

The classification of subfactors with index at most $5\frac{1}{4}$

Narjess Afzaly, Scott Morrison, and David Penneys

In memory of Uffe Haagerup

Abstract

Subfactor standard invariants encode quantum symmetries. The small index subfactor classification program has been a rich source of interesting quantum symmetries. We give the complete classification of subfactor standard invariants to index $5\frac{1}{4}$, which includes $3 + \sqrt{5}$, the first interesting composite index.

1 Introduction

The classification of small index subfactors is an essential part of the search for exotic quantum symmetries. A quantum symmetry is a non-commutative analogue of the representation category of a finite group. There is no single best axiomatization: choices include standard invariants of finite index subfactors [Pop95a; Jon99] or fusion categories [ENO05]. We focus on standard invariants here.

Topological field theories and topological phases of matter have revolutionized our understanding of symmetry in physics: these systems do not have a group of symmetries in the classical sense, but rather possess quantum symmetries, described by a higher categorical structure. [NSSFDS08; BBCW14]

What, then, do quantum symmetries look like? The basic examples come either from finite group theory (possibly with cohomological data) or from quantum enveloping algebras at roots of unity. Many are also realized from conformal field theories. While there are a number of constructions producing new quantum symmetries from old, we are far from having a good structure theory. We are still at the phenomenological phase of studying quantum symmetries, and understanding the range of examples is an essential problem.

We now have several instances of quantum symmetries that do not come from the basic examples, even allowing these constructions. Indeed, the strangest and least understood of all known quantum symmetries were discovered in exhaustive classifications of subfactors at small index [Haa94].

A critical next step in our understanding of quantum symmetries will be developing structure theory. (See, for example, Question 5 below.) This article lays essential groundwork for this, by completing the classification of small index subfactors beyond the first interesting composite index (that is, a product of smaller allowed indices), namely $3+\sqrt{5}$. Initially, it was expected that the classification at index $3 + \sqrt{5}$ would be very complicated, with a profusion of examples built by composites and other constructions from basic examples at smaller indices. These composite planar algebras were classified by [Liu15], contradicting that expectation.

There are relatively few subfactor standard invariants in the range we study, suggesting that the as yet unknown structure theory of quantum symmetries will strongly constrain possible examples.

Theorem. *There are exactly 15 subfactor standard invariants with index in $(5, 5\frac{1}{4}]$, besides the Temperley-Lieb-Jones A_∞ and the reducible $A_\infty^{(1)}$ standard invariants at every index. (See Theorem A below.)*

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1.1 Quantum symmetries and the Galois correspondence

Subfactors are universal hosts for quantum symmetries. In this section we say just enough to explain our slogan, before explaining precisely what a subfactor is in the next.

The first astonishing fact (of several!) about subfactors is Jones' index rigidity theorem [Jon83]. A subfactor $A \subset B$ has an index measuring the relative sizes of the factors, and this index is *quantized*:

$$[B : A] \in \{4 \cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty].$$

The Jones index has been understood from the beginning as a non-commutative analogue of the index of a field extension. The first sign that this analogy is important comes from a subfactor version of the Galois correspondence for field extensions. Given a finite group G , there is an essentially unique action of G on R , the hyperfinite II_1 -factor [Jon80]. We obtain a subfactor $R \subset R \rtimes G$, whose intermediate subfactors $R \subset P \subset R \rtimes G$ are all of the form $P = R \rtimes H$ for some subgroup $H \subset G$ [NT60].

This analogy runs even deeper. The *standard invariant* of a subfactor is a collection of finite dimensional vector spaces equipped with algebraic operations. It can equivalently be axiomatized via Popa's λ -lattices [Pop95a], Ocneanu's paragroups [Ocn88], or Jones' planar algebras [Jon99]. The essential feature is that this standard invariant plays the same role as the Galois group of a field extension — it describes the quantum symmetries of the subfactor.

We can now justify our initial slogan. The 'even half' of a standard invariant is a rigid C^* -tensor category, and when the subfactor is 'finite depth', this is a fusion category. Conversely, given a unitary fusion category \mathcal{C} , there is a hyperfinite subfactor which we should think of as $R \subset R \rtimes \mathcal{C}$, whose even half is exactly \mathcal{C} , and moreover this subfactor is essentially unique [Pop90], [FR13, Theorem 4.1].

2 Background

In this section we introduce subfactors, their standard invariants, and their role as quantum symmetries. Readers for whom this is familiar can skip ahead to Section 3 for the overview of our new results.

2.1 Subfactors and their standard invariants

A factor is a von Neumann algebra with trivial centre, and a subfactor is a unital inclusion of factors. Factors are classified into three types; we will be interested throughout in II_1 -subfactors, which are infinite dimensional and have a tracial state. (Most of what we describe below extends to type III subfactors, cf. [Kos86; Izu93; Pop95b].)

To prove the index restriction, Jones introduced the *basic construction* [Jon83]. After taking the GNS completion $L^2(B)$ of B with respect to the tracial state, we have the orthogonal projection e_A with range $L^2(A)$. The basic construction applied to $A \subset B$ is

the new factor $\langle B, e_A \rangle$, containing B as a subfactor. When $[B : A] < \infty$, we get a new II_1 -subfactor $B \subset \langle B, e_A \rangle$ with the same index.

Iterating the basic construction for $A = A_0 \subset A_1 = B$, we obtain the *Jones tower*

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots.$$

The first sign that something genuinely interesting is happening is that the Jones projections e_i satisfy the *Temperley-Lieb-Jones relations* [Jon83]:

- (1) $e_i = e_i^2 = e_i^*$,
- (2) $e_i e_j = e_j e_i$ when $|i - j| > 1$, and
- (3) $e_i e_{i \pm 1} e_i = [A_1 : A_0]^{-1} e_i$.

From the Jones tower, we extract two towers of finite dimensional centralizer algebras [GHJ89]:

$$\begin{array}{ccccccc} A'_0 \cap A_0 & \subset & A'_0 \cap A_1 & \subset & A'_0 \cap A_2 & \subset & A'_0 \cap A_3 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & A'_1 \cap A_1 & \subset & A'_1 \cap A_2 & \subset & A'_1 \cap A_3 & \subset & \cdots \end{array}$$

There's much more structure present here than just the $*$ -algebra structures and their inclusions – in particular there are also the restrictions of the conditional expectations $E_i : A_i \rightarrow A_{i-1}$ (obtained by restricting the Jones projection e_i on $L^2(A_i)$ to A_i) and the Jones projections e_j , all interacting according to intricate algebraic relations.

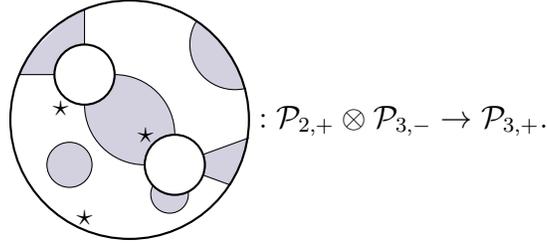
Our preferred way to axiomatize all this data is as a *subfactor planar algebra*, which we briefly define here. (Recall the alternatives are λ -lattices [Pop95a] or paragroups [Ocn88].) These are the main objects of study of this article, and they correspond under Theorem 2.1 below to subfactors. More detail, and a summary of the important techniques for analyzing a subfactor planar algebra, can be found in the survey article [JMS14].

A *shaded planar algebra* [Jon99] is a collection of complex vector spaces $\mathcal{P}_\bullet = (\mathcal{P}_{n,\pm})_{n \geq 0}$, together with an action of the operad of shaded planar tangles. A shaded planar tangle consists of a disc with several sub-discs removed, a collection of non-intersecting strings in the complementary region (whose endpoints lie on the boundary circles), with an alternating shading of the regions between the strings, and a marked interval on each boundary circle. For the careful definition of a shaded planar tangle, see [Jon99; Pet10]; we settle for giving an illustrative example below.

Suppose we have a shaded planar tangle T . We number the output circle 0, and number the input circles 1 through r . Suppose there are $2k_i$ points on the i -th circle, and the marked interval of the i -th circle is either unshaded or shaded according to a sign \pm_i respectively. Then the structure of a planar algebra assigns to this tangle T a multilinear map

$$\mathcal{P}(T) : \mathcal{P}_{k_1, \pm_1} \otimes \cdots \otimes \mathcal{P}_{k_r, \pm_r} \rightarrow \mathcal{P}_{k_0, \pm_0}.$$

For example,



We require that the identity tangle acts as the identity map. We can glue tangles one inside the other (we require that the distinguished intervals marked with \star match), and we require that gluing corresponds to composition of multilinear maps. Tangles which are isotopic must give the same linear map. (Contrary to the usual situation in quantum algebra, isotopies may move the boundary, although as each boundary circle has a marked interval this just means that 2π rotations act as the identity.)

Moreover, to be a *subfactor planar algebra*, we require that:

- each $\mathcal{P}_{n,\pm}$ is finite dimensional,
- \mathcal{P}_\bullet is *evaluable*, that is $\mathcal{P}_{0,\pm}$ is 1-dimensional,
- each $\mathcal{P}_{n,\pm}$ has an involution $*$ which is compatible with reflection of tangles,
- \mathcal{P}_\bullet is *positive*, in the sense that the sesquilinear form $\langle x, y \rangle = \text{tr}(y^*x)$ on each $\mathcal{P}_{n,\pm}$ is positive definite, where multiplication of elements in $\mathcal{P}_{n,\pm}$ is stacking, and

$$\text{tr} = \star \left(\star \left(\text{circle with nested circles and dots} \right) \right) : \mathcal{P}_{n,\pm} \rightarrow \mathcal{P}_{0,\pm} \cong \mathbb{C},$$

- and \mathcal{P}_\bullet is *spherical*, that is closed diagrams are invariant under spherical isotopy.

We sometimes also talk about *non-spherical* or *non-extremal* subfactor planar algebras, which do not satisfy this last axiom.

Theorem 2.1 ([Pop95a; Jon99]). *Given a finite index II_1 -subfactor, its standard invariant forms a subfactor planar algebra. Conversely, given a subfactor planar algebra \mathcal{P}_\bullet , there is a II_1 -subfactor whose standard invariant is \mathcal{P}_\bullet .*

Under this correspondence, the planar algebra is spherical if and only if the subfactor is extremal [Pop02]. We say a subfactor planar algebra is irreducible if $\dim(\mathcal{P}_{1,\pm}) = 1$. Irreducible subfactor planar algebras correspond to irreducible subfactors, i.e. those $A \subset B$ where $A' \cap B = \mathbb{C}$.

From the Jones tower for $A_0 \subset A_1$, we define the associated planar algebra \mathcal{P}_\bullet as follows. The vector spaces $\mathcal{P}_{n,\pm}$ are defined as the two towers of centralizer algebras:

$\mathcal{P}_{n,+} = A'_0 \cap A_n$ and $\mathcal{P}_{n,-} = A'_1 \cap A_{n+1}$. The planar algebra structure is given in [Jon99] or [JP11, Section 2.3]. We note that as proven in [JP11, Section 2.6], the planar algebra structure is completely determined by the following, where $\delta = [A_1 : A_0]^{1/2}$:

- Stacking elements in $\mathcal{P}_{n,\pm}$ is multiplication in the centralizer algebra.
- The involution $*$ on $\mathcal{P}_{n,\pm}$ is the involution on the centralizer algebra.
- The n -th Jones projection is given by $e_n = \delta^{-1} \left[\begin{array}{c} \square \\ \cdots \\ \square \end{array} \right] \in \mathcal{P}_{n+1,+} = A'_0 \cap A_{n+1}$.
- Adding a string on the right is the inclusion $\mathcal{P}_{n,+} = A'_0 \cap A_n \hookrightarrow A'_0 \cap A_{n+1} = \mathcal{P}_{n+1,+}$.
- Adding a string on the left is the inclusion $\mathcal{P}_{n,-} = A'_1 \cap A_{n+1} \hookrightarrow A'_1 \cap A_{n+1} = \mathcal{P}_{n+1,+}$.
- Capping on the right is δ times the restriction of the conditional expectation $A_n \rightarrow A_{n-1}$ to

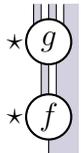
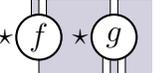
$$\mathcal{P}_{n,+} = A'_0 \cap A_n \rightarrow A'_0 \cap A_{n-1} = \mathcal{P}_{n-1,+}.$$
- Capping on the left is δ times the restriction of the conditional expectation $A'_0 \rightarrow A'_1$ to

$$\mathcal{P}_{n,+} = A'_0 \cap A_n \rightarrow A'_1 \cap A_n = \mathcal{P}_{n-1,-}.$$

Since $\mathcal{P}_{0,\pm}$ is 1-dimensional, the shaded and unshaded closed loops are multiples of the empty diagram. The spherical axiom ensures they are the same multiple, $\delta = [A_1 : A_0]^{1/2}$. We define the index of a subfactor planar algebra by the quantity δ^2 . As an exercise, the reader can check that the diagrammatic e_n is an idempotent with respect to stacking.

A planar tangle with no input discs and $2n$ boundary points gives a map $\mathbb{C} \rightarrow \mathcal{P}_{n,\pm}$, and can be thought of as an element of $\mathcal{P}_{n,\pm}$. The span of these elements forms the Temperley-Lieb-Jones planar subalgebra $\mathcal{TLJ}(\delta)_\bullet$ present inside any planar algebra \mathcal{P}_\bullet with index δ^2 . Indeed every index value allowed by Jones' restriction is realized by a subfactor whose standard invariant is 'trivial', in the sense that it is no bigger than the Temperley-Lieb-Jones planar algebra [Pop93].

From a subfactor planar algebra \mathcal{P}_\bullet , we may define a strict pivotal 2-category \mathcal{C} . An equivalent construction is described in detail in [BP14, Section 2.1]. As a first step, we define a preliminary 2-category $\hat{\mathcal{C}}$. It has two objects, called '+' (or 'unshaded') and '-' (or 'shaded'), and the 1-morphisms are natural numbers, even for 1-morphisms that do not change the shading, and odd for those that do. (We use the \otimes symbol to denote horizontal composition; on composable 1-morphisms we have $n \otimes m = n + m$.) The 2-morphisms are given by $\text{Hom}(n \rightarrow m) = \mathcal{P}_{\frac{1}{2}(n+m),\pm}$, with the sign determined by the source of n and m . The structure as a pivotal 2-category is readily provided by the planar algebra operations. For example,

- if $f \in \text{Hom}_{+-}(1 \rightarrow 3)$ and $g \in \text{Hom}_{+-}(3 \rightarrow 5)$, we have $g \circ f =$

- if $f \in \text{Hom}_{+-}(1 \rightarrow 3)$ and $g \in \text{Hom}_{--}(4 \rightarrow 2)$, we have $f \otimes g =$

- evaluation is given by the cap $\text{ev} : +1_- \otimes -1_+ \rightarrow 0 =$
, while coevaluation is the cup $\text{coev} : 0 \rightarrow +1_- \otimes -1_+$, and similarly for the other shading.

Finally, we declare \mathcal{C} to be the idempotent completion of $\hat{\mathcal{C}}$.

Remark 2.2. When the subfactor planar algebra comes from a finite index subfactor $A \subset B$, this 2-category is a purely algebraic model of the 2-category of $A - A$, $A - B$, $B - A$, and $B - B$ L^2 -bimodules (or at least, those generated by the bimodule ${}_A L^2(B)_B$) [Jon08].

Definition 2.3. A subfactor planar algebra is *finite depth* if \mathcal{C} has only finitely many isomorphism classes of 1-morphisms.

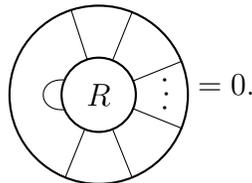
Definition 2.4. The *even part* of a subfactor planar algebra is the \otimes -category obtained as the endomorphisms of the unshaded object, in the associated 2-category.

A critical invariant of a subfactor planar algebra is its *supertransitivity* [Jon12].

Definition 2.5. The supertransitivity of a subfactor planar algebra \mathcal{P}_\bullet is the least integer k such that $\dim \mathcal{P}_{k+1, \pm} > \dim \mathcal{T}\mathcal{L}\mathcal{J}_{k+1, \pm}$.

Every subfactor planar algebra can be seen as a representation of the annular Temperley-Lieb-Jones algebra (this is spanned by the planar tangles with one input disc), and decomposed into a direct sum of irreducible representations. These have been described in [GL98; Jon01].

Capturing slightly less information than the full decomposition into irreducibles, we can look at the *low weight spaces* $\mathcal{W}_{n, \pm} \subset \mathcal{P}_{n, \pm}$ of the planar algebra: those vectors which are annihilated by capping any two adjacent strings:



Definition 2.6. The *annular multiplicity sequence* of the planar algebra is the sequence $(\dim \mathcal{W}_{n, \pm})_{n \geq 0}$.

The Fourier transform (1-click rotation) tangle gives a vector space isomorphism $\mathcal{W}_{n,+} \cong \mathcal{W}_{n,-}$. This sequence can be computed [Jon12, p.1], according to the formula appearing in [MS12, Definition 2.5], which is obtained by inverting the generating function for the dimensions of the annular Temperley-Lieb-Jones irreps given in [Jon01, Corollary 5.4].

$$\dim \mathcal{W}_{n,\pm} = \sum_{r=0}^n (-1)^{r-n} \frac{2n}{n+r} \binom{n+r}{n-r} \dim P_{r,\pm}. \quad (2.1)$$

For an evaluable k -supertransitive planar algebra, the sequence of annular multiplicities necessarily starts with 10^k . When we say below that a standard invariant has annular multiplicities $*ab$, we mean that while we may not know the supertransitivity k yet, the annular multiplicity sequence begins with $10^k ab$.

2.2 Towards classification

We now turn towards classifying subfactor standard invariants. Classical results [Ocn88; GHJ89] give us an ADE classification when the index is less than 4. To explain how these Coxeter-Dynkin diagrams arise, we introduce the *principal graph* $\Gamma(\mathcal{P}_\bullet)$ of a subfactor planar algebra \mathcal{P}_\bullet .

Recall in the associated pivotal 2-category \mathcal{C} we have a generating 1-morphism $1 : + \rightarrow -$. (When we start with a subfactor $A \subset B$ this is the bimodule ${}_A L^2(B)_B$.) We denote it as X , and its dual as X^* .

Definition 2.7. The vertices of $\Gamma(\mathcal{P}_\bullet)$ are the isomorphisms classes of simple 1-morphisms in \mathcal{C} . If vertices Y and Z have the same shadings on their sources, and Y 's target is unshaded while Z 's target is shaded, the number of edges between Y and Z is $\dim \text{Hom}_{\mathcal{C}}(Y \otimes X \rightarrow Z) = \dim \text{Hom}_{\mathcal{C}}(Z \otimes X^* \rightarrow Y)$.

The vertices of the principal graph come in 4 types, according to the shadings of their sources and targets. The principal graph has two components, according to the shadings of the sources, and each component is bipartite, according to the shadings of the targets.

Each component of the graph is pointed, with the basepoints being the identity 1-morphisms. We say the *depth* of a vertex is its distance to the basepoint in that component.

For our purposes, we nearly always consider principal graphs equipped also with the involution recording the duals of simple 1-morphisms (contragredients of bimodules in the subfactor setting¹). It is easy to see that duality preserves depth on the principal

¹ Recall that the vertices of Γ_\pm correspond to 4 different flavors of L^2 -bimodules generated by $L^2(B)$: the $A - A$, $A - B$, $B - A$, and $B - B$. Each bimodule has a contragredient, or *dual*, which is the complex conjugate Hilbert space with the conjugate action. For example, given ${}_A Q_B$, the dual ${}_B \bar{Q}_A = \{\bar{\xi} \mid \xi \in Q\}$ with action given by $b \cdot \bar{\xi} \cdot a = \overline{a^* \xi b^*}$. The dual of an $A - A$ bimodule is again an $A - A$ bimodule and similarly for $B - B$ bimodules, but the dual of an $A - B$ bimodule is a $B - A$ bimodule.

This gives us a very powerful tool – graph norms are increasing under graph inclusion (strictly increasing for connected finite graphs), and so we obtain easy lower bounds for the index of a subfactor with a given principal graph.

When the index is at least 4, the principal graph of the Temperley-Lieb-Jones planar algebra \mathcal{TLJ}_\bullet is two copies of the graph A_∞ (with graph norm 2). When the index is $4 \cos^2(\pi/n)$ for some $n \geq 3$, the principal graph of \mathcal{TLJ}_\bullet is two copies of the Coxeter-Dynkin diagram A_{n-1} .

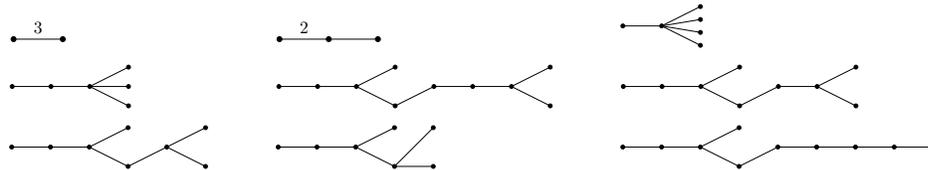
The mere fact that the only bipartite graphs with graph norm less than 2 are the ADE Coxeter-Dynkin diagrams gives us the start of the classification. It is relatively straightforward to see that both components of the principal graph must be the same Coxeter-Dynkin diagram. The full classification of subfactor planar algebras with index at most 4 was developed by Jones [Jon87] and Ocneanu [Ocn88], with many of the details provided by others [GHJ89; BN91; Izu91; Izu94; Kaw95b] (see also [Jon01] for an independent approach using annular tangles). As is well-known by now, there is a unique subfactor planar algebra for each A_n and D_{2n} Coxeter-Dynkin diagram, two distinct subfactor planar algebras for each of E_6 and E_8 , and no subfactor planar algebras for D_{2n+1} or E_7 .

The classification at index exactly 4 was given by Popa [GHJ89; Pop94; IK93]; the principal graphs are all *affine* Coxeter-Dynkin diagrams.

