_S(x) \quad \textit{whenever } x \in \mathbf{H}[S] \textit{ and } y \in \mathbf{H}[T], \tag{6}$$

$$\Delta_{S,T}\big(\mathrm{s}_I(z)\big) = \sum \mathrm{s}_S(z/_T) \otimes \mathrm{s}_T(z|_T) \quad \textit{whenever } z \in \mathbf{H}[I]. \tag{7}$$

PROOF. See [**2**, Proposition 1.22.(iii)]. □

More generally, let $F = (S_1, \dots, S_k)$ be a composition of I. Denote the reverse composition by $\overline{F} = (S_k, \dots, S_1)$. Let the *switch map*

$$\mathrm{sw}_F : \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \to \mathbf{H}[S_k] \otimes \cdots \otimes \mathbf{H}[S_1]$$

reverse the tensor factors; that is,

$$\mathrm{sw}_F(x_1 \otimes \cdots \otimes x_k) = x_k \otimes \cdots \otimes x_1$$

whenever $x_i \in \mathbf{H}[S_i]$ for $1 \le i \le k$. Let

$$\mathrm{s}_F = \mathrm{s}_{S_1} \otimes \cdots \otimes \mathrm{s}_{S_k} : \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k] \to \mathbf{H}[S_1] \otimes \cdots \otimes \mathbf{H}[S_k].$$

PROPOSITION 1.1.16. *Let* $\mathbf{H}$ *be a Hopf monoid. For any composition* $F = (S_1, \dots, S_k)$ *of* I*,*

$$\mathrm{s}_I\, \mu_F = \mu_{\overline{F}}\, \mathrm{sw}_F\, \mathrm{s}_F, \qquad \Delta_F\, \mathrm{s}_I = \mathrm{s}_F\, \mathrm{sw}_{\overline{F}}\, \Delta_{\overline{F}} \tag{8}$$

PROOF. For $k = 2$, this is a restatement of Proposition 1.1.15. For $k \ge 2$, this is the result of iterating Proposition 1.1.15. □

PROPOSITION 1.1.17. *Let* $\mathbf{H}$ *be a Hopf monoid that is either commutative or cocommutative. For any finite set* I*,*

$$\mathrm{s}_I^2 = \mathrm{id}. \tag{9}$$

If $\mathbf{H}$ *is commutative, then* $\mathbf{H}$ *and its co-opposite* $\mathbf{H}^{\mathrm{cop}}$ *share the same antipode.*

If $f : \mathbf{H} \to \mathbf{K}$ is a morphism of Hopf monoids, then

$$f_I \, \mathrm{s}_I = \mathrm{s}_I \, f_I. \tag{10}$$

PROOF. See [**2**, Propositions 1.16 and 1.22, Corollary 1.24]. □

REMARK 1.1.18. A more general result than Propositions 1.1.15 and 1.1.16 is given in [**5**, Lemma 12.12]. More general results than those in Proposition 1.1.17 are given in [**5**, Lemmas 12.2, 12.15, 12.17].

EXAMPLE 1.1.19. Consider the Hopf monoid $\mathbf{L}$ of linear orders in vector species. Problem 1.1.13 asks for an explicit expression for $\mathrm{s}_I(\ell)$, where ℓ is a linear order on a finite set I. Takeuchi's formula (5) yields a very large alternating sum of linear orders, but many cancellations take place. It turns out that only one term survives:

$$\mathrm{s}_I(i_1 i_2 \dots i_n) \;=\; (-1)^n \, i_n \dots i_2 i_1.$$

In other words, up to a sign, the antipode simply reverses the linear order.

Here is a simple proof. When I is a singleton, this follows readily from (5). When $|I| \geq 2$, Proposition 1.1.16 tells us that the antipode reverses products, which implies that $\mathrm{s}_I(i_1 i_2 \dots i_n) = \mathrm{s}_{\{i_n\}}(i_n) \cdot \dots \cdot \mathrm{s}_{\{i_1\}}(i_1) \;=\; (-i_n) \cdots (-i_1) = (-1)^n \, i_n \dots i_2 i_1$, as desired.

For more complicated Hopf monoids, obtaining such an explicit description for the antipode is often difficult. It requires understanding the cancellations that occur in a large alternating sum indexed by combinatorial objects; the antipode problem is therefore of a clear combinatorial nature.

Several instances of the antipode problem are solved in [**2**, Chapters 11–12]. In Section 1.6 of this monograph we offer a unified framework that solves this problem for many other Hopf monoids of interest, as outlined in the table in the introduction. We describe a few consequences of these formulas in Sections 2.4, 4.1, 4.3, and 5.7.

1.1.10. From Hopf monoids to Hopf algebras. Our results on Hopf monoids have counterparts for Hopf algebras, thanks to the *Fock functor*[1] $\overline{\mathcal{K}}$ that takes Hopf monoids in species to graded Hopf algebras. We only employ Hopf algebras briefly in this monograph, but we provide this brief discussion for the benefit of the interested reader. See [**2**, Section 15.1.1] for further details.

First let us see how a connected Hopf monoid on set species H gives rise to a graded Hopf algebra $\overline{\mathcal{K}}(\mathrm{H})$. Let $\mathrm{H}[n] = \mathrm{H}[\{1, \dots, n\}]$ for $n \in \mathbb{N}$. Say $h_1, h_2 \in \mathrm{H}[n]$ are *isomorphic* if they are in the same S_n-orbit; that is, if there exists a bijection $\sigma : [n] \to [n]$ such that $\sigma(h_1) = h_2$. Let

$$\overline{\mathcal{K}}(\mathrm{H}) := \bigoplus_{n \geq 0} H_n \quad \text{where} \quad H_n := \mathrm{span}\{\text{isomorphism classes of elements of } \mathrm{H}[n]\}$$

The operations of the Hopf algebra $\overline{\mathcal{K}}(\mathrm{H})$ are induced from those of the Hopf monoid $\mathbf{H}$ by means of the processes of *shifting* and *standardization*:

- The product of $[h_1] \in H_{k_1}$ and $[h_2] \in H_{k_2}$ is

$$[h_1] \cdot [h_2] = [h_1 \cdot h_2^{+k_1}] \;\in\; H_{k_1+k_2},$$

where $h_2^{+k_1}$ is the image of h_2 under the order-preserving bijection from $[k_2]$ to $\{k_1 + 1, \dots, k_1 + k_2\}$.

[1] In fact this is only one of four Fock functors; see [**2**, Sections 15.2, 17].

• The coproduct of $[h] \in H_n$ is

$$\Delta([h]) = \sum_{[n]=S\sqcup T} [\mathrm{std}(h|_S)] \otimes [\mathrm{std}(h/_S)] \ \in \ \bigoplus_{k=0}^{n} H_k \otimes H_{n-k},$$

where $\mathrm{std}(h|_S)$ and $\mathrm{std}(h/_S)$ are the images of $h|_S$ and $h/_S$ under the unique order preserving bijections from S to $[|S|]$ and from T to $[|T|]$, respectively.

More generally, a connected Hopf monoid in vector species $\mathbf{H}$ gives rise to a graded Hopf algebra $\overline{\mathcal{K}}(\mathbf{H})$. The symmetric group S_n acts on $\mathbf{H}[n]$, and we define

$$\overline{\mathcal{K}}(\mathbf{H}) := \bigoplus_{n\geq 0} H_n \qquad \text{where} \qquad H_n := \mathbf{H}[n] \,/\, \mathrm{span}\{w \cdot h - h \mid w \in S_n, h \in \mathbf{H}[n]\}.$$

The graded component H_n is known as the space of S_n-*coinvariants* of $\mathbf{H}[n]$.

A morphism of species commutes with the symmetric group actions and hence descends to coinvariants. In this manner, $\overline{\mathcal{K}}$ acts on morphisms.

THEOREM 1.1.20 ([**2**, Proposition 3.50, Theorem 15.12]). *If* $\mathbf{H}$ *is a Hopf monoid in species, then* $\overline{\mathcal{K}}(\mathbf{H})$ *is a graded Hopf algebra. Furthermore, if* s *is the antipode of* $\mathbf{H}$*, then* $\overline{\mathcal{K}}(\mathrm{s})$ *is the antipode of* $\overline{\mathcal{K}}(\mathrm{H})$*.*

EXAMPLE 1.1.21. We invite the reader to verify that $\overline{\mathcal{K}}(\mathbf{L}) = \Bbbk[x]$, the polynomial Hopf algebra on one generator, with x^n corresponding to the unique S_n-orbit on $\mathrm{L}[n]$. We have

$$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i} \quad \text{and} \quad \mathrm{s}(x^n) = (-1)^n x^n.$$

Figure 1 shows antipode formulas for the Hopf algebras associated to some of the Hopf monoids treated later in this work. They are to be compared with the formulas in Figure 1, from which these formulas are derived. Isomorphic structures in $\mathbf{H}$ represent the same basis element in $\overline{\mathcal{K}}(\mathbf{H})$, so their coefficients in Figure 1 combine into one coefficient in Figure 1. According to Theorem 1.6.1, the signs appearing in the former figure only depend on the dimension of the polytope modeling the combinatorial structure. Hence, isomorphic structures occur with the same sign, and cancellations do not take place in this passage. For other Hopf monoids not related to generalized permutahedra, cancellations may occur.

1.2. G, M, P, Π, F: Graphs, matroids, posets, set partitions, partitions into paths

In this section, we illustrate the previous definitions with five examples of Hopf monoids built from combinatorial structures. Some of these and many others appear in [**2**, Chapter 13]. Important ideas leading to these constructions are due to Joni and Rota [**60**], Schmitt [**84**], and many others; additional references are given below.

1.2.1. G**: Graphs.** A *graph with vertex set* I consists of a multiset of edges. Each edge is a subset of I of cardinality 1 or 2; in the former case we call it a *half-edge*.

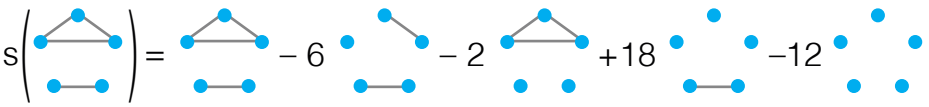
Let $\mathrm{G}[I]$ denote the set of all graphs with vertex set I. One may use a bijection $\sigma : I \to J$ to relabel the vertices of a graph $g \in \mathrm{G}[I]$ and turn it into a graph $\mathrm{G}[\sigma](g) \in \mathrm{G}[J]$. Thus, G is a species, which we now turn into a Hopf monoid. Fix I and a decomposition $I = S \sqcup T$.

Graphs $\overline{\mathcal{K}}(\mathbf{G})$:

Matroids $\overline{\mathcal{K}}(\mathbf{M})$:

Posets $\overline{\mathcal{K}}(\mathbf{P})$:

Partitions $\overline{\mathcal{K}}(\mathbf{\Pi})$:



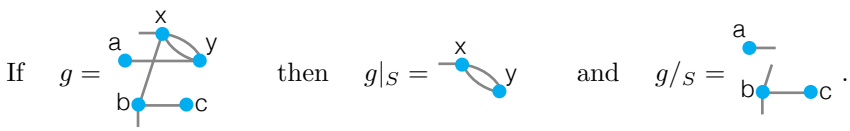
Paths $\overline{\mathcal{K}}(\mathbf{F})$:

FIGURE 1. Antipode calculations in Hopf algebras

• The product of two graphs $g_1 \in \mathrm{G}[S]$ and $g_2 \in \mathrm{G}[T]$ is the graph $g_1 \cdot g_2$ with vertex set I and edge set the union of the edge sets of g_1 and those of g_2. Since the vertex sets of g_1 and g_2 are disjoint, so are the edge sets. Thus, an edge of $g_1 \cdot g_2$ is an edge of exactly one of g_1 or g_2.

• The coproduct of a graph $g \in \mathrm{G}[I]$ is $\Delta_{S,T}(g) = (g|_S, g/_S)$ defined as follows. We let $g|_S \in \mathrm{G}[S]$ be the graph with vertex set S consisting of the edges and half-edges of g which are incident to S only. The edges incident to T (on at least one vertex) are removed. By contrast, $g/_S \in \mathrm{G}[T]$ is the graph with vertex set T consisting of all edges and half-edges of g incident to T on at least one vertex: an edge $\{t, s\}$ in g joining $t \in T$ and $s \in S$ becomes a half-edge at t in $g/_S$.

An example follows. Let $I = \{a, b, c, x, y\}$, $S = \{x, y\}$, and $T = \{a, b, c\}$.

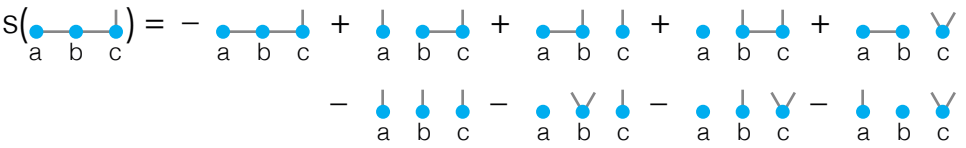


The Hopf monoid axioms are easily verified.

A simpler version of this Hopf monoid (disallowing half-edges) is discussed in [**2**, Section 13.2]; this in turn elaborates on work of Schmitt [**84**].

EXAMPLE 1.2.1. Consider the antipode s for the linearization $\mathbf{G}$ of the Hopf monoid of graphs. For a graph on 3 vertices, Takeuchi's formula (5) returns an

alternating sum of 13 graphs on the same vertex set, corresponding to the 13 compositions of a 3-element set. An explicit calculation yields



Cancellations took place which resulted in a cancellation-free and combination-free sum of only 9 graphs. The antipode problem 1.1.13 for the Hopf monoid **G** asks for an understanding of this phenomenon. This problem is solved in Section 3.2.

1.2.2. M**: Matroids.** Let I be a finite set. A *matroid on ground set* I is a nonempty collection m of subsets of I which is closed under inclusion and satisfies the following axiom: if A and B are in m and $|A| = |B| + 1$, there exists $a \in A - B$ such that $B \cup \{a\}$ is in m.

The sets in the collection are called *independent*; the remaining subsets of I are called *dependent*. The maximal independent sets are called *bases*. Matroids abstract the notion of independence, and arise naturally in many fields of mathematics. Three key examples are the following.

(1) *Linear matroids:* If I is a set of vectors linearly spanning a vector space V, the collection of subsets of I which are linearly independent is a matroid. The bases are the subsets of I which are linear bases of V.
(2) *Graphical matroids:* If I is the set of edges of a graph g, the collection of subsets of I containing no cycles is a matroid. The bases are the subsets of I which constitute a spanning forest of g.
(3) *Algebraic matroids:* If I is a set of elements which generate a field extension $\mathbb{K}$ of $\mathbb{F}$, the collection of subsets of I which are algebraically independent over $\mathbb{F}$ is a matroid. The bases are the subsets of I which constitute a transcendence basis for $\mathbb{K}$ over $\mathbb{F}$.

We review a number of basic operations on matroids; for more details on these and other notions related to matroids, we refer the reader to [**72**, **100**].

Consider a matroid m on I and a decomposition $I = S \sqcup T$. The *restriction* of m to S is the matroid on ground set S defined as

$$m|_S = \{A \subseteq S \mid A \in m\}.$$

The *contraction* of S from m is the matroid on ground set T defined as

$$m/_S = \{B \subseteq T \mid \text{there is a basis } A \text{ of } m|_S \text{ such that } A \cup B \in m\}.$$

Let m_1 and m_2 be matroids on ground set S and T, respectively, and $I = S \sqcup T$. Their *direct sum* is the matroid on ground set I defined as

$$m_1 \oplus m_2 = \{A_1 \cup A_2 \mid A_1 \in m_1, A_2 \in m_2\}.$$

Let $\mathrm{M}[I]$ be the set of matroids on ground set I. Again, M is a species, which we now turn into a Hopf monoid. Fix I, S and T as above.

- The product of $m_1 \in \mathrm{M}[S]$ and $m_2 \in \mathrm{M}[T]$ is their direct sum $m_1 \oplus m_2$.
- The coproduct of $m \in \mathrm{M}[I]$ is $\Delta_{S,T}(m) = (m|_S, m/_S)$.

The Hopf monoid axioms boil down to familiar properties relating direct sums, restriction, and contraction of matroids.

The (linearization **M** of the) Hopf monoid M is discussed in [**2**, Section 13.8]. The crucial idea of assembling these matroid operations into an algebraic structure goes back to Joni and Rota [**60**, Section XVII] and Schmitt [**85**, Section 15]. In fact the terms restriction, contraction, and minor, which we employ for an arbitrary Hopf monoid in set species, originate in this example.

EXAMPLE 1.2.2. We consider the antipode of the Hopf algebra $\overline{\mathcal{K}}(\mathbf{M})$, where isomorphic matroids are identified. Let m be the matroid on $\{a, b, c, d\}$ whose bases are ab, ac, ad, bc, and bd. Takeuchi's formula (5) expresses the antipode $\mathrm{s}(m)$ as an alternating sum of 73 matroids, but after extensive cancellation, one obtains:

where we are representing isomorphism classes of matroids by affine diagrams [**94**]; points represent elements and the following represent dependent sets: any four points, three points on a line, two points above each other, and one hollow point. In particular, hollow points represent *loops*. The antipode problem 1.1.13 for the Hopf monoid **M** asks for an understanding of this cancellation. This problem is solved in Section 3.3.

1.2.3. P**: Posets.** A *poset* p on a finite set I is a relation $p \subseteq I \times I$, denoted $\leq$, which is reflexive, antisymmetric and transitive.

Let $\mathrm{P}[I]$ denote the set of all posets on I and $\mathbf{P}[I]$ its linearization; that is, the vector space with basis $\mathrm{P}[I]$. Then P is a set species and **P** is a vector species. We turn **P** into a Hopf monoid in vector species as follows. Fix $I = S \sqcup T$.

• The product of $p_1 \in \mathbf{P}[S]$ and $p_2 \in \mathbf{P}[T]$ is the poset $p_1 \cdot p_2$ on I which as a subset of $I \times I$ is simply the (disjoint) union of the sets $p_1 \subseteq S \times S$ and $p_2 \subseteq T \times T$. In $p_1 \cdot p_2$ there are no relations between elements of S and elements of T.

• The coproduct $\Delta_{S,T} : \mathbf{P}[I] \to \mathbf{P}[S] \otimes \mathbf{P}[T]$ is the linear map determined by

$$\Delta_{S,T}(p) = \begin{cases} p|_S \otimes p|_T & \text{if } S \text{ is a lower set of } p, \\ 0 & \text{otherwise.} \end{cases}$$

We say S is a *lower set* or *order ideal* of p if no element of T is less than an element of S, and we let

$$p|_S = p \cap (S \times S)$$

be the induced poset on S.

The Hopf monoid axioms are easily verified.

The Hopf monoid **P** is commutative but not cocommutative. It is discussed in [**2**, Section 13.1]. Work of Malvenuto [**69**] and of Schmitt [**85**, Section 16] is at the root of this construction. Work of Gessel [**45**] is also relevant.

WARNING. The vector species **P** is the linearization of the set species P, so each space $\mathbf{P}[I]$ carries the canonical basis $\mathrm{P}[I]$. Note, however, that $\Delta_{S,T} : \mathbf{P}[I] \to \mathbf{P}[S] \otimes \mathbf{P}[T]$ does not always send a basis element in $\mathrm{P}[I]$ to a basis element in $\mathrm{P}[S] \times \mathrm{P}[T]$. In other words, the Hopf monoid structure on **P** is not the linearization of a Hopf monoid structure on P.

EXAMPLE 1.2.3. We consider the antipode of the Hopf algebra of posets $\overline{\mathcal{K}}(\mathbf{P})$, where isomorphic posets are identified, and represent isomorphism classes by means

of unlabeled Hasse diagrams. For the class shown below (of a poset on four elements), Takeuchi's alternating sum of 73 posets simplifies to:

The antipode problem 1.1.13 for posets is solved in Section 3.4.

1.2.4. Π: Set partitions. A *partition* π of a finite set I is a covering of I by nonempty and pairwise disjoint subsets: $\bigsqcup_{B\in\pi} B = I$. The sets $B \in \pi$ are called the *parts* or *blocks* of π.

Neither the blocks nor the elements within each block come in any specified order. To display a set partition we arrange the blocks and the elements within each block in an arbitrary order. For instance, $\{ab, cde\}$ denotes the partition $\pi = \{B, C\}$ with blocks $B = \{a, b\}$ and $C = \{c, d, e\}$.

Let $\Pi[I]$ denote the set of all set partitions on I. Then Π is a set species. We turn it into a Hopf monoid. Let $I = S \sqcup T$ be a decomposition.

• The product $\pi \cdot \rho \in \Pi[I]$ of $\pi \in \Pi[S]$ and $\rho \in \Pi[T]$ is the union of the two collections. Thus, a block of $\pi \cdot \rho$ is either a block of π or of ρ.

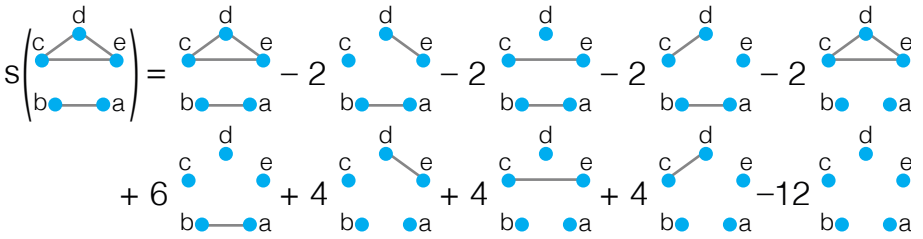
• The coproduct of $\pi \in \Pi[I]$ is $\Delta_{S,T}(\pi) = (\pi|_S, \pi|_T)$ where $\pi|_S$ is the collection of nonempty intersections $B \cap S$ for $B \in \pi$.

For example, if $I = \{a, b, c, d, e\}$, $S = \{a, b, d\}$, $T = \{c, e\}$ and

$$\pi = \{ab, cde\} \quad \text{then} \quad \pi|_S = \{ab, d\} \quad \text{and} \quad \pi/_S = \pi|_T = \{ce\}.$$

The Hopf monoid axioms are easily verified. The Hopf monoid Π is both commutative and cocommutative. The associated Hopf algebra $\overline{\mathcal{K}}(\mathbf{\Pi})$ is the classical Hopf algebra of symmetric functions. See [**2**, Sections 12.6 and 17.4] and [**3**, Section 9.3].

EXAMPLE 1.2.4. For the antipode $\mathrm{s}(\{ab, cde\})$, Takeuchi's formula (5) returns an alternating sum of 530 set partitions, which simplifies as shown below.



Here we represent a partition of I by a graph on I whose edges are the pairs of elements in the same block. This graph is a union of complete graphs, one for each block of the partition.

The antipode of $\mathbf{\Pi}$ is fully described in [**2**, Theorem 12.47]. The result is rederived here in Section 5.6. In Section 2.4 we apply this result to the calculation of the multiplicative inverse of a formal power series.

1.2.5. F**: Paths and partitions into paths.** Let I be a finite set. A *path* on I is an equivalence class of linear orders on I under reversal. For example, the linear orders abc and cba represent the same path p on $\{a, b, c\}$.

A *partition of* I *into paths* is a partition of the set I together with a path p_B on each block B of the partition. We display partitions into paths in the same manner as set partitions, but the elements within each block B are now listed by employing one of the two linear orders that represent the path p_B. For example, $\{ac, bde\}$, $\{ca, bde\}$, $\{ac, edb\}$, and $\{ca, edb\}$ all represent the same partition into paths.

Let $\mathrm{C}[I]$ denote the set of paths on I and $\mathrm{F}[I]$ denote the set of partitions of I into paths. They define set species C and F.

Let $I = S \sqcup T$ be a decomposition. Given a path $p \in \mathrm{C}[I]$, we define a path $p|_S \in \mathrm{C}[S]$ by erasing the elements of T from p and splicing the resulting pieces together into one path, so that elements of S that bound a run of consecutive elements of T in p become consecutive in $p|_S$. We also define a partition into paths $p/_S \in \mathrm{F}[T]$ by simply erasing the elements of S from p, so that the path p breaks into the partition of T whose paths are the maximal runs of elements of T in p. For example, if $I = \{a, b, c, d, e, f\}$, $S = \{b, c, f\}$, $T = \{a, d, e\}$, and

$$p = abcdef, \quad \text{then} \quad p|_S = bcf \quad \text{and} \quad p/_S = \{a, de\}.$$

Both operations extend to partitions α into paths, by applying them to each path p in α. Thus, if $\alpha \in \mathrm{F}[I]$, we obtain two partitions $\alpha|_S \in \mathrm{F}[S]$ and $\alpha/_S \in \mathrm{F}[T]$.

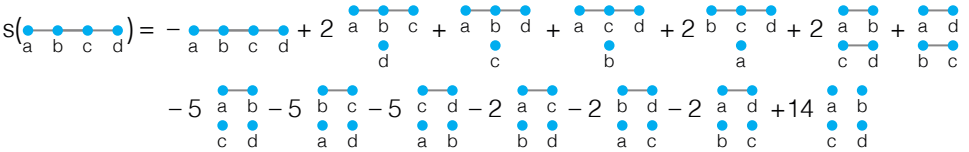
We employ these constructions to turn the species F into a Hopf monoid.

- The product $\alpha \cdot \beta \in \mathrm{F}[I]$ of $\alpha \in \mathrm{F}[S]$ and $\beta \in \mathrm{F}[T]$ is the union of the two collections of paths. Thus, a path of $\alpha \cdot \beta$ is either a path of α or of β.
- The coproduct of $\alpha \in \mathrm{F}[I]$ is $\Delta_{S,T}(\alpha) = (\alpha|_S, \alpha/_S)$ defined as above.

The Hopf monoid axioms are easily verified.

We may embed $\mathrm{C}[I]$ into $\mathrm{F}[I]$ by viewing each path as a partition into a single path. In this manner, the commutative monoid F is freely generated by the species C.

Example 1.2.5. Consider the single path $abcd \in \mathrm{F}[\{a, b, c, d\}]$. For the antipode $\mathrm{s}(abcd)$, Takeuchi's formula (5) returns an alternating sum of 73 partitions into paths which simplifies as shown below.



Here paths are represented by graphs, so that each pair of consecutive elements are joined by an edge.

In Section 5.7 we solve the antipode problem 1.1.13 for **F**. In particular, we explain why every coefficient in this formula is a Catalan number, and the number of terms is also a Catalan number. We will also discuss how F is closely related to the associahedron, the Faà di Bruno Hopf algebra, and the calculation the compositional inverse of a formal power series.

1.3. Generalized permutahedra

The permutahedron is a ubiquitous polytope. Its vertices are in bijection with the set of permutations of a finite set. We are interested in its deformations, known

as *generalized permutahedra*. This family of polytopes is special enough to welcome combinatorial analysis, and general enough to model many combinatorial families of interest. It is also precisely the family of polytopes which are amenable to the algebraic techniques of this monograph, as Section 1.5 will show.

We now recall some basic facts about permutahedra and their deformations. These results and other background on polyhedra may be found in [**36**, **41**, **77**, **87**, **98**, **103**].

1.3.1. Normal fans of polyhedra. Let V be a *Euclidean space*. Thus, V is a finite dimensional real vector space endowed with an inner product $\langle -, - \rangle$.

Let $\mathfrak{p}$ be a *polytope* (bounded polyhedron) in V and $v \in V$ a vector. We refer to the set

$$\mathfrak{p}_v = \{p \in \mathfrak{p} \mid \langle p, v \rangle \geq \langle q, v \rangle \text{ for all } q \in \mathfrak{p}\}$$

as the *v-maximum face* of $\mathfrak{p}$. The set $\mathfrak{p}_v$ is then a face of $\mathfrak{p}$, the functional $\langle -, v \rangle$ is constant on it, and greater than on the rest of $\mathfrak{p}$. In other words, the face $\mathfrak{p}_v$ is the locus of $\mathfrak{p}$ where the functional $\langle -, v \rangle$ achieves its maximum.

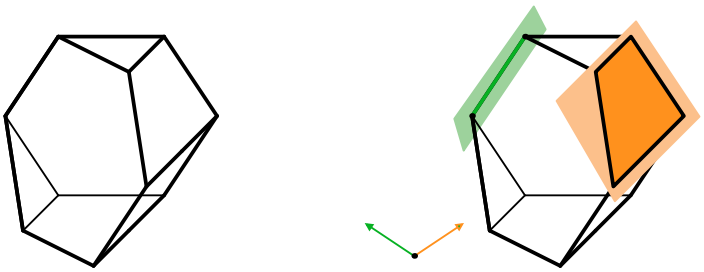


FIGURE 2. a) A generalized permutahedron $\mathfrak{p}$. b) Two directions v and w and the corresponding maximal faces $\mathfrak{p}_v$ and $\mathfrak{p}_w$.

The face $\mathfrak{p}_v$ only depends on the direction of v: dilating v by a positive scalar results on the same face. If we intersect $\mathfrak{p}$ with affine hyperplanes orthogonal to v, $\mathfrak{p}_v$ is the last nonempty intersection we encounter as we travel outward in the direction of v.

The same definition applies more generally when $\mathfrak{p}$ is a (possibly unbounded) polyhedron. In this case, the functional $\langle -, v \rangle$ may not achieve a maximum on it; equivalently, the set $\mathfrak{p}_v$ may be empty.

We define the (open and closed) *normal cones* of a face $\mathfrak{q}$ of a polyhedron $\mathfrak{p}$ by

$$\mathcal{N}_\mathfrak{p}^\circ(\mathfrak{q}) = \{v \in V \mid \mathfrak{p}_v = \mathfrak{q}\},$$
$$\mathcal{N}_\mathfrak{p}(\mathfrak{q}) = \{v \in V \mid \mathfrak{q} \text{ is a face of } \mathfrak{p}_v\},$$

respectively. They are polyhedral cones (open and closed, respectively), $\mathcal{N}_\mathfrak{p}(\mathfrak{q})$ is the closure of $\mathcal{N}_\mathfrak{p}^\circ(\mathfrak{q})$, $\dim \mathcal{N}_\mathfrak{p}(\mathfrak{q}) = \dim V - \dim \mathfrak{q}$, and $\mathfrak{q}_1$ is a face of $\mathfrak{q}_2$ if and only if $\mathcal{N}_\mathfrak{p}(\mathfrak{q}_2)$ is a face of $\mathcal{N}_\mathfrak{p}(\mathfrak{q}_1)$.

The *normal fan* $\mathcal{N}_\mathfrak{p}$ of $\mathfrak{p} \subseteq V$ consists of the normal cones $\mathcal{N}_\mathfrak{p}(\mathfrak{q})$ for all faces $\mathfrak{q}$ of $\mathfrak{p}$. Its support is the cone of directions with respect to which $\mathfrak{p}$ is bounded above. In particular, it is a convex subset of V. If $\mathfrak{p}$ is a polytope, the support is the whole of V, and the fan $\mathcal{N}_\mathfrak{p}$ is *complete*.

Two polyhedra $\mathfrak{p}$ and $\mathfrak{p}'$ in V are *normally equivalent* if they have the same normal fan:

$$\mathfrak{p} \equiv \mathfrak{p}' \iff \mathcal{N}_{\mathfrak{p}} = \mathcal{N}_{\mathfrak{p}'};$$

that is, if $\mathcal{N}_{\mathfrak{p}}$ and $\mathcal{N}_{\mathfrak{p}'}$ consist of exactly the same cones. A polyhedron is normally equivalent to any of its translates or nonzero dilations.

We say a polyhedron $\mathfrak{q}$ is a *deformation* of a polyhedron $\mathfrak{p}$ if the normal fan $\mathcal{N}_{\mathfrak{q}}$ is a coarsening of the normal fan $\mathcal{N}_{\mathfrak{p}}$; that is, every cone of $\mathcal{N}_{\mathfrak{p}}$ is a subset of a cone of $\mathcal{N}_{\mathfrak{q}}$. We say $\mathfrak{q}$ is an *extended deformation* of $\mathfrak{p}$ if the normal fan $\mathcal{N}_{\mathfrak{q}}$ is a coarsening of a convex subfan of the normal fan $\mathcal{N}_{\mathfrak{p}}$.

When $\mathfrak{p}$ is a simple polytope, it is shown in [**76**, Theorem 15.3] that we may think of the deformations of $\mathfrak{p}$ equivalently as being obtained by any of the following three procedures:

- moving the vertices of $\mathfrak{p}$ while preserving the direction of each edge, or
- changing the edge lengths of $\mathfrak{p}$ while preserving the direction of each edge, or
- moving the facets of $\mathfrak{p}$ while preserving their directions, without allowing a facet to move past a vertex.

The *Minkowski sum* of polyhedra $\mathfrak{p}$ and $\mathfrak{q}$ in V is

$$\mathfrak{p} + \mathfrak{q} = \{p + q \mid p \in \mathfrak{p},\ q \in \mathfrak{q}\} \subseteq V.$$

The normal fan $\mathcal{N}_{\mathfrak{p}+\mathfrak{q}}$ is the coarsest common refinement of the normal fans $\mathcal{N}_{\mathfrak{p}}$ and $\mathcal{N}_{\mathfrak{q}}$. Its cones are the nonempty intersections between cones in $\mathcal{N}_{\mathfrak{p}}$ and cones in $\mathcal{N}_{\mathfrak{q}}$.

A *zonotope* is a Minkowski sum $Z(A) = \sum_{a \in A} a$ of a finite set of segments A. The deformations of zonotopes can be described more simply as follows.

PROPOSITION 1.3.1. [**8**, Prop. 2.6] *Let A be a finite set of vectors in a vector space V and $Z(A)$ the corresponding zonotope. A polytope is a deformation of the zonotope $Z(A)$ if and only if every edge is parallel to some vector in A. More generally, a polyhedron P in V is an extended deformation of $Z(A)$ if and only if every face affinely spans a parallel translate of span(S) for some $S \subseteq A$.*

1.3.2. Lemmas from polyhedral geometry. For proofs of the results collected here we refer to [**103**, Chapter 7].

Let V and W be Euclidean spaces and endow $V \times W$ with the inner product

$$\langle (v, w), (v', w') \rangle = \langle v, v' \rangle + \langle w, w' \rangle.$$

Given polyhedra $\mathfrak{p} \subseteq V$ and $\mathfrak{q} \subseteq W$, let $\mathfrak{p} \times \mathfrak{q} \subseteq V \times W$ be their Cartesian product. Maximum faces of a product are products of maximum faces, as follows.

LEMMA 1.3.2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be polytopes in V and W, respectively. Let $v \in V$ and $w \in W$. Then*

$$(\mathfrak{p} \times \mathfrak{q})_{(v,w)} = \mathfrak{p}_v \times \mathfrak{q}_w. \tag{11}$$

In particular,

$$(\mathfrak{p} \times \mathfrak{q})_{(v,0)} = \mathfrak{p}_v \times \mathfrak{q} \quad \text{and} \quad (\mathfrak{p} \times \mathfrak{q})_{(0,w)} = \mathfrak{p} \times \mathfrak{q}_w. \tag{12}$$

Lemma 1.3.2 holds as well for polyhedra, assuming that v and w lie in the support of $\mathcal{N}_{\mathfrak{p}}$ and $\mathcal{N}_{\mathfrak{q}}$, respectively. This is equivalent to assuming that (v, w) lies in the support of $\mathcal{N}_{\mathfrak{p}\times\mathfrak{q}}$.

The following describes the result of computing maximum faces iteratively.

LEMMA 1.3.3. *Let $\mathfrak{p}$ be a polytope in V. Let v_1 and $v_2 \in V$. There exist $\lambda_1, \lambda_2 > 0$ such that*

$$(\mathfrak{p}_{v_1})_{v_2} = \mathfrak{p}_{\lambda_1 v_1 + \lambda_2 v_2}. \tag{13}$$

In fact, there exists $r > 0$ such that (13) holds for all $\lambda_1, \lambda_2 > 0$ with $\lambda_1/\lambda_2 > r$.

We abbreviate the above conditions on λ_1 and λ_2 by writing $\lambda_1 >> \lambda_2 > 0$.

Lemma 1.3.3 holds as well for polyhedra, assuming that v_1 and v_2 lie in the support of $\mathcal{N}_{\mathfrak{p}}$, which implies that $\lambda_1 v_1 + \lambda_2 v_2$ also does.

The following is a consequence of Lemma 1.3.2.

LEMMA 1.3.4. *We have*

$$\mathcal{N}_{\mathfrak{p}\times\mathfrak{q}} = \mathcal{N}_{\mathfrak{p}} \times \mathcal{N}_{\mathfrak{q}}. \tag{14}$$

More precisely, the faces of $\mathfrak{p} \times \mathfrak{q}$ are products of faces $\mathfrak{p}_1$ of $\mathfrak{p}$ and $\mathfrak{q}_1$ of $\mathfrak{q}$, and

$$\mathcal{N}_{\mathfrak{p}\times\mathfrak{q}}(\mathfrak{p}_1 \times \mathfrak{q}_1) = \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_1) \times \mathcal{N}_{\mathfrak{q}}(\mathfrak{q}_1),$$

so that the cones in the normal fan of the product $\mathfrak{p} \times \mathfrak{q}$ identify with pairs of cones in the normal fans of $\mathfrak{p}$ and $\mathfrak{q}$.

LEMMA 1.3.5. *Let $\mathfrak{p}, \mathfrak{p}'$ be polyhedra in V and $\mathfrak{q}, \mathfrak{q}'$ be polyhedra in W. Then*

$$\mathfrak{p} \equiv \mathfrak{p}' \quad \textit{and} \quad \mathfrak{q} \equiv \mathfrak{q}' \iff \mathfrak{p} \times \mathfrak{q} \equiv \mathfrak{p}' \times \mathfrak{q}'. \tag{15}$$

This follows from (14).

LEMMA 1.3.6. *Let $\mathfrak{p} \equiv \mathfrak{p}'$ be normally equivalent polyhedra in V. Let v be a vector in the support of $\mathcal{N}_{\mathfrak{p}} = \mathcal{N}_{\mathfrak{p}'}$. Then*

$$\mathfrak{p}_v \equiv \mathfrak{p}'_v. \tag{16}$$

1.3.3. Standard Euclidean spaces. Let I be a finite set I and $\mathbb{R}^I$ the real vector space whose vectors are I-tuples of real numbers:

$$\mathbb{R}^I = \{(a_i)_{i\in I} \mid a_i \in \mathbb{R}\}.$$

Let $\{e_i \mid i \in I\}$ be the standard basis of $\mathbb{R}^I$. For any subset S of I, let

$$e_S = \sum_{i\in S} e_i.$$

We endow $\mathbb{R}^I$ with the standard inner product

$$\langle x, y\rangle = \sum_{i\in I} x_i y_i.$$

The standard basis $\{e_i\}_{i\in I}$ is then orthonormal, and for all $x \in \mathbb{R}^I$,

$$\langle x, e_S\rangle = \sum_{i\in S} x_i.$$

Let $I = S \sqcup T$ a decomposition. Then

$$\mathbb{R}^S \times \mathbb{R}^T = \mathbb{R}^I$$

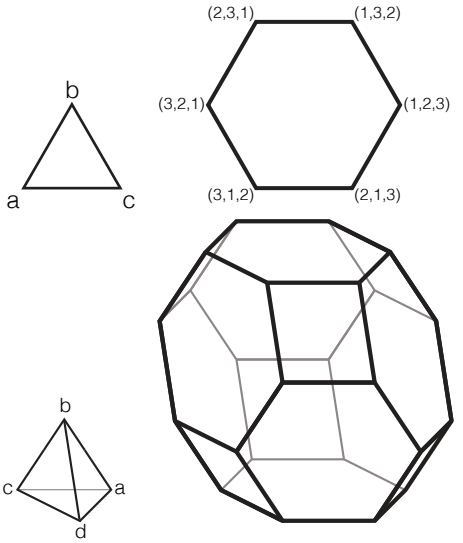
as Euclidean spaces. Indeed, the coordinates of a vector $x \in \mathbb{R}^I$ split into a pair (y, z) where the coordinates of y are indexed by S and those of z by T.

1.3.4. Standard permutahedra. Let I be a nonempty finite set and $n = |I|$. The *standard permutahedron* π_I is the convex hull of the points in $\mathbb{R}^I$ whose coordinates consist precisely of the elements $1, \ldots, n$, listed in any order:

$$\pi_I = \operatorname{conv}\{\, (a_i)_{i\in I} \mid \{a_i\}_{i\in I} = [n] \,\} \subseteq \mathbb{R}^I.$$

It is a convex polytope of dimension $n-1$. We let $\pi_n = \pi_{[n]}$ denote the standard permutahedron in $\mathbb{R}^n$.

For example, $\pi_{\{a,b,c\}}$ is a regular hexagon lying on the plane $x_a + x_b + x_c = 6$, while $\pi_{\{a,b,c,d\}}$ is a truncated octahedron on the hyperplane $x_a + x_b + x_c + x_d = 10$ in $\mathbb{R}^{\{a,b,c,d\}}$. These polytopes are shown below, in each case next to the standard simplex in $\mathbb{R}^I$.



The permutahedron π_I may also be described as the set of solutions $(x_i)_{i\in I} \in \mathbb{R}^I$ to the following system of (in)equalities:

$$\sum_{i\in I} x_i = \binom{n+1}{2}, \tag{17}$$

$$\sum_{i\in S} x_i \leq \binom{n+1}{2} - \binom{t+1}{2}, \tag{18}$$

for all compositions (S, T) of I, where $t = |T| = n - |S|$.

The facial structure of the permutahedron admits a simple description.

• (Dimension 0.) The vertices of π_I are in bijection with the linear orders on I. The vertex corresponding to the order ℓ has i coordinate equal to the position of i in the reversal of ℓ. For example, the linear order abc corresponds to the vertex in the hexagon with coordinates $x_a = 3$, $x_b = 2$, $x_c = 1$. More plainly, the vertices of π_n are the $n!$ permutations of $(1, 2, \ldots, n)$.

