Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-021-10061-x

Symmetry Properties of Sign-Changing Solutions to Nonlinear Parabolic Equations in Unbounded Domains

Juraj Földes[1] · **Alberto Saldaña**[2] · **Tobias Weth**[3]

Received: 10 April 2021 / Revised: 19 July 2021 / Accepted: 30 July 2021

Abstract
We study the asymptotic (in time) behavior of positive and sign-changing solutions to nonlinear parabolic problems in the whole space or in the exterior of a ball with Dirichlet boundary conditions. We show that, under suitable regularity and stability assumptions, solutions are asymptotically (in time) foliated Schwarz symmetric, i.e., all elements in the associated omega-limit set are axially symmetric with respect to a common axis passing through the origin and are nonincreasing in the polar angle. We also obtain symmetry results for solutions of Hénon-type problems, for equilibria (i.e. for solutions of the corresponding elliptic problem), and for time periodic solutions.

Keywords Asymptotic symmetry · Nodal solutions · Exterior domains

Mathematics Subject Classification 35B40 · 35B30 · 35B07

✉ Alberto Saldaña
alberto.saldana@im.unam.mx

Juraj Földes
foldes@virginia.edu

Tobias Weth
weth@math.uni-frankfurt.de

[1] Department of Mathematics, University of Virginia 322 Kerchof Hall, Charlottesville, VA 22904-4137, USA

[2] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, 04510 Coyoacán, Ciudad de México, Mexico

[3] Institut für Mathematik, Johann Wolfgang Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60629 Frankfurt, Germany

Published online: 20 August 2021

1 Introduction

In this paper we study the asymptotic behavior of (possibly sign-changing) classical uniformly bounded solutions of

$$\begin{aligned} u_t - \Delta u &= f(t, |x|, u) && (x,t) \in \Sigma \times (0,\infty), \\ u(x,t) &= 0 && (x,t) \in \partial\Sigma \times (0,\infty), \\ u(x,0) &= u_0(x) && x \in \Sigma, \end{aligned} \tag{1.1}$$

where Σ is either the whole space $\mathbb{R}^N$ or the complement of a ball $\mathbb{R}^N \setminus B_R(0)$ in dimension $N \geq 2$ for some $R > 0$. The initial profile u_0 and the nonlinearity f satisfy some regularity and stability assumptions detailed below.

A solution u is said to be asymptotically symmetric if all the elements in the ω-limit set $\omega(u)$, defined as

$$\omega(u) := \{z \in C_0(\overline{\Sigma}) : z(x) = \lim_{n\to\infty} u(x, t_n) \text{ for } x \in \Sigma \text{ and some } t_n \to \infty\}, \tag{1.2}$$

share some symmetry. Here, $C_0(\overline{\Sigma})$ is the space of continuous functions which decay to zero at infinity and vanish on $\partial\Sigma$, equipped with the supremum norm. Standard parabolic estimates (see Lemma 2.3) yield that

$$\lim_{t\to\infty} \operatorname{dist}_{C_0(\overline{\Sigma})}(u(\cdot,t), \omega(u)) = 0\,, \tag{1.3}$$

and therefore the asymptotic symmetry implies that the solution is more and more symmetric as $t \to \infty$.

We are interested in a particular kind of symmetry sometimes referred to as foliated Schwarz symmetry. We say that a function $u \in C(\Sigma)$ is *foliated Schwarz symmetric with respect to some unit vector* $p \in \mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$, if u is axially symmetric with respect to the axis $\mathbb{R}p$ and nonincreasing in the polar angle $\theta := \arccos(\frac{x}{|x|} \cdot p) \in [0, \pi]$. If u is strictly decreasing in θ, then we say that u is *strictly foliated Schwarz symmetric*.

From the asymptotic symmetry point of view, the results we present in this paper are, as far as we know, the first to consider sign-changing solutions in unbounded domains, non-monotone spatial dependences on the nonlinearity, and unbounded domains different from $\mathbb{R}^N$ (we give an account of previously known results below). In this more general setting, however, we need to impose a geometric assumption on the initial profile u_0 to guarantee that all functions in the ω-limit set are foliated Schwarz symmetric.

Before we state our theorems in full generality, we illustrate our results with two paradigmatic particular cases. First, we exploit the (non-monotone) spatial dependence of the coefficients of a nonautonomous Hénon-type problem to guarantee a symmetrizing effect as $t \to \infty$.

Theorem 1.1 *Let Σ be either $\mathbb{R}^N$ or $\mathbb{R}^N \setminus B_1(0)$, $a, b \in C([0,\infty)) \cap L^\infty([0,\infty))$, and let $\eta > 0$ be such that*

$$\inf_{t>0} b \geq \eta. \tag{1.4}$$

Let $p > 1$, $0 \leq \alpha < \beta$, and let $u \in C^{2,1}(\Sigma \times (0,\infty)) \cap C(\overline{\Sigma} \times [0,\infty)) \cap L^\infty(\Sigma \times [0,\infty))$ be a solution of

$$\begin{aligned} u_t - \Delta u &= a(t)|x|^\alpha |u|^{p-1}u - b(t)|x|^\beta u && (x,t) \in \Sigma \times (0,\infty), \\ u &= 0 && (x,t) \in \partial\Sigma \times (0,\infty), \\ u(x,0) &= u_0(x) && x \in \Sigma, \end{aligned} \tag{1.5}$$

where $u_0 \in L^\infty(\Sigma) \cap C(\overline{\Sigma})$ *satisfies that*

$$\begin{aligned} &u_0(x_1, x_2, \dots, x_N) \geq u_0(-x_1, x_2, \dots, x_N) \quad \text{for all } x \in \Sigma \text{ with } x_1 > 0, \\ &u_0(x_1, x_2, \dots, x_N) > u_0(-x_1, x_2, \dots, x_N) \quad \text{for some } x \in \Sigma \text{ with } x_1 > 0. \end{aligned} \tag{1.6}$$

Then, u *is asymptotically foliated Schwarz symmetric, that is, there is* $p \in \mathbb{S}^{N-1}$ *such that* z *is foliated Schwarz symmetric with respect to* p *for all* $z \in \omega(u)$.

Observe that the direction of the symmetry axis p is not determined by the equation, which is invariant under rotations, p is determined by u_0. Theorem 1.1 is a consequence of the more general Theorem 1.3 below.

Our results also cover translationally invariant problems, where we can show that (possibly sign-changing) solutions are asymptotically signed and radially symmetric with respect to *some* center. Unlike for Hénon-type problems, a center of symmetry is not fixed a priori if the equation is translationally invariant; and therefore a suitable assumption on the initial condition u_0 plays a key role to guarantee the asymptotic symmetry. In the next section we present more general conditions on u_0; however, to simplify the presentation and to illustrate the main ideas, in the next theorem we assume that the supports of the positive and the negative parts of u_0 are strictly separated by the cones $K^+ := \{x = (x_1, x') \in \mathbb{R}^N : x_1 > 1 \text{ and } |x'| < |x_1 - 1|\}$ and $K^- := \{x = (x_1, x') \in \mathbb{R}^N : x_1 < -1 \text{ and } |x'| < |x_1 + 1|\}$, see Fig. 1 below.

Theorem 1.2 *Let* $p > 1$ *and* $a, b \in C([0,\infty)) \cap L^\infty([0,\infty))$ *such that (1.4) holds. Let* $u \in C^{2,1}(\mathbb{R}^N \times (0,\infty)) \cap C(\mathbb{R}^N \times [0,\infty)) \cap L^\infty(\mathbb{R}^N \times [0,\infty))$ *be a solution of*

$$\begin{aligned} u_t - \Delta u + b(t)\, u &= a(t)\, |u|^{p-1} u && (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) &= u_0(x) && x \in \mathbb{R}^N, \end{aligned} \tag{1.7}$$

where $u_0 \in C_0(\mathbb{R}^N) \backslash \{0\}$ *is such that*

$$\{x \in \mathbb{R}^N : u_0(x) > 0\} \subset K^+, \quad \{x \in \mathbb{R}^N : u_0(x) < 0\} \subset K^-. \tag{1.8}$$

Assume furthermore that u *has uniform decay at spatial infinity*

$$\lim_{|x| \to \infty} \sup_{t > 0} u(x,t) = 0. \tag{1.9}$$

If $0 \notin \omega(u)$, *then* u *is asymptotically signed and radially symmetric, that is, there is* $q \in \mathbb{R}^N \backslash \{x_1 = 0\}$ *such that all elements in* $\omega(u)$ *are radially symmetric with respect to* q *and*

1. $z > 0$ *in* $\mathbb{R}^N$ *for all* $z \in \omega(u)$, *if* $q \in \{x_1 > 0\}$,
2. $z < 0$ *in* $\mathbb{R}^N$ *for all* $z \in \omega(u)$, *if* $q \in \{x_1 < 0\}$.

Note that, in this setting, there is an additional complication if 0 belongs to the ω-limit set. If $0 \in \omega(u)$, then for a sequence of times $t_n \to \infty$ the solution tends uniformly to zero, and therefore the (symmetrizing) influence from the initial condition weakens and the asymptotic symmetry of solutions becomes unclear. This is why Theorem 1.2 poses an alternative: either $0 \in \omega(u)$ or the solution u is asymptotically signed and radially symmetric. Theorem 1.2 is a consequence of the more general Theorem 1.8 below together with an argument involving the translational invariance of the equation.

Before we present our more general results in the following section, we mention that proving foliated Schwarz symmetry simplifies the analysis of solutions of evolution problems of type (1.1). For example, if $u : \mathbb{R}^N \to \mathbb{R}$ is foliated Schwarz symmetric with respect to $p = (1, 0, \dots, 0)$, then u is also axially symmetric with respect to the x_1-axis and therefore

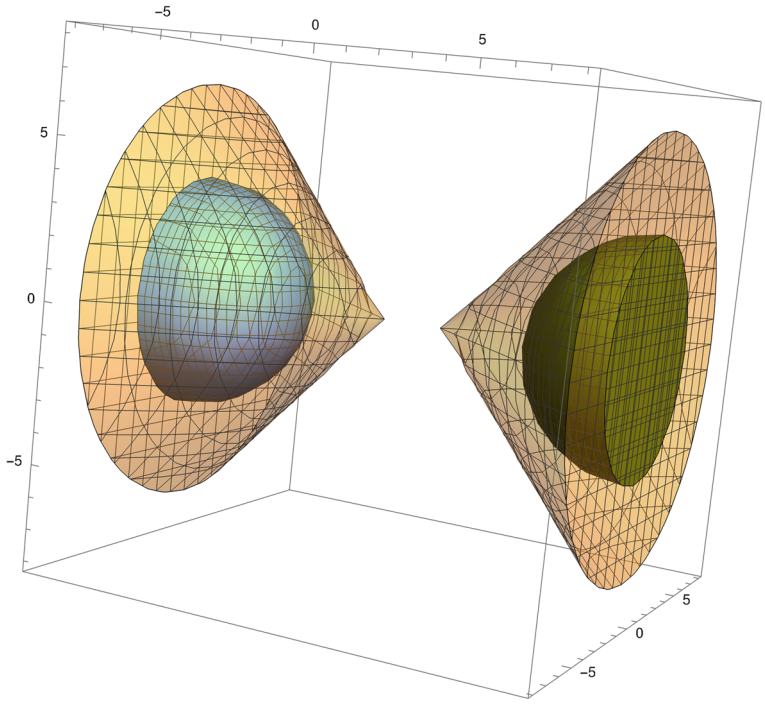

Fig. 1 Geometry of the condition (1.8) in $\mathbb{R}^3$, where the support of the positive part of u_0 (represented by the dark region on the right) lies inside a cone in the halfspace $\{x_1 > 1\}$ and the support of the negative part of u_0^- (represented by the dark region on the left) is inside a cone in the halfspace $\{x_1 < -1\}$

u only depends on the two variables $x_1 \in \mathbb{R}$ and $\rho = |(x_2, \ldots, x_N)| > 0$. Moreover, (1.1) can be written in these space variables, which simplifies the implementation of numerical approximation methods. In the context of a related elliptic problem, numerical methods have been applied, for example, in [2] to approximate foliated Schwarz symmetric solutions. We also mention that there are results and techniques for nonlinear second order equations in two dimensions that are not available in general, see for example [16].

Main Results

In the following, we give a more abstract framework to state general results from which Theorems 1.1 and 1.2 can be deduced. Our results are complementary to the papers [17,18], where asymptotic radial symmetry of positive solutions is proved in bounded and unbounded domains. More specifically, the conclusions in [17,18] are stronger (radial versus foliated Schwarz symmetry) and there are no geometric assumptions on the initial condition; however, [17,18] require positivity of solutions and monotone dependence of f on $|x|$ (we remark that the theorems in [17] do not consider spatial dependences on f, but the proof can be adjusted to include nonincreasing radial spatial dependences). The main tool in [17,18] is the *parabolic moving plane method* (MPM), which is a perturbation technique strongly based on maximum principles, Harnack inequalities, parabolic regularity, and the construction of suitable subsolutions. The lack of compactness in unbounded domains is the main difficulty when trying to characterize the symmetry of solutions of (1.1), and in fact, symmetry-breaking phe-

nomena in unbounded domains are known, see for example [19]. For other results regarding reflectional symmetry and monotonicity in bounded domains via MPM, see [6–12].

In this paper we consider sign-changing solutions, spatially non-monotone nonlinearities, and exterior domains. In this setting, a standard MPM cannot be used; instead, we use a variant sometimes called *rotating plane method* (RPM). We explain the differences between these two techniques below. For similar results in bounded domains for parabolic equations and systems, we refer to [22–25], and for elliptic problems to [13,26].

We introduce some notation to formulate our results. Recall that Σ can be the whole space without a ball of radius $R > 0$, $\Sigma = \mathbb{R}^N \setminus B_R(0)$, or the whole space $\Sigma = \mathbb{R}^N$, in which case we set $R = 0$. For a vector $e \in \mathbb{S}^{N-1}$, we consider the hyperplane $H(e) := \{x \in \mathbb{R}^N : x \cdot e = 0\}$ and the half domain $\Sigma(e) := \{x \in \Sigma : x \cdot e > 0\}$. We also write $\sigma_e : \Sigma \to \Sigma$ to denote reflection with respect to $H(e)$, i.e. $\sigma_e(x) := x - 2(x \cdot e)e$ for each $x \in \Sigma$.

We consider a classical global solution $u \in C^{2,1}(\Sigma \times (0,\infty)) \bigcap C(\overline{\Sigma} \times [0,\infty))$ of (1.1) satisfying the following assumptions.

(U_0) *(Initial reflectional inequality)* The initial profile u_0 belongs to $C_0(\overline{\Sigma})$ and there exists $e \in \mathbb{S}^{N-1}$ such that

$$u_0(x) \geq u_0(x^e) \text{ for all } x \in \Sigma(e) \text{ and } u_0 \not\equiv u_0 \circ \sigma_e.$$

(U_1) *(Uniform decay)* $\lim_{|x|\to\infty} \sup_{t>0} u(x,t) = 0$.

(U_2) *(Uniform boundedness)* $||u||_{L^\infty(\Sigma\times(0,\infty))} =: M_1 < \infty$.

Define $I := [R,\infty) = \overline{\{|x| : x \in \Sigma\}}$. We now state our first assumption on f.

(f_0) *(Boundedness at zero)* For every $r > 0$ there is $K_r > 0$ such that

$$\sup_{s\in[R,r],t>0} |f(t,s,0)| < K_r.$$

Assumption (f_0) is used to obtain the equicontinuity of u, see Lemma 2.3.

Next, we impose some standard regularity on f and a crucial sign assumption on f_u outside a compact set. This last assumption has two important variants (see $(f_2)'$ and (f_2) below), and we divide our results accordingly.

Strong Stability Outside Compact Sets

Let λ_1 denote the principal eigenvalue of the Dirichlet Laplacian on the unit ball $B_1(0) \subset \mathbb{R}^N$. Assume that

$(f_1)'$ *(Regularity)* the nonlinearity $f : [0,\infty) \times I \times \mathbb{R} \to \mathbb{R}$, $(t,r,u) \mapsto f(t,r,u)$ is continuous in t, r, and locally uniformly Lipschitz in u, that is, for every bounded interval $J \subset I$ and $K > 0$ there is $C = C(K,J) > 0$ such that

$$\sup_{r\in J, t>0, u,v\in[-K,K]} |f(t,r,u) - f(t,r,v)| \leq C|u - v|.$$

In particular, $f_u(t,r,u)$ exists for almost every u and it is locally bounded.

$(f_2)'$ *(Strong stability)* given $M > 0$ there are $\rho > R$, $\varepsilon > 0$, and an interval $J \subset (R,\rho)$ such that

$$\max_{r>\rho, u\in[-\varepsilon,\varepsilon]} f_u(t,r,u) < -\max_{\substack{r\in J,\\ u\in[-M,M]}} |f_u(t,r,u)| - 4\frac{\lambda_1}{|J|^2} \quad \text{for all } t > 0. \tag{1.10}$$

If the bound M_1 from (U_2) is known, then it suffices to suppose that $(f_2)'$ is satisfied for $M = M_1$. Observe that the assumption $(f_2)'$ can only be satisfied by nonlinearities with a spatial dependence. This is one of the novelties in Theorem 1.3 below: we show that a condition on the spatial dependence of the nonlinearity (given by (1.10)) yields a symmetrizing effect on the solution for large times.

Our main result in this setting is the following.

Theorem 1.3 *Let $u \in C^{2,1}(\Sigma \times (0,\infty)) \bigcap C(\overline{\Sigma} \times [0,\infty))$ be a classical global solution of (1.1) satisfying (U_0)-(U_2) and assume that (f_0), $(f_1)'$, and $(f_2)'$ hold. Then u is asymptotically foliated Schwarz symmetric, that is, there is $p \in \mathbb{S}^{N-1}$ such that z is foliated Schwarz symmetric with respect to p for all $z \in \omega(u)$.*

Note that the axis of symmetry $\mathbb{R}p$ is fixed for all elements in $\omega(u)$. A typical situation where $(f_2)'$ is satisfied is the case of a bounded nonlinearity plus a suitable potential. The next corollary shows an example.