It is a remarkable fact that once we decide to ignore subfactors with trivial standard invariant and reducible subfactors, the index is actually also quantized *above* 4. It is straightforward to see that there is a gap between 4 and $\lambda(E_{10})^2 \sim 4.0264$. This gap is an easy consequence of the following exercise, which is an excellent introduction to the genre.

Exercise ([CDG82]). Show that every bipartite graph is either

- (1) a Coxeter-Dynkin diagram,
- (2) an affine Coxeter-Dynkin diagram,
- (3) A_∞ , $A_\infty^{(1)}$, or D_∞ , or
- (4) contains one of the following as a subgraph:



Then, by calculating the graph norms of the finitely many exceptions, show that the last has the lowest graph norm. Thus $\lambda(E_{10})^2$ gives a lower bound on the index of any subfactor above index 4, leaving aside subfactors with trivial standard invariant and reducible subfactors. \square

2.3 Some first obstructions

Enumerating possible graph pairs above index 4 gets difficult quickly. Over the years, a number of important obstructions to a graph pair being realized as the principal graph of a subfactor have been developed. We recall these briefly in this section. Many of these will be incorporated into the algorithm for enumerating potential principal graphs of subfactors which we describe in Section 4.

We saw above that $\dim(\mathcal{P}_{n,\pm})$ is given by the number of loops of length $2n$ beginning at the basepoint of the \pm component. We also saw that the Fourier transform (1-click rotation) tangle gives a vector space isomorphism $\mathcal{P}_{n,+} \cong \mathcal{P}_{n,-}$, which restricts to a vector space isomorphism $\mathcal{W}_{n,+} \cong \mathcal{W}_{n,-}$. This gives us the following easy constraints.

Fact 2.12 (Dimension constraints [GHJ89; Jon01]).

- (1) *The numbers of based loops of length $2n$ on either component Γ_{\pm} must agree.*
- (2) *Both Γ_+ and Γ_- have the same supertransitivity.*
- (3) *The annular multiplicity sequences of Γ_{\pm} must agree by Equation (2.1).*

From the principal graphs $\Gamma = (\Gamma_+, \Gamma_-)$, we can already deduce a lot of information about the associated strict pivotal 2-category \mathcal{C} . One strong constraint comes from associativity of composition of 1-morphisms. The Ocneanu 4-partite graph $\mathcal{O}(\Gamma)$ encodes the same information as Γ , but allows us to verify associativity of certain tensor products easily.

Definition 2.13. Suppose $A \subset B$ is a finite index subfactor with standard invariant \mathcal{P}_{\bullet} and principal graphs $\Gamma = (\Gamma_+, \Gamma_-)$. Taking 2 copies of each of Γ_{\pm} , they fit together in the *Ocneanu 4-partite graph* $\mathcal{O}(\Gamma)$:

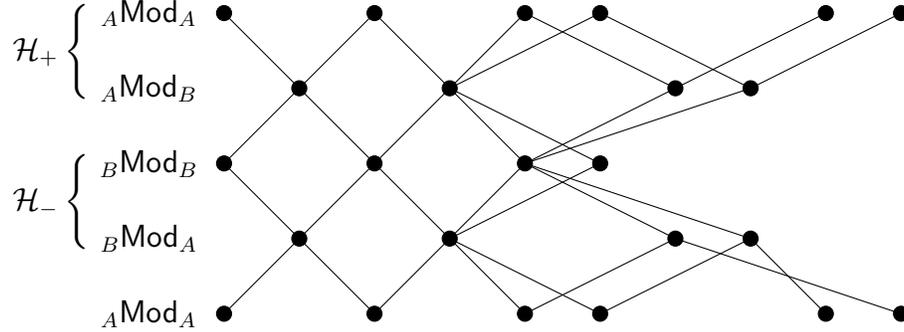
$$\begin{array}{ccc}
 V_{00} = \{A - A \text{ bimodules}\} & \xrightarrow[\Gamma_+]{{}_B L^2(B) \otimes_A -} & V_{10} = \{B - A \text{ bimodules}\} \\
 - \otimes_A L^2(B)_B \Big| \Gamma_+ & & \Gamma_- \Big| - \otimes_A L^2(B)_B \\
 V_{01} = \{A - B \text{ bimodules}\} & \xrightarrow[{}_B L^2(B) \otimes_A -]{\Gamma_-} & V_{11} = \{B - B \text{ bimodules}\}
 \end{array}$$

We note that the right hand graph is exactly Γ_+ , but the top copy of Γ_+ , while abstractly isomorphic to Γ_+ , has different vertex labels. We note vertices ${}_A P_A$ and ${}_B S_A$ are connected by

$$\dim(\text{Hom}({}_B L^2(B) \otimes_A P_A \rightarrow {}_B S_A)) = \dim(\text{Hom}({}_A \bar{P} \otimes_A L^2(B)_B \rightarrow {}_A \bar{S}_B)) \quad (2.2)$$

edges. Similarly, the right copy of Γ_- is exactly Γ_- , and the bottom copy is twisted using the dual data. Thus the graph pair $\Gamma = (\Gamma_+, \Gamma_-)$ with dual data is exactly the same data as $\mathcal{O}(\Gamma)$.

For example, the Ocneanu 4-partite graph of the Haagerup subfactor is given by



Since composition of 1-morphisms in \mathcal{C} is associative, we must have that for every simple ${}_A P_A$,

$$({}_B L^2(B) \otimes_A P_A) \otimes_A L^2(B)_B \cong {}_B L^2(B) \otimes_A ({}_A P \otimes_A L^2(B)_B).$$

There is a similar condition starting with each of the other 3 flavors of bimodules. We deduce:

Fact 2.14 (Associativity constraint [Ocn88; EK98]). *Given two vertices v and w on opposite corners of the Ocneanu 4-partite graph $\mathcal{O}(\Gamma)$, there are the same number of paths between v and w going either way around.*

The two ways of going around the Ocneanu 4-partite graph correspond to:

- (1) *moving to a neighbour on the principal graph, taking dual, moving to a neighbour, and taking dual again, or*
- (2) *taking the dual, moving to a neighbour, taking dual, and finally moving to a neighbour again.*

Ocneanu's axiomatization of the standard invariant of a finite depth subfactor used connections on 4-partite graphs.

Definition 2.15. A *connection* on Γ is a pair (W, \dim) where \dim is a dimension function on the vertices of $\mathcal{O}(\Gamma)$ satisfying the Frobenius-Perron condition, and W is a complex valued function on the loops of length 4 of $\mathcal{O}(\Gamma)$ which include a vertex of each color. The connection is said to be *bi-unitary* if the following two axioms hold:

- (Unitarity) For every P, R on diagonally opposite corners of $\mathcal{O}(\Gamma)$, the matrix $W(P, -, R, -)$ is unitary, i.e.,

$$\sum_S W(P, Q, R, S) \overline{W(P, Q', R, S)} = \delta_{Q, Q'}$$

- (Renormalization) For all P, Q, R, S , we have

$$W(P, Q, R, S) = \sqrt{\frac{\dim(Q) \dim(S)}{\dim(P) \dim(R)}} W(Q, P, S, R)$$

Fact 2.16 (Existence of connection [Ocn88; EK98]). *A necessary condition for Γ to be the principal graph of a subfactor is that $\mathcal{O}(\Gamma)$ must have a bi-unitary connection.*

Ocneanu found the first triple point obstruction, which is a simple consequence of the existence of a bi-unitary connection. This obstruction was used by Haagerup to classify principal graphs to index $3 + \sqrt{3}$ [Haa94]. In the case of initial triple points, improvements were made subsequently by [Jon12; Sny13; Pen15]. These obstructions were invaluable to the previous classification to index 5 [MS12; MPSS12], and the new obstruction [Pen15] is vital to this classification (see Section 6 below).

Fact 2.17 (Ocneanu’s triple point obstruction [Haa94]). *Let $A \subset B$ be a finite index subfactor with principal graph $\Gamma = (\Gamma_+, \Gamma_-)$. Suppose we have two 3-valent vertices v on Γ_+ and w on Γ_- at the same depth, and there are exactly 6 paths between v and w on $\mathcal{O}(\Gamma)$ (3 in each direction around the square). If there is a dimension preserving bijection β between the neighbors of v on Γ_+ and the neighbors of w on Γ_- such that*

- *for every pair of neighbors v' of v and w' of w such that $\beta(v') \neq w'$, there are exactly 2 paths on $\mathcal{O}(\Gamma)$ from v' to w' (one each way around the square),*

then $[B : A] \leq 4$.

A reader who (understandably) finds that formulation hard to digest may find working through the following exercise helpful.

Exercise ([Haa94]). Show that the obvious bijection between the neighbors of the triple points on the graph pair

$$(\mathcal{H}_+, \mathcal{H}_+) = \left(\begin{array}{c} \text{---} \bullet \text{---} \left\langle \begin{array}{l} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right. \text{---} \bullet \text{---} \end{array} \right), \left(\begin{array}{c} \text{---} \bullet \text{---} \left\langle \begin{array}{l} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right. \text{---} \bullet \text{---} \end{array} \right)$$

satisfies the bulleted condition in Ocneanu’s triple point obstruction 2.17. Conclude that $(\mathcal{H}_+, \mathcal{H}_+)$ is not the principal graph of a subfactor.

Popa’s principal graph stability gives a strong constraint on extensions of graph pairs. We denote the truncation of Γ_{\pm} to depth n by $\Gamma_{\pm}(n)$.

Definition 2.18. A graph Γ_{\pm} is called *stable at depth n* if Γ_{\pm} does not merge, split, or have multiple edges between depths n and $n + 1$. We say $\Gamma = (\Gamma_+, \Gamma_-)$ is stable at depth n if both Γ_{\pm} are stable at depth n .

Fact 2.19 (Stability constraint [Pop95a; BP14]). *Suppose $\delta > 2$ and $\Gamma_{\pm}(n) \neq A_{n+1}$.*

- (1) If the graph $\Gamma = (\Gamma_+, \Gamma_-)$ is stable at depth n , then Γ is stable for all depths $k \geq n$, and Γ is finite. (This means $\Gamma_{\pm} \setminus \Gamma_{\pm}(n)$ must be a disjoint union of finite type A Coxeter-Dynkin diagrams.)
- (2) If the graph Γ_+ is stable at depths n and $n + 1$, then $\Gamma = (\Gamma_+, \Gamma_-)$ is stable at depth $n + 1$.

A final easy obstruction comes from duality.

Fact 2.20 (Duality constraint [MS12, Lemma 3.6]). *Suppose Γ has supertransitivity $n - 1$ (so that depth n is one past the branch) with n even. If the graphs $\Gamma_{\pm}(n)$ are both simply laced, then the number of self-dual vertices on Γ_+ at depth n is equal to the number of self-dual vertices on Γ_- at depth n .*

3 The main theorem

Over the last few years, we've made considerable progress in understanding small index subfactor standard invariants. Haagerup initiated the classification of subfactors above index 4, leaving aside reducible subfactors and the poorly understood non-amenable subfactors with Temperley-Lieb-Jones standard invariant. Haagerup gave the classification up to index $3 + \sqrt{3}$ [Haa94], with components proved in [Bis98; AY09; BMPS12]. Following this, the next major step was the classification of subfactors up to index $5 = 3 + \sqrt{4}$. This was completed in a series of articles [MS12; MPPS12; IJMS12; PT12; IMPPS15], with some additional number theoretic ingredients in [CMS11]. There is now a survey paper summarising this work [JMS14]. Unfortunately, the techniques developed there struggle beyond index 5.

For the special case of 1-supertransitive standard invariants, it has nevertheless been possible to extend the classification further: up to index $3 + \sqrt{5}$ in [MP14c], and then 1-supertransitive standard invariants without intermediates up to index $6\frac{1}{5}$ in [LMP15].

In this article we give the complete classification of subfactor standard invariants with index at most $5\frac{1}{4}$. At every index above 4, we have the Temperley-Lieb-Jones A_{∞} standard invariant, as well as the reducible $A_{\infty}^{(1)}$ standard invariant (see Lemma 3.2 below).

Theorem A. *The only subfactor standard invariants in the index range $(5, 5\frac{1}{4}]$, besides the A_{∞} and $A_{\infty}^{(1)}$ standard invariants, are the following standard invariants at index ~ 5.04892 , the largest root of $x^3 - 6x^2 + 5x - 1$:*

- the unique subfactor planar algebra coming from the irreducible 3-dimensional representation of the quantum group $\mathfrak{su}(2)_5$ [Wen88; MP14c] and
- the unique subfactor planar algebra coming from (either of) the irreducible 3-dimensional representation of the quantum group $\mathfrak{su}(3)_4$ [Wen88; MP14c],

and the following standard invariants at index $3 + \sqrt{5}$:

- the Fuss-Catalan $A_3 * A_4$ subfactor planar algebra and its dual, constructed by Bisch and Jones [BJ97],
- the 3 finite quotients of the Fuss-Catalan $A_3 * A_4$ subfactor planar algebra [BH94; Liu15; IMP13] (the first is the self-dual tensor product; the other two are not self-dual),
- the unique 2D2 subfactor planar algebra and its dual [MP14b],
- Izumi’s unique symmetrically self-dual $3^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ subfactor planar algebra [Izu16; MP15],
- Izumi’s unique $3^{\mathbb{Z}/4\mathbb{Z}}$ subfactor planar algebra and its dual [Izu16; PP15], and
- the unique symmetrically self-dual 4442 subfactor planar algebra [MP15; MP14b].

(Our arguments supersede the earlier combinatorial arguments required for index at most $3 + \sqrt{3}$ and at most 5, but still rely on many obstructions and existence results proved by other authors.)

As in previous classification efforts, the problem essentially divides into three parts:

- (1) First, we enumerate all possible principal graph pairs, satisfying certain combinatorial constraints, with graph norm up to the square root of the index.
- (2) Second, we rule out several ‘difficult’ families of such graphs using quite complicated arguments specific to each family, and we reduce many ‘easy’ families of such graphs down to finitely many cases using a standard number theoretic approach.
- (3) Finally, we completely classify all subfactor planar algebras with the given principal graph in each of the remaining cases.

In general, one expects all parts to suffer when increasing the bound on the index; the combinatorial enumeration problem will be harder for the computer, and moreover it will produce more results, leaving a rapidly growing workload for the humans.

To overcome these problems, this article brings three essential new tools to bear.

- (1) We use a completely new technique for tackling the combinatorial enumeration problem. Previous methods produced many isomorphic copies of a single graph during the enumeration, and removing the corresponding redundancies in the search tree became unrealistically computationally expensive. We now use McKay’s approach of ‘construction by canonical paths’ [McK98] to perform this enumeration in a manner which avoids pairwise isomorphism checking between graphs.
- (2) A recent new result [Pen15] using quadratic tangles tightly constrains certain parameters for principal graphs with annular multiplicity sequence $*11$ (see Definition 2.6 above). Happily this constraint applies to many of the new potential principal graphs we find.

- (3) Calegari-Guo [CG15] developed a new number theoretic tool for cylinders (weeds which are stable in the sense of [Pop95a; BP14]) similar to the uniform treatment of vines afforded by [CMS11]. We use this in one particular case (and anticipate it would be very useful in any future classifications) and obtain a generalisation which we apply to a certain ‘periodic’ weed.

The complete list of all non- A_∞ , non- $A_\infty^{(1)}$ subfactor standard invariants with index in $(4, 5\frac{1}{4}]$ appears in Appendix A.3. In Appendix A.4, we give the current ‘map of subfactors’, showing the overlapping ranges of the classification results to date.

3.1 The future

In this section, we suggest some open questions to guide our future exploration of quantum symmetries.

Question 1. *How far can the classification of small index subfactors go?*

Perhaps somewhat surprisingly, there is not a clear ‘wall’ after which classification becomes too hard. Index 6 remains the distant goal; there we know many new phenomena arise (subfactors not classifiable by their standard invariants [BNP07; BV15], as well as infinitely many non-isomorphic finite depth Bisch-Haagerup subfactors [BH96] associated to finite quotients of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$). A first step would be $5\frac{1}{3}$, the largest index of any 4-spoke. Perhaps the next goal should be $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$, which is the minimum index for an extremal, reducible subfactor [PP86, Corollary 4.6].

The new combinatorial enumerator seems to be able to look some distance above our present cut-off of $5\frac{1}{4}$. However we quickly find graphs which we don’t know how to deal with. We present the reader with the following weed which appears above index 5.27.



There are many possible avenues for attacking this weed – a quadratic tangles approach to quadruple points, number theoretic obstructions based on the repeating unit in the tail, or analyzing possible connections – but so far we have had no success. The absence of a doubly one-by-one connection entry means we can’t use the techniques of Section 5.2 below.

Question 2. *Is there a global bound on supertransitivity for standard invariants above index 4?*

Leaving aside the subfactors of index less than 4, high supertransitivity is exceedingly rare. At this time, the record is the extended Haagerup subfactor [BMPS12] with supertransitivity 7, and the second highest is the 5-supertransitive Asaeda-Haagerup subfactor. The third is the 4-supertransitive 4442 subfactor at index $3 + \sqrt{5}$ [MP15], discovered as part of the project leading up to this paper.

As supertransitivity is a subfactor analog of transitivity of group actions, one hopes by analogy that for sufficiently large k there are only finitely many non-trivial k -supertransitive standard invariants. At this point, we have no evidence that this is not the case for $k \geq 5$.

At present we have two methods for proving bounds on supertransitivity for families of graphs. First, the number-theoretic results of [CMS11] show that only finitely many translations of a fixed graph can be principal graphs of subfactors, giving an explicit bound on the supertransitivity. These results are further generalized in [CG15] and Section 5.3 below, allowing us to eliminate all finite depth translated extensions of a certain graph ((f)(9) in Theorem 3.1), by first proving a bound on supertransitivity in Lemma 5.19.

Second, quadratic tangles inequalities sometimes give bounds on supertransitivity, and we see this while eliminating the weeds (f)(7) and (f)(8) from Theorem 3.1, in Section 6.3. One hopes that this technique can be improved by a deeper understanding of quadratic tangles (see Question 4 below).

Question 3. *Can Liu’s results showing there are only finitely many composites of A_3 and A_4 be generalized?*

Following Bisch-Haagerup’s classification of the possible principal graphs of composites of A_3 and A_4 [BH94] (see also [IMP13]), the one might expect to find an infinite family of finite depth composite subfactor standard invariants whose principal graphs converge to the $A_3 * A_4$ Fuss-Catalan [BJ97] principal graph, parallel to the situation at index 4, with the $D_{n+2}^{(1)}$ standard invariants ‘converging’ to the D_∞ standard invariant.

Liu’s result [Liu15] came as quite a surprise, now suggesting that quotients of the free product Fuss-Catalan standard invariants are also rare.

This is analogous to the situation in Question 2, where we observe that high supertransitivity is rare. A planar algebra \mathcal{P}_\bullet with supertransitivity k looks like its Temperley-Lieb-Jones subalgebra along with certain extra elements which only appear in the space $\mathcal{P}_{n,\pm}$ for $n > k$. The observation that high supertransitivity is rare can be reformulated as the ‘difficulty’ of adding a large generator to the Temperley-Lieb-Jones algebra. Similarly, as any composite planar algebra contains (more or less by definition) the corresponding Fuss-Catalan planar algebra, we could define the *Fuss-Catalan supertransitivity* as the first n so $\mathcal{P}_{n,\pm}$ contains elements beyond Fuss-Catalan. Liu’s result can then be interpreted as saying that it is hard to add large generators to a Fuss-Catalan planar algebra.

Is this a general phenomenon? Are there number theoretic, or even algebraic geometric, constraints limiting possible quotients of free products?

Question 4. *How can we effectively develop the theory of quadratic tangles?*

Understanding the representation theory of the annular Temperley-Lieb-Jones algebras led to the well-developed theory of annular tangles [GL98; Jon01]. As annular tangles together with quadratic tangles generate the entire planar operad, analyzing quadratic tangles systematically is an extremely difficult task. However, we are rewarded

with strong constraints and structure theorems which tightly restrict the structure of subfactor planar algebras [Jon12; Sny13; Pen15].

Certainly there are more constraints which can be obtained by analyzing higher annular consequences. A systematic treatment of the space of second annular consequences was given in [PP15], but deeper analysis is needed to extract useful information.

Moreover, we lack a good description of the fusion of low weight representations above index 4, along with the possible quotients which occur in subfactor planar algebras. Again, we may see tight restrictions which lead to quadratic tangles obstructions.

Question 5. *Is there a good extension theory for fusion categories?*

At this time, we have the theories of G -graded extensions of fusion categories for finite groups G [ENO10], (de)equivariantization (the notion goes back to [Kaw95b; Kaw95c]; see also [DGNO10, Section 4] and [BN13]), and exact sequences of fusion categories [BN11; BN14]. However, there are certain examples arising from the small index subfactor classification program which behave somewhat like extensions but do not fit into these theories.

The 4442 subfactor at index $3 + \sqrt{5}$ [MP15] was discovered in exhaustive enumeration of principal graphs. While this subfactor appears as an equivariantization of Izumi's $3^{\mathbb{Z}/2 \times \mathbb{Z}/2}$ subfactor [Izu16], its even half also resembles a 'non-graded' extension of $\text{Rep}(A_4)$. It has the form $\mathcal{C} \oplus \mathcal{M}$ where $\mathcal{C} = \text{Rep}(A_4)$, and \mathcal{M} is \mathcal{C} as a module over itself, while the tensor product structure on \mathcal{M} is stranger.

Question 6. *Where do the quadratic categories come from?*

A quadratic category has a group G of invertible objects, together with one other orbit of simple objects. At this point, thanks to [Izu01; EG11; EG14], there is a well-developed theory of quadratic fusion categories using endomorphisms of Cuntz algebras for classification and construction. These categories include the near group categories [Sie03] and a possible infinite family generalizing the even half of the exotic Haagerup subfactor [AH99]. Recently, [GIS15] gave a new construction of the exotic Asaeda-Haagerup subfactor, showing it is related to a quadratic category.