• (Dimension 1.) There is an edge between two vertices x and y if and only if their coordinates can be obtained from each other by swapping two consecutive values. Thus $x_i = r$ and $x_j = r+1$ become $y_i = r+1$ and $y_j = r$ for some i and j in I, r in $[n]$, while $y_k = x_k$ for $k \neq i, j$. The edge joining x and y is then a parallel translate of the vector $e_i - e_j$.

• (Dimension $n-2$.) The 2^n-2 facets of π_I are in bijection with the compositions of I into 2 parts. The facet corresponding to (T,S) is obtained by turning (18) into an equality, and keeping the remaining (in)equalities.
• (Arbitrary dimension.) The $(n-k)$-dimensional faces of π_I are in bijection with the compositions of I into k parts. For each composition $F=(S_1,\dots,S_k)\vDash I$, the corresponding face π_F has as vertices the permutations $x\in\mathbb{R}^I$ such that the coordinates $\{x_i \mid i\in S_1\}$ are the largest $|S_1|$ numbers in $[n]$, the coordinates $\{x_i \mid i\in S_2\}$ are the next largest $|S_2|$ numbers in $[n]$, and so on. This description shows that the face π_F is a parallel translate of the product of permutahedra $\pi_{S_1}\times\cdots\times\pi_{S_k}$ in $\mathbb{R}^{S_1}\times\cdots\times\mathbb{R}^{S_k}=\mathbb{R}^I$.
• (Face containment.) Given compositions $F=(S_1,\dots,S_k)$ and $G=(T_1,\dots,T_l)$, we say that F *refines* G if each T_i is a union of consecutive S_j's. It follows from the preceding discussion that π_F is contained in π_G if and only if F refines G.

There is another convenient representation of the permutahedron. If we let Δ_{ij} be the segment connecting e_i and e_j in $\mathbb{R}^I$, then we can represent the standard permutahedron as the zonotope

$$\pi_I=\sum_{i\neq j\in I}\Delta_{ij}+e_I. \tag{19}$$

Note that the summand e_I simply affords a translation by the vector $(1,\dots,1)$.

1.3.5. The braid arrangement. The *braid arrangement* $\mathcal{B}_I$ consists of the $\binom{n}{2}$ hyperplanes in $\mathbb{R}^I$ with equations

$$y_i=y_j,\qquad i,j\in I,\ i\neq j.$$

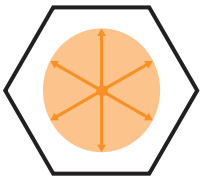
If $n=1$, the arrangement is empty.

The *faces* of the braid arrangement are in bijection with compositions of I, with $F=(S_1,\dots,S_k)$ labeling the face defined by the inequalities

$$y_i=y_j \text{ if } i,j\in S_a \quad\text{and}\quad y_i\geq y_j \text{ if } i\in S_a,\ j\in S_b,\ a<b.$$

The vectors y lying in the relative interior of this face of $\mathcal{B}_I$ are precisely those for which the y-maximum face of the standard permutahedron π_I is π_F.

In other words, the faces of the braid arrangement $\mathcal{B}_I$ are precisely the cones $\mathcal{N}_{\pi_I}(\pi_F)$ in the normal fan of the permutahedron π_I. We call this fan the *braid fan*.



The vector $e_I=(1,\dots,1)$ is orthogonal to π_I and the line it spans is the lineality space of the braid fan $\mathcal{N}_{\pi_I}$ (the minimum cone in the fan). Two vectors congruent modulo e_I lie in the same cone of the fan.

Assume S and T are proper and nonempty. Then the vector e_S lies in the open face of $\mathcal{B}_I$ labeled by the composition (S,T). More generally, for F as above and any positive scalars λ_i, the vector

$$\lambda_1 e_{S_1}+\lambda_2 e_{S_1\sqcup S_2}+\cdots+\lambda_k e_{S_1\sqcup\cdots\sqcup S_k} \tag{20}$$

lies in the open face labeled by F. Among these there is the vector

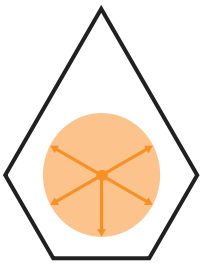
$$e_F=e_{S_1}+e_{S_1\sqcup S_2}+\cdots+e_{S_1\sqcup\cdots\sqcup S_k} \tag{21}$$

Note that $e_{S,T} = e_S + e_I$ and

$$e_{R\sqcup S,T} + e_{R,S\sqcup T} \equiv e_{R,S,T} \mod e_I. \tag{22}$$

1.3.6. Generalized permutahedra. Recall that a fan $\mathcal{G}$ *coarsens* another fan $\mathcal{F}$ (or $\mathcal{F}$ *refines* $\mathcal{G}$) if every cone of $\mathcal{F}$ is contained in a cone of $\mathcal{G}$ or, equivalently, if every cone of $\mathcal{G}$ is a union of cones of $\mathcal{F}$. We are now ready to define our main object of study.

DEFINITION 1.3.7. A *generalized permutahedron on* I is a deformation of the permutahedron π_I; that is, a polytope $\mathfrak{p} \subseteq \mathbb{R}^I$ whose normal fan $\mathcal{N}_\mathfrak{p}$ coarsens the braid fan $\mathcal{N}_{\pi_I}$.



Let $\mathfrak{p}$ be a generalized permutahedron in $\mathbb{R}^I$. To a composition F of I, one may attach a face of $\mathfrak{p}$, as follows. Recall that F determines a face π_F of π_I. By assumption, the open cone $\mathcal{N}^\circ_{\pi_I}(\pi_F)$ is contained in a unique open cone of the form $\mathcal{N}^\circ_\mathfrak{p}(\mathfrak{q})$, where $\mathfrak{q}$ is a face of $\mathfrak{p}$. We denote this face $\mathfrak{q}$ by $\mathfrak{p}_F$. Thus,

$$\mathcal{N}^\circ_\mathfrak{p}(\mathfrak{q}) = \bigsqcup_{F:\, \mathfrak{p}_F = \mathfrak{q}} \mathcal{N}^\circ_{\pi_I}(\pi_F).$$

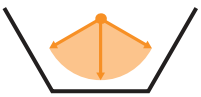
Equivalently, $\mathfrak{p}_F$ is the y-maximum face of $\mathfrak{p}$ for any y lying in the open face of $\mathcal{B}_I$ labeled by F. In particular,

$$\mathfrak{p}_F = \mathfrak{p}_{e_F}, \tag{23}$$

where e_F is as in (21) Every face of $\mathfrak{p}$ arises in this manner, in general for several compositions F.

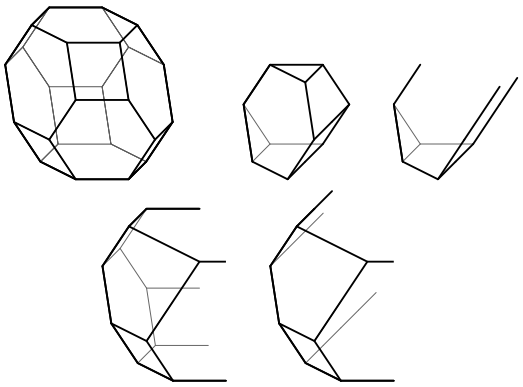
We wish to consider more general (unbounded) polyhedra. For this we allow the support of the normal fan to be smaller than the ambient space $\mathbb{R}^I$.

DEFINITION 1.3.8. An *extended generalized permutahedron on* I is an extended deformation of the permutahedron π_I; that is, a polytope $\mathfrak{p} \subseteq \mathbb{R}^I$ whose normal fan $\mathcal{N}_\mathfrak{p}$ coarsens a subfan of the braid fan $\mathcal{N}_{\pi_I}$.



The support of such a subfan coincides with the support of $\mathcal{N}_\mathfrak{p}$, and hence must be convex (in fact, a cone).

Since the permutahedron is a simple polytope, a generalized permutahedron is obtained from it by shifting the facets while preserving their directions, without letting a facet go past a vertex. To deform a permutahedron into an extended generalized permutahedron, vertices can be moved off to infinity, and facet hyperplanes can be erased. The figure below shows the standard permutahedron in $\mathbb{R}^4$ and four of its deformations.



Since the permutahedron π_I is a parallel translate of the zonotope of the root system $A_I = \{e_i - e_j : i, j \in I\}$, the following is a special case of Proposition 1.3.1.

COROLLARY 1.3.9. *A polytope $\mathfrak{p}$ is a generalized permutahedron if and only if every edge is parallel to a vector of the form $e_i - e_j$ for $i, j \in I$. More generally, a polyhedron $\mathfrak{p}$ in $\mathbb{R}^I$ is an extended generalized permutahedron if and only if every face affinely spans a parallel translate of span(S) for some $S \subseteq A_I = \{e_i - e_j : i, j \in I\}$.*

REMARK 1.3.10. Generalized permutahedra were introduced by Postnikov in [**77**], but have emerged in various forms and guises in the work of many authors.

Up to translation, generalized permutahedra are equivalent to base polytopes of *polymatroids*, which were defined earlier by Edmonds [**36**]; see Section 3.1.3. Generalized permutahedra are also equivalent to *submodular functions* [**36**, **41**, **71**, **87**]; we discuss this further in Section 3.1. We have extended these definitions to allow for unbounded polyhedra; similar generalizations were considered by Fujishige [**41**] and Derksen and Fink [**32**].

Complete fans coarsening $\mathcal{N}_{\pi_I}$ appear in [**76**] as *complete fans of posets* and in [**71**] as *convex rank tests*. Not every such coarsening is the normal fan of a polytope. When it is, the polytope is (by Definition 1.3.7) a generalized permutahedron. Those polytopal fans are the *submodular rank tests* of [**71**]. It is shown in [**71**, Theorem 9] that convex rank tests are in bijection with *semigraphoids*, a concept arising in nonparametric statistics [**30**, **73**, **97**].

1.4. GP: The Hopf monoid of generalized permutahedra

We turn the collection of generalized permutahedra into a Hopf monoid in set species. We focus on polytopes initially, and treat the unbounded case (extended generalized permutahedra) in Section 1.4.5.

1.4.1. Cartesian product, restriction, and contraction. We introduce suitable operations on generalized permutahedra. Let I be a finite set and $I = S \sqcup T$ a decomposition. Recall that

$$\mathbb{R}^S \times \mathbb{R}^T = \mathbb{R}^I.$$

Assume that S and T are proper and nonempty.

PROPOSITION 1.4.1. *If $\mathfrak{p} \subseteq \mathbb{R}^S$ and $\mathfrak{q} \subseteq \mathbb{R}^T$ are generalized permutahedra, then $\mathfrak{p} \times \mathfrak{q} \subseteq \mathbb{R}^I$ is a generalized permutahedron.*

PROOF. By assumption, $\mathcal{N}_{\pi_S}$ and $\mathcal{N}_{\pi_T}$ refine $\mathcal{N}_{\mathfrak{p}}$ and $\mathcal{N}_{\mathfrak{q}}$, respectively. Employing (14) we deduce that $\mathcal{N}_{\pi_S \times \pi_T} = \mathcal{N}_{\pi_S} \times \mathcal{N}_{\pi_T}$ refines $\mathcal{N}_{\mathfrak{p}} \times \mathcal{N}_{\mathfrak{q}} = \mathcal{N}_{\mathfrak{p} \times \mathfrak{q}}$. In turn, the braid fan $\mathcal{N}_{\pi_I}$, which is cut out by the hyperplanes $x_i = x_j$ for $i, j \in I$, refines the product $\mathcal{N}_{\pi_S} \times \mathcal{N}_{\pi_T}$ of the braid fans in $\mathbb{R}^S$ and $\mathbb{R}^T$, which is cut out by the hyperplanes $x_i = x_j$ for $i, j \in S$ or $i, j \in T$. It follows that $\mathcal{N}_{\pi_I}$ refines $\mathcal{N}_{\mathfrak{p} \times \mathfrak{q}}$. □

Let $F = (S, T)$. This is a composition of I. Given a generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^I$, the face $\mathfrak{p}_F$ of $\mathfrak{p}$ is defined (Section 1.3.6). It is the e_S-maximum face of $\mathfrak{p}$, and more generally the y-maximum face for any y lying in the open face of $\mathcal{B}_I$ labeled by F.

PROPOSITION 1.4.2. *There exist generalized permutahedra* $\mathfrak{p}|_S \subseteq \mathbb{R}^S$ *and* $\mathfrak{p}/_S \subseteq \mathbb{R}^T$ *such that*

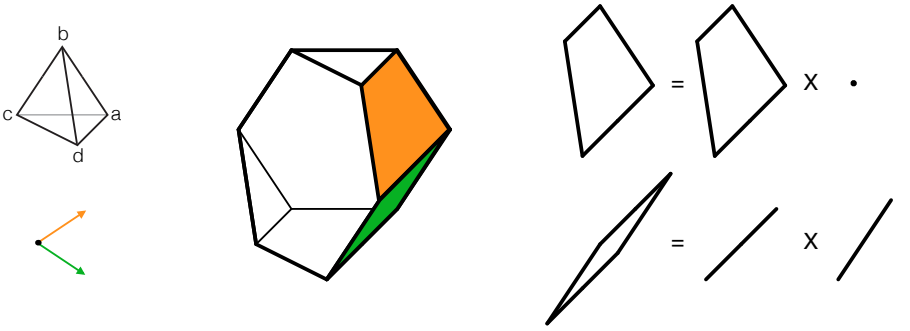
$$\mathfrak{p}_{S,T} = \mathfrak{p}|_S \times \mathfrak{p}/_S. \tag{24}$$

PROOF. This result appears in [**41**, Theorem 3.15]. It may be also derived from the proof of Theorem 3.1.3. □

We call $\mathfrak{p}|_S$ the *restriction* of $\mathfrak{p}$ to S and $\mathfrak{p}/_S$ the *contraction* of S from $\mathfrak{p}$.

The figure below shows a generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^{abcd}$ and its faces

$$\mathfrak{p}_{abd,c} = \mathfrak{p}|_{abd} \times \mathfrak{p}/_{abd} \subseteq \mathbb{R}^{abd} \times \mathbb{R}^c \qquad \text{and} \qquad \mathfrak{p}_{ad,bc} = \mathfrak{p}|_{ad} \times \mathfrak{p}/_{ad} \subseteq \mathbb{R}^{ad} \times \mathbb{R}^{bc}.$$



1.4.2. The Hopf monoid GP of generalized permutahedra. For each finite set I, let $\mathrm{GP}[I]$ denote the set of generalized permutahedra on I. We agree that $\mathrm{GP}[\emptyset]$ consists of a single polytope, the only polytope (the origin) in the 0-dimensional space $\mathbb{R}^\emptyset$. A bijection $\sigma : I \to J$ induces a linear isomorphism

$$\mathbb{R}^I \to \mathbb{R}^J, \quad x \mapsto y \tag{25}$$

where $y_j = x_{\sigma^{-1}(j)}$ for each $j \in J$. This sends π_I and its faces bijectively onto π_J and its faces. Therefore, it sends a generalized permutahedron on I to another on J.

In this manner, GP is a connected set species. We turn it into a Hopf monoid as follows. Let (S, T) be a composition of I.

• The product of $\mathfrak{p} \in \mathrm{GP}[S]$ and $\mathfrak{q} \in \mathrm{GP}[T]$ is

$$\mathfrak{p} \cdot \mathfrak{q} := \mathfrak{p} \times \mathfrak{q} \in \mathrm{GP}[I].$$

• The coproduct of $\mathfrak{p} \in \mathrm{GP}[I]$ is

$$\Delta_{S,T}(\mathfrak{p}) = (\mathfrak{p}|_S, \mathfrak{p}/_S),$$

where the restriction $\mathfrak{p}|_S \in \mathrm{GP}[S]$ and contraction $\mathfrak{p}/_S \in \mathrm{GP}[T]$ are defined in Proposition 1.4.2.

THEOREM 1.4.3. *These operations turn the set species* GP *into a (connected) Hopf monoid.*

PROOF. We verify two of the axioms; the others are straightforward.

Coassociativity. Let $\mathfrak{p} \subseteq \mathbb{R}^I$ be a generalized permutahedron. We need to show that for any decomposition $I = R \sqcup S \sqcup T$,

$$(\mathfrak{p}|_{R\sqcup S})|_R = \mathfrak{p}|_R, \qquad (\mathfrak{p}|_{R\sqcup S})/_R = (\mathfrak{p}/_R)|_S, \qquad \mathfrak{p}/_{R\sqcup S} = (\mathfrak{p}/_R)/_S.$$

We may assume R, S and T are nonempty. We employ (12) and (24) to calculate

$$\begin{aligned}(\mathfrak{p}_{R\sqcup S,T})_{R,S\sqcup T} &= (\mathfrak{p}|_{R\sqcup S} \times \mathfrak{p}/_{R\sqcup S})_{R,S\sqcup T}\\ &= (\mathfrak{p}|_{R\sqcup S})_{R,S} \times \mathfrak{p}/_{R\sqcup S} = (\mathfrak{p}|_{R\sqcup S})|_R \times (\mathfrak{p}|_{R\sqcup S})/_R \times \mathfrak{p}/_{R\sqcup S},\\ (\mathfrak{p}_{R,S\sqcup T})_{R\sqcup S,T} &= (\mathfrak{p}|_R \times \mathfrak{p}/_R)_{R\sqcup S,T}\\ &= \mathfrak{p}|_R \times (\mathfrak{p}/_R)_{S,T} = \mathfrak{p}|_R \times (\mathfrak{p}/_R)|_S \times (\mathfrak{p}/_R)/_S.\end{aligned}$$

Thus, it suffices to prove the equality of the polytopes $(\mathfrak{p}_{R\sqcup S,T})_{R,S\sqcup T}$ and $(\mathfrak{p}_{R,S\sqcup T})_{R\sqcup S,T}$. For this, we employ (13) and (20) to find that for $\lambda >> \mu > 0$ and $\lambda' >> \mu' > 0$,

$$\begin{aligned}(\mathfrak{p}_{R\sqcup S,T})_{R,S\sqcup T} &= (\mathfrak{p}_{e_{R\sqcup S}})_{e_R} = \mathfrak{p}_{\lambda e_{R\sqcup S} + \mu e_R} = \mathfrak{p}_{R,S,T},\\ (\mathfrak{p}_{R,S\sqcup T})_{R\sqcup S,T} &= (\mathfrak{p}_{e_R})_{e_{R\sqcup S}} = \mathfrak{p}_{\lambda' e_R + \mu' e_{R\sqcup S}} = \mathfrak{p}_{R,S,T}.\end{aligned}$$

The last equations in each step above follow from the fact that $\lambda e_{R\sqcup S} + \mu e_R$ and $\lambda' e_R + \mu' e_{R\sqcup S}$ lie in the open face of the braid fan labeled by the composition (R, S, T).

Compatibility. Fix decompositions $S \sqcup T = I = S' \sqcup T'$ and let A, B, C, D be the pairwise intersections, as in (3). Let $\mathfrak{p} \subseteq \mathbb{R}^S$ and $\mathfrak{q} \subseteq \mathbb{R}^T$ be generalized permutahedra. We need to verify that

$$(\mathfrak{p} \times \mathfrak{q})|_{S'} = \mathfrak{p}|_A \times \mathfrak{q}|_C \quad \text{and} \quad (\mathfrak{p} \times \mathfrak{q})/_{S'} = \mathfrak{p}/_A \times \mathfrak{q}/_C.$$

If any of A, B, C, D is empty, these hold trivially. Assume they are not. We calculate employing (12) and (24), noting that $e_{S'} = e_A + e_C$:

$$\begin{aligned}(\mathfrak{p} \times \mathfrak{q})|_{S'} \times (\mathfrak{p} \times \mathfrak{q})/_{S'} &= (\mathfrak{p} \times \mathfrak{q})_{e_{S'}} = \mathfrak{p}_{e_A} \times \mathfrak{q}_{e_C}\\ &= \mathfrak{p}|_A \times \mathfrak{p}/_A \times \mathfrak{q}|_C \times \mathfrak{q}/_C = (\mathfrak{p}|_A \times \mathfrak{q}|_C) \times (\mathfrak{p}/_A \times \mathfrak{q}/_C).\end{aligned}$$

The desired equalities follow. □

The Hopf monoid $\mathfrak{p}$ is commutative but not cocommutative.

Having verified (co)associativity, we are now able to describe higher (co)products in GP in geometric terms. Recall from (21) that if $F = (S_1, \ldots, S_k)$ then $e_F = e_{S_1} + e_{S_1\sqcup S_2} + \cdots + e_{S_1\sqcup\cdots\sqcup S_k}$, where e_S is the indicator vector of S in $\mathbb{R}^E$ for $S \subseteq E$.

PROPOSITION 1.4.4. *Let $F = (S_1, \ldots, S_k)$ be a composition of I. In the Hopf monoid* GP*, the higher product and coproduct associated to F are as follows:*

• *For generalized permutahedra $\mathfrak{p}_1 \ldots, \mathfrak{p}_k$ in $\mathbb{R}^{S_1}, \ldots, \mathbb{R}^{S_k}$, $\mu_F(\mathfrak{p}_1, \ldots, \mathfrak{p}_k) = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_k$.*

• *For a generalized permutahedron* $\mathfrak{p}$ *in* $\mathbb{R}^I$, $\Delta_F(\mathfrak{p}) = (\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ *where* $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ *are the generalized permutahedra in* $\mathbb{R}^{S_1}, \ldots, \mathbb{R}^{S_k}$, *respectively, such that the* e_F*-maximal face of* $\mathfrak{p}$ *is* $\mathfrak{p}_F = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_k$.

Consequently, we have

$$\mu_F \Delta_F(\mathfrak{p}) = \mathfrak{p}_F. \tag{26}$$

PROOF. The expression for the higher product follows readily by associativity. We prove the expression for the higher coproduct by induction on the number of parts of F, recalling from (21) and (23) that $\mathfrak{p}_F = \mathfrak{p}_{e_F}$. When F has one part, e_F is orthogonal to $\mathfrak{p}$, so $\mathfrak{p}_F = \mathfrak{p}$ and the result holds. When F has two or more parts, let S be the first part, T the union of the remaining parts, and G the composition of T consisting of those parts. By the induction hypothesis we have $\Delta_G(\mathfrak{p}/_S) = (\mathfrak{p}_2, \ldots, \mathfrak{p}_k)$ for the generalized permutahedra $\mathfrak{p}_2, \ldots, \mathfrak{p}_k$ in $\mathbb{R}^{S_2}, \ldots, \mathbb{R}^{S_k}$ such that $(\mathfrak{p}/_S)_G = \mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_k$. Now, $e_F = ke_S + e_G$ lies in the same open face of the braid fan as $\lambda e_S + \mu e_G$ when $\lambda, \mu > 0$. In light of Lemma 1.3.3 and (12), whenever $\lambda >> \mu > 0$ we have

$$\mathfrak{p}_F = \mathfrak{p}_{\lambda e_S + \mu e_G} = (\mathfrak{p}_{e_S})_{e_G} = (\mathfrak{p}|_S \times \mathfrak{p}/_S)_{e_G} = \mathfrak{p}|_S \times (\mathfrak{p}/_S)_{e_G} = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \ldots \cdot \mathfrak{p}_k$$

where $\mathfrak{p}_1 = \mathfrak{p}|_S \in \mathrm{GP}[S]$. By coassociativity,

$$\Delta_F(\mathfrak{p}) = (\mathrm{id}, \Delta_G)\Delta_{S,T}(\mathfrak{p}) = (\mathrm{id}, \Delta_G)(\mathfrak{p}|_S, \mathfrak{p}/_S) = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k)$$

as desired.

The descriptions of μ_F and Δ_F imply that $\mu_F \Delta_F(\mathfrak{p}) = \mathfrak{p}_F$. □

REMARK 1.4.5. In the language of polymatroids and submodular functions, equivalent definitions of restriction and contraction were given by Edmonds in [**36**]. A similar Hopf algebraic structure on polymatroids was defined independently by Derksen and Fink [**32**] at about the same time that our results were announced. In our work we emphasize the polytopal perspective, which allows us to obtain many new results.

1.4.3. Normal equivalence: the Hopf monoid $\overline{\mathrm{GP}}$. Let $\overline{\mathrm{GP}}[I]$ denote the quotient of the set $\mathrm{GP}[I]$ in which normally equivalent generalized permutahedra in $\mathbb{R}^I$ are identified. This defines a quotient species $\overline{\mathrm{GP}}$ of GP.

PROPOSITION 1.4.6. *The Hopf monoid structure of* GP *descends to* $\overline{\mathrm{GP}}$.

PROOF. The product descends to the quotient in view of (15). For the coproduct, consider two normally equivalent generalized permutahedra $\mathfrak{p} \equiv \mathfrak{p}'$ in $\mathbb{R}^I$ and let (S, T) be a composition of I. It suffices to show that $\mathfrak{p}_{S,T} \equiv \mathfrak{p}'_{S,T}$, again by (15). This follows from (16) applied to the vector e_S. □

We obtain a quotient Hopf monoid

$$\mathrm{GP} \twoheadrightarrow \overline{\mathrm{GP}}.$$

The elements of $\overline{\mathrm{GP}}[I]$ are in one-to-one correspondence with *polytopal* coarsenings of the normal fan of π_I (coarser fans which arise as normal fans of a polytope in $\mathbb{R}^I$). There is a finite number of such coarsenings. It follows from the discussion in [**71**, Section 4] that for $|I| = 1, 2, 3, 4$ there are 1, 2, 22, 22108 elements in $\overline{\mathrm{GP}}[I]$. Figure 3 shows the 22 classes for $|I| = 3$, together with the corresponding fans.

One may also define an intermediate quotient Hopf monoid in which generalized permutahedra are identified only up to translations and dilations.

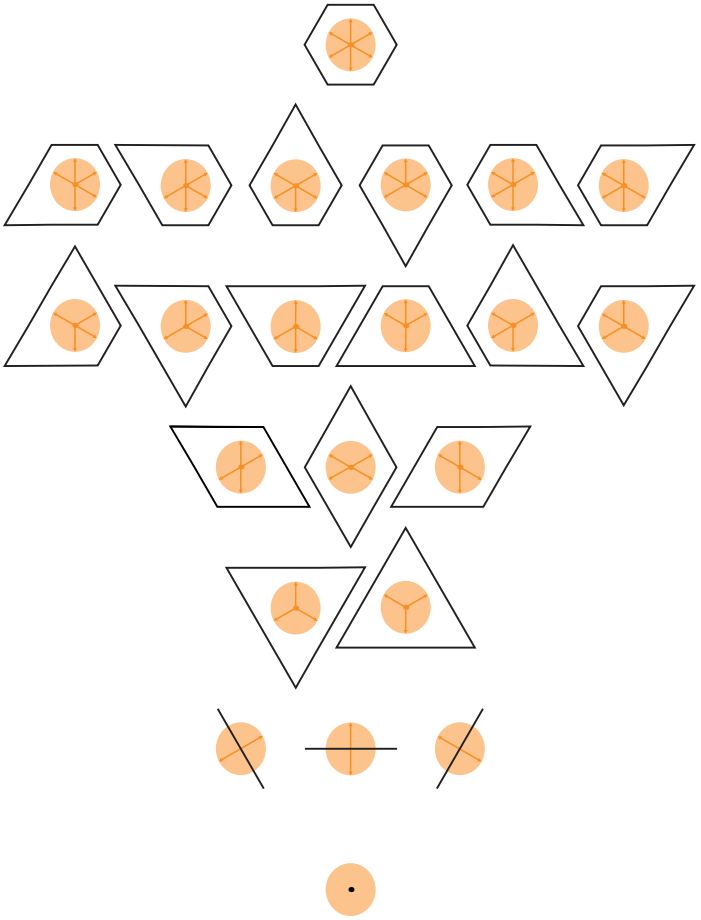


FIGURE 3. The pantheon of generalized permutahedra on $\{a, b, c\}$.

1.4.4. Central symmetry. Given a polyhedron $\mathfrak{p}$, its opposite is

$$\overline{\mathfrak{p}} = \{-x \mid x \in \mathfrak{p}\}.$$

We call $\mathfrak{p}$ *centrally symmetric* if $\overline{\mathfrak{p}} = \mathfrak{p}$.

For any face $\mathfrak{q}$ of $\mathfrak{p}$, we have

$$\mathcal{N}_{\overline{\mathfrak{p}}}(\overline{\mathfrak{q}}) = \overline{\mathcal{N}_{\mathfrak{p}}(\mathfrak{q})}. \tag{27}$$

A fan is centrally symmetric if for each cone in the fan, the opposite cone belongs to the fan. It follows from (27) that if $\mathfrak{p}$ is centrally symmetric, so is its normal fan. Hence the same is true if $\mathfrak{p}$ is normally equivalent to its opposite.

The standard permutahedron π_I is normally equivalent to its opposite, since its translate to the origin (by means of $-\binom{n+1}{2}e_I$) is centrally symmetric. It follows that the fan $\mathcal{N}_{\pi_I}$ is centrally symmetric, and then that if $\mathfrak{p}$ is a generalized permutahedron in $\mathbb{R}^I$, so is $\overline{\mathfrak{p}}$. We obtain a bijective map

$$\mathrm{GP}[I] \to \mathrm{GP}[I], \quad \mathfrak{p} \mapsto \overline{\mathfrak{p}},$$

and then an isomorphism of species $\mathrm{GP} \to \mathrm{GP}$.

PROPOSITION 1.4.7. *The above map is an isomorphism of Hopf monoids* $\mathrm{GP}^{\mathrm{cop}} \to \mathrm{GP}$.

PROOF. The product is preserved since $\overline{\mathfrak{p} \times \mathfrak{q}} = \overline{\mathfrak{p}} \times \overline{\mathfrak{q}}$. To verify that the coproduct is reversed, pick a generalized permutahedron $\mathfrak{p}$ in $\mathbb{R}^I$ and a composition (S, T) of I. Note that

$$e_T = \overline{e_S} + e_I.$$

We calculate using (24):

$$\overline{\mathfrak{p}|_S} \times \overline{\mathfrak{p}/_S} = \overline{\mathfrak{p}_{e_S}} = \overline{\mathfrak{p}}_{\overline{e_S}} = \overline{\mathfrak{p}}_{e_T} = \overline{\mathfrak{p}}|_T \times \overline{\mathfrak{p}}/_T.$$

The third equality holds since e_I is orthogonal to $\mathfrak{p}$. This says that $\overline{\Delta_{S,T}(\mathfrak{p})} = \Delta_{T,S}(\overline{\mathfrak{p}})$. □

The map $\mathfrak{p} \mapsto \overline{\mathfrak{p}}$ is well-defined on classes under normal equivalence by (27), so it descends to the quotient yielding an isomorphism of Hopf monoids $\overline{\mathrm{GP}}^{\mathrm{cop}} \to \overline{\mathrm{GP}}$.

1.4.5. The Hopf monoid $\mathbf{GP}^+$ of extended generalized permutahedra. For each finite set I, let $\mathrm{GP}^+[I]$ be the set of extended generalized permutahedra on I. Then GP^+ is a set species in the same manner as GP is. Let $\mathbf{GP}^+$ denote its linearization. We proceed to turn the latter into a Hopf monoid in vector species: certain components of the coproduct are not defined set-theoretically. Let (S, T) be a composition of I.

• The product of $\mathfrak{p} \in \mathrm{GP}^+[S]$ and $\mathfrak{q} \in \mathrm{GP}^+[T]$ is

$$\mathfrak{p} \cdot \mathfrak{q} := \mathfrak{p} \times \mathfrak{q} \in \mathrm{GP}^+[I].$$

This operation is set theoretic, and extends linearly to $\mu_{S,T} : \mathbf{GP}^+[S] \otimes \mathbf{GP}^+[T] \to \mathbf{GP}^+[I]$.

• The coproduct is defined on a basis element $\mathfrak{p} \in \mathrm{GP}[I]$ by

$$\Delta_{S,T}(\mathfrak{p}) = \begin{cases} \mathfrak{p}|_S \otimes \mathfrak{p}/_S & \text{if } e_S \text{ lies in the support of } \mathcal{N}_{\mathfrak{p}}, \\ 0 & \text{otherwise,} \end{cases}$$

and extended linearly to $\Delta_{S,T} : \mathbf{GP}^+[I] \to \mathbf{GP}^+[S] \otimes \mathbf{GP}^+[T]$. This operation is not set theoretic.

Propositions 1.4.1 and 1.4.2 hold for extended generalized permutahedra (the latter in the case that e_S lies in the support of the normal fan). This makes the above operations well-defined.

Proposition 1.4.4 holds in the following form:

$$\mu_F \Delta_F(\mathfrak{p}) = \begin{cases} \mathfrak{p}_F & \text{if } F \text{ lies in the support of } \mathcal{N}_{\mathfrak{p}}, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

THEOREM 1.4.8. *These operations turn the vector species* $\mathbf{GP}^+$ *into a (connected) Hopf monoid.*

PROOF. We verify coassociativity. Let (R, S, T) be a composition of I. The key fact is this:

$$e_{R,S\sqcup T} \text{ and } e_{R\sqcup S,T} \text{ lie in the support of } \mathcal{N}_{\mathfrak{p}} \iff e_{R,S,T} \text{ does.}$$

The forward implication holds by (22) and the comments following Lemma 1.3.3. To show the backward implication, recall that the fan $\mathcal{N}_{\mathfrak{p}}$ refines a subfan Σ of $\mathcal{N}_{\pi_I}$. Since $e_{R,S,T}$ is interior to the cone of the braid fan spanned by $e_{R,S\sqcup T}$ and $e_{R\sqcup S,T}$,

if that subfan Σ contains $e_{R,S,T}$, it must contain the generating rays $e_{R,S\sqcup T}$ and $e_{R\sqcup S,T}$.

This fact guarantees that if one encounters 0 in calculating $(\mathrm{id}\times\Delta_{S,T})\Delta_{R,S\sqcup T}(\mathfrak{p})$, then one also encounters 0 in calculating $(\Delta_{R,S}\times\mathrm{id})\Delta_{R\sqcup S,T}(\mathfrak{p})$. When 0 is not encountered, coassociativity holds by the same argument as in Theorem 1.4.3. The compatibility axiom requires a similar check. □

The Hopf monoid $\mathbf{GP}^+$ contains $\mathbf{GP}$ as a Hopf submonoid.

Repeating the construction of Section 1.4.3, we let $\overline{\mathbf{GP}^+}$ denote the Hopf monoid of extended generalized permutahedra modulo normal equivalence. We obtain the commutative diagram of Hopf monoids below.

$$\begin{array}{ccc} \mathbf{GP} & \hookrightarrow & \mathbf{GP}^+ \\ \downarrow & & \downarrow \\ \overline{\mathbf{GP}} & \hookrightarrow & \overline{\mathbf{GP}^+} \end{array}$$

1.5. Maximality of GP

In Section 1.4 we endowed the family of generalized permutahedra with the structure of a Hopf monoid in set species. The operations capture natural geometric features of this family of polytopes. One may wonder if other families of polytopes may lend themselves to the same treatment. We show here that this is not the case: generalized permutahedra constitute the largest such family.

Suppose P is a connected Hopf monoid in set species such that for every finite set I and composition (S,T) of I, the following properties hold.

- The elements of $\mathrm{P}[I]$ are polytopes in $\mathbb{R}^I$.
- The action of a bijection $I\to J$ on polytopes is induced from the map $\mathbb{R}^I\to\mathbb{R}^J$ in (25).
- The product $\mu_{S,T}(\mathfrak{p},\mathfrak{q})\in\mathrm{P}[I]$ of two polytopes $\mathfrak{p}\in\mathrm{P}[S]$ and $\mathfrak{q}\in\mathrm{P}[T]$ is their cartesian product $\mathfrak{p}\times\mathfrak{q}\subseteq\mathbb{R}^S\times\mathbb{R}^T=\mathbb{R}^I$.
- If we write the coproduct of $\mathfrak{p}\in\mathrm{P}[I]$ as

$$\Delta_{S,T}(\mathfrak{p})=(\mathfrak{p}|_S,\mathfrak{p}/_S)\in\mathrm{P}[S]\times\mathrm{P}[T],$$

then the polytope $\mathfrak{p}|_S\times\mathfrak{p}/_S\subseteq\mathbb{R}^S\times\mathbb{R}^T=\mathbb{R}^I$ is the maximum face of $\mathfrak{p}$ in the direction of e_S.

THEOREM 1.5.1. *Suppose* P *is as above. Then every polytope in* $\mathrm{P}[I]$ *is a generalized permutahedron on* I*, and* P *is a Hopf submonoid of* GP*.*

PROOF. Connectedness means that $\mathrm{P}[\emptyset]$ consists of a single polytope, the only polytope (the origin) in the 0-dimensional space $\mathbb{R}^\emptyset$. Denote it 1.

First notice that for any polytope $\mathfrak{p}\in\mathrm{P}[I]$ we have, by counitality for P,

$$\mathfrak{p}_{e_I}=\mathfrak{p}|_I\times\mathfrak{p}/_I=\mathfrak{p}\times 1=\mathfrak{p}.$$

It follows that $\langle -,e_I\rangle$ is constant on $\mathfrak{p}$. Therefore $\mathfrak{p}$ is not full-dimensional, and the direction e_I is in the lineality space of its normal fan $\mathcal{N}_\mathfrak{p}$. In other words, e_I belongs to every cone in $\mathcal{N}_\mathfrak{p}$.

Write $\mathfrak{p}_{S,T}=\mathfrak{p}|_S\times\mathfrak{p}/_S$ when (S,T) is a composition of I. As in the proof of Theorem 1.4.3, coassociativity for P implies that

$$(\mathfrak{p}_{R\sqcup S,T})_{R,S\sqcup T}=(\mathfrak{p}_{R,S\sqcup T})_{R\sqcup S,T}$$

when (R, S, T) is a composition of I. Let us denote this polytope $\mathfrak{p}_{R,S,T}$. Then by (13) we have

$$\mathfrak{p}_{R,S,T} = \mathfrak{p}_{\lambda e_{R\sqcup S} + \mu e_R} = \mathfrak{p}_{\lambda' e_R + \mu' e_{R\sqcup S}}$$

for $\lambda >> \mu > 0$ and $\lambda' >> \mu' > 0$. It follows that $\lambda e_{R\sqcup S} + \mu e_R$ and $\lambda' e_R + \mu' e_{R\sqcup S}$ are both in the normal cone $\mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_{R,S,T})$. Since that cone is closed, we may take the limits in which μ/λ and μ'/λ' approach 0 and obtain that $e_{R\sqcup S}$ and e_R are in $\mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_{R,S,T})$. Since the braid cone $\mathcal{N}_{\pi_I}(\pi_{R,S,T})$ is spanned by $\{e_R, e_{R\sqcup S}, e_{R\sqcup S\sqcup T} = e_I\}$, it must be contained in the normal cone $\mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_{R,S,T})$. We conclude that every three-dimensional cone of the braid fan $\mathcal{N}_{\Pi_I}$ is contained in a cone of the normal fan $\mathcal{N}_{\mathfrak{p}}$ of $\mathfrak{p}$.

We now use higher coassociativity to carry out the analogous argument for any composition $F = (S_1, \ldots, S_k)$ of I, using Proposition 1.4.4. The higher coproduct $\Delta_{S_1,\ldots,S_k}(\mathfrak{p}) = (\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ may be computed by iterating the coproduct maps $\Delta_{S,T}$ in any meaningful way. Write $\mathfrak{p}_F = \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_k$. Each way gives rise to an expression for this face of $\mathfrak{p}$. One of them is

$$\mathfrak{p}_F = (\cdots((\mathfrak{p}_{S_1\sqcup\cdots\sqcup S_{k-1}, S_k})_{S_1\sqcup\cdots\sqcup S_{k-2}, S_{k-1}\sqcup S_k})\ldots)_{S_1, S_2\sqcup\cdots\sqcup S_k}.$$

This implies, by (13), that $\lambda_1 e_{S_1\sqcup\cdots\sqcup S_{k-1}} + \lambda_2 e_{S_1\sqcup\cdots\sqcup S_{k-2}} + \cdots + \lambda_{k-1} e_{S_1}$ lies in the normal cone $\mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_F)$ for any $\lambda_1 >> \lambda_2 >> \cdots >> \lambda_{k-1} > 0$. By sending $\lambda_{k-1}/\lambda_{k-2}, \ldots, \lambda_3/\lambda_2, \lambda_2/\lambda_1 \to 0$ in that order, we obtain $e_{S_1\sqcup\cdots\sqcup S_{k-1}} \in \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_F)$. By computing the coproduct in different ways, we similarly obtain $e_{S_1\sqcup\cdots\sqcup S_j} \in \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_F)$ for any $1 \leq j \leq k-1$. We already know that $e_{S_1\sqcup\cdots\sqcup S_k} = e_I \in \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_F)$ as well. Therefore, $\mathcal{N}_{\mathfrak{p}}(\mathfrak{p}_F)$ contains the cone spanned by the vectors $e_{S_1}, e_{S_1\sqcup S_2}$, $\ldots, e_{S_1\sqcup\cdots\sqcup S_k}$, and this cone is $\mathcal{N}_{\pi_I}(\pi_F)$.

It follows that every cone in the braid fan $\mathcal{N}_{\pi_I}$ is contained in a cone of the normal fan $\mathcal{N}_{\mathfrak{p}}$. By definition, this means that $\mathfrak{p}$ is a generalized permutahedron.

This shows that, for each I, $\mathrm{P}[I]$ is a subset of $\mathrm{GP}[I]$, and the condition on the action of bijections guarantees that P is a subspecies of GP. The remaining conditions state that the operations on P are the restriction of those of GP. It follows that P is a Hopf submonoid of GP, as desired. □

A similar result holds for Hopf monoids built out of (possibly unbounded) polyhedra. They are necessarily Hopf submonoids of $\mathbf{GP}^+$. We leave the details to the reader.

1.6. The antipode of GP

In this section we derive a remarkably simple formula for the antipode of the Hopf monoid of generalized permutahedra. This is the best possible formula in that it involves no cancellations or repeated terms.