Corollary 1.4 *Let f satisfy $(f_1)'$ and be such that*

$$\sup_{r\geq R, t>0, s\in\mathbb{R}} |f_u(t,r,s)| =: C_0 < \infty.$$

Let $u \in C^{2,1}(\Sigma \times (0,\infty)) \bigcap C(\Sigma \times [0,\infty))$ be a classical solution satisfying (U_0)–(U_2) of

$$\begin{aligned} u_t - \Delta u + V(|x|)u &= f(t,|x|,u) && \text{in } \Sigma \times (0,\infty), \\ u(x,0) &= 0 && \text{on } \partial\Sigma \times (0,\infty), \\ u(x,0) &= u_0(x) && \text{for } x \in \Sigma, \end{aligned} \tag{1.11}$$

where $V : [0,\infty) \to \mathbb{R}$ is a continuous function such that

$$\min_{(\rho,\infty)} V > \max_{\mathcal{J}} |V| + 4\frac{\lambda_1}{|\mathcal{J}|^2} + 2C_0, \tag{1.12}$$

for some $\rho > R$ and some interval $\mathcal{J} \subset [R,\rho)$. Then u is asymptotically foliated Schwarz symmetric.

Observe that (1.12) is always satisfied if $V(r) \to \infty$ as $r \to \infty$, and there is no restriction on the behavior of V in $[0,\rho)\backslash I$. We also remark that $0 < \lambda_1 \leq \frac{N}{2}\left(\sqrt{\frac{N}{2}+1}+1\right)^2$, (see [5]) where N is the dimension.

The proof of Theorem 1.3 is based on an RPM which extends the ideas from the bounded domain case [18,24] to unbounded domains. Similarly as the MPM, the RPM can be divided in three parts: the start, the perturbation step, and the characterization of symmetry. The start relies strongly on assumption (U_0), maximum principles, and a suitable linearization of (1.1) (see Lemma 2.1). The perturbation step is based on maximum principles (Lemma 2.5), Harnack inequalities (Lemma 2.4), parabolic regularity (Lemma 2.3), and the construction of suitable subsolutions (Lemmas 2.2 and 3.6). A crucial aspect in this part is the fine use of constants which have precise dependences in order to construct and control a suitable perturbation. This is particularly delicate when considering unbounded domains and nonlinearities with a spatial dependence. Here hypothesis (U_1), (U_2), and $(f_2)'$ allow to control the solution outside large compact sets, and this is essential to compensate for the lack of compactness in unbounded domains. The last part, the characterization of symmetry, relies on a geometric characterization of foliated Schwarz symmetry in terms of reflectional inequalities (Lemma 2.6), such characterizations were first presented in [3] in the study of elliptic variational problems.

Weak Stability Outside Compact Sets

Next we use a set of assumptions which is closer to that of [17]. Assume that

(f_1) (Regularity) $f : [0,\infty) \times I \times \mathbb{R} \to \mathbb{R}$, $(t,r,u) \mapsto f(t,r,u)$ is continuous in t, r, continuously differentiable in u, and satisfies that

$$\lim_{u \to v} \sup_{r \in I, t>0} |f_u(t,r,u) - f_u(t,r,v)| = 0 \qquad \text{for every } v \in \mathbb{R},$$

$$\sup_{r \in J, t>0, s \in [-K,K]} |f_u(t,r,s)| < \infty \qquad \text{for every } K > 0 \text{ and } J \subset\subset I.$$

(f_2) *(Stability)* there are a constants $\rho, \gamma, \varepsilon > 0$ such that

$$f_u(t,|x|,s) < -\gamma \qquad \text{for all } |x| > \rho, t \geq 0, \text{ and } s \in (-\varepsilon, \varepsilon).$$

Here $J \subset\subset I$ means that J is *compactly contained* in I, that is, $\overline{J}$ is compact and a subset of I. The difference between $(f_2)'$ and (f_2) is that $\gamma > 0$ in (f_2) can be arbitrarily small. This allows to consider, for example, a problem as in (1.11), where V is possibly unbounded from below but $V(r_n) = -\gamma$ for a sequence $r_n \to \infty$ and some $\gamma \in (0,1)$ small. This weaker assumption (f_2) implies, however, a weaker control on the solution at spatial infinity and forces the RPM to use a different subsolution given by Lemma 3.6, which in turn requires some knowledge on the elements in $\omega(u)$. As a consequence, our main result under the weaker assumption (f_2) describes an alternative.

Theorem 1.5 *Assume (f_0), (f_1), (f_2), and let $u \in C^{2,1}(\Sigma \times (0,\infty)) \bigcap C(\overline{\Sigma} \times [0,\infty))$ be a classical global solution of (1.1) satisfying (U_0), (U_1), and (U_2). Then one of the following alternatives happens:*

1. *u is asymptotically foliated Schwarz symmetric with respect to some $p \in \mathbb{S}^{N-1}$, that is, all elements in $\omega(u)$ are foliated Schwarz symmetric with respect to p. Moreover, all elements in $\omega(u)$ are strictly decreasing in the polar angle.*
2. *There exists $z \in \omega(u)$ such that $z \equiv z \circ \sigma_e$ with e is in (U_0).*

The proof of Theorem 1.5 follows an RPM and some ideas from [17]. We emphasize that, in general, the second alternative in Theorem 1.5 can occur; for example, if $u_0 \geq 0$, then [17, Theorem 1.1] implies that all the elements in $\omega(u)$ are radially symmetric with respect to some center. Note that the first alternative rules out radial symmetry (with respect to the origin) because of the monotonicity properties in the polar angle. In Sect. 5.1, we show an example of a solution u for which $\omega(u)$ only has nodal strictly foliated Schwarz functions. This example also shows that the set of solutions satisfying the assumptions of Theorem 1.5 is nonempty. Furthermore, we show in Sect. 5.2 that the symmetry in the second alternative in Theorem 1.5 may not propagate to the other elements in $\omega(u)$; more precisely, we show in Sect. 5.2 two examples (one in a bounded domain and one in an unbounded domain) of problems whose solution u has both radially symmetric and strictly foliated Schwarz functions in $\omega(u)$; in particular, this shows that the presence of a radially symmetric function in $\omega(u)$ does not imply, in general, that all elements in $\omega(u)$ are radially symmetric.

Whenever the second alternative happens, this creates an obstacle to start the method, since one cannot use the subsolution in Lemma 3.6. This kind of obstacles do not appear in a MPM framework (as in [17]), because the starting step in the method relies on the positivity of solutions, the uniform decay at spatial infinity, and on the stability of 0 given by assumption (f_2).

The following are direct corollaries for elliptic and periodic parabolic equations, where the second alternative can be discarded using the maximum principle.

Corollary 1.6 *Assume* (f_0)–(f_2), *and let* $u \in C^{2,1}(\Sigma \times (0,\infty)) \bigcap C(\overline{\Sigma} \times [0,\infty))$ *be a classical bounded periodic solution of (1.1) that satisfies* (U_0) *and that* $\lim_{|x|\to\infty} u(x,t) = 0$ *for* $t \geq 0$. *Then there exists* $p \in \mathbb{S}^{N-1}$ *such that, for all* $t \geq 0$, $u(\cdot,t)$ *is foliated Schwarz symmetric with respect to* p *and* u *is strictly decreasing in the polar angle.*

Corollary 1.7 *Let* $f : I \times \mathbb{R} \to \mathbb{R}$, $(r,u) \mapsto f(r,u)$ *be continuous in* r *and continuously differentiable in* u, *with* f_u *uniformly continuous and bounded with respect to* r. *Moreover, assume that there are constants* $\rho, \gamma > 0$ *such that* $f_u(|x|,0) < -\gamma$ *for all* $|x| > \rho$. *Let* $u \in C^2(\Sigma) \bigcap C(\overline{\Sigma})$ *be a classical uniformly bounded solution of*

$$-\Delta u = f(|x|,u) \quad \text{in } \Sigma, \qquad u(x) = 0 \quad \text{on } \partial\Sigma, \qquad \lim_{|x|\to\infty} u(x) = 0.$$

If there is $e \in \mathbb{S}^{N-1}$ *such that* $u(x) \geq u(\sigma_e(x))$ *in* $\Sigma(e)$ *and* $u \not\equiv u \circ \sigma_e$, *then* u *is foliated Schwarz symmetric with respect to some* $p \in \mathbb{S}^{N-1}$ *and* u *is strictly decreasing in the polar angle.*

Similar corollaries can be stated for Theorem 1.3. In fact, for elliptic and periodic parabolic problems a slightly simpler proof can be done, where assumption (f_1) can be weakened (as in [24], since Lemma 3.6 is not needed).

Under a more "geometrically stable" assumption on u_0, we can prove that if some $z \in \omega(u)$ has $H(e)$ as symmetry hyperplane, then z is radial. This assumption is the following:

$(U_0)'$ The initial profile u_0 belongs to $C_0(\overline{\Sigma})$ and there exists an open set $U \subset \mathbb{S}^{N-1}$ such that

$$u_0 \geq u_0 \circ \sigma_e \quad \text{in } \Sigma(e) \text{ for all } e \in U.$$

Theorem 1.8 *Assume the same hypothesis as in Theorem 1.5 but with* $(U_0)'$ *instead of* (U_0). *Then, only one of the following alternatives happen:*

1. u *is asymptotically foliated Schwarz symmetric with respect to some* $p \in \mathbb{S}^{N-1}$, *i.e. all elements in* $\omega(u)$ *are strictly foliated Schwarz symmetric with respect to* p; *that is, all the elements in* $\omega(u)$ *are axially symmetric and strictly decreasing in the polar angle.*
2. *There exists at least one* $z \in \omega(u)$ *such that* z *is radially symmetric with respect to the origin.*

Functions that satisfy $(U_0)'$ are, for example, positive functions with support in a one-sided cone, or sign changing functions with support in two (disjoint) cones being positive in one side and negative in the other (see Fig. 2 and Sect. 5.1). The proof of Theorem 1.8 uses Theorem 1.5 plus some geometric results that characterize the rigidity that reflectional inequalities impose on a function (see Lemmas 2.10 and 2.7).

Finally, we remark that Theorem 1.5 can be extended to the quasilinear setting as in [17]. The main obstacle for a generalization to fully nonlinear problems is, as noted in [17, Sect. 4], the generalization of the subsolution given in Lemma 3.6. However, Theorem 1.3 can be extended to fully nonlinear problems as in the bounded domain case [18,24]. Our results consider only the semilinear setting for simplicity and to make the main ideas more transparent.

On the Uniform Decay Assumption

In Theorem 1.1, the assumption (U_1) is not stated as a hypothesis, since it can be obtained using the particular form of the equation and comparison principles with suitable supersolutions. In this part of the introduction, we discuss some other conditions that can be used to guarantee that (U_1) holds.

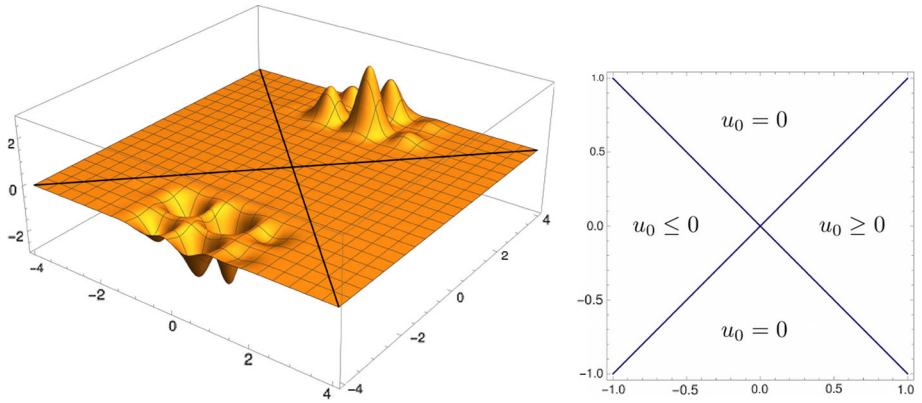

Fig. 2 Example of a function u_0 satisfying $(U_0)'$

In [4, Corollary 1.2] it is proved that (U_1) is satisfied if the initial condition has compact support and the nonlinearity does not depend on $|x|$. Following the same idea, we give below sufficient conditions to achieve the uniform decay assumption (U_1) in settings where nonradial solutions (and nonradial limit profiles) could be expected, for example, if the nonlinearity $f : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is such that, for some $R_0 > 0$,

$$f(t, r_1, u) \leq f(t, r_2, u), \quad \text{for all } 0 \leq r_1 \leq R_0 \leq r_2 \text{ or } R_0 \leq r_1 \leq r_2, \tag{1.13}$$

and all $t > 0$, $u \in \mathbb{R}$. In particular, f can be non-monotone inside a ball of radius R_0. Then, we have the following lemmas.

Lemma 1.9 *Let $\Sigma = \mathbb{R}^N$ and u a solution of (1.1), assume that $\partial_u f(t, r, u)$ is bounded uniformly in t and r, and that $f(t, r, u)$ satisfies (1.13). If*

1. *$u_0 \geq 0$,*
2. *u_0 has compact support,*
3. *$||u(\cdot, t)||_{L^\infty(\mathbb{R}^N)} \leq C$ and $||u(\cdot, t)||_{L^q(\mathbb{R}^N)} \leq C$ for some $C > 0$, $1 \leq q < \infty$ and all $t > 0$,*

then u satisfies (U_1).

Lemma 1.10 *Let $\Sigma = \mathbb{R}^N$ and assume $\partial_u f(t, r, u)$ is bounded uniformly in t and r, and that $f(t, r, u)$ satisfies (1.13). If*

1. *u_0 has compact support,*
2. *the problems*

$$v_t - \Delta v = f(t, |x|, v) \quad \text{in } \mathbb{R}^N \times (0, \infty), \qquad v(x, 0) = u_0^{\pm}(x) \quad \text{for } x \in \mathbb{R}^N, \tag{1.14}$$

where u_0^+ and u_0^- denote the positive and negative part of u_0 respectively, have global solutions v_1 and v_2 such that $||v_i(\cdot, t)||_{L^\infty(\mathbb{R}^N)} \leq C$ and $||v_i(\cdot, t)||_{L^q(\mathbb{R}^N)} \leq C$ for some $C > 0$, $1 \leq q < \infty$, and all $t > 0$, $i = 1, 2$,

then any solution u of (1.1) satisfies (U_1).

Organization of the Paper

The paper is organized as follows. In Sect. 2 we collect several useful results regarding a linearization of (1.1), regularity, Harnack inequalities, maximum principles, the existence of suitable explicit subsolutions, and some geometric lemmas which are used in the characterization of symmetry. Section 3 is devoted to the proofs of our main symmetry results including Theorems 1.1 and 1.2. The proofs of Lemmas 1.9 and 1.10 can be found in Sect. 4. Finally, in Sect. 5, we include a series of examples to show some of the different possible behaviors of the elements in the ω-limit set; in particular, we exhibit an ω-limit set with only strictly foliated Schwarz symmetric functions and an ω-limit set with a strictly foliated Schwarz symmetric function and a radially symmetric element.

2 Auxiliary Lemmas

2.1 Linearization

Let $R \geq 0$ be such that

$$\Sigma = \mathbb{R}^N \backslash B_R, \tag{2.1}$$

where $B_R = B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$ is the open ball of radius R centered at the origin if $R > 0$ and $B_0 := \emptyset$, that is, $\mathbb{R}^N \backslash B_0 = \mathbb{R}^N$. Using the notation given in the introduction, define $\Sigma_1 := \Sigma(e_1)$, where $e_1 = (1, 0, \ldots, 0)$.

Throughout this section we assume that f satisfies $(f_1)'$, which follows from (f_1), and (f_2) which follows from $(f_2)'$. Let u be a classical solution of (1.1) satisfying (U_1) and (U_2).

Fix $e \in \mathbb{S}^{N-1}$ and let $\Gamma_e : \Sigma_1 \to \Sigma(e)$ be a rotation that maps e_1 to e. Then,

$$w_e(x,t) := u(\Gamma_e(x), t) - u(\sigma_e(\Gamma_e(x)), t) \qquad \text{for } (x,t) \in \overline{\Sigma}_1 \times [0,\infty),$$

is a classical solution of

$$\begin{aligned} \partial_t w_e - \Delta w_e - c^e(x,t) w_e &= 0 && \text{in } \Sigma_1 \times (0,\infty), \\ w_e(x,t) &= 0 && \text{on } \partial\Sigma_1 \times (0,\infty), \\ w_e(x,0) &= u_0(\Gamma_e(x)) - u_0(\sigma_e(\Gamma_e(x))) && \text{for } x \in \Sigma_1, \end{aligned} \tag{2.2}$$

satisfying (U_1) and (U_2) with u replaced by w_e, where

$$c^e(x,t) := \int_0^1 \partial_u f(t, |x|, su(\Gamma_e(x), t) + (1-s)u(\sigma_e(\Gamma_e(x)), t))ds$$

is well defined by (U_2) and $(f_1)'$. Moreover, since u is uniformly bounded in time and the Lipschitz constant of f is time independent, for every bounded subset $U \subset \Sigma_1$, there is $\beta_U > 0$ (independent of e) such that

$$\sup_{(x,t) \in U \times [0,\infty)} |c^e(x,t)| < \beta_U \,. \tag{2.3}$$

Moreover, by (U_1), (f_1), and (f_2) (or (U_1), $(f_1)'$, and $(f_2)'$), there exists $\rho_1 \geq \rho$ (see (f_2) or $(f_2)'$ for the definition of ρ) such that

$$\sup_{(x,t) \in (\Sigma_1 \backslash B_{\rho_1}(0)) \times [0,\infty)} c^e(x,t) < -\gamma, \tag{2.4}$$

where γ is given by (f_2) or

$$\gamma := \max_{\substack{r\in J,\\ u\in[-M_1,M_1]}} |f_u(t,r,u)| + 4\frac{\lambda_1}{|J|^2} \qquad \text{if } (f_2)' \text{ is assumed,} \tag{2.5}$$

with M_1 as in (U_2) and J as in $(f_2)'$. Observe that γ and ρ_1 are independent of $e \in \mathbb{S}^{N-1}$.