The theory of quadratic categories provides a classification (for a fixed group of invertible objects, and orbit structure for the non-invertible objects) in terms of the finitely many solutions of an explicit system of polynomials. (Analogously, one could attempt directly solving the pentagon equations; there, however, there is a large gauge group, and categories correspond to orbits. In the theory of quadratic categories there is no gauge group.) Nevertheless, constructing quadratic categories by solving these equations leaves something to be desired. The simple structure of these categories led Evans and Gannon to conjecture that they should arise from conformal field theories, and that there is some unknown underlying construction.

On the planar algebra side, we know two possible skein theoretic approaches to quadratic categories, based on jellyfish relations [BP14; MP15; PP15] or Yang-Baxter relations [LP15]. Although both appear promising, Cuntz algebra techniques have been much more successful to date. We are far from fully understanding this situation.

Question 7. *Can we find further sources of infinite families of examples?*

At this point, almost every known quantum symmetry is related to quantum groups or quadratic categories via known constructions. In fact, the extended Haagerup subfactor [BMPS12] stands alone as the only quantum symmetry not arising in this way! While it is certainly important to study this example further (e.g. [MW14]), in the hopes that maybe it too is related to quantum groups or quadratic categories, what we really want is new infinite families of examples.

One way to find new families may be to continue the search for quantum symmetries which are ‘small’ by some metric besides the index of the standard invariant, for example rank or global dimension. For example, we have a full classification of rank 2 fusion categories [Ost03], rank 3 pivotal categories [Ost13], and a partial classification of rank 4 pseudo-unitary categories [Lar14]. There are only finitely many modular categories of a given rank [BNRW13], and they have been completely classified up to rank 5 [BNRW15].

Perhaps it would be interesting to look at certain families of graphs, e.g., spoke graphs. Are there only finitely many higher spoke graphs with high valence and high supertransitivity? There is good number theoretic evidence that beyond certain families, very few (possibly only finitely many) spoke graphs have an index which is a cyclotomic integer.

Question 8. *What can we say about hyperfinite $A_3 * A_4$ subfactors?*

To fully understand non- A_∞ irreducible hyperfinite subfactors with index at most $5\frac{1}{4}$, we would still like to know what $A_3 * A_4$ subfactors can exist. We note that there is a unique $A_3 * A_3 = D_\infty$ subfactor, coming from the fact that this standard invariant is amenable [Pop94]. However, the higher Fuss-Catalan standard invariants [BJ97] are not amenable [HI98], and we conjecture that $A_3 * A_4$ already exhibits the same unclassifiability phenomenon seen for hyperfinite subfactors with $A_3 * D_4$ standard invariant [BV15]. As a proof of concept, it would be interesting to approach $A_3 * A_5$ hyperfinite subfactors, which can be obtained by composing subfactors associated to groups.

3.2 Proof of the main theorem

The proof of the main theorem splits naturally in many parts, most of which are independent of each other. We begin with a combinatorial enumeration that shows that every principal graph of an irreducible subfactor with index at most $5\frac{1}{4}$ must be represented by one of a certain list of weeds or one of a certain list of vines. This appears as Theorem 3.1 below. This calculation is closely analogous to the calculation performed in [MS12], but with more advanced combinatorial techniques, described in Section 4.

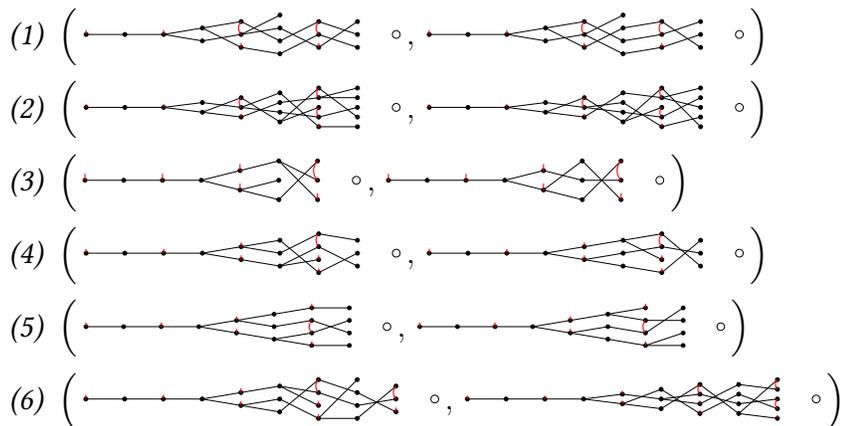
Theorem 3.1. *The principal graph of any subfactor with index in $(4, 5\frac{1}{4}]$ must either be*

- (a) A_∞ (in which case the subfactor is non-amenable)
- (b) reducible, i.e. there are multiple edges between depths 0 and 1,

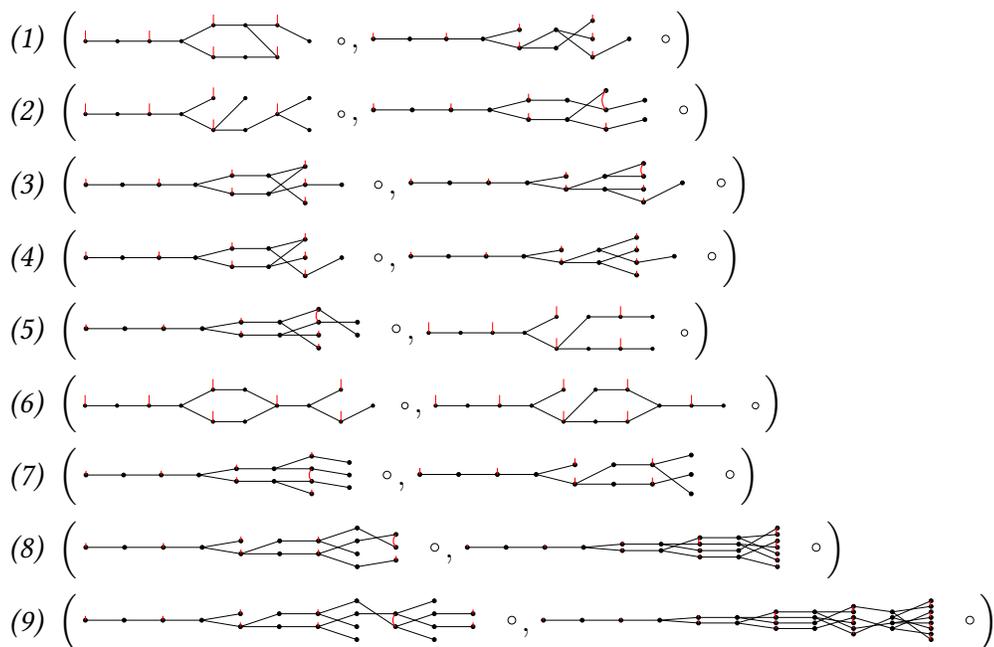
(c) exactly 1-supertransitive,

(d) not simply laced,

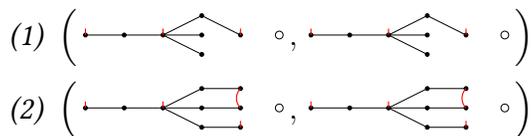
(e) a translated extension of one of the following 'weeds' with annular multiplicities *11,

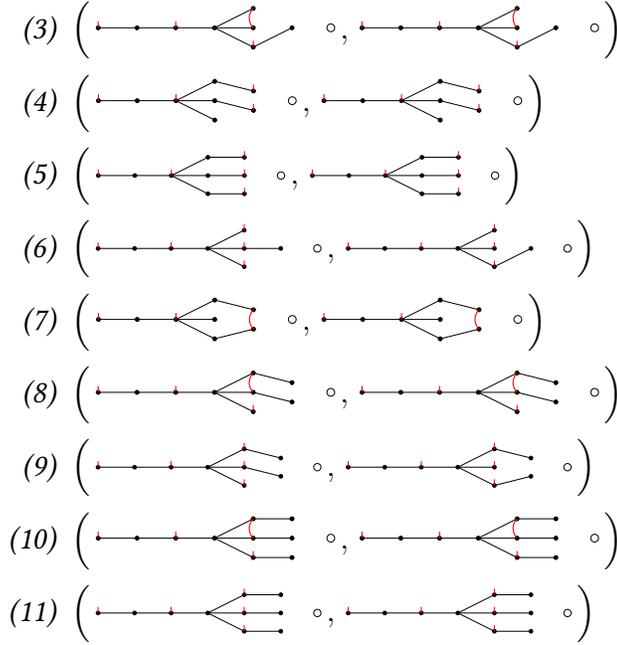


(f) a translated extension of one of the following 'weeds' with annular multiplicities *10,



(g) a translated extension of one of the following stable graphs with annular multiplicities *20,





(h) or a translation of one of the ‘vines’ listed in Section 8.

The proof of this appears in Section 4.4, after we describe the underlying algorithm in Section 4.1 and our implementations in Section 4.3. Unfortunately this classification looks at first quite discouraging compared with the corresponding classification up to index 5.

First, we give the folklore result which deals with everything in case (b), by describing all reducible subfactors in the index range $(4, 3 + 2\sqrt{2})$.

Lemma 3.2. *The only reducible subfactor standard invariants with index in $(4, 3 + 2\sqrt{2})$ are the $A_\infty^{(1)}$ standard invariants at every such index. Moreover these are all non-spherical, and can be obtained as perturbations of the spherical index 4 $A_\infty^{(1)}$ standard invariant.*

Proof. By [Pop02] (see also [DGG14, Section 3]), a non-spherical reducible subfactor standard invariant can be *perturbed* without altering the principal graph. Amongst these perturbations, there is a unique one with minimal index [Kos86; Lon89], and this perturbation is necessarily spherical. By [PP86, Corollary 4.6], a spherical reducible subfactor standard invariant with index in $(4, 8)$ must have index $(1 + 2\cos(\pi/n))^2$, for $n \geq 3$. The least such index is $3 + 2\sqrt{2}$.

Thus a reducible subfactor standard invariant \mathcal{P}_\bullet with index in $(4, 3 + 2\sqrt{2})$ must be a perturbation of some reducible spherical subfactor standard invariant \mathcal{Q}_\bullet at index 4. The only such standard invariants are $A_{2n-1}^{(1)}$ and $A_\infty^{(1)}$. The $A_{2n-1}^{(1)}$ standard invariants, being finite depth, admit no non-spherical perturbations [EK98, Proposition 10.4]. Thus \mathcal{Q}_\bullet is the unique $A_\infty^{(1)}$ standard invariant, and necessarily \mathcal{P}_\bullet also has principal graph $A_\infty^{(1)}$. \square

We next note that case (c) from Theorem 3.1, the 1-supertransitive principal graphs, has already been completely classified for index less than 6 in [LMP15]. Case (d) from Theorem 3.1, the non-simply laced graphs, can be handled as follows.

Lemma 3.3. *There are no subfactors with index in $(4, 5.3]$ whose principal graphs are not simply-laced.*

Proof. First, we note

$$\begin{aligned} \|\bullet \xrightarrow{3} \bullet\|^2 = 9, \quad \|\bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet\|^2 = 8, \quad \|\bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet\|^2 > 6.56, \quad \left\| \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \end{array} \right\|^2 = 6, \\ \|\bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet\|^2 > 5.82, \quad \left\| \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \end{array} \right\|^2 > 5.56, \quad \|\bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet\|^2 > 5.3. \end{aligned}$$

The only non-simply laced graphs not containing one of these as a subgraph are $\bullet \xrightarrow{2} \bullet$, $\bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet$, and $\bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet \xrightarrow{2} \bullet$. The first is at index 4. In the other two cases, the dimension of the non-trivial object in the even half is not 1 or $\tau = \frac{1+\sqrt{5}}{2}$, so these graphs cannot be principal graphs of subfactors by [Ost03]. More generally, we note that [SV93] shows that the graphs β_{2n+1} (that is, the $2n - 2$ -translate of $\bullet \xrightarrow{2} \bullet$) are not principal graphs of subfactors by showing that the fusion rules must have coefficients of the form $\frac{2k+1}{2}$ for $k \in \mathbb{N}_{\geq 0}$. \square

The remaining cases (e), (f), and (g) constitute ‘weeds’, i.e. translated extensions of some finite set of possibilities, and the cases in (h) are ‘vines’, i.e. translations of some finite set of possibilities.

While there are significantly more vines than in previous classifications, this itself is no problem, as we by now have a uniform and rather efficient mechanism for dealing with vines. This is described in Theorem 8.1, and all the vines from part (h) are analyzed in Section 8.

The *11 graphs can be dealt with by the techniques of [Pen15], and we prove the requisite inequalities for all but one of these in Section 6.2. The remaining *11 graph (case (e)(5) in Theorem 3.1) is treated in Section 6.1, and requires additional analysis using doubly one-by-one connection entries, since it has undetermined relative dimensions.

Nearly all the *10 graphs have branch factor not equal to 1, and can be dealt with by the techniques of [Jon12; MPPS12]. We do this in Section 6.4. Two graphs (namely (f)(7) and (f)(8) in Theorem 3.1) require additional arguments in Section 6.3 using doubly one-by-one connection entries. We note that both these exceptional *10 weeds require using Morrison’s hexagon obstruction [Mor14] in an essential way. Of the remaining *10 weeds, we get one candidate graph (from the first graph in either (f)(3) or (f)(4) from Theorem 3.1) which is eliminated using the formal codegree obstruction (see Section 6.5).

Additionally, there are three exceptional *10 graphs, with branch factor 1. The first (namely (f)(5) in Theorem 3.1) is the depth 2 truncation of one of the weeds ruled out in [MPPS12], and we give an easy argument based on stability to rule it out in Theorem 5.1.

according to the novelty of the methods. The really interesting stuff comes first: the new combinatorics of the enumerator in Section 4, our new approach to infinite periodic graphs in Section 5, Penneys’ obstructions to $*11$ graphs in Section 6, applications of Morrison’s hexagon obstruction in Section 6.3, and eliminating one candidate via formal codegrees in Section 6.5. We defer Section 7 on 4-spokes and Section 8 on vines to the end of the article; these sections are essentially applications of already well-known arguments.

4 Better combinatorics for graph enumeration

In this section we describe our new technique for enumerating principal graphs, based on McKay’s method of construction by canonical paths. The classification statement we prove using this technique has appeared above as Theorem 3.1. Its proof appears below, in Section 4.4.

We begin with the precise definition of the objects we enumerate.

Definition 4.1. A *principal graph pair (PGP)* is a tuple $(\Gamma_+, \Gamma_-, \text{depth}, \bar{\cdot}, n)$ where

- Γ_{\pm} are finite bipartite graphs,
- depth is a graph homomorphism from $\Gamma_{\pm} \rightarrow \mathbb{N}$, the graph with one edge between n and $n + 1$ for all $n \in \mathbb{N}$, such that $\text{depth}^{-1}(0)$ intersects every connected component of Γ_{\pm} ,
- $\bar{\cdot}$ is an involution on the vertices of $\Gamma_+ \sqcup \Gamma_-$, which
 - (1) takes an even depth vertex on Γ_{\pm} to a vertex at the same depth on Γ_{\pm} , and
 - (2) takes an odd depth vertex on Γ_{\pm} to a vertex at the same depth on Γ_{\mp} ,
- and n , called the ‘working depth’, is a non-negative integer which is either equal to the maximum distance of a vertex from the base vertex, or to the maximum distance plus one,

satisfying the following constraints, which are described below:

- the PGP associativity constraint,
- the PGP triple point obstruction, and
- the PGP duality constraint.

There is an obvious notion of isomorphism of PGPs.

The *index* of a PGP is $\max\{\lambda_+^2, \lambda_-^2\}$, where λ_{\pm} is the Frobenius-Perron eigenvalue for Γ_{\pm} , that is, the largest eigenvalue of the adjacency matrix.

It is important to note here that we are not allowing multiple edges between vertices, so we only see ‘simply-laced’ principal graphs. This is not an essential restriction, but it makes the implementations so much simpler that it is worthwhile having to deal with non simply-laced principal graphs separately.

Definition 4.2. If $\text{nbhd}(v)$ denotes the set of neighbours of a vertex v , *associativity between vertices v and w* is the condition

$$\left| \overline{\text{nbhd}(v)} \cap \text{nbhd}(\bar{w}) \right| = \left| \text{nbhd}(\bar{v}) \cap \overline{\text{nbhd}(w)} \right|. \quad (4.1)$$

This condition is symmetric in v and w , and trivially satisfied unless $\text{depth}(v)$ and $\text{depth}(w)$ differ by -2 , 0 , or 2 . The *PGP associativity constraint* is associativity between all pairs of vertices v and w such that at least one of v or w is at depth $n - 2$ or less.²

Definition 4.3. We first define the *combinatorial dimension relation*, on the set of vertices of a PGP Γ . This is the weakest equivalence relation such that

- $v \sim w$ if $v = \bar{w}$, and
- $v \sim w$ whenever v and w have depth at most $n - 2$, and there is a bijection $\alpha : \text{nbhd}(v) \rightarrow \text{nbhd}(w)$ such that $u \sim \alpha(u)$ for all $u \in \text{nbhd}(v)$.

(Clearly if Γ is the principal graph of a subfactor, and $v \sim w$, then $\dim(v) = \dim(w)$.)

A PGP Γ satisfies the *PGP triple point obstruction* if it satisfies Ocneanu’s triple point obstruction 2.17 for all pairs of triple points with depth at most $n - 2$, replacing the condition that the bijection β preserves dimensions with the condition that $\beta(u) \sim u$ for all u .

Remark 4.4. The classification to index 5 only used the triple point obstruction for initial triple points [MS12]. Our use of the combinatorial dimension relation allows us to apply the triple point obstruction at non-initial other triple points, which is an essential improvement, without which the algorithm described here would not terminate.

The *PGP duality constraint* is identical to the duality constraint 2.20.

Facts 4.5.

- (1) *No extension of a PGP can satisfy the associativity constraint 2.14 unless the PGP satisfies the PGP associativity constraint.*
- (2) *No extension of a PGP can satisfy Ocneanu’s triple point obstruction 2.17 unless the PGP satisfies the PGP triple point obstruction.*
- (3) *No extension of a PGP can satisfy the duality constraint 2.20 unless the PGP satisfies the PGP duality constraint.*

² Since we may subsequently add vertices at the working depth n , we cannot require this identity to hold for pairs of vertices both at depth at least $n - 1$, because later vertices at depth n may change either side of the equation. When v is at depth at most $n - 2$, on the other hand, the sizes of the sets in the above equation for a vertex w , at any depth, will not change when further vertices are introduced. Nevertheless, there is an inequality one may impose for pairs of vertices at depth $n - 1$, which we address later.

4.1 Generation by canonical construction paths for principal graphs

We now review McKay's method of enumeration by construction by canonical paths [McK98]. Although our description here is a very close parallel of his, we have slightly specialized his framework, and allowed ourselves the use of slightly more sophisticated mathematical language (particularly of groupoids). We hope that our exposition is accessible, and simultaneously explains the general picture and the particular instance used in this paper. McKay's running example is triangle free graphs, while our running example will be principal graph pairs (PGPs). Throughout this section, the general method is described in the main text, and the specialization to the PGP example is illustrated in highlighted boxes.

The general setup for McKay's method is that we have a (countable) groupoid of combinatorial objects \mathcal{O} , and we would like to enumerate the isomorphism classes.

The groupoid \mathcal{O} is all PGPs with index bounded by some fixed constant L , with PGP isomorphisms.

Our initial plan is to come up with a finite collection of seed objects and generating steps so that a representative of each class can be reached from a seed in a finite number of steps. This plan ensures that we can enumerate all isomorphism classes. The problem, of course, is that we may produce many representatives of the same class.

For example, any PGP Γ with index at most L may be reached from the trivial PGP $(\emptyset, \emptyset, id, 0)$ using finitely many applications of these two generating steps.

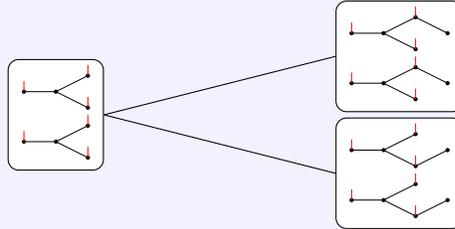
- (1) We can increase the working depth at which we add vertices.
- (2) We may join new vertices at the working depth n to vertices at the previous depth. The details differ depending on whether n is even or odd.
 - (a) If n is odd, given two sets of vertices $S_+ \subset V(\Gamma_+)$ and $S_- \subset V(\Gamma_-)$ at depth $n - 1$, we may add a new vertex v to Γ_+ and a new vertex \bar{v} to Γ_- , both at depth n , joined to the vertices of S_+ and S_- respectively.
 - (b) If n is even, there are two ways to add vertices to either Γ_+ or Γ_- .
 - (i) Given a set of vertices $S \subset V(\Gamma_{\pm})$ at depth $n - 1$, we may add a new self dual vertex v at depth n to Γ_{\pm} , joined by an edge to each vertex of S .
 - (ii) Given two sets of vertices $S_1, S_2 \subset V(\Gamma_{\pm})$ at depth $n - 1$, we may add two new vertices v and \bar{v} at depth n to Γ_{\pm} , joined by edges to S_1 and S_2 respectively.

In fact, there are restrictions on when these steps may be applied (ensuring associativity and staying below the index limit). We postpone discussing these until later in this section.

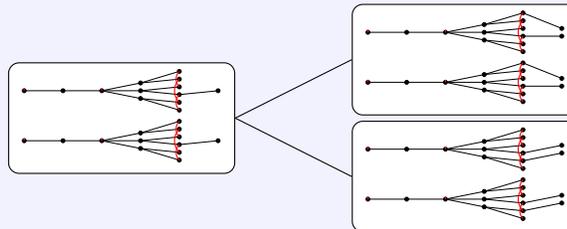
In particular, there are three ways in which this process would produce representatives in the same isomorphism class.

- (1) If two generating steps starting from the same object are equivalent under the automorphism group of that object, the results will be isomorphic.
- (2) Two inequivalent generating steps applied to the same object can yield isomorphic objects.
- (3) Starting with two non-isomorphic objects and applying a generating step can result in isomorphic objects.