If $\mathfrak{p} \subseteq \mathbb{R}^I$ is a generalized permutahedron, then so is every face $\mathfrak{q}$ of $\mathfrak{p}$ by Corollary 1.3.9.

THEOREM 1.6.1. *The antipode of the Hopf monoid* $\mathbf{GP}$ *of generalized permutahedra is given by the following* ***cancellation-free*** *and* ***combination-free*** *formula. If* $\mathfrak{p}$ *is a generalized permutahedron on* I*, then*

$$\mathrm{s}_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim \mathfrak{q}}\, \mathfrak{q}, \tag{29}$$

where the sum is over all faces $\mathfrak{q}$ *of* $\mathfrak{p}$*. The same formula holds for the antipode of the Hopf monoid* $\mathbf{GP}^+$ *of extended generalized permutahedra.*

PROOF. Takeuchi's formula (5) together with (26) give us

$$\mathrm{s}_I(\mathfrak{p}) = \sum_{F \vDash I} (-1)^{\ell(F)}\, \mu_F \Delta_F(\mathfrak{p}) = \sum_{F \vDash I} (-1)^{\ell(F)}\, \mathfrak{p}_F.$$

This is indeed a linear combination of faces of $\mathfrak{p}$. Collecting the coefficient

$$\alpha_{\mathfrak{q}} = \sum_{F \vDash I:\, \mathfrak{p}_F = \mathfrak{q}} (-1)^{\ell(F)}$$

of each face $\mathfrak{q}$ of $\mathfrak{p}$, we have

$$\mathrm{s}_I(\mathfrak{p}) = \sum_{\mathfrak{q} \le \mathfrak{p}} \alpha_{\mathfrak{q}}\, \mathfrak{q},$$

and we are left with the task of proving that $\alpha_{\mathfrak{q}} = (-1)^{|I| - \dim \mathfrak{q}}$.

Since the fan $\mathcal{N}_{\mathfrak{p}}$ refines the fan $\mathcal{N}^\circ_{\pi_I}$, we have

$$\mathfrak{p}_F = \mathfrak{q} \iff \mathcal{N}^\circ_{\pi_I}(\pi_F) \subseteq \mathcal{N}^\circ_{\mathfrak{p}}(\mathfrak{q}).$$

Define

$$\mathcal{C}_{\mathfrak{q}} = \{F \vDash I \mid \mathcal{N}^\circ_{\pi_I}(\pi_F) \subseteq \mathcal{N}^\circ_{\mathfrak{p}}(\mathfrak{q})\} \quad \text{and} \quad \overline{\mathcal{C}_{\mathfrak{q}}} = \{F \vDash I \mid \mathcal{N}^\circ_{\pi_I}(\pi_F) \subseteq \mathcal{N}_{\mathfrak{p}}(\mathfrak{q})\}.$$

Noting that $\ell(F) = \dim \mathcal{N}^\circ_{\pi_I}(\pi_F)$, we have

$$\alpha_{\mathfrak{q}} = \sum_{F \in \mathcal{C}_{\mathfrak{q}}} (-1)^{\dim \mathcal{N}^\circ_{\pi_I}(\pi_F)}.$$

We would like to interpret this sum as an Euler characteristic, but as F varies in $\mathcal{C}_{\mathfrak{q}}$, the set of cones $\mathcal{N}^\circ_{\pi_I}(\pi_F)$ does not constitute a polyhedral complex, since it is not closed under subfaces. To remedy this, we observe that the cones indexed by $\overline{\mathcal{C}_{\mathfrak{q}}}$ as well as those indexed by $\overline{\mathcal{C}_{\mathfrak{q}}} - \mathcal{C}_{\mathfrak{q}}$ do constitute polyhedral complexes. We may then rewrite the previous equation as

$$\begin{aligned}\alpha_{\mathfrak{q}} &= \sum_{F \in \overline{\mathcal{C}_{\mathfrak{q}}}} (-1)^{\dim \mathcal{N}^\circ_{\pi_I}(\pi_F)} - \sum_{F \in \overline{\mathcal{C}_{\mathfrak{q}}} - \mathcal{C}_{\mathfrak{q}}} (-1)^{\dim \mathcal{N}^\circ_{\pi_I}(\pi_F)} \\ &= \overline{\chi}(\overline{\mathcal{C}_{\mathfrak{q}}}) - \overline{\chi}(\overline{\mathcal{C}_{\mathfrak{q}}} - \mathcal{C}_{\mathfrak{q}}),\end{aligned}$$

where $\overline{\chi}$ denotes the reduced Euler characteristic. We employ it since we want to count the composition (I) which belongs to both complexes.

Let us intersect the cones in $\mathcal{N}_{\pi_I}$ with the sphere $\mathcal{S} = \{x \in \mathbb{R}^I \mid \sum x_i = 0, \sum x_i^2 = 1\}$. The resulting cells form a CW-decomposition of $\mathcal{S}$, namely, the Coxeter complex of type A_I. The cells indexed by $\overline{\mathcal{C}_{\mathfrak{q}}}$ form a CW-decomposition of $\mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) \cap \mathcal{S}$, while the cells indexed by $\overline{\mathcal{C}}_{\mathfrak{q}} - \mathcal{C}_{\mathfrak{q}}$ form a CW-decomposition of $\partial \mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) \cap \mathcal{S}$. So we have

$$\alpha_{\mathfrak{p}} = \overline{\chi}(\mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) \cap \mathcal{S}) - \overline{\chi}(\partial \mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) \cap \mathcal{S}).$$

We now observe that if $\mathfrak{q}$ is a proper face of $\mathfrak{p}$, $\mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) \cap \mathcal{S}$ is a ball of dimension $\dim \mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) - 2$, and $\partial \mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) \cap \mathcal{S}$ is a sphere of dimension $\dim \mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) - 3$. Therefore, in this case,

$$\alpha_{\mathfrak{q}} = 0 - (-1)^{\dim \mathcal{N}_{\mathfrak{p}}(\mathfrak{q}) - 3} = (-1)^{|I| - \dim \mathfrak{q}}.$$

On the other hand, $\mathcal{N}_{\mathfrak{p}}(\mathfrak{p}) \cap \mathcal{S}$ is a sphere of dimension $\dim \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}) - 2$, and $\partial \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}) \cap \mathcal{S}$ is empty. So in this case

$$\alpha_{\mathfrak{p}} = (-1)^{\dim \mathcal{N}_{\mathfrak{p}}(\mathfrak{p}) - 2} - 0 = (-1)^{|I| - \dim \mathfrak{p}}$$

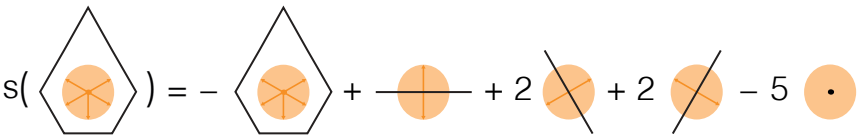
as well. This completes the proof of (29).

The formula is cancellation-free and combination-free since distinct polytopes are linearly independent in $\mathbf{GP}$.

Formula (29) holds for the Hopf monoid $\mathbf{GP}^+$, as stated. In the proof, we employ (28) in place of (26), and the rest of the argument goes through unchanged. □

REMARK 1.6.2. Let $\mathbf{P} = \Bbbk\mathrm{P}$ be any linearized Hopf monoid. The coefficients of the antipode on the basis P *always* admit a description in terms of the reduced Euler characteristic of a pair of complexes. See [**4**, Section 7.7] and [**5**, Section 12.9].

The formula also holds for the quotients $\overline{\mathbf{GP}}$ and $\overline{\mathbf{GP}^+}$. At this level it is no longer combination-free, since normally equivalent faces may occur. It is still cancellation-free, since normally equivalent polytopes have the same dimension and hence only terms of the same sign may combine in (29). For example, the 11 faces of the pentagon in $\overline{\mathbf{GP}}[a, b, c]$ combine in the antipode formula as follows.



The Hopf monoid $\mathbf{GP}$ is commutative, so by Proposition 1.1.17 the antipode is involutory. The reader may enjoy verifying from (29) that this boils down to the fact that in the poset of faces of the polytope $\mathfrak{p}$, the Möbius function satisfies

$$\mu(\mathfrak{q}, \mathfrak{p}) = (-1)^{\dim \mathfrak{p} - \dim \mathfrak{q}}.$$

This holds since the poset of faces of a polytope is *Eulerian*.

CHAPTER 2

Permutahedra, associahedra, and inversion

2.1. The group of characters of a Hopf monoid

We return to the general setting of Hopf monoids of Section 1.1. We define the notion of characters on a Hopf monoid, and discuss how they assemble into a group. We use this general construction to settle a question of Loday [**66**] and a conjecture of Humpert and Martin [**58**] in Sections 2.4 and 3.2, respectively.

2.1.1. Characters.

DEFINITION 2.1.1. Let $\mathbf{H}$ be a connected Hopf monoid in vector species. A *character* ζ on $\mathbf{H}$ is a collection of linear maps

$$\zeta_I : \mathbf{H}[I] \to \Bbbk,$$

one for each finite set I, subject to the following axioms.

NATURALITY. For each bijection $\sigma : I \to J$ and $x \in \mathbf{H}[I]$, we have $\zeta_J\big(\mathrm{H}[\sigma](x)\big) = \zeta_I(x)$.

MULTIPLICATIVITY. For each $I = S \sqcup T$, $x \in \mathbf{H}[S]$ and $y \in \mathbf{H}[T]$, we have $\zeta_I(x \cdot y) = \zeta_S(x)\zeta_T(y)$.

UNITALITY. The map $\zeta_\emptyset : \mathbf{H}[\emptyset] \to \Bbbk$ sends $1 \in \Bbbk = \mathbf{H}[\emptyset]$ to $1 \in \Bbbk$: we have $\zeta_\emptyset(1) = 1$.

In most examples that interest us, naturality and unitality are trivial, and we can think of characters simply as multiplicative functions. When $\mathbf{H}$ is the linearization of a Hopf monoid H over set species, the characters ζ are constructed easily: one chooses arbitrarily the value $\zeta_I(h)$ for each object $h \in \mathrm{H}[I]$ that is indecomposable under multiplication, and then extend those values multiplicatively to all objects.

2.1.2. The character group. The characters of a connected Hopf monoid $\mathbf{H}$ have the structure of a group, called the *character group* $\mathbb{X}(\mathbf{H})$.

THEOREM 2.1.2. *Let $\mathbf{H}$ be a connected Hopf monoid in vector species. The set $\mathbb{X}(\mathbf{H})$ of characters of $\mathbf{H}$ is a group under the convolution product, defined by*

$$(\varphi\psi)_I(x) = \sum_{I=S\sqcup T} \varphi_S(x|_S)\psi_T(x/_S) \tag{30}$$

for characters φ and ψ. The identity ϵ is given by $\epsilon_I = 0$ if $I \neq \emptyset$ and $\epsilon_\emptyset(1) = 1$. The inverse of a character ζ is $\zeta \circ \mathrm{s}$, its composition with the antipode s of $\mathbf{H}$.

PROOF. We need to check that the convolution product of characters φ and ψ is indeed a character. Let $I = S \sqcup T$ be a decomposition and $z = x \cdot y$ for $x \in \mathbf{H}[S]$

and $y \in \mathbf{H}[T]$. Then, using the notation of (3) and the compatibility of the product and coproduct, we get

$$\begin{aligned}(\varphi\psi)_I(x \cdot y) &= \sum_{I=S'\sqcup T'} \varphi_{S'}((x \cdot y)|_{S'})\psi_{T'}((x \cdot y)/_{S'}) = \sum_{I=S'\sqcup T'} \varphi_{S'}(x|_A \cdot y|_C)\psi_{T'}(x/_A \cdot y/_C) \\ &= \sum_{\substack{S=A\sqcup B \\ T=C\sqcup D}} \varphi_A(x|_A)\varphi_C(y|_C)\psi_B(x/_A)\psi_D(y/_C) = (\varphi\psi)_S(x) \cdot (\varphi\psi)_T(y)\end{aligned}$$

as desired. It is easy to check that ϵ is indeed the identity, and the description of the inverse follows from [**2**, Definition 1.15]. □

There is a well-known analogous notion for Hopf algebras. If H is a Hopf algebra, then a character is a function from H to $\Bbbk$ that is multiplicative and unital. The characters of H form a group under the convolution $\phi\psi = m(\phi \otimes \psi)\Delta$. In this group, the inverse of a character ϕ is $\phi \circ \mathrm{S}$, where S is the antipode.

We mentioned in Section 1.1.8 that the antipode of a Hopf monoid plays the role of the inverse function in a group. The previous theorem is a concrete manifestation of that analogy. The following is another fundamental question.

PROBLEM 2.1.3. *Find an explicit description for the character group of a given Hopf monoid.*

We will now answer Problem 2.1.3 for permutahedra and associahedra, in Sections 2.2 and 2.3 respectively. This will establish the connection between these Hopf-theoretic structures on polytopes and the inversion of power series, as described in the introduction.

2.2. $\overline{\Pi}$: Permutahedra and the multiplication of power series

In this section we consider the Hopf monoid of permutahedra, and show that its character group is the group of formal power series under multiplication.

Recall that π_I is the standard permutahedron in $\mathbb{R}^I$. Let $\overline{\Pi}$ be the Hopf submonoid of $\overline{\mathrm{GP}}$ generated by the standard permutahedra, where the Hopf monoid $\overline{\mathrm{GP}}$ is the quotient of GP where we identify generalized permutahedra with the same normal fan.

LEMMA 2.2.1. *The coproduct of $\overline{\Pi}$ is given by*

$$\Delta_{S,T}(\pi_I) = (\pi_S\,,\, \pi_T).$$

for each decomposition $I = S \sqcup T$.

PROOF. From the description of the faces of permutahedron $\pi_I \subseteq \mathbb{R}^I$ in Section 1.3.4 we know that the maximal face of π_I in the direction of $\mathbf{1}_S$ is $\pi_{S,T} = \pi_I|_S \times \pi_I/_S$ where $\pi_I|_S$ is a translation of π_S and $\pi_I/_S$ is equal to π_T. The result follows. □

This implies, in particular, that the corresponding Hopf monoid in vector species is given by

$$\overline{\mathbf{\Pi}}[I] = \mathrm{span}\{\pi_{S_1} \times \cdots \times \pi_{S_k} \mid I = S_1 \sqcup \cdots \sqcup S_k\}. \tag{31}$$

We can now prove the main result of this section.

THEOREM 2.2.2. *The group of characters $\mathbb{X}(\overline{\mathbf{\Pi}})$ of the Hopf monoid of permutahedra is isomorphic to the group of exponential formal power series*

$$\left\{1 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots \mid a_1, a_2, \ldots \in \Bbbk\right\}$$

under multiplication.

PROOF. Since characters are multiplicative and invariant under relabeling, a character ζ of $\overline{\mathbf{\Pi}}$ is uniquely determined by the sequence $(1, z_1, z_2, \ldots)$ of values that it takes on the standard permutahedra of order $0, 1, 2, \ldots$. Here $z_n = \zeta_I(\pi_I)$ for $|I| = n$. (Recall that any character has $z_0 = \zeta_\emptyset(1) = 1$.) We encode this sequence in the exponential generating function $\zeta(t) = 1 + z_1 t + z_2 t^2/2! + z_3 t^3/3! + \cdots$. Conversely, any such formal power series determines a character of $\overline{\mathbf{\Pi}}$.

Now suppose that two characters φ, ψ and their convolution product $\varphi\psi$ give rise to sequences $(1, a_1, a_2, \ldots)$, $(1, b_1, b_2, \ldots)$, and $(1, c_1, c_2, \ldots)$, respectively. Consider any I with $|I| = n$. By (30) we have

$$c_n = (\varphi\psi)_I(\pi_I) = \sum_{I=S\sqcup T} \varphi_S(\pi_S)\psi_T(\pi_T) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

This is equivalent to

$$\varphi\psi(x) := \sum_{n\geq 0} c_n \frac{x^n}{n!} = \left(\sum_{k\geq 0} a_k \frac{x^k}{k!}\right)\left(\sum_{l\geq 0} b_l \frac{x^l}{l!}\right) =: \varphi(x)\psi(x),$$

as desired. □

Using Lemma 2.2.1 it is not difficult to see that the Hopf monoid of permutahedra $\overline{\Pi}$ is isomorphic to the Hopf monoid of set partitions Π. Theorem 1.6.1 then gives us a combinatorial formula for the antipode of the Hopf monoid of set partitions Π. We will carry out this computation in Section 5.6, and explain why the Fock functor $\overline{\mathcal{K}}$ takes the Hopf monoid or permutahedra $\overline{\mathbf{\Pi}}$ to the Hopf algebra of symmetric functions Λ.

2.3. $\overline{\mathcal{K}}(\overline{\mathrm{A}})$: Associahedra and the composition of power series

In this section we consider the Hopf algebra $\overline{\mathcal{K}}(\overline{\mathrm{A}})$ of Loday associahedra, and show that its character group is the group of formal power series under composition.

2.3.1. Loday's associahedron. The *associahedron* is "a mythical polytope whose face structure represents the lattice of partial parenthesizations of a sequence of variables" [**52**]. Stasheff [**95**] constructed it as an abstract cell complex in the context of homotopy theory and Milnor suggested that it could be realized as a polytope. There are now many different polytopal realizations due to Tamari, Stasheff, Haiman, Lee, and others; see [**27**] for a survey. We will focus on the following construction due to Loday [**65**] and, in this formulation, to Postnikov [**77**].

DEFINITION 2.3.1. Let I be a finite set and ℓ be a linear order on I. *Loday's associahedron* $\mathfrak{a}_\ell$ is the Minkowski sum

$$\mathfrak{a}_\ell = \sum_{i\leq j} \Delta_{[i,j]_\ell}$$

where $[i,j]_\ell = \{m \in I \mid i \le m \le j \text{ in } \ell\}$ is the interval from i to j for $i \le j$ in ℓ.

We let $\mathfrak{a}_n$ denote the Loday associahedron for the natural order of $[n]$. We state the following theorem for completeness, but the connection between the associahedron and parenthesizations will be irrelevant for now. We will return to this connection and its combinatorial consequences in Section 5.7.

THEOREM 2.3.2 ([**65**,**77**]). *Loday's associahedron $\mathfrak{a}_\ell$ is a simple polytope whose face poset is isomorphic to the poset of partial parenthesizations of a sequence of $n+1$ variables ordered by refinement. In particular, the number of vertices is the* Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

A key property of Loday's associahedron is the following.

LEMMA 2.3.3. *Let I be a finite set and ℓ a linear order on I. Let $I = S \sqcup T$ be a decomposition and let $T = T_1 \sqcup \cdots \sqcup T_k$ be the decomposition of T into maximal subintervals of ℓ. Then*

$$\mathfrak{a}_\ell\,|_S \equiv \mathfrak{a}_{\ell\,|S}, \qquad \mathfrak{a}_\ell/_S = \mathfrak{a}_{\ell\,|T_1} \times \cdots \times \mathfrak{a}_{\ell\,|T_k}$$

where $\equiv$ denotes normal equivalence and for each subset $U \subseteq I$, $\ell\,|U$ denotes the restriction of the linear order ℓ to U.

PROOF. Let us write $[i,j]$ for $[i,j]_\ell$ for simplicity. The maximal face of a Minkowski sum $P+Q$ in direction v is $(P+Q)_v = P_v + Q_v$ [**51**]. Therefore the $\mathbf{1}_S$-maximal face of $\mathfrak{a}_\ell$ is

$$(\mathfrak{a}_\ell)_{S,T} = (\mathfrak{a}_\ell)_{\mathbf{1}_S} = \sum_{i\le j} \big(\Delta_{[i,j]}\big)_{\mathbf{1}_S} = \sum_{i\le j \mid [i,j]\cap S\neq\emptyset} \Delta_{[i,j]\cap S} + \sum_{i\le j \mid [i,j]\subseteq T} \Delta_{[i,j]},$$

where the first summand lives in $\mathbb{R}^S$ and the second lives in $\mathbb{R}^T$, so they are $(\mathfrak{a}_\ell)|_S$ and $(\mathfrak{a}_\ell)/_S$, respectively. In $\mathbb{R}^T$ we have

$$(\mathfrak{a}_\ell)/_S = \sum_{i\le j \mid [i,j]\subseteq T} \Delta_{[i,j]} = \sum_{t=1}^{k} \sum_{i\le j \mid [i,j]\subseteq T_l} \Delta_{[i,j]} = \sum_{t=1}^{k} \mathfrak{a}_{\ell\,|T_t} = \mathfrak{a}_{\ell\,|T_1} \times \cdots \times \mathfrak{a}_{\ell\,|T_k}$$

as desired. In $\mathbb{R}^S$ we get

$$(\mathfrak{a}_\ell)|_S = \sum_{i\le j \mid [i,j]\cap S\neq\emptyset} \Delta_{[i,j]\cap S}.$$

Now notice that $[i,j]\cap S$ is always a subinterval of S with respect to the induced order $\ell\,|S$, and every such subinterval equals $[i,j]\cap S$ for some choice of $i \le j$ in ℓ. It follows that the Minkowski sum above involves the same summands as the Minkowski sum defining $\mathfrak{a}_{\ell\,|S}$ – possibly with different coefficients.

We now recall the fact that the normal fan $\mathcal{N}(P+Q)$ is the common refinement of $\mathcal{N}(P)$ and $\mathcal{N}(Q)$, while $\mathcal{N}(\lambda P) = \mathcal{N}(P)$ for any $\lambda > 0$ [**51**]. Therefore the normal fan of a Minkowski sum of scaled polytopes $\sum_i \lambda_i P_i$ does not depend on the scaling factors λ_i as long as they are all positive. This implies that $(\mathfrak{a}_\ell)|_S \equiv \mathfrak{a}_{\ell\,|S}$ as desired. □

The above description of $(\mathfrak{a}_\ell)_{S,T} = (\mathfrak{a}_\ell)|_S \times (\mathfrak{a}_\ell)/_S$ has a nice pictorial description. It is natural to arrange the summands of $\mathfrak{a}_n = \sum_{1\le i\le j\le n} \Delta_{[i,j]}$ into a staircase of size n, as shown in the left panel of Figure 1 for $n = 9$. To get the $\mathbf{1}_S$-maximal face $(\mathfrak{a}_n)_{S,T}$ we replace each summand $\Delta_{[i,j]}$ with $(\Delta_{[i,j]})_{\mathbf{1}_S}$. We can separate the resulting summands into a staircase above each one of the T_is – which

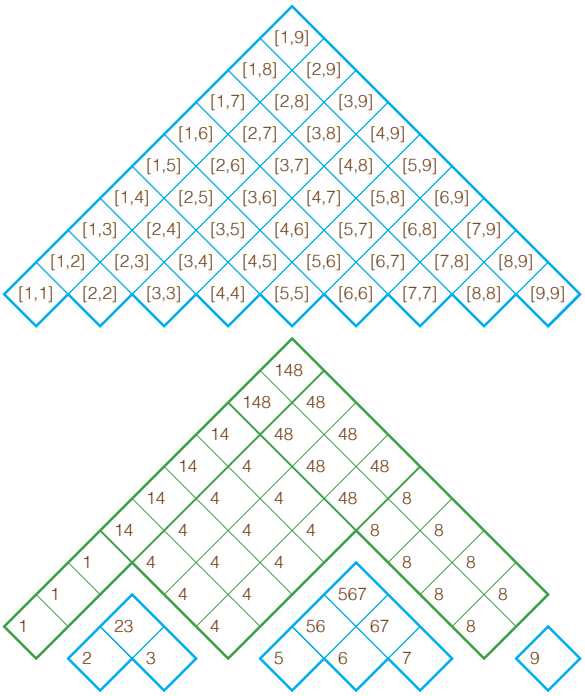


FIGURE 1. The Minkowski sum decompositions of $\mathfrak{a}_9$ and $(\mathfrak{a}_9)_{148,235679}$.

give the associahedra $\mathfrak{a}_{\ell|T_1}, \ldots, \mathfrak{a}_{\ell|T_k}$ – and a (fattened) staircase above S which gives a polytope normally equivalent to $\mathfrak{a}_{\ell|S}$. This is illustrated in the right panel of Figure 1 for the decomposition $[9] = \{1, 4, 8\} \sqcup \{2, 3, 5, 6, 7, 9\}$.

2.3.2. The Hopf algebra of Loday associahedra and its character group. We consider the Hopf monoid $\overline{\mathrm{A}}$ generated by associahedra inside the Hopf monoid $\overline{\mathrm{GP}}$ of generalized permutahedra modulo normal equivalence. We also consider the *Hopf algebra of associahedra* $\overline{\mathcal{K}}(\overline{\mathrm{A}})$ obtained by applying the Fock functor $\overline{\mathcal{K}}$ to the Hopf monoid $\overline{\mathrm{A}}$. Recall that the Fock functor $\overline{\mathcal{K}}$ identifies elements of $\mathrm{H}[n]$ in the same S_n orbit. Since every linear order ℓ on $[n]$ has a bijection to $[n]$, every corresponding Loday associahedron $\mathfrak{a}_\ell$ in $\mathbb{R}^n$ is identified with the *standard* Loday associahedron $\mathfrak{a}_n$ in the Hopf algebra $\overline{\mathcal{K}}(\overline{\mathrm{A}})$. Lemma 2.3.3 may be restated algebraically as follows.

COROLLARY 2.3.4. *The coproduct of a Loday associahedron in* $\overline{\mathrm{GP}}$ *is given by*

$$\Delta_{S,T}(\mathfrak{a}_\ell) = (\mathfrak{a}_{\ell|S}\,,\ \mathfrak{a}_{\ell|T_1} \times \cdots \times \mathfrak{a}_{\ell|T_k}).$$

for each linear order ℓ *on* I *and each decomposition* $I = S \sqcup T$*, where* $T = T_1 \sqcup \cdots \sqcup T_k$ *is the decomposition of* T *into maximal intervals of* ℓ*.*

In particular, it follows that $\overline{\mathrm{A}}[I]$ consists of products of associahedra:

$$\overline{\mathrm{A}}[I] = \{\mathfrak{a}_{\ell_1} \times \cdots \times \mathfrak{a}_{\ell_k} \mid \ell_i \text{ is a linear order on } S_i \text{ for } I = S_1 \sqcup \cdots \sqcup S_k\}. \tag{32}$$

We can now prove the main result of this section.

THEOREM 2.3.5. *The group of characters* $\mathbb{X}(\overline{\mathcal{K}}(\overline{\mathrm{A}}))$ *of the Hopf algebra of associahedra is isomorphic to the group of ordinary formal power series*

$$\left\{x + a_1x^2 + a_2x^3 + \cdots \mid a_1, a_2, \ldots \in \Bbbk\right\}$$

under composition.

PROOF. A character ζ of $\overline{\mathcal{K}}(\overline{\mathrm{A}})$ is uniquely determined by the sequence $(1, z_1, z_2, \ldots)$ where $z_n = \zeta_{[n]}(\mathfrak{a}_n)$. We encode that character in the formal power series $\zeta(t) = t + z_1t^2 + z_2t^3 + \cdots$. Conversely, any such formal power series gives a character of $\overline{\mathcal{K}}(\overline{\mathrm{A}})$.

Now suppose that two characters φ, ψ and their convolution product $\varphi\psi$ give sequences $(1, a_1, a_2, \ldots)$, $(1, b_1, b_2, \ldots)$, and $(1, c_1, c_2, \ldots)$, respectively. By (30) and Corollary 2.3.4,

$$c_{n-1} = (\varphi\psi)_{[n-1]}(\mathfrak{a}_{n-1}) = \sum_{[n-1]=S\sqcup T} \varphi_S(\mathfrak{a}_S)\psi_{T_1}(\mathfrak{a}_{T_1})\cdots\psi_{T_k}(\mathfrak{a}_{T_k}).$$

where $T = T_1 \sqcup \cdots \sqcup T_k$ is the decomposition of T into maximal subintervals of $[n-1]$, and $S, T_1, \ldots, T_k$ are listed in their standard linear order.

Each $(k-1)$-subset $S \subseteq [n-1]$ determines a "gap sequence" $i_1, \ldots, i_k$ where $i_j = |T_j|$ is the number of elements of $[n-1]$ in the gap between the $(j-1)$th and the jth elements of S. These non-negative integers satisfy $i_1 + \cdots + i_k + (k-1) = n-1$, and it is clear how to recover S from them. Since $\mathfrak{a}_{n-1}|_S \equiv \mathfrak{a}_{k-1}$ and $\mathfrak{a}_{n-1}/_S \equiv \mathfrak{a}_{i_1} \times \cdots \times \mathfrak{a}_{i_k}$ by Lemma 2.3.3, we may rewrite the above equation as

$$c_{n-1} = \sum_{k=1}^{n} \sum_{\substack{i_1,\ldots,i_k\geq 0 \\ i_1+\cdots+i_k+(k-1)=n-1}} a_{k-1}b_{i_1}\cdots b_{i_k}$$

which is equivalent to

$$\phi\psi(t) := \sum_{n\geq 1} c_{n-1}x^n = \sum_{k\geq 1} a_{k-1}\left(\sum_{i\geq 0} b_i x^{i+1}\right)^k =: \phi(\psi(t)),$$

as desired. □

A similar Hopf-theoretical result, without the connection to associahedra, is due to Doubilet, Rota, and Stanley [**34**].

In light of Corollary 2.3.4, Theorem 1.6.1 gives us a combinatorial formula for the antipode of the Hopf monoid of paths A. We will carry out this computation in Section 5.7, and relate the Hopf algebra of associahedra $\overline{\mathcal{K}}(\overline{\mathrm{A}})$ to the Faà di Bruno Hopf algebra F.

2.4. Inversion of formal power series and Loday's question

In this section we will show how the formulas for multiplicative and compositional inverses of formal power series follow directly from the Hopf algebraic structures $\overline{\Pi}$ and $\overline{\mathcal{K}}(\overline{\mathrm{A}})$ on permutahedra and associahedra, respectively.[1]

[1]More symmetrically, and slightly more complicatedly, we could use $\overline{\mathcal{K}}(\overline{\Pi})$ instead of $\overline{\Pi}$ here.

2.4.1. Multiplicative Inversion Formulas. As illustrated in the Introduction, the multiplicative inversion of power series is precisely given by the facial structure of permutahedra. We now explain this phenomenon.

THEOREM 2.4.1. (Multiplicative Inversion, Polytopal Version) *The multiplicative inverse of*

$$A(x) = 1+a_1x+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+\cdots \qquad \textit{is} \qquad \frac{1}{A(x)} = B(x) = 1+b_1x+b_2\frac{x^2}{2!}+b_3\frac{x^3}{3!}+\cdots$$

where

$$b_n = \sum_{F \text{ face of } \pi_n} (-1)^{n-\dim F} a_F$$

and we write $a_F = a_{f_1}\cdots a_{f_k}$ *for each face* $F \cong \pi_{f_1}\times\cdots\times\pi_{f_k}$ *of the permutahedron* π_n.

PROOF. Theorem 2.2.2 allows us to identify the formal power series $A(x) = \sum a_n x^n/n!$ and $1/A(x) = B(x) = \sum b_n x^n/n!$ with the characters α and β of the Hopf monoid $\overline{\mathbf{\Pi}}$ determined uniquely by

$$\alpha_{[n]}(\pi_n) = a_n, \qquad \beta_{[n]}(\pi_n) = b_n,$$

where π_n is the standard permutahedron in $\mathbb{R}[n]$. By Theorem 2.2.2, since $B(x) = 1/A(x)$, these characters are inverses of each other in the character group $\mathbb{X}(\overline{\Pi})$.

Recall that the inverse in the character group of any Hopf monoid is given by $\beta = \alpha\circ \mathrm{s}$ where s is the antipode. For $\overline{\Pi}$, this antipode is given by Theorem 1.6.1. Therefore

$$\begin{aligned} b_n = \beta_{[n]}(\pi_n) = (\alpha\circ \mathrm{s})_{[n]}(\pi_n) &= \alpha_{[n]}\left(\sum_{F \text{ face of } \pi_n} (-1)^{n-\dim F} F\right) \\ &= \sum_{F \text{ face of } \pi_n} (-1)^{n-\dim F} a_F, \end{aligned}$$

using the multiplicativity of the character α. □

THEOREM 2.4.2. (Multiplicative Inversion, Enumerative Version) *The multiplicative inverse of*

$$A(x) = 1+a_1x+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+\cdots \qquad \textit{is} \qquad \frac{1}{A(x)} = B(x) = 1+b_1x+b_2\frac{x^2}{2!}+b_3\frac{x^3}{3!}+\cdots,$$

where

$$b_n = \sum_{\langle 1^{m_1}2^{m_2}\cdots\rangle \vdash n} (-1)^{|m|} \binom{n}{\underbrace{1,1,\ldots}_{m_1},\underbrace{2,2,\ldots}_{m_2},\ldots} \binom{|m|}{m_1,m_2,\ldots} a_1^{m_1}a_2^{m_2}\cdots$$

summing over all partitions $\langle 1^{m_1}2^{m_2}\cdots\rangle = \underbrace{11\ldots}_{m_1}\underbrace{22\ldots}_{m_2}\cdots$ *of* n, *where* $|m| = m_1 + m_2 + \cdots$.

PROOF. Recall from Section 1.3.4 that the faces of π_n are in bijection with the compositions $(S_1,\ldots,S_k)$ of $[n]$, where the face $F = \pi_{S_1,\ldots,S_k} \cong \pi_{S_1}\times\cdots\times\pi_{S_k}$ corresponds to the composition $(S_1,\ldots,S_k)$. If we let m_i be the number of S_js of size i, then $n - \dim F = k = |m|$ and $a_F = a_1^{m_1}a_2^{m_2}\cdots$. Therefore the coefficient of this monomial is the number of compositions leading to block sizes $\langle 1^{m_1}2^{m_2}\cdots\rangle$.

There are $\binom{m_1+m_2+\cdots}{m_1,m_2,\ldots}$ ways of assigning these sizes to the parts $S_1,\ldots,S_k$ in some order. Having fixed that order, there are then $\binom{n}{1,1,\ldots,2,2,\ldots}$ ways of partitioning the elements of I into parts $S_1,\ldots,S_k$ of those respective sizes. The desired result follows. □

2.4.2. Compositional inversion formulas. Just as the facial structure of permutahedra tells us exactly how to compute the multiplicative inverse of a formal power series, the facial structure of associahedra tell us how to compute the compositional inverse.

THEOREM 2.4.3. (Lagrange Inversion, polytopal version) *The compositional inverse of*

$$C(x) = x + c_1x^2 + c_2x^3 + \cdots \qquad \textit{is} \qquad C^{\langle -1\rangle}(x) = D(x) = x + d_1x^2 + d_2x^3 + \cdots,$$

where

$$d_n = \sum_{F \text{ face of } \mathfrak{a}_n} (-1)^{n-\dim F} c_F$$

and we write $c_F = c_{f_1}\cdots c_{f_k}$ *for each face* $F \cong \mathfrak{a}_{f_1}\times\cdots\times\mathfrak{a}_{f_k}$ *of the associahedron* $\mathfrak{a}_n$.

PROOF. We proceed exactly as in the proof of Theorem 2.4.1. We identify the formal power series $C(x) = \sum c_{n-1}x^n$ and $C^{\langle -1\rangle}(x) = D(x) = \sum d_{n-1}x^n$ with the characters γ and δ of the Hopf algebra of associahedra $\overline{\mathcal{K}}(\overline{\mathrm{A}})$ determined uniquely by

$$\gamma_{[n]}(\mathfrak{a}_n) = c_n, \qquad \delta_{[n]}(\mathfrak{a}_n) = d_n$$

for the standard Loday associahedron $\mathfrak{a}_n$. By Theorem 2.3.5, since $C^{\langle -1\rangle}(x) = D(x)$, these characters are inverses in the character group $\mathbb{X}(\overline{\mathcal{K}}(\overline{\mathrm{A}}))$. Therefore $\delta = \gamma\circ\mathrm{S}$, and the result now follows from the antipode formula of Theorem 1.6.1. □

THEOREM 2.4.4. (Lagrange Inversion, enumerative version) *The compositional inverse of*

$$C(x) = x + c_1x^2 + c_2x^3 + \cdots \qquad \textit{is} \qquad C^{\langle -1\rangle}(x) = D(x) = x + d_1x^2 + d_2x^3 + \cdots,$$

where

$$d_n = \sum_{\langle 1^{m_1}2^{m_2}\cdots\rangle \vdash n} (-1)^{|m|} \frac{(n+|m|)!}{(n+1)!\,m_1!\,m_2!\cdots} c_1^{m_1}c_2^{m_2}\cdots$$

summing over all partitions $\langle 1^{m_1}2^{m_2}\cdots\rangle$ *of* n, *where* $|m| = m_1 + m_2 + \cdots$.

PROOF. This follows from Theorem 2.4.3 and the known correspondence between faces of associahedra and trees, which we reprove in a more general setting in Section 5.5. More precisely, the $(n-|m|)$-dimensional faces of the associahedron $\mathfrak{a}_n$ of type $\mathfrak{a}_1^{m_1}\times\mathfrak{a}_2^{m_2}\times\cdots$ are in bijection with the plane rooted trees that have $n+1$ leaves and m_i vertices of down-degree i for each $i \geq 1$. The result then follows from the fact [**92**, Theorem 5.3.10] that there are $(n+|m|)!/((n+1)!\,m_1!\,m_2!\cdots)$ such plane rooted trees. □

2.4.3. Loday's question and Schmitt's remark. It has long been known that Lagrange inversion is closely related to the enumeration of trees (or, equivalently, parenthesizings). In turn, this enumeration is related to the associahedron; see for example [**6**, **92**]. However, in 2005, Loday [**66**] asked for a direct explanation of the connection between Lagrange inversion and the associahedra:

> "There exists a short operadic proof of the [Lagrange inversion] formula which explicitly involves the parenthesizings, but it would be interesting to find one which involves the topological structure of the associahedron."

The associahedral statement and proof of the Lagrange inversion formula in Theorem 2.4.3 may be regarded as an answer to Loday's question. It is a combinatorics-free approach. Aside from the basic Hopf monoid architecture, it relies only on two key ingredients:

- our topological proof for the antipode of the associahedron (Theorem 1.6.1)
- the structure of Loday's associahedron with respect to the $\mathbf{1}_S$ directions (Lemma 2.3.3)

Interestingly, there are many other realizations of the associahedron as a generalized permutahedron [**28**, **29**, **54**, **56**, **57**, **64**, **74**, **75**]. These have isomorphic face posets, but they lead to different Hopf structures and different character groups. Surprisingly, to answer Loday's question within this algebro-polytopal context, Loday's realization of the associahedron is precisely the one that we need!

Relatedly, in the closing remarks to his 1987 paper [**83**], Schmitt wrote about the cancellation of 1s and -1s that leads to his Hopf algebraic proof of the Lagrange inversion formula [2]:

> "We believe that an understanding of exactly how these cancellations take place will not only provide a direct combinatorial proof of the Lagrange inversion formula, but may well yield analogous formulas for the antipodes of [...] other [...] Hopf algebras."

Schmitt's suggestion is very close to the philosophy of this project, though our approach is more geometric and topological than combinatorial. Applying the same point of view to other families of polytopes, we will obtain optimal formulas for the antipodes of many Hopf monoids throughout this monograph.

[2] According to Schmitt [**84**], the connection between antipodes of Hopf algebras and inversion of formal power series was foreshadowed by Joni in 1979.

CHAPTER 3

Submodular functions, graphs, matroids, and posets

3.1. SF: Submodular functions and generalized permutahedra

Generalized permutahedra arise in a multitude of settings, and can be used to model many combinatorial objects: graphs, matroids, posets, set partitions, paths, and many others. In this section we present one reason for the ubiquity of these polyhedra: generalized permutahedra are equivalent to submodular functions, which are central objects in optimization. These functions occur in numerous mathematical and real-world contexts, since they are characterized by a *diminishing returns* property that is natural in many settings.

3.1.1. Boolean functions. Let 2^I denote the collection of subsets of a finite set I. A *Boolean function* on I is an arbitrary function $z : 2^I \to \mathbb{R}$ such that $z(\emptyset) = 0$.

Let $\mathrm{BF}[I]$ denote the set of Boolean functions on I. To turn the species BF into a connected Hopf monoid, we first notice that $\mathrm{BF}[\emptyset]$ is indeed a singleton. Now fix a decomposition $I = S \sqcup T$. We make the following definitions.

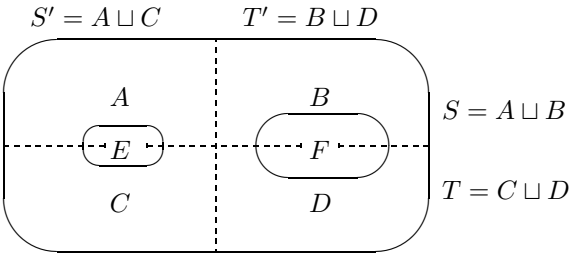
• The product of two Boolean functions $u \in \mathrm{BF}[S]$ and $v \in \mathrm{BF}[T]$ is the function $u \cdot v \in \mathrm{BF}[I]$ given by

$$(u \cdot v)(E) := u(E \cap S) + v(E \cap T) \text{ for } E \subseteq I. \tag{33}$$

• The coproduct of a Boolean function $z \in \mathrm{BF}[I]$ is $(z|_S, z/_S) \in \mathrm{BF}[S] \times \mathrm{BF}[T]$, where

$$z|_S(E) := z(E) \text{ for } E \subseteq S \quad \text{and} \quad z/_S(E) := z(E \cup S) - z(S) \text{ for } E \subseteq T. \tag{34}$$

The Hopf monoid axioms of Definition 1.1.5 are easily verified. To illustrate this, we check the compatibility between products and coproducts. Consider two compositions $I = S \sqcup T$ and $I = S' \sqcup T'$ as described in (3) and illustrated below, and choose $u \in \mathrm{BF}[S]$, $v \in \mathrm{BF}[T]$.