For the proofs of our main results we need c^e to be negative not only far from the origin, but also near $\partial\Sigma_1$. This is achieved by a modification of w_e using appropriate subsolutions, as in [1,17,18]; however, since we only consider radial domains, the proof is simpler and more explicit. We use χ_J to denote the characteristic function of an interval $J \subset \mathbb{R}$.

Lemma 2.1 *For every $e \in \mathbb{S}^{N-1}$, let w_e be a classical solution of (2.2) with c^e satisfying (2.3) and (2.4). For ρ_1 as in (2.4) and $R > 0$, fix $\delta > 0$ such that*

$$\delta < \min\left\{\frac{1}{\gamma + \beta_{B_{\rho_1}(0)}}, \frac{R}{8R + 2(N-1)}\right\}, \tag{2.6}$$

where $\beta_{B_{\rho_1}(0)}$ is given by (2.3). If $R = 0$, we take

$$\delta < \frac{1}{\gamma + \beta_{B_{\rho_1}(0)}}. \tag{2.7}$$

Then, there are measurable functions $\hat{b}_i$, $\hat{c}^e$, and a strong solution $\hat{w}_e$ of

$$\begin{aligned} (\hat{w}_e)_t - \Delta\hat{w}_e - \sum_{i=1}^{N} \hat{b}_i(x)(\hat{w}_e)_{x_i} - \hat{c}^e(x,t)\hat{w}_e = 0 \qquad & \text{in } \Sigma_1 \times (0,\infty), \\ \hat{w}_e = 0 \qquad & \text{on } \partial\Sigma_1 \times (0,\infty), \end{aligned} \tag{2.8}$$

such that, for each $x \in \Sigma_1$ the sign of $w_e(x)$ is the same as the sign of $\hat{w}_e(x)$ and for any subset $Q \subset \Sigma_1 \times (0,\infty)$,

$$\frac{1}{4}||\hat{w}_e||_{L^\infty(Q)} \le ||w_e||_{L^\infty(Q)} \le ||\hat{w}_e||_{L^\infty(Q)}. \tag{2.9}$$

Furthermore, for $x \in \Sigma_1$ and R as in (2.1),

$$\begin{aligned} |\hat{b}_1(x_1)| &\le 4\chi_{[0,\delta]}(x_1) + 4\chi_{[R,R+\delta]}(|x|), \\ |\hat{b}_i(x_1)| &\le 4\chi_{[R,R+\delta]}(|x|), \\ |\hat{c}^e(x)| &\le |c^e| + \frac{2}{\delta}\chi_{[0,\delta]}(x_1) + 8\chi_{[0,\delta]}(x_1)\chi_{[R,R+\delta]}(|x|) \\ &\quad + \left(\frac{2}{\delta} + \frac{2(N-1)}{R}\right)\chi_{[R,R+\delta]}(|x|), \end{aligned} \tag{2.10}$$

where all terms containing R are dropped if $R = 0$. In addition,

$$\hat{c}^e < -\gamma \quad \text{in } \mathbb{R}^N \backslash G, \qquad G := \{x \in \Sigma_1 \,:\, \operatorname{dist}(x, \partial\Sigma_1) > \delta \text{ and } |x| < \rho_1\}. \tag{2.11}$$

Proof Let $R \ge 0$ be as in (2.1), $\delta > 0$ be as in (2.6), and recall that $I := [R, \infty)$.

For $R > 0$ let $h : [0,\infty) \to \mathbb{R}$ and $g : [R,\infty) \to \mathbb{R}$ be given by

$$\begin{aligned} h(t) &= \left(-\frac{1}{2\delta}(t-\delta)^2 + \frac{\delta}{2} + \frac{1}{2}\right)\chi_{[0,\delta]}(t) + \left(\frac{\delta}{2} + \frac{1}{2}\right)\chi_{(\delta,\infty]}(t), \\ g(r) &= \left(-\frac{1}{2\delta}(r-R-\delta)^2 + \frac{\delta}{2} + \frac{1}{2}\right)\chi_{[R,R+\delta]}(r) + \left(\frac{\delta}{2} + \frac{1}{2}\right)\chi_{(R+\delta,\infty]}(r). \end{aligned}$$

If $R = 0$, we keep the definition of h and set $g = \frac{1}{2}$. Observe that h and g are differentiable, non-decreasing, concave, piecewise C^2 functions with

$$\frac{1}{2} \le g < 1 \qquad \text{and} \qquad \frac{1}{2} \le h < 1 \qquad \text{in } \Sigma_1, \tag{2.12}$$

and

$$\begin{aligned} |h'(t)| &\le \chi_{[0,\delta]}(t)\,, \qquad |g'(r)| \le \chi_{[R,R+\delta]}(r)\,, \\ |h''(t)| &\le \frac{1}{\delta}\chi_{[0,\delta]}(t)\,, \qquad |g''(r)| \le \frac{1}{\delta}\chi_{[R,R+\delta]}(r)\,. \end{aligned} \tag{2.13}$$

Then, if $R > 0$ a direct calculation shows that

$$\hat{w}_e(x,t) := \frac{w_e(x,t)}{g(|x|)h(x_1)}, \qquad (x,t) \in \Sigma_1 \times (0,\infty),$$

belongs to $C^2(\Sigma_1 \backslash S)$, where $S := \{x : x_1 = \delta \text{ or } |x| = R + \delta\}$ and $\hat{w}_e$ satisfies (2.8) outside of $\Sigma_1 \backslash S$ with

$$\hat{b}_i(x) = \begin{cases} 2\frac{h'(x_1)}{h(x_1)} + 2\frac{g'(|x|)}{g(|x|)}\frac{x_1}{|x|}, & \text{if } i = 1, \\ 2\frac{g'(|x|)}{g(|x|)}\frac{x_i}{|x|}, & \text{if } i \ne 1, \end{cases}$$

$$\hat{c}^e(x) = \frac{h''(x_1)}{h(x_1)} + 2\frac{h'(x_1)}{h(x_1)}\frac{g'(|x|)}{g(|x|)}\frac{x_1}{|x|} + \frac{g''(|x|) + \frac{N-1}{|x|}g'(|x|)}{g(|x|)} + c^e(x)\,.$$

Note that $|x| \ge R > 0$ and denominators do not vanish. If $R = 0$, $\hat{w}_e$ satisfies (2.8) and $\hat{b}_i$, $\hat{c}^e$ as above with all terms containing g' or g'' removed. In addition, $\hat{w}_e$ has bounded second derivatives on $\Sigma_1 \backslash S$ and, in particular, $\hat{w}_e$ belongs to $W^{1,\infty}(\Sigma_1)$. Thus, $\hat{w}_e$ is a strong solution of (2.8).

By (2.12), we have that (2.9) holds and, by using (2.13) and $\frac{|x_i|}{|x|} \le 1$, we obtain that (2.10) holds. Furthermore, since $R + \delta < \rho_1$, $g' = g'' = 0$ on $[\rho_1, \infty)$ and h is concave, it follows that

$$\hat{c}^e < -\gamma \qquad \text{if} \quad |x| > \rho_1\,. \tag{2.14}$$

If $x_1 < \delta$ and $|x| < \rho_1$, then by (2.6)

$$\hat{c}^e \le -\frac{1}{\delta} + \left(8 + \frac{2(N-1)}{R} - \frac{1}{\delta}\right)\chi_{[R,R+\delta]}(|x|) + \sup_{|x|<\rho_1} |c^e| < -\gamma\,, \tag{2.15}$$

where, as above, we drop the term with $\chi_{[R,R+\delta]}(|x|)$ if $R = 0$. Finally, in the case $R > 0$, if $x_1 > \delta$ and $|x| \in (R, R+\delta)$, we have, by (2.6), that

$$\hat{c}^e \le 8 + \frac{2(N-1)}{R} - \frac{1}{\delta} < -\gamma \tag{2.16}$$

and (2.11) follows. □

Next, using $(f_2)'$, we construct a suitable subsolution inside G. Let G be as in (2.11) and γ as in (2.5). Let $J = (a,b)$ be the interval given by $(f_2)'$ for $M = M_1$ (with M_1 as in (U_2)) and for some $R < a < b < \rho$. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$.

Lemma 2.2 *Let $0 \le \tau < T \le \infty$, $r = \frac{|I|}{2}$, $B = B_r(\frac{b+a}{2}e_1) \subset G$, and let η denote the positive principal eigenfunction of the Laplacian with Dirichlet boundary conditions on the*

unit ball $B_1(0)$. Then $\varphi(x,t) := e^{-\gamma t}\eta(\frac{x-x_0}{r})$ is a (strict) subsolution of (2.8) in $B\times(\tau,T)$, namely,

$$\varphi_t < \Delta\varphi + \sum_{i=1}^{N} \hat{b}_i\,\varphi_{x_i} + \hat{c}^e\,\varphi \quad \text{in } B\times(\tau,T), \quad \varphi = 0 \ \text{ on } \partial B\times(\tau,T)\,, \tag{2.17}$$

where $\hat{b}_i$ and $\hat{c}^e$ are defined in the proof of Lemma 2.1.

Proof Since $\hat{c}^e = c^e$ and $\hat{b}_i^e = 0$ in G it suffices to verify that

$$\varphi_t < \Delta\varphi + c^e\varphi \ \text{ in } B\times(\tau,T), \quad \varphi = 0 \ \text{ on } \partial B\times(\tau,T). \tag{2.18}$$

However, by the definition, $\varphi(x) = 0$ on ∂B and, in addition, from (2.5) (with $J = \{|x| : x\in B\}$) we have that

$$\varphi_t - \Delta\varphi - c^e\varphi = \left(-\gamma + \frac{\lambda_1}{r^2} + c^e\right)\varphi < 0 \quad \text{in } B\times(\tau,T). \tag{2.19}$$

□

2.2 Regularity of Solutions

In this section we show that u is locally equicontinuous. We extend the proof of [23, Lemma 3.1] to unbounded domains. Fix $\alpha\in(0,1]$ and a domain $\Omega\subset\mathbb{R}^N$. Set $Q := \Omega\times(\tau,T)$ for $0\le\tau<T$. Following [21, p. 4] we define $C^{1+\alpha,\frac{1+\alpha}{2}}(Q) := \{f : |f|_{1+\alpha;Q} < \infty\}$, where

$$\begin{aligned} [f]_{\alpha;Q} &:= \sup\left\{\frac{|f(x,t)-f(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}} \ : \ (x,t),(y,s)\in Q,\ (x,t)\neq(y,s)\right\},\\ |f|_{1+\alpha;Q} &:= \sum_{|\beta|\le 1}\sup_Q |D_x^\beta f| + \sum_{|\beta|=1}[D_x^\beta f]_{\alpha;Q}, \end{aligned} \tag{2.20}$$

and D_x^β denotes spatial derivatives of order $\beta\in\mathbb{N}_0^N$.

Lemma 2.3 *Let u be a classical solution of (1.1) and assume that (U_1), (U_2), (f_0), and (f_1) or $(f_1)'$ hold. If $\tilde{\gamma} = \frac{1}{N+3}$, then for every $R_1 > R$ there is $\tilde{C} > 0$ satisfying*

$$|u|_{1+\tilde{\gamma};\overline{B_{R_1}(0)\cap\Sigma}\times[s,s+2]} \le \tilde{C} \qquad \text{for all } s > 2.$$

Proof Let $R\ge 0$ be as in (2.1), $M_1 > 0$ as in (U_2), K_{R_1} as in (f_0), and $C = C(M_1,[R,R_1])$ as in $(f_1)'$ (or implicitly given by (f_1)). Then $|u|\le M_1$ in $\Sigma\times(0,\infty)$ and

$$|f_u(t,r,v)| < C \qquad \text{for } t>0,\ r\in[R,R_1],\ v\in[-M_1,M_1].$$

Furthermore, $f(t,r,0) < K_{R_1}$ for $t>0$ and $r\in[R,R_1]$. Fix $s>2$. Then, u satisfies that

$$|u_t - \Delta u| = |f(t,|x|,u)| = \left|\int_0^1 \partial_u f(t,|x|,su)ds\ u + f(t,|x|,0)\right| < M_1C + K_{R_1} =: C_1$$

for $(x,t)\in Q := (B_{R_1}(0)\cap\Sigma)\times(s,s+2)$. Recall that, if $R>0$, then $u=0$ on $\partial B_R(0)$. Then, by [14, Theorem 7.22 or Theorem 7.30], there is $C_2 = C_2(R,C,M_1,K_{R_1},N,R_1) > 0$ such that $\|D^2u\|_{L^{N+3}(Q)} + \|u_t\|_{L^{N+3}(Q)} \le C_2$. By a standard interpolation argument (see [14, Lemma 7.20]), $\|u\|_{W^{2,1}_{N+3}(Q)} \le C_3$ for some constant $C_3 = C_3(R,C,M_1,K_{R_1},N,R_1) > 0$. By Sobolev embeddings (see, for example, [21, embedding (1.2)] and the references therein),

we then have that $u \in C^{1+\tilde{\gamma},(1+\tilde{\gamma})/2}(Q)$ for $\tilde{\gamma} = \frac{1}{N+3} \in (0,1)$, and there is a constant $C_4 = C_4(R, C, M_1, K_{R_1}, N, R_1) > 0$ such that $|u|_{1+\tilde{\gamma};Q} \le C_4 \|u\|_{W^{2,1}_{N+3}(Q)} \le C_4 C_3 =: \tilde{C}$. □

From Lemma 2.3 and the uniform decay assumption (U_1) it follows that $\omega(u)$ (as defined in (1.2)) is a nonempty and compact set in $C_0(\Sigma)$ and that (1.3) holds.

2.3 Estimates of Solutions of Linear Equations

In this subsection we prove bounds for general linear parabolic equations needed for the symmetry results. Fix a domain $\Omega \subset \mathbb{R}^N$ (possibly unbounded), $0 \le \tau < T \le \infty$, and denote $Q := \Omega \times (\tau, T)$. For $i = 1, \dots, N$, let $b_i, c : Q \to \mathbb{R}$ be measurable functions such that, for every bounded subdomain $\omega \subset \Omega$, there is $\beta_\omega > 0$ such that

$$|b_i|, |c| < \beta_\omega \qquad \text{in } \omega \times (\tau, T). \tag{2.21}$$

We formulate the following Harnack inequality for sign-changing solutions proved in [18]. Let $v^- := -\min\{0, v\} \ge 0$ and $v^+ := \max\{0, v\} \ge 0$ denote the negative and positive parts of v, respectively. We define $\partial_P Q$ to be the parabolic boundary of a cylindrical domain $Q = U \times (\tau, T)$, that is, $\partial_P Q = (\partial U \times (\tau, T)) \bigcup (\bar{U} \times \{\tau\})$.

Lemma 2.4 [18, Lemma 3.4] *Let Ω be a bounded domain. Given $d > 0, \theta > 0$ there is a positive constant κ determined only by N, $\mathrm{diam}(\Omega)$, β_Ω, d, and θ with the following property. If D, U are domains in Ω with $D \subset\subset U$, $\mathrm{dist}(\overline{D}, \partial U) \ge d$, and v is a solution of*

$$v_t = \Delta v + \sum_{i=1}^{N} b_i(x,t) v_{x_i} + c(x,t) v \qquad \text{in } U \times (\tau, \tau + 4\theta),$$

then

$$\inf_{D \times (\tau+3\theta, \tau+4\theta)} v(x,t) \ge \kappa ||v^+||_{L^\infty(D \times (\tau+\theta, \tau+2\theta))} - e^{4m\theta} \sup_{\partial_P (U \times (\tau, \tau+4\theta))} v^-,$$

where $m = \sup_{U \times (\tau, \tau+4\theta)} c$.

The next lemma is the weak maximum principle in unbounded domains, see, for example, [17, Lemma 2.1], [14,20].

Lemma 2.5 *Let $U \subset \mathbb{R}^N$ be a (possibly unbounded) domain and let v be such that*

$$v_t \ge \Delta v + \sum_{i=1}^{N} b_i(x,t) v_{x_i} + c(x,t) v \quad \text{in } U \times (0, \infty)$$

with

$$m := \sup_{U \times (0,\infty)} c(x,t) < \infty \tag{2.22}$$

and

$$\lim_{|x| \to \infty, x \in U} v(x,t) = 0 \qquad \text{for all } t > 0. \tag{2.23}$$

Then, for each $(x,t) \in U \times (0, \infty)$ one has

$$e^{-mt} v^-(x,t) \le \sup_{(y,s) \in \partial_p (U \times (0,\infty))} e^{-ms} v^-(y,s).$$

Note that assumption (2.23) can be weakened, see for example [21, Proposition 52.4].

2.4 Geometric Lemmas

We use the following geometric characterization of foliated Schwarz symmetry proved in [24]. Recall that, for $e \in \mathbb{S}^{N-1}$, $\Sigma(e) := \{x \in \Sigma : x \cdot e > 0\}$ and $\sigma_e(x) := x - 2(x \cdot e)e$.