- (1) Adding a vertex in two different locations to the same graph will give isomorphic results, if those locations are equivalent under the automorphism group action.



- (2) Two inequivalent ways of adding vertices to the same graph pair may yield isomorphic results.

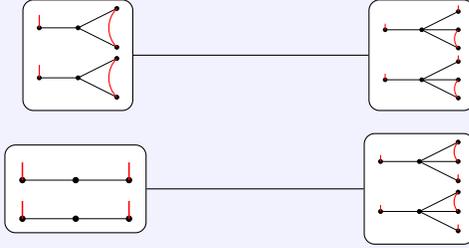


These extensions are not equivalent: in the first case, but not the second, the new vertices are connected to vertices which are dual to vertices at distance 3 from the old vertices at the working depth.

The resulting PGPs are equivalent by permuting the vertices at successive depths in each graph in the lower pair by $(1), (1), (1), (231), (436512), (12)$.

- (3) Adding vertices to two inequivalent graphs pairs may yield isomorphic re-

sults.



McKay’s method of construction by canonical paths avoids these problems. First, however, we need to endow our groupoid of objects with the following pieces of extra structure.

Definition 4.6. A McKay groupoid is a countable groupoid \mathcal{O} with the extra structure $(\ell, U, L, \hat{\cdot}, (\cdot)^{-1}, \phi)$ described in conditions (C1)-(C6) below.

(C1) The groupoid \mathcal{O} should be graded by \mathbb{N} ; that is, every object $o \in \mathcal{O}$ has a isomorphism invariant ‘level’, which we write as $\ell(o)$.

For PGPs, we use the following slightly complicated level function. Let a be the working depth of o , let b be the number of self-dual vertices in o , and let c be the number of pairs of dual vertices in o . Then $\ell(o) = a + b + c$.

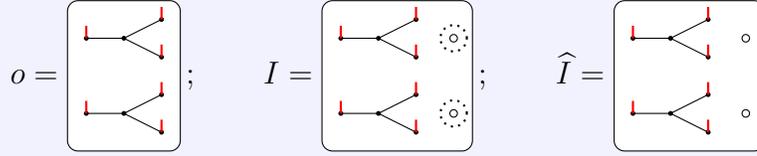
$$\ell \left(\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \right) = 6 + 8 + 5 = 19$$

(C2) Associated to an object $o \in \mathcal{O}$ we define a new set, the ‘upper objects’ $U(o)$ for o . The elements of $U(o)$ consist of an object in \mathcal{O} along with certain extra data. (It is important to point out here that ‘upper’ is not a property of an object, but rather an upper object is some extra structure associated with an object. We keep this terminology for consistency with [McK98].) If $u \in U(o)$ is an upper object for o , we write \hat{u} for the resulting object in \mathcal{O} . We must have $\ell(\hat{u}) > \ell(o)$. Essentially, the upper object u records the generating step that produces \hat{u} from o .

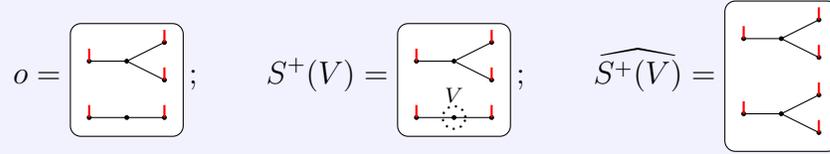
We have three types of upper objects, called ‘increasing the depth’ I , ‘adding a self-dual vertex’ $S^+(V)$, and ‘adding a pair of dual vertices’ $P^+(V_1, V_2)$.

- There is only one way to increase the depth, and the underlying object \hat{I} is just o with the working depth incremented by one. We denote \hat{I}

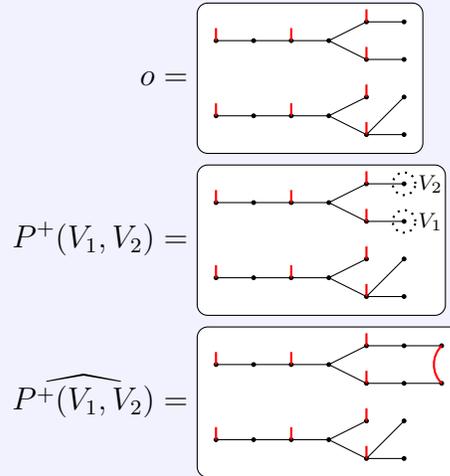
by adding a white shaded vertex at the next depth, and we denote I by drawing a dotted circle around this white vertex.



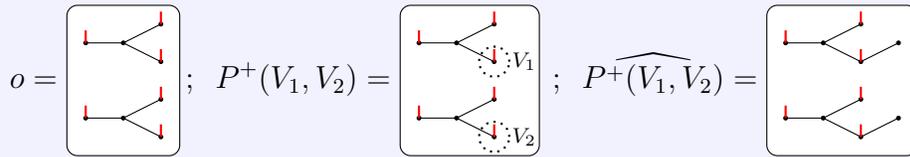
- When we add a self-dual vertex, $S^+(V) \in U(o)$, V denotes a subset of the vertices of o at depth $n - 1$, all on one graph. The underlying object $\widehat{S^+(V)}$ is the PGP obtained by adding a new self-dual vertex at depth n , connected by an edge to each vertex in V .



- When we add a pair of dual vertices $P^+(V_1, V_2) \in U(o)$, the sets V_1 and V_2 denote two collections of vertices of o at depth $n - 1$. When the working depth n is even, both must be collections of vertices of the same graph, and we do not distinguish between $P^+(V_1, V_2)$ and $P^+(V_2, V_1)$.



When the working depth n is odd, each is a subset of vertices on different graphs.



The resulting object $P^+(\widehat{V_1, V_2})$ is the PGP obtained by adding a pair of vertices, dual to each other at depth n (either on the same graph, or different graphs, as appropriate), with the first connected by an edge to each vertex in V_1 , and the second connected to each vertex in V_2 .

For a given PGP o :

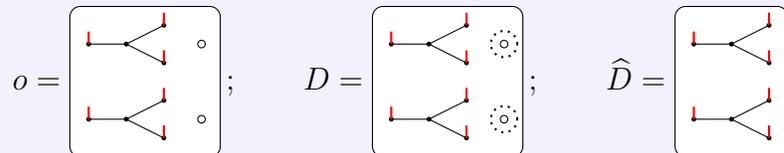
- We include $I \in U(o)$ if and only if there is at least one vertex on each graph at the current working depth, and the associativity condition holds between all pairs of vertices at depth $n - 1$. (This ensures that \widehat{I} is associative.)
- We include $S^+(V)$ for every subset V such that $\widehat{S^+(V)}$ would have index below our cutoff, and the associativity condition holds between all vertices at depth $n - 2$ and the new vertex in $\widehat{S^+(V)}$.
- Similarly we include $P^+(V_1, V_2)$ if it has small enough index and the associativity condition holds between all vertices at depth $n - 2$ and both of the new vertices in $\widehat{P^+(V_1, V_2)}$.

Note that associativity between any pair of vertices v and w is checked precisely once.

(C3) Associated to an object $o \in \mathcal{O}$, we define a new set, the ‘lower objects’ $L(o)$ for o . The elements of $L(o)$ consist of an object in \mathcal{O} along with certain extra data. (As with upper object, again take care to note that lower objects are not just objects satisfying a property.) If $l \in L(o)$ is a lower object for o , we write \widehat{l} for the resulting object in \mathcal{O} . We must have $\ell(\widehat{l}) < \ell(o)$. Again, a lower object l essentially records the generating step that produces o from \widehat{l} .

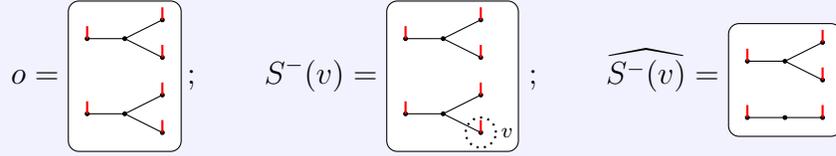
We insist that $L(o)$ is empty if and only if $\ell(o) = 0$.

This is completely parallel to the description of upper objects. We have three types of lower objects, called ‘decreasing the depth’ D ,

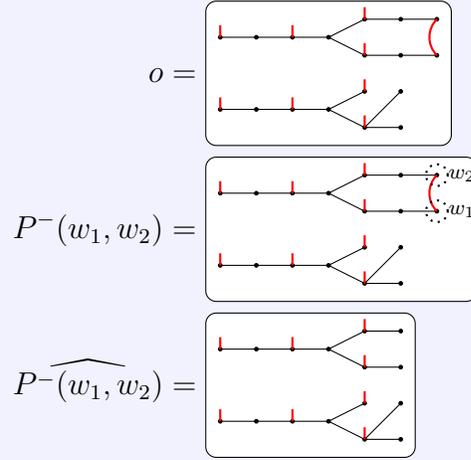


‘deleting a self-dual vertex’ $S^-(v)$, where v denotes any self-dual vertex at

the working depth,



and ‘deleting a pair of dual vertices’ $P^-(w_1, w_2)$, where w_1, w_2 denote a pair of dual vertices at the working depth.



The resulting objects are the PGPs which are obtained by deleting the marked vertices.

We have $D \in L(o)$ if and only if there are no vertices on either graph at the working depth. We have $S^-(v) \in L(o)$ and $P^-(w_1, w_2) \in L(o)$ for all valid choices of v or w_1, w_2 .

- (C4) The groupoid \mathcal{O} acts on the bundle of sets U . That is, for each o , $U(o)$ carries an action of the automorphism group $\text{Aut}(o)$, and moreover we have a coherent family of bijections $U(o) \cong U(o')$ for each isomorphism $o \cong o'$. Similarly \mathcal{O} acts on the bundle of sets L .

For example, the following upper objects in $U(o)$ are isomorphic:

$$o = \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array}; \quad P^+(V_1, V_2) = \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array} \cong \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array} = P^+(V'_1, V'_2)$$

The automorphism of o which swaps the vertices V_1, V'_1 and the vertices V_2, V'_2 induces the above isomorphism.

Remark 4.7. When we write $u \cong u'$, for $u \in U(o)$ and $u' \in U(o')$, we mean an isomorphism $o \cong o'$ carrying u to u' (and similarly for lower objects).

- (C5) Each upper object $u \in U(o)$ must have an inverse lower object, denoted u^{-1} , in $L(\widehat{u})$, such that $\widehat{u^{-1}} = o$. We require that when $u_1^{-1} \cong u_2^{-1}$, it is also the case that $u_1 \cong u_2$.

For o and u the first two graphs below, we see that $\widehat{u^{-1}} = o$:

$$\begin{array}{cc} o = \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array} & u = \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array} \\ \widehat{u} = \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array} & u^{-1} = \begin{array}{|c|} \hline \begin{array}{c} \text{---} \diagup \text{---} \\ \text{---} \diagdown \text{---} \end{array} \\ \hline \end{array} \end{array}$$

An isomorphism $g : \widehat{u_1} \xrightarrow{\cong} \widehat{u_2}$ carrying $u_1^{-1} \in L(\widehat{u_1})$ to $u_2^{-1} \in L(\widehat{u_2})$ can be restricted to the vertices which are not being deleted, obtaining an automorphism $g' : o \xrightarrow{\cong} o$. This automorphism then carries $u_1 \in U(o)$ to $u_2 \in U(o)$.

- (C6) Finally, for each o such that $L(o)$ is non-empty, we have a chosen orbit $\phi(o) \in L(o)/\text{Aut}(o)$ called the canonical reduction orbit of o . These choices must be coherent with respect to the groupoid action, that is for $g : o \cong o'$, $\phi(g(o)) = g(\phi(o))$.

(The choice of canonical reduction is the critical optimization step for this algorithm; see below.)

When o is a PGP, we have many potential criteria to choose the canonical reduction orbit in $L(o)$. Here we explain the general framework for such criteria, deferring the actual choice we make to Section 4.3.

If there are no vertices at the working depth, $L(o)$ is the singleton containing the lower object which decreases the working depth, and there is no choice to make. Otherwise, every lower object deletes either a self-dual vertex or a pair of dual vertices at the working depth. (Note that if the working depth is odd, there can not be any self-dual vertices.)

We may make choices such as to prefer deleting vertices from Γ_- over Γ_+ , deleting dual pairs of vertices over deleting self-dual vertices, and so on, as long as these choices are invariant under the groupoid (because we are choosing an orbit, not a particular lower object). Our two implementations make slightly different choices here. After expressing these preferences, there may still be alternatives, and indeed in the simplest case, where we have no such preferences, all the orbits are alternatives!

The `nauty` [MP14a] package provides an algorithm for canonically labelling the vertices of a vertex-coloured graph. Since the data of a lower object for a PGP can be encoded as a vertex-coloured graph, preserving automorphisms, it is always possible to use this canonical labelling to make a choice of orbit of lower objects.

The precise details differ in our two implementations; we describe these in Section 4.3.

This concludes the definition of a McKay groupoid (and simultaneously how we see PGPs as an example).

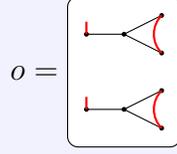
Definition 4.8. Given an upper object $u \in U(o)$, we say it is *genuine* exactly if u^{-1} is contained in the canonical reduction orbit $\phi(\widehat{u})$.

Observe that this property is preserved by the groupoid action: if $g : o \cong o'$, then u is genuine if and only if $g(u)$ is genuine.

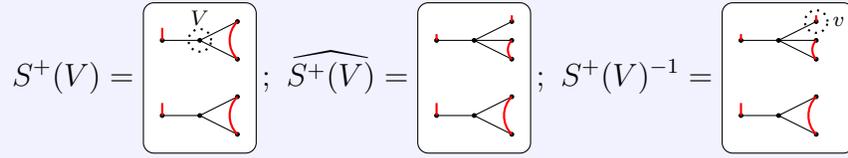
Remark 4.9. Making a clever choice of canonical reduction ϕ amongst the automorphism orbits in $L(o)$ can provide a significant speed-up. The key fact is that when preparing the upper objects as above, we may omit any upper object that we know in advance can not possibly be genuine.

If the choice function simply relies on canonical labellings from `nauty`, it is essentially a black box, and it is not possible to make such predictions. As such, it would be impossible to prune the list of upper objects. Moreover, calls to `nauty` can be computationally expensive. We find that it is possible to specify ϕ in a way that drastically reduces the number of calls needed; this is described in Section 4.3.

We consider the following PGP (with Γ_+ the upper graph, Γ_- the lower graph).



We give an example of an upper object $S^+(V) \in U(o)$ which is not genuine since $S^+(V)^{-1} \notin \phi(\widehat{S^+(V)})$, for the choice function ϕ used in the `Scala` implementation, given in detail in Section 4.3 below.



Because the choice function ϕ prefers to delete vertices from Γ_- , we have

$$\phi(\widehat{S^+(V)}) = \left\{ \begin{array}{|c} \hline \text{Diagram 1} \\ \hline \text{Diagram 2} \\ \hline \end{array} \right\}.$$

Our strategy now, described in detail in the next section, is to build a tree of objects in \mathcal{O} , such that the children of an object o are the resulting objects of representatives of the $\text{Aut}(o)$ orbits of genuine upper objects.

4.2 Exhaustivity and uniqueness

We write $K(\mathcal{O})$ for the set of isomorphism classes of the groupoid \mathcal{O} , and $K(\mathcal{O}_1)$ for the set of isomorphism classes of elements $o \in \mathcal{O}$ such that $L(o)$ is not empty.

Lemma 4.10 (cf. [McK98, Lemma 1]³). *There is a unique function $\pi : K(\mathcal{O}_1) \rightarrow K(\mathcal{O})$ such that for each*

- $o \in [o] \in K(\mathcal{O}_1)$,
- $l \in \phi(o)$, and
- $o' \in \mathcal{O}$ and $u \in U(o')$ such that $u^{-1} \cong l$,

³The statement of Lemma 1 in [McK98] is slightly incorrect; it should say that for each element $\hat{X} \in f(\check{X})$ there is a $Y \in p(S)$ such that $\hat{X} \in U(Y)$, rather than that all elements of $f(\check{X})$ are in a single such $U(Y)$.

we have $o' \in \pi([o])$.

Proof. We just need to show that for any allowed choices (o_1, l_1, o'_1, u_1) and (o_2, l_2, o'_2, u_2) with $o_1 \cong o_2$, we have $o'_1 \cong o'_2$.

Condition (C6) shows $l_1 \cong l_2$. Now $u_1^{-1} \cong u_2^{-1}$, and by Condition (C5), $u_1 \cong u_2$, which in particular (recall Remark 4.7) means $o'_1 \cong o'_2$. \square

As in [McK98, p. 6], this is called the parent function. An ancestor of an isomorphism class $[o]$ is an $[o']$ such that $\pi^k([o]) = [o']$ for some k . There is an obvious notion of the children and descendants of $[o]$. Because π reduces the level by one by Condition (C3), it is clear that every isomorphism class is a descendant of some ‘progenitor’ $[o]$ with $L(o)$ empty.

Given a McKay groupoid \mathcal{O} satisfying Conditions (C1)–(C6), we define a forest \mathcal{F} of elements. Theorems 4.13 and 4.14 below shows that this forest consists of a single representative of each isomorphism class $[o] \in K(\mathcal{O})$.

Definition 4.11. We first define a tree $\mathcal{T}_{[r]} \subset \mathcal{O}$ for any isomorphism class $[r]$. The root is an arbitrarily chosen representative r of $[r]$, and the children of any node $o \in \mathcal{T}_{[r]}$ are then obtained as follows:

- for each orbit in $U(o)/\text{Aut}(o)$,
- pick a representative u , and
- accept \hat{u} if u is genuine (i.e. $u^{-1} \in \phi(\hat{u})$), rejecting otherwise.

The forest \mathcal{F} is the union of all the trees $\mathcal{T}_{[r]}$ where $[r]$ varies over all isomorphism classes for which $L(r)$ is empty.

Remark 4.12. This definition is a close parallel of the procedure $\text{scan}(X, n)$ [McK98, p. 6]. Our setup is less general, and in particular we use u^{-1} as the canonical choice of $\hat{Y} \in f'(\hat{X})$ in $\text{scan}(X, n)$.

Theorem 4.13. *For any $r \in \mathcal{O}$, if $[o]$ is descended from $[r]$, then some $o \in [o]$ appears in $\mathcal{T}_{[r]}$.*

Proof. We say a descendant $[o]$ of $[r]$ is in generation i if $\pi^i([o]) = [r]$. We induct on the generation. The base case is trivial.

Consider $[o]$ descended from $[r]$ in generation $i + 1$. Pick o , a representative of $[o]$, some $l \in \phi(o)$, and u such that $l \cong u^{-1}$. The upper object u is an upper object for some o' . By the definition of π , $[o'] = \pi([o])$, and $[o']$ is also a descendant of $[r]$, but in generation at most i . By induction, some other $o'' \cong o'$ appears in $\mathcal{T}_{[r]}$. As U is equivariant by Condition (C4), there is a $u' \in U(o'')$, with $u' \cong u$. We see that $u'^{-1} \cong l$, and so $\hat{u}' \cong o$.

Finally, we need to check that $u'^{-1} \in \phi(\hat{u}')$, so that we accept \hat{u}' . As ϕ is also equivariant by Condition (C6), this follows from $l \in \phi(o)$. \square

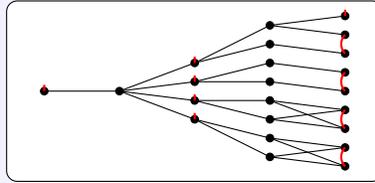
- If the PGP has both self-dual vertices and pairs of dual vertices at the working depth, then the canonical reduction will be a lower object which removes a dual pair. The canonical reduction only removes a self-dual vertex if all vertices at the working depth are self-dual.
- Subject to these constraints, the canonical reduction is a lower object which removes a set (either a single vertex or a dual pair) of vertices with least total degree amongst the lower objects.
- Finally, ties are broken using canonical labellings from `nauty`, according to one of the two strategies described here. The C implementation uses the first strategy, while the Scala implementation uses the second strategy.

In both, we need to encode PGPs as vertex-coloured graphs, in order to be able to use `nauty`. From the underlying pair (Γ_+, Γ_-) for o , we apply a vertex-colouring according to depth, and additionally add new (depth-preserving) edges between pairs of dual objects. Then the automorphism group of this graph, which we denote $G(\Gamma_+, \Gamma_-)$ is exactly the automorphism group of the PGP o .

- (1) We observe that all the lower objects we are considering are described by some subset of vertices (of size 1 or 2) on a fixed graph. Only some of these subsets are allowed according to the choices described above. We use `nauty` [MP14a] to compute a canonical labelling of the vertices of the graph $G(\Gamma_+, \Gamma_-)$, as well as the action of the automorphism group on subsets of the appropriate size (in some cases, the C implementation shortcuts this calculation, using the action on vertices to quickly deduce the action of subsets of size 2). We identify which subset has least canonical labelling, and then choose the orbit of lower objects consisting of the images of this subset under the automorphism action.
- (2) For each orbit $[l]$ of lower objects satisfying our preferences, we pick a representative $l \in L(o)$, and construct a single vertex-coloured graph \mathcal{G}_l encoding l , in an $\text{Aut}(o)$ equivariant manner. This is the same vertex-coloured graph as that described in the first alternative, with the addition of an extra vertex-colour for the vertices to be deleted by l . We then call `nauty` to canonically label the vertices of \mathcal{G}_l , obtaining a vertex-coloured graph $\mathcal{G}_{[l]}$ which did not depend on the representative l . We then define a total ordering on vertex-coloured graphs (e.g. dictionary order on a textual representation), and declare that our chosen orbit is the one with the least $\mathcal{G}_{[l]}$.

Note the second approach may require several calls to `nauty` when the first requires just one; the Scala implementation is certainly less efficient.

For simplicity, we consider the following graph (not a graph pair) at an even working depth, whose lower objects consist of deleting vertices at the working depth.