For any $E \subseteq S'$ we have

$$\begin{aligned}(u \cdot v)|_{S'}(E) &= (u \cdot v)(E) = u(E \cap S) + v(E \cap T) = u(E \cap A) + v(E \cap C) \\ &= u|_A(E \cap A) + v|_C(E \cap C) = \big(u|_A \cdot v|_C\big)(E),\end{aligned}$$

and for any $F \subseteq T'$ we have

$$\begin{aligned}(u \cdot v)/_{S'}(F) &= (u \cdot v)(F \cup S') - (u \cdot v)(S') \\ &= u\big((F \cup S') \cap S\big) + v\big((F \cup S') \cap T\big) - u(S' \cap S) - v(S' \cap T) \\ &= u\big((F \cap B) \cup A\big) - u(A) + v\big((F \cap D) \cup C\big) - v(C) \\ &= (u/_A)(F \cap B) + (v/_C)(F \cap D) = \big((u/_A) \cdot (v/_C)\big)(F).\end{aligned}$$

Thus $(u \cdot v)|_{S'} = (u|_A) \cdot (v|_C)$ and $(u \cdot v)/_{S'} = (u/_A) \cdot (v/_C)$, as needed.

3.1.2. Submodular functions and diminishing returns. A Boolean function z on I is *submodular* if

$$z(A \cup B) + z(A \cap B) \leq z(A) + z(B) \tag{35}$$

for every $A, B \subseteq I$. Submodular functions arise in many contexts in mathematics and applications, partly because submodularity is equivalent to a natural *diminishing returns property* that we now describe.

Suppose the Boolean function z measures some quantifiable *benefit* $z(A)$ associated to each subset $A \subseteq I$. Then the contraction $z/_S$ has a natural interpretation: for $e \notin S$,

$$z/_S(e) = z(S \cup e) - z(S) = \textit{marginal return} \text{ of adding } e \text{ to } S.$$

THEOREM 3.1.1 ([**87**, Theorem 44.1], Diminishing returns). *A Boolean function z on I is submodular if and only if for every $e \in I$ we have*

$$z/_S(e) \geq z/_T(e) \qquad \textit{for } S \subseteq T \subseteq I - e \qquad \textit{(diminishing returns)} \tag{36}$$

that is, the marginal return $z/_S(e)$ decreases as we add more elements to S.

From the algebraic point of view, submodular functions have a Hopf monoid structure because they are closed under products and coproducts.

THEOREM 3.1.2. *Let* $\mathrm{SF}[I]$ *denote the set of submodular functions on I. Then* SF *is a Hopf submonoid of* BF*, with the product and coproduct given by* (33) *and* (34)*.*

PROOF. It suffices to show that submodular functions are closed under the product (33) and coproduct (34) of Boolean functions as defined above. This is well known [**72**] and follows from Theorem 3.1.1; the details are left to the reader. □

3.1.3. Submodular functions and generalized permutahedra. The *base polytope* of a given Boolean function $z : 2^I \to \mathbb{R}$ is the set[1]

$$\mathcal{P}(z) := \{x \in \mathbb{R}^I \mid \sum_{i \in I} x_i = z(I) \text{ and } \sum_{i \in A} x_i \leq z(A) \text{ for all } A \subseteq I\}. \tag{37}$$

[1]It is worth remarking that, in Postnikov's work on generalized permutahedra [**77**], he writes the defining inequalities as $\sum_{i \in A} x_i \geq z'(A)$. The difference is unimportant thanks to the equality $\sum_{i \in I} x_i = z(I)$. Our convention affords a cleaner connection between generalized permutahedra and submodular functions.

For $x \in \mathbb{R}^I$ and $A \subseteq I$, we denote

$$x(A) = \sum_{i \in A} x_i.$$

We say the inequality $x(A) \leq z(A)$ is *optimal* for $\mathcal{P}(z)$ if $z(A)$ is the minimum value for which this inequality holds; that is, if $z(A)$ equals the maximum value of $x(A)$ over all x in the polytope $\mathcal{P}(z)$.

The following theorem collects several results from the literature, and plays a central role in this monograph.

THEOREM 3.1.3 ([**32**, **41**, **71**, **77**, **87**]). *For a polytope $\mathfrak{p}$ in $\mathbb{R}^I$, the following conditions are equivalent.*

(1) *The polytope $\mathfrak{p}$ is a generalized permutahedron.*
(2) *The normal fan $\mathcal{N}_\mathfrak{p}$ is a coarsening of the braid arrangement $\mathcal{B}_I$.*
(3) *Every edge of $\mathfrak{p}$ is parallel to the vector $e_i - e_j$ for some $i, j \in I$.*
(4) *There exists a submodular function $z : 2^I \to \mathbb{R}$ such that $\mathfrak{p} = \mathcal{P}(z)$.*

Furthermore, when these conditions hold, the submodular function z of part 4 is unique, and every definining inequality in (37) *is optimal.*

We will extend this result to possibly unbounded objects in Theorem 3.1.6, and provide references and a complete proof there. We are now ready to prove an important result about the Hopf monoid GP.

THEOREM 3.1.4. *The collection of maps*

$$\mathrm{SF}[I] \to \mathrm{GP}[I], \quad z \mapsto \mathcal{P}(z)$$

is an isomorphism of Hopf monoids in set species $\mathrm{SF} \cong \mathrm{GP}$.

PROOF. Theorem 3.1.3 shows that each one of those maps is bijective. It is not difficult to check that the products on SF and GP agree. To prove that the coproducts agree, we now check that restriction and contraction coincide in SF and GP.

Let $\mathfrak{p} = \mathcal{P}(z)$ be a generalized permutahedron in $\mathbb{R}^I$ and let $I = S \sqcup T$ be a decomposition. We need to show that the maximal face in direction $\mathbf{1}_S$ is $\mathfrak{p}_{S,T} = \mathcal{P}(z|_S) \times \mathcal{P}(z/_S)$. We prove the two inclusions.

$\supseteq$: First consider any point $x = (x_S, x_T) \in \mathcal{P}(z|_S) \times \mathcal{P}(z/_S)$. For any $A \subseteq I$ let $A_S = A \cap S$ and $A_T = A \cap T$, so that $A = A_S \sqcup A_T$. Then

$$\begin{aligned} x(A) &= x_S(A_S) + x_T(A_T) \leq z|_S(A_S) + z/_S(A_T) \\ &= z(A_S) + z(A_T \cup S) - z(S) = z(A \cap S) + z(A \cup S) - z(S) \leq z(A) \end{aligned}$$

by submodularity. In particular, for $A = I$ we get

$$x(I) = x_S(S) + x_T(T) = z|_S(S) + z/_S(T) = z(S) + z(T \cup S) - z(S) = z(I).$$

Therefore $x \in \mathfrak{p}$. On the other hand, for $A = S$ we get

$$x(S) = x_S(S) + x_T(\emptyset) = z|_S(S) + 0 = z(S)$$

which, in view of (37), implies that x is $\mathbf{1}_S$-maximal in $\mathfrak{p}$, that is, $x \in \mathfrak{p}_{S,T}$.

$\subseteq$: In the other direction, let $x \in \mathfrak{p}_{S,T}$. By Theorem 3.1.3, x attains the $\mathbf{1}_S$-optimal value $x(S) = z(S)$. Letting $x = (x_S, x_T) \in \mathbb{R}^S \times \mathbb{R}^T$, we then have

$$x_S(S) = x(S) = z(S) = z|_S(S),$$
$$x_T(T) = x(T) = x(I) - x(S) = z(I) - z(S) = z/_S(T).$$

Furthermore, for any $A \subseteq S$ and $B \subseteq T$,

$$x_S(A) = x(A) \leq z(A) = z|_S(A),$$
$$x_T(B) = x(B) = x(B \cup S) - x(S) \leq z(B \cup S) - z(S) = z/_S(B).$$

These observations imply that $x_S \in \mathcal{P}(z|_S)$ and $x_T \in \mathcal{P}(z/_S)$ as desired. □

A *polymatroid* is a submodular function $f : 2^I \to \mathbb{R}$ with $f(\emptyset) = 0$ that is non-negative ($f(S) \geq 0$ for all $\emptyset \neq S \subseteq I$) and non-decreasing ($f(S) \leq f(T)$ for all $S \subseteq T \subseteq I$) [**36**]. Its *base polytope* is the corresponding generalized permutahedron $\mathcal{P}(f)$. One can verify that a submodular function f is a polymatroid if and only if $\mathcal{P}(f)$ is in the positive orthant. Therefore, as polytopes modulo translation, the family of generalized permutahedra is the same as the family of base polytopes of polymatroids.

Definitions of restriction and contraction for polymatroids are given in [**36**] and [**32**]. They correspond to restriction of contraction of Boolean functions and of generalized permutahedra from Sections 1.4.1 and 3.1.1.

PROPOSITION 3.1.5. *Let* PM[I] *denote the set of polymatroids on* I. *Then* PM *is a Hopf submonoid of* SF.

PROOF. One readily verifies that polymatroids are closed under the product (33) and coproduct (34) of Boolean functions, from which the result follows. □

3.1.4. GP$^+$: Extended generalized permutahedra and extended submodular functions. We now extend the previous constructions to allow for unbounded polyhedra. Most of the results of this section may be found in Fujishige [**41**].

Let an *extended* Boolean function be a function $z : 2^I \to \mathbb{R} \cup \{\infty\}$ with $z(\emptyset) = 0$ and $z(I) \neq \infty$. We say z is *submodular* if

$$z(A \cup B) + z(A \cap B) \leq z(A) + z(B) \quad \text{for all } A, B \subseteq I \text{ such that } z(A), z(B) \text{ are finite.}$$

Extended submodular functions are also called *submodular systems* [**41**]. The *base polyhedron* of z is

(38)
$$\mathcal{P}(z) := \{x \in \mathbb{R}^I \mid \sum_{i \in I} x_i = z(I) \text{ and } \sum_{i \in A} x_i \leq z(A) \text{ for all } A \subseteq I \text{ with } z(A) < \infty\}.$$

Theorem 3.1.3 extends to this setting, providing a bijective correspondence between extended submodular functions and extended generalized permutahedra. We now survey this correspondence in Theorem 3.1.6, providing proofs for some statements which we were not able to find in the literature.

Define a *braid cone* to be a cone in $(\mathbb{R}^I)^* \cong \mathbb{R}^I$ cut out by inequalities of the form $y(i) \geq y(j)$ for $i, j \in I$. Define a *root subspace* of $\mathbb{R}^I$ to be a subspace spanned by vectors of the form $e_i - e_j$ for $i, j \in I$; these vectors are the *roots* of the root system $A_I = \{e_i - e_j \mid i, j \in I\}$ in the sense of Lie theory [**59**]. Define an *affine root subspace* of $\mathbb{R}^I$ to be a translate of a root subspace.

THEOREM 3.1.6 ([**41**, **77**, **87**]). *For a polyhedron* $\mathfrak{p}$ *in* $\mathbb{R}^I$*, the following are equivalent.*

(1) *The polyhedron* $\mathfrak{p}$ *is an extended generalized permutahedron.*
(2) *The normal fan* $\mathcal{N}_\mathfrak{p}$ *is a coarsening of* $(\mathcal{B}_I)|_C$*, the restriction of the braid arrangement* $\mathcal{B}_I$ *to some braid cone* C*.*
(3) *The affine span of every face of* $\mathfrak{p}$ *is an affine root subspace.*
(4) *There exists an extended submodular function* $z : 2^I \to \mathbb{R} \cup \{\infty\}$ *such that* $\mathfrak{p} = \mathcal{P}(z)$*.*

Furthermore, when these conditions hold, the extended submodular function z *of part 4 is unique, and every definining inequality in* (38) *is optimal.*

PROOF. We proceed in several steps.

1 $\Leftrightarrow$ **2**: This is Definition 1.3.8.

3 $\Leftrightarrow$ **4**: This is anticipated by Fujishige in [**41**, Thms. 3.15, 3.18, 3.22] and proved explicitly by Derksen and Fink in [**32**, Proposition 2.9] for *megamatroids*, where the function z is integral; their proof works for general z. In condition 3 they include the additional hypothesis that the polyhedron $\mathfrak{p}$ lies on a hyperplane of the form $\sum_{i\in I} x_i = r$ for some $r \in \mathbb{R}$, but this follows from the assumption that the affine span of $\mathfrak{p}$ is an affine root subspace.

2 $\Rightarrow$ **3**: Assume $\mathfrak{p}$ satisfies 2. Since $\mathbf{1} \in \mathbb{R}^I$ is in every braid cone, it is also in $\mathcal{N}_\mathfrak{p}(\mathfrak{p})$, so $\mathbf{1}(x) = \sum_{i\in I} x_i$ is constant on $\mathfrak{p}$.

Now let $\mathfrak{q}$ be any d-dimensional face of $\mathfrak{p}$ and write $\operatorname{aff}(\mathfrak{q}) = v + W$ for a vector v and a subspace W. We need to show that W is a root subspace. The normal face $\mathcal{N}_\mathfrak{p}(\mathfrak{q})$ contains a face F of the braid arrangement $\mathcal{B}_I$ of its same dimension, so $\operatorname{span}(\mathcal{N}_\mathfrak{p}(\mathfrak{q})) = \operatorname{span}(F)$ is the intersection of d independent hyperplanes $y(i_k) = y(j_k)$ for $1 \le k \le d$. We claim that $W = \operatorname{span}\{e_{i_k} - e_{j_k} \mid 1 \le k \le d\}$. Since both of these vector spaces are d-dimensional, it suffices to show that $e_{i_k} - e_{j_k} \in W$ for each k.

We have the following inequality description of $\mathcal{N}_\mathfrak{p}(\mathfrak{q})$:

$$\mathcal{N}_\mathfrak{p}(\mathfrak{q}) = \{y \in \mathbb{R}^I \mid y(q_1) = y(q_2) \text{ for } q_1, q_2 \in \mathfrak{q},\ \ y(q) \ge y(p) \text{ for } q \in \mathfrak{q}, p \in \mathfrak{p}\}$$

Since $y \in \mathcal{N}_\mathfrak{p}(\mathfrak{q})$ implies that $y(i_k) = y(j_k)$, $e_{i_k} - e_{j_k}$ must be a linear combination of vectors of the form $q_1 - q_2$ for $q_1, q_2 \in \mathfrak{q}$. But every such vector is in W, so $e_{i_k} - e_{j_k} \in W$ as desired.

(**3** + **4**) $\Rightarrow$ **2**: Let $\mathfrak{p}$ satisfy 3 and 4.

First we show that the support of the normal fan $\mathcal{N}_\mathfrak{p}$

$$C = \operatorname{supp}(\mathcal{N}_\mathfrak{p}) = \{y \in \mathbb{R}^I \mid \max_{p\in\mathfrak{p}} y(p) \text{ is finite }\},$$

is a braid cone. Let $D = \mathcal{N}_\mathfrak{p}(\mathfrak{q})$ be a codimension 1 face of $\mathcal{N}_\mathfrak{p}$ on the boundary of $\mathcal{N}_\mathfrak{p}$. Say $\mathfrak{q}$ is d-dimensional, and, in light of 3, let the affine span of $\mathfrak{q}$ be a translate of the subspace $W = \operatorname{span}\{e_{i_1} - e_{j_1}, \ldots, e_{i_d} - e_{j_d}\}$. We claim that $\operatorname{span}(D)$ is the intersection of the hyperplanes $y(i_k) = y(j_k)$ for $1 \le k \le d$. Since both subspaces have codimension d, it is enough to prove one inclusion. To do that, observe that if $y \in D$, then $y(q)$ is constant for $q \in \mathfrak{q}$, so $y(w) = 0$ for $w \in W$ and therefore $y(i_k) = y(j_k)$. The same statement is then true for any $y \in \operatorname{span}(D)$. We conclude that C can be described by inequalities of the form $y(i) \ge y(j)$, as desired.

Now that we know that $\mathcal{N}_{\mathfrak{p}}$ is supported on a braid cone C, we need to show that it is refined by the braid arrangement; that is, that for $y \in C$, the relative order of the coordinates of $y \in \mathbb{R}^I$ is enough to determine the maximum face $\mathfrak{p}_y$. But condition 4 tells us that $\mathfrak{p} = \mathcal{P}(z)$ for an extended submodular function z, and Fujishige showed that this family of functions may be optimized using the greedy algorithm, which only pays attention to the relative order of the coordinates of y [**41**, Thms. 3.15, 3.18]. The result follows.

Having proved the equivalence of 1, 2, 3, and 4, it remains to remark that the uniqueness and optimality of the defining equations (38) of $\mathfrak{p}$ are implicit in [**41**, Section 3]. □

REMARK 3.1.7. When $\mathfrak{p}$ is bounded, Theorem 3.1.6 reduces to Theorem 3.1.3. Condition 3 looks different in these two statements, but in this setting, the seemingly weaker condition that every edge is parallel to a root $e_i - e_j$ implies that every face spans an affine root subspace. The reason for this is that in a bounded polytope, every face is spanned by its edges. This is not true in general; some unbounded polytopes do not even have one-dimensional faces.

Let $\mathrm{SF}^+[I]$ be the set of extended submodular functions on I. To construct a connected Hopf monoid, we use essentially the same operations as in BF and SF. The only difference is that the contraction $z/_S$ of $z \in \mathrm{SF}^+[I]$ is no longer defined when $z(S) = \infty$. Therefore, we need to modify the coproduct by defining

$$\Delta_{S,T}(z) = \begin{cases} z|_S \otimes z/_S & \text{if } z(S) \neq \infty \\ 0 & \text{if } z(S) = \infty. \end{cases}$$

for a decomposition $I = S \sqcup T$. This definition forces us to work in the context of vector species. It is now straightforward to extend Theorem 3.1.4 to this context.

THEOREM 3.1.8. *The collection of maps*

$$\mathbf{SF}^+[I] \to \mathbf{GP}^+[I], \quad z \mapsto \mathcal{P}(z)$$

is an isomorphism of Hopf monoids in vector species $\mathbf{SF}^+ \cong \mathbf{GP}^+$.

3.2. G: Graphs and graphic zonotopes

In this section we revisit the Hopf monoid of graphs of Section 1.2.1, now taking a geometric perspective: we realize G as a submonoid of GP. The key idea is that every graph g is modeled by a generalized permutahedron Z_g called its graphic zonotope, and this model respects the Hopf structure of graphs. This geometric interpretation of the Hopf monoid $\mathbf{G}$ readily gives us the optimal formula for its antipode – obtained independently by Humpert and Martin [**58**] – and allows us to prove their conjecture from [**58**, Section 5].

3.2.1. Graphic zonotopes. Let g be a graph with vertex set I. Given $A \subseteq I$ and an edge e of g, we say that e is *incident* to A if either endpoint of e belongs to A. Consider the *incidence* function

$$\begin{aligned} \mathrm{inc}_g &: 2^I \to \mathbb{Z} \\ \mathrm{inc}_g(A) &= \text{number of edges and half-edges of } g \text{ incident to } A. \end{aligned}$$

For example, the incidence function of the graph 

is given by

$$\mathrm{inc}_g(\emptyset) = 0, \quad \mathrm{inc}_g(\{x\}) = 3, \quad \mathrm{inc}_g(\{y\}) = 2, \quad \text{and} \quad \mathrm{inc}_g(\{x,y\}) = 3.$$

The following result is well-known.

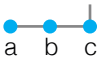
PROPOSITION 3.2.1. *For any graph g, the incidence function inc_g is submodular.*

PROOF. By Theorem 3.1.1 it suffices to observe that the marginal benefit of adding e to S:

$$(\mathrm{inc}_g)/_S(e) = \#\text{ of edges of } g \text{ incident to } e \text{ and not to } S$$

diminishes as we add elements to S. □

By Theorem 3.1.3 and (37), the submodular function inc_g gives rise to a generalized permutahedron $\mathcal{P}(\mathrm{inc}(g)) = Z_g$ which is called the *graphic zonotope* of g.

EXAMPLE 3.2.2. Revisiting Example 1.2.1, if g is the graph 

then the graphic zonotope $Z_g = \mathcal{P}(\mathrm{inc}_g)$ is given by

$$x_a+x_b+x_c = 3, \quad x_a+x_b \le 2, \quad x_b+x_c \le 3, \quad x_a+x_c \le 3, \quad x_a \le 1, \quad x_b \le 2, \quad x_c \le 2$$

and is shown in Figure 1. Note that the third and fifth inequalities are optimal but redundant.

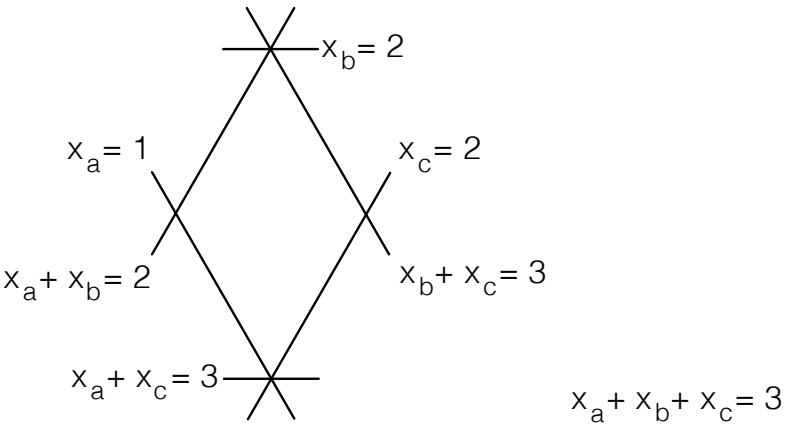


FIGURE 1. A graphic zonotope.

There is a useful alternative description of the zonotope of a graph.

PROPOSITION 3.2.3. [**77**, Proposition 6.3] *The zonotope $Z_g \subseteq \mathbb{R}^I$ of a graph g on I equals the Minkowski sum*

$$Z_g = \sum_{\{i\} \text{ half-edge of } g} \Delta_i + \sum_{\{i,j\} \text{ edge of } g} \Delta_{\{i,j\}}. \tag{39}$$

In particular, the zonotope of the complete graph K_I on the set I is a translation of the standard permutahedron π_I:

$$Z_{K_I} = \pi_I - e_I.$$

Note that the right hand side of (39) may have repeated summands.

The facial structure of graphic zonotopes can be described combinatorially [**77**, **90**] as we now recall. A *flat* f of a graph g is a set of edges with the property that for any cycle of g consisting of edges $e_1, \dots, e_k$, if $e_1, \dots, e_{k-1} \in f$ then $e_k \in f$.

For each flat f of g and each acyclic orientation o of g/f, let $g(f, o)$ be the graph obtained from g by keeping f intact, and replacing each edge $\{i, j\}$ not in f by the half-edge $\{i\}$ where $i \to j$ in the orientation o of g/f. The following result is essentially known [**90**].

LEMMA 3.2.4. *Let g be a graph with vertex set I. The faces of the zonotope $Z_g \subseteq \mathbb{R}^I$ are in bijection with the pairs of a flat f of g and an acyclic orientation o of g/f. The face corresponding to flat f and orientation o is $Z_{g(f,o)}$, and it is a translation of Z_f.*

PROOF. By (39), the maximal face of Z_g in the direction of $y \in \mathbb{R}^I$ is

$$(Z_g)_y = \sum_{\{i\} \in g} \Delta_i + \sum_{\substack{\{i,j\} \in g \\ y(i) = y(j)}} \Delta_{\{i,j\}} + \sum_{\substack{\{i,j\} \in g \\ y(i) > y(j)}} \Delta_i + \sum_{\substack{\{i,j\} \in g \\ y(i) < y(j)}} \Delta_j. \tag{40}$$

The vector y determines a flat f_y consisting of the edges $\{i, j\}$ of g such that $y(i) = y(j)$. It also determines an acyclic orientation o_y of g/f obtained by giving the edge $\{i, j\}$ the orientation $i \to j$ if $y(i) > y(j)$ or $i \leftarrow j$ if $y(i) < y(j)$. Clearly the maximal face $(Z_g)_y$ depends only on f_y and o_y. Furthermore, different choices of f_y and o_y determine different faces of $(Z_g)_y$, and every choice of a flat f of g and an acyclic orientation o of g/f can be realized by some vector y. This proves the desired one-to-one correspondence. It follows from (40) that $(Z_g)_y = Z_{g(f_y, o_y)}$ and that this is a translation of Z_{f_y}, as desired. □

3.2.2. Graphs as a submonoid of generalized permutahedra. Recall that G is the Hopf monoid of graphs, where $\mathrm{G}[I]$ is the set of graphs with vertex set I, where repeated edges and half-edges are allowed. For a decomposition $I = S \sqcup T$, the product of two graphs $g_1 \in \mathrm{G}[S]$ and $g_2 \in \mathrm{G}[T]$ is their disjoint union. The coproduct of $g \in \mathrm{G}[S]$ is $(g|_S, g/_S) \in \mathrm{G}[S] \times \mathrm{G}[T]$, where the restriction $g|_S \in \mathrm{G}[S]$ is the induced subgraph on S, while the contraction $g/_S \in \mathrm{G}[T]$ is obtained by keeping all edges incident to T, converting each edge from T to S into a half-edge on T.

Let $\mathrm{G}^{\mathrm{cop}}$ be the Hopf monoid co-opposite to G, as defined at the end of Section 1.1.2.

PROPOSITION 3.2.5. *The map* $\mathrm{inc} : \mathrm{G}^{\mathrm{cop}} \to \mathrm{SF} \xrightarrow{\cong} \mathrm{GP}$ *is an injective morphism of Hopf monoids.*

PROOF. We first check that inc is a morphism of Hopf monoids. Let $I = S \sqcup T$. Choose $g_1 \in \mathrm{G}[S]$ and $g_2 \in \mathrm{G}[T]$. Since there are no edges connecting S to T in $g_1 \cdot g_2$, an edge of $g_1 \cdot g_2$ incident to $A \subseteq I$ is either incident to $A \cap S$ or to $A \cap T$, but not both. Hence,

$$\mathrm{inc}_{g_1 \cdot g_2}(A) = \mathrm{inc}_{g_1}(A \cap S) + \mathrm{inc}_{g_2}(A \cap T) = (\mathrm{inc}_{g_1} \cdot \mathrm{inc}_{g_2})(A).$$

Thus, inc preserves products.

Let us now show that inc reverses coproducts. Choose $g \in \mathrm{G}[I]$. If $A \subseteq T$, then for any edge e of g incident to A there is a corresponding edge e' of $g/_S$ incident to A (possibly a half-edge, if the other endpoint of e belongs to S). Since every edge of $g/_S$ arises in this manner from an edge of g, we have

$$\mathrm{inc}_{g/_S}(A) = \mathrm{inc}_g(A) = (\mathrm{inc}_g)|_T(A).$$

Now, if $A \subseteq S$, notice that an edge of g incident to $A \cup T$ is either incident to T, or has both endpoints in S (and at least one endpoint in A), in which case it is an edge of $g|_S$. Therefore $\mathrm{inc}_g(A \cup T) = \mathrm{inc}_g(T) + \mathrm{inc}_{g|_S}(A)$, so

$$\mathrm{inc}_{g|_S}(A) = \mathrm{inc}_g(A \cup T) - \mathrm{inc}_g(T) = (\mathrm{inc}_g)/_T(A).$$

It follows that inc reverses restrictions and contractions, as desired.

To prove injectivity, note that if a and b are two distinct vertices of a graph g, then the number of edges of g between a and b is $\mathrm{inc}_g(\{a\}) + \mathrm{inc}_g(\{b\}) - \mathrm{inc}_g(\{a, b\})$. Also, the number of half-edges at a is $\mathrm{inc}_g(I) - \mathrm{inc}_g(I \setminus \{a\})$. These numbers determine g entirely. □

Remark 3.2.6. In graph theory one also considers the *cut function* cut_g defined by

$$\mathrm{cut}_g(A) = \text{ the number of edges of } g \text{ joining } A \text{ to } I \setminus A,$$

The map $g \mapsto \mathrm{cut}_g$ is not a morphism of Hopf monoids $\mathrm{G} \to \mathrm{SF}$: neither restrictions nor contractions are preserved. However, we do have $\mathrm{cut}_g(A) = 2 \cdot \mathrm{inc}_g(A) - \sum_{i \in A} \deg_g(i)$, where the *degree* $\deg_g(i)$ is the number of edges incident to vertex i. It follows from this that cut_g is submodular (a known result) and its generalized permutahedron $\mathcal{P}(\mathrm{cut}_g)$ is a scaling of $\mathcal{P}(\mathrm{inc}_g) = Z_g$ followed by a translation by the vector $-\deg_g \in \mathbb{R}^I$. Therefore the map $g \mapsto \mathrm{cut}_g$ does give a morphism of Hopf monoids $\mathrm{G} \to \overline{\mathrm{GP}}$; but since $\mathcal{P}(\mathrm{inc}_g)$ and $\mathcal{P}(\mathrm{cut}_g)$ are normally equivalent, this morphism does not teach us anything new about the Hopf monoid of graphs.

3.2.3. The antipode of graphs. In view of Proposition 3.2.5 and Theorem 1.6.1, the antipode of $\mathbf{G}$ is given by the facial structure of graphic zonotopes, as described in Lemma 3.2.4.

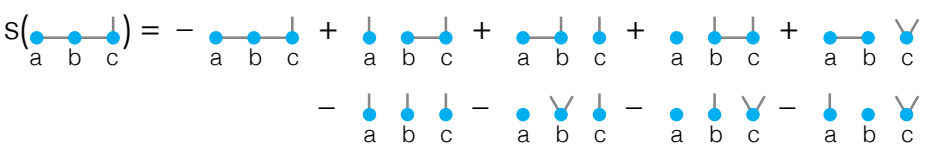
Corollary 3.2.7. *The antipode of the Hopf monoid of graphs* $\mathbf{G}$ *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If g is a graph on I then*

$$\mathrm{s}_I(g) = \sum_{f,o} (-1)^{c(f)} g(f, o),$$

summing over all pairs of a flat f of g and an acyclic orientation o of g/f, where $c(f)$ is the number of connected components of f.

Proof. This follows from Theorem 1.6.1 and Lemma 3.2.4, and the observation that the dimension of the zonotope Z_f is $|I| - c(F)$. □

Example 3.2.8. Let us revisit Example 1.2.1. The formula



is the algebraic manifestation of the face structure of the graphic zonotope of Example 3.2.2 which consists of one parallelogram, four edges, and four vertices. These nine faces are the graphic zonotopes of the nine graphs occurring in the expression above.

3.2.4. Simple graphs. A graph is *simple* if it has no half-edges or multiple edges. Let $\mathrm{SG}[I]$ denote the set of all simple graphs with vertex set I. Then SG is a subspecies of G, but it is not a Hopf submonoid because a contraction of a simple graph need not be simple.

To remedy this situation, consider the *simplification* map

$$\mathrm{G}[I] \to \mathrm{SG}[I], \quad g \mapsto g'.$$

which removes half-edges and edge multiplicities: in g' there is a unique edge joining two vertices a and b if and only if $a \neq b$ and there is at least one edge joining a and b in g. This defines a surjective morphism of species

$$\mathrm{G} \twoheadrightarrow \mathrm{SG}.$$

The Hopf monoid structure of G descends to SG via this map, so that SG is a quotient Hopf monoid of G. In SG, products and contractions have the same description as in G, while restrictions now coincide with contractions. Therefore SG is cocommutative.

The (linearization of the) Hopf monoid SG appears (with different notation) in [**2**, Section 13.2]. A closely related structure was first considered by Schmitt [**84**, Example 3.3.(3)].

PROPOSITION 3.2.9. *There is a commutative diagram of morphisms of Hopf monoids as follows.*

$$\begin{array}{ccc} \mathrm{G}^{\mathrm{cop}} & \hookrightarrow & \mathrm{GP} \\ \downarrow & & \downarrow \\ \mathrm{SG}^{\mathrm{cop}} & \hookrightarrow & \overline{\mathrm{GP}} \end{array}$$

PROOF. Simplification gives the vertical map $\mathrm{G}^{\mathrm{cop}} \twoheadrightarrow \mathrm{SG}^{\mathrm{cop}}$ while the map $\mathrm{GP} \twoheadrightarrow \overline{\mathrm{GP}}$ identifies generalized permutahedra with the same normal fan. The top map $\mathrm{G}^{\mathrm{cop}} \hookrightarrow \mathrm{GP}$ is given by Proposition 3.2.5, while the bottom map $\mathrm{SG}^{\mathrm{cop}} \hookrightarrow \overline{\mathrm{GP}}$ sends a simple graph to the normal equivalence class of its zonotope. To verify that the diagram commutes, we need to show that if g is a graph and g' is its simplification, then Z_g and $Z_{g'}$ are normally equivalent.

By (39), the normal fan $\mathcal{N}(Z_g)$ is the common refinement of the fans $\mathcal{N}(\Delta_{\{i\}})$ for all half-edges $\{i\}$ and $\mathcal{N}(\Delta_{\{i,j\}})$ for all edges $\{i,j\}$. This common refinement is unaffected by the removal of the former fans (which are trivial) and by the removal of repetitions of the latter fans. Therefore $\mathcal{N}(Z_g) = \mathcal{N}(Z_{g'})$ as desired. □

COROLLARY 3.2.10. *The antipode of the Hopf monoid of simple graphs* SG *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If g is a simple graph on I then*

$$\mathrm{s}_I(g) = \sum_{f \text{ flat of } g} (-1)^{c(f)} a(g/f)\, f$$

where $a(g/f)$ is the number of acyclic orientations of the contraction g/f and $c(f)$ is the number of connected components of f.

PROOF. This follows from Corollary 3.2.7 and the observation that when g is a simple graph, the simplification of $g(f, o)$ is f. □

An equivalent formula for Hopf algebras was also obtained by Humpert and Martin [**58**] through a clever inductive argument. In the context of Hopf algebras isomorphic graphs are identified, so to find the coefficient of a particular graph h in $\mathrm{s}_I(g)$ one has to overcome the additional problem of identifying all flats of g isomorphic to h. This is one reason to prefer working with Hopf monoids instead of Hopf algebras in combinatorial contexts. See also Remark 3.3.7.

3.2.5. Characters of complete graphs and a conjecture of Humpert and Martin. For each $k \in \mathbb{C}$ let ξ_k be the character on $\mathbf{G}$ given by $(\xi_k)_I(g) = k^{|I|}$ for any graph g on vertex set I. Let ζ be the character on $\mathbf{G}$ where $\zeta_I(g)$ equals 1 if g has no edges and 0 otherwise. For each $k \in \mathbb{C}$ and $c \in \mathbb{Z}$ let $\xi_k\zeta^c$ denote the convolution product of ξ_k and ζ^c in $\mathbf{G}$.

Recall that a *derangement* of I is a permutation of I without fixed points, and an *arrangement* is a permutation of a subset of I. The following formulas were conjectured by Humpert and Martin [**58**].

THEOREM 3.2.11 ([**58**, Conjecture (27)]). *Let K_n be the complete graph on n vertices. Then*

$$\sum_{n\geq 0}(\xi_k\zeta^c)(K_n)\frac{x^n}{n!} = e^{kx}(1+x)^c$$

for any complex number k and integer c. In particular,

$$(\xi_1\zeta^{-1})(K_n) = (-1)^n D_n \qquad (\xi_{-1}\zeta^{-1})(K_n) = (-1)^n A_n$$

where D_n and A_n are the numbers of derangements and arrangements of $[n]$ respectively.

PROOF. Since the graphic zonotope of a complete graph K_I is a translation of the standard permutahedron π_I by Proposition 3.2.3, the Hopf submonoid of $\mathbf{G}$ generated by complete graphs is isomorphic to the Hopf submonoid $\overline{\mathbf{\Pi}}$ of $\overline{\mathbf{GP}}$ generated by standard permutahedra, considered in Section 2.2. We may then regard ξ_k and ζ as characters on $\overline{\mathbf{\Pi}}$. This allows us to carry out the required computations in the character group $\mathbb{X}(\overline{\mathbf{\Pi}})$, where they become straightforward.

By Theorem 2.2.2, convolution of characters of $\overline{\mathbf{\Pi}}$ corresponds to multiplication of their exponential generating functions; therefore

$$\sum_{n\geq 0}(\xi_k\zeta^c)(\pi_n)\frac{x^n}{n!} = \left(\sum_{n\geq 0}\xi_k(\pi_n)\frac{x^n}{n!}\right)\left(\sum_{n\geq 0}\zeta(\pi_n)\frac{x^n}{n!}\right)^c = e^{kx}(1+x)^c$$

as desired. By comparing this with the generating functions

$$\sum_{n\geq 0}(-1)^n D_n\frac{x^n}{n!} = \sum_{n\geq 0}\left[(-1)^n\left(\sum_{i=0}^{n}(-1)^i\frac{n!}{i!}\right)\frac{x^n}{n!}\right] = \left(\sum_{i\geq 0}\frac{x^i}{i!}\right)\left(\sum_{j\geq 0}(-1)^j x^j\right)$$
$$= e^x(1+x)^{-1}$$

and

$$\sum_{n\geq 0}(-1)^n A_n \frac{x^n}{n!} = \sum_{n\geq 0}\left[(-1)^n\left(\sum_{i=0}^{n}\frac{n!}{i!}\right)\frac{x^n}{n!}\right] = \left(\sum_{i\geq 0}\frac{(-1)^i x^i}{i!}\right)\left(\sum_{j\geq 0}(-1)^j x^j\right)$$
$$= e^{-x}(1+x)^{-1}$$

we obtain the remaining two formulas. □

3.3. M: Matroids and matroid polytopes

Similarly to graphs, matroids also have a polyhedral model called its *matroid polytope*, due to Edmonds [**36**]. This model respects the Hopf-algebraic structure of matroids, introduced in 1982 by Joni and Rota [**60**] and further studied by Schmitt [**84**]. We now employ the geometric perspective to compute, for the first time, the optimal formula for the antipode of matroids.

3.3.1. Matroid polytopes. Let m be a matroid on ground set I. The *rank* of $A \subseteq I$ in m, denoted $\text{rank}_m(A)$, is the cardinality of any maximal independent set of m contained in A. The matroid axioms guarantee that this is well-defined and moreover, that the function

$$\text{rank}_m : 2^I \to \mathbb{N}$$

is submodular [**72**, Lemma 1.3.1]; indeed, the marginal benefit of adding e to S

$$(\text{rank}_m)/_S(e) = \begin{cases} 1 & \text{if } e \text{ is independent of } S, \\ 0 & \text{if } e \text{ is dependent on } S \end{cases}$$

weakly decreases as we add elements to S.

By Theorem 3.1.3 and (37), the submodular function rank_m gives rise to a generalized permutahedron $\mathcal{P}(\text{rank}_m) = \mathcal{P}(m)$ which is called the *matroid polytope* of m. This polytope has an elegant vertex description.

PROPOSITION 3.3.1. [**36**, **44**] *The matroid polytope* $\mathcal{P}(\text{rank}_m) = \mathcal{P}(m)$ *of a matroid* m *on* I *is given by*

$$\mathcal{P}(m) = \text{conv}\,\{e_{b_1} + \cdots + e_{b_r} \mid \{b_1, \ldots, b_r\} \text{ is a basis of } m\} \subseteq \mathbb{R}^I,$$

where $\{e_i \mid i \in I\}$ *is the standard basis. Furthermore, every basis gives a vertex of* $\mathcal{P}(m)$.

This construction goes back to Edmonds [**36**] in optimization, and later to Gel'fand, Goresky, MacPherson, and Serganova [**44**] in algebraic geometry. In what follows, we will sometimes identify a matroid m with its matroid polytope $\mathcal{P}(m)$.

EXAMPLE 3.3.2. Revisiting Example 1.2.2, let m be the matroid of rank 2 on $\{a, b, c, d\}$ whose only non-basis is $\{c, d\}$. The matroid polytope $\mathcal{P}(m)$, shown in Figure 3.3.1, is given by the inequalities:

$$x_a + x_b + x_c + x_d = 2, \quad x_a + x_b, x_a + x_c, x_a + x_d, x_b + x_c, x_b + x_d \leq 2,$$
$$x_c + x_d \leq 1, \quad x_a, x_b, x_c, x_d \leq 1$$

There does not seem to be a simple and purely combinatorial indexing for the faces of the matroid polytope $\mathcal{P}(m)$. For a non-bijective description of these faces, see [**10**, Proposition 2] or [**22**, Problem 1.26].

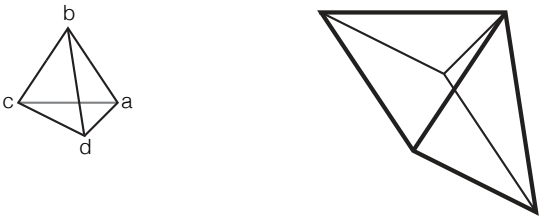


FIGURE 2. The matroid polytope of the matroid of Example 3.3.2.

3.3.2. Matroids as a submonoid of generalized permutahedra. Recall that M is the Hopf monoid of matroids, where $\mathrm{M}[I]$ is the set of matroids on ground set I. For a decomposition $I = S \sqcup T$, the product of two matroids $m_1 \in \mathrm{M}[S]$ and $m_2 \in \mathrm{M}[T]$ is their direct sum $m_1 \oplus m_2 \in \mathrm{M}[I]$. The coproduct of a matroid $m \in \mathrm{M}[I]$ is $(m|_S, m/_S)$, where $m|_S \in \mathrm{M}[S]$ and $m/_S \in \mathrm{M}[T]$ are the restriction and contraction of m with respect to S, respectively.

PROPOSITION 3.3.3. *The map* $\mathrm{rank} : \mathrm{M} \to \mathrm{SF} \xrightarrow{\cong} \mathrm{GP}$ *is an injective morphism of Hopf monoids.*

PROOF. The descriptions for the rank function of the direct sum, restriction and contraction of matroids in [**72**, Propositions 3.1.5, 3.1.7, 4.2.17] imply that rank is a morphism of Hopf monoids. Injectivity holds since the rank function determines the matroid uniquely. □

More widely, by Proposition 3.1.5, basis polytopes of polymatroids also form a submonoid of generalized permutahedra.