Proposition 2.6 [24, Proposition 3.3] *Let $\mathcal{U}$ be a set of continuous functions defined on a radial domain $\Sigma \subset \mathbb{R}^N$, $N \geq 2$ such that*

$$\mathcal{M} := \{e \in \mathbb{S}^{N-1} \mid z(x) \geq z(\sigma_e(x)) \text{ for all } x \in \Sigma(e) \text{ and } z \in \mathcal{U}\} \neq \emptyset\,.$$

Fix $\tilde{e} \in \mathcal{M}$ and assume that for any two dimensional subspaces $P \subseteq \mathbb{R}^N$ containing $\tilde{e}$ there exist $p_1 \neq p_2$ in a connected component of $\mathcal{M} \cap P$ such that $z \equiv z \circ \sigma_{p_1}$ and $z \equiv z \circ \sigma_{p_2}$ for every $z \in \mathcal{U}$. Then, there exists $p \in \mathbb{S}^{N-1}$ such that every $z \in \mathcal{U}$ is foliated Schwarz symmetric with respect to p.

Next, we show the following lemma, which is used to show Theorem 1.8.

Lemma 2.7 *Let $U \subset \mathbb{S}^{N-1}$ be an open set, $\Sigma \subset \mathbb{R}^N$ a radial domain and $z \in C(\Sigma)$. If*

$$z(x) \geq z(\sigma_e(x)) \qquad \text{for all } x \in \Sigma(e) \text{ and } e \in U\,, \tag{2.24}$$

$$z \equiv z \circ \sigma_p \qquad \text{for some } p \in U\,, \tag{2.25}$$

then z is radially symmetric.

The proof of Lemma 2.7 is based on the following results.

Lemma 2.8 [24, Lemma 3.1] *Let $v \in C(\mathbb{R})$ be an even and 2π-periodic function, and denote $\mathcal{R} := \{\eta \in \mathbb{R} : v(2\eta - \varphi) = v(\varphi) \text{ for each } \varphi \in \mathbb{R}\}$ the set of points of reflectional symmetry of v. If for some $\eta \in \mathbb{R}$,*

$$\begin{aligned} v(\eta + \varphi) &\geq v(\eta - \varphi) \quad \text{for all } \varphi \in [0, \pi] \text{ and} \\ v(\eta + \varphi_0) &> v(\eta - \varphi_0) \quad \text{for some } \varphi_0 \in (0, \pi), \end{aligned} \tag{2.26}$$

then $\mathcal{R} = \{n\pi \ : \ n \in \mathbb{Z}\}$.

Lemma 2.9 *Let $v \in C(\mathbb{R})$ be a 2π periodic function. If*

(i) (symmetry)

$$v(\theta) = v(-\theta) \qquad \text{for all } \theta \in \mathbb{R}, \tag{2.27}$$

(ii) (local reflectional inequalities) There is $\varepsilon > 0$ such that

$$v(\eta + \theta) \geq v(\eta - \theta) \qquad \text{for all } \eta \in (0, \varepsilon),\ \theta \in (0, \pi)\,, \tag{2.28}$$

then v is nondecreasing in $(0, \pi)$ and also

$$v(\eta + \theta) \geq v(\eta - \theta) \qquad \text{for all } \eta, \theta \in (0, \pi). \tag{2.29}$$

Moreover, if the inequality in (2.28) is strict, then (2.29) is also strict and v is strictly increasing in $(0, \pi)$.

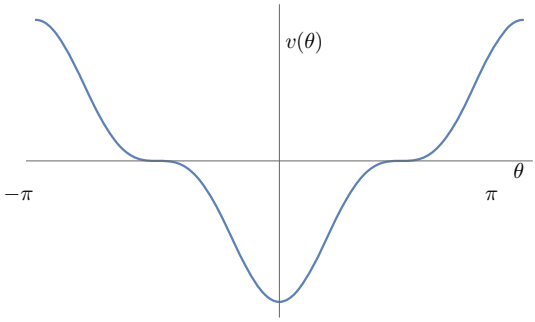

Fig. 3 Example of a 2π-periodic function v with reflectional symmetry at 0 and satisfying the reflectional inequalities (2.26) and (2.29)

Proof If (2.28) holds with equality for all $\eta \in (0, \varepsilon)$ and $\theta \in (0, \pi)$, then v is constant and the conclusion follows trivially. Now, assume that there is $\eta_0 \in (0, \varepsilon)$ and $\theta_0 \in (0, \pi)$ such that (2.28) is strict, then Lemma 2.8 implies that 0 and π are points of reflectional symmetry of v, that is, $v(\theta) = v(-\theta) = v(2\pi - \theta)$ for each θ.

Clearly (2.28) implies that v is non-decreasing on $(0, \pi)$. Indeed, fix any $\theta_1, \theta_2 \in (0, \pi)$ such that $\alpha := \theta_2 - \theta_1 \in (0, \varepsilon)$. Set $\eta := \alpha/2 \in (0, \varepsilon)$ and $\varphi_0 := \theta_1 + \alpha/2 \in (0, \pi)$. Then, $\theta_2 = \eta + \varphi_0$ and $-\theta_1 = \eta - \varphi_0$, and therefore by (2.28)

$$v(\theta_2) = v(\eta + \varphi_0) \geq v(\eta - \varphi_0) = v(-\theta_1) = v(\theta_1). \tag{2.30}$$

Moreover, if the inequality in (2.28) is strict then v is strictly increasing.

To prove (2.29), fix $\eta, \theta \in (0, \pi)$ and since v is even, it suffices to prove $v(\eta + \theta) \geq v(|\eta - \theta|)$. If $\eta + \theta \leq \pi$, then $\eta + \theta \geq |\eta - \theta|$ and (2.29) follows from the monotonicity of v. If $\eta + \theta > \pi$, then $\pi \geq 2\pi - (\eta + \theta) \geq |\theta - \eta|$ and, by the symmetry and monotonicity of v, $v(\eta + \theta) = v(2\pi - (\eta + \theta)) \geq v(|\theta - \eta|)$, as required. The case with strict inequality follows analogously. □

For $e \in \mathbb{S}^{N-1}$ let

$$z_e(x) := z(\Gamma_e(x)) - z(\sigma_e(\Gamma_e(x))), \quad x \in \Sigma_1, z \in \omega(u)\,, \tag{2.31}$$

where Γ_e is a rotation that maps e_1 to e (introduced in Sect. 2.1). Recall $\Sigma_1 := \{x \in \Sigma : x_1 > 0\}$.

Lemma 2.10 *Let $z \in C(\Sigma)$, $\varepsilon \in (0, \pi)$, and let*

$$e(\eta) := (\cos\eta, \sin\eta, 0, \ldots, 0) \in \mathbb{R}^N \quad \text{for any } \eta \in (0, 2\pi).$$

If $z_{e(0)} \equiv 0$ in Σ_1 and $z_{e(\eta)} > 0$ in Σ_1 for all $\eta \in (0, \varepsilon)$ then

$$z_{e(\eta)} > 0 \quad \text{in } \Sigma_1 \quad \text{for all } \eta \in (0, \pi), \qquad z_{e(\eta)} < 0 \quad \text{in } \Sigma_1 \quad \text{for all } \eta \in (-\pi, 0). \tag{2.32}$$

Proof For every $x \in \mathbb{R}^N$ $(N \geq 2)$ we write $x = (x_1, x_2, x')$ for some $x' \in \mathbb{R}^{N-2}$. Fix $x' \in \mathbb{R}^{N-2}$, $r > 0$, and let $v : (0, 2\pi) \to \mathbb{R}$ be given by $v(\theta) := z(r\sin\theta, r\cos\theta, x')$. With this notation, (2.31) can be rewritten for $x_1 = r\sin\theta$ and $x_2 = r\cos\theta$ as

$$z_{e(\eta)}(x) = v(\theta + \eta) - v(\eta - \theta). \tag{2.33}$$

Note that we are not using the usual polar coordinates. Since $z_{e(0)} \equiv 0$ in Σ_1 and $z_{e(\eta)} > 0$ in Σ_1 for all $\eta \in (0, \varepsilon)$ we have that $v(\theta) = v(-\theta)$ for $\theta \in (0, \pi)$ and $v(\eta + \theta) > v(\eta - \theta)$ for all $\theta \in (0, \pi)$ and $\eta \in (0, \varepsilon)$. By Lemma 2.9 we have that $v(\eta + \theta) > v(\eta - \theta)$ for $\theta, \eta \in (0, \pi)$. Using the symmetry with respect to 0 and the (2π)-periodicity we also have

that $v(\eta-\theta) > v(\eta+\theta)$ for $\theta \in (0,\pi)$ and $\eta \in (-\pi, 0)$. Since this holds for $x' \in \mathbb{R}^{N-2}$ and $r > 0$ arbitrary we obtain that (2.32) holds. □

We are ready to show Lemma 2.7.

Proof of Lemma 2.7 Let $p \in U$ be as in (2.25), $q \in \mathbb{S}^{N-1}\backslash\{p\}$ and $P := \text{span}\{q, p\}$. Without loss of generality we assume that $p = (1, 0, \dots, 0)$ and $P := \{(x_1, x_2, 0, \dots, 0) \mid x_1, x_2 \in \mathbb{R}\}$. Using polar coordinates, let $\tilde{v}(r, \varphi, x') := z(r\cos\varphi, r\sin\varphi, x') = z(x)$ with $x' = (x_3, \dots, x_N) \in \mathbb{R}^{N-2}$, $(x_1, x_2, x') = x \in \Sigma$, $\varphi \in \mathbb{R}$, and $r = |x|$. Fix $r > 0$, $x' \in \mathbb{R}^{N-2}$ and define $v : \mathbb{R} \to \mathbb{R}$, $v(\varphi) := \tilde{v}(r, \varphi, x')$. Since U is open, there exists $\varepsilon > 0$ such that $v(\varphi) = v(-\varphi)$ for $\varphi \in \mathbb{R}$ and $v(\eta+\varphi) \geq v(\eta-\varphi)$ for $\eta \in (-\varepsilon, \varepsilon)$ and $\varphi \in (0, \pi)$. Then, by Lemma 2.9 we have that $v(\eta+\varphi) \geq v(\eta-\varphi)$ for $\eta \in (-\varepsilon, \pi)$, $\varphi \in (0, \pi)$, which yields that $v(\eta+\varphi) = v(\eta-\varphi)$ for $\eta \in (-\varepsilon, 0)$, $\varphi \in (0, \pi)$, and consequently v is constant, that is, $\tilde{v}$ does not depend in φ. Since $r > 0$ and $x' \in \mathbb{R}^{N-2}$ were arbitrary, we have $z \equiv z \circ \sigma_e$ for all $e \in P$, and in particular $z \equiv z \circ \sigma_q$. Since q was chosen arbitrarily in $\mathbb{S}^{N-1}$ the radial symmetry follows. □

3 Symmetry Results

3.1 Strong Stability Outside Compact Sets

In this section we assume that (f_0), $(f_1)'$, $(f_2)'$ hold and u is a classical solution of (1.1) satisfying (U_0)–(U_2). We recall some previously used notation. Let $\Sigma = \mathbb{R}^N\backslash B_R$, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ if $R > 0$ and $B_0 := \emptyset$; furthermore, as defined in the introduction, $\Sigma_1 = \Sigma(e_1) = \{x \in \Sigma : x \cdot e_1 > 0\}$, with $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$. Let G be given by (2.11) and U be a bounded domain such that

$$G \subset\subset U \subset\subset \Sigma_1. \tag{3.1}$$

If $J = (a, b)$ is the interval given in (1.10), by $(f_2)'$, then

$$B = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}e_1\right) \subset\subset G \tag{3.2}$$

as in Lemma 2.2, see Fig. 4. For $e \in \mathbb{S}^{N-1}$, let $\hat{w}_e$ be as in Lemma 2.1 and note that, by (2.10), there is $\hat{\beta}_U > 0$ (independent of e) such that

$$|\hat{c}^e| + |\hat{b}_i| < \hat{\beta}_U \qquad \text{in } U \times (0, \infty). \tag{3.3}$$

The next proposition extends the strategy in [18, Theorem 3.7] to unbounded domains.

Proposition 3.1 *Let γ be as in (2.5). There is $\mu > 0$ such that, if there is $\tau > 0$ and $e \in \mathbb{S}^{N-1}$ with*

$$\begin{aligned} \hat{w}_e &> 0, \qquad \text{in } \overline{G} \times [\tau, \tau+1), \\ ||\hat{w}_e^-(\cdot, \tau)||_{L^\infty(\Sigma_1\backslash G)} &\leq \mu ||\hat{w}_e||_{L^\infty(G\times(\tau+\frac{1}{8}, \tau+\frac{1}{4}))}, \end{aligned} \tag{3.4}$$

then $w_e > 0$ in $\overline{G} \times [\tau, \infty)$ and $\lim_{t\to\infty} |w_e^-(\cdot, t)||_{L^\infty(\Sigma_1)} = 0$.

Proof For $\theta = \frac{1}{8}$, $D = B$ as in (3.2), $\Omega = G$, and U as in (3.1), $\hat{\beta}_U$ as in (3.3), and γ as in (2.5), let $\kappa > 0$ be given by Lemma 2.4, and let $\mu > 0$ be such that

$$\mu < \frac{\kappa^2}{\kappa+1} < \kappa. \tag{3.5}$$

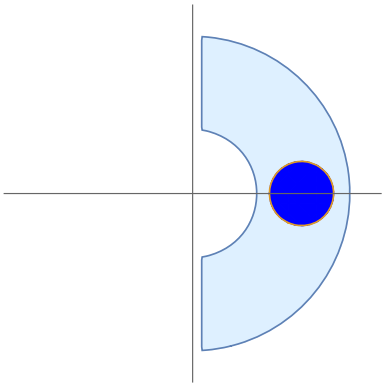

Fig. 4 The set G with $\Sigma = \mathbb{R}^2 \backslash B_R$ and the ball $B = B_{\frac{b-a}{2}}(\frac{a+b}{2}e_1)$

We argue by contradiction. Assume (by the maximum principle) that there is $T \geq \tau + 1$ such that

$$\begin{aligned} \hat{w}_e &> 0 \qquad \text{in } G \times [\tau, T), \\ \hat{w}_e(x^*, T) &= 0 \qquad \text{for some } x^* \in \partial G. \end{aligned} \tag{3.6}$$

Define $r_0 := ||(\hat{w}_e)^-(\cdot, \tau)||_{L^\infty(\Sigma_1 \backslash G)}$. By Lemma 2.1, $m := \sup_{\Sigma_1 \backslash G \times (\tau, \infty)} c^e \leq -\gamma$ and $\hat{w}_e \geq 0$ in $\partial(\Sigma_1 \backslash G) \times [\tau, T]$. Then, by Lemma 2.5,

$$\begin{aligned} ||(\hat{w}_e)^-(\cdot, t)||_{L^\infty(\Sigma_1)} &= ||(\hat{w}_e)^-(\cdot, t)||_{L^\infty(\Sigma_1 \backslash G)} && (3.7) \\ &\leq e^{mt} \sup_{(s,y) \in \partial_p(\Sigma_1 \backslash G \times [\tau, T])} e^{-ms}(\hat{w}_e)^-(y) \\ &= e^{m(t-\tau)} ||(\hat{w}_e)^-(\cdot, \tau)||_{L^\infty(\Sigma_1 \backslash G)} && (3.8) \\ &\leq r_0, && (3.9) \end{aligned}$$

for all $t \in [\tau, T)$. Hence, (3.4), (3.5), (3.9), and Lemma 2.4 imply, for $(x, t) \in \overline{G} \times [\tau + 3\theta, \tau + 4\theta]$, that

$$\begin{aligned} \hat{w}_e(x, t) &\geq \kappa ||(\hat{w}_e)^+||_{L^\infty(G \times (\tau+\theta, \tau+2\theta))} - e^{4m\theta} \sup_{\partial_P(U \times (\tau, \tau+4\theta))} \hat{w}_e^- \\ &\geq \kappa ||(\hat{w}_e)^+||_{L^\infty(G \times (\tau+\theta, \tau+2\theta))} - e^{4m\theta} r_0 \geq r_0 \left(\frac{\kappa}{\mu} - 1 \right) =: r_1 > 0 \,. \end{aligned} \tag{3.10}$$

Furthermore, by Lemma 2.2, there is $\varphi(x, t) = e^{-\gamma t} \eta(\frac{x - x_0}{r})$ satisfying (2.17) (with $x_0 = \frac{a+b}{2} e_1$). By (3.10),

$$\begin{aligned} \hat{w}_e &> 0 \qquad \text{in } G \times [\tau + 4\theta, T), \\ \varphi &= 0 \qquad \text{on } \partial B \times [\tau + 4\theta, T), \\ \hat{w}_e(x, \tau + 4\theta) &\geq r_1 \frac{\varphi(x, \tau + 4\theta)}{||\varphi(\cdot, \tau + 4\theta)||_{L^\infty(B)}} \qquad \text{for } x \in B. \end{aligned}$$

Then, by comparison,

$$\hat{w}_e(x, t) \geq r_1 \frac{\varphi(x, t)}{||\varphi(\cdot, \tau + 4\theta)||_{L^\infty(B)}} \geq r_1 e^{-\gamma(t - \tau - 4\theta)} \frac{\eta\left(\frac{x - x_0}{r}\right)}{||\eta||_{L^\infty(B)}}$$

for $x \in B$ and $t \in [\tau + 4\theta, T]$. Using Lemma 2.4 in $[\tau + 4\theta, T]$ and (3.8),

$$\begin{aligned}\hat{w}_e(x,T) &\geq \kappa ||\hat{w}_e||_{L^\infty(G\times(T-3\theta,T-2\theta))} - e^{4m\theta} \sup_{\partial_P(U\times(T-4\theta,T))} \hat{w}_e^- \\ &\geq \kappa r_1 e^{-\gamma(T-\tau-4\theta)} - e^{4m\theta} e^{m(T-\tau-4\theta)} r_0 \\ &= r_0 e^{-\gamma(T-\tau-4\theta)} \left(\kappa \left(\frac{\kappa}{\mu} - 1 \right) - e^{4m\theta} e^{(m+\gamma)(T-\tau-4\theta)} \right) \\ &\geq r_0 e^{-\gamma(T-\tau-4\theta)} \left(\frac{\kappa^2}{\mu} - \kappa - 1 \right) > 0,\end{aligned}$$

for all $x \in \overline{G}$, which contradicts (3.6). Therefore, $\hat{w}_e > 0$ in $\overline{G} \times [\tau, \infty)$, and consequently by letting $t \to \infty$ in (3.8), we have that $\lim_{t\to\infty} ||\hat{w}_e^-(\cdot, t)||_{L^\infty(\Sigma_1)} = 0$. □

For $e \in \mathbb{S}^{N-1}$, let z_e be as in (2.31) and define

$$\mathcal{M} := \{e \in \mathbb{S}^{N-1} \ : \ z_e(x) \geq 0 \text{ in } \Sigma_1 \text{ for all } z \in \omega(u)\}. \tag{3.11}$$

We are ready to show Theorem 1.3 via a rotating-plane method. We split the proof in several lemmas.