There are 4 orbits of lower objects under the $\text{Aut}(o)$ action:

- (1) deleting one of the first two dual pairs (counting from the bottom),
- (2) deleting the third dual pair,
- (3) deleting the fourth dual pair, and
- (4) deleting the self-dual vertex.

Here, the canonical reduction orbit must be deleting a dual pair with least total degree amongst dual pairs, so it must be deleting either the third or fourth dual pair. We then call `nauty` to produce a canonical labeling to break the tie.

We note the following differences between the implementations. The optimizations made in the C implementation are described in detail in Afzaly's Ph.D. thesis [Afz15].

- The C implementation only enumerates simply-laced principal graphs (as reflected in the definition of PGPs above), while the Scala implementation can also produce non-simply-laced graphs (requiring the obvious modifications to the definitions above). Given Lemma 3.3, this is not a significant difference for the purposes of this paper.
- The C implementation uses certain shortcuts for estimating graph norms, while the Scala implementation uses the straightforward heuristic of bounding a graph norm below by $\|A^{n+1}v\|/\|A^n v\|$ for any chosen n and v (taking $n = 10$ and v the vector which is 1 on every vertex is good enough).
- The C implementation differs slightly from the description above in that it treats the graphs (Γ_+, Γ_-) in a PGP as an unordered pair. Thus the output from the Scala implementation contains (Γ_+, Γ_-) and (Γ_-, Γ_+) separately whenever $\Gamma_+ \neq \Gamma_-$.
- The Scala implementation assumes that there is exactly one vertex on each graph in depth 0, while the C implementation allows arbitrarily many. This is

a more significant limitation, as this freedom is essential for the ‘tail enumerator’ described below, and used in Section 5.4.

- The C implementation does not natively implement the triple point obstruction; we filter its output using an implementation of the triple point obstruction written in `Mathematica` (thus the two overall implementations maintain separate codebases, although both triple point obstructions were implemented by the second and third authors).
- Finally, the Scala implementation includes one additional inequality, which sometimes rules out a PGP on the basis of having no possible associative descendants. Let $\text{nbhd}^+(v)$ and $\text{nbhd}^-(v)$ denote the neighbours of a vertex v at the next and the previous depth respectively. We can decompose both sides of the associativity constraint in Equation (4.1) as a sum of two positive terms. For vertices v and w at the same depth (and in particular depth $n - 1$) we have

$$A + B = C + D$$

where

$$\begin{aligned} A &= \left| \overline{\text{nbhd}^+(v)} \cap \text{nbhd}^+(\bar{w}) \right| & B &= \left| \overline{\text{nbhd}^-(v)} \cap \text{nbhd}^-(\bar{w}) \right| \\ C &= \left| \text{nbhd}^+(\bar{v}) \cap \overline{\text{nbhd}^+(w)} \right| & D &= \left| \text{nbhd}^-(\bar{v}) \cap \overline{\text{nbhd}^-(w)} \right|. \end{aligned}$$

We are interested in the specific case where n is even, and there are already vertices on Γ_- at depth n . Consider v on Γ_+ at depth $n - 1$, and w on Γ_- at depth $n - 1$. We know that in any genuine child the additional vertices at depth n will also be on Γ_- . Thus three out of the four terms above will not change: only $C = \left| \text{nbhd}^+(\bar{v}) \cap \overline{\text{nbhd}^+(w)} \right|$ will increase as we look at genuine children. Thus we can discard any graph in this situation for which $A + B - C - D$ is already negative. (In fact, this apparently rather specific check saves a huge amount of effort, reducing the total number of graphs considered in our application from 239710 to 17360!) The C implementation similarly implements associativity checks as early as possible [Afz15], although the details are different because of the symmetry between Γ_+ and Γ_- there.

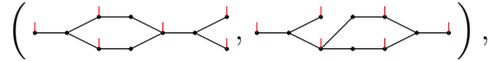
The tail enumerator. We can apply the above algorithm starting with a PGP which does not represent the entirety of a subfactor principal graph up to some depth, but rather a ‘block’ \mathcal{B} of a principal graph containing only the vertices at depths between some specified limits a and $a + k$. Using this, we can enumerate all possible blocks \mathcal{C} such that $\mathcal{A}\mathcal{B}\mathcal{C}$ could be the principal graph of subfactor, where $\mathcal{A}\mathcal{B}\mathcal{C}$ represents a principal graph given by some unknown \mathcal{A} from depth 0 to depth a (with a itself perhaps also unknown), the fixed block \mathcal{B} from depth a to depth $a + k$, and the block \mathcal{C} from depth

we deal with using the algorithms from [PT12]. The only cyclotomic translates are when $(a, b) \in \{(0, 0), (1, 1)\}$. This calculation is performed in the Mathematica notebook `CalegariGuoSmallCases.nb` bundled with the arXiv source.

We now eliminate the two remaining cases. First, $(a, b) = (0, 0)$, corresponding to the graph



is not possible, since the branch point does not occur at an odd depth, which is necessary by Ocneanu's triple point obstruction [Haa94] (see also [Jon12, Theorem 5.1.11]). To eliminate

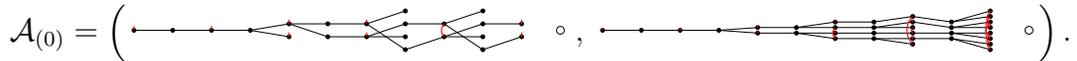


we note that this graph pair has index 5, which is not a composite index. However, by looking at the dual graph, we see that there is a normalizer, which would give rise to an intermediate subfactor [PP86, Proposition 1.7], a contradiction. (A planar algebraic proof of this easy fact is available at [IMP13, Lemma 4.7].) \square

5.1 The remaining $r = 1$ weed

In the remainder of this section, we eliminate the final weed with branch factor $r = 1$, (f)(9) from Theorem 3.1.

Theorem 5.3. *No subfactor planar algebra has principal graph a translated extension of*



(Note that $\mathcal{A}_{(0)}$ is isomorphic to the graph in part (f)(9) of Theorem 3.1.)

We define $\mathcal{A}_{(2t)}$ to be the translation of $\mathcal{A}_{(0)}$ by $2t$.

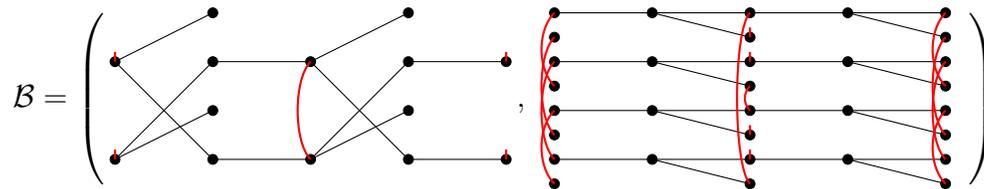
Theorem 5.3 follows from the following four lemmas.

Lemma 5.4. *There are no finite depth subfactor planar algebras with principal graph a translated extension of $\mathcal{A}_{(0)}$.*

This lemma is a striking application of number theory; while previously we've used the cyclotomicity of the index for finite depth subfactors to rule out arbitrary translations of a fixed graph, this lemma is the first case in which we are able to rule out arbitrary translations and *all finite extensions*. Its proof appears in Section 5.3.

We prove the next two lemmas, which both rely on graph enumeration arguments, in Section 5.4.

Lemma 5.5. *Any subfactor principal graph which is an infinite depth extension of $\mathcal{A}_{(2t)}$ must be an extension of $\mathcal{A}_{(2t)}\mathcal{B}$ where*



Lemma 5.6. *The only possible subfactor principal graph which is an infinite depth extension of $\mathcal{A}_{(2t)}\mathcal{B}$ is $\mathcal{A}_{(2t)}\mathcal{B}^\infty$.*

Finally, the remaining possibilities are ruled out by the next lemma, proved in Section 5.5.

Lemma 5.7. *There is no subfactor planar algebra with principal graph $\mathcal{A}_{(2t)}\mathcal{B}^\infty$.*

5.2 Relative dimensions and doubly one-by-one connection entries

For some weeds it is possible to determine the dimensions of all the vertices as functions of q and t , where t is the translation. If we can do so, we call these dimensions the *relative dimensions* of the vertices [MPPS12, Section 4.1].

To compute the relative dimensions, we combine and solve the following three types of equations.

- (1) Since the trivial vertex marked \star always has dimension 1, and the quantum dimension of the vertex at depth 1 is $[2] = q + q^{-1}$, we set the dimension of the leftmost vertex of our weed equal to $[t + 1] = \frac{q^t - q^{-t}}{q - q^{-1}}$.
- (2) If two vertices are dual to each other, they must have the same dimension.
- (3) For every vertex v for which we know all its neighbors w , we have the equation $[2] \dim(v) = \sum_{\text{neighbors } w} \dim(w)$.

When we can determine all the relative dimensions, it is possible to use certain branch factor inequalities to try to rule out the weed, as in Section 6 below. However, in many cases, we have too few equations to determine all the relative dimensions. In extremely fortunate situations, we can calculate an additional unknown dimension by using connection techniques, namely looking for doubly one-by-one connection entries.

By Fact 2.16, a necessary condition for a PGP Γ to be the principal graph of a subfactor is that it must have a bi-unitary connection on the Ocneanu 4-partite graph $\mathcal{O}(\Gamma)$. We can form this 4-partite graph for an arbitrary PGP Γ parallel to Definition 2.13, and associativity of Γ directly corresponds to the associativity constraint for $\mathcal{O}(\Gamma)$.

Remark 5.8. A PGP Γ will not have any extension which is the principal graph of a subfactor unless there is a ‘partial connection’ on Γ .

A partial connection on a PGP with working depth n is a pair (W, \dim) as in Definition 2.15, with $W(P, Q, R, S)$ only defined when at least one of $\{P, R\}$ and at least one of $\{Q, S\}$ have depth at most $n - 2$. The unitarity and renormalization conditions must be satisfied whenever all the relevant entries are defined. (The restriction on the defined entries ensures that the conditions we impose for a partial connection on a PGP Γ are implied by the conditions for a partial connection on any extension of Γ .)

In certain nice conditions, we can use the existence of a bi-unitary connection to specify certain unknown relative dimensions.

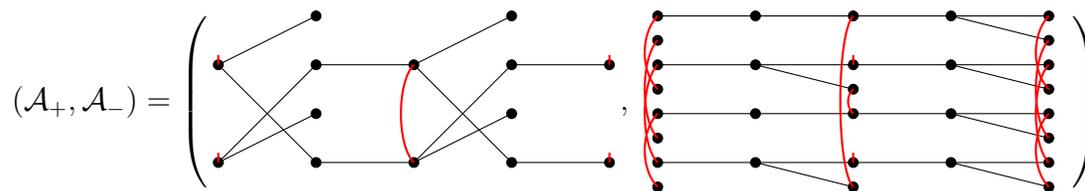
Definition 5.9. Suppose Γ is a PGP with a bi-unitary connection on $\mathcal{O}(\Gamma)$. A *doubly one-by-one connection entry* is a loop of length 4 $\ell = (P, Q, R, S)$ around $\mathcal{O}(\Gamma)$ which is the unique such loop passing through the pairs (P, R) and (Q, S) . Such an ℓ exists if and only if there are exactly 2 paths of length 2 on $\mathcal{O}(\Gamma)$ between P and R and exactly 2 paths of length 2 between Q and S . For an example, see the proof of Lemma 5.10 below.

If we have a doubly one-by-one connection entry, then the one-by-one matrix $W(P, -, R, -)$ is unitary by the Unitarity Axiom. Also, by the Renormalization Axiom, so is the one-by-one matrix $W(Q, -, S, -)$. Hence we must have the following identity of dimensions:

$$\dim(P) \dim(R) = \dim(Q) \dim(S).$$

The presence of a doubly one-by-one connection entry sometimes allows us to solve for one relative dimension in terms of three other known relative dimensions.

We now return to our example at hand. Looking at the section of $\mathcal{A}_{(0)}$ between depths 6 and 10, which we will denote $(\mathcal{A}_+, \mathcal{A}_-)$, we get \mathcal{B} with two missing univalent vertices at depth 8:



Recall that $\mathcal{A}_{(2t)}$ is the $2t$ -translation of $\mathcal{A}_{(0)}$.

Lemma 5.10. *There is a doubly one-by-one entry of the connection for any principal graph which is an extension of $\mathcal{A}_{(2t)}$, and so*

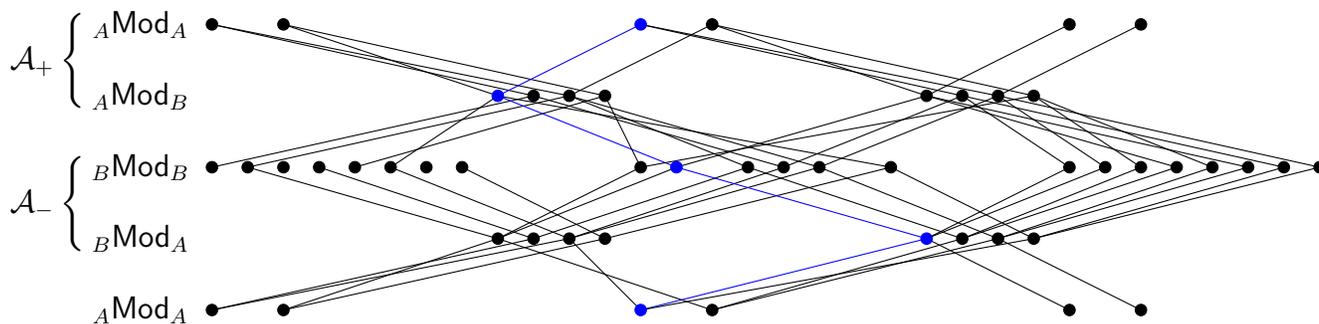
$$a^2 X_1(q) + X_0(q) + a^{-2} X_1(q^{-1}) = 0 \tag{5.1}$$

where $a = q^{2t}$ and

$$\begin{aligned} X_1(q) &= q^{16} - 2q^{14} - q^{12} - 2q^{10} - 3q^8 - 4q^6 - q^4 \\ X_0(q) &= -2q^4 + 8q^2 + 12 + 8q^{-2} - 2q^{-4}. \end{aligned}$$

Proof. The loop $(V_{2t+6,1}^p, V_{2t+5,1}^p, V_{2t+6,2}^d, V_{2t+7,1}^d)$ (in the notation of [MPPS12]: the p and d stand for ‘principle’ and ‘dual’ to specify which graph, and the subscript i, j gives the depth and height) in $\mathcal{O}(\mathcal{A}_+, \mathcal{A}_-)$, outlined in blue below, gives a doubly one-by-one

connection entry.



Happily, we can compute the dimensions of these vertices in the principal graph $\mathcal{A}_{(2t)}\mathcal{C}$, as functions of q and $a = q^{2t}$, irrespective of tail \mathcal{C} . The formula in the statement is now obtained from the Renormalization Axiom. \square

5.3 A variation on Calegari-Guo

In this section we adapt the arguments of [CG15] to prove the following number theoretic result, with a very helpful consequence.

Theorem 5.11. *Suppose $t \geq 0$, and $q > 1$ satisfies*

$$P_t(q) = q^{2t}X_1(q) + X_0(q) + q^{-2t}X_1(q^{-1}) = 0,$$

where X_0 and X_1 are the Laurent polynomials defined in Lemma 5.10. Then the quantity $(q + q^{-1})^2$ is not a totally real cyclotomic integer.

Remark 5.12. We write q_t for the largest real root of P_t , and q_∞ for the largest real root of X_1 . It is clear that $q_t \rightarrow q_\infty$, but we will need careful control of the convergence.

Before we prove this theorem, we use it to prove Lemma 5.4.

Proof of Lemma 5.4. Suppose (Γ_+, Γ_-) is a finite translated extension of $\mathcal{A}_{(2t)}$ which occurs as a subfactor principal graph. By [Jon87], the index of this subfactor must be equal to $\|\Gamma_\pm\|^2$. By the uniqueness of the Frobenius-Perron dimension function on the vertices, together with Lemma 5.10, $\|\Gamma_\pm\|^2 = (q + q^{-1})^2$ where q is a solution to Equation (5.1) which is greater than 1. Now by [CG94; ENO05], the index of a finite depth subfactor must be a totally real cyclotomic integer. But by Theorem 5.11, $(q + q^{-1})^2$ is not a totally real cyclotomic integer, a contradiction. \square

The proof of Theorem 5.11 occupies the remainder of this section. Some calculations are performed in the Mathematica notebook `CalegariGuoAdaptation.nb`, and the interested reader should look there for more details.

Recall Theorem 3.3 of [CG15] which we restate here with the parameter L fixed as $L = 3.16826$ (for which $B(L^2) < 0$).

Theorem. Let β be a totally real algebraic integer such that

- (1) β^2 is not Galois conjugate to $(\zeta_t + \zeta_t^{-1})^2$ for any $t \leq 52$.
- (2) The largest conjugate $|\bar{\beta}|$ of β is less than $L = 3.16826$.
- (3) At most M conjugates of β^2 lie outside the interval $[0, 4]$.

Then, $\mathcal{M}(\beta) = \frac{\text{Tr}_{\mathbb{Q}(\beta^2)/\mathbb{Q}}(\beta^2)}{[\mathbb{Q}(\beta^2):\mathbb{Q}]} < \frac{14}{5}$ or $[\mathbb{Q}(\beta^2) : \mathbb{Q}] < \frac{260}{11} \cdot M$.

Our plan is to apply this theorem with $\beta_t = (q_t + q_t^{-1})^2 - 2$ (recall q_t was defined in Remark 5.12). After showing that we can take $M = 1$, this will give us a dichotomy: either $\mathcal{M}(\beta_t) < \frac{14}{5}$ or $[\mathbb{Q}(\beta_t^2) : \mathbb{Q}] \leq 23$. In the first case, there are not many possibilities for β_t^2 being a cyclotomic integer, controlled by later results of [CG15]. In the second case, analysis of the rate of the convergence of $|\bar{\beta}_t|$ as $t \rightarrow \infty$ will give a bound on t , after which we have reduced the problem to a finite number of cases.

Lemma 5.13. The polynomial $P_t(q)$ has a unique root $q > 1$.

Proof. By Descartes' rule of signs, $P_t(q)$ has either 4, 2, or 0 positive roots. Easily, $q = 1$ is a double root. The polynomial $P_t(q)$ is reciprocal, so there is at most one root $q > 1$. Since $P_t(q) \rightarrow \infty$ as $q \rightarrow \infty$ and $P_t''(1) = -32(3t^2 + 24t + 52) < 0$, there is exactly one such root. \square

Lemma 5.14. The largest roots q_t of $P_t(q)$ are always smaller than q_∞ .

Proof. By definition $X_1(q_\infty) = 0$, so we just calculate $P_t(q_\infty) = X_0(q_\infty) + q_\infty^{-2t} X_1(q_\infty) = X_0(q_\infty) > 0$ (this can just be done numerically: $q_\infty \sim 1.67712$, and $X_0(q_\infty) \sim 21.2704$). The previous lemma, and the fact that $P_t(q)$ is eventually positive, gives the result. \square

Remark 5.15. Consider the function $f(q) = ((q + q^{-1})^2 - 2)^2$. This maps points q on the unit circle to $[0, 4]$, and points $q > 1$ to $(4, \infty)$. Complex numbers off the unit circle or real line are sent to points off the real line.

Since $\beta_t \in \mathbb{Q}(q_t)$, f gives a surjection from the Galois conjugates of q_t to the Galois conjugates of β_t^2 , and in particular from the Galois conjugates of q_t greater than 1 to the Galois conjugates of β_t^2 greater than 4, and the last two lemmas establish that condition (2) holds and condition (3) holds with $M = 1$ in Calegari-Guo's theorem.

Moreover, if q_t has Galois conjugates that are neither real nor on the unit circle, β_t cannot be totally real.

Lemma 5.16. The largest roots q_t converge monotonically to q_∞ .

Proof. Recall $P_t(q)$ has a unique root q_t greater than 1. It suffices to show $P_t(q_{t-1}) < 0$. We first observe that $X_0(q)$ and $X_1(q^{-1})$ are positive for $1 < q < q_\infty$, and then calculate

$$\begin{aligned} P_t(q_{t-1}) &= q^2 q^{2t-2} X_1(q_{t-1}) + X_0(q_{t-1}) + q^{-2t} X_1(q_{t-1}^{-1}) \\ &= -q^2 (X_0(q_{t-1}) + q^{-2t+2} X_1(q_{t-1}^{-1})) + X_0(q_{t-1}) + q^{-2t} X_1(q_{t-1}^{-1}) \\ &= (1 - q^2) X_0(q_{t-1}) + q^{-2t} (1 - q^4) X_1(q_{t-1}^{-1}) \\ &< 0. \end{aligned} \quad \square$$

Although we're staying purely in the number theoretic context for now, recall that $q_t + q_t^{-1}$ is actually the largest ℓ^2 -eigenvalue of the graph $\mathcal{A}_{(2t)}\mathcal{B}^\infty$, which easily gives both of the last two lemmas.

Condition (1) of Calegari-Guo's theorem is trivial as $|\zeta_t^k + \zeta_t^{-k}|^2 \leq 4$, while $|\beta_t^2| > |\beta_0^2| > 9$ by monotonicity.

We have now established that Theorem 3.3 of [CG15] applies to β_t .