3.3.3. The antipode of matroids. We now give a formula for the antipode of matroids in Proposition 3.3.3 and Theorem 1.6.1. The result is expressed in terms of the facial structure of matroid polytopes.

Every matroid m has a unique maximal decomposition as a direct sum of smaller matroids. Let $c(m)$ be the number of summands, which are called the *connected components* of m [**72**, Section 4].

THEOREM 3.3.4. *The antipode of the Hopf monoid of matroids* M *is given by the following* ***cancellation-free*** *and* ***combination-free*** *formula. If m is a matroid on I, then*

$$\mathrm{s}_I(m) = \sum_{n \leq m} (-1)^{c(n)}\, n, \tag{41}$$

where we sum over all the nonempty faces n of the matroid polytope of m.

PROOF. This is an immediate consequence of Theorem 1.6.1, taking into account that the dimension of a matroid polytope $\mathcal{P}(m)$ on I equals $|I| - c(m)$ [**36**]. □

As mentioned earlier, there seems to be no simple combinatorial indexing of the faces of a matroid polytope, and hence no purely combinatorial counterpart of this formula.

The (discrete and algebraic) geometric point of view on matroids, initiated in [**36**] and [**44**], has evolved into a central component of matroid theory thanks to the natural appearances of matroid polytopes in various settings in optimization,

algebraic geometry, and tropical geometry. Theorem 3.3.4 shows that this geometric point of view also plays an essential role here: if one wishes to fully understand the Hopf algebraic structure of matroids, it becomes indispensable to view them as polytopes.

3.3.4. The Hopf algebra of matroids. The Fock functor sends the Hopf monoid M to the Hopf algebra of (isomorphism classes of) matroids defined by Joni and Rota [**60**, Section XVII] and also studied by Schmitt [**85**, Section 15]. Theorem 3.3.4 answers the open question of determining the optimal formula for the antipode of a matroid: it is simply the signed sum of the faces of the matroid polytope.

THEOREM 3.3.5. *In the Hopf algebra of (isomorphism classes of) matroids, the antipode of a matroid m is*

$$\mathrm{s}(m) = \sum_n (-1)^{c(n)} a(m:n)\, n, \tag{42}$$

where $c(n)$ is the number of components of a matroid n and $a(m:n)$ is the number of faces of the matroid polytope $\mathcal{P}(m)$ which are congruent to $\mathcal{P}(n)$.

PROOF. This is an immediate consequence of Theorem 3.3.4. □

EXAMPLE 3.3.6. Let us revisit Example 1.2.2. The formula is the algebraic

manifestation of the face structure of the corresponding matroid polytope, which is a square pyramid. It has one full-dimensional face, 5 two-dimensional faces (in matroid isomorphism classes of sizes 2, 1, 2), 8 edges (in one isomorphism class), and 5 vertices (in one isomorphism class).

REMARK 3.3.7. Theorems 3.3.4 and 3.3.5 illustrate an important advantage of working with Hopf monoids instead of Hopf algebras.

To try to discover (42), we might compute a few small examples and try to find a pattern. After witnessing unexpected cancellations and unexplained groupings of equal terms, we are left with coefficients $a(m:n)$ that are very hard to identify; in fact, we do not know any enumerative properties of these coefficients.

If, instead, we work in the context of Hopf monoids, a coefficient equal to 5 in (42) comes from a sum 1+1+1+1+1 in (41) where each 1 is indexed combinatorially; this additional granularity allows us to identify each term contributing to (41), and to then combine them to obtain (42).

However, for matroids, the geometric lens is crucial – even in the context of Hopf monoids. It is not easy to identify the individual terms of (41) if one is not thinking about the matroid polytope, whose faces have no simple combinatorial description.

A cancellation-free but not combination-free formula for the antipode of the Hopf algebra of matroids appears in [**24**, **25**].

3.3.5. Graphical matroids and another Hopf monoid of graphs. Any family of matroids which is closed under direct sums, restriction, and contraction forms a Hopf submonoid of M. Many important families of matroids satisfy these properties and have the structure of a Hopf monoid; for instance: linear matroids over a fixed field, graphical matroids, algebraic matroids over a fixed field, gammoids, and lattice path matroids [**20**, **72**, **101**]. In particular, the Hopf monoid of graphical matroids is closely related to a third Hopf monoid of graphs, which we now describe.

For a finite set I, let $\Gamma[I]$ be the set of graphs with edges labeled by I, with unlabeled vertices, and without isolated vertices. To define a product and coproduct on Γ, let $I = S \sqcup T$ be a decomposition. The product $\gamma_1 \cdot \gamma_2 \in \Gamma[I]$ is the (disjoint) union of the graphs $\gamma_1 \in \Gamma[S]$ and $\gamma_2 \in \Gamma[T]$. The coproduct $\Delta_{S,T}(\gamma) = (\gamma|_S, \gamma/_S)$ is given by the standard notions of restriction and contraction from graph theory, which are defined as follows. The restriction $\gamma|_S \in \Gamma[S]$ is obtained from $\gamma \in \Gamma[I]$ by removing all edges in T and all vertices not incident to S. The contraction $\gamma/_S \in \Gamma[S]$ is obtained by contracting each edge e in S from $\gamma \in \Gamma[I]$ – removing e and identifying its endpoints – and removing any isolated vertices that remain.

The two Hopf monoids of graphs Γ and G that we have discussed are not directly related; in fact, they differ already as species. Instead, we have a morphism of Hopf monoids

$$\Gamma \to \mathrm{M}$$

mapping each graph γ to its graphical matroid, which is the set of spanning trees of γ [**72**]. We do not know further properties of the Hopf monoid Γ, in particular, because we are not aware of any results on graphical matroid polytopes.

3.4. P: Posets and poset cones

Similarly to graphs and matroids, posets also have a polyhedral model that respects the Hopf algebra structure introduced by Schmitt in 1994 [**85**]. We use this geometric model to give an optimal combinatorial formula for the antipode of posets.

3.4.1. Poset cones. A $\{0, \infty\}$ *function* on I is a Boolean function $z : 2^I \to \{0, \infty\}$ such that $z(\emptyset) = z(I) = 0$. Its *support* is $\mathrm{supp}(z) = \{J \subseteq I \mid z(J) = 0\}$. For a $\{0, \infty\}$ *function* z on I,

$$z \text{ is submodular} \iff \text{if } A, B \in \mathrm{supp}(z) \text{ then } A \cup B, A \cap B \in \mathrm{supp}(z). \tag{43}$$

For each poset p on I we define the *lower set* function

$$\mathrm{low}_p : 2^I \mapsto \mathbb{R} \cup \{\infty\}, \quad \mathrm{low}_p(J) = \begin{cases} 0 & \text{if } J \text{ is a lower set of } p, \\ \infty & \text{if } J \text{ is not a lower set of } p. \end{cases}$$

This is an extended submodular function since the family of lower sets of p is closed under unions and intersections.

By Theorem 3.1.6 and (38), the submodular function low_p gives rise to an extended generalized permutahedron

$$\mathcal{P}(p) := \mathcal{P}(\mathrm{low}_p) = \{x \in \mathbb{R}^I \mid \sum_{i \in I} x_i = 0 \text{ and } \sum_{a \in A} x_a \leq 0 \text{ for every lower set } A \text{ of p}\}$$

which we call the *poset cone* of p. This cone has an elegant description in terms of generators. Dobbertin proved an analogous result for a related polytope in [**33**].

PROPOSITION 3.4.1. *The poset cone of a poset p is given by*

$$\mathcal{P}(p) = \text{cone}\{e_i - e_j \mid i > j \text{ in } p\}$$

where $\{e_i \mid i \in I\}$ is the standard basis of $\mathbb{R}^I$. The generating rays of $\mathcal{P}(p)$ are given by the roots $e_i - e_j$ corresponding to the cover relations $i \gtrdot j$ of p.

PROOF. Recall the notation

$$x(A) = \sum_{i \in A} x_i$$

for $x \in \mathbb{R}^I$ and $A \subseteq I$. We prove both containments:

$\supseteq$: Let $i > j$ in p. Every order ideal A that contains i must also contain j, so $e_i - e_j$ satisfies $x(A) \leq 0$. This implies that $\text{cone}\{e_i - e_j \mid i > j \in p\} \subseteq \mathcal{P}(p)$.

$\subseteq$: We will need the following lemma.

LEMMA 3.4.2. *Let $x \in \mathcal{P}(p)$. Let i be a maximal element of p such that $x_i \neq 0$, and let $i_1, \ldots, i_k$ be the elements covered by i in p. We can write*

$$x = x' + \lambda(e_i - e_j)$$

for some $x' \in \mathcal{P}(p)$, some element $j \lessdot i$, and some number $\lambda > 0$ which is a linear combination of the x_is.

PROOF OF LEMMA 15.2. The maximality of i and the fact that $p_{\geq i} := \{j \in p \mid j \geq i\}$ is an upper set imply that

$$x_i = \sum_{j \in p_{\geq i}} x_j = x(p_{\geq i}) > 0. \tag{44}$$

Let $i_1, \ldots, i_k$ be the elements covered by i in p. We claim that

(45) there exists an index $1 \leq a \leq k$ with $x(I_a) < 0$ for every lower set $I_a \ni i_a$.

We prove this claim by contradiction. If that was not the case, then for every $1 \leq a \leq k$ we would have a lower set $I_a \ni i_a$ such that $x(I_a) = 0$. Now, we observe that

(46)
if A and B are lower sets with $x(A) = x(B) = 0$, then $x(A \cup B) = x(A \cap B) = 0$.

This observation follows from the fact that $A \cup B$ and $A \cap B$ are lower sets, so they satisfy $x(A \cup B) \leq 0$ and $x(A \cap B) \leq 0$, while also satisfying $x(A \cup B) + x(A \cap B) = x(A) + x(B) = 0$. Applying (46) repeatedly, we see that $I_1 \cup \cdots \cup I_k$ is a lower set with $x(I_1 \cup \cdots \cup I_k) = 0$. But then we observe that $I_1 \cup \cdots \cup I_k \cup i$ is also a lower set, so we get

$$x_i = x(I_1 \cup \cdots \cup I_k \cup i) \leq 0,$$

contradicting (44).

Having proved (44) and (45), let $1 \leq a \leq k$ be as in (45). Then

$$\lambda := \min(\{x_i\} \cup \{-x(I_a) \mid I_a \text{ is a lower set containing } i_a\}) > 0,$$

and define $x' = x - \lambda(e_i - e_{i_a})$ as required. To conclude, it remains to prove that $y \in \mathcal{P}(p)$. To do this, let J be any lower set of p. If J contains both i_a and i, or if it contains neither i_a nor i, then we have $x'(J) = x(J) \leq 0$. On the other hand, if J contains i_a but not i, then $x'(J) = x(J) + \lambda \leq 0$ by the definition of λ, since J is a lower set containing i_a. It follows that $x' \in \mathcal{P}(p)$, concluding the proof of the lemma. □

Now we need to prove that any $x \in \mathcal{P}(p)$ is a positive linear combination of vectors of the form $e_i - e_j$ such that $i < j$ in p. Since the rationals are a dense subset of the reals and the cones we are considering are closed, it suffices to prove this when all entries of x are rational. We proceed by induction on the number of positive entries of x.

Let i be a maximal element of p with $x_i > 0$. Write $x = x' + \lambda(e_i - e_j)$ for $\lambda > 0$ and $i \gtrdot j$ as in the lemma, and note that $x'_i < x_i$. If $x'_i > 0$, use the lemma again to write $x' = x'' + \lambda'(e_i - e_{j'})$ for $\lambda' > 0$ and $i \gtrdot j'$, and note that $x''_i < x'_i < x_i$. We can continue applying the lemma in this way while $x_i^{''\cdots'} > 0$. In each step, the ith coordinate decreases by a positive linear combination of the original x_is. Since the x_is are rational, the ith coordinate is decreasing discretely, and must reach 0 eventually. We will then have written $x = y + c$ for a linear combination $c \in \text{cone}\{e_i - e_j \mid i > j \in p\}$ and a vector $y \in \mathcal{P}(p)$ with one fewer positive entry, since $y_i = 0$. The induction hypothesis now gives $y \in \text{cone}\{e_i - e_j \mid i > j \in p\}$, which implies $x \in \text{cone}\{e_i - e_j \mid i > j \in p\}$ as well. The desired result follows by induction.

Having proved that $\mathcal{P}(p)$ is generated by the vectors $e_i - e_j$ where $i > j$, let us observe that if $i > j$ then there is a sequence of cover relations $i \gtrdot k_1 \gtrdot \cdots \gtrdot k_r \gtrdot j$, which implies that $e_i - e_j = (e_i - e_{k_1}) + (e_{k_1} - e_{k_2}) + \cdots + (e_{k_r} - e_j)$. Therefore the vectors $e_i - e_j$ with $i \gtrdot j$ generate $\mathcal{P}(p)$. By a similar argument one sees that they generate $\mathcal{P}(p)$ irredundantly. □

The faces of poset polytopes were described (for the cones dual to poset cones) by Postnikov-Reiner-Williams [**76**, Proposition 3.5] (for order polytopes) by Geissinger [**43**] and Stanley [**91**], and (for oriented matroids) by Las Vergnas [Proposition 9.1.2][**18**]. Our presentation follows Las Vergnas, interpreting his general criterion in this special case.

Define a *circuit* of p to be a cyclic sequence $i_1, \ldots, i_n$ of elements of p where every consecutive pair is comparable in p. Circuits consist of *up-edges* where $i_j < i_{j+1}$ in p and *down-edges* where $i_j > i_{j+1}$ in p. We will say that a subposet q of p is *positive*[2] if the following conditions hold for every circuit X:

(1) if all the down-edges of a circuit X are in q, then all the up-edges of X are in q, and

(2) if all the up-edges of a circuit X are in q, then all the down-edges of X is in q.

LEMMA 3.4.3. *Let p be a poset on I. The faces of the poset cone $\mathcal{P}(p) \subseteq \mathbb{R}^I$ are precisely the poset cones $\mathcal{P}(q)$ as q ranges over the positive subposets of p.*

PROOF. In this proof we will assume some basic facts about oriented matroid theory; see [**9**, **19**] for the relevant definitions. Let $\mathcal{M}$ be the (acyclic) oriented matroid of the set of vectors $\{e_i - e_j \mid i > j \text{ in } p\}$. The faces of the poset cone $\mathcal{P}(p)$ are the cones generated by the *positive flats* of the Las Vergnas face lattice of $\mathcal{M}$. By [**19**, Proposition 9.1.2], these are the subsets F of $\mathcal{M}$ such that for every signed circuit X of $\mathcal{M}$, $X^+ \subseteq F$ implies $X^- \subseteq F$.

The oriented matroid $\mathcal{M}$ is isomorphic to the graphical oriented matroid of the graph of p on I, whose directed edges $i \to j$ correspond to the order relations $i > j$ in p. Therefore the signed circuits of $\mathcal{M}$ correspond to the cycles of the graph; they

[2]this terminology comes from the theory of oriented matroids

are the sets of the form:

$$X = \{e_{i_k} - e_{i_{k+1}} \mid i_1, \ldots, i_n \text{ is a circuit of } p\}$$

where $i_{n+1} = i_1$. Each circuit X comes with two orientations. One of them is given by $X^+ = \{e_{i_k} - e_{i_{k+1}} \mid i_k > i_{k+1} \text{ in } p\}$ and $X^- = \{e_{i_k} - e_{i_{k+1}} \mid i_k < i_{k+1} \text{ in } p\}$ and the other one is its reverse.

Now let $F = q \subseteq p$ be a subposet of p. In the first orientation of X, the condition that $X^+ \subseteq F$ implies $X^- \subseteq F$ says that if every down-edge is in q then every up-edge must be in q. In the other orientation, this condition is reversed. It follows that the positive flats of $\mathcal{M}$ are in bijection with the positive subposets of p, as desired. □

EXAMPLE 3.4.4. Let p be the poset on $\{a, b, c, d\}$ given by the cover relations $a < c$, $b < c$, $a < d$, $b < d$. The poset cone of p is shown in Figure 3. The positive subposets $q \neq p$ are those which do not contain both vertical cover relations $a < c$ and $b < d$, and do not contain both diagonal cover relations $a < d$ and $b < c$. There are nine such subposets, corresponding to the nine proper faces of $\mathcal{P}(p)$.

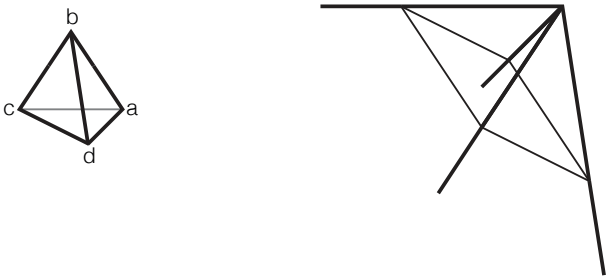


FIGURE 3. The poset cone for the poset of Example 15.4.

REMARK 3.4.5. Let us give some additional intuition for the definition of positive subposets. We will need preposets; see Section 3.4.4 for a definition.

A *poset contraction* is a preposet obtained from p by successively *contracting* order relations $i < j$ of p and replacing them by equivalence relations $i \sim j$. Since we need to keep the preposet transitive, contracting the up-edges of a circuit forces us to also contract the down-edges, and viceversa. For instance, in Example 3.4.4, if we contract $a < c$ and $b < d$, we get the contradictory relations $a \sim c > b \sim d > a$; to remedy this, we are forced to contract $b < c$ and $a < d$ into $b \sim c$ and $a \sim d$ as well.

In conclusion, the positive subposets of p are precisely the contracted subposets for the contractions of p.

3.4.2. Posets as a submonoid of extended generalized permutahedra. Recall that $\mathbf{P}$ is the Hopf monoid (in vector species) of posets. For $I = S \sqcup T$, the product of two posets p_1 on S and p_2 on T is their disjoint union $p_1 \cdot p_2$ regarded as a poset on I. The coproduct $\Delta_{S,T} : \mathbf{P}[I] \to \mathbf{P}[S] \otimes \mathbf{P}[T]$ is

$$\Delta_{S,T}(p) = \begin{cases} p|_S \otimes p|_T & \text{if } S \text{ is a lower set of } p, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 3.4.6. *The map* $\mathrm{low} : \mathbf{P} \to \mathbf{SF}^+ \xrightarrow{\cong} \mathbf{GP}^+$ *is an injective morphism of Hopf monoids in vector species.*

PROOF. To check that low preserves the product, let $I = S \sqcup T$ be a decomposition. Let p_1 and p_2 be posets on S and T, and $p_1 \cdot p_2$ be their product. A subset $J \subseteq I$ is a lower set of $p_1 \cdot p_2$ if and only if $J \cap S$ and $J \cap T$ are lower sets of p_1 and p_2, respectively. It follows that

$$\mathrm{low}_{p_1 \cdot p_2}(J) = \mathrm{low}_{p_1}(J \cap S) + \mathrm{low}_{p_2}(J \cap T) = (\mathrm{low}_{p_1} \cdot \mathrm{low}_{p_2})(J),$$

so low preserves products.

To check that low preserves the coproduct, let $I = S \sqcup T$ and let p be a poset on I. We need to consider two cases:

1. Suppose S is not a lower set of p. Then $\Delta_{S,T}(p) = 0$. In this case we also have $\mathrm{low}_p(S) = \infty$ so $\Delta_{S,T}(\mathrm{low}_p) = 0$ by the definition of the coproduct in $\mathbf{SF}^+$. It follows that low trivially respects the coproduct in this case.

2. Suppose S is a lower set of p. Then the restriction and contraction of p with respect to S are $p|_S$ and $p|_T$, respectively. Also $\mathrm{low}_p(S) = 0$. To see that low is compatible with restriction, notice that for $R \subseteq S$ we have $(\mathrm{low}_p)|_S(R) = \mathrm{low}_p(R)$, so

$$\mathrm{low}_{p|_S}(R) = \begin{cases} 0 & \text{if } R \text{ is a lower set of } p|_S \\ \infty & \text{otherwise} \end{cases},$$

$$(\mathrm{low}_p)|_S(R) = \begin{cases} 0 & \text{if } R \text{ is a lower set of } p \\ \infty & \text{otherwise.} \end{cases}$$

Since R is a lower set of $p|_S$ if and only if it is a lower set of p, we have $\mathrm{low}_{p|_S} = (\mathrm{low}_p)|_S$.

On the other hand, to see that low is compatible with contraction, notice that for $R \subseteq T$ we have $\mathrm{low}_{p/_S}(R) = \mathrm{low}_{p|_T}(R) = \mathrm{low}_p(R)$ and $(\mathrm{low}_p)/_S(R) = \mathrm{low}_p(R \cup S)$, so

$$\mathrm{low}_{p/_S}(R) = \begin{cases} 0 & \text{if } R \text{ is a lower set of } p|_T \\ \infty & \text{otherwise} \end{cases},$$

$$(\mathrm{low}_p)/_S(R) = \begin{cases} 0 & \text{if } R \cup S \text{ is a lower set of } p \\ \infty & \text{otherwise.} \end{cases}$$

Since R is a lower set of $p|_T$ if and only if it $R \cup S$ is a lower set of p, we have $\mathrm{low}_{p/_S} = (\mathrm{low}_p)/_S$.

We conclude that low is a morphism of monoids. Injectivity follows from the fact that we can recover a poset p from its collection of lower sets as follows: two elements i, j of p satisfy $i < j$ if and only if every lower set containing j also contains i. □

3.4.3. The antipode of posets. In view of Proposition 3.4.6 and Theorem 1.6.1, the antipode of $\mathbf{P}$ is given by the facial structure of poset polytopes, as described in Lemma 3.4.3. This allows us to give the optimal combinatorial formula for the antipode of the Hopf monoid of posets.

Recall that the *Hasse diagram* of a poset p is the graph whose vertices correspond to the elements of p and whose edges $x \to y$, which are always drawn with x lower than y, correspond to the cover relations $x \lessdot y$ of p.

COROLLARY 3.4.7. *The antipode of the Hopf monoid of posets* $\mathbf{P}$ *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If p is a poset on I then*

$$\mathrm{s}_I(p) = \sum_q (-1)^{c(q)} q,$$

summing over all positive subposets q of p, where $c(q)$ is the number of connected components of the Hasse diagram of q.

PROOF. This follows from Theorem 1.6.1 and Lemma 3.4.3, and the observation that the dimension of the poset cone $\mathcal{P}(p)$ is $|I| - c(p)$. □

EXAMPLE 3.4.8. Let us revisit Example 1.2.3. This example takes place in the Hopf algebra of posets P, where isomorphic posets are identified. The formula

is the algebraic manifestation of the face structure of the corresponding poset cone, which is the cone over a square shown in Figure 3. It has one full-dimensional face, 4 two-dimensional faces (in poset isomorphism classes of sizes 2 and 2), 4 rays (in one isomorphism class), and 1 vertex. Combinatorially, the summands correspond to the positive subposets of the poset in question, as described in Example 3.4.4.

3.4.4. Preposets and preposet cones. One may wonder whether there are other interesting submonoids of **GP** consisting of cones, or (almost equivalently) submonoids of **SF** consisting of $\{0, \infty\}$ functions. In Theorem 3.4.9 and Proposition 3.4.6 we show that, essentially, there aren't. We prove that $\{0, \infty\}$ submodular functions are equivalent to the slightly larger class of preposets, which may be viewed as posets in their own right.

A *preposet* on I is a binary relation $q \subseteq I \times I$, denoted $\leq$, which is reflexive ($x \leq x$ for all $x \in q$) and transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in q$). A preposet is not necessarily antisymmetric, and we define an equivalence relation by setting

$$x \sim y \text{ when } x \leq y \text{ and } y \leq x.$$

Let $p = q/\!\sim$ be the set of equivalence classes of p. The relation $\leq$ induces a relation $\leq$ on $q/\!\sim$ which is still reflexive and transitive, and is also antisymmetric; *i.e.*, it defines a poset.

It follows that we may think of preposets as posets whose elements are labeled by nonempty and pairwise disjoint sets. More precisely, we may equivalently define a *preposet* on I to be a set partition $\pi = \{I_1, \ldots, I_k\}$ of I together with a poset p on π.

If p' is a lower set of the poset $p = q/\!\sim$, then we say $q' = \bigcup_{K \in p'} K$ is a *lower set* of the preposet q. As before, we define the lower set function of q to be

$$\mathrm{low}_q : 2^I \mapsto \mathbb{R} \cup \{\infty\}, \quad \mathrm{low}_q(J) = \begin{cases} 0 & \text{if } J \text{ is a lower set of } q, \\ \infty & \text{otherwise.} \end{cases}$$

THEOREM 3.4.9. *A Boolean function $z : I \to \{0, \infty\}$ is submodular if and only if $z = \text{low}_q$ is the lower set function of a preposet q on I.*

PROOF. The backward direction is straightforward: If q is a preposet then its collection of lower sets is closed under union and intersection. It follows from (43) that low_q is submodular.

The forward direction will require more work. Suppose z is a submodular $\{0, \infty\}$ function on I and let

$$L := \text{supp}(z).$$

We need to show that L is the collection of lower sets of a preposet q on I.

Thanks to (43) we know that $L = \text{supp}(z)$ is a lattice under the operations of union and intersection. These operations are distributive, so Birkhoff's fundamental theorem of distributive lattices [**94**, Theorem 3.4.1] applies: If L_{irred} is the subposet of join-irreducible elements of L, and if $J(L_{\text{irred}})$ is the poset of lower sets of L_{irred} ordered by inclusion, then

$$L \cong J(L_{\text{irred}}).$$

We reinterpret L_{irred} as a preposet on I as follows. For each set $A \in L_{\text{irred}}$ let

$$\text{ess}(A) = A - \bigcup_{\substack{B \in L_{\text{irred}} \\ B < A}} B.$$

be the *essential set* of A, consisting of the *essential elements* which are in no lesser join-irreducible. Consider the collection of essential sets

$$q := \{\text{ess}(A) \mid A \in L_{\text{irred}}\},$$

endowed with the partial order inherited from L_{irred}. We will now show that:

1. q is a preposet on I, and
2. L is the collection of lower sets of q.

These two statements will complete the proof.

Before we prove these two statements, let us illustrate this construction with an example. The left panel of Figure 4 shows a distributive lattice L of subsets of $I = \{a, b, c, d, e, f, g, h, i, j\}$. We only label the join-irreducible elements; the label of every other set is the union of the join-irreducibles less than it in L. The right hand side panel shows the subposet L_{irred}. For each join-irreducible set $A \in L_{\text{irred}}$ we have indicated its essential set $\text{ess}(A)$ in bold. These essential sets partition I, allowing us to think of this object q as a preposet on I.

Step 1. q is a preposet on I: We need to show that the sets in q form a set partition of I. Each essential set $\text{ess}(A)$ is nonempty because A is join-irreducible. To prove that the essential sets are pairwise disjoint, assume contrariwise that $x \in \text{ess}(A)$ and $x \in \text{ess}(B)$ for some $A \neq B \in L_{\text{irred}}$. Then $A \cap B \in L$ and $x \in A \cap B$, so $x \in C$ for some join irreducible $C \in L_{\text{irred}}$ with $C \subseteq A \cap B \subsetneq A$. This contradicts the assumption that x is an essential element of A.

The following lemma completes the proof of Step 1.

LEMMA 3.4.10. *For all $A \in L$,*

$$A = \bigsqcup_{\substack{B \in L_{\text{irred}} \\ B \leq A}} \text{ess}(B).$$

In particular, $\{\text{ess}(B) \mid B \in L_{\text{irred}}\}$ is a partition of I.

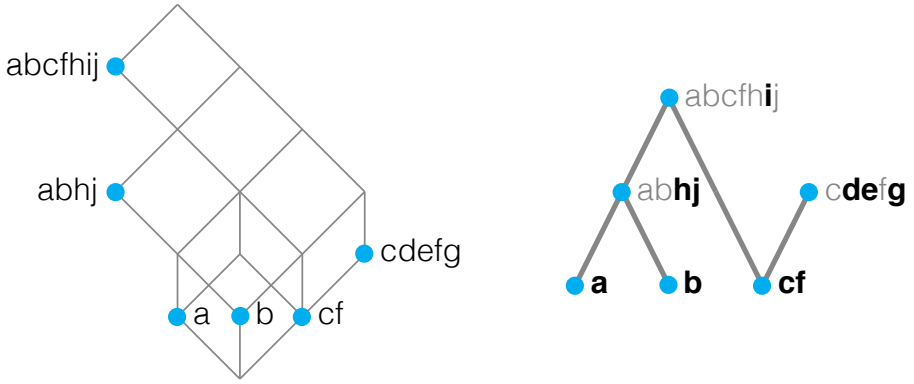


FIGURE 4. A distributive lattice L of subsets of $I = \{a, b, c, d, e, f, g, h, i, j\}$ and its poset of join-irreducibles. The essential sets of L_{irred} are shown in boldface; they give rise to a preposet q on I, whose lower sets are precisely the sets in L.

PROOF OF LEMMA 3.4.10. First we prove that the lemma holds for each join-irreducible $A \in L_{\text{irred}} \subseteq L$, proceeding by induction. This statement is clearly true for the minimal elements of L_{irred}. Also, if it holds for all elements $B < A$ in L_{irred}, then using the definition of $\operatorname{ess}(A)$ and the induction hypothesis,

$$A = \operatorname{ess}(A) \sqcup \bigcup_{\substack{B \in L_{\text{irred}} \\ B < A}} B = \operatorname{ess}(A) \sqcup \bigcup_{\substack{B \in L_{\text{irred}} \\ B < A}} \bigsqcup_{\substack{C \in L_{\text{irred}} \\ C \leq B}} \operatorname{ess}(C) = \bigsqcup_{\substack{C \in L_{\text{irred}} \\ C \leq A}} \operatorname{ess}(C)$$

so the claim holds for A as well. Therefore the lemma holds for all $A \in L_{\text{irred}}$.

Now we can prove Lemma 3.4.10 holds for all $A \in L$. The backward inclusion is clear. To prove the forward inclusion, let $x \in A$. Since A is the union of the join-irreducibles less than it in L, we have $x \in C$ for some $C \in L_{\text{irred}}$ with $C \leq A$. By the previous paragraph, $x \in \operatorname{ess}(D)$ for some $D \in L_{\text{irred}}$ with $D \leq C$; but then $D \leq A$ also, so x is in one of the essential sets on the right hand side. The desired result follows.

The last statement follows by recalling that $z(I) = 0$ and applying the lemma to $A = I$, which is the maximum element of the lattice L. This completes the proof of Lemma 3.4.10 and of Step 1 of this proof. □

Step 2. L is the collection of lower sets of q: By Birkhoff's theorem and Lemma 3.4.10, $A \in L$ if and only if there is a down set $J \subseteq L_{\text{irred}}$ with

$$A = \bigcup_{B \in J} B = \bigsqcup_{B \in J} \operatorname{ess}(B);$$

that is, if and only if A is a lower set of q. □

We now state an algebraic counterpart of Theorem 3.4.9. Let $\mathrm{Q}[I]$ be the set of preposets on I. Preposets become a Hopf monoid in vector species $\mathbf{Q}$ with the same operations of the Hopf monoid of posets $\mathbf{P}$. Let $\mathbf{SF}_{\{0,\infty\}}$ be the submonoid of $\mathbf{SF}$ consisting of $\{0, \infty\}$ functions. Let $\overline{\mathbf{GP}}_{\text{cone}}$ be the submonoid of $\overline{\mathbf{GP}}$ consisting of cones.

PROPOSITION 3.4.11. *The maps* low : $\mathbf{Q} \xrightarrow{\cong} \mathbf{SF}_{\{0,\infty\}} \xrightarrow{\cong} \overline{\mathbf{GP}}_{cone}$ *are isomorphisms of Hopf monoids in vector species.*

PROOF. The first isomorphism is an immediate consequence of Theorem 3.4.9. For the second one, notice that every cone $c \in \mathbf{GP}$ is a translate of a unique cone c' that contains the origin. The submodular function z_c such that $c' = \mathcal{P}(z_c)$ is a $\{0, \infty\}$ function, and the correspondence $c \mapsto z_c$ gives the desired isomorphism. □

In the correspondence between preposets and generalized permutahedra which are cones, posets on I correspond to cones of the maximum possible dimension $|I|-1$. The antipode formula for preposets $\mathbf{Q}$ is essentially the same as the antipode formula for posets $\mathbf{P}$.

CHAPTER 4

Characters, polynomial invariants, and reciprocity

4.1. Invariants of Hopf monoids and reciprocity

Once again, we set aside the combinatorial examples of earlier sections and return to the general setting of Hopf monoids of Section 1.1, where we begin to develop the basics of character theory. Similar results for Hopf algebras can be found in [**1**, **17**].

This section shows that each character on a Hopf monoid gives rise to an associated polynomial invariant. There are two main results. Proposition 4.1.1 shows that the polynomial invariant is indeed polynomial and invariant. Proposition 4.1.5 relates the values of the invariant on an integer and on its negative by means of the antipode of the Hopf monoid.

This abstract framework has concrete combinatorial consequences. For instance, we will see in Section 4.3 that the simplest non-zero characters on the Hopf monoids $\mathbf{G}, \mathbf{P}, \mathbf{M}$ give rise to three important combinatorial polynomials: the chromatic polynomial of a graph, the strict order polynomial of a poset, and the BJR polynomial of a matroid. Furthermore, this Hopf-theoretic framework shows that the celebrated reciprocity theorems for these three polynomials, due to Stanley [**90**, Theorem 1.2], [**88**, Theorem 3] and Billera, Jia, and Reiner [**17**, Theorem 6.3], are specific instances of the same general fact about extended generalized permutahedra.

4.1.1. The polynomial invariant of a character. Recall from Section 2.1 the notion of a character ζ on a Hopf monoid in vector species $\mathbf{H}$. In the examples that interest us, $\mathbf{H}$ is a Hopf monoid coming from a family of combinatorial objects, and ζ is a multiplicative function on our objects which is invariant under relabelings of the ground set.

Throughout this section, we fix a connected Hopf monoid $\mathbf{H}$ and a character $\zeta : \mathbf{H} \to \Bbbk$. Define, for each element $x \in \mathbf{H}[I]$ and each natural number $n \in \mathbb{N}$, the scalar

$$\chi_I(x)(n) := \sum_{I = S_1 \sqcup \cdots \sqcup S_n} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(x), \tag{47}$$

summing over *all* decompositions of I into n disjoint subsets which are allowed to be empty. For fixed I and x, the function $\chi_I(x)$ is defined for $n \in \mathbb{N}$ and takes values on $\Bbbk$. Note that

$$\chi_I(x)(0) = \begin{cases} \zeta_\emptyset(x) & \text{if } I = \emptyset, \\ 0 & \text{otherwise,} \end{cases} \qquad \chi_I(x)(1) = \zeta_I(x). \tag{48}$$

PROPOSITION 4.1.1. (Polynomial invariants) *Let* $\mathbf{H}$ *be a connected Hopf monoid,* $\zeta : \mathbf{H} \to \Bbbk$ *be a character, and* χ *be defined by* (47). *Fix a finite set* I *and an element* $x \in \mathbf{H}[I]$.

(1) *For each* $n \in \mathbb{N}$ *we have*

$$\chi_I(x)(n) \ = \sum_{k=0}^{|I|} \chi_I^{(k)}(x) \binom{n}{k}$$

where, for each $k = 0, \ldots, |I|$,

$$\chi_I^{(k)}(x) = \sum_{(T_1,\ldots,T_k)\vDash I} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x) \ \in \Bbbk.$$

summing over all compositions $(T_1, \ldots, T_k)$ *of* I. *Therefore,* $\chi_I(x)$ *is a polynomial function of* n *of degree at most* $|I|$.

(2) *Let* $\sigma : I \to J$ *be a bijection,* $x \in \mathbf{H}[I]$ *and* $y := \mathbf{H}[\sigma](x) \in \mathbf{H}[J]$. *Then* $\chi_I(x) = \chi_J(y)$.

PROOF. 1. Given a decomposition $I = S_1 \sqcup \cdots \sqcup S_n$, let $(T_1, \ldots, T_k)$ be the composition of I obtained by removing the empty S_is and keeping the remaining ones in order. In view of the unitality of Δ and ζ, we have

$$(\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(x) = (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x).$$

Note that $k \leq |I|$ and the number of decompositions $I = S_1 \sqcup \cdots \sqcup S_n$ which give rise to a given composition $(T_1, \ldots, T_k)$ is $\binom{n}{k}$. It follows that

$$\chi_I(x)(n) \ = \sum_{k=0}^{|I|} \left(\sum_{(T_1,\ldots T_k)\vDash I} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\ldots,T_k}(x) \right) \binom{n}{k}.$$

as desired. Since each $\binom{n}{k}$ is a polynomial function of n of degree k, $\chi_I(x)$ is polynomial of degree at most $|I|$.

2. This follows from the naturality of Δ and ζ. □

Let $\Bbbk[t]$ denote the polynomial algebra. Proposition 4.1.1 states that each character ζ gives rise to a family of polynomials $\chi_I(x) \in \Bbbk[t]$ associated to each structure $x \in \mathbf{H}[I]$, whose values on nonnegative integers n are given by (47). Furthermore, it says that two isomorphic structures have the same associated polynomial. Thus, the function $\chi_I(x)$ is a *polynomial invariant* of the structure x (canonically associated to the Hopf monoid $\mathbf{H}$ and the character ζ).

4.1.2. Properties of the polynomial invariant of a character. We now collect some useful properties of these polynomial invariants.

PROPOSITION 4.1.2. *Let* $\mathbf{H}$ *be a connected Hopf monoid,* $\zeta : \mathbf{H} \to \Bbbk$ *be a character, and* χ *be the associated polynomial invariant, defined by* (47). *Let* I *be a finite set.*

(i) χ_I *is a linear map from* $\mathbf{H}[I]$ *to* $\Bbbk[t]$.

(ii) *Let* $I = S \sqcup T$ *be a decomposition. For any* $x \in \mathbf{H}[S]$ *and* $y \in \mathbf{H}[T]$, *we have the equality of polynomials*

$$\chi_I(x \cdot y) = \chi_S(x)\chi_T(y)$$

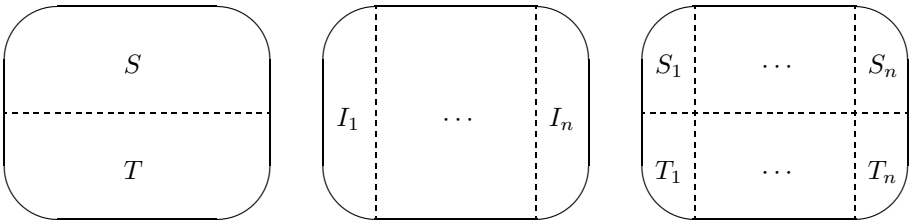
(iii) $\chi_\emptyset(1) = 1$, *the constant polynomial.*

(iv) *For any $x \in \mathbf{H}[I]$ and scalars n and m,*

$$\chi_I(x)(n+m) = \sum_{I=S\sqcup T} \chi_S(x|_S)(n)\chi_T(x/_S)(m).$$

PROOF. Property (i) follows from the linearity of Δ and ζ.

Property (ii) follows from the compatibility between μ and Δ and the multiplicativity of ζ. We provide the details. First, decompositions $I = I_1 \sqcup \cdots \sqcup I_n$ into n parts are in bijection with pairs of decompositions $S = S_1 \sqcup \cdots \sqcup S_n$ and $T = T_1 \sqcup \cdots \sqcup T_n$, where $S_i = I_i \cap S$ and $T_i = I_i \cap T$.



The compatibility between μ and Δ and the associativity of the latter imply that if we write

$$\Delta_{S_1,\ldots,S_n}(x) = \sum x_1 \otimes \cdots \otimes x_n \quad \text{and} \quad \Delta_{T_1,\ldots,T_n}(y) = \sum y_1 \otimes \cdots \otimes y_n,$$

in Sweedler's notation, as described in Section 1.1.5, then

$$\Delta_{I_1,\ldots,I_n}(x \cdot y) = \sum (x_1 \cdot y_1) \otimes \cdots \otimes (x_n \cdot y_n)$$

The above, together with the multiplicativity of ζ, yield that $\chi_I(x \cdot y)(n)$ equals

$$\begin{aligned}
&\sum_{I=I_1\sqcup\cdots\sqcup I_n} (\zeta_{I_1} \otimes \cdots \otimes \zeta_{I_n}) \circ \Delta_{I_1,\ldots,I_n}(x \cdot y) \\
&= \sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n \\ T=T_1\sqcup\cdots\sqcup T_n}} \sum \zeta_{I_1}(x_1 \cdot y_1) \cdots \zeta_{I_n}(x_n \cdot y_n) \\
&= \sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n \\ T=T_1\sqcup\cdots\sqcup T_n}} \sum \zeta_{S_1}(x_1)\zeta_{T_1}(y_1) \cdots \zeta_{S_n}(x_n)\zeta_{T_n}(y_n) \\
&= \Big(\sum_{S=S_1\sqcup\cdots\sqcup S_n} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(x) \Big) \\
&\quad \times \Big(\sum_{T=T_1\sqcup\cdots\sqcup T_n} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_n}) \circ \Delta_{T_1,\ldots,T_n}(y) \Big) \\
&= \chi_S(x)(n)\, \chi_T(y)(n).
\end{aligned}$$

Thus $\chi_I(x \cdot y) = \chi_S(x)\chi_T(y)$ as polynomials, since they agree at every natural number n.

Property (iii) follows from unitality of Δ and ζ.