Lemma 3.2 *Let $e \in \mathbb{S}^{N-1}$ be as in (U_0) and $P \subset \mathbb{R}^N$ be a two dimensional subspace such that $e \in P$. Then, there exists $\varepsilon > 0$ such that $e' \in \mathcal{M}$ for all $e' \in \mathbb{S}^{N-1} \bigcap P$ with $|e' - e| < \varepsilon$.*

Proof Due to rotational invariance, we can assume without loss of generality that $e = e_1 = (1, 0, \ldots, 0)$ and $P = \{(x_1, 0, \ldots, 0, x_N) \mid x_1, x_N \in \mathbb{R}\}$. By Lemma 2.5,

$$e^{-mt} w_e^-(x,t) \leq \sup_{(y,s)\in\partial_P(\Sigma_1\times(0,\infty))} e^{-ms} w_e^-(y,s) = \sup_{\Sigma_1} w_e^-(\cdot, 0) = 0,$$

for all (x, t) in $\Sigma_1 \times (0, \infty)$ and with $m := \sup_{\Sigma_1\times(0,\infty)} c^e$. Then $w_e \geq 0$ in $\Sigma_1 \times (0, \infty)$ and, since $w_e(x, 0) \not\equiv 0$, the strong maximum principle (see, for example, [20, Theorem 5 on p. 173 and Remark 2 on p. 175]) implies that

$$w_e(x,t) > 0 \quad \text{in } \Sigma_1 \times (0,\infty). \tag{3.12}$$

By (3.12), there is $\eta > 0$ such that $\hat{w}_e > 2\eta > 0$ in $\overline{G} \times [1, 2]$, where $\hat{w}_e$ is as in Lemma 2.1. Then, by continuity and (U_1), there exists $\varepsilon > 0$ such that any $e' \in \mathbb{S}^{N-1} \bigcap P$ with $|e - e'| < \varepsilon$ satisfies that

$$\hat{w}_{e'} > \eta > 0 \quad \text{in } \overline{G} \times [1,2], \qquad ||\hat{w}_{e'}^-||_{L^\infty(\Sigma(e_1)\times(1,2))} \leq \mu\eta \leq \mu ||\hat{w}_{e'}||_{L^\infty(G\times(1,2))},$$

where μ is given by Proposition 3.1. Then, by Proposition 3.1, $\lim_{t\to\infty} ||\hat{w}_{e'}^-(\cdot, t)||_{L^\infty(\Sigma(e'))} = 0$, that is, $e' \in \mathcal{M}$. □

Lemma 3.3 *Let $e \in \mathcal{M}$ and let $P \subset \mathbb{R}^N$ be a two dimensional subspace with $e \in P$. If there is $\tilde{z} \in \omega(u)$ such that $\tilde{z}_e \not\equiv 0$, then there exists $\varepsilon > 0$ such that $e' \in \mathcal{M}$ for all $e' \in \mathbb{S}^{N-1} \bigcap P$ with $|e - e'| < \varepsilon$.*

Proof Recall that U and G were already defined at the beginning of the section. As above, we can assume without loss of generality that $e = e_1 = (1, 0, \ldots, 0)$ and $P = \{(x_1, 0, \ldots, 0, x_N) \mid x_1, x_N \in \mathbb{R}\}$ and we use the same notation as in Lemma 3.2.

Since, by assumption, $\tilde{z}_e \not\equiv 0$, there is $\alpha > 0$ and $x_0 \in \Sigma_1$ such that $\tilde{z}_e(x_0) \geq 4\alpha > 0$. Without loss of generality we can assume that $x_0 \in G = G(\rho_1, \delta)$ (making ρ_1 bigger or δ smaller if necessary). Fix $\{t_n\}_{n=0}^\infty \subset \mathbb{R}$ so that $\lim_{n\to\infty} u(x, t_n) = \tilde{z}(x)$ for all $x \in \mathbb{R}^N$.

There exists $\mathcal{N} \in \mathbb{N}$ such that

$$\hat{w}_e(x_0, t_n) > \alpha \qquad \text{for all } n \geq \mathcal{N}.$$

Since $z_e \geq 0$ in Σ_1 for all $z \in \omega(u)$ and $\lim_{t\to\infty} \operatorname{dist}_{C_0(\mathbb{R}^N)}(\omega(u), u(\cdot, t)) = 0$, there is $T > 0$ such that

$$||\hat{w}_e^-(x,t)||_{L^\infty(\Sigma_1)} < \frac{\mu\kappa\alpha}{8} e^{-4\hat{\beta}_U} \qquad \text{for all } t > T,$$

where $\kappa = \kappa(N, \operatorname{diam}(U), \hat{\beta}_U, \operatorname{dist}(U, G), 1) > 0$, is given by Lemma 2.4, μ as in Proposition 3.1, and $\hat{\beta}_U$ as in (3.3).

Set $\tau := t_{n_0} - 1$, with $n_0 \geq \mathcal{N}$ and $t_{n_0} \geq T$. Then, by Lemma 2.4,

$$\begin{aligned}\inf_{G\times(\tau+3,\tau+4)} \hat{w}_e(x,t) &\geq \kappa ||\hat{w}_e^+||_{L^\infty(D\times(\tau+1,\tau+2))} - e^{4\hat{\beta}_U} \sup_{\partial_P(U\times(\tau,\tau+4))} \hat{w}_e^-, \\ &\geq \kappa\alpha - \frac{\mu\kappa\alpha}{8} \geq \frac{\kappa\alpha}{2} =: \eta > 0.\end{aligned}$$

Then, by continuity and uniform decay (see (U_1)), there is $\varepsilon > 0$ such that, for all $e' \in \mathbb{S}^{N-1} \bigcap P$ with $|e - e'| < \varepsilon$,

$$||\hat{w}_e(x,t) - \hat{w}_{e'}(x,t)||_{L^\infty(\Sigma_1)} < \frac{\eta\mu}{4} \quad \text{for all } t \in [\tau, \tau+4].$$

Hence, if $\tilde{\tau} := \tau + 3$,

$$\begin{aligned}\inf_{D\times(\tilde{\tau},\tilde{\tau}+1)} \hat{w}_{e'}(x,t) &\geq \frac{\eta}{2} > 0, \\ ||\hat{w}_{e'}^-(x,\tilde{\tau})||_{L^\infty(\Sigma_1)} &\leq ||\hat{w}_e^-(x,\tilde{\tau}) - \hat{w}_{e'}^-(x,\tilde{\tau})||_{L^\infty(\Sigma_1)} + ||\hat{w}_e^-(x,\tilde{\tau})||_{L^\infty(\Sigma_1)} \\ &< \frac{\eta\mu}{4} + \frac{\mu\kappa\alpha}{8} e^{-4\hat{\beta}_U} < \frac{\mu\eta}{4} + \frac{\mu\eta}{4} = \mu\frac{\eta}{2} \\ &\leq \mu ||\hat{w}_{e'}^+(x,t)||_{L^\infty(D\times(\tilde{\tau},\tilde{\tau}+1))}.\end{aligned}$$

In particular, the hypotheses of Proposition 3.1 are satisfied, and we have that $\lim_{t\to\infty} ||\hat{w}_{e'}^-(x,t)||_{L^\infty(\Sigma_1)} = 0$, which yields the result. □

Proof of Theorem 1.3 Let $e \in \mathbb{S}^{N-1}$ be as in (U_0). Then, by Lemma 3.2, there is some $\varepsilon > 0$ such that

$$e' \in \mathcal{M} \qquad \text{for all } |e' - e| < \varepsilon. \tag{3.13}$$

Let P be any $2D$-plane that contains e and the origin. Without loss of generality, we may assume that $e = (1, 0, \ldots, 0)$ and $P = \{x = (x_1, x_2, 0, \ldots, 0) \mid x_1, x_2 \in \mathbb{R}\}$. Define

$$\begin{aligned}e_\theta &:= (\cos(\theta), \sin(\theta), 0, \ldots, 0), \\ w_\theta(x,t) &:= w_{e_\theta}(x,t), \quad (x,t) \in \Sigma_1 \times (0,\infty), \\ z_\theta(x) &:= z_{e_\theta}(x), \quad x \in \Sigma_1,\end{aligned}$$

for $\theta \in [-\pi, \pi]$, and

$$\begin{aligned}\Theta_1 &:= \sup\{\theta \in [0,\pi) \mid e_\varphi \in \mathcal{M} \text{ for all } 0 \leq \varphi \leq \theta\}, \\ \Theta_2 &:= \inf\{\theta \in (-\pi, 0] \mid e_\varphi \in \mathcal{M} \text{ for all } \theta \leq \varphi \leq 0\}.\end{aligned}$$

We show that the assumptions of Proposition 2.6 are satisfied. Clearly $e_{\Theta_1}, e_{\Theta_2} \in \mathcal{M} \cap P$ and, by Lemma 3.3, necessarily $z_{\Theta_1} \equiv z_{\Theta_2} \equiv 0$ for all $z \in \omega(u)$. Then, $H(e_{\Theta_1})$ and $H(e_{\Theta_2})$ are symmetry hyperplanes for all elements in $\omega(u)$ and, by (3.13), $\Theta_2 < \Theta_1$.

If $\Theta_2 \neq -\pi$ or $\Theta_1 \neq \pi$, then $e_{\Theta_1} \neq e_{\Theta_2}$ and $e_\varphi \in \{e \in \mathbb{S}^{N-1} \mid z_e \geq 0 \text{ in } \Sigma_1\}$ for all $\varphi \in (\Theta_2, \Theta_1)$ by the definition of $\mathcal{M}$. In this case, let $p_1 = e_{\Theta_1}$ and $p_2 = e_{\Theta_2}$.

If $\Theta_2 = -\pi$ and $\Theta_1 = \pi$, then (since $z_e = -z_{-e}$) we have that $z_{e_\varphi} \equiv 0$ for all $\varphi \in (-\pi, \pi)$ and for all $z \in \omega(u)$. In this case, let $p_1 = e_{\pi/2}$ and $p_2 = e_{-\pi/2}$.

Since P is arbitrarily chosen, the claim now follows from Proposition 2.6 using these choices of p_1 and p_2. □

Proof of Theorem 1.1 Let Σ, η, p, a, b, α, and β as in the statement. We begin by showing that u satisfies the uniform decay condition (U_1) in $\Sigma \times (1, \infty)$. Let $0 < \gamma < \min\{\beta, 1\}$,

$$r_1 > \max\left\{\left(\frac{2}{\eta}(\gamma^2+1)\right)^{\frac{1}{\beta-\gamma}}, \left(\frac{2}{\eta}\|a\|_\infty \|u\|_\infty^{p-1}\right)^{\frac{1}{\beta-\alpha}}, 1\right\}, \tag{3.14}$$

and $v(x,t) := \|u\|_\infty e^{r_1^\gamma - |x|^\gamma t}$. Then, $v_t - \Delta v + (b(t)|x|^\beta - a(t)|x|^\alpha |u|^{p-1})v = c(x,t)v$, where, for $(x,t) \in \Sigma \backslash B_{r_1}(0) \times (0,1)$,

$$\begin{aligned} c(t,x) &:= -|x|^\gamma - \gamma^2 t^2 |x|^{2\gamma-2} + \gamma t(\gamma+N-2)|x|^{\gamma-2} + b(t)|x|^\beta - a(t)|x|^\alpha |u|^{p-1} \\ &\geq |x|^\beta \left(\eta - \frac{\gamma^2}{r_1^{\beta+2(1-\gamma)}} - \frac{1}{r_1^{\beta-\gamma}} - \frac{\|a\|_\infty \|u\|_\infty^{p-1}}{r_1^{\beta-\alpha}}\right) \\ &\geq |x|^\beta \left(\eta - \frac{\gamma^2+1}{r_1^{\beta-\gamma}} - \frac{\|a\|_\infty \|u\|_\infty^{p-1}}{r_1^{\beta-\alpha}}\right) > 0, \end{aligned}$$

by (3.14). Note that

$$|u(t,x)| \leq \|u\|_\infty \leq \|u\|_\infty e^{r_1^\gamma - |x|^\gamma t} = v(x,t) \quad \text{in } (\overline{B_{r_1}(0) \cap \Sigma}) \times [0,1]. \tag{3.15}$$

As a consequence, $\widetilde{v} := v - u$ satisfies that $\widetilde{v}_t - \Delta \widetilde{v} + (b(t)|x|^\beta - a(t)|x|^\alpha |u|^{p-1})\widetilde{v} = c\, v > 0$ in $(\Sigma \backslash B_{r_1}(0)) \times (0,1]$ and $\widetilde{v} \geq 0$ on the parabolic boundary $\partial_P(\Sigma \backslash B_{r_1}(0) \times (0,1))$. Then, by the maximum principle [21, Proposition 52.4], $\widetilde{v} > 0$ in $\Sigma \backslash B_{r_1}(0) \times (0,1)$, and by (3.15), $\widetilde{v} \geq 0$ in $\Sigma \times [0,1]$. Arguing similarly with $\widetilde{v} = v + u$ yields that

$$|u(x,t)| \leq \|u\|_\infty e^{r_1^\gamma} e^{-|x|^\gamma t} \quad \text{in } \Sigma \times [0,1]. \tag{3.16}$$

For $(x,t) \in (\Sigma \backslash B_{r_1}) \times [1,\infty)$, let $\widehat{v}(x,t) := v(x,1) - u(x,t)$. Then, for $(x,t) \in (\Sigma \backslash B_{r_1}) \times [1,\infty)$,

$$\begin{aligned} \widehat{v}_t - \Delta \widehat{v} + (b(t)|x|^\beta - a(t)|x|^\alpha |u|^{p-1})\widehat{v} &= |x|^\beta \left(-\frac{\gamma^2}{|x|^{\beta+2(1-\gamma)}} + \frac{\gamma(\gamma+N-2)}{|x|^{\beta-\gamma+2}}\right. \\ &\quad \left. + b(t) - \frac{a(t)|u|^{p-1}}{|x|^{\beta-\alpha}}\right) v(x,1) \\ &\geq |x|^\beta \left(\eta - \frac{\gamma^2}{r_1^{\beta-\gamma}} - \frac{\|a\|_\infty \|u\|_\infty^{p-1}}{r_1^{\beta-\alpha}}\right) v(x,1) > 0, \end{aligned}$$

by (3.14). On the other hand,

$$|u(t,x)| \leq \|u\|_\infty \leq \|u\|_\infty e^{r_1^\gamma - |x|^\gamma} = v(x,1) \quad \text{in } (\overline{B_{r_1}(0) \cap \Sigma}) \times [1,\infty). \tag{3.17}$$

By (3.16) and (3.17), $\widehat{v} > 0$ on $\partial_P((\Sigma \backslash B_{r_1}(0)) \times (1,\infty))$; and then, by the maximum principle [21, Proposition 52.4] and (3.17), $\widehat{v} > 0$ in $\Sigma \times (1,\infty)$. Arguing similarly with

$\widehat{v} = v + u$ yields that

$$|u(x,t)| \le Me^{-|x|^\gamma} \quad \text{in } \Sigma \times [1,\infty).$$

As a consequence, u satisfies the uniform decay assumption (U_1) in $\Sigma \times (1,\infty)$. Note also that, by the assumption (1.6), the linearization procedure (2.2), and the maximum principle [21, Proposition 52.4], one has that (U_0) is satisfied by $u(\cdot,t)$ for all $t > 0$; in particular, $u(\cdot,1)$ satisfies (U_0) and (U_1).

Then, by Theorem 1.3, it suffices to show that (f_0), $(f_1)'$, and $(f_2)'$ are satisfied. Let

$$f(t,r,u) := a(t)r^\alpha |u|^{p-1}u - b(t)r^\beta u, \qquad f_u(t,r,u) := p\,a(t)r^\alpha |u|^{p-1} - b(t)r^\beta.$$

Then $f(t,s,0) = 0$ and (f_0) is satisfied. Assumption $(f_1)'$ also holds, because for $K > 0$ and $k > 1$,

$$\sup_{r\in[0,k],t>0,u\in[-K,K]} |f_u(t,r,u)| \le pK^{p-1}k^\alpha \|a\|_\infty + k^\beta \|b\|_\infty =: C,$$

where $C > 0$ depends only on K, k, and the fixed data of the problem p, α, $\|a\|_\infty$, $\|b\|_\infty$, and β.