Lemma 5.17. *We have $|q_\infty - q_t| < \frac{23}{7077}(2.802)^{-t} < \exp(-5 - t)$.*

Proof. By monotonicity, $q_t \in [q_0, q_\infty)$. We now follow the proof of [CG15, Lemma 6.3]. In this interval, we have the inequalities

$$\begin{aligned} |X_1(q)| &> 7077|q - q_\infty| \\ |X_0(q)| &< 22 \\ |X_1(q^{-1})| &< 1. \end{aligned}$$

As $q_t^{2t}X_1(q_t) + X_0(q_t) + q_t^{-2t}X_1(q_t^{-1}) = 0$, by the triangle inequality we obtain

$$q_0^{2t}7077|q_t - q_\infty| < q_t^{2t}|X_1(q_t)| \leq |X_0(q_t)| + q_t^{-2t}|X_1(q_t^{-1})| < 23,$$

giving the result, since $2.802 < q_0^2$. \square

Lemma 5.18. *If $\mathcal{M}(\beta_t) < 14/5$ then β_t is not a totally real cyclotomic integer.*

Proof. By [CG15, Proposition 4.3], a totally real cyclotomic integer β with $\mathcal{M}(\beta) < 14/5$ must either be a sum of two roots of unity (and hence less than 2) or $|\beta|$ must be on an explicit finite list. By Lemma 5.17, once $t \geq 2$, q_t is within 0.001 of q_∞ , and so β_t is within 0.01 of 3.17, and there are no such numbers on the list. \square

Lemma 5.19. *If $[\mathbb{Q}(\beta_t^2) : \mathbb{Q}] \leq 23$ and β_t is totally real, then $t \leq 66$.*

Proof. As q_t is not Galois conjugate to q_∞ , $X_1(q_t)$ is a non-zero algebraic number, and so

$$\prod_{\sigma \in \text{Gal}(\mathbb{Q}(\beta_t^2)/\mathbb{Q})} |\sigma X_1(q_t)| \geq 1,$$

being a positive integer.

For σ non-trivial, $|\sigma q_t| \leq 1$ by Remark 5.15, so $|\sigma X_1(q_t)| \leq 14$, and so

$$|X_1(q_t)| \geq \frac{1}{14^{22}}.$$

Applying Lemma 5.17, we see

$$\begin{aligned} |X_1(q_t)| &= |X_1(q_t) - X_1(q_\infty)| \\ &\leq 14|q_t^{16} - q_\infty^{16}| \\ &= 14|q_\infty - q_t||q_\infty^{15} + q_\infty^{14}q_t + \cdots + q_t^{15}| \\ &\leq 14 \cdot 16 \cdot q_\infty^{15} \cdot \exp(-5 - t). \end{aligned}$$

This gives

$$14 \cdot 16 \cdot q_\infty^{15} \cdot \exp(-5 - t) \geq \frac{1}{14^{22}}$$

and hence the result. \square

Finally we check the cases $t \leq 66$ individually, verifying that the totally real roots are not cyclotomic integers. This completes the proof of Theorem 5.11. \square

5.4 The tail enumerator and periodicity

Proof of Lemma 5.5. By Lemma 5.10, any subfactor with principal graph which is a translated extension of $\mathcal{A}_{(2t)}$ must have $(a = q^{2t}, q)$ satisfying Equation (5.1). By Lemma 5.16, $q < q_\infty$, and so the index of this subfactor must be at most $(q_\infty + q_\infty^{-1})^2 < 5.1683$.

We thus run the enumerator described in Section 4.1 on $\mathcal{A}_{(0)}$, up to index 5.1683, stopping whenever we reach depth 17. We find 6 vines (which aren't relevant now, as we're only interested in infinite depth extensions), and 20 graphs at depth 17. All of these 20 graphs truncate back to depth 14 as $\mathcal{A}_{(0)}\mathcal{B}$. \square

By running the tail enumerator described in Section 4.3 on the basic block \mathcal{B} for 12 depths, we observe the following.

Lemma 5.20. *Any extension of the basic block \mathcal{B} must either*

- *start with another basic block,*
- *be one of several cylinders, or*
- *terminate before depth 16, and in particular be finite depth.*

These computations are performed in the Mathematica notebook `r1TailEnumerator.nb` bundled with the `arXiv` source.

Corollary 5.21. *Any principal graph of the form $\mathcal{A}_{(2t)}\mathcal{B}\mathcal{C}$ where \mathcal{B} is the basic block has the form*

- $\mathcal{A}_{(2t)}\mathcal{B}^\infty$,
- $\mathcal{A}_{(2t)}\mathcal{B}^k\mathcal{C}$, for some k , where \mathcal{C} is a cylindrical tail, or
- $\mathcal{A}_{(2t)}\mathcal{B}^k\mathcal{D}$, for some k , where \mathcal{D} is a finite tail.

Combined with Lemma 5.4 we have proved Lemma 5.6. \square

5.5 Ruling out the infinite depth graph

We now consider the family of infinite depth graphs $\mathcal{A}_{(2t)}\mathcal{B}^\infty$. Suppose that we have a subfactor with principal graph $\mathcal{A}_{(2t)}\mathcal{B}^\infty$, with index $(q + q^{-1})^2$ (which may well be greater than the square of the graph norm of $\mathcal{A}_{(2t)}\mathcal{B}^\infty$ itself).

We introduce $a = q^{2t}$. Happily, the dimension of every bimodule may be computed as a rational function in a and q .

Lemma 5.22. *We have the following explicit formula for a^2 :*

$$a^2 = \frac{q^{20} - 4q^{18} - 6q^{16} - 4q^{14} + q^{12} - \sqrt{q^{20}(q^2 - 1)^2(q^2 + 1)^6(q^8 - 2q^6 - q^4 - 2q^2 + 1)}}{q^{20}(q^{12} - 2q^{10} - q^8 - 2q^6 - 3q^4 - 4q^2 - 1)} \quad (5.2)$$

Proof. By Lemma 5.10, we see that a satisfies Equation (5.1). Applying the quadratic formula, we get two possible solutions for a^2 . Only the one in Equation (5.2) is greater than 1 in the relevant range for q , i.e., $1.65 < q < q_\infty$, where 1.65 is less than the q such that $q + q^{-1}$ is the graph norm of $\mathcal{A}_{(0)}$, and q_∞ is as in Section 5.3. \square

Let α_m denote the dimension of the two bimodules on the first graph at depth $6 + 2t + 2m$. One easily sees that these satisfy the recursion $\alpha_{m+1} = ((q + q^{-1})^2 - 3)\alpha_m - \alpha_{m-1}$, and thus

$$\alpha_m = c_+ \left(\frac{1 - q^2 + q^4 + \sqrt{1 - 2q^2 - q^4 - 2q^6 + q^8}}{2q^2} \right)^m + c_- \left(\frac{1 - q^2 + q^4 - \sqrt{1 - 2q^2 - q^4 - 2q^6 + q^8}}{2q^2} \right)^m$$

for some constants c_+ and c_- (depending on t). We can solve for these constants (cf. the notebook code/r=1.nb), using

$$\alpha_0 = \frac{a^2 q^{16} - a^2 q^{14} - a^2 q^{12} - a^2 q^{10} + q^6 + q^4 + q^2 - 1}{2a q^6 (q^2 - 1)(q^2 + 1)}$$

and

$$\alpha_1 = \frac{a^2 q^{20} - 2a^2 q^{18} - a^2 q^{16} - a^2 q^{14} - a^2 q^{10} + q^{10} + q^6 + q^4 + 2q^2 - 1}{2a q^8 (q^2 - 1)(q^2 + 1)},$$

obtaining

$$c_+ = \frac{f_+(a, q)}{8a q^{10} (q^4 - 1) z} \quad \text{and} \quad c_- = \frac{f_-(a, q)}{2a q^6 (q^2 - 1)(q^2 + 1) z (-q^4 + q^2 + z - 1)}$$

where

$$\begin{aligned}
z &= \sqrt{q^8 - 2q^6 - q^4 - 2q^2 + 1} \\
f_+(a, q) &= z^2 (a^2 q^{10} (-q^{10} + 2q^8 + q^6 + q^4 + 1) - q^{10} - q^6 - q^4 - 2q^2 + 1) \\
&\quad + z (2a^2 q^{14} (q^6 - q^4 - q^2 - 1) + 2q^4 (q^6 + q^4 + q^2 - 1)) \\
&\quad + a^2 q^{10} (q^{18} - 4q^{16} + 4q^{14} - 3q^{12} + 2q^{10} - 2q^8 + 3q^6 - 2q^4 + 2q^2 - 1) \\
&\quad + q^{18} - 2q^{16} + 2q^{14} - 3q^{12} + 2q^{10} - 2q^8 + 3q^6 - 4q^4 + 4q^2 - 1 \\
f_-(a, q) &= z (a^2 q^{10} (-q^{10} + 2q^8 + q^6 + q^4 + 1) - q^{10} - q^6 - q^4 - 2q^2 + 1) \\
&\quad + a^2 q^{10} (q^{14} - 3q^{12} + 2q^6 + q^2 - 1) \\
&\quad + q^{14} - q^{12} - 2q^8 + 3q^2 - 1
\end{aligned}$$

Finally, substituting a from Equation (5.2) and simplifying, we obtain $c_+ = 0$, and note that for $q > \frac{1}{2}(1 + \sqrt{5})$, i.e. index greater than 5, we have

$$\frac{1 - q^2 + q^4 - \sqrt{1 - 2q^2 - q^4 - 2q^6 + q^8}}{2q^2} < 1,$$

so α_m is eventually less than 1, for large enough m . Thus there is no subfactor with principal graph $\mathcal{A}_{(2t)}\mathcal{B}^\infty$. This completes the proof of Lemma 5.7. \square

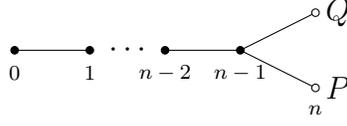
6 Ruling out weeds using branch factor inequalities

The results of [Jon12; Sny13; Pen15] give, for certain graph pairs $\Gamma = (\Gamma_+, \Gamma_-)$, a rational function $p(a, q)$ such that for a t -translated extension $\Gamma' = (\Gamma'_+, \Gamma'_-)$ of Γ to be the principal graph of a subfactor with index $(q + q^{-1})^2$, we must have $p_\Gamma(q^t, q) \leq 0$. Now, the index of a subfactor with principal graph Γ' must be at least $\|\Gamma'_\pm\|^2 \geq \|\Gamma_\pm\|^2$. Thus, if p_Γ is positive for all $q > q_0$ where $\|\Gamma\| = q_0 + q_0^{-1}$, then there are no possible subfactors with principal graph a translated extension of Γ . In this case, we say the result rules out the weed Γ .

In each of these results, as long as the annular multiplicities match some pattern, we obtain an inequality involving $q, a = q^{2t}$, and ratios of the relative dimensions of certain vertices on the graph $\Gamma = (\Gamma_+, \Gamma_-)$, called the *relative branch factors*. In fortunate circumstances, all the relative dimensions can be computed as functions of a and q , so we can easily write down the inequality $p_\Gamma(a, q) \leq 0$. In less fortunate circumstances, there are undetermined relative dimensions; sometimes, nevertheless, we can compute these relative dimensions from a doubly one-by-one connection entry. See Sections 6.1 and 6.3 for such examples. This often gives these unknown dimensions as functions of a, q which are no longer rational, and it is usually more work to apply the inequality.

We now recall the three results on branch factors. We assume that $\Gamma = (\Gamma_+, \Gamma_-)$ has

an initial triple point at depth $n - 1$:



We label the projections at depth n of Γ_+ by P and Q . The new low-weight rotational eigenvector perpendicular to $\mathcal{TLJ}_{n,+}$ is given by $A = rP - Q$, where $r = \text{Tr}(Q)/\text{Tr}(P)$ is the branch factor. The rotational eigenvalue is denoted ω_A .

Remark 6.1. If (Γ_+, Γ_-) is k -supertransitive, then the t -translated graph has $n = t + k + 1$.

Theorem 6.2 ([Jon12, Theorem 5.1.11]). *Suppose (Γ_+, Γ_-) is a $(n - 2)$ -translated extension of*



Then

$$\omega_A + \omega_A^{-1} = (r + r^{-1} - 2)[n][n + 2] - 2. \quad (6.1)$$

Theorem 6.3 ([Sny13, Theorem 3]). *Suppose (Γ_+, Γ_-) has an initial triple point at depth $n - 1$ where n is even, and Γ_- has a univalent vertex at depth n . Then Equation (6.1) holds.*

Corollary 6.4 (Triple-single branch factor inequality). *If (Γ_+, Γ_-) has an initial triple point at depth $n - 1$ where n is even, and Γ_- has a univalent vertex at depth n , then*

$$-4 \leq \omega_A + \omega_A^{-1} - 2 = (r + r^{-1} - 2)[n][n + 2] - 4 \leq 0.$$

The main result of [Pen15] is strictly stronger than those of [Jon12; Sny13], but we include only what we need for the annular multiplicity *11 weeds in this article. For the following theorem, Γ_{\pm} are both $(n - 2)$ -translated extensions of

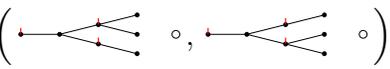


and P is chosen to be the bivalent vertex at depth n of Γ_+ (regardless of $\text{Tr}(P)$ and $\text{Tr}(Q)$). When n is even, we choose \check{P} to be the trivalent vertex at depth n of Γ_- , and when n is odd, we choose \check{P} to be the bivalent vertex at depth n of Γ_- .

Again, $A = rP - Q$, and $\check{A} = \check{r}\check{P} - \check{Q}$ where $r = \text{Tr}(Q)/\text{Tr}(P)$ and $\check{r} = \text{Tr}(\check{Q})/\text{Tr}(\check{P})$. We must have that the one click rotation of A is equal to $\frac{\sqrt{r}}{\sqrt{\check{r}}}\sigma_A\check{A}$ where σ_A is some $2n$ -th root of unity with $\sigma_A^2 = \omega_A$.

Theorem 6.5 ([Pen15, Theorem 3.10]). *Suppose Γ_{\pm} are both $(n - 2)$ -translated extensions of*



(1) *If n is even, (Γ_+, Γ_-) is a translated extension of*  *and*

$$\sigma_A + \sigma_A^{-1} = \frac{\sqrt{\check{r}}}{\sqrt{r}}[n + 2] - \frac{\sqrt{r}}{\sqrt{\check{r}}}[n].$$

(2) If n is odd, (Γ_+, Γ_-) is a translated extension of $\left(\begin{array}{c} \circ \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \circ, \begin{array}{c} \circ \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \circ \right)$,
and

$$\sigma_A + \sigma_A^{-1} = \frac{[n+2]}{r} - r[n].$$

Remark 6.6. The two formulas in Theorem 6.5 are really the same using an alternate definition of the branch factor. If we let s_{\pm} be the trace of the trivalent vertex of Γ_{\pm} divided by the trace of the bivalent vertex of Γ_{\pm} , then both formulas can be written as

$$\sigma_A + \sigma_A^{-1} = \frac{[n+2]}{\sqrt{s_+ s_-}} - \sqrt{s_+ s_-} [n]. \quad (6.2)$$

Indeed, we always have $s_+ = r$; when n is even $s_- = 1/\check{r}$, and when n is odd $s_- = s_+ = r$.

Squaring both sides of Equation (6.2), we have

$$\frac{[n+2]^2}{s_+ s_-} - 2[n+2][n] + s_+ s_- [n]^2 = \omega_A + \omega_A^{-1} + 2 \in [0, 4], \quad (6.3)$$

so subtracting 4, we get the following:

Corollary 6.7 (Non-univalent *11 branch factor inequality).

$$-4 \leq \omega_A + \omega_A^{-1} - 2 = \frac{[n+2]^2}{s_+ s_-} - 2[n+2][n] + s_+ s_- [n]^2 - 4 \leq 0.$$

Corollary 6.7 was used in [Pen15, Theorem 3.17] to eliminate weed (2) in part (e) of Theorem 3.1, which has annular multiplicities *11 (see Section 6.2). Corollary 6.4 was used in [MPPS12, Section 4] to eliminate two weeds in part (f) of Theorem 3.1, which have annular multiplicities *10 (see Section 6.4).

These weeds were eliminated by determining the relative dimensions of the vertices as functions of q and $a = q^{2t}$. Next, the relative branch factors were computed, which are the expressions for r, \check{r} as functions of a and q . Finally, we use Corollaries 6.4 and 6.7 to write $\omega_A + \omega_A^{-1} - 2 = p_{\Gamma}(a, q)$, and we show p_{Γ} is always positive for $a \geq q^{2t_0}$ and $q > q_0$ where q_0 is determined by the graph norm of a $2t_0$ -translate of the weed in question. This eliminates all $2t$ -translates of extensions of the weed for $t \geq t_0$. For the *11 weed eliminated in [Pen15, Theorem 3.17], $t_0 = 0$, which eliminates all translated extensions. However, for the two *10 weeds eliminated in [MPPS12, Section 4], $t_0 = 1, 2$ respectively, and additional arguments were supplied for these small translates.

The following trick was used in [Pen15] to verify positivity for certain polynomials $p(a, q)$.

Definition 6.8. Given a polynomial p in variables a and q

$$p(a, q) = \sum_{i=0}^k p_i(q) a^i$$

we say that p is *obviously positive* for $q > q_0$ if each single variable polynomial

$$\sum_{i=j}^k p_i(q) \text{ for } j = 0, \dots, k$$

has positive leading coefficient, and the largest root of any of them is at most q_0 .

We say a rational function in a, q is obviously positive for $q > q_0$ if each irreducible factor, of either the numerator or denominator, is obviously positive.

This condition is sufficiently straightforward to check that we make such claims without explicit proofs. The following lemma is ‘obvious’.

Lemma 6.9. *If p is obviously positive for $q \geq q_0$, then $p(a, q)$ is positive for all $q > q_0$ and $a \geq 1$.*

In the subsequent subsections, we suppress most of the calculations, which are straightforward. These calculations are performed in the Mathematica notebook `Weeds.nb`, bundled with the `arXiv` source.

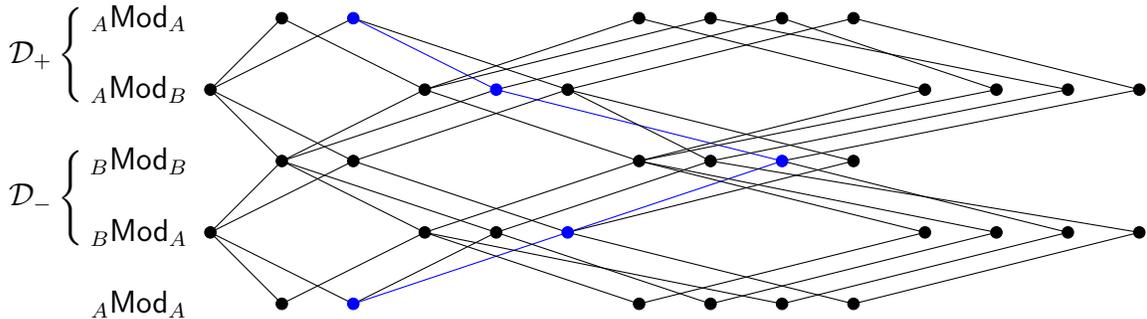
6.1 Ruling out a particular *11 weed

We begin by eliminating a particularly difficult *11 weed: number (5) of part (e) of Theorem 3.1.

Theorem 6.10. *There is no subfactor whose principal graphs are a translated extension of*

$$\mathcal{D} = (\mathcal{D}_+, \mathcal{D}_-) = \left(\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \circ, \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \circ \right).$$

We would like to eliminate this weed using the technique outlined in the previous section, but when solving for the relative dimensions, we have one unknown parameter: $\dim(V_{5+2t,3}^d)$. Luckily, as in Section 5.2, the loop $(V_{2t+4,2}^p, V_{2t+5,2}^p, V_{2t+6,3}^d, V_{2t+5,3}^d)$ outlined in blue below in $\mathcal{O}(\mathcal{D})$ between depths $2t + 3$ and $2t + 7$ corresponds to a doubly one-by-one connection entry.



This doubly one-by-one entry gives us the following formula for our undetermined relative dimension:

$$\dim(V_{5+2t,3}^d) = \frac{1 + K - 2q^2 + 3q^4 + q^8 + a^2(-q^8 - 3q^{12} + 2q^{14} - q^{16})}{2a(-1 + q)q^3(1 + q)(1 + 4q^4 + q^8)}$$

where $K > 0$ such that

$$\begin{aligned} K^2 = & a^4 (q^{32} + 4q^{30} + 14q^{28} + 24q^{26} + 27q^{24} + 20q^{22} + 10q^{20} + 4q^{18} + q^{16}) \\ & + a^2 (-2q^{24} - 8q^{22} - 20q^{20} - 44q^{18} - 62q^{16} - 44q^{14} - 20q^{12} - 8q^{10} - 2q^8) \\ & + q^{16} + 4q^{14} + 10q^{12} + 20q^{10} + 27q^8 + 24q^6 + 14q^4 + 4q^2 + 1. \end{aligned}$$

Remark 6.11. There is another solution for $\dim(V_{5+2t,3}^d)$ with $-K$ instead of $+K$; this is always negative for $a, q \geq 1$, which is impossible.