For property (iv), note that decompositions of I into $n+m$ parts are in bijection with tuples

$$(S, S_1, \ldots, S_n, T, T_1, \ldots, T_m)$$

where $I = S \sqcup T$, $S = S_1 \sqcup \cdots \sqcup S_n$, and $T = T_1 \sqcup \cdots \sqcup T_m$. In addition, associativity of Δ implies that

$$\Delta_{S_1,\ldots,S_n,T_1,\ldots,T_m} = \big(\Delta_{S_1,\ldots,S_n} \otimes \Delta_{T_1,\ldots,T_m}\big) \circ \Delta_{S,T}.$$

Therefore, $\chi_I(x)(n+m)$ is equal to

$$\begin{aligned}
&\sum_{I=S_1\sqcup\cdots\sqcup S_n\sqcup T_1\sqcup\cdots\sqcup T_m} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n} \otimes \zeta_{T_1} \otimes \cdots \otimes \zeta_{T_m}) \circ \Delta_{S_1,\ldots,S_n,T_1,\ldots,T_m}(x) \\
&= \sum_{I=S\sqcup T} \sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n \\ T=T_1\sqcup\cdots\sqcup T_m}} (\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n} \otimes \zeta_{T_1} \otimes \cdots \otimes \zeta_{T_m}) \\
&\qquad \circ \big(\Delta_{S_1,\ldots,S_n} \otimes \Delta_{T_1,\ldots,T_m}\big) \circ \Delta_{S,T}(x) \\
&= \sum_{I=S\sqcup T} \sum_{\substack{S=S_1\sqcup\cdots\sqcup S_n \\ T=T_1\sqcup\cdots\sqcup T_m}} \Big((\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \\
&\qquad \circ \Delta_{S_1,\ldots,S_n}(x|_S)\Big)\Big((\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_m}) \circ \Delta_{T_1,\ldots,T_m}(x/_S)\Big) \\
&= \sum_{I=S\sqcup T} \chi_S(x|_S)(n)\, \chi_T(x/_S)(m).
\end{aligned}$$

The above yields the desired equality when n and m are nonnegative integers. Since both sides of the equation are polynomial functions of (n, m) in view of Proposition 4.1.1, the result then follows for arbitrary scalars n and m. □

The following result states that if two characters are related by a morphism of Hopf monoids, then the same relation holds for the corresponding polynomial invariants.

PROPOSITION 4.1.3. *Let* $\mathbf{H}$ *and* $\mathbf{K}$ *be two Hopf monoids. Suppose* $\zeta^{\mathbf{H}}$ *is a character on* $\mathbf{H}$, $\zeta^{\mathbf{K}}$ *is a character on* $\mathbf{K}$, *and* $f : \mathbf{H} \to \mathbf{K}$ *is a morphism of Hopf monoids such that*

$$\zeta^{\mathbf{K}}_I\big(f_I(x)\big) = \zeta^{\mathbf{H}}_I(x)$$

for every I *and* $x \in \mathbf{H}[I]$. *Let* $\chi^{\mathbf{H}}$ *and* $\chi^{\mathbf{K}}$ *be the polynomial invariants corresponding to* $\zeta^{\mathbf{H}}$ *and* $\zeta^{\mathbf{K}}$, *respectively. Then*

$$\chi^{\mathbf{K}}_I\big(f_I(x)\big) = \chi^{\mathbf{H}}_I(x)$$

for every I *and* $x \in \mathbf{H}[I]$.

PROOF. Since f preserves coproducts, we have $\Delta_{S,T}\big(f_I(x)\big) = (f_S \otimes f_T)\big(\Delta_{S,T}(x)\big)$ and a similar fact for iterated coproducts. This and the hypothesis give the result. □

REMARK 4.1.4. Most of the results in this section hold under weaker hypotheses (different ones for each result). For instance, Proposition 4.1.1 holds for any collection of linear maps $\zeta_I : \mathbf{H}[I] \to \Bbbk$ which is unital (with the same proof). If n and m are nonnegative integers, statement (iv) in Proposition 4.1.2 holds for any collection of linear maps $\zeta_I : \mathbf{H}[I] \to \Bbbk$. Proposition 4.1.3 holds for any morphism of comonoids which preserves the characters.

4.1.3. From Hopf monoids to reciprocity theorems. For a character ζ on a Hopf monoid $\mathbf{H}$, the construction of Section 4.1.1 produces a polynomial invariant χ whose values on natural numbers are well understood in terms of $\mathbf{H}$ and ζ. What about the values on negative integers? The antipode provides an answer to this question.

PROPOSITION 4.1.5. (Reciprocity for polynomial invariants) *Let $\mathbf{H}$ be a connected Hopf monoid, $\zeta : \mathbf{H} \to \Bbbk$ be a character, and χ be the associated polynomial invariant, defined by* (47)*. Let* s *be the antipode of $\mathbf{H}$. Then*

$$\chi_I(x)(-1) = \zeta_I\big(\mathrm{s}_I(x)\big). \tag{49}$$

More generally, for every scalar n,

$$\chi_I(x)(-n) = \chi_I\big(\mathrm{s}_I(x)\big)(n). \tag{50}$$

PROOF. Since $\binom{-1}{k} = (-1)^k$, Proposition 4.1.1 implies

$$\chi_I(x)(-1) = \sum_{k=0}^{|I|} \left(\sum_{(T_1,\dots T_k)\vDash I} (\zeta_{T_1} \otimes \cdots \otimes \zeta_{T_k}) \circ \Delta_{T_1,\dots,T_k}(x) \right) (-1)^k.$$

Using multiplicativity of ζ and Takeuchi's formula (5), this may be rewritten as

$$\begin{aligned} \chi_I(x)(-1) &= \sum_{k=0}^{|I|} \left(\sum_{(T_1,\dots T_k)\vDash I} \zeta_I \circ (\mu_{T_1} \otimes \cdots \otimes \mu_{T_k}) \circ \Delta_{T_1,\dots,T_k}(x) \right) (-1)^k \\ &= \zeta_I \left(\sum_{k\geq 0} (-1)^k \sum_{(T_1,\dots T_k)\vDash I} \mu_{T_1,\dots,T_k} \circ \Delta_{T_1,\dots,T_k}(x) \right) = \zeta_I\big(\mathrm{s}_I(x)\big), \end{aligned}$$

which proves (49).

To prove (50) one may assume that the scalar n is a nonnegative integer, since both sides are polynomial functions of n. We make this assumption and proceed by induction on $n \in \mathbb{N}$.

When $n = 0$ the result holds in view of (48) and the fact that $\mathrm{s}_\emptyset = \mathrm{id}$. When $n = 1$ it follows from (48) and (49). For $n \geq 2$ we apply Proposition 4.1.2(iv) as follows:

$$\chi_I(x)(-n) = \chi_I(x)(-n+1-1) = \sum_{I=S\sqcup T} \chi_S(x|_S)(-n+1)\, \chi_T(x/_S)(-1).$$

Using the induction hypothesis, and then reversing the roles of S and T, this equals

$$\begin{aligned} &\sum_{I=S\sqcup T} \chi_S\big(\mathrm{s}_S(x|_S)\big)(n-1)\, \chi_T\big(\mathrm{s}_T(x/_S)\big)(1) \\ &= \sum_{I=S\sqcup T} \chi_S\big(\mathrm{s}_S(x/_T)\big)(1)\, \chi_T\big(\mathrm{s}_T(x|_T)\big)(n-1). \end{aligned}$$

Applying Proposition 4.1.2(iv) to $\mathrm{s}_I(x)$, and using the fact (7) that the antipode reverses coproducts, we see that this equals

$$\chi_I\big(\mathrm{s}_I(x)\big)(1+n-1) = \chi_I\big(\mathrm{s}_I(x)\big)(n),$$

as needed. □

Formulas (49) and (50) are *reciprocity results* of a very general nature. They gives us another reason to be interested in an explicit antipode formula: such a formula allows for knowledge of the values of *all* polynomial invariants at negative integers. The antipode acts as a universal link between the values of the invariants at positive and negative integers. We now apply this approach to **GP** in Section 4.2. This will allow us to unify several important reciprocity results in combinatorics and to obtain new ones in Section 4.3.

4.2. The basic character and the basic invariant of GP

In this section we return to specifics, focusing on the Hopf monoids of generalized permutahedra **GP** and $\mathbf{GP}^+$. We will prove the results in this section for **GP** but they also hold in $\mathbf{GP}^+$; see Remark 4.2.6.

We introduce the (almost trivial) *basic character* β and its associated *basic invariant* χ on the Hopf monoid of generalized permutahedra **GP**. We use the algebraic structure of **GP** and β to obtain combinatorial formulas for $\chi(n)$ and $\chi(-n)$ for $n \in \mathbb{N}$ in Propositions 4.2.3 and 4.2.4; these were also obtained in [**17**]. In Section 4.3 we will see that several important combinatorial facts about graphs, posets, and matroids are straightforward consequences of this setup.

DEFINITION 4.2.1. The *basic character* β of **GP** is given by

$$\beta_I(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ is a point} \\ 0 & \text{otherwise.} \end{cases}$$

for a generalized permutahedron $\mathfrak{p} \in \mathbb{R}^I$. The *basic invariant* χ of **GP** is the polynomial invariant associated to β by Proposition 4.1.1 and (47).

Note that β is indeed a character because the product of two polytopes $\mathfrak{p} \times \mathfrak{q}$ is a point if and only if both $\mathfrak{p}$ and $\mathfrak{q}$ are points.

4.2.1. A lemma on directionally generic faces. Given a generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^I$ and a linear functional $y \in \mathbb{R}^I$, say $\mathfrak{p}$ is *directionally generic in the direction of* y if the y-maximal face $\mathfrak{p}_y$ is a point. If this is the case, we will also say that y is $\mathfrak{p}$-generic and that $\mathfrak{p}$ is *y-generic*. See Figure 1.

We will need the following technical lemma about directionally generic faces.

LEMMA 4.2.2. *For any generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^I$ and linear functional $y \in \mathbb{R}^I$, the following equations hold as formal sums of polytopes.*

(1)
$$\sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim \mathfrak{q}} \mathfrak{q}_y = \sum_{\mathfrak{q} \leq \mathfrak{p}_{-y}} (-1)^{\dim \mathfrak{q}} \mathfrak{q}$$

(2)
$$\sum_{\substack{\mathfrak{q} \leq \mathfrak{p}: \\ y \text{ is } \mathfrak{q}\text{-generic}}} (-1)^{\dim \mathfrak{q}} = (-1)^{|I|} \text{ (number of vertices of } \mathfrak{p}_{-y}).$$

PROOF. 1. Let us express both sides of the equation Hopf-theoretically. Let F be the face of the braid arrangement that y belongs to, and say it corresponds to the decomposition $I = S_1 \sqcup \cdots \sqcup S_k$, as described in Section 1.3.5. Also recall from Section 1.1.9 that we denote $\mu_F = \mu_{S_1,\ldots,S_k}$, $\Delta_F = \Delta_{S_1,\ldots,S_k}$, and $\mathrm{s}_F = \mathrm{s}_{S_1} \otimes \cdots \otimes \mathrm{s}_{S_k}$.

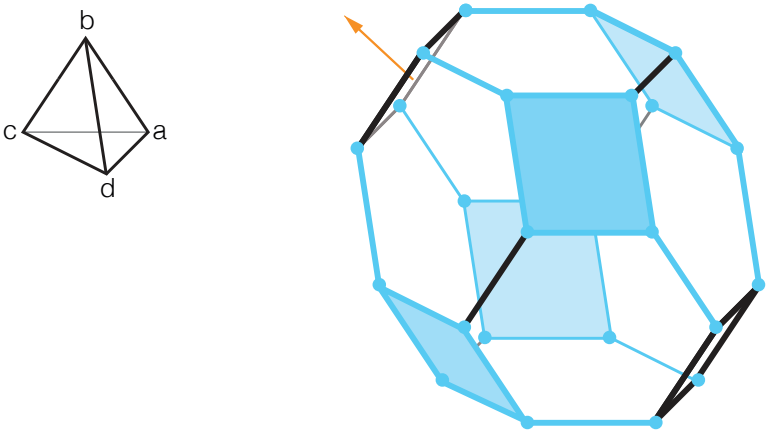


FIGURE 1. The y-generic faces of the permutahedron π_4 for $y = (0, 1, 1, 0)$ are shaded.

For any generalized permutahedron $\mathfrak{r} \subseteq \mathbb{R}^I$ we have $\mathfrak{r}_y = \mathfrak{r}_F = \mu_F \Delta_F(\mathfrak{r})$ by Proposition 1.4.4. It then follows from the formula for the antipode of **GP** in Theorem 1.6.1 that

$$(-1)^{|I|} \mu_F \Delta_F \, \mathrm{s}_I(\mathfrak{p}) = \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim \mathfrak{q}} \mathfrak{q}_y.$$

Now let $\overline{F}$ be the opposite face of F, corresponding to the decomposition $I = S_k \sqcup \cdots \sqcup S_1$. Then $\overline{F}$ contains $-y$ so $\mu_{\overline{F}} \Delta_{\overline{F}}(\mathfrak{p}) = \mathfrak{p}_{-y}$, and

$$(-1)^{|I|} \, \mathrm{s}_I \, \mu_{\overline{F}} \Delta_{\overline{F}}(\mathfrak{p}) = \sum_{\mathfrak{q} \leq \mathfrak{p}_{-y}} (-1)^{\dim \mathfrak{q}} \mathfrak{q}.$$

To prove that these expressions equal each other, recall Proposition 1.1.16, which holds for any Hopf monoid in vector species:

$$\mathrm{s}_I \, \mu_F = \mu_{\overline{F}} \, \mathrm{s}_{\overline{F}} \, \mathrm{sw}_F, \qquad \Delta_F \, \mathrm{s}_I = \mathrm{s}_F \, \mathrm{sw}_{\overline{F}} \, \Delta_{\overline{F}}.$$

where we have rewritten the first equation, using that $\mathrm{s}_{\overline{F}} \mathrm{sw}_F = \mathrm{sw}_F \, \mathrm{s}_F$. Applying the second equation to F and then the first equation to $\overline{F}$, we obtain

$$\mu_F \Delta_F \, \mathrm{s}_I = \mu_F \, \mathrm{s}_F \, \mathrm{sw}_{\overline{F}} \, \Delta_{\overline{F}} = \mathrm{s}_I \, \mu_{\overline{F}} \Delta_{\overline{F}},$$

which gives the desired result.

2. This follows by applying the character χ_I to both sides of the equation of part 1. □

4.2.2. The basic invariant and the basic reciprocity theorem of GP. Recall that the basic invariant χ of **GP** is the polynomial invariant that Proposition 4.1.1 associates to the basic character β of Definition 4.2.1.

PROPOSITION 4.2.3. [**17**, Def. 2.3, Thm 9.2.(v)] *At a natural number n, the* basic invariant χ *of a generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^I$ is given by*

$$\chi_I(\mathfrak{p})(n) = \ \textit{(number of } \mathfrak{p}\textit{-generic functions } y : I \to [n]).$$

PROOF. First notice that each summand in (47) comes from a decomposition $I = S_1 \sqcup \cdots \sqcup S_n$, which bijectively corresponds to a function $y : I \to [n]$ defined by $y(i) = k$ for each $i \in S_k$. The corresponding summand for $\chi_I(\mathfrak{p})(n)$ is

$$(\zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(\mathfrak{p}) = \zeta_{S_1}(\mathfrak{p}_1) \cdots \zeta_{S_n}(\mathfrak{p}_n)$$

where the y-maximal face $\mathfrak{p}_y$ factors as $\mathfrak{p}_y = \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_n$ for $\mathfrak{p}_i \in \mathbb{R}S_i$. This term contributes to the sum if and only if every $\mathfrak{p}_i$ is a point, that is, if and only if $\mathfrak{p}_y$ is a point; and in that case, it contributes 1. The desired result follows. □

PROPOSITION 4.2.4. [**17**, Theorems 6.3 and 9.2.(v)] *(Basic invariant reciprocity.) At a negative integer $-n$, the basic invariant χ of a generalized permutahedron $\mathfrak{p} \subseteq \mathbb{R}^I$ is given by*

$$(-1)^{|I|}\chi_I(\mathfrak{p})(-n) = \sum_{y:I\to[n]} \textit{(number of vertices of } \mathfrak{p}_y\textit{)}$$

where $\mathfrak{p}_y$ is the y-maximum face of p.

PROOF. Using the general reciprocity formula for characters of Proposition 4.1.5 and the formula for the antipode of Theorem 1.6.1 of GP we obtain

$$\chi_I(\mathfrak{p})(-n) = \chi_I(\mathrm{s}_I(\mathfrak{p}))(n) = (-1)^{|I|}\sum_{\mathfrak{q}\leq\mathfrak{p}}(-1)^{\dim\mathfrak{q}}\chi_I(\mathfrak{q})(n).$$

Proposition 4.2.3 and Lemma 4.2.2 then give

$$\begin{aligned}\chi_I(\mathfrak{p})(-n) &= (-1)^{|I|}\sum_{\mathfrak{q}\leq\mathfrak{p}}(-1)^{\dim\mathfrak{q}}(\#\text{ of }\mathfrak{q}\text{-generic functions } y : I \to [n]) \\ &= (-1)^{|I|}\sum_{y:I\to[n]}\ \sum_{\substack{\mathfrak{q}\leq\mathfrak{p}:\\ y \text{ is } \mathfrak{q}\text{-generic}}} (-1)^{\dim\mathfrak{q}} = \sum_{y:I\to[n]} (\text{number of vertices of } \mathfrak{p}_{-y}).\end{aligned}$$

This gives the desired result since $\mathfrak{p}_{-y} = \mathfrak{p}_{(n+1,\ldots,n+1)-y}$, and $(n+1,\ldots,n+1) - y$ maps I to $[n]$ if and only if y maps I to $[n]$. □

REMARK 4.2.5. Propositions 4.2.3 and 4.2.4 were also obtained by Billera, Jia, and Reiner in [**17**]; their proof of the basic invariant reciprocity of Proposition 4.2.4 relies on Stanley's combinatorial reciprocity theorem for P-partitions. Our approach is different: we choose to give Hopf-theoretic proofs of these results. This will allow us to give straightforward derivations of various combinatorial reciprocity theorems, using only the Hopf-theoretic structure of **GP**; we do this in the following section.

REMARK 4.2.6. The results of this section also hold for the Hopf monoid $\mathbf{GP}^+$ of possibly unbounded generalized permutahedra. In that setting, we must set $\mathfrak{p}_y = 0$ whenever the polyhedron $\mathfrak{p}$ is unbounded above in the direction of y. For a linear functional y to be $\mathfrak{p}$-generic, we must require that the polyhedron $\mathfrak{p}$ is bounded above in the direction of y, and that $\mathfrak{p}_y$ is a point.

4.3. Combinatorial reciprocity for graphs, matroids, and posets

We now show how characters on Hopf monoids naturally give rise to numerous reciprocity theorems in combinatorics; some old, some new. We would like to emphasize one benefit of this approach: this algebraic framework allows us to discover and prove reciprocity theorems automatically. All we have to do is define

a character on a Hopf monoid, and the general theory will produce a polynomial invariant and a reciprocity theorem satisfied by it. In this section we will use some of the simplest possible characters to obtain several theorems of interest.

The ideas in this section are closely related to those in [**1**,**17**] and in [**2**, Chapters 11 and 13].

4.3.1. The basic invariant of graphs is the chromatic polynomial. Given a graph g, an n-coloring of the vertices of g is an assignment of a color in $[n]$ to each vertex of g. A coloring is *proper* if any two vertices connected by an edge have different colors.

PROPOSITION 4.3.1. *Let ζ be the character on the Hopf monoid of graphs* $\mathbf{G}$ *defined by*

$$\zeta_I(g) = \begin{cases} 1 & \text{if } g \text{ has no edges, and} \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding polynomial invariant is the chromatic polynomial*, which equals*

$$\chi_I(g)(n) \;=\; \text{number of proper colorings of } g \text{ with } n \text{ colors.}$$

for $n \in \mathbb{N}$.

PROOF. The zonotope Z_g is a point if and only if g has no edges. Therefore, thanks to the inclusion $\mathbf{G}^{\mathrm{cop}} \hookrightarrow \mathbf{GP}$ of Proposition 3.2.5, when we restrict the basic character β of $\mathbf{GP}$ to graphic zonotopes, we obtain the character ζ of graphs. It follows that $\chi_I(g)$ is the basic invariant of the graphic zonotope Z_g, and Proposition 4.2.3 then tells us that $\chi_I(g)(n)$ is the number of Z_g-generic functions $y : I \to [n]$. By (40), a function $y : I \to [n]$ is Z_g-generic if and only if $y(i) \neq y(j)$ whenever $\{i, j\}$ is an edge of g; that is, if and only if y is a proper coloring of g. The result follows. □

We say that an n-coloring y of g and an acyclic orientation o of the edges of g are *compatible* if we have $y(i) \geq y(j)$ for every directed edge $i \to j$ in the orientation o.

COROLLARY 4.3.2. (Stanley's reciprocity for graphs [**90**, Theorem 1.2]) *Let g be a graph on vertex set I, and $n \in \mathbb{N}$. Then $(-1)^{|I|}\chi_I(g)(-n)$ equals the number of compatible pairs of an n-coloring and an acyclic orientation of g. In particular, $(-1)^{|I|}\chi_I(g)(-1)$ is the number of acyclic orientations of g.*

PROOF. This result is a special case of Proposition 4.2.4. To see this, regard an n-coloring y of g as a linear functional $y : I \to [n]$ on the zonotope Z_g. This coloring induces a partial orientation o_y of the edges of g, assigning an edge $\{i, j\}$ the direction $i \to j$ whenever $y(i) > y(j)$. By (40), the vertices of $(Z_g)_y$ correspond to the acyclic orientations that extend o_y; these are precisely the acyclic orientations of g compatible with y. □

4.3.2. The basic invariant of matroids is the Billera-Jia-Reiner polynomial. Given a matroid m on I, say a function $y : I \to [n]$ is *m-generic* if m has a unique *y-maximum basis* $\{b_1, \ldots, b_r\}$ maximizing $y(b_1) + \cdots + y(b_r)$.

PROPOSITION 4.3.3. *Let ζ be the character on the Hopf monoid of matroids* $\mathbf{M}$ *defined by*

$$\zeta_I(m) = \begin{cases} 1 & \text{if } m \text{ has only one basis, and} \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding polynomial invariant is the Billera-Jia-Reiner polynomial *of a matroid, which equals*

$$\chi_I(m)(n) \ := \ \text{number of } m\text{-generic functions } y : I \to [n]$$

for $n \in \mathbb{N}$.

PROOF. The matroid polytope of m is a point if and only if m has only one basis. Therefore, thanks to the inclusion $\mathbf{M} \hookrightarrow \mathbf{GP}$ of Proposition 3.3.3, when we restrict the basic character β of $\mathbf{GP}$ to matroid polytopes, we obtain the character ζ of matroids. It follows that $\chi_I(m)$ The result now follows by applying Proposition 4.2.3 to matroid polytopes. □

COROLLARY 4.3.4. (Billera-Jia-Reiner's reciprocity for matroids [**17**, Theorem 6.3.]) *Let m be a matroid on I and $n \in \mathbb{N}$. Then*

$$(-1)^{|I|}\chi_I(m)(-n) \ = \ \sum_{y: I \to [n]} \text{(number of } y\text{-maximum bases of } m).$$

PROOF. This is the result of applying Proposition 4.2.4 to matroid polytopes. □

4.3.3. The basic invariant of posets is the strict order polynomial. Given a poset p, say a map $y : p \to [n]$ is *order-preserving* if $y(i) \leq y(j)$ whenever $i < j$ in p. Say y is *strictly order-preserving* if $y(i) < y(j)$ whenever $i < j$ in p.

PROPOSITION 4.3.5. *Let ζ be the character on the Hopf monoid of posets* $\mathbf{P}$ *defined by*

$$\zeta_I(p) = \begin{cases} 1 & \text{if } p \text{ is an antichain, and} \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding polynomial invariant is the strict order polynomial*, which equals*

$$\chi_I(p)(n) \ := \ \text{number of strictly order-preserving maps } p \to [n].$$

for $n \in \mathbb{N}$.

PROOF. The poset cone $\mathcal{P}(p)$ is a point if and only if p is an antichain. Therefore, thanks to the inclusion $\mathbf{P} \hookrightarrow \mathbf{GP}^+$ of Proposition 3.4.6 (see Remark 4.2.6), when we restrict the basic character β of $\mathbf{GP}^+$ to poset cones, we obtain the character ζ of posets. It follows that $\chi_I(p)(n)$ is the number of $\mathcal{P}(p)$-generic functions $y : p \to [n]$. Now, thanks to Proposition 3.4.1, the normal fan to $\mathcal{P}(p)$ is a single cone cut out by the inequalities $y(i) \leq y(j)$ for $i > j$ in p, so the p-generic functions are precisely the strictly order-reversing maps. It remains to note that there is a natural bijection between order-reversing maps $I \to [n]$ and order-preserving maps $I \to [n]$. □

COROLLARY 4.3.6. (Stanley's reciprocity for posets [**88**, Theorem 3]) *Let p be a poset on I and $n \in \mathbb{N}$. Then $(-1)^{|I|}\chi_I(p)(-n)$ is the* order polynomial *of p, that is,*

$$(-1)^{|I|}\chi_I(p)(-n) \ = \ \text{number of order-preserving maps } p \to [n].$$

PROOF. This is a consequence of Proposition 4.2.4 and the following observations. The poset cone $\mathcal{P}(p)$ only has one vertex, namely, the origin. If $y : p \to [n]$ is order-reversing, then there is a y-maximum face $\mathcal{P}(p)_y$, and it contains that single vertex. If y is not order-reversing, then $\mathcal{P}(p)$ is not bounded above in the direction of y. □

4.3.4. The Bergman polynomial of a matroid. A *loop* in a matroid is an element which is not contained in any basis.

DEFINITION 4.3.7. The *Bergman character* γ of the Hopf monoid of matroids $\mathbf{M}$ is given by

$$\gamma_I(m) = \begin{cases} 1 & \text{if } m \text{ has no loops} \\ 0 & \text{otherwise.} \end{cases}$$

for a matroid m on I. The *Bergman polynomial* $B(m)$ of a matroid m is the invariant associated to γ by Proposition 4.1.1 and (47).

Note that γ is indeed a character, because a direct sum of matroids $m \oplus n$ is loopless if and only if m and n are both loopless. To study the Bergman polynomial, we need some definitions. A *flat* is a set F of elements such that $r(F \cup i) > r(F)$ for every $i \notin F$. When m is the matroid of a collection of vectors A in a vector space V, the flats correspond to the subspaces of V spanned by subsets of A. The flats form a lattice L under inclusion, and the Möbius number $\mu_L(\widehat{0}, \widehat{1})$ of this lattice (see [**6**], [**94**, Chapter 3]) is also called the *Möbius number of the matroid* $\mu(m)$.

We call $B(m)$ the *Bergman polynomial* because it is related to the *Bergman fan*

$$\mathcal{B}(m) = \{y \in \mathbb{R}^I \mid m_y \text{ has no loops}\}$$

where m_y is the matroid whose bases are the y-maximum bases of m. Notice that the matroid polytope of m_y is the y-maximum face of the matroid polytope of m; that is, $\mathcal{P}(m_y) = \mathcal{P}(m)_y$. Therefore $\mathcal{B}(m)$ is a polyhedral fan: it is a subfan of the normal fan of the matroid polytope $\mathcal{P}(m)$, consisting of the faces $\mathcal{N}_m(n)$ normal to the loopless faces n of m.

Note also that $\mathcal{B}(m)$ is invariant under translation by $\mathbf{1}$ and under scaling by a positive constant. Therefore, nothing is lost by intersecting it with the hyperplane $\sum_i x_i = 0$ and the sphere $\sum_i x_i^2 = 1$, to obtain the *Bergman complex* $\widetilde{\mathcal{B}}(m)$.

Bergman fans of matroids are central objects in tropical geometry, because they are the tropical analog of linear spaces [**10**, **99**]. Two central results are the following combinatorial and topological descriptions.

THEOREM 4.3.8. [**10**] *Let m be a matroid of rank r on I. The Bergman fan $\mathcal{B}(m)$ has a triangulation into cones of the braid arrangement $\mathcal{B}_I$, consisting of the cones $\mathcal{B}_{S_1,\ldots,S_r}$ such that $S_1 \sqcup \cdots \sqcup S_i$ is a flat of m for $i = 1, \ldots, r$.*

THEOREM 4.3.9. [**10**] *The Bergman complex of a matroid m of rank r is homeomorphic to a wedge of $(-1)^r \mu(m)$ spheres of dimension $r - 2$, where $\mu(m)$ is the Möbius number of m.*

We now describe some of the combinatorial properties of the Bergman polynomial. The first one is essentially equivalent to [**23**, Example 4.15]. Define a *flag of flats* of m to be an increasing chain of flats under containment $\emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = \widehat{1}$. We call n the length of the flag. Similarly, a *weak flag of flats* to be a weakly increasing chain of flats.

PROPOSITION 4.3.10. *At a natural number n, the* Bergman polynomial $B(m)$ *of a matroid m is given by*

$$B(m)(n) = \textit{number of weak flags of flats of } m \textit{ of length } n = \sum_{k=0}^{r} c_d \binom{n}{d},$$

where c_d is the number of flags of flats of m of length d. Its degree is the rank r of m.

PROOF. We use the inclusion $\mathbf{M} \hookrightarrow \mathbf{GP}$ to proceed geometrically. Let $\mathfrak{p} = \mathcal{P}(m)$ be the matroid polytope of m. The summand of $B_I(\mathfrak{p})(n)$ in (47) corresponding to a decomposition $I = S_1 \sqcup \cdots \sqcup S_n$ equals

$$(\gamma_{S_1} \otimes \cdots \otimes \gamma_{S_n}) \circ \Delta_{S_1,\ldots,S_n}(\mathfrak{p}) = \gamma_{S_1}(\mathfrak{p}_1) \cdots \gamma_{S_n}(\mathfrak{p}_n) = \gamma_I(\mathfrak{p}_{T_1,\cdots,T_d})$$

where $I = T_1 \sqcup \cdots \sqcup T_d$ is the composition obtained by removing all empty parts, and $\mathfrak{p}_{T_1,\cdots,T_d}$ is the y-maximal face of $\mathfrak{p}$ for any $y \in \mathcal{B}_{T_1,\ldots,T_d}$. This term contributes 1 to the sum if $\mathfrak{p}_{T_1,\cdots,T_d}$ is loopless and 0 otherwise.

By Theorem 4.3.8, $\mathfrak{p}_{T_1,\cdots,T_d}$ is loopless if and only if $\mathcal{B}_{T_1,\ldots,T_d}$ is in the Bergman fan of m, and this is the case if and only if $\emptyset \subsetneq T_1 \subsetneq T_1 \cup T_2 \subsetneq \cdots \subsetneq T_1 \cup \cdots \cup T_d = I$ is a flag of flats. For fixed n and d there are c_d choices for that flag of flats, and $\binom{n}{d}$ ways to enlarge the resulting composition $I = T_1 \sqcup \cdots \sqcup T_d$ into a decomposition $I = S_1 \sqcup \cdots \sqcup S_n$ by adding empty parts. This results in a weak flag of flats $\emptyset \subseteq S_1 \subseteq S_1 \cup S_2 \subseteq \cdots \subseteq S_1 \cup \cdots \cup S_n = I$ of length n.

Since $\binom{n}{d}$ is a polynomial in n of degree d, the degree of $B(m)$ is the largest possible length of a flag of flats of m, which is the rank r of m. □

PROPOSITION 4.3.11. (Bergman polynomial reciprocity.) *The Bergman polynomial of a matroid m of rank r satisfies*

$$B(m)(-1) = (-1)^r \mu(m)$$

where $\mu(m)$ is the Möbius number of m.

PROOF. Using Proposition 4.1.5 and Theorem 1.6.1 we get

$$\begin{aligned} B(m)(-1) &= \gamma_I(\mathrm{s}_I(m)) = \sum_{n \text{ face of } m} (-1)^{|I|-\dim n} \gamma(n) \\ &= \sum_{\substack{n \text{ face of } m \\ n \text{ loopless}}} (-1)^{|I|-\dim n} = \sum_{\substack{F = \mathcal{N}_m(n) \\ \text{face of } \mathcal{B}(m)}} (-1)^{\dim F} = \overline{\chi}(\widetilde{\mathcal{B}}(m)), \end{aligned}$$

the reduced Euler characteristic of the Bergman complex of m. The result now follows from Theorem 4.3.9. □

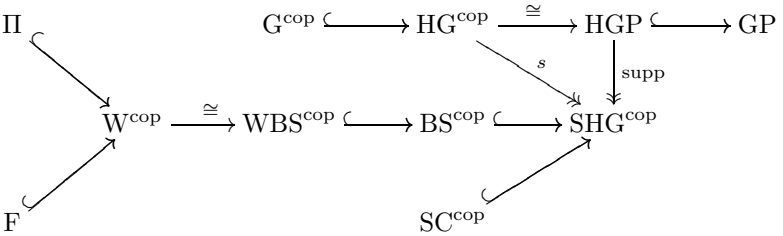
CHAPTER 5

Hypergraphs, simplicial complexes, and building sets

For the remainder of this manuscript, when P and Q are polytopes, we will write $P + Q$ for the Minkowski sum of P and Q. This is not to be confused with the formal sum of polytopes entering in earlier formulas such as (29).

5.1. HGP: Minkowski sums of simplices, hypergraphs, Rota's question

In this section we focus on a large family of generalized permutahedra which we call *hypergraphic polytopes* or *Minkowski sums of simplices*. The polytopes in this family conserve the Hopf algebraic structure of GP while featuring additional combinatorial structure, which makes them very useful for combinatorial applications, as we will see in Sections 5.2, 5.3, 5.4, 5.5, 5.6, and 5.7. In fact, HGP is a useful source of old and new Hopf monoids: we start with some important subfamilies of generalized permutahedra – namely hypergraphic polytopes, graphic zonotopes, simplicial complex polytopes, nestohedra, graph associahedra, permutahedra, and associahedra – and we let them give rise to several interesting (and mostly new) Hopf monoids of a more combinatorial nature, denoted $\mathrm{HG}, \mathrm{SHG}, \mathrm{G}, \mathrm{SC}, \mathrm{BS}, \mathrm{WBS}, \mathrm{W}, \Pi, \mathrm{F}$, which consist of hypergraphs, simple hypergraphs, graphs, simplicial complexes, building sets, graphical building sets, simple graphs, set partitions, and paths, respectively. As we will see in the upcoming sections, these Hopf monoids are related to each other by the following morphisms:



5.1.1. Minkowski sums of simplices. We briefly mentioned in earlier sections that permutahedra, Loday's associahedra, and graphic zonotopes may be expressed as Minkowski sums of simplices. We now place these statements into a broader context, following Postnikov [**77**].

Recall that the *Minkowski sum* of two polytopes P and $Q \subseteq \mathbb{R}^I$ is

$$P + Q := \{p + q \mid p \in P,\ q \in Q\} \subseteq \mathbb{R}^I.$$

For the remainder of this monograph, $P + Q$ will always denote this Minkowski sum.

Normal fans of polytopes behave well under scaling and Minkowski sums: the polytopes P and λP have the same normal fan for $\lambda > 0$, while the normal fan of $P + Q$ (and hence of $\lambda P + \mu Q$ for $\lambda, \mu > 0$) is the coarsest common refinement of the normal fans of P and Q [**99**]. It follows that if P and Q are generalized permutahedra, then so is $\lambda P + \mu Q$ for $\lambda, \mu \geq 0$.

Recalling from Theorem 3.1.3 that every generalized permutahedron $\mathfrak{p}$ is associated to a unique submodular function z such that $\mathfrak{p} = \mathcal{P}(z)$, the previous statement has the following counterpart. If z and z' are submodular functions, then so is $\lambda z + \mu z'$ for $\lambda, \mu \geq 0$, and

$$\lambda \mathcal{P}(z) + \mu \mathcal{P}(z') = \mathcal{P}(\lambda z + \mu z'). \tag{51}$$

Let $\Delta_I = \operatorname{conv}\{e_i \mid i \in I\}$ be the *standard simplex* in $\mathbb{R}^I$. Let

$$\Delta_J = \operatorname{conv}\{e_i \mid i \in J\} \qquad \text{for } J \subseteq I$$

be the faces of Δ_I; note that the face Δ_J is itself the standard simplex in $\mathbb{R}^J$. The following proposition is a consequence of (51).

PROPOSITION 5.1.1 ([**77**, Proposition 6.3]). *If $y : 2^I \to \mathbb{R}_{\geq 0}$ is a non-negative Boolean function then the Minkowski sum $\sum_{J \subseteq I} y(J)\Delta_J$ of dilations of faces of the standard simplex in $\mathbb{R}^I$ is a generalized permutohedron. We have*

$$\sum_{J \subseteq I} y(J)\Delta_J = \mathcal{P}(z), \tag{52}$$

where z is the submodular function given by

$$z(J) = \sum_{K \cap J \neq \emptyset} y(K) \quad \text{for each } J \subseteq I.$$

Furthermore, if a polytope can be written in the form (52)*, then there is a unique choice of y that makes this equation hold.*[1]

DEFINITION 5.1.2. A generalized permutahedron $\mathfrak{p}$ is *y-positive* if it is given by (52) for a non-negative Boolean function $y : 2^I \to \mathbb{R}_{\geq 0}$. If, additionally, $y(J)$ is an integer for all $J \subseteq I$, we call $\mathfrak{p}$ a *Minkowski sum of simplices* or a *hypergraphic polytope*.

We should say a word about this nomenclature. A *hypergraph* $\mathcal{H}$ on I is a collection of (possibly repeated) subsets of I, called the *multiedges* of $\mathcal{H}$. Our convention will be that the empty set appears exactly once in $\mathcal{H}$. Then there is a natural bijection between hypergraphs and hypergraphic polytopes: to a hypergraph $\mathcal{H}$ on I containing $y(J)$ copies of the subset $J \subseteq I$, we associate the hypergraphic polytope $\Delta_{\mathcal{H}} = \sum_{H \in \mathcal{H}} \Delta_H = \sum_{J \subseteq I} y(J)\Delta_J$.

REMARK 5.1.3. We saw in Theorem 3.1.3 that there is a one-to-one correspondence between generalized permutahedra in $\mathbb{R}^n$ and submodular functions, which naturally form a polyhedral cone in $\mathbb{R}^{2^n - 1}$. The y-positive generalized permutahedra form a polyhedral subcone of this submodular cone, which is full-dimensional

[1] In fact, every generalized permutahedron can be expressed uniquely as a *signed Minkowski sum* $\sum_{J \subseteq I} y(J)\Delta_J$ where $y(J)$ is allowed to be negative, but the definitions become more subtle. We will not pursue this point of view here; for more information, see [**7**, Proposition 2.3].

since it is parameterized by $2^n - 1$ independent parameters. The inequalities defining this subcone will be given in Proposition 5.1.4.2.

Many polytopes of interest are hypergraphic, although that is not always apparent at the outset. For example, graphic zonotopes, permutahedra, and associahedra turn out to be hypergraphic, but this is not clear from their definitions. We will see many other examples in the upcoming sections.

5.1.2. Relations, hypergraphic polytopes, and Rota's question. A relation $R \subseteq I \times J$ gives rise to a function $f_R : 2^I \to \mathbb{N}$ defined by

$$f_R(A) = |R(A)| = |\{b \in B \mid (a,b) \in R \text{ for some } a \in A\}| \qquad \text{for } A \subseteq I.$$

Let us call such a function *relational*. One may verify that every relational function is submodular, and Rota [**63**, Problem 2.4.1(d)] asked for a characterization of these *relational submodular functions*:

> There is an interesting open question which ought to have been worked out, and that I ought to have worked out, but I haven't: Characterize those submodular set functions that come from a relation in this way [**80**, Exercise 18.1].

It is likely that Rota knew how to do this, but we have not been able to find a precise statement in the literature. We offer the following characterizations.

PROPOSITION 5.1.4. *A submodular function $f : 2^I \to \mathbb{R}$ is relational if and only if either of the following conditions hold:*

(1) *Its associated polytope $\mathcal{P}(f)$ is hypergraphic.*
(2) *$f(\emptyset) = 0$ and for all $A \subseteq I$ we have $f(A) \in \mathbb{Z}$ and*

$$\sum_{K \supseteq A} (-1)^{|K-A|} f(K) \leq 0.$$

PROOF. 1. A relation $R \subseteq I \times J$ naturally gives rise to a hypergraph $\mathcal{H}_R$ on I whose hyperedges $h_j = \{i \mid (i,j) \in R\}$ for $j \in J$ are given by the columns of R. Clearly any hypergraph on I arises in this way from a relation. If $y_R(K)$ is the multiplicity of hyperedge K in $\mathcal{H}_R$ then

$$f_R(A) = \sum_{K \cap A \neq \emptyset} y_R(K) \tag{53}$$

for all $A \subseteq I$. Proposition 5.1.1 then gives

$$\mathcal{P}(f_R) = \sum_{J \subseteq I} y_R(K) \Delta_K.$$

which is a Minkowski sum of simplices. Conversely, given such a Minkowski sum, we can use its coefficients as the multiplicities of a hypergraph which gives rise to the desired relation.

2. The submodular function of a relation R clearly satisfies $f_R(\emptyset) = 0$. We rewrite (53) as $f_R(A) = |J| - \sum_{K \subseteq I-A} y_R(K)$ and use the inclusion-exclusion formula to obtain

$$y_R(B) = \sum_{K \subseteq B} (-1)^{|B-K|} (|J| - f_R(I-K)) = -\sum_{K \subseteq B} (-1)^{|B-K|} f_R(I-K)$$

for $B \neq \emptyset$. Therefore

$$y_R(I - A) = -\sum_{K \supseteq A} (-1)^{|K-A|} f_R(K) \geq 0 \tag{54}$$

Conversely, for any integral function f satisfying the given inequalities, (54) gives us a non-negative function $y : 2^I \to \mathbb{Z}$. We then construct the desired relation $R \subseteq I \times J$ as in part 1: for each $K \subseteq I$ we include $y(K)$ elements j in J such that $h_j = K$. □

We wish to study these objects further, following the philosophy of Joni and Rota's paper [**60**]: we will describe their Hopf algebraic structure in Sections 5.1.3 and 5.2. This will turn out to be a crucial ingredient for the rest of this monograph.