To prove $(f_2)'$, note that for any $M > 0$

$$\max_{r\in[1,2],u\in[-M,M]} |f_u(t,r,u)| \le 2^\beta \max\{\|a\|_\infty, \|b\|_\infty\} M^{p-1} =: C_M.$$

Define $\varepsilon := \left(\frac{\eta}{p\max\{\|a\|_\infty,1\}}\right)^{\frac{1}{p-1}}$, and let $\rho_M > 1$ be such that $\eta(\rho_M^\alpha - \rho_M^\beta) \le -C_M - 4\lambda_1$. Then, since $\alpha < \beta$

$$\begin{aligned}
&\max_{r>\rho_M,u\in[-\varepsilon,\varepsilon]} f_u(t,r,u)\\
&= \max_{r>\rho_M,u\in[-\varepsilon,\varepsilon]} p|u|^{p-1}a(t)r^\alpha - b(t)r^\beta\\
&\le \max_{r>\rho_M,u\in[-\varepsilon,\varepsilon]} p\varepsilon^{p-1}\|a\|_\infty r^\alpha - \eta r^\beta \le \eta \max_{r>\rho_M}(r^\alpha - r^\beta) = \eta(\rho_M^\alpha - \rho_M^\beta)\\
&\le -C_M - 4\lambda_1 < -\max_{r\in[1,2],u\in[-M,M]} |f_u(t,r,u)| - 4\lambda_1,
\end{aligned}$$

and $(f_2)'$ holds. The result now follows from Theorem 1.3. □

3.2 Weak Stability Outside Compact Sets

Let u be a classical solution of (1.1) and assume that (U_0)–(U_2), (f_0), and (f_1) from Theorem 1.5 hold. For any $e \in \mathbb{S}^{N-1}$, define w_e as in Sect. 2.1, let z_e be as in (2.31), and $\mathcal{M}$ as in (3.11).

Lemma 3.4 *If $e \in \mathcal{M}$ and $\bar{z} \not\equiv 0$ for some $\bar{z} \in \omega(u)$, then $\bar{z}_e > 0$ in Σ.*

Proof Let $\{t_n\} \subset [0,\infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $u(x,t_n) \to \bar{z}(x)$ for all $x \in \mathbb{R}^N$. By hypothesis there exist $x_0 \in \Sigma_1$ and $\alpha > 0$ such that $\bar{z}_e \ge 4\alpha$. Then there is $\mathcal{N} \in \mathbb{N}$ such that

$$w_e(x_0,t_n) > 2\alpha > 0, \qquad \text{for all } n \ge \mathcal{N}.$$

By Lemma 2.3, there exists $\theta > 0$ such that

$$w_e(x_0,s) > \alpha, \qquad \text{for all } s \in [t_n - 4\theta, t_n] \text{ and } n \ge \mathcal{N}.$$

Fix D and U such that $x_0 \in D \subset\subset U \subset\subset \Sigma$. By Lemma 2.4,

$$w_e(x,t_n) \geq \kappa ||w_e^+||_{L^\infty(D\times(t_n-3\theta,t_n-2\theta))} - e^{4m\theta} \sup_{\partial_P(U\times(t_n-4\theta,t_n))} w_e^-$$
$$> \kappa\alpha - e^{4m\theta} \sup_{\partial_P(U\times(t_n-4\theta,t_n))} w_e^- \tag{3.18}$$

for $x \in D$ and $n \geq \mathcal{N}$, where $m = \sup_{U\times(t_n-4\theta,t_n)} c^e$. Since $z_e \geq 0$ in Σ_1 for all $z \in \omega(u)$, then $||w_e^-(\cdot,t)||_{L^\infty(\Sigma)} \to 0$ as $t \to \infty$. Therefore, passing $n \to \infty$ in (3.18) yields $\bar{z}_e(x) \geq \kappa\alpha > 0$ for all $x \in D$. Since D can be arbitrarily big (with $\kappa > 0$ depending on D), the result follows. □

Lemma 3.5 *Fix any $e \in \mathbb{S}^{N-1}$ and suppose that $z_e > 0$ in Σ_1 for all $z \in \omega(u)$. Then, there is $\varepsilon > 0$ such that $z_{e'} > 0$ in Σ_1 for all $z \in \omega(u)$ and $e' \in \mathbb{S}^{N-1}$ with $|e'-e| < \varepsilon$.*

Proof Let ρ_1, δ, and $G \subset\subset \Sigma_1$ be as in (2.11). Since $z_e > 0$ for all $z \in \omega(u)$ and G is a compact set, we have, by (1.3), that there are $\alpha, T > 0$ such that $w_e(x,t) > 2\alpha > 0$ $x \in G \times (T,\infty)$. By Lemma 2.3, there is $\varepsilon > 0$ such that, for $e' \in \mathbb{S}^{N-1}$ with $|e'-e| < \varepsilon$, one has

$$w_{e'}(x,t) > \alpha > 0 \text{ for all } x \in G \times (T,\infty). \tag{3.19}$$

Let $\hat{w}_{e'}$ be as in Lemma 2.1, then (3.19) implies that $\hat{w}_{e'} > 0$ in $G \times (T,\infty)$, and by (2.9), (2.4), and Lemma 2.5,

$$||w_{e'}^-(x,t)||_{L^\infty(\Sigma)} \leq ||\hat{w}_{e'}^-(x,t)||_{L^\infty(\Sigma\setminus G)}$$
$$\leq e^{-\gamma(t-T)}||\hat{w}_{e'}^-(x,T)||_{L^\infty(\Sigma\setminus G)} \leq 4e^{-\gamma(t-T)}||w_{e'}^-(x,T)||_{L^\infty(\Sigma)}$$

for $t > T$, where we used that $\hat{w}^- \equiv 0$ and $c \leq -\gamma$ on G.

Letting $t \to \infty$ we have that $z_{e'} \geq 0$ in Σ_1 and $z_{e'} \not\equiv 0$ by (3.19) for all $z \in \omega(u)$. The claim now follows by Lemma 3.4. □

The next Lemma is an adaptation of [17, Lemma 3.8] to our setting and to the rotating plane method using the notation of Lemma 2.1, we give a proof for completeness.

Lemma 3.6 *Fix $e \in \mathbb{S}^{N-1}$ and assume that $z_e > 0$ in Σ_1 for all $z \in \omega(u)$, then there are $\varphi \in C^{2,1}(\Sigma_1 \times (0,\infty))$ and $T(e) = T > 0$ such that*

1. *If G is as in (2.11), there is a bounded domain D with $G \subset\subset D \subset\subset \Sigma_1$ such that $\varphi(x,t) < 0$ in $\partial D \times (T,\infty)$ and $\varphi(x,t) > 0$ in $G \times (T,\infty)$,*
2. *For all $T < s < t$ there exists some constant $C > 0$ independent of t and s such that*

$$\frac{||\varphi(x,t)||_{L^\infty(D)}}{||\varphi(x,s)||_{L^\infty(G)}} \geq Ce^{-\gamma(t-s)},$$

3. *There is $\varepsilon > 0$ independent of e such that $\varphi_t < \Delta\varphi + \sum_{i=1}^N \hat{b}_i\varphi_{x_i} + \hat{c}^{e'}\varphi$ in $D \times (T,\infty)$ for all $e' \in \mathbb{S}^{N-1}$ with $|e-e'| < \varepsilon$. Recall that $\hat{b}$ and $\hat{c}^e$ were defined in Lemma 2.1*

Proof Let

$$\delta \in (0, \gamma/2). \tag{3.20}$$

By (f_1) and Lemma 2.3 there exists $\varepsilon > 0$ such that

$$||c^{e'}(\cdot,t) - c^e(\cdot,t)||_{L^\infty(\Sigma_1)} < \delta \tag{3.21}$$

for all $t > 0$ and $e' \in \mathbb{S}^{N-1}$ with $|e - e'| < \varepsilon$. Observe that for any $z \in \omega(u)$, one has $z_e > 0$ in Σ_1 and $z_e(x) = 0$ for $x \in \partial\Sigma_1$ or if $|x| \to \infty$. Hence, the compactness of $\omega(u)$ and the uniform convergence of $u(\cdot, t)$ to $\omega(u)$ as $t \to \infty$ (see (U2) implies that there exist $\alpha, t_0 > 0$ such that

$$w_e(x, t) > \alpha > 0 \text{ in } G \times (t_0, \infty). \tag{3.22}$$

In addition, if we define

$$\begin{aligned} M &:= \sup\{|w_e(x, t)| : e \in \mathbb{S}^{N-1}, x \in \Sigma_1, t \in (0, \infty)\}, \\ m &:= \sup\{|c^e(x, t)| : e \in \mathbb{S}^{N-1}, x \in G, t \in (0, \infty)\}, \\ 0 < s &< \min\left\{\frac{\gamma\alpha}{2(m + \gamma)}, \alpha\right\}, \end{aligned} \tag{3.23}$$

then there is bounded domain D with $G \subset\subset D \subset\subset \Sigma_1$ such that if we increase t_0 if necessary, then

$$w_e(x, t) - s < 0 \text{ in } \partial D \times (t_1, \infty)\,. \tag{3.24}$$

Let $\hat{w}_e$ be as in Lemma 2.1, then Lemma 2.1 yields that (3.22) and (3.24) holds with w_e replaced by $\hat{w}_e$. If we define $\varphi(x, t) := e^{-\gamma t}(\hat{w}_e(x, t) - s)$, by (3.22), (3.23), and (3.24), claim 1. follows. Next, note that

$$e^{\gamma t}||\varphi(x, t)||_{L^\infty(D)} \le 4M + s \quad \text{and} \quad e^{\gamma t}||\varphi(x, t)||_{L^\infty(G)} \ge \alpha - s$$

for $t > T$, which implies Claim 2. Finally, for $(x, t) \in D\backslash G \times (T, \infty)$ by the definition of the coefficients $\hat{c}^e$ in Lemma 2.1, (2.4), (3.20), (3.21), and (3.22) yield

$$\begin{aligned} \varphi_t - \Delta\varphi - \hat{b}_i\varphi_{x_i} - \hat{c}^{e'}\varphi &= e^{-\gamma t}[(\hat{c}^e - \hat{c}^{e'})\hat{w}_e - \gamma\hat{w}_e + s\gamma + s\hat{c}^{e'}] \\ &\le e^{-\gamma t}[(\delta - \gamma)\hat{w}_e + s\gamma - s\gamma] < 0. \end{aligned}$$

For $(x, t) \in G \times (T, \infty)$ by (2.4), (3.20), (3.21), (3.22), and (3.23) we have

$$\begin{aligned} \varphi_t - \Delta\varphi - \hat{b}_i\varphi_{x_i} - \hat{c}^{e'}\varphi &= e^{-\gamma t}[(\hat{c}^e - \hat{c}^{e'})\hat{w}_e - \gamma\hat{w}_e + s\gamma + s\hat{c}^{e'}] \\ &\le e^{-\gamma t}\left[\left(\delta - \frac{\gamma}{2}\right)\hat{w}_e - \frac{\gamma}{2}\hat{w}_e + s(m + \gamma)\right] \\ &\le e^{-\gamma t}\left[-\frac{\gamma\alpha}{2} + s(m + \gamma)\right] < 0. \end{aligned}$$

This implies claim 3. □

Proof of Theorem 1.5 We prove that the first alternative holds assuming that the second one does not hold. Let $e \in \mathbb{S}^{N-1}$ as in (U_0) and assume that $z_e \not\equiv 0$ for all $z \in \omega(u)$. Then, by Lemma 3.4 and Lemma 3.5, there is $\varepsilon_0 > 0$ such that

$$z_{e'} > 0 \quad \text{in } \Sigma_1 \text{ for all } z \in \omega(u) \text{ and } e' \in \mathbb{S}^{N-1} \text{ with } |e' - e| < \varepsilon_0. \tag{3.25}$$

Now, let P be any two dimensional subspace of $\mathbb{R}^N$ containing e. Without loss of generality, we may assume that $e = (1, 0, \ldots, 0)$ and $P = \{x = (x_1, x_2, 0, \ldots, 0) \mid x_1, x_2 \in \mathbb{R}\}$. Define

$$e_\eta := (\cos\eta, \sin\eta, 0, \ldots, 0), \quad w_\eta(x, t) := w_{e_\eta}(x, t), \quad z_\eta(x) := z_{e_\eta}(x)$$

for $\theta \in \mathbb{R}$ and

$$\Theta_1 := \inf\{\eta < 0 \,:\, z_\varphi > 0 \text{ in } \Sigma_1 \text{ for all } z \in \omega(u), 0 \le \varphi \le \eta\},$$
$$\Theta_2 := \sup\{\eta > 0 \,:\, z_\varphi > 0 \text{ in } \Sigma_1 \text{ for all } z \in \omega(u), 0 \le \varphi \le \eta\}.$$

We note that $\Theta_1 < 0 < \Theta_2$ by (3.25). We claim that $\Theta_2 - \Theta_1 = \pi$. By contradiction, assume that $\Theta_2 - \Theta_1 < \pi$; then, by continuity and Lemma 3.4, we can assume that there are $\bar{z}, \tilde{z} \in \omega(u)$ such that

$$\tilde{z}_{\Theta_2} \equiv 0 \quad \text{and} \quad \bar{z}_{\Theta_1} \equiv 0 \qquad \text{in } \Sigma_1.$$

Fix $\varepsilon_1 > 0$ as in Lemma 3.6 part 3 and let

$$0 < \varepsilon < \min\left\{\pi - (\Theta_2 - \Theta_1), \frac{\varepsilon_1}{2}, \varepsilon_0\right\}.$$

Fix $\theta \in (\Theta_1 - \varepsilon, \Theta_1)$. By Lemma 2.10,

$$\tilde{z}_\theta > 0 \quad \text{and} \quad \bar{z}_\theta < 0 \qquad \text{in } \Sigma_1. \tag{3.26}$$

Let $\bar{t}_n, \tilde{t}_n \to \infty$ such that $\tilde{t}_n < \bar{t}_n$ for all $n \in \mathbb{N}$ and

$$w_\theta(x, \tilde{t}_n) \to \tilde{z}_\theta(x), \qquad w_\theta(x, \bar{t}_n) \to \bar{z}_\theta(x) \qquad \text{as } n \to \infty. \tag{3.27}$$

Now we conclude the proof arguing as in [17, Lemma 3.7], we include the details for completeness. Recall the change of variables detailed in Lemma 2.1, where the coefficients $\hat{b}_i$, $\hat{c}^e$, the function $\hat{w}_e$ and the set $G \subset \Sigma$ were defined. For any $\eta \in [0, 2\pi)$ we also define $\hat{w}_\eta(x,t) := \hat{w}_{e_\eta}(x,t)$ for $(x,t) \in \Sigma_1 \times (0,\infty)$.

Since $z_\psi > 0$ for each $z \in \omega(u)$ and $\psi > \Theta_1$ sufficiently close to Θ_1, Lemma 3.6 gives the existence of $C, T > 0$, a bounded domain D with $G \subset\subset D \subset\subset \Sigma$ and a function $\varphi \in C^{2,1}(D \times (T,\infty))$, that satisfies for $\vartheta \in (\Theta_1 - \varepsilon, \Theta_1)$ that

$$\begin{aligned} \varphi_t - \Delta\varphi - \hat{b}_i \varphi_{x_i} - \hat{c}^{e_\vartheta}\varphi < 0 \quad &\text{in } D \times (T,\infty), \\ \varphi < 0 \quad &\text{on } \partial D \times (T,\infty), \\ \varphi > 0 \quad &\text{in } G \times (T,\infty), \\ \frac{||\varphi(\cdot,t)||_{L^\infty(D)}}{||\varphi(\cdot,s)||_{L^\infty(D)}} \ge Ce^{-\gamma(t-s)} \quad &\text{for } T < s < t, \end{aligned}$$

where we used that $|\vartheta - \psi| < 2\varepsilon \le \varepsilon_1$. By (3.26), (3.27), and the definition of $\hat{w}_e$ there exists $\alpha > 0$ such that

$$\hat{w}_\theta(\cdot, \tilde{t}_n) > 2\alpha > 0 \quad \text{and} \quad \hat{w}_\theta(\cdot, \bar{t}_n) < 0 \quad \text{in } D$$

for any sufficiently large n. Then, there is $T_n \in (\tilde{t}_n, \bar{t}_n)$ such that

$$\hat{w}_\theta(x,t) > 0 \text{ in } G \times (\tilde{t}_n, T_n), \tag{3.28}$$
$$\hat{w}_\theta(x^*, T_n) = 0 \text{ for some } x^* \in \overline{G}. \tag{3.29}$$

From Lemma 2.5 and (2.4) follows

$$||\hat{w}_\theta^-(\cdot,t)||_{L^\infty(\Sigma)} \le e^{-\gamma(t-\tilde{t}_n)}||\hat{w}_\theta^-(\cdot,\tilde{t}_n)||_{L^\infty(\Sigma)} \quad \text{for } t \in (\tilde{t}_n, T_n). \tag{3.30}$$

By Lemma 2.3, there is $\theta > 0$ independent of n such that

$$||\hat{w}_\theta(\cdot,t) - \hat{w}_\theta(\cdot,s)||_{L^\infty(\Sigma)} < \alpha \text{ for } |t-s| < 4\theta.$$

This means that $T_n - \tilde{t}_n > 4\theta$ for all sufficiently large n. Let n be sufficiently large such that $\tilde{t}_n > T$. Since φ satisfies (3.28) with $\vartheta = \theta$, by the comparison principle [21, Proposition 52.6],

$$\hat{w}_\theta(x,t) > \alpha \frac{\varphi(x,t)}{||\varphi(\cdot,\tilde{t}_n)||_{L^\infty(D)}} \quad \text{for } (x,t) \in D \times (\tilde{t}_n, T_n). \tag{3.31}$$

And, by (3.26), (3.27), (3.28), (3.30), (3.31), and Lemma 2.4, there are $\kappa, C_1, C_2, m > 0$ independent of n such that

$$\begin{aligned}\inf_{x\in G} \hat{w}_\theta(x,T_n) &> \kappa ||\hat{w}_\theta^+||_{L^\infty(G\times(T_n-3\theta,T_n-2\theta))} - e^{4m\theta} \sup_{\partial_P(D\times(T_n-4\theta,T_n))} \hat{w}_\theta^- \\ &\geq \kappa\alpha C e^{-\gamma(T_n-\tilde{t}_n-2\theta)} - e^{4m\theta} e^{-\gamma(T_n-\tilde{t}_n-4\theta)} ||\hat{w}_\theta^-(\cdot,\tilde{t}_n)||_{L^\infty(\Sigma)} \\ &\geq e^{-\gamma(T_n-\tilde{t}_n)}[C_1 - C_2||\hat{w}_\theta^-(\cdot,\tilde{t}_n)||_{L^\infty(\Sigma)}] > 0\end{aligned}$$

for n sufficiently big, a contradiction to (3.29).