Now applying the technique of the previous section, Corollary 6.7 tells us the following inequality must be satisfied:

$$\frac{4(aq^5 + 1)^2(aq^5 - 1)^2(\alpha(a, q) - K\beta(a, q))(\alpha(-a, q) - K\beta(-a, q))}{a^2(q + 1)^2(q - 1)^2q^{10} \prod_{i=1}^4 (\gamma_i(a, q) - K\delta_i(a, q))} \leq 0 \quad (6.4)$$

where

$$\begin{aligned} \alpha(a, q) = & a^4 (q^{42} + q^{40} + 4q^{38} - 7q^{36} - 5q^{34} - 13q^{32} + q^{30} + 8q^{28} + 17q^{26} + 16q^{24} + 8q^{22} + 4q^{20} + q^{18}) \\ & + a^3 (-2q^{36} - 4q^{34} - 14q^{32} - 14q^{30} - 16q^{28} + 10q^{26} + 38q^{24} + 28q^{22} + 26q^{20} + 14q^{18} + 4q^{16} + 2q^{14}) \\ & + a^2 (-q^{34} - 2q^{32} - 8q^{30} - 17q^{28} - 17q^{26} - 14q^{24} - 25q^{22} + 25q^{20} + 14q^{18} + 17q^{16} + 17q^{14} + 8q^{12} + 2q^{10} + q^8) \\ & + a (-2q^{28} - 4q^{26} - 14q^{24} - 26q^{22} - 28q^{20} - 38q^{18} - 10q^{16} + 16q^{14} + 14q^{12} + 14q^{10} + 4q^8 + 2q^6) \\ & - q^{24} - 4q^{22} - 8q^{20} - 16q^{18} - 17q^{16} - 8q^{14} - q^{12} + 13q^{10} + 5q^8 + 7q^6 - 4q^4 - q^2 - 1 \end{aligned}$$

$$\begin{aligned} \beta(a, q) = & a^2 (-q^{26} + q^{24} - q^{22} + 6q^{20} - 3q^{18} + 6q^{16} + q^{14} + 2q^{12} + q^{10}) \\ & + a (2q^{20} + 8q^{16} + 2q^{14} + 2q^{12} + 8q^{10} + 2q^6) \\ & + q^{16} + 2q^{14} + q^{12} + 6q^{10} - 3q^8 + 6q^6 - q^4 + q^2 - 1 \end{aligned}$$

$$\gamma_1(a, q) = a^2 (q^{16} + 2q^{14} + 5q^{12} + q^8) - q^8 - 5q^4 - 2q^2 - 1$$

$$\gamma_2(a, q) = a^2 (2q^{20} + 5q^{18} + 12q^{16} + 9q^{14} + 4q^{12} + q^{10}) - q^{10} - 4q^8 - 9q^6 - 12q^4 - 5q^2 - 2$$

$$\gamma_3(a, q) = a^2 (2q^{20} + q^{18} + 6q^{16} + 3q^{14} + 2q^{12} + q^{10}) - q^{10} - 2q^8 - 3q^6 - 6q^4 - q^2 - 2$$

$$\begin{aligned} \gamma_4(a, q) = & a^2 (2q^{24} + 4q^{22} + 9q^{20} + 14q^{18} + 17q^{16} + 10q^{14} + 5q^{12} + 2q^{10}) \\ & - 2q^{14} - 5q^{12} - 10q^{10} - 17q^8 - 14q^6 - 9q^4 - 4q^2 - 2 \end{aligned}$$

$$\delta_1(a, q) = -1$$

$$\delta_2(a, q) = -2q^4 - q^2 - 2$$

$$\delta_3(a, q) = q^2$$

$$\delta_4(a, q) = 2q^6 + q^4 + 2q^2$$

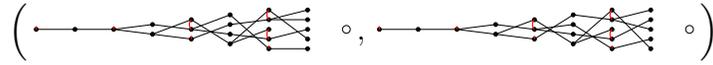
Lemma 6.12. *When $a \geq 1$ and $q \geq 1.65$, every factor in both the numerator and the denominator of Inequality (6.4) is positive. Hence for $a \geq 1$ and $q \geq 1.65$, Inequality (6.4) does not hold.*

Proof. This calculation is straightforward, and performed in Weeds . nb. □

Proof of Theorem 6.10. If such a subfactor existed, then by Corollary 6.7, Inequality (6.4) must hold. We note that the q for $(\mathcal{D}_+, \mathcal{D}_-)$ must be larger than 1.65 by looking at the graph norm. But by Lemma 6.12, Inequality (6.4) is never satisfied for the relevant range of a and q , a contradiction. □

6.2 Ruling out the remaining *11 weeds

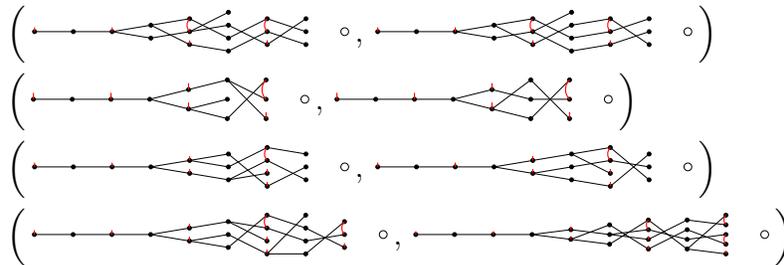
Recall that the weed



was eliminated in [Pen15, Theorem 3.17]. The argument there is a straightforward application of the method outlined in Section 6 by showing that each factor in the numerator and each factor in the denominator of the relative branch factor are obviously positive (see Lemma 6.9) for $a \geq 1$ and $q \geq 1.6789$, which is a lower bound for the q for the above weed. Hence no subfactor exists with principal graphs a translated extension, since Corollary 6.7 would not hold.

The argument for each weed in the following theorem is identical to [Pen15, Theorem 3.17], mutatis mutandis. The interested reader can view the necessary calculations in Weeds . nb.

Theorem 6.13. *No subfactor has principal graphs a translated extension of any of*

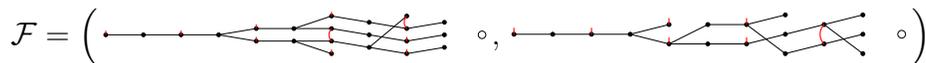


Now the previous theorem, [Pen15, Theorem 3.17], and Theorem 6.10 rule out all the *11 weeds in part (e) of Theorem 3.1. □

6.3 Ruling out two particular *10 weeds with doubly one-by-ones

6.3.1 A truncation of \mathcal{F} from [MPPS12]

In [MPPS12, Section 4.5], the problematic weed



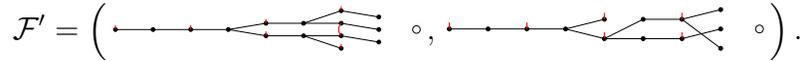
was ruled out using both branch factor inequalities and doubly one-by-one connection entries.

First, the relative dimensions can be computed in terms of q and $a = q^{2t}$, where $2t$ is the translation, which gives a formula for the relative branch factor. Then Branch Factor Inequality (6.1) shows that $t \leq 1$. Now for each of $t = 0$ and $t = 1$, we know ω_A is a $(2t + 4)$ -th root of unity. This allows us to solve directly for q in each case.

Second, there is a doubly one-by-one connection entry, which allows us to exactly solve for q in the cases $t = 0, 1$. It turns out that these values of q are incompatible with the values of q obtained from Branch Factor Inequality (6.1) for any allowed ω_A . Hence no translated extension of \mathcal{F} can be the principal graphs of a subfactor.

Working a bit harder, we can use the recent advances [Mor14; IMPPS15] to rule out the truncation \mathcal{F}' of \mathcal{F} by two depths.

Theorem 6.14. *There are no subfactors whose principal graphs are a translated extension of*



(This is case (f)(7) from Theorem 3.1.)

For \mathcal{F}' , we can no longer solve for all the relative dimensions; we have one unknown parameter: $\dim(V_{2t+5,2}^p)$. By the proof of [MPPS12, Proposition 4.17], we still have the same doubly-one-by-one connection entry, corresponding to the loop $(V_{2t+6,2}^p, V_{2t+5,1}^p, V_{2t+4}^d, V_{2t+5,2}^d)$. This yields the following equation for our unknown parameter:

$$\dim(V_{2t+5,2}^p) = \frac{-2 - q^2 - Kq^2 + q^6 + a^2(-q^{10} + q^{14} + 2q^{16})}{2a(-1 + q)q^5(1 + q)(1 + q^2)^2}$$

where $K > 0$ is the positive square root of

$$K^2 = 5 + 4q^2 + 6q^4 + 4q^6 + q^8 + a^2(-2q^8 - 8q^{10} - 20q^{12} - 8q^{14} - 2q^{16}) + a^4(q^{16} + 4q^{18} + 6q^{20} + 4q^{22} + 5q^{24}).$$

Remark 6.15. As in Remark 6.11 in Section 6.1, we get two solutions for $\dim(V_{2t+5,2}^p)$, where the other solution has $+K$ instead of $-K$. This time, both formulas give positive values for $\dim(V_{2t+5,2}^p)$, but this second solution with $+K$ gives a negative value for

$$\dim(V_{2t+5,1}^p) = \frac{a^2q^{10} - aq^5 \dim(V_{2t+5,2}^p) + 1}{aq^5}$$

when $q, a \geq 1$.

Applying the relative branch factor technique, Corollary 6.4 tells us that the following inequality must be satisfied:

$$\frac{4(-1 + a^2q^{10})^2(\alpha(a, q) - q^2K)(\alpha(-a, q) - q^2K)}{a^2(-1 + q)^2q^{10}(1 + q)^2(\gamma(a, q) + K)(\delta(a, q) - q^2K)} \leq 0 \quad (6.5)$$

where

$$\begin{aligned}\alpha(a, q) &= -1 - q^4 + a(q^4 + 2q^6 - 2q^{10} - q^{12}) + a^2(q^{12} + q^{16}) \\ \gamma(a, q) &= -3 - 2q^2 - q^4 + a^2(q^8 + 2q^{10} + 3q^{12}) \\ \delta(a, q) &= -2 - 3q^2 - 4q^4 - q^6 + a^2(q^{10} + 4q^{12} + 3q^{14} + 2q^{16})\end{aligned}$$

It is straightforward to calculate that every factor in both the numerator and the denominator of Inequality (6.5) is positive for $q > 1.567$ and $a \geq q^6$ (see Weeds .nb for more details). This immediately gives us the following lemma.

Lemma 6.16. *Any subfactor with principal graphs a $2t$ -translate of an extension of \mathcal{F}' must have $t \leq 2$.*

Proof. Suppose we have such a subfactor with $t \geq 3$. Then by Corollary 6.4, Inequality (6.5) must hold. We note that the q for \mathcal{F}' translated by 6 is larger than 1.567. Now all the factors in Equation (6.5) are positive when $a \geq q^6$ and $q > 1.567$. This shows that Inequality (6.5) is never satisfied for the relevant range of $a \geq q^6$ and $q > 1.567$, a contradiction. Hence $t \leq 2$. \square

We now deal with the cases that $t \leq 2$ by hand as in [MPPS12, Proposition 4.16], noting that there are only finitely many possibilities for the rotational eigenvalue ω_A in Corollary 6.4.

Proposition 6.17. *Any subfactor with principal graphs a translated extension of \mathcal{F}' either has*

- (1) $t = 0$ and $\text{index}(q + q^{-1})^2 = 3 + \sqrt{5}$, or
- (2) $t = 1, 2$ and $\text{index}(q + q^{-1})^2 \leq 5$.

Proof. Using the notation $D = \dim(V_{2t+5,1}^p)$ Equation (6.1) simplifies to

$$\frac{f(a, q, \omega)}{a^2 q^{10} (-1 + a^2 q^8 + aD(-q^3 + q^5)) (-1 + a^2 q^{12} + aD(q^5 - q^7))} = 0 \quad (6.6)$$

where

$$\begin{aligned}f(a, q, \omega) &= (a^2 q^8 - 1)(a^2 q^{12} - 1)(a^2 q^{10} - \omega)(a^2 q^{10} - \omega^{-1}) \\ &\quad + aDq^5(-1 + aq^5(D - aq^5)) \left(4 + 4a^4 q^{20} + a^2 q^8 \left((-1 + q^2)^2 (\omega + \omega^{-1}) - 2(1 + q^2)^2\right)\right)\end{aligned}$$

When $t = 0, 1, 2$, $a = 1, q^2, q^4$ respectively. We note that we must have $q > 1.558$, which comes from solving $\|\mathcal{F}'\| = q + q^{-1}$. Recall that if ω is a primitive k -th root of unity, then $2k \mid (2t + 4)$ by [Jon12, Theorem 5.1.11]. Using the fact that Equation (6.6) is symmetric in ω and ω^{-1} , it remains to look at the cases

$$(t, \omega) \in \{(0, 1), (0, -1), (2, 1), (2, e^{2\pi i/3}), (4, 1), (4, -1), (4, i)\}.$$

Proof. The root of $-4 + 15x - 8x^2 + x^3$ which is about 5.3234 is not a cyclotomic integer. \square

In fact, this weed can be completely ruled out up to index $5\frac{1}{4}$ only using enumeration! Of course this lemma is sufficient for the main result of this article, but we can get the much stronger result of Theorem 6.18 with some more work.

Lemma 6.20. *A subfactor with index at most $5\frac{1}{4}$ cannot have principal graphs a translated extension of \mathcal{G} .*

Proof. See the section titled *The (f)(8) weed from Theorem 3.1* in the `Mathematica notebook enumerator.nb`. \square

Together with the quadratic tangles technique and the main result of [Mor14] (Theorem 6.3.1), Lemma 6.20 actually reduces \mathcal{G} to the single case left in Theorem 6.18. Again, a quadratic tangles argument shows us that $t \leq 2$, and Theorem 6.3.1 says $t > 0$. However, the upper bounds for q when $t = 2$ or when $t = 1$ and $\omega = e^{\pm 2\pi i/3}$ are between 5 and $5\frac{1}{4}$, so Lemma 6.20 does the remaining work. While we could just ignore \mathcal{G} for now due to index considerations and Lemma 6.20, we will perform the extra analysis required to reduce \mathcal{G} to the one remaining case for future enumeration considerations.

As in the previous subsection, there are undetermined relative dimensions, in particular $\dim(V_{6,4}^p)$. (In this case, there is actually more than one undetermined dimension, but we need only determine one of them!) Luckily, there is a doubly-one-by-one entry of the connection corresponding to the loop $(V_{6,3}^p, V_{5,2}^p, V_{4,1}^d, V_{5,1}^d)$.

This gives us the following formula for one of our undetermined dimensions:

$$\dim(V_{6,4}^p) = \frac{-2 + 4q^2 + q^4 + 3q^6 + q^8 + K(-1 + q^2 - q^4) + a^2(-q^8 - 3q^{10} - q^{12} - 4q^{14} + 2q^{16})}{2a(-1 + q)q^4(1 + q)(1 + q^2)^3}$$

where $K > 0$ is the positive square root of

$$K^2 = 4 + 4q^4 + 4q^6 + q^8 + a^2(-4q^{10} - 18q^{12} - 4q^{14}) + a^4(q^{16} + 4q^{18} + 4q^{20} + 4q^{24}).$$

Remark 6.21. As in Remark 6.11, the other solution for $\dim(V_{6,4}^p)$ with $-K$ instead of K is impossible, since it is always negative.

Again, applying the relative branch factor technique, Corollary 6.4 tells us that the following inequality must be satisfied:

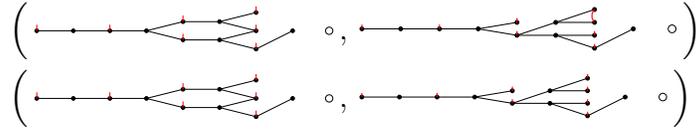
$$\frac{4(-1 + a^2q^{10})^2(\alpha(a, q) - q^2K)(\alpha(-a, q) - q^2K)}{a^2(-1 + q)^2q^{10}(1 + q)^2(\gamma(a, q) + K)(\delta(a, q) - q^2K)} \leq 0 \quad (6.7)$$

where

$$\begin{aligned} \alpha(a, q) &= -1 + q^2 - q^4 + a(-q^4 - 2q^6 + 2q^{10} + q^{12}) + a^2(q^{12} - q^{14} + q^{16}) \\ \gamma(a, q) &= -4 - 2q^2 - q^4 + a^2(q^8 + 2q^{10} + 4q^{12}) \\ \delta(a, q) &= -2 - 2q^2 - 4q^4 - q^6 + a^2(q^{10} + 4q^{12} + 2q^{14} + 2q^{16}) \end{aligned}$$

We now consider the case $t = 0$. Notice that this weed is actually stable, so that any extension must end with A_{finite} tails. By [LMP15, Proposition 1.17], these tails must not be longer than the initial arm of length 3, so there are only two possible extensions. The trivial extension is not possible for a number of reasons. For example, the index is not a cyclotomic integer, and there is a vertex dimension between 1 and 2 which is not of the form $2 \cos(\pi/k)$ for $k \geq 3$. The only other possibility is the $S_4 \subset S_5$ principal graph. \square

Theorem 6.23. *No subfactor has principal graphs a translated extension of either of*



Proof. In fact, these two weeds can be ruled out simultaneously. Each one has the same relative branch factor. Similar to the proof of Theorem 6.22, we can rule out all $2t$ -translated extensions for $t \geq 2$ by showing that Corollary 6.4 cannot hold. The interested reader can view this calculation in Weeds .nb.

We must now consider the case that $t \leq 1$. Again, these weeds are stable, so any extension must end with A_{finite} tails, and by [LMP15, Proposition 1.17], the tail cannot be longer than the initial arm. A simple calculation shows that the norm squared is only a cyclotomic integer when $t = 1$ and we extend stably by 3. At this point, we look at the common graph for both graph pairs:

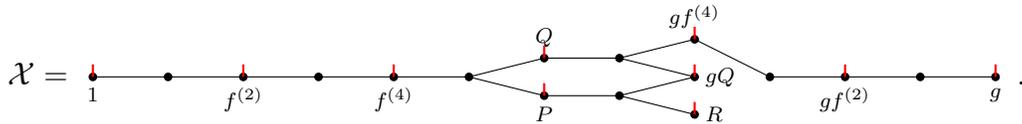


This graph cannot be the principal graph of a subfactor by Theorem 6.31 in the next section. \square

Now Theorems 5.1, 5.2 (with [CG15]), 5.3, 6.14, 6.18, 6.22, 6.23, and [MPPS12, Theorem 4.10] together rule out all the *10 weeds in part (f) of Theorem 3.1.

6.5 Ruling out a graph with formal codegrees

In this section, we use formal codegrees to rule out the graph



(We performed a graph isomorphism to make the fusion matrices in Appendix A.2 easier to interpret.)

Definition 6.24. Let \mathcal{C} be a fusion category over \mathbb{C} , and let $K_0(\mathcal{C})$ be its (Grothendieck) fusion ring. A dimension function on $K_0(\mathcal{C})$ is a ring homomorphism $K_0(\mathcal{C}) \rightarrow \mathbb{C}$. (Note that a dimension function gives a character on $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$, which is semi-simple [Lus87, 1.2(a)] and hence a multi-matrix algebra.)

Given a dimension function $\dim : K_0(\mathcal{C}) \rightarrow \mathbb{C}$, its *formal codegree* is

$$f_{\dim} = \sum_{X \in \text{Irr}(\mathcal{C})} |\dim(X)|^2,$$

where $\text{Irr}(\mathcal{C})$ is the set of isomorphism classes of simple objects of \mathcal{C} .

Remark 6.25. As $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a multi-matrix algebra, we can represent a fusion ring as a collection of matrices L_X for $X \in \text{Irr}(\mathcal{C})$ acting in the left regular representation, in the basis corresponding to $\text{Irr}(\mathcal{C})$. Suppose there is a common eigenvector v for all the fusion matrices L_X , i.e., $L_X v = \lambda_X v$ for some $\lambda_X \in \mathbb{C}$ for all $X \in \text{Irr}(\mathcal{C})$. We normalize v so that $v_1 = 1$. Letting $\varphi : \mathbb{C}v \rightarrow \mathbb{C}$ by $\lambda v \mapsto \lambda$, we get a dimension function by the formula

$$\dim(X) = \varphi(L_X v) = \lambda_X,$$

which is easily seen to be a ring homomorphism. Denoting the basis for $\text{Irr}(\mathcal{C})$ by $\{e_X\}$ with dual basis $\{e_X^*\}$, we have

$$\lambda_X = \lambda_X v_1 = \lambda_X e_1^*(v) = e_1^*(\lambda_X v) = e_1^*(L_X v) = e_{X^*}^*(v) = v_{X^*}.$$

In particular, if all objects of $\text{Irr}(\mathcal{C})$ are self-dual, then $v_X = \lambda_X$ for all $X \in \text{Irr}(\mathcal{C})$.

Example 6.26. For every fusion ring, there is unique simultaneous eigenvector for the left fusion matrices L_X such that every entry is strictly positive, normalized so that the entry corresponding to 1 is equal to 1. This gives rise to the Frobenius-Perron dimension function FPdim . The Frobenius-Perron dimensions of the even vertices of \mathcal{X} are given lexicographically by depth and height (bottom to top) by

$$(1, 2 + \sqrt{5}, 6 + 3\sqrt{5}, 10 + 4\sqrt{5}, 9 + 4\sqrt{5}, 4 + 2\sqrt{5}, 9 + 4\sqrt{5}, 6 + 3\sqrt{5}, 2 + \sqrt{5}, 1).$$

Remark 6.27. There are also formal codegrees for arbitrary irreducible representations of the fusion ring over \mathbb{C} [Ost09], but we only need the one-dimensional case here.

There are strong number theoretic properties of formal codegrees of fusion categories [Ost09; Ost13]. To eliminate the graph \mathcal{X} , we need the following result.

Theorem ([Ost13, Corollary 2.15]). *Let \mathcal{C} be a spherical fusion category. Then every formal codegree of $K_0(\mathcal{C})$ belongs to the number field generated by the dimensions of the objects of \mathcal{C} .*

In particular, if \mathcal{C} is pseudo-unitary, then the formal codegrees belong to the number field generated by the Frobenius-Perron dimensions of the objects of \mathcal{C} .

Example 6.28. Using the dimension function FPdim from Example 6.26, we see that the number field generated by the Frobenius-Perron dimensions of the even vertices of \mathcal{X} is $\mathbb{Q}(\sqrt{5})$.

We now find a dimension function on $K_0(\frac{1}{2}\mathcal{X})$, the fusion ring of the even part of \mathcal{X} whose formal codegree does not belong to $\mathbb{Q}(\sqrt{5})$.