5.1.3. The Hopf monoid of hypergraphic polytopes.

PROPOSITION 5.1.5. *The hypergraphic polytopes form a submonoid* HGP *of the Hopf monoid of generalized permutahedra* GP.

PROOF. Let $I = S \sqcup T$ be a decomposition. To prove HGP is a submonoid of GP we need to prove two things:

• If polytopes $\mathfrak{p}$ and $\mathfrak{q}$ are hypergraphic in $\mathbb{R}^S$ and $\mathbb{R}^T$, then $\mathfrak{p} \times \mathfrak{q}$ is hypergraphic in $\mathbb{R}^I$.

• If $\mathfrak{p}$ is hypergraphic in $\mathbb{R}^I$, then $\mathfrak{p}|_S$ and $\mathfrak{p}/_S$ are hypergraphic in $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively.

For the first statement, if $\mathfrak{p} = \sum_{J \subseteq S} y_1(J)\Delta_J \subseteq \mathbb{R}^S$ and $\mathfrak{q} = \sum_{K \subseteq T} y_2(K)\Delta_K \subseteq \mathbb{R}^T$ are Minkowski sums of simplices, then

$$\mathfrak{p} \times \mathfrak{q} = \mathfrak{p} + \mathfrak{q} = \sum_{J \subseteq S} y_1(J)\Delta_J + \sum_{K \subseteq T} y_2(K)\Delta_K \subseteq \mathbb{R}^I \tag{55}$$

is also a Minkowski sum of simplices.

For the second one, we use that $(P+Q)_v = P_v + Q_v$ for any polytopes $P, Q \subseteq \mathbb{R}^I$ and any linear functional $v \in \mathbb{R}^I$. Now, the maximal face of the simplex Δ_J in direction $\mathbf{1}_S$ is

$$(\Delta_J)_{S,T} = \begin{cases} \Delta_{J \cap S} & \text{if } J \cap S \neq \emptyset \\ \Delta_J & \text{if } J \cap S = \emptyset. \end{cases}$$

Therefore if $\mathfrak{p} = \sum_{J \subseteq I} y(J)\Delta_J \subseteq \mathbb{R}^I$ is a hypergraphic polytope, then its $\mathbf{1}_S$-maximal face is $\mathfrak{p}_{S,T} = \mathfrak{p}|_S + \mathfrak{p}/_S$ where

$$\mathfrak{p}|_S = \sum_{J \cap S \neq \emptyset} y(J)\Delta_{J \cap S} \subseteq \mathbb{R}^S, \qquad \mathfrak{p}/_S = \sum_{J \cap S = \emptyset} y(J)\Delta_J \subseteq \mathbb{R}^T. \tag{56}$$

Therefore $\mathfrak{p}|_S$ and $\mathfrak{p}/_S$ are hypergraphic, as desired. □

Since HGP is a Hopf submonoid of GP, Theorem 1.6.1 gives us a formula for the antipode of **HGP**. We write it down in Theorem 5.2.5 in terms of hypergraphs.

5.2. HG: Hypergraphs

Recall that a *hypergraph* with vertex set I is a collection $\mathcal{H}$ of (possibly repeated) subsets of I. We will use the convention that there is always a single copy of $\emptyset$ in $\mathcal{H}$.[2] We can think of each subset H in $\mathcal{H}$ as a *multiedge* which can now connect any number of vertices.

[2]This is the opposite of the usual convention that $\emptyset \notin \mathcal{H}$.

5.2.1. The Hopf monoid of hypergraphs. Let $HG[I]$ be the set of all hypergraphs with vertex set I. Clearly HG is a species, which we now turn into a Hopf monoid.

Let $I = S \sqcup T$ be a decomposition.

• For $\mathcal{H}_1 \in HG[S]$ and $\mathcal{H}_2 \in HG[T]$, define their product $\mathcal{H}_1 \cdot \mathcal{H}_2 \in HG[I]$ to be the disjoint union $\mathcal{H}_1 \sqcup \mathcal{H}_2$ as a hypergraph on I.

• The coproduct of $\mathcal{H} \in HG[I]$ is $(\mathcal{H}|_S, \mathcal{H}/_S)$, where the restriction and contraction of $\mathcal{H}$ with respect to S are the multisets

$$\begin{aligned} \mathcal{H}|_S &:= \{H \mid H \in \mathcal{H},\, H \subseteq S\} \\ \mathcal{H}/_S &:= \{H \cap T \mid H \in \mathcal{H},\, H \not\subseteq S\} \cup \{\emptyset\}. \end{aligned}$$

Each multiedge H_S of $\mathcal{H}|_S$ has the same multiplicity that it had in $\mathcal{H}$, while the multiplicity of a nonempty multiedge H_T of $\mathcal{H}/_S$ is the sum of the multiplicities of the edges $H \in \mathcal{H}$ such that $H \cap T = H_T$.

The Hopf monoid axioms are easily verified.

EXAMPLE 5.2.1. For the hypergraph $\mathcal{H} = \{\emptyset, 1, 2, 3, 12, 23, 123\}$ on $I = [3]$, we have

$$\begin{aligned} \mathcal{H}|_{13} &= \{\emptyset, 1, 3\}, & \mathcal{H}/_{13} &= \{\emptyset, 2, 2, 2, 2\}, \\ \mathcal{H}|_2 &= \{\emptyset, 2\}, & \mathcal{H}/_2 &= \{\emptyset, 1, 1, 3, 3, 13\}. \end{aligned}$$

We omit the brackets from the individual multiedges in $\mathcal{H}$ for clarity.

5.2.2. Hypergraphs as a submonoid of generalized permutahedra. Recall that the *hypergraphic polytope* of a hypergraph $\mathcal{H}$ on I is the Minkowski sum

$$\Delta_{\mathcal{H}} = \sum_{H \in \mathcal{H}} \Delta_H$$

where Δ_H is the standard simplex in $\mathbb{R}^H \subseteq \mathbb{R}^I$.

EXAMPLE 5.2.2. The hypergraphic polytope for the hypergraph

$$\mathcal{H} = \{\emptyset, 1, 2, 3, 12, 23, 123\}$$

is

$$\Delta_{\mathcal{H}} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_{12} + \Delta_{23} + \Delta_{123},$$

as shown in Figure 1.

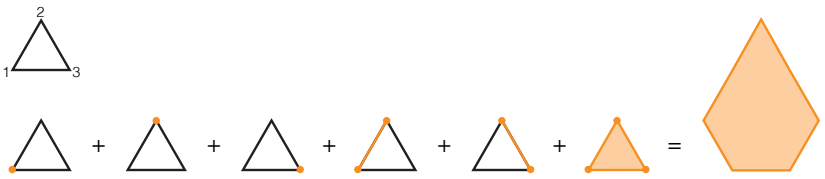


FIGURE 1. The hypergraphic polytope of the hypergraph $\mathcal{H} = \{\emptyset, 1, 2, 3, 12, 23, 123\}$.

Let $\mathrm{HG}^{\mathrm{cop}}$ be co-opposite to the Hopf monoid of hypergraphs HG, as defined in Section 1.1.2; it has the same product and the reverse coproduct of HG.

PROPOSITION 5.2.3. *The map $\mathcal{H} \mapsto \Delta_H$ gives an isomorphism* $\mathrm{HG}^{cop} \xrightarrow{\cong} \mathrm{HGP}$ *between* HG^{cop} *and the Hopf monoid of hypergraphic polytopes* HGP.

PROOF. We know that the map is bijective. The equation (55) says that the map preserves the product and (56), which may be rewritten as $(\Delta_{\mathcal{H}})|_S = \Delta_{\mathcal{H}/_T}$ and $(\Delta_{\mathcal{H}})/_S = \Delta_{\mathcal{H}|_T}$, says that the map reverses the coproduct. □

EXAMPLE 5.2.4. For the hypergraphic polytope of Example 5.2.2 and Figure 1, the northwest edge and southwest vertex are described by

$$\begin{aligned}(\Delta_{\mathcal{H}})_{13,2} &= (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_1 + \Delta_3 + \Delta_{13}) \\ &= \Delta_{\{\emptyset,1,1,3,3,13\}} \times \Delta_{\{\emptyset,2\}} = \Delta_{\mathcal{H}/_2} \times \Delta_{\mathcal{H}|_2}\end{aligned}$$

$$\begin{aligned}(\Delta_{\mathcal{H}})_{2,13} &= (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_2 + \Delta_2 + \Delta_2) \\ &= \Delta_{\{\emptyset,2,2,2,2\}} \times \Delta_{\{\emptyset,1,3\}} = \Delta_{\mathcal{H}/_{13}} \times \Delta_{\mathcal{H}|_{13}},\end{aligned}$$

in (co-opposite) agreement with Example 5.2.2.

THEOREM 5.2.5. *The antipode of the Hopf monoid of hypergraphs* **HG** *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If $\mathcal{H}$ is a hypergraph on I then*

$$\mathrm{s}_I(\mathcal{H}) = \sum_{\Delta_{\mathcal{G}} \leq \Delta_{\mathcal{H}}} (-1)^{c(\mathcal{G})} \mathcal{G},$$

summing over all faces $\Delta_{\mathcal{G}}$ of the hypergraphic polytope $\Delta_{\mathcal{H}}$ of $\mathcal{H}$, where $c(\mathcal{G})$ is the number of connected components of the hypergraph $\mathcal{G}$.

PROOF. This is the result of applying Theorem 1.6.1 to the submonoid **HGP** of **GP**, taking into account the identification of **HGP** and **HG** of Proposition 5.2.3 and the observation that $\dim \Delta_{\mathcal{G}} = |I| - c(\mathcal{G})$. There is no cancellation or grouping in the right hand side of this equation because $\Delta_{\mathcal{G}} = \Delta_{\mathcal{G}'}$ implies $\mathcal{G} = \mathcal{G}'$. □

EXAMPLE 5.2.6. The antipode of the hypergraph $\mathcal{H} = \{\emptyset, 1, 2, 3, 12, 23, 123\}$ in **HG** is given by the hypergraphic polytope of Figure 1, namely:

$$\begin{aligned}\mathrm{s}_{[3]}(\mathcal{H}) = \quad & \{\emptyset,1,2,3,12,23,123\} - \{\emptyset,1,2,3,1,23,1\} - \{\emptyset,1,2,3,1,3,13\} \\ & -\{\emptyset,1,2,3,12,3,3\} - \{\emptyset,1,2,3,2,23,23\} - \{\emptyset,1,2,3,12,2,12\} \\ & +\{\emptyset,1,2,3,1,2,1\} + \{\emptyset,1,2,3,1,3,1\} + \{\emptyset,1,2,3,1,3,3\} \\ & +\{\emptyset,1,2,3,2,3,3\} + \{\emptyset,1,2,3,2,2,2\}.\end{aligned}$$

5.2.3. Graphs, revisited. We now give another explanation of the inclusion of $\mathrm{G}^{\mathrm{cop}}$ into GP shown in Proposition 3.2.5.

PROPOSITION 5.2.7. *The map $g \mapsto Z_g$ is an injective morphism of Hopf monoids* $\mathrm{G}^{\mathrm{cop}} \hookrightarrow \mathrm{GP}$.

PROOF. Since the graph operations of G defined in Section 1.2.1 are special cases of the hypergraph operations of HG defined in Section 5.2.4, we have an inclusion of Hopf monoids, $\mathrm{G} \hookrightarrow \mathrm{HG}$, which gives an inclusion $\mathrm{G}^{\mathrm{cop}} \hookrightarrow \mathrm{HG}^{\mathrm{cop}}$. Proposition 5.2.3 tells us that the map $\mathcal{H} \mapsto \Delta_{\mathcal{H}}$ is an isomorphism $\mathrm{HG}^{\mathrm{cop}} \cong \mathrm{HGP}$. By Proposition 3.2.3, the composition of these maps is the map $\mathrm{G}^{\mathrm{cop}} \to \mathrm{HGP} \hookrightarrow \mathrm{GP}$ given by $g \mapsto Z_g$. □

5.2.4. Simple hypergraphs and simplification. A hypergraph is *simple* if it has no repeated multiedges.[3] In the applications we have in mind, we are only interested in simple hypergraphs. Unfortunately, simple hypergraphs are not closed under the contraction map of HG, so the Hopf structure that we define on them requires a slightly different contraction map. Let $\mathrm{SHG}[I]$ be the set of all simple hypergraphs with vertex set I.

Let $I = S \sqcup T$ be a decomposition.

- The product of $\mathcal{H}_1 \in \mathrm{SHG}[S]$ and $\mathcal{H}_2 \in \mathrm{SHG}[T]$ is their disjoint union $\mathcal{H}_1 \sqcup \mathcal{H}_2$.
- The coproduct of $\mathcal{H} \in \mathrm{SHG}[I]$ is $(\mathcal{H}|_S, \mathcal{H}/_S)$, where the restriction and contraction of $\mathcal{H}$ with respect to S are:

$$\begin{aligned} \mathcal{H}|_S &:= \{H \mid H \in \mathcal{H},\, H \subseteq S\} \\ \mathcal{H}/_S &:= \{H \cap T \mid H \in \mathcal{H},\, H \not\subseteq S\} \cup \{\emptyset\} = \{B \subseteq T \mid A \sqcup B \in \mathcal{H} \text{ for some } A \subseteq S\}, \end{aligned}$$

now regarded as sets without repetition.

One easily verifies that the *simplification* maps, which remove any repetitions of multiedges in a hypergraph, give a morphism of Hopf monoids $s : \mathrm{HG} \twoheadrightarrow \mathrm{SHG}$. We now show that this map behaves reasonably well with respect to the corresponding polytopes. Define $\overline{\mathrm{HGP}} \subseteq \overline{\mathrm{GP}}$ to be the quotient of HGP obtained by identifying hypergraphic polytopes with the same normal fan.

Proposition 5.2.8. *We have a commutative diagram of Hopf monoids as follows.*

$$\begin{array}{ccc} \mathrm{HG}^{\mathrm{cop}} & \xleftarrow{\ \cong\ } & \mathrm{HGP} \\ {\scriptstyle s}\downarrow & & \downarrow \\ \mathrm{SHG}^{\mathrm{cop}} & \twoheadrightarrow & \overline{\mathrm{HGP}} \end{array}$$

Proof. The two vertical maps are defined in the previous paragraph, while the top map is $\mathcal{H} \mapsto \Delta_{\mathcal{H}}$. It remains to verify that the bottom map that makes this diagram commute is well-defined: if $\mathcal{H}$ is a hypergraph, the normal fan $\mathcal{N}_{\Delta_{\mathcal{H}}}$ is the common refinement of $\mathcal{N}_{\Delta_H}$ as we range over all $H \in \mathcal{H}$; this only depends on the simplification of $\mathcal{H}$. □

Remark 5.2.9. The bottom map $\mathrm{SHG}^{\mathrm{cop}} \hookrightarrow \overline{\mathrm{HGP}}$ of Proposition 5.2.8 is not an isomorphism. For example, $\Delta_{\{\emptyset,12,13,23\}}$ and $\Delta_{\{\emptyset,12,13,23,123\}}$ are hexagons with the same normal fan. More generally, for any simple hypergraph $\mathcal{H}$ on I containing all pairs $\{i.j\}$ with $i, j \in I$, the hypergraphic polytope $\Delta_{\mathcal{H}}$ is normally equivalent to the standard permutahedron π_I. To see this, notice that the normal fan of $\Delta_{\mathcal{H}}$ coarsens the braid arrangement (since $\Delta_{\mathcal{H}}$ is a generalized permutahedron) and refines the braid arrangement (since it has $\pi_I = \sum_{\{i,j\} \subseteq I} \Delta_{\{i,j\}}$ as a Minkowski summand).

5.2.5. The support maps. The *support maps* $\mathrm{supp}_I : \mathrm{HGP}[I] \to \mathrm{SHG}[I]$ will be an important tool in what follows; they take a hypergraphic polytope $\mathfrak{p} = \Delta_{\mathcal{H}} = \sum_{J \subseteq I} y(J)\Delta_J \subseteq \mathbb{R}^I$ to the simple hypergraph supporting it:

$$\mathrm{supp}_I(\mathfrak{p}) := \{J \subseteq I \mid y(J) > 0\} \cup \{\emptyset\}.$$

[3] We allow simple hypergraphs to contain singletons, slightly against the usual convention.

Under the isomorphism $\mathrm{HGP} \cong \mathrm{HG}^{\mathrm{cop}}$ of Proposition 5.2.3 which identifies $\mathfrak{p}$ with its corresponding hypergraph $\mathcal{H}$, the support $\mathrm{supp}_I(\mathfrak{p})$ is the *simplification* of $\mathcal{H}$.

THEOREM 5.2.10. *The support maps* $\mathrm{supp}_I : \mathrm{HGP}[I] \to \mathrm{SHG}[I]$ *give a surjective morphism of Hopf monoids* $supp : \mathrm{HGP} \twoheadrightarrow \mathrm{SHG}^{\mathrm{cop}}$.

PROOF. This morphism is the composition of the top isomorphism with the simplification map s in Proposition 5.2.8. □

THEOREM 5.2.11. *The antipode of the Hopf monoid of simple hypergraphs* **SHG** *is given by the following* ***cancellation-free*** *expression. If* $\mathcal{H}$ *is a simple hypergraph on* I *then*

$$\mathrm{s}_I(\mathcal{H}) = \sum_{F \leq \Delta_{\mathcal{H}}} (-1)^{c(F)} \mathrm{supp}_I(F),$$

summing over all faces F *of the hypergraphic polytope* $\Delta_{\mathcal{H}}$ *of* $\mathcal{H}$*, where* $c(F) = |I| - \dim F$ *is the number of connected components of the hypergraph* $\mathrm{supp}_I(F)$.

PROOF. Thanks to Proposition 1.1.17, the surjective maps supp turn Theorem 5.2.5, our formula for the antipode of $\mathbf{HG}^{\mathrm{cop}} \cong \mathbf{HGP}$, into a formula for the antipode of **SHG**. The formula is cancellation free because faces of different dimension must have different support. □

EXAMPLE 5.2.12. The antipode of the hypergraph $\mathcal{H} = \{\emptyset, 1, 2, 3, 12, 23, 123\}$ in **SHG** is also given by the hypergraphic polytope of Figure 1, but the result is now the simplification of the one in Example 5.2.6:

$$\mathrm{s}_{[3]}(\mathcal{H}) = \quad \mathcal{H} - 2\{\emptyset, 1, 2, 3, 23\} - 2\{\emptyset, 1, 2, 3, 12\} - \{\emptyset, 1, 2, 3, 12\} + 5\{\emptyset, 1, 2, 3\}$$

As in the case of matroids, we have no simple combinatorial labeling of the faces of a general hypergraphic polytope, so we do not have a way of simplifying the formula of Theorem 5.2.11. This shows that hypergraphic polytopes are fundamental in the Hopf structure of hypergraphs.

However, we do know a few families of hypergraphic polytopes whose combinatorial structure we can describe more explicitly; they give rise to interesting combinatorial families which inherit Hopf monoid structures from their polytopes. In the remaining sections of this monograph, we will describe the resulting Hopf monoids and use Theorem 5.2.11 to describe their antipodes.

5.3. SC: Simplicial complexes, graphs, and Benedetti et al.'s formula

Benedetti, Hallam, and Machacek [**14**] constructed a combinatorial Hopf algebra of simplicial complexes, and obtained a formula for its antipode through a clever combinatorial argument. Surprisingly, the formula is almost identical to Humpert and Martin's formula for the antipode of the Hopf algebra of graphs [**58**]. In this section, by modeling simplicial complexes polytopally, we are able to offer a simple geometric explanation of this phenomenon.

A(n abstract) *simplicial complex* on a finite set I is a collection $\mathcal{C}$ of subsets of I, called *faces*, such that any subset of a face is a face; that is, if $J \in \mathcal{C}$ and $K \subseteq J$ then $J \in \mathcal{C}$. For a subset $J \subseteq I$, the *induced simplicial complex* $\mathcal{C}|_J$ consists of the faces of $\mathcal{C}$ which are subsets of J.

5.3.1. The Hopf monoid of simplicial complexes. Let $\mathrm{SC}[I]$ denote the set of all simplicial complexes on I. We turn the set species SC into a commutative and cocommutative Hopf monoid with the following structure.

Let $I = S \sqcup T$ be a decomposition.

- The product of two simplicial complexes $\mathcal{C}_1 \in \mathrm{SC}[S]$ and $\mathcal{C}_2 \in \mathrm{SC}[T]$ is their disjoint union.
- The coproduct of a simplicial complex $\mathcal{C} \in \mathrm{SC}[I]$ is $(\mathcal{C}|_S, \mathcal{C}|_T)$.

The Hopf monoid axioms are easily verified.

At first sight, this Hopf monoid – which is cocommutative – does not seem related to the Hopf monoids of hypergraphs – which are not cocommutative. However, it turns out that SC lives inside the cocommutative part of SHG.

PROPOSITION 5.3.1. *The Hopf monoid of simplicial complexes* SC *is a submonoid of the Hopf monoid of simple hypergraphs* SHG.

PROOF. Simplicial complexes are simple hypergraphs, and the product and restriction operations for these two families coincide. The contraction operations are defined slightly differently. However, if $\mathcal{C}$ is a simplicial complex and $I = S \sqcup T$ is a decomposition, one may verify that the contraction $\mathcal{C}/_S$ in the sense of simple hypergraphs coincides with the restriction $\mathcal{C}|_T$ in the sense of simplicial complexes. □

5.3.2. Simplicial complex polytopes. Each simplicial complex $\mathcal{C}$, being a hypergraph, has a corresponding hypergraphic polytope $\Delta_{\mathcal{C}} := \sum_{C \in \mathcal{C}} \Delta_C$. Unlike general hypergraphic polytopes, this family of polytopes have a simple combinatorial facial structure.

Recall that the *one-skeleton* $\mathcal{C}^{(1)}$ of a simplicial complex on I is the graph on I whose edges are the sets in $\mathcal{C}$ of size 2.

LEMMA 5.3.2. *For any simplicial complex $\mathcal{C}$, the hypergraphic polytope $\Delta_{\mathcal{C}}$ is normally equivalent to the graphic zonotope $Z_{\mathcal{C}^{(1)}}$ of its one-skeleton $\mathcal{C}^{(1)}$.*

PROOF. We use the central fact from Proposition 5.2.8 that the normal equivalence class of a hypergraphic polytope $\Delta_{\mathcal{H}}$ depends only on the support $\mathrm{supp}(\Delta_{\mathcal{H}})$.

Let $\mathcal{C}$ be a simplicial complex on I. Since they have the same support, the simplicial complex polytope $\Delta_{\mathcal{C}} = \sum_{F \in \mathcal{C}} \Delta_F$ is normally equivalent to the polytope

$$P_1 = \sum_{G \in \mathcal{C}} \sum_{F \subseteq G} \Delta_F = \sum_{G \in \mathcal{C}} \pi'_G.$$

where we define $\pi'_G := \sum_{F \subseteq G} \Delta_F$ for each set $G \in \mathcal{C}$. By Remark 5.2.9, π'_G is normally equivalent to the standard permutahedron π_G in $\mathbb{R}^I$. Therefore the polytope P_1 is normally equivalent to

$$P_2 = \sum_{G \in \mathcal{C}} \pi_G = \sum_{G \in \mathcal{C}} \sum_{\{i,j\} \subseteq G} \Delta_{\{i,j\}}$$

using (19). In turn, P_2 is normally equivalent to $Z_{\mathcal{C}^{(1)}} = \sum_{\{i,j\} \in \mathcal{C}} \Delta_{\{i,j\}}$ since they have the same support. □

As a consequence, the simplicial complex polytope $\Delta_{\mathcal{C}}$ has the same facial structure as the zonotope Z_g for $g = \mathcal{C}^{(1)}$, as described by Lemma 3.2.4. It would be interesting to further study these simplicial complex polytopes.

5.3.3. The antipode of simplicial complexes. Since simplicial complexes form a submonoid of simple hypergraphs by Proposition 5.3.1, we may use Theorem 5.2.11 to compute the antipode of **SC**, thus recovering the formula of Benedetti, Hallam, and Machacek [**14**]. We now carry this out.

Let $\mathcal{C}$ be a simplicial complex $\mathcal{C}$ on I and let f be a flat of the 1-skeleton $\mathcal{C}^{(1)}$ of $\mathcal{C}$. The flat f is a subgraph of $\mathcal{C}^{(1)}$, and its connected components form a partition $\pi = \{\pi_1, \dots, \pi_k\}$ of its vertex set I. As before, we let $c(f) = k$ denote the number of connected components of f. We define $\mathcal{C}(f) = \mathcal{C}|_{\pi_1} \sqcup \cdots \sqcup \mathcal{C}|_{\pi_k}$ to be the subcomplex of $\mathcal{C}$ consisting of the faces which are contained in a connected component of f.

Corollary 5.3.3 ([**14**]). *The antipode of the Hopf monoid of simplicial complexes* **SC** *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If $\mathcal{C}$ is a simplicial complex on I then*

$$\mathrm{s}_I(\mathcal{C}) = \sum_f (-1)^{c(f)} a(g/f)\, \mathcal{C}(f),$$

summing over all flats f of the 1-skeleton $g = \mathcal{C}^{(1)}$ of $\mathcal{C}$, where $a(g/f)$ is the number of acyclic orientations of the contraction g/f.

Proof. By Theorem 5.2.11, the antipode of $\mathcal{C}$ is given by the face structure of the polytope $\Delta_{\mathcal{C}}$, which is equivalent to the face structure of the zonotope Z_g by Lemma 5.3.2. Lemma 3.2.4 tells us that the faces of these polytopes are in bijection with the pairs of a flat f of g and an acyclic orientation o of g/f. Recall from that proof that the maximal face $(\Delta_{\mathcal{C}})_y$ in a direction $y \in \mathbb{R}^I$ depends only on a flat $f = f_y$ of g and an orientation $o = o_y$ of g/f determined by y. The flat f of g consists of the edges ij such that $y(i) = y(j)$; the acyclic orientation o of g/f will be irrelevant here.

The corollary will now follow from the claim that the support of the $(|I| - c)$-dimensional face $(\Delta_{\mathcal{C}})_y$ equals $\mathcal{C}(f)$, independently of the choice of o. To prove this claim, we will use the following expressions:

$$(\Delta_{\mathcal{C}})_y = \sum_{C \in \mathcal{C}} (\Delta_C)_y, \qquad \Delta_{\mathcal{C}(f)} = \sum_{\substack{C \in \mathcal{C}: \\ y \text{ is constant on } C}} \Delta_C. \tag{57}$$

We will show that they have the same summands, possibly with different multiplicities.

$\longrightarrow$: For each $C \in \mathcal{C}$ we have $(\Delta_C)_y = \Delta_{C_{max}}$ where $C_{max} = \{c \in C \,|\, y(c) \text{ is maximum}\}$. Clearly y is constant on C_{max}, so this is a summand of $\Delta_{\mathcal{C}(f)}$.

$\longleftarrow$: For any summand Δ_C of $\Delta_{\mathcal{C}(f)}$, C is a face of the simplicial complex $\mathcal{C}$ where y is constant, so $\Delta_C = \Delta_{C_{max}} = (\Delta_C)_y$ is a summand of $(\Delta_{\mathcal{C}})_y$.

This proves the claim that $\mathrm{supp}(\Delta_{\mathcal{C}})_y = \mathcal{C}(f)$, and the desired result follows. □

The proof above gives a simple geometric explanation for the striking similarity between the antipode formulas for the Hopf algebra of graphs **G** and the Hopf algebra of simplicial complexes **SC**: these formulas have the same combinatorial structure because they are controlled by polytopes that are normally equivalent.

5.4. BS: Building sets and nestohedra

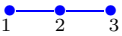
In this section we study *building sets*, a second family of hypergraphs whose hypergraphic polytope has an elegant combinatorial structure. This allows us to describe the Hopf theoretic structure of building sets very explicitly.

Building sets were introduced independently and almost simultaneously in two very different contexts by De Concini and Procesi [**31**] in their construction of the wonderful compactification of a hyperplane arrangement, and by Schmitt [**86**] (who called them *Whitney systems*) in an effort to abstract the notion of connectedness. We follow [**77**]; see also [**37**, **38**, **49**].

DEFINITION 5.4.1. A collection $\mathcal{B}$ of subsets of a set I is a *building set* on I if it satisfies the following conditions:
• If $J, K \in \mathcal{B}$ and $J \cap K \neq \emptyset$ then $J \cup K \in \mathcal{B}$
• For all $i \in I$, $\{i\} \in \mathcal{B}$.
We call the sets in $\mathcal{B}$ *connected*.

We call the maximal sets of a building set $\mathcal{B}$ its *connected components*; one may show that they form a partition of I. If $I \in \mathcal{B}$ then we say $\mathcal{B}$ is *connected*.

One prototypical example of a building set comes from a graph w on vertex set I. The connected sets are the subsets $J \subseteq I$ for which the induced subgraph of w on J is connected. This family of *graphical building sets* is the subject of Section 5.5.

EXAMPLE 5.4.2. The graphical building set for the path 

on $[3]$ is the hypergraph $\{\emptyset, 1, 2, 3, 12, 23, 123\}$ of Example 5.2.2.

Another example of a building set comes from a matroid m on I. The connected sets of m form a building set on I. We recall that a subset $J \subseteq I$ of a matroid is *connected* if for every pair of elements $x, y \in J$ there exists a circuit C (a minimal set with $r(C) < |C|$) such that $\{x, y\} \subseteq C \subseteq J$.

5.4.1. The Hopf monoid of building sets. Let $\mathrm{BS}[I]$ denote the species of building sets on I. The species BS becomes a Hopf monoid with the following additional structure.

Let $I = S \sqcup T$ be a decomposition.

• The product of two building sets $\mathcal{B}_1 \in \mathrm{BS}[S]$ and $\mathcal{B}_2 \in \mathrm{BS}[T]$ is their disjoint union.

• The coproduct of a building set $\mathcal{B} \in \mathrm{BS}[I]$ is $(\mathcal{B}|_S, \mathcal{B}/_S) \in \mathrm{BS}[S] \times \mathrm{BS}[T]$, where the restriction and contraction of $\mathcal{B}$ with respect to S are defined as

$$\begin{aligned} \mathcal{B}|_S &= \{B \mid B \in \mathcal{B},\, B \subseteq S\} \\ \mathcal{B}/_S &= \{B \subseteq T \mid A \sqcup B \in \mathcal{B} \text{ for some } A \subseteq S\}. \end{aligned}$$

One may check that these two collections are indeed building sets, and that the operations defined above satisfy the axioms of Hopf monoid.

PROPOSITION 5.4.3. *The Hopf monoid of building sets* BS *is a submonoid of the Hopf monoid of simple hypergraphs* SHG.

PROOF. Building sets are simple hypergraphs, and the product, restriction, and contraction operations for these two families are defined identically. □

Note that this Hopf structure is essentially the same as the one defined by Grujić in [**48**, **50**], but different from the (cocommutative) Hopf algebras of building sets defined in [**49**, **86**].

5.4.2. Nestohedra. Since each building set $\mathcal{B}$ is a hypergraph, we can model it polytopally using its hypergraphic polytope, which is called the *nestohedron*

$$\Delta_{\mathcal{B}} = \sum_{J \in \mathcal{B}} \Delta_J.$$

Unlike general hypergraphic polytopes, there is an explicit combinatorial description of the faces of the nestohedron $\Delta_{\mathcal{B}}$; they are in bijection with the *nested sets* for $\mathcal{B}$ and with the $\mathcal{B}$-*forests*, two equivalent families of objects which we now define.

DEFINITION 5.4.4. [**38**, **77**] A *nested set* $\mathcal{N}$ for a building set $\mathcal{B}$ is a subset $\mathcal{N} \subseteq \mathcal{B}$ such that:
(N1) If $J, K \in \mathcal{N}$ then $J \subseteq K$ or $K \subseteq J$ or $J \cap K = \emptyset$.
(N2) If $J_1, \ldots, J_k \in \mathcal{N}$ are pairwise incomparable and $k \geq 2$ then $J_1 \cup \cdots \cup J_k \notin \mathcal{B}$.
(N3) All connected components of $\mathcal{B}$ are in $\mathcal{N}$.
The nested sets of $\mathcal{B}$ form a simplicial complex, called the *nested set complex* of $\mathcal{B}$.

EXAMPLE 5.4.5. The collection $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$ is a nested set for the graphical building set of the graph shown in Figure 2(a); see also Figure 4.

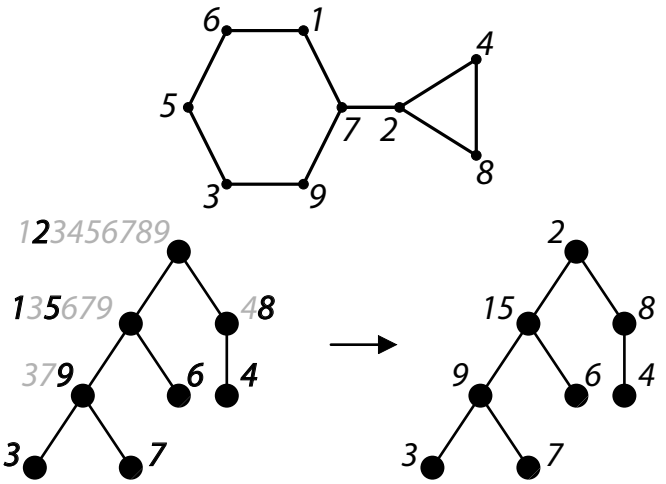


FIGURE 2. (a) A graph w. (b) A nested set for the graphical building set $\mathcal{B}$ of w and the corresponding $\mathcal{B}$-forest.

As shown in [**38**, **77**] and illustrated in Figure 2(b), nested sets for $\mathcal{B}$ are in bijection with a family of objects called $\mathcal{B}$-forests, as follows. We may regard a nested set $\mathcal{N}$ as a poset ordered by containment. We then relabel each node by removing all elements which appear in nodes below it; the result is the corresponding $\mathcal{B}$-forest. We now define these objects more precisely.

DEFINITION 5.4.6. [**38**,**77**] Given a building set $\mathcal{B}$ on I, a $\mathcal{B}$-*forest* $\mathcal{N}$ is a rooted forest whose vertices are labeled with nonempty sets partitioning I such that:
(F1) For any node S, $\mathcal{N}_{\leq S} \in \mathcal{B}$.
(F2) If $S_1, \ldots, S_k$ are pairwise incomparable and $k \geq 2$, $\bigcup_{i=1}^{k} \mathcal{N}_{\leq S_i} \notin \mathcal{B}$.
(F3) If $R_1, \ldots, R_r$ are the roots of $\mathcal{N}$, then the sets $\mathcal{N}_{\leq R_1}, \ldots, \mathcal{N}_{\leq R_r}$ are precisely the connected components of $\mathcal{B}$.

Here $\leq$ denotes the partial order on the nodes of the forest where all branches are directed up towards the roots. Also we denote $\mathcal{N}_{\leq S} := \bigsqcup_{T \leq S} T$.

PROPOSITION 5.4.7. [**38**, **77**] *For any building set $\mathcal{B}$ on I, there is a bijection between the nested sets for $\mathcal{B}$ and the $\mathcal{B}$-forests.*

As the notation suggests, we will make no distinction between a nested set and its corresponding $\mathcal{B}$-forest.

Each $\mathcal{B}$-forest $\mathcal{N}$ gives rise to a building set

$$\mathcal{B}(\mathcal{N}) := \bigsqcup_{S \text{ node of } \mathcal{N}} \mathcal{B}[\mathcal{N}_{<S}, \mathcal{N}_{\leq S}] \tag{58}$$

where for $X \subseteq Y \subseteq I$ we define $\mathcal{B}[X, Y] := (\mathcal{B}|_Y)/X = (\mathcal{B}/X)|_{Y-X}$ on $Y - X$.

THEOREM 5.4.8. [**38**, **77**] *Let $\mathcal{B}$ be a building set. There is an order-reversing bijection between the faces of the nestohedron $\Delta_{\mathcal{B}}$ and the nested sets of $\mathcal{B}$. If $\mathcal{N}$ is a nested set of $\mathcal{B}$ and $F_{\mathcal{N}}$ is the corresponding face of $\Delta_{\mathcal{B}}$, then* $\dim F_{\mathcal{N}} = |I| - |\mathcal{N}|$ *and* $\operatorname{supp}_I(F_{\mathcal{N}}) = \mathcal{B}(\mathcal{N})$.

PROOF. This is implicit in the proofs of [**11**, Proposition 3.5] and [**77**, Theorem 7.4, 7.5]. □

In other words, the nestohedron $\Delta_{\mathcal{B}}$ is a simple polytope whose dual simplicial complex is isomorphic to the nested set complex of $\mathcal{B}$. An example is illustrated in Figure 3.

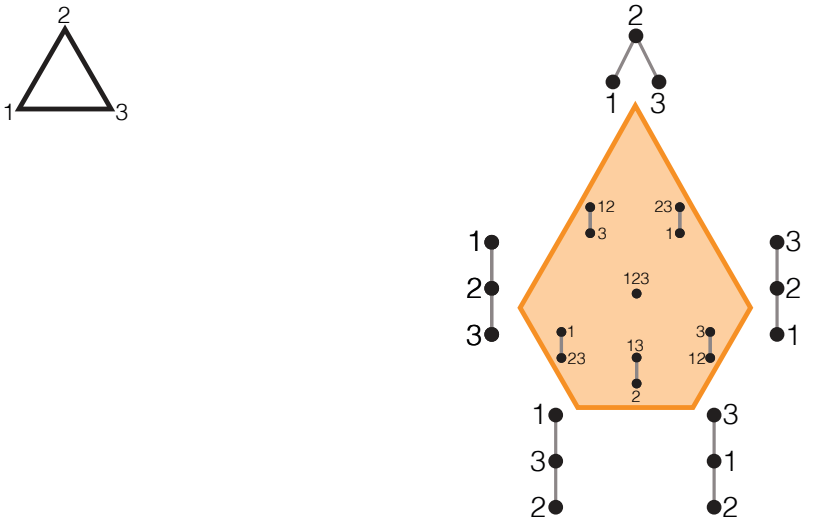


FIGURE 3. The hypergraphic polytope of Figure 1 is the nestohedron for the building set $\mathcal{B} = \{\emptyset, 1, 2, 3, 12, 23, 123\}$; its faces are labeled by the $\mathcal{B}$-forests.

5.4.3. The antipode of building sets. Since building sets form a submonoid of simple hypergraphs by Proposition 5.4.3, we may use Theorem 5.2.11 to compute the antipode of **BS**.

COROLLARY 5.4.9. *The antipode of the Hopf monoid of building sets* **BS** *is given by the following* ***cancellation-free*** *expression. If $\mathcal{B}$ is a building set on I then*

$$s_I(\mathcal{B}) = \sum_{\mathcal{B}-forests\ \mathcal{N}} (-1)^{|\mathcal{N}|} \mathcal{B}(\mathcal{N})$$

where for each $\mathcal{B}$-forest $\mathcal{N}$, $|\mathcal{N}|$ is the number of vertices of $\mathcal{N}$ and $\mathcal{B}(\mathcal{N})$ is defined in (58).

PROOF. By Theorem 5.2.11, the antipode of **BS** is given by the face structure of the nestohedron $\Delta_{\mathcal{B}}$. It remains to invoke Theorem 5.4.8 which tells us the dimension and the building set supporting each face of $\Delta_{\mathcal{B}}$. The formula is cancellation-free since faces of different dimensions have different supports. □

Note that the formula of Corollary 5.4.9 is not combination-free. For example, all vertices of $\Delta_{\mathcal{B}}$ map to the trivial building set $\{\{i\}, i \in I\} \cup \{\emptyset\}$.

EXAMPLE 5.4.10. Let us return to the building set $\{\emptyset, 1, 2, 3, 12, 23, 123\}$ of Example 5.4.2. We computed its antipode in Example 5.2.12:

$$\mathrm{s}_{[3]}(\mathcal{H}) = \quad \mathcal{H} - 2\{\emptyset, 1, 2, 3, 23\} - 2\{\emptyset, 1, 2, 3, 12\} - \{\emptyset, 1, 2, 3, 12\} + 5\{\emptyset, 1, 2, 3\}$$

and we now encourage the reader to compare this with the expression in Corollary 5.4.9.

5.5. W: Simple graphs, ripping and sewing, and graph associahedra

In Section 5.4 we briefly mentioned how connectivity in graphs was one of the motivations to study building sets. In this section we focus on the *graphical building sets* that arise in this way, which give rise to a new Hopf monoid W on graphs. This *ripping and sewing* Hopf monoid should not be confused with the monoids G, SG, and Γ of Sections 3.2 and 3.3.5.

DEFINITION 5.5.1. Let w be a simple graph whose vertex set is I. A subset $J \subseteq I$ is a *tube* if the induced subgraph of w on J is connected. The set of tubes of w is a building set; we denote it tubes(w) and call it the *graphical building set of* w.

Let WBS[I] be the set of graphical building sets on I. We will see in Proposition 5.5.3 that graphical building sets form a submonoid of BS, which we now describe directly in terms of the graphs.

5.5.1. The ripping and sewing Hopf monoid of simple graphs.

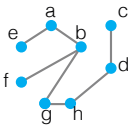
DEFINITION 5.5.2. Given a simple graph w whose vertex set is I, and a partition $I = S \sqcup T$, an *S-thread* is a path in w whose initial and final vertices are in T, and all of whose intermediate vertices (if any) are in S.

Define the operations of *ripping and sewing* as follows.

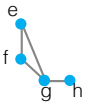
• *ripping out T*: $w|_S$ is the induced subgraph on S, obtained by "ripping out" every vertex of T and every edge incident to T.

• *sewing through S*: $w/_S$ is the simple graph on T where we add or "sew in" an edge uv between vertices $u, v \in T$ if the graph w contains an *S-thread* from u to v. Note that this includes all edges of $w|_T$.

For example, let $I = \{a, b, c, d, e, f, g, h\}$, $S = \{a, b, c, d\}$, and $T = \{e, f, g, h\}$. For the graph

$w =$ 

, we have $w|_S =$ 

and $w/_S =$ 

.