Therefore $\Theta_2 - \Theta_1 = \pi$, which in particular implies that $z_{\Theta_1} \equiv z_{\Theta_2} \equiv 0$ in Σ for all $z \in \omega(u)$. Since this can be done for all 2D planes P that contain e and the origin, Proposition 2.6 yields the asymptotic symmetry of u. The strict monotonicity follows from the fact that $z_\eta > 0$ for all $\eta \in (\Theta_1, \Theta_2)$. □

We are ready to show Theorem 1.8.

Proof of Theorem 1.8 By hypothesis $(U_0)'$ we have that $z_e \geq 0$ in Σ_1 for all $z \in \omega(u)$ and all $e \in U$. If there exists $e \in U$ such that $z_e \not\equiv 0$ for all $z \in \omega(u)$ then Theorem 1.5 yields the first alternative in the statement of the theorem. On the other hand, if for each $e \in U$ there is $z \in \omega(u)$ such that $z_e \equiv 0$; then, Lemma 2.7 implies that the second alternative in Theorem 1.8 holds. □

Proof of Corollaries 1.7 and 1.6 Corollaries 1.7 and 1.6 follow directly from Theorem 1.5, since in these cases, the strict initial reflectional inequality and the maximum principle discard the second alternative in Theorem 1.5. □

Proof of Theorem 1.2 Let $f(t,u) := a(t)|u|^{p-1}u - b(t)u$. Then $f_u(t,u) := p\,a(t)|u|^{p-1} - b(t)$ and (f_1) holds because

$$\lim_{u\to v} \sup_{t>0} |f_u(t,u) - f_u(t,v)| \leq p\|a\|_\infty \lim_{u\to v} \left| |u|^{p-1} - |v|^{p-1} \right| = 0$$

and, for every $K > 0$,

$$\sup_{r\in J, t>0, s\in[-K,K]} |f_u(t,s)| \leq p \max\{\|a\|_\infty, \|b\|_\infty\}|K^{p-1} + 1| < \infty.$$

Moreover, if $\eta > 0$ is as in (1.4) and $\varepsilon := \left(\frac{\eta}{2p\max\{\|a\|_\infty,1\}}\right)^{\frac{1}{p-1}}$, then $\sup_{u\in(-\varepsilon,\varepsilon)} |p\,a(t)|u|^{p-1}| \leq \frac{\eta}{2}$ and

$$f_u(t,u) \leq |p\,a(t)|u|^{p-1}| - \eta = -\frac{\eta}{2} \quad \text{for all } t > 0 \text{ and } u \in (-\varepsilon,\varepsilon),$$

which implies that (f_2) is satisfied. Finally, by (1.8), condition $(U_0)'$ holds. Then, by Theorem 1.8, either there is $z \in \omega(u)$ radially symmetric with respect to the origin, or u is asymptotically strictly foliated Schwarz symmetric.

It remains to prove that

(a) If $z \in \omega(u)$ is radially symmetric, then $z \equiv 0$.
(b) If $0 \notin \omega(u)$, then there is $q \in \mathbb{R}^N$ such that, for all $z \in \omega(u)$, z is radially symmetric with respect to q and all elements in $\omega(u)$ are either all negative or all positive.

Proof of (a): For a contradiction, assume there is $z \in \omega(u)$ radially symmetric and $x_0 \in \mathbb{R}^N$ such that $z(x_0) \neq 0$. Using the radiality and continuity of z we may without loss of generality assume that $x_0 = re_1 = (r, 0, \dots, 0)$ for some $r > 0$ and $z(re_1) < 0$. For $\lambda \in (-1, 1)$, denote $\mathbb{R}^N_\lambda := \{x \in \mathbb{R}^N : x_1 > \lambda\}$ and $u_\lambda(x, t) := u(x, t) - u(2\lambda e_1 - x, t)$ for $x \in \mathbb{R}^N_\lambda$ and $t \geq 0$. By (1.8), $u_\lambda(x, 0) \geq 0$ and $u_\lambda(x, 0) \not\equiv 0$ for any $\lambda \in (-1, 1)$, and therefore, by the maximum principle, $u_\lambda > 0$ in $\mathbb{R}^N_\lambda \times (0, \infty)$. Hence, $z(x) \geq z(2\lambda e_1 - x)$ for all $x \in \mathbb{R}^N_\lambda$ and $\lambda \in (-1, 1)$. Since z is radial and in particular even, for any fixed $0 < \lambda < \min\{r, 1\}$ one has $z_{-\lambda} \geq 0$ in $\mathbb{R}^N_{-\lambda}$, and then

$$0 > z(re_1) \geq z(-2\lambda e_1 - re_1) = z((2\lambda + r)e_1). \tag{3.32}$$

Iterating this procedure, we obtain that $0 > z(re_1) \geq z((2k\lambda + r)e_1)$ for all $k \in \mathbb{N}$. Since $|(2k\lambda + r)e_1| \to \infty$ as $k \to \infty$ we obtain a contradiction to the uniform decay assumption (1.9). As a consequence, $z \equiv 0$ in $\mathbb{R}^N$.

Proof of (b): Assume that $0 \notin \omega(u)$. Then, as shown above, there is $p_0 \in \mathbb{S}^{N-1}$ such that, for all $z \in \omega(u)$, z is strictly foliated Schwarz symmetric with respect to p_0. Let $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, …, $e_N = (0, 0, \dots, 1)$ denote a basis for $\mathbb{R}^N$. For $\alpha \in \mathbb{R}$ and $i \neq 1$, consider $u_{\alpha,i}(x) := u(x + \alpha e_i)$ and note that $u_{\alpha,i}$ solves the same equation as u (which is translationally invariant) and also satisfies $(U_0)'$. Then, arguing as before, there is $p_{\alpha,i} \in \mathbb{S}^{N-1}$ such that, for all $z \in \omega(u_{\alpha,i})$, z is foliated Schwarz symmetric with respect to $p_{\alpha,i}$; namely, every $z \in \omega(u)$ is strictly foliated Schwarz symmetric with respect to the axes $P_{i,\alpha} := \mathbb{R}p_{\alpha,i} + \alpha e_i$.

Fix $\bar{z} \in \omega(u)$ and assume without loss of generality that $\bar{z}^+ \not\equiv 0$. Since $\bar{z}$ is strictly monotone with respect to the polar angle and it decays uniformly to zero at infinity, then the maximum of z in $\mathbb{R}^N$ must lie in an intersection between $P_{i,\alpha}$ and P_0, $q := P_{i,\alpha} \cap P_0$. By varying $\alpha \in \mathbb{R}$, we can obtain an irrational angle between $P_{i,\alpha}$ and P_0, which then easily implies that $\bar{z}$ must be a radially symmetric function with respect to q. But the axis P_0 and $P_{i,\alpha}$ are symmetry axes for *all* elements $z \in \omega(u)$, and therefore $\omega(u)$ can only have radially symmetric elements with respect to q.

Finally, for a contradiction, assume that there is a sign-changing $z \in \omega(u)$. Assume first that $q \in \{x_1 \geq 0\}$. We argue as in (3.32) on the line $q + \mathbb{R}e_1$. Indeed, since z changes sign and z is radially symmetric and continuous, there is $r > 0$ such that $z(q + re_1) < 0$. But then, for $0 < \lambda < \min\{r, 1\}$,

$$0 > z(q + re_1) \geq z(-2\lambda e_1 - q - re_1) = z(q + (2\lambda + r)e_1).$$

Iterating this procedure, we obtain a contradiction to the uniform decay assumption (1.9) as before. In fact, we can deduce that if $q \in \{x_1 \geq 0\}$, then necessarily $z \geq 0$ in $\mathbb{R}^N$ for all $z \in \omega(u)$, and the strict positivity follows from Lemma 2.4 using that $0 \notin \omega(u)$. Similarly, if $q \in \{x_1 \leq 0\}$, then, since z changes sign and z is radially symmetric and continuous, $z(q - re_1) > 0$ for some $r > 0$ and, for $0 < \lambda < \min\{r, 1\}$,

$$0 < z(q - re_1) \leq z(2\lambda e_1 - q + re_1) = z(q - (2\lambda + r)e_1).$$

Iterating this procedure, we obtain again a contradiction to (1.9) and, as before, we conclude that if $q \in \{x_1 \leq 0\}$, then necessarily $z \leq 0$ in $\mathbb{R}^N$ for all $z \in \omega(u)$ and the strict negativity follows from Lemma 2.4 applied to $-u$. Finally, observe that if $q \in \{x_1 = 0\}$, then necessarily

$z \equiv 0$ for all $z \in \omega(u)$, which cannot happen since we assumed that $0 \notin \omega(u)$. Therefore, $q \notin \{x_1 = 0\}$ and this ends the proof. □

4 Uniform Decay Assumption

Proof of Lemma 1.9 The proof follows closely the ideas of [4, Corollary 1.2]. We first show that there is some $R_0 > 0$ such that

$$\nabla u(x,t) \cdot e < 0 \qquad \text{for all } x \in \mathbb{R}^N,\ x \cdot e > R_0,\ t > 0,\ e \in \mathbb{S}^{N-1}. \tag{4.1}$$

To this end, we use a moving plane method. Let $R > 0$ as in (1.13) and $e \in \mathbb{S}^{N-1}$, without loss of generality we may assume that $e = e_1 = (1, 0, \ldots, 0)$. For $\lambda > R$ and $(x_1, x') = x \in \mathbb{R}^N$ define $x \mapsto x^\lambda = (2\lambda - x_1, x')$ the reflection in the hyperplane $H_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}$ and let $V_\lambda u(x,t) := u(x^\lambda, t) - u(x,t)$ for $x \in \Sigma_\lambda := \{x \in \mathbb{R}^N : x_1 > \lambda\}$ and $t > 0$. Then, by (1.13), we have that

$$\begin{aligned}(V_\lambda u(x,t))_t - \Delta V_\lambda u(x,t) &= f(t, |x^\lambda|, u^\lambda) - f(t, |x|, u) \\ &\geq f(t, |x|, u^\lambda) - f(t, |x|, u) = c^\lambda(x,t) V_\lambda u(x,t)\end{aligned}$$

for all $(x,t) \in \Sigma_\lambda \times (0,\infty)$, where $c^\lambda(x,t) := \int_0^1 \partial_u f(t, |x|, su(x^\lambda, t) + (1-s)u(x,t))ds$. Note that $c^\lambda \in L^\infty(\Sigma_\lambda \times (0,\infty))$. Let $R_0 \geq R$ be such that $\operatorname{supp}(u_0) \subset B_{R_0}(0)$, then, for $\lambda > R_0$ we have that $V_\lambda u(x,0) \geq 0$ in Σ_λ and $V_\lambda u(x,0) \not\equiv 0$. Then, since u is globally bounded, we can apply the parabolic maximum principle to show that $V_\lambda u(x,t) > 0$ in $\Sigma_\lambda \times (0,\infty)$ for all $\lambda > R_0$ and $V_\lambda u(x,t) \equiv 0$ in $H_\lambda \times (0,\infty)$. Then, by the Hopf's Lemma we have that $-2\nabla u(x,t) \cdot e = -2\partial_e u(x,t) = \partial_e V_\lambda u(x,t) > 0$ on $H_\lambda \times (0,\infty)$ and (4.1) follows. Next, to prove the uniform decay property. We proceed by contradiction. Assume there is some sequence $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathbb{R}^N \times (0,\infty)$ with $|x_n|, t_n \to \infty$ as $n \to \infty$ such that

$$u(x_n, t_n) \geq k \tag{4.2}$$

for some constant $k > 0$. Passing to a subsequence, we may assume (using rotations if necessary) that $x_k/|x_k|$ tends to the unitary vector e_1. By (4.1) we have that $\partial_{x_1} u(x,t) < 0$ for any $t > 0$ and any $x = (x_1, x') \in \mathbb{R}^N$ with x_1 big enough. Then, the assumed L^∞ bound and standard parabolic regularity estimates imply that $|\nabla u|$ is uniformly bounded. Then, we can find $\varepsilon > 0$ such that $u(x, t_n) \geq \frac{k}{2}$ for all $x \in B_\varepsilon(x_n)$. Since $|x_n| \to \infty$ and $\partial_{x_1} u(x,t) < 0$ we obtain a contradiction to the assumed L^q-bound. □

Proof of Lemma 1.10 We proceed by comparison. Consider the solution of the problem

$$v_t - \Delta v = f(t, |x|, v) \quad \text{in } \mathbb{R}^N \times (0,\infty), \qquad v(x,0) = u_0^+(x) \quad \text{for } x \in \mathbb{R}^N;$$

then, by Lemma 1.9, $\lim_{|x|\to\infty} \sup_{t>0} v(x,t) = 0$. On the other hand, let $w(x,t) := v(x,t) - u(x,t)$, then w satisfies the linearization

$$w_t - \Delta w = \left[\int_0^1 \partial_u f(t, |x|, sv + (1-s)u)ds\right] w \quad \text{in } \mathbb{R}^N \times (0,\infty)$$

and $w(x,0) \geq 0$ for $x \in \mathbb{R}^N$. Since the coefficient of w is bounded, the parabolic maximum principle implies that $v \geq u$ in $\mathbb{R}^N \times (0,\infty)$. Then, $\lim_{|x|\to\infty} \sup_{t>0} u(x,t) \leq 0$. Now, using the solution of

$$v_t - \Delta v = f(t, |x|, v) \quad \text{in } \mathbb{R}^N \times (0,\infty), \qquad v(x,0) = u_0^-(x) \quad \text{for } x \in \mathbb{R}^N,$$

and repeating the argument we obtain that $\lim_{|x|\to\infty} \sup_{t>0} u(x,t) \geq 0$ and the claim follows. □

5 Examples

5.1 Example 1: An ω-Limit Set with Only Strictly Foliated Schwarz Symmetric Elements

Here we exhibit a problem which does not have any radially symmetric[1] element in the ω-limit set of the solution $\omega(u)$, but that satisfies the assumptions of Theorem 1.8, and therefore all elements in $\omega(u)$ are foliated Schwarz symmetric with respect to the same axis.

First, fix any ball $B \subset \mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_1 > 0\}$ and let $\lambda_1(B)$ denote the first eigenvalue of the Dirichlet Laplacian in B. Define $g(v) = v^p - \lambda v$, where $1 < p < (N+2)/(N-2)$ and $0 < \lambda < \lambda_1(B)$. Then, by standard variational arguments (see e.g. [21, Theorem 6.2]) there exists a positive solution ζ of the problem

$$-\Delta\zeta = g(\zeta) \quad \text{in } B, \qquad \zeta = 0 \quad \text{on } \partial B\,. \tag{5.1}$$

Let $M = 4\sup_B \zeta$ and define f such that $f(|x|, v) = g(v)$ for $x \in B$ and $0 \leq v < M/2$. For $v > M/2$ and $x \in \mathbb{R}^N \backslash B$, $f : [0,\infty) \times \mathbb{R}$ is extended as a smooth function such that it satisfies (f_0) and (f_1). We assume also that

(f_3) f is an odd function in u, that is, $f(\cdot, u) = -f(\cdot, -u)$ for all $u \in \mathbb{R}$,
(f_4) There is $M^* > M$ such that

$$uf(r,u) < 0 \qquad \text{for any } r > 0 \text{ and } |u| \geq M^*, \tag{5.2}$$

(f_5) There is $R^* > 1$ and $\Lambda > 0$ such that

$$f(r,0) = 0 \qquad \text{and} \qquad f_u(r,u) < -\Lambda \qquad \text{for any } |u| \leq M^* \text{ and } r > R^*\,. \tag{5.3}$$

Let $u_0 \in C(\mathbb{R}^N)$ be an odd function with respect to x_1, that is, $u_0(x_1, x') = -u_0(-x_1, x')$ for all $x = (x_1, x') \in \mathbb{R}^N$, and assume that

$$\operatorname{supp}(u_0) = A_1 \cup A_2, \tag{5.4}$$

where A_1 and A_2 are two disjoint compact sets such that $B \subset A_1$,

$$A_1 \subset\subset \{x \in \mathbb{R}^N \,:\, |x| < R^*, x_1 > 0,\ x_i > 0\ i = 2, \ldots, N\}, \tag{5.5}$$

and A_2 is symmetric to A_1 with respect to the hyperplane $\{x \in \mathbb{R}^N : x_1 = 0\}$. Finally, assume that

$$u_0 \geq \zeta \quad \text{in } B, \qquad |u_0| < M^* \quad \text{in } \mathbb{R}^N, \tag{5.6}$$

and u_0 is positive in A_1 and negative in A_2.

Let us first derive a priori estimates for classical solutions u of

$$u_t - \Delta u = f(|x|, u) \quad \text{in } \mathbb{R}^N \times (0,\infty), \qquad u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^N\,. \tag{5.7}$$

[1] with respect to the origin or with any other point $x_0 \in \mathbb{R}^N$.

Note that, by (f_4),

$$(u-M^*)_t-\Delta(u-M^*)-\frac{f(|x|,u)-f(|x|,M^*)}{u-M^*}(u-M^*)$$
$$=f(|x|,M^*)<0 \quad \text{in } \mathbb{R}^N\times(0,\infty),$$

where, by (f_4), $\sup_{x\in\mathbb{R}^N,t>0}\frac{f(|x|,u)-f(|x|,M^*)}{u-M^*}<\infty$. Then, by (5.6) and the maximum principle, $u\le M^*$ in $\mathbb{R}^N\times(0,\infty)$. Arguing similarly with M^*+u, we obtain that

$$|u|\le M^* \quad \text{in } \mathbb{R}^N\times(0,\infty). \tag{5.8}$$

Then, by standard arguments (see, e.g., [15, Proposition 7.3.1]), there exists a global bounded, smooth solution u of (5.7).