Let $\mathrm{W}[I]$ be the set of simple graphs on vertex set I. We turn the species W into the *ripping and sewing Hopf monoid* with the following operations.

Let $I = S \sqcup T$ be a decomposition.

- The product of two simple graphs $w_1 \in \mathrm{W}[S]$ and $w_2 \in \mathrm{W}[T]$ is their disjoint union.
- The coproduct of a simple graph $w \in \mathrm{W}[I]$ is $(w|_S, w/_S) \in \mathrm{W}[S] \times \mathrm{W}[T]$ where $w|_S$ and $w/_S$ are obtained from w by ripping out T and sewing through S, respectively.

One easily checks that this is indeed a Hopf monoid.

PROPOSITION 5.5.3. *The species* WBS *of graphical building sets is a submonoid of the Hopf monoid of building sets. Furthermore, the tube maps* $w \mapsto \mathrm{tubes}(w)$ *give an isomorphism of Hopf monoids* $\mathrm{W} \cong \mathrm{WBS} \hookrightarrow \mathrm{BS}$.

PROOF. We first prove that the map $\mathrm{tubes} : \mathrm{W} \to \mathrm{BS}$ is a morphism of Hopf monoids. We do know that the set $\mathrm{tubes}(w)$ is a building set for any w. Also tubes preserves products because $\mathrm{tubes}(w_1 \sqcup w_2) = \mathrm{tubes}(w_1) \sqcup \mathrm{tubes}(w_2)$ for $w_1 \in \mathrm{W}[S]$ and $w_2 \in \mathrm{W}[T]$. It remains to check that the map tubes preserves coproducts; that is,

$$\mathrm{tubes}(w)|_S = \mathrm{tubes}(w|_S), \qquad \mathrm{tubes}(w)/_S = \mathrm{tubes}(w/_S)$$

for any simple graph w on I and any subset $S \subseteq I$.

The first statement is clear: the connected sets in w which are subsets of S are precisely the connected sets in $w|_S$, the induced subgraph on S. Let us prove the second one.

$\subseteq$: Suppose $B \in \mathrm{tubes}(w)/_S$, so $A \sqcup B$ is a tube of w for some subset $A \subseteq S$. To show $B \in \mathrm{tubes}(w/_S)$, we need to show that for any $u, v \in B$ there is a path from u to v in $w/_S$.

We do have a path P from u to v inside the induced subgraph $A \sqcup B$ of w, since this is a tube in w. This path may contain vertices of S and T; let $u = t_0, t_1, \ldots, t_{k-1}, t_k = v$ be the vertices of T that it visits, in that order. Now, for each $0 \leq i \leq k-1$, the path P contains an S-thread $t_i s_1 \ldots s_l t_{i+1}$ from t_i to t_{i+1} for some $l \geq 0$, so $t_i t_{i+1}$ is an edge of $w/_S$. It follows that $t_0 t_1 \ldots t_{k-1} t_k$ is our desired path from u to v in $w/_S$. We conclude that $B \in \mathrm{tubes}(w/_S)$.

$\supseteq$: Conversely, suppose $B \in \mathrm{tubes}(w/_S)$. For each edge uv in $w/_S$, choose an S-thread from u to v; let $S_{uv} \subseteq S$ be the set of vertices on that S-thread other than u and v. Let $A \subseteq S$ be the union of the sets S_{uv} as we range over all edges uv of $w/_S$. We claim that $A \sqcup B$ is a tube in w. To show this, first note that any two vertices u, v of B are connected by an S-thread inside $A \sqcup B$ by construction. Furthermore, any vertex of A belongs to the set S_{uv} for some $u, v \in B$, and hence is connected to u and v by a path in $A \sqcup B$. It follows that $A \sqcup B$ is a tube of w and $B \in \mathrm{tubes}(w)/_S$ as desired.

Thus we have proved that tubes : W → BS is a morphism of Hopf monoids, and hence that its image WBS is a submonoid of BS. It remains to prove that the surjective map tubes : W ↠ WBS is also injective. To see this, notice that we can easily recover a simple graph $w \in \mathrm{W}[I]$ from its graphical building set tubes(w): the edges of w are precisely the tubes of size 2. □

5.5.2. Graph associahedra. For a simple graph w on I we define the *graph associahedron* $\Delta_w \subseteq \mathbb{R}^I$ to be

$$\Delta_w := \sum_{\tau \in \mathrm{tubes}(w)} \Delta_\tau.$$

Graph associahedra are the nestohedra corresponding to graphical building sets. Let us recall their combinatorial structure, as described in [**26**, **77**].

Definition 5.5.4. Let w be a simple graph. A *tubing* is a set t of tubes such that:
- any two tubes τ_1 and τ_2 in t are disjoint or nested: we have $\tau_1 \subseteq \tau_2, \tau_1 \supseteq \tau_2$, or $\tau_1 \cap \tau_2 = \emptyset$.
- if $\tau_1, \ldots, \tau_k$ are pairwise disjoint tubes in t, then $\tau_1 \cup \cdots \cup \tau_k$ is not a tube of w.
- every connected component of w is a tube in t.

Comparing this with Definition 5.4.4 we see that the tubings of w are precisely the nested sets for the graphical building set tubes(w). An example is shown in Figure 4.

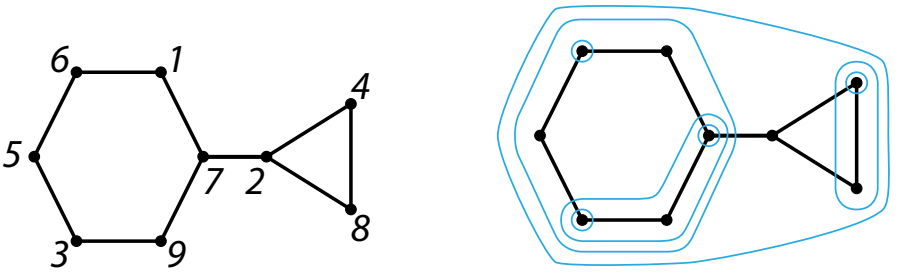


Figure 4. The nested set $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$ of Figure 2, now drawn as a tubing.

For each tube τ in a tubing t, let $t_{<\tau}$ be the union of the tubes of t that are strictly contained in τ, and let the *essential set* of τ be $\mathrm{ess}(\tau) = \tau - t_{<\tau}$. As τ ranges over the tubes of t, the essential sets $\mathrm{ess}(\tau)$ partition I.

Each tubing t of w gives rise to a simple graph

$$w(t) := \bigsqcup_{\tau \text{ tube of } t} w[t_{<\tau}, \tau], \tag{59}$$

where $w[t_{<\tau}, \tau] := (w|_\tau)/_{t_{<\tau}}$ is the simple graph on $\mathrm{ess}(\tau)$ obtained by restricting w to τ and then sewing through the tubes strictly inside of τ. Since the essential sets of τ partition I, $w(t)$ is a simple graph on I.

THEOREM 5.5.5. [**26**, **77**] *Let w be a simple graph. There is an order-reversing bijection between the faces of the graph associahedron Δ_w and the tubings of w. If t is a tubing of w and F_t is the corresponding face of Δ_w, then* $\dim F_t = |I| - |t|$ *and* $\operatorname{supp}_I(F_t) = w(t)$.

PROOF. This is the result of specializing Theorem 5.4.8 to graphical building sets and graph associahedra. □

An example is illustrated in Figure 5.

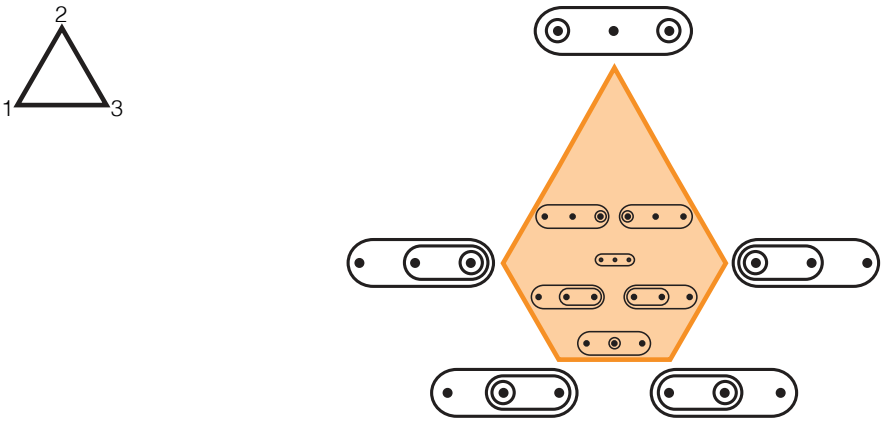


FIGURE 5. The nestohedron of Figures 1 and 3 is the graph associahedron for the path of length 3; its faces are labeled by the tubings of the path.

5.5.3. The antipode of the ripping and sewing Hopf monoid.

THEOREM 5.5.6. *The antipode of the ripping and sewing Hopf monoid of simple graphs* $\mathbf{W}$ *is given by the following* ***cancellation-free*** *expression. If w is a simple graph on I then:*

$$s_I(w) = \sum_{t \text{ tubing}} (-1)^{|t|} w(t)$$

where $|t|$ is the number of tubes of t and $w(t)$ is defined in (59).

PROOF. Since $\mathbf{W}$ is isomorphic to the Hopf monoid of graphical building sets **WBS**, which is a submonoid of the Hopf monoid of simple hypergraphs **SHG**, its antipode is given by Theorem 5.2.11. It remains to invoke Theorem 5.5.5, and to remark again that faces of different dimension map to different supports. □

Note that the formula above is not combination-free. For example, for every maximal tubing t, $w(t)$ is the graph with no edges.

EXAMPLE 5.5.7. The antipode of the path of length 3 in $\mathbf{W}$ is dictated by its graph associahedron, which again is the polytope of Figures 1, 3, and 5. The result is now:

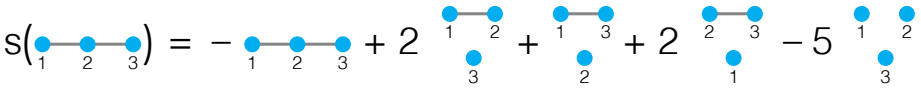


FIGURE 6. The antipode of a path of length 3 in **W**.

5.6. Π: Set partitions and permutahedra, revisited

DEFINITION 5.6.1. A *clique* is a complete graph. A *cliquey graph* is a disjoint union of complete graphs.

Let K[I] be the set of cliquey graphs on I. There is a natural bijection between cliquey graphs on I and set partitions of I: the cliquey graph w on I corresponds to the set partition $\pi(w)$ formed by its connected components.

PROPOSITION 5.6.2. *The species* K *of cliquey graphs is a submonoid of the ripping and sewing Hopf monoid of simple graphs* W. *Furthermore,* K *is isomorphic to the Hopf monoid of set partitions* Π.

PROOF. Since the disjoint union of cliquey graphs is cliquey, K is closed under multiplication. Also if K_I is the clique on I then $(K_I)|_S = K_S$ and $(K_I)/_S = K_T$, so K is also closed under comultiplication, proving the first assertion. The map $\pi : \mathrm{K} \to \Pi$ sending a cliquey graph w to $\pi(w)$ gives the desired isomorphism; it clearly preserves products, and since

$$\pi(K_I)|_S = \{I\}|_S = \{S\} = \pi(K_S) = \pi(K_I|_S) \text{ and}$$
$$\pi(K_I)/_S = \{I\}/_S = \{T\} = \pi(K_T) = \pi(K_I/_S),$$

it also preserves coproducts. □

Since Π is cocommutative, we also have $\Pi \cong K \hookrightarrow W^{\mathrm{cop}}$, as shown in the commutative diagram at the beginning of Section 5.1.

5.6.1. The antipode of set partitions.

THEOREM 5.6.3 ([**2**, Theorem 12.47]). *The antipode of the Hopf monoid of set partitions* **Π** *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If π is a set partition on I,*

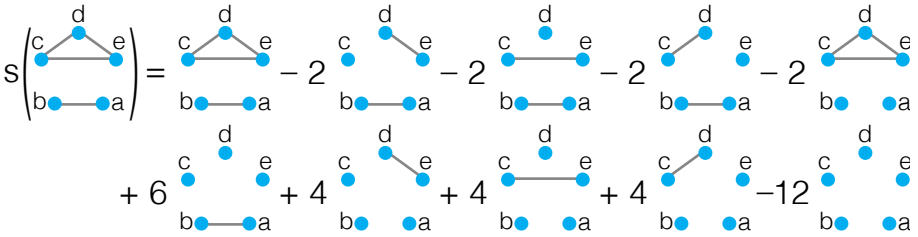
$$s_I(\pi) = \sum_{\rho | \pi \leq \rho} (-1)^{b(\rho)} (\pi : \rho)! \, \rho$$

summing over all partitions ρ that refine π. Here $b(\rho)$ denotes the number of blocks of ρ, and $(\pi : \rho)! = \prod_{p_i \in \pi} n_i!$ where n_i is the number of blocks of ρ that partition the block p_i of π.

PROOF. Let w be a cliquey graph and $\pi = \{p_1, \ldots, p_k\}$ be the corresponding set partition. A tube on w is a subset of one of the parts p_i. A tubing t on w cannot contain two disjoint subsets of the same p_i; thus t consists of a flag $t^i_\bullet$ of subsets $\emptyset = \tau^i_0 \subset \cdots \subset \tau^i_{n_i} = p_i$ for each part p_i. The flag $t^i_\bullet$ gives rise to a composition $p_i = \rho^i_1 \sqcup \cdots \sqcup \rho^i_{n_i}$ where $\rho^i_j = \tau^i_j - \tau^i_{j-1}$. If we let $\rho(t) = \{\rho^i_j \,|\, 1 \leq i \leq k, 1 \leq j \leq n_i\}$ as an unordered set partition, then $\rho(t)$ is the partition corresponding to the graph $w(t)$ of (59). Clearly $\rho(t) \geq \pi$ and $|t| = b(\rho(t))$.

It remains to observe that the map from a tubing t to the partition $\rho(t)$ is a $(\pi : \rho)$!-to-1 map, because there are $n_i!$ linear orders for the partition $\{\rho_1^i, \cdots, \rho_{n_i}^i\}$ of p_i for $1 \leq i \leq k$, which give rise to different choices of the tubing t. □

As an example, let us revisit the cancellation-free formula for the antipode of the set partition $\{ab, cde\}$ shown in the introduction.



As should be clear by now, our derivation of Theorem 5.6.3 is controlled by a polytope; for the set partition π with blocks $p_1, \ldots, p_k$, it is the graph associahedron

$$\Delta_\pi = \pi'_{p_1} \times \cdots \pi'_{p_k} \equiv \pi_{p_1} \times \cdots \pi_{p_k},$$

where $\pi'_I := \sum_{J \subseteq I} \Delta_J$ is normally equivalent to the standard permutahedron π_I.

Thus the antipode of $\pi = \{ab, cde\}$ is an algebraic shadow of the face structure of the hexagonal prism $\pi'_{\{a,b\}} \times \pi'_{\{c,d,e\}}$: it has one 3-face, eight 2-faces (in normal equivalence classes of size 2, 2, 2, 2), eighteen edges (in equivalence classes of sizes 6, 4, 4, 4) and twelve vertices (in one equivalence class of size 12).

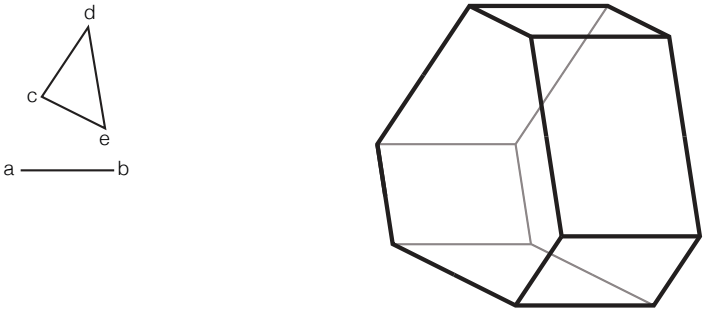


FIGURE 7. The product $\pi'_{\{a,b\}} \times \pi'_{\{c,d,e\}}$ in $\mathbb{R}^{\{a,b,c,d,e\}}$.

5.6.2. Permutahedra, set partitions, and the Hopf algebra of symmetric functions. We conclude this section by precisely stating connections between permutahedra, set partitions, and symmetric functions

PROPOSITION 5.6.4. *The Hopf monoid of permutahedra* $\overline{\Pi}$ *is isomorphic to the Hopf monoid of set partitions* Π.

PROOF. The Hopf monoid $\overline{\Pi}$ is generated multiplicatively by the standard permutahedra π_I, with coproduct given by $\Delta_{S,T}(\pi_I) = (\pi_S, \pi_T)$ as observed in Lemma 2.2.1. Comparing this with the definition of the Hopf monoid Π gives the isomorphism. □

Recall that $\overline{\mathcal{K}}$ is the Fock functor that associates a Hopf algebra $\overline{\mathcal{K}}(\mathbf{H})$ to any Hopf monoid in vector species $\mathbf{H}$.

PROPOSITION 5.6.5. *The Hopf algebra of permutahedra $\overline{\mathcal{K}}(\overline{\mathbf{\Pi}})$ is isomorphic to the Hopf algebra of symmetric functions Λ.*

PROOF. This proof requires some basic facts about symmetric functions; see [**68**] and [**92**, Section 7]. The Hopf algebra of symmetric functions $\Lambda = \Bbbk[x_1, x_2, \ldots]^{S_\infty}$ is most easily described in terms of the *homogeneous* and *elementary* symmetric functions:

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}, \qquad e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

As an algebra, $\Lambda = \Bbbk[e_1, e_2, \ldots]$ is simply the polynomial algebra on the e_i, while the coproduct and antipode of Λ are

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j, \qquad \mathrm{s}(e_n) = (-1)^n h_n.$$

for $n \geq 0$, where $e_0 = 1$.

The Fock functor $\overline{\mathcal{K}}$ maps $\overline{\mathbf{\Pi}}$ to the graded Hopf algebra $\overline{\mathcal{K}}(\overline{\mathbf{\Pi}})$; let it take the permutahedron $\pi_I \in \overline{\mathbf{\Pi}}[I]$ to the element $n!g_n \in \Pi_n$ where $n = |I|$. Then (31) tells us that as an algebra $\overline{\mathcal{K}}(\overline{\mathbf{\Pi}}) = \Bbbk[g_1, g_2, \ldots]$ while Lemma 2.2.1 tells us that the coproduct of $\overline{\mathcal{K}}(\overline{\mathbf{\Pi}})$ is given by

$$\Delta(g_n) = \sum_{i+j=n} g_i \otimes g_j.$$

It follows that the map $g_n \mapsto e_n$ preserves the product and coproduct. Since the antipode of a graded Hopf algebra is unique, this map also also preserves the antipode. This gives the desired isomorphism $\overline{\mathcal{K}}(\overline{\mathbf{\Pi}}) \cong \Lambda$. □

It is instructive to compare the antipodes of $\overline{\mathcal{K}}(\overline{\mathbf{\Pi}})$ and Λ. In $\overline{\mathbf{\Pi}}$ the antipode of $n!g_n$ is given by the face structure of the permutahedron π_n, as described in Section 1.3.4. This gives:

$$\mathrm{s}(g_n) = \sum_{\lambda_1 + \cdots + \lambda_k = n} (-1)^k g_{\lambda_1} \cdots g_{\lambda_k},$$

while the antipode of Λ is given by $\mathrm{s}(e_n) = (-1)^n h_n$. Comparing these expressions, we obtain a polyhedral algebraic proof of the expression of the homogeneous symmetric function h_n in the elementary basis:

$$h_n = \sum_{\lambda_1 + \cdots + \lambda_k = n} (-1)^{n-k} e_{\lambda_1} \cdots e_{\lambda_k}.$$

5.7. F: Paths and associahedra, revisited

Recall that a *partition into paths* on I is a graph whose connected components are paths, and $\mathrm{F}[I]$ denotes the collection of partitions into paths on I. Recall the Hopf monoid F defined in Section 1.2.5. The product of two partitions into paths is their disjoint union. If s is a path and $I = S \sqcup T$ is a decomposition, then $s|_S$ is the path on S with the order inherited from s, whereas $s/_S$ is the induced subgraph on T.

PROPOSITION 5.7.1. *The Hopf monoid* F *of paths is a submonoid of the cooppposite* $\mathrm{W}^{\mathrm{cop}}$ *of the ripping and sewing Hopf monoid* W.

PROOF. This follows readily from the observation that the product operations on F and W coincide, while the coproducts are co-opposite to each other. □

In light of this statement and the fact that W and $\mathrm{W}^{\mathrm{cop}}$ share the same antipode by Proposition 1.1.17, Theorem 5.5.6 immediately gives us a combinatorial formula for the antipode of the Hopf monoid of paths F. This formula has several interesting combinatorial variants, which we explore in the remaining sections.

5.7.1. The antipode of paths. If l is a linear graph and t is a tubing of l, define the *linear graph of* t, denoted $l(t)$, as follows. Each tube τ of t gives a path $l(\tau)$ consisting of the vertices which are in τ and in no smaller tube of t, in the order they appear in τ. The union of these paths is $l(t)$. This procedure is illustrated in Figure 8.

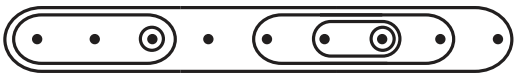


FIGURE 8. A tubing t of the path 123456789; its linear graph is $l(t) = 12|3|49|58|6|7$. The labels and edges of the path have been omitted for clarity.

PROPOSITION 5.7.2. *The antipode of the Hopf monoid of paths* **F** *is given by the following* ***cancellation-free*** *expression. If l is a linear graph on I then*

$$s_I(l) = \sum_{t \text{ tubing}} (-1)^{|t|}\, l(t)$$

summing over all tubings t of l, where $l(t)$ is the linear graph of t.

PROOF. This is a direct consequence of Theorem 5.5.6 because for a linear graph $w = l$, the graph $w(t)$ given by (59) is the linear graph $l(t)$. □

There are natural bijections between tubings on a path p_n of length n, valid parenthesizations of the expression $x_0x_1\cdots x_n$, and plane rooted trees with $n+1$ unlabeled leaves [**26**] [**92**, Chapter 6]. This bijection allows us to state Proposition 5.7.2 in terms of parenthesizations or plane rooted trees as well. We leave the details to the interested reader.

We can obtain a more useful formula by grouping equal terms in Proposition 5.7.2 as follows. As we range over the tubes τ of a tubing t, the components of the linear graph $l(t)$ form a set partition of I, which we call $\pi = \pi(t)$. We also write $l(\pi) = l(t)$.

Notice that $\pi = \pi(t)$ is a *noncrossing partition* of l; that is, if we let $<$ denote (either of) the (two) linear order(s) on I imposed by l, then π does not contain blocks $p_i \neq p_j$ and elements $a < b < c < d$ such that $a, c \in p_i$ and $b, d \in p_j$. It remains to describe the coefficient of $l(\pi)$ for each noncrossing partition π in the expression of Proposition 5.7.2.

Let $NC(l)$ be the set of noncrossing partitions of l. If $|l| = n$, then

$$|NC(l)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the n-th *Catalan number* [**62**]. We define the *linear graph of a noncrossing partition* $\pi \in NC(l)$ to be the graph on I containing one path for each part of π with the order induced by l.

To simplify the discussion we let $I = [n]$ and l be the path $12\cdots n$. For a noncrossing partition π of I, let the *adjacent closure* $\overline{\pi}$ be the partition obtained from π by successively merging any two *adjacent* blocks S_1 and S_2 such that $\max S_1 = b$ and $\min S_2 = b + 1$ for some b.

EXAMPLE 5.7.3. The adjacent closure of the noncrossing partition $\pi = 1|26|3|45|78$ in $NC(8)$ is $\overline{\pi} = 12678|345$.

THEOREM 5.7.4. *The antipode of the Hopf monoid of paths* F *is given by the following* ***cancellation-free*** *and* ***combination-free*** *expression. If l is a path on I,*

$$s_I(l) = \sum_{\pi \in NC(l)} (-1)^{|\pi|} C_{(\overline{\pi}:\pi)}\, l(\pi)$$

summing over all the noncrossing partitions π of l. Here $l(\pi)$ denotes the linear graph of π, $\overline{\pi} = \{p_1, \ldots, p_k\}$ is the adjacent closure of π, and $C_{(\overline{\pi}:\pi)} = \prod_{p_i \in \overline{\pi}} C_{n_i}$ where n_i is the number of blocks of π refining block p_i of $\overline{\pi}$.

PROOF. For a noncrossing partition π, the coefficient of $l(\pi)$ in the expression of Proposition 5.7.2 is equal to the number of tubings u of l with $\pi(u) = \pi$. We claim that this number equals $C_{(\overline{\pi}:\pi)}$.

Let π be a noncrossing partition of $[n]$, and consider the set t of tubes $\tau_i = [\min p_i, \max p_i]$ for all blocks p_i of π. Notice that $\tau_i \subset \tau_j$, $\tau_i \supset \tau_j$, or $\tau_i \cap \tau_j = \emptyset$ for $i \neq j$; if that were not the case, without loss of generality we would have $\min p_i < \min p_j < \max p_i < \max p_j$, which would contradict the assumption that π is noncrossing. However, t is not necessarily a tubing because it may contain adjacent tubes.

Let $\overline{t}$ be the tubing obtained from t by successively merging any two adjacent tubes of the form $[a, b]$ and $[b + 1, c]$. It follows from the definitions that the noncrossing partition associated to $\overline{t}$ is $\overline{\pi}$.

For each tube of $\overline{t}$, let us remember the tubes in t that constituted it by drawing vertical dotted lines separating them. This process is shown in Figure 9. Notice that if part p_i of $\overline{\pi}$ contains n_i parts of π, then the corresponding tube $\overline{t_i}$ of $\overline{t}$ contains n_i tubes of t.

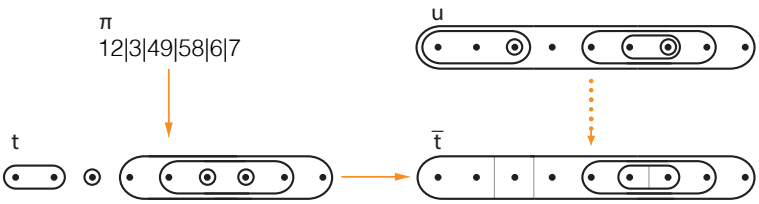
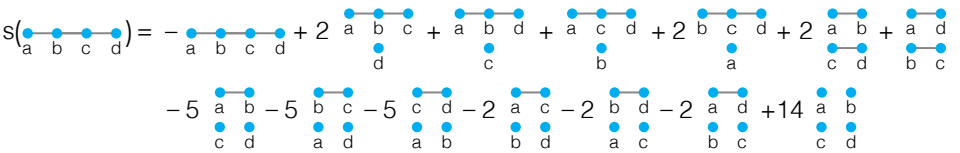


FIGURE 9. The process to go from a noncrossing partition $\pi = 12|3|49|58|6|7$ to a tubing u such that $\pi(u) = \pi$. The step $\pi \mapsto t$ is bijective and the map $t \mapsto t'$ is defined uniquely; we draw the vertical lines in t' are a visual aid, but they are not part of $\overline{t}$. The partial tubing $\overline{t}$ has $\prod_{p_i \in \overline{\pi}} C_{n_i} = C_3 C_2 = 10$ possible preimages u, corresponding to resolving the two tubes having 3 and 2 vertical compartments, respectively.

Any tubing u such that $\pi(u) = \pi$ is obtained from the set t of tubes – which is usually not a tubing – by "resolving" any maximal sequence of adjacent tubes, making them nested. To do this, we consider each tube $\overline{\pi_i}$ of $\overline{t}$, treat the n_i tubes of t that it contains as singletons, and replace them with a maximal tubing of size n_i; there are C_{n_i} such tubings for each i. This explains why there are $C_{(\overline{\pi}:\pi)}$ tubings u of l with $\pi(u) = \pi$, completing the proof. □

Since F is commutative, its antipode is multiplicative. This gives a similar cancellation-free and combination-free formula for $\mathrm{s}_I(\alpha)$ for any partition into paths α on I.

EXAMPLE 5.7.5. For the path $abcd$, Theorem 5.7.4 gives the formula from the introduction:



Theorem 5.7.4 explains the double appearance of Catalan numbers in the formula for the antipode of a linear graph: each coefficient is a products of Catalan numbers, and the number of terms (14 in this case) is the number of noncrossing partitions, which is also a Catalan number.

5.7.2. Associahedra and paths. As we have already anticipated, our formulas for the antipode of the Hopf monoid of paths F are controlled by Loday's associahedra. We now make this connection precise.

We begin with a technical lemma. Recall that the Loday associahedron $\mathfrak{a}_\ell$ of a linear order ℓ of I is the Minkowski sum $\mathfrak{a}_\ell = \sum_J \Delta_J$, where we sum over all the intervals J of the linear order ℓ.

LEMMA 5.7.6. *If $\ell_1 \neq \ell_2$ are linear orders on I, then $\mathfrak{a}_{\ell_1}$ and $\mathfrak{a}_{\ell_2}$ are normally equivalent if and only if ℓ_2 is the reversal of ℓ_1.*

PROOF. If ℓ_2 is the reversal of ℓ_1 then ℓ_1 and ℓ_2 have the same intervals, so $\mathfrak{a}_{\ell_1} = \mathfrak{a}_{\ell_2}$.

Conversely, suppose we know the normal fan $\mathcal{N} := \mathcal{N}(\mathfrak{a}_\ell)$ of the associahedron of a linear order ℓ. Then we know which hyperplanes of the form $y(i) = y(j)$ for $i, j \in I$ are contained in (the codimension 1 subcomplex of) $\mathcal{N}$. The hyperplane $y(i) = y(j)$ can only arise if $\mathfrak{a}_\ell$ has Δ_{ij} as a Minkowski summand. In turn, that summand appears if and only if i and j are adjacent in the linear order ℓ. It follows that $\mathcal{N}$ determines the set of adjacent pairs of ℓ, and these completely determine the linear order ℓ up to reversal. The desired result follows. □

PROPOSITION 5.7.7. *The Hopf monoid of paths* F *is isomorphic to the Hopf monoid of associahedra* $\overline{\mathrm{A}}$.

PROOF. The injective maps $\mathrm{F} \hookrightarrow \mathrm{W}^{\mathrm{cop}} \cong \mathrm{WBS}^{\mathrm{cop}} \hookrightarrow \mathrm{BS}^{\mathrm{cop}} \hookrightarrow \mathrm{SHG}^{\mathrm{cop}}$ of Propositions 5.7.1, 5.5.3, and 5.4.3 allow us to identify a path $l \in \mathrm{F}[I]$ with the set $\mathrm{tubes}(l) \in \mathrm{SHG}[I]$. Together with the surjection $\mathrm{SHG}^{\mathrm{cop}} \twoheadrightarrow \overline{\mathrm{HGP}}$ of Proposition 5.2.8, this gives a map $\mathfrak{a} : \mathrm{F} \to \overline{\mathrm{HGP}}$ which sends a path l to the associahedron $\mathfrak{a}_l$. The image of this map is $\overline{\mathrm{A}} \subseteq \overline{\mathrm{HGP}}$. Furthermore, $\mathfrak{a}$ is injective thanks to Lemma 5.7.6, keeping in mind that a path and its reverse are identified in F. The desired result follows. □

5.7.3. Associahedra and Faà di Bruno. The *Faà di Bruno Hopf algebra* $\mathcal{F}$, introduced by Joni and Rota [**60**] and anticipated by many others, appears naturally in several areas of mathematics and physics [**35**, **39**]. In this section we show that the Fock functor relates the Hopf monoid of associahedra $\overline{\mathrm{A}}$ (or equivalently the Hopf monoid of paths F) to the Faà Bruno Hopf algebra $\mathcal{F}$.

As an algebra, the Faà di Bruno Hopf algebra $\mathcal{F}$ is freely generated as a graded commutative algebra by $\{x_2, x_3, \ldots\}$ with $\deg x_n = n - 1$. It is convenient to write $x_1 = 1$. The coproduct is given by

$$\Delta(x_n) = \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1! \lambda_2! \cdots 1!^{\lambda_1} 2!^{\lambda_2} \cdots} x_1^{\lambda_1} x_2^{\lambda_2} \cdots \otimes x_k$$

summing over all sequences $\lambda = (1, 1, \ldots; 2, 2, \ldots; \ldots) = (1^{\lambda_1}, 2^{\lambda_2}, \ldots)$ of length k and total sum n, so $\lambda_1 + \lambda_2 + \lambda_3 + \cdots = k$ and $\lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots = n$.

The grading and the formulas are cleaner when we present F in terms of the generators $a_{n-1} = x_n/n!$; it is useful to write $a_0 = 1$. Then we have

$$\Delta(a_{n-1}) = \sum_{k=1}^{n} \sum_{\mu} \binom{k}{\mu_0, \mu_1, \mu_2, \ldots} a_1^{\mu_1} a_2^{\mu_2} \cdots \otimes a_{k-1}$$

summing over all sequences $\mu = (0, 0, \ldots; 1, 1, \ldots; 2, 2, \ldots; \ldots) = (0^{\mu_0}, 1^{\mu_1}, 2^{\mu_2}, \ldots)$ of length k and total sum $n-k$, so $\mu_0+\mu_1+\mu_2+\mu_3+\cdots = k$ and $\mu_1+2\mu_2+3\mu_3+\cdots = n - k$.

PROPOSITION 5.7.8. *The Fock functor* $\overline{\mathcal{K}}$ *maps the co-opposite* $\overline{\mathrm{A}}^{\mathrm{cop}}$ *of the Hopf monoid of associahedra* $\overline{\mathrm{A}}$ *to the Faà di Bruno Hopf algebra* $\mathcal{F}$.

PROOF. Let the Fock functor $\overline{\mathcal{K}}$ take the associahedron $\mathfrak{a}_\ell$ to the element a_n where $n = |\ell|$. Then (32) tells us that as an algebra $\overline{\mathcal{K}}(\overline{\mathrm{A}}^{\mathrm{cop}}) = \Bbbk[a_0, a_1, \ldots]$ while

Lemma 2.3.4 tells us that the coproduct of $\overline{\mathcal{K}}(\overline{\mathrm{A}}^{\mathrm{cop}})$ is given by

$$\Delta(a_{n-1}) = \sum_{[n-1]=S\sqcup T} a_{|T_1|}\cdots a_{|T_k|}\otimes a_{|S|}$$

where if $S=\{s_1,\dots,s_{k-1}\}$ then T_i is the interval of integers strictly between s_i and s_{i+1}, with the convention that $s_0=0$ and $s_k=n$.

A decomposition $[n-1]=S\sqcup T$ contributes to the term $a_1^{\mu_1}a_2^{\mu_2}\cdots\otimes a_{k-1}$ in $\Delta(a_{n-1})$ when $|S|=k-1$ and the k gaps $|T_1|,\dots,|T_k|$ between consecutive elements of S, including the initial and final gap, have sizes $0,0,\dots$ (μ_0 times), $1,1,\dots$ (μ_1 times), $2,2,\dots$ (μ_2 times), etcetera. For example, for the decomposition $[12]=\{1,2,4,7,8,12\}\sqcup\{3,4,5,9,10,11\}$, the gaps between consecutive elements of $S=\{1,2,4,7,8,12\}$ have sizes $0,0,1,2,0,3,0$ in that order.

Now it remains to observe that there are $\binom{k}{\mu_0,\mu_1,\mu_2,\dots}$ different ways of assigning the gap sizes $0,0,\dots$ (μ_0 times), $1,1,\dots$ (μ_1 times), $2,2,\dots$ (μ_2 times), etcetera to their k slots accordingly. Furthermore, these determine the possible choices for S and T that contribute to the term $a_1^{\mu_1}a_2^{\mu_2}\cdots\otimes a_{k-1}$ in $\Delta(a_{n-1})$, as desired. □

5.7.4. Three antipode formulas for the associahedron. At this point we have given formulas for the antipode of Loday's associahedron $\mathfrak{a}_n$ in three different Hopf algebraic structures: the Hopf monoids **GP** and $\overline{\mathbf{GP}}$ and the Hopf algebra $\overline{\mathcal{K}}(\overline{\mathbf{GP}})$.

In **GP**, Theorem 1.6.1 gives

$$\mathrm{s}(\mathfrak{a}_n) = \sum_{F \text{ face of } \mathfrak{a}_n} (-1)^{n-\dim F} F \tag{60}$$

where every face F of $\mathfrak{a}_n$ is normally equivalent to a product of Loday associahedra.

In $\overline{\mathbf{A}}\subseteq\overline{\mathbf{GP}}$, thanks to the isomorphism $\mathrm{F}\cong\overline{\mathrm{A}}$, Theorem 5.7.4 gives

$$\mathrm{s}(\mathfrak{a}_n) = \sum_{\pi\in NC(n)} (-1)^{|\pi|} C_{(\overline{\pi}:\pi)}\, \mathfrak{a}_{p_1}\cdots\mathfrak{a}_{p_k} \tag{61}$$

summing over the noncrossing partitions π of $[n]$; here $\overline{\pi}=\{p_1,\dots,p_l\}$ is the adjacent closure of π, and $C_{(\overline{\pi}:\pi)}=C_{n_1}\cdots C_{n_l}$ where n_i is the number of blocks of π refining block p_i of $\overline{\pi}$.

In $\overline{\mathcal{K}}(\overline{\mathbf{A}})\subseteq\overline{\mathcal{K}}(\overline{\mathbf{GP}})$ the proofs of Theorems 2.4.3 and 2.4.4 give

$$\mathrm{s}(\mathfrak{a}_n) = \sum_{\langle 1^{m_1}2^{m_2}\cdots\rangle\vdash n} (-1)^{|m|}\frac{(n+|m|)!}{(n+1)!\,m_1!\,m_2!\cdots}\mathfrak{a}_1^{m_1}\mathfrak{a}_2^{m_2}\cdots \tag{62}$$

summing over all partitions $\langle 1^{m_1}2^{m_2}\cdots\rangle$ of n, where $|m|=m_1+m_2+\cdots$.

Each formula coarsens the previous one under the projection maps $\mathbf{GP}\twoheadrightarrow\overline{\mathbf{GP}}\twoheadrightarrow\overline{\mathcal{K}}(\overline{\mathbf{GP}})$. In the first formula all faces of the associahedron are distinct. In the second formula, faces of the associahedron are grouped together according to their normal equivalence classes, which in turn correspond to their combinatorial type and position with respect to the axes. In the third formula, normal equivalence classes of faces of the associahedron are grouped according to their orbits under the symmetric group, which correspond to their combinatorial type.

Example 5.7.9. Let us consider the contribution of the 6 pentagonal faces of the associahedron $\mathfrak{a}_4$ to the three versions of the antipode $\mathrm{s}(\mathfrak{a}_4)$:

- In **GP**, each one of these six pentagonal faces is a separate term of $\mathrm{s}(\mathfrak{a}_4)$.
- In $\overline{\mathbf{GP}}$, these six faces group into four normal equivalence classes: the noncrossing

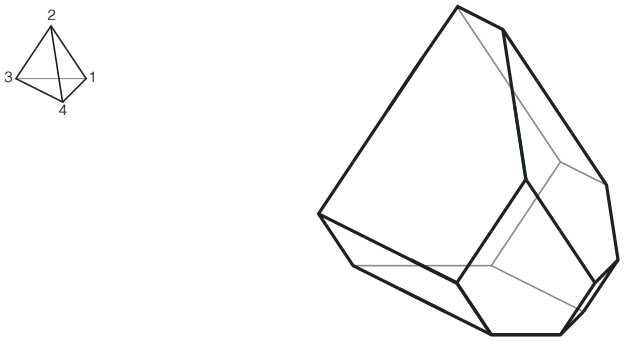


FIGURE 10. The three-dimensional associahedron $\mathfrak{a}_4$.

partitions $\{123, 4\}$ and $\{1, 234\}$ contribute two pentagons each, while the noncrossing partitions $\{134, 2\}$ and $\{124, 3\}$ contribute one pentagon each.
• In $\overline{\mathcal{K}}(\overline{\mathbf{GP}})$, these six faces are all grouped together into the coefficient of $\mathfrak{a}_3\mathfrak{a}_1$, which is equal to $(-1)^2(4+2)!/(4+1)!1!1! = 6$.

These observations have two interesting enumerative corollaries.

COROLLARY 5.7.10. *The number of normal equivalence classes of faces of Loday's associahedron $\mathfrak{a}_n$ is the Catalan number C_n.*

PROOF. The projection $\mathbf{GP} \twoheadrightarrow \overline{\mathbf{GP}}$ takes (60) to (61), mapping the faces of $\mathfrak{a}_n$ onto their normal equivalence classes. The result follows from the fact that the terms of (61) are in bijection with the noncrossing partitions of $[n]$ which are counted by the Catalan number C_n. □

COROLLARY 5.7.11. *Let $\mu = \langle 1^{m_1} 2^{m_2} \cdots \rangle$ be a partition of n and write $|m| = m_1 + m_2 + \cdots$. Let $NC(\mu)$ be the set of noncrossing partitions of n having type μ; that is, having m_i blocks of size i for $i = 1, 2, \ldots$. Then, in the notation of Theorem* 5.7.4,

$$\sum_{\pi \in NC(\mu)} C_{(\overline{\pi}:\pi)} = \frac{(n+|m|)!}{(n+1)!\, m_1!\, m_2! \cdots}.$$

PROOF. The map $\overline{\mathbf{GP}} \twoheadrightarrow \overline{\mathcal{K}}(\overline{\mathbf{GP}})$ takes (61) to (62). It maps each normal equivalence class of faces, which is labeled by a noncrossing partition of $[n]$, to its combinatorial type, which is the corresponding partition of n. It then remains to observe that the noncrossing partitions of type μ are the ones that map to the partition μ, so their contributions to (61) must add up to the contribution of μ to (62). □

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