Observe that $\widehat{w}(x,t):=u(x,t)+u(x^{e_1},t)$ solves

$$\widehat{w}_t-\Delta\widehat{w}=\widehat{c}(x,t)\widehat{w} \quad \text{in } \mathbb{R}^N\times(0,\infty),$$

where $\widehat{c}(x,t):=\frac{f(u(x,t))+f(u(x^{e_1},t))}{\widehat{w}(x,t)}$ if $\widehat{w}(x,t)\neq 0$ and $\widehat{c}(x,t)=0$ if $\widehat{w}(x,t)=0$. Since $\widehat{w}(x,0)=u_0(x)+u_0(x^{e_1})\equiv 0$ in $\mathbb{R}^N_+:=\{x\in\mathbb{R}^N: x_1>0\}$ we have that $\widehat{w}\equiv 0$ in $\mathbb{R}^N\times[0,\infty)$. In particular, it means that $u(\cdot,t)$ is odd in x_1 for all $t>0$. Therefore, $u(0,x',t)=0$ for all $t>0$ and $x'\in\mathbb{R}^{N-1}$. Since u_0 is nonnegative and nontrivial in $\mathbb{R}^N_+$, the maximum principle implies that

$$u>0 \quad \text{in } \mathbb{R}^N_+\times(0,\infty). \tag{5.9}$$

Next, let $s(x):=M^*e^{-\theta(|x|-R^*)}$ with $\theta^2<\Lambda$; then,

$$s_t-\Delta s=\left(\theta\frac{N-1}{|x|}-\theta^2\right)s \qquad \text{for } |x|\ge R^*. \tag{5.10}$$

Therefore, by (f_5), $w:=s-u$ satisfies

$$w_t-\Delta w-\int_0^1 f_u(|x|,\theta u)\,d\theta\, w$$
$$=\left(\theta\frac{N-1}{|x|}-\theta^2-\int_0^1 f_u(|x|,su)\,ds\right)s\ge\left(\Lambda-\theta^2\right)s>0$$

in $\mathbb{R}^N\setminus B_{R^*}$. Since $w=M^*-u\ge 0$ for $x\in\partial B_{R^*}$ and $w(x,0)=s(x)-u_0(x)=s(x)$ for $x\in\mathbb{R}^N\setminus B_{R^*}$, the maximum principle yields that $w>0$ on $\mathbb{R}^N\setminus B_{R^*}\times(0,\infty)$. Thus,

$$0\le u\le M^*e^{-\theta(|x|-R)} \quad \text{in } \mathbb{R}^N\setminus B_{R^*}\times(0,\infty). \tag{5.11}$$

Since f coincides with g on the range of ζ and $u>0$ on $\partial B\times(0,\infty)\subset\mathbb{R}^N_+\times(0,\infty)$, we have that

$$(u-\zeta)_t-\Delta(u-\zeta)-\frac{g(u)-g(\zeta)}{u-\zeta}(u-\zeta)=0 \quad \text{in } B\times(0,\infty).$$

Moreover, by (5.9), $u-\zeta>0$ on $\partial B\times(0,\infty)$ and, by (5.6), $u(\cdot,0)-\zeta\ge 0$ in B. Then, by the maximum principle, $u\ge\zeta$ in $B\times(0,\infty)$; in particular, $0\notin\omega(u)$. Furthermore, since u is odd in x_1, any $z\in\omega(u)$ is odd in x_1 as well, and consequently z is not radially symmetric.

Finally, note that all the assumptions of Theorem 1.8 are satisfied; indeed, $(f0)$ and $(f1)$ hold by construction, (f_2) follows from (f_5), $(U_0)'$ from (5.4), (U_1) from (5.11), and (U_2) from (5.8). Therefore, we obtain that all the elements in $\omega(u)$ are strictly foliated Schwarz symmetric.

5.2 Example 2: An ω-Limit Set with a Strictly Foliated Schwarz Symmetric Function and a Radially Symmetric Element

In this example we show that the presence of a nonzero radially symmetric element in $\omega(u)$ does *not* imply that all the elements in $\omega(u)$ are radially symmetric. Although we work in an abstract setting we provide concrete examples.

Let D be a smooth radial domain, bounded or unbounded, in $\mathbb{R}^N$ and assume that a linear operator $\mathcal{L}$ acting on smooth functions on D has eigenvalues $0 < \lambda_1 < \lambda_2$ corresponding to eigenfunctions φ_1 and φ_2 such that φ_1 is foliated Schwarz symmetric and φ_2 is radially symmetric. Such choice is possible since there exist radial eigenfunctions corresponding to arbitrarily large eigenvalues. Note that we require the existence of a radial eigenfunction corresponding to a larger eigenvalue than the foliated Schwarz symmetric one. To satisfy our assumptions we allow $\mathcal{L}$ to depend only on $|x|$. Let $u(x,t) := \alpha(t)\varphi_1(x) + \beta(t)\varphi_2(x)$ for $x \in D$ and $t > 0$ and, for fixed $\mu \in (\lambda_1, \lambda_2)$ and define

$$f(t,x,u) = \mu\zeta(t)u + \psi(t)\varphi_2(x) \qquad x \in D,\ t > 0, \tag{5.12}$$

where $\alpha, \beta, \zeta, \psi : (0,\infty) \to \mathbb{R}$ are chosen below. Note that, since φ_2 is radially symmetric, f depends on x only through $|x|$.

First we require that $\alpha(0) = 1$ and $\beta(0) = 0$; therefore, $u_0(x) = \varphi_1(x)$ satisfies (U_0). Also we assume that u is a solution of

$$u_t + \mathcal{L}u = f(t,x,u) \quad \text{in } D \times (0,\infty), \qquad u(x,t) = 0 \quad \text{for } x \in \partial D\,,$$

where $u = 0$ on ∂D is interpreted as $u(x) \to 0$ when $|x| \to \infty$ if D is unbounded.

Since the operator is linear and φ_i is an eigenfunction we immediately have that u is the desired solution if

$$\alpha' + \lambda_1\alpha = \mu\zeta\alpha, \qquad \beta' + \lambda_2\beta = \mu\zeta\beta + \psi, \qquad \alpha(0) = 1, \quad \beta(0) = 0\,. \tag{5.13}$$

Consequently, by the variation of parameters formula, for any $t > t_0 \geq 0$ we have that

$$\alpha(t) = \alpha(t_0)\exp\left(\int_{t_0}^{t} \mu\zeta(s) - \lambda_1\, ds\right), \tag{5.14}$$

$$\beta(t) = \int_{t_0}^{t} \psi(s)\exp\left(\int_{s}^{t} \mu\zeta(r) - \lambda_2\, dr\right) ds + \beta(t_0)\exp\left(\int_{t_0}^{t} \mu\zeta(s) - \lambda_2\, ds\right). \tag{5.15}$$

We finish the argument by constructing sequences $(T_k)_{k\in\mathbb{N}}$, $(\bar{T}_k)_{k\in\mathbb{N}}$ such that

$$\alpha(T_k) = 1, \quad |\beta(T_k)| \leq \frac{1}{2^k}, \qquad 0 < |\alpha(\bar{T}_k)| \leq \frac{1}{2^k}, \quad |\beta(\bar{T}_k) - 1| \leq \frac{1}{2^k}\,. \tag{5.16}$$

Indeed, then we obtain

$$\lim_{k\to\infty} u(\cdot, T_k) = \varphi_1 \quad \lim_{k\to\infty} u(\cdot, \bar{T}_k) = \varphi_2 \tag{5.17}$$

and the statement follows.

To prove (5.16), we proceed by induction and in addition we show that $\zeta(T_n) = 1$ and $\psi(T_n) = 0$ for every n. Set $T_1 = 0$ and assume that we already constructed $T_1 < \bar{T}_1 < T_2 < \cdots < \bar{T}_{n-1} < T_n$ with the desired properties (5.16).

First introduce a short transition period (of order) to shift the values of ζ and ψ to 0 and λ_2 respectively. We use linear functions, but if the smoothness at the endpoints is required, the transition can me made smooth at the cost of less explicit expressions. Define

$$\zeta(t) = 1 + T_n - t, \qquad \psi(t) = \lambda_2(t - T_n) \qquad t \in [T_n, T_n + 1) \tag{5.18}$$

and, by (5.14) and (5.15),

$$\alpha(T_n+1)=\alpha(T_n)e^{\frac{\mu}{2}-\lambda_1}\,,\quad \beta(T_n+1)=c_1+\beta(T_n)e^{\frac{\mu}{2}-\lambda_2}\,, \tag{5.19}$$

where c_1 is a universal constant independent of n. Then, for fixed A_1 specified below, define

$$\zeta(t)=0,\qquad \psi(t)=\lambda_2,\qquad t\in[T_n+1,T_n+A_1+1)\,, \tag{5.20}$$

and set $\bar{T}_n=T_n+1+A_1$. Consequently, by (5.14) and (5.15)

$$\alpha(\bar{T}_n)=\alpha(T_n+1)e^{-\lambda_1 A_1},\qquad \beta(\bar{T}_n)=1-e^{-\lambda_2 A_1}+\beta(T_n+1)e^{-\lambda_2 A_1}\,. \tag{5.21}$$

Thus, for any sufficiently large A_1, $\alpha(\bar{T}_n)$ and $\beta(\bar{T}_n)$ satisfy (5.16). Next, we introduce the second transition period that connects $(\zeta(\bar{T}_n),\psi(\bar{T}_n))=(0,\lambda_2)$ to $(1,0)$. Set

$$\zeta(t)=t-\bar{T}_n,\qquad \psi(t)=\lambda_2(\bar{T}_n+1-t),\qquad t\in[\bar{T}_n,\bar{T}_n+1)\,, \tag{5.22}$$

and therefore

$$\alpha(\bar{T}_n+1)=\alpha(\bar{T}_n)e^{\mu/2-\lambda_1},\qquad \beta(\bar{T}_n+1)=\beta(\bar{T}_n)c_2+c_3\,, \tag{5.23}$$

where c_2, c_3 are universal constants independent of n or A_1 (which can be evaluated explicitly). Then, set

$$\zeta(t)=1,\qquad \psi(t)=0,\qquad t\in[\bar{T}_n+1,\bar{T}_n+1+A_2)\,, \tag{5.24}$$

where A_2, depending on A_1 is specified below. Denote $T_{n+1}=\bar{T}_n+1+A_2$ and we obtain

$$\alpha(T_{n+1})=\alpha(T_n)e^{-\lambda_1 A_1}e^{\mu-2\lambda_1}e^{(\mu-\lambda_1)A_2}, \tag{5.25}$$

$$\beta(T_{n+1})=\beta(\bar{T}_n+1)e^{(\mu-\lambda_2)A_2}\,. \tag{5.26}$$

Since $\mu>\lambda_1$, we can set

$$A_2=\frac{\lambda_1 A_1+2\lambda_1-\mu}{\mu-\lambda_1}>0 \tag{5.27}$$

to obtain $\alpha(T_{n+1})=\alpha(T_n)=1$. Also, by making A_1 larger if necessary, $\mu<\lambda_2$ yields that $\beta(T_{n+1})$ satisfies (5.16). Clearly, $\zeta(T_{n+1})=1$ and $\psi(T_{n+1})=0$ and the proof is complete.

Example 5.1 Let $B\subset\mathbb{R}^N$ with $N\geq 2$ be a ball and let $\mathcal{L}=-\Delta$. It is standard to show that such $\mathcal{L}$ has an increasing sequence of eigenvalues diverging to infinity corresponding to the radial eigenfunctions. Furthermore, the first non-radial eigenfunction is foliated Schwarz symmetric. Hence, the assumptions are satisfied and we can find f such that the ω-limit set contains both radial and foliated Schwarz symmetric functions.

Example 5.2 If $D=\mathbb{R}^N$ with $N\geq 2$, we can set $\mathcal{L}=-\Delta+V(|x|^2)$, where $V\geq 0$ and $V(r)\to\infty$ as $r\to\infty$. If we denote $L^2(\mathbb{R}^N,V)$ the L^2 space with weight V, then, due to compactness of the embedding $L^2(\mathbb{R}^N)\hookrightarrow H^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,V)$ and the spectral theorem, one obtains that $\mathcal{L}$ has only discrete spectrum and the eigenvectors span the whole space.

Also, the method of separation of variables implies that the eigenvalue problem

$$\mathcal{L}u=\lambda u \tag{5.28}$$

with $u(r,\theta)=R(r)\Theta(\theta)$, $r\in[0,\infty)$, $\theta\in\mathbb{S}^{N-1}$ can be written as

$$-\Delta_r R+V(r^2)R=\lambda_1 R,\qquad R'(0)=0 \tag{5.29}$$

and

$$-\Delta_\theta \Theta = \lambda_2 \Theta\,, \tag{5.30}$$

where Δ_r is radial Laplacian and Δ_θ is Laplace-Beltrami operator on $\mathbb{S}^{N-1}$.

It is easy to show that such $\mathcal{L}$ has an increasing sequence of eigenvalues diverging to infinity corresponding to the radial eigenfunctions. Furthermore, the first non-radial eigenfunction is foliated Schwarz symmetric. Hence, the assumptions are satisfied and we can find f such that ω-limit set contains both radial and foliated Schwarz symmetric functions.

Acknowledgements J. Földes is partly supported by the National Science Foundation under the grant NSF-DMS-1816408. A. Saldaña is supported by UNAM-DGAPA-PAPIIT grant IA101721, Mexico.

References

1. Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method. Bol. Soc. Brasil. Mat. (N.S.) **22**(1), 1–37 (1991)
2. Bonheure, D., Grumiau, C., Troestler, C.: Multiple radial positive solutions of semilinear elliptic problems with Neumann boundary conditions. Nonlinear Anal. **147**, 236–273 (2016)
3. Brock, F.: Symmetry and monotonicity of solutions to some variational problems in cylinders and annuli. Electron. J. Differ. Equ. **2003**(108), 20 (2003)
4. Busca, J., Jendoubi, M.A., Polacik, P.: Convergence to equilibrium for semilinear parabolic problems in $\mathbb{R}^N$. Commun. Partial Differ. Equ. **27**(9–10), 1793–1814 (2002)
5. Chambers, L.G.: An upper bound for the first zero of Bessel functions. Math. Comput. **38**(158), 589–591 (1982)
6. Földes, J.: On symmetry properties of parabolic equations in bounded domains. J. Differ. Equ. **250**(12), 4236–4261 (2011)
7. Földes, J.: Symmetry of positive solutions of asymptotically symmetric parabolic problems on $\mathbb{R}^N$. J. Dyn. Differ. Equ. **23**(1), 45–69 (2011)
8. Földes, J.: On Serrin's symmetry result in nonsmooth domains and its applications. Adv. Differ. Equ. **18**(5–6), 523–548 (2013)
9. Földes, J., Polacik, P.: On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry. Discrete Contin. Dyn. Syst. **25**(1), 133–157 (2009)
10. Földes, J., Polacik, P.: Convergence to a steady state for asymptotically autonomous semilinear heat equations on $\mathbb{R}^N$. J. Differ. Equ. **251**(7), 1903–1922 (2011)
11. Földes, J., Polacik, P.: On asymptotically symmetric parabolic equations. Netw. Heterog. Media **7**(4), 673–689 (2012)
12. Földes, J., Polacik, P.: Equilibria with a nontrivial nodal set and the dynamics of parabolic equations on symmetric domains. J. Differ. Equ. **258**(6), 1859–1888 (2015)
13. Gladiali, F., Pacella, F., Weth, T.: Symmetry and nonexistence of low Morse index solutions in unbounded domains. J. Math. Pures Appl. (9) **93**(5), 536–558 (2010)
14. Lieberman, G.M.: Second Order Parabolic Differential Equations. World Scientific Publishing Co. Inc., River Edge, NJ (1996)
15. Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel (1995). 2013 reprint of the 1995 original
16. Nirenberg, L.: On nonlinear elliptic partial differential equations and Hölder continuity. Commun. Pure Appl. Math. **6**, 103–156 (1953). addendum, 395
17. Polacik, P.: Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$. I. Asymptotic symmetry for the Cauchy problem. Commun. Partial Differ. Equ. **30**(10–12), 1567–1593 (2005)
18. Polacik, P.: Estimates of solutions and asymptotic symmetry for parabolic equations on bounded domains. Arch. Ration. Mech. Anal. **183**(1), 59–91 (2007)
19. Poláčik, P., Yanagida, E.: Nonstabilizing solutions and grow-up set for a supercritical semilinear diffusion equation. Differ. Integral Equ. **17**(5–6), 535–548 (2004)
20. Protter, M.H., Weinberger, H.F.: Maximum Principles in Differential Equations. Springer-Verlag, New York (1984). Corrected reprint of the 1967 original

21. Quittner, P., Souplet, P.: Superlinear Parabolic Problems. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel (2007). Blow-up, global existence and steady states
22. Saldaña, A.: Partial symmetry of solutions to parabolic problems via reflection methods. Rend. Sem. Mat. Univ. Politec. Torino **74**(2), 105–112 (2016)
23. Saldaña, A.: Qualitative properties of coexistence and semi-trivial limit profiles of nonautonomous nonlinear parabolic Dirichlet systems. Nonlinear Anal. **130**, 31–46 (2016)
24. Saldaña, A., Weth, T.: Asymptotic axial symmetry of solutions of parabolic equations in bounded radial domains. J. Evolut. Equ. **12**(3), 697–712 (2012)
25. Saldaña, A., Weth, T.: On the asymptotic shape of solutions to Neumann problems for non-cooperative parabolic systems. J. Dyn. Differ. Equ. **27**(2), 307–332 (2015)
26. Weth, T.: Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods. Jahresber. Dtsch. Math.-Ver. **112**(3), 119–158 (2010)