We note that since all even vertices of \mathcal{X} are self-dual, we must have that $K_0(\frac{1}{2}\mathcal{X})$ is commutative. Also, the vertex g at depth 12 has dimension 1, so we must have that tensoring with g gives us a $\mathbb{Z}/2\mathbb{Z}$ -symmetry on the vertices. Thus giving the fusion rules amongst the vertices $1, f^{(2)}, f^{(4)}, P, Q, R$, all other fusion rules can be determined by commutativity together with $g^2 = 1$. Thus we have the following lemma:

Lemma 6.29. *The fusion ring of the even half of \mathcal{X} is completely determined by the fusion matrices given in Appendix A.2.*

Proof. To determine $K_0(\frac{1}{2}\mathcal{X})$, we use the `FusionAtlas` function `FindFusionRules`, and we keep the only solution with non-negative entries. This calculation is performed in `Weeds.nb`. \square

Lemma 6.30. *The map $\text{dim} : K_0(\frac{1}{2}\mathcal{X}) \rightarrow \mathbb{R}$ by*

$$\left(1, f^{(2)}, f^{(4)}, P, Q, R, gP, gf^{(4)}, gf^{(2)}, g\right) \mapsto \left(1, 1 + \sqrt{2}, 1 + \sqrt{2}, 1, 0, 0, -1, -1 - \sqrt{2}, -1 - \sqrt{2}, -1\right)$$

defines a dimension function on $K_0(\frac{1}{2}\mathcal{X})$.

Proof. The right hand side gives a simultaneous eigenvector for all the fusion matrices listed in Appendix A.2. We are finished by the discussion in Remark 6.25. \square

Theorem 6.31. *There is no subfactor whose principal graph is \mathcal{X} .*

Proof. We see the formal codegree of the dimension function in Lemma 6.30 is given by $f_{\text{dim}} = 16 + 8\sqrt{2} \notin \mathbb{Q}(\sqrt{5})$, which is the number field generated by the Frobenius-Perron dimensions of the vertices of \mathcal{X} by Example 6.28. Thus by [Ost13, Corollary 2.15], $K_0(\frac{1}{2}\mathcal{X})$ is not categorifiable, and thus \mathcal{X} is not the principal graph of a subfactor. \square

7 Ruling out 4-spokes

A 4-spoke, called a 4-star in [Sch90; IMPPS15], is a simply laced graph with a single central vertex with valence 4 and with all other vertices having valence at most 2. We denote a 4-spoke by $S(a, b, c, d)$, which has arms with a, b, c, d edges respectively connected to the central 4-valent vertex. If a 4-spoke is a component of the principal graph of a subfactor, then the distinguished vertex marked by \star must be on the end of the longest arm, cf. [LMP15, Proposition 1.17].

In [Sch90], Schou gave a complete list of 4-spokes Γ such that (Γ, Γ) has a biunitary connection. This is a necessary condition for (Γ, Γ) to be the principal graph pair of a subfactor.

Theorem 7.1 ([Sch90, p. 41]). *If $\Gamma = S(a, b, c, d)$ is a 4-spoke such that (Γ, Γ) has a biunitary connection, then Γ must be $S(1, 2, 2, 5)$ or one of:*

- $S(j, j, k, k)$ for $1 \leq j \leq k$
- $S(j, j + 1, j + 1, j + m)$ for $1 \leq j$ and $1 \leq m \leq 3$
- $S(j, j + 1, j + 2, j + m)$ for $1 \leq j$ and $2 \leq m \leq 4$
- $S(j, j + 2, j + 2, j + 2)$ for $1 \leq j$

Lemma 7.2. *Of the 4-spokes with biunitary connections, only the following have index in $(5, 5\frac{1}{4}]$:*

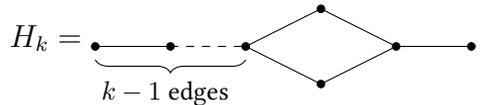
- $S(2, 2, k, k)$ for $k \geq 3$ and $S(3, 3, 3, 3)$,
- $S(2, 3, 3, 3)$, $S(2, 3, 3, 4)$, $S(2, 3, 3, 5)$,
- $S(2, 3, 4, 4)$, $S(2, 3, 4, 5)$, $S(2, 3, 4, 6)$,
- $S(2, 4, 4, 4)$

Proof. Recall that if Γ is a subgraph of Γ' , then $\|\Gamma\| \leq \|\Gamma'\|$. Since $\|S(3, 3, 3, 4)\|^2 > 5.25$, this gives an upper bound on each of the 4 families. For the lower bound of the first family, we note that $\|S(1, 1, k, k)\| < 5$ for all $k \in \mathbb{N}$. In fact, these 4-spokes were treated in [IJMS12]. For the lower bound of the second two families, we note that the norm of $S(1, 2, 3, 5)$ is less than 5. Finally, for the fourth family, we note that the norm of $S(1, 3, 3, 3)$ is exactly 5.

These computations are performed in the Mathematica notebook `4Spokes.nb`. □

Lemma 7.3. *Of the 4-spokes with biunitary connections and index in $(5, 5\frac{1}{4}]$, only $S(3, 3, 3, 3)$, and $S(2, 4, 4, 4)$ have norm squared which is a cyclotomic integer (and indeed, these both have index $3 + \sqrt{5}$).*

Proof. The only difficult case is the family $S(2, 2, k, k)$, but this is readily treated by the same argument as in [IJMS12, Section 4]. We see these graphs have the same norm c_k as the graphs



The field $\mathbb{Q}(c_k^2)$ is not cyclotomic for any $k \geq 3$; the argument in Section 8 shows that the adjacency matrix of H_k has a multiplicity free eigenvalue λ_k with $\mathbb{Q}(\lambda_k^2)$ not cyclotomic for all $k \geq 3$. Now the characteristic polynomials P_k for the adjacency matrix of H_k satisfy

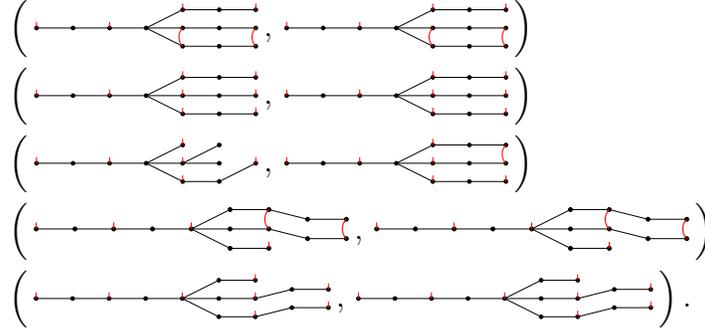
$$P_k(t + t^{-1})(t - t^{-1}) = t^k A(t) - t^{-k} A(t^{-1})$$

with $A(t) = t^7 - t^5 - 4t^3 - 3t - t^{-1}$. This polynomial has just two real roots with magnitude greater than 1, namely the square roots of the real root of $\mu^3 - 2\mu^2 - 2\mu - 1$, and so by Remark 10.1.7 of [CMS11], the polynomial $P_k(x)$ is $S(x^2)$ times a product of cyclotomic polynomials where S is a Salem polynomial. Thus λ_k must be Galois conjugate to c_k , giving the result.

The interested reader can view this calculation in the `Salem 4-spoke` section of the Mathematica notebook `4Spokes.nb`. \square

Finally, we need to analyze the possible dual data for $S(3, 3, 3, 3)$ and $S(2, 4, 4, 4)$. The function `FindGraphPartners` in the `FusionAtlas` package computes all possible graphs pairs with dual data containing a specified single graph without dual data (by applying the graph enumerator one depth at a time, discarding all branches which do not agree with the specified graph up to the current depth). We obtain

Lemma 7.4. *A subfactor principal graph containing an $S(3, 3, 3, 3)$ or $S(2, 4, 4, 4)$ must be amongst*



Here we don't care about the third case, as one of the graphs is not a 4-spoke (in fact, this graph is the principal graph of the $3^{\mathbb{Z}/4\mathbb{Z}}$ subfactor, which appears in Section 8). We easily rule out the fifth case, as the dimension 1 vertices would form a group of invertible bimodules of order three, with all objects involutions. The first case is ruled out by the following.

Lemma 7.5. *The first graph in Lemma 7.4 cannot be the principal graph of a subfactor, because it does not satisfy the conditions of [Pen15, Theorem 4.5].*

Proof. Let P be the self-dual vertex at depth 4 of Γ_+ , and let P' be the vertex at depth 5 of Γ_+ connected to P . Then $\overline{P'}$ is only connected to the self-dual vertex at depth 4 of Γ_- . By [Pen15, Theorem 4.5], we must have that $\delta \leq 2$, a contradiction. \square

Summarizing this section, we have

Theorem 7.6. *The only subfactor planar algebras with principal graphs both 4-spokes, and index in the interval $(5, 5\frac{1}{4}]$, are Izumi's $3^{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ planar algebra [Izu16; MP15] and the 4442 planar algebra [MP15; MP14b]. In each case, the principal graph is realized by just a single planar algebra.*

This finishes our treatment of case (g) from Theorem 3.1, and gives two of the subfactor standard invariants described in Theorem A.

8 Cyclotomicity of vines

We now perform the analysis of [CMS11; PT12] to determine which translations of the vines enumerated in Theorem 3.1 may be principal graphs of subfactors.

We say a graph is *cyclotomic* if for every multiplicity free eigenvalue λ of the adjacency matrix, the quantity λ^2 is a cyclotomic integer. (This is stronger than the requirement that the square of the graph norm is a cyclotomic integer, and is necessary for a graph to be the principal graph of a subfactor, by [CG94; ENO05].)

For each vine (Γ_+, Γ_-) , we have a bound $N(\Gamma) = \min\{N(\Gamma_+), N(\Gamma_-)\}$, where $N(\Gamma_{\pm})$ is a bound on the total number of vertices a translation of Γ_{\pm} may have and still potentially be cyclotomic. It is calculated according to the results of [CMS11], using the algorithm described in [PT12].⁴

We then look at each of the finitely many translates remaining, and check whether each has cyclotomic index. There are in every case very few translations which may have cyclotomic index, and all are significantly smaller than the bounds given by $N(\Gamma)$. We rule out all the other translations by explicitly finding a witness prime, modulo which the minimal polynomial factors into irreducible factors with different degrees. If we fail to find such a prime amongst the first 500 primes we say that the index may be cyclotomic. Although in this case we don't certify cyclotomicity, in practice these exceptions are always cyclotomic (and are either ruled out by easy obstructions or realized as principal graphs of subfactors).

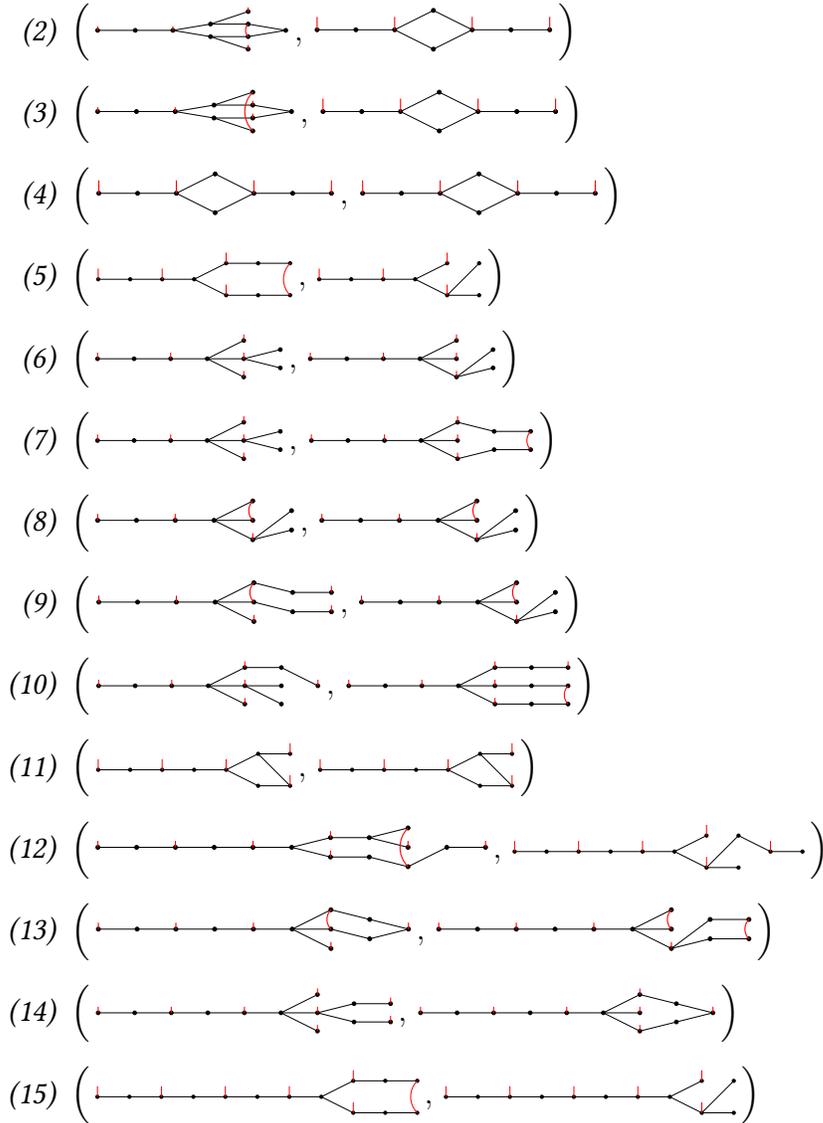
Certain optimizations are necessary to efficiently find all the minimal polynomials up to the bound. Observe that the minimal polynomial of the index is a factor of the characteristic polynomial of $A^t A$, with A the adjacency matrix of the graph. In fact, in practice we see that the set of irreducible factors of the quotient of this characteristic polynomial by the minimal polynomial of the index is periodic in the translation, although we do not know a proof. Nevertheless, this gives an efficient practical method for finding the minimal polynomials; we compute the first few minimal polynomials directly, observe the factors appearing in the quotient, and then for the tail we merely remove these factors from the easily computed characteristic polynomial, and verify the irreducibility of what remains.

Finally, we take each of the cyclotomic translates and run a few simple tests on the graphs, allowing us to rule out most of them as principal graphs of subfactors. All these computations are detailed in Appendix A.1, and give the following result.

Theorem 8.1. *The only possible principal graphs arising from the vines enumerated in Theorem 3.1, with index in the interval $(4, 21/4]$ are the following.*

$$(1) \left(\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \right), \left(\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\ \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \end{array} \right)$$

⁴Since the publication of [PT12], we discovered a potentially significant error in the code used in that paper, in particular in the BoundR1 function. While fixing this error was essential for the following calculations, fortuitously it did not change any of the claims of the original paper.



We note that for some of these principal graphs there is already a complete classification of subfactors realizing them. We summarize these here. The notation ‘2d’ in the ‘# of subfactors’ column indicates that there are 2 non-isomorphic subfactors which are dual to each other.

| principal graph | name | # of subfactors | citation |
|-----------------|--|-----------------|-----------------------|
| (1) | $U_{\exp(2\pi i/14)}(\mathfrak{su}_3)$ | 1 | [Wen88; Wen98; MP14c] |
| (2) | 2D2 | 2d | [MP14b] |
| (5) | the Haagerup | 2d | [AH99] |
| (7) | $A_4 \subset A_5$ | 2d | [IMPPS15] |
| (10) | $3^{\mathbb{Z}/4\mathbb{Z}}$ | 2d | [Izu16; PP15] |
| (12) | the Asaeda-Haagerup | 2d | [AH99] |
| (15) | the extended Haagerup | 2d | [BMPS12] |

Further, some of the graphs in Theorem 8.1 have already been ruled out as principal graphs of subfactors in other papers.

Theorem ([MP14b]). *There is a unique subfactor with principal graph $\Gamma_+ = \begin{array}{c} \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \\ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \end{array}$, and it must have dual graph $\Gamma_- = \begin{array}{c} \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \\ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \end{array}$. Thus the graph pairs numbered (3) and (4) above, are not principal graphs of subfactors.*

Theorem 8.2 ([IMPPS15, Lemma 3.10, Lemma 3.11, Proposition 3.12]). *There are no subfactors with the principal graphs numbered (6), (8), or (9).*

Finally a calculation based on checking associativity of the fusion ring concludes our treatment of these vines.

Lemma 8.3. *The remaining three graphs, numbered (11), (13), and (14) above, cannot be the principal graphs of subfactors, as there are no compatible fusion rings.*

A Appendices

A.1 Cyclotomicity bounds

We now display the cyclotomicity bounds for the vines discussed in Section 8, along with the potentially cyclotomic translates and the results of running simple tests on these. This table, and the computations underlying it, are constructed in the Mathematica notebook `processing-vines.nb` available with the arXiv sources of this article.

The first column shows the vine (Γ_+, Γ_-) . In the second column, we give the upper bound $\max N(\Gamma_{\pm})$ on the number of vertices appearing in a cyclotomic translate. The third column, named ‘CT’ for ‘cyclotomic translates’, shows those translations up to that bound which may be cyclotomic. The fourth column, named ‘Obstr.’ for ‘obstructions’, indicates if a simple obstruction can rule out each of the potentially cyclotomic translates.

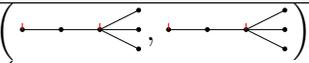
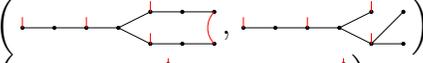
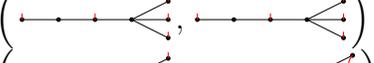
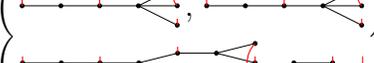
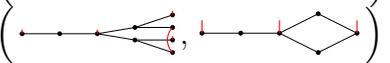
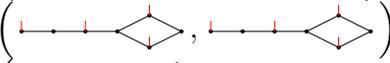
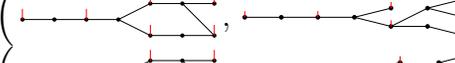
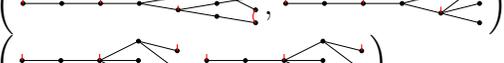
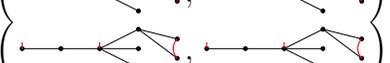
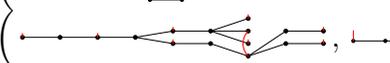
These obstructions are labelled as follows, with the most elementary ones coming first:

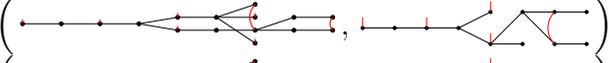
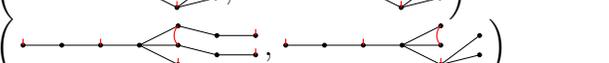
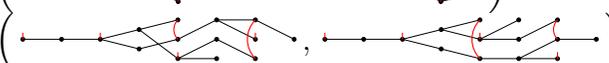
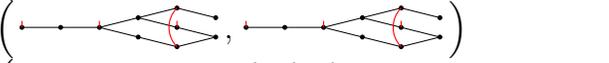
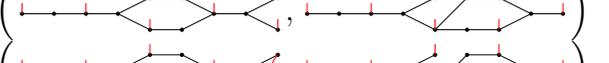
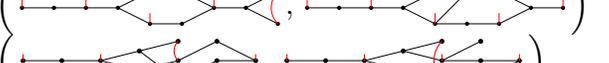
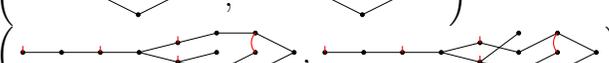
- (a) Some bimodule has dimension less than 1. (The index of a subfactor is bounded below by 1.)
- (b) Some bimodule has a dimension which is not an algebraic integer. (The index of a finite depth subfactor is an eigenvalue of an integer matrix.)
- (c) Some bimodule with dimension less than 2 has dimension not of the form $2 \cos(\pi/n)$ for $n \geq 3$, which is impossible by [Jon83].
- (d) Some low weight space would have negative dimension, as computed according to [Jon01, p. 33].

- (e) The global dimension of the even part (that is, the sum of the squares of the dimensions of vertices at even depths) is not an Ostrik d -number, which is a necessary condition by [Ost09].

Any potentially cyclotomic translate which is not ruled out by one of these obstructions is marked with a ‘?’.

In the above, we use the fact that the dimension of any bimodule is the square root of the index of the associated reduced subfactor [Jon87], and if the principal graph is finite depth then all the associated reduced subfactors are finite depth.

| vine | $N(\Gamma)$ | CT | Obstr. |
|---|-------------|-------------|-------------|
|  | 76 | {0} | {c} |
|  | 87 | {0,4} | {?,?} |
|  | 76 | {0} | {d} |
|  | 76 | {0} | {d} |
|  | 76 | {0} | {d} |
|  | 96 | {2} | {?} |
|  | 70 | {0,2} | {e,e} |
|  | 123 | {0,2} | {a,e} |
|  | 123 | \emptyset | \emptyset |
|  | 75 | {0} | {b} |
|  | 75 | {0} | {b} |
|  | 75 | {0} | {e} |
|  | 75 | {0} | {e} |
|  | 122 | {0} | {e} |
|  | 122 | {0} | {e} |
|  | 101 | \emptyset | \emptyset |
|  | 119 | \emptyset | \emptyset |
|  | 119 | \emptyset | \emptyset |
|  | 119 | \emptyset | \emptyset |
|  | 104 | \emptyset | \emptyset |
|  | 121 | \emptyset | \emptyset |
|  | 121 | \emptyset | \emptyset |

| | | | |
|---|-----|-------------|-------------|
|  | 121 | \emptyset | \emptyset |
|  | 121 | \emptyset | \emptyset |
|  | 225 | \emptyset | \emptyset |
|  | 131 | \emptyset | \emptyset |
|  | 119 | $\{0\}$ | $\{?\}$ |
|  | 119 | $\{0\}$ | $\{?\}$ |
|  | 119 | $\{0\}$ | $\{?\}$ |
|  | 119 | $\{0\}$ | $\{?\}$ |
|  | 200 | $\{0,2\}$ | $\{a,e\}$ |
|  | 200 | \emptyset | \emptyset |
|  | 113 | $\{0\}$ | $\{?\}$ |
|  | 216 | \emptyset | \emptyset |
|  | 113 | \emptyset | \emptyset |
|  | 120 | \emptyset | \emptyset |
|  | 120 | \emptyset | \emptyset |
|  | 147 | \emptyset | \emptyset |
|  | 261 | \emptyset | \emptyset |
|  | 111 | \emptyset | \emptyset |
|  | 111 | \emptyset | \emptyset |
|  | 111 | \emptyset | \emptyset |
|  | 109 | \emptyset | \emptyset |
|  | 109 | \emptyset | \emptyset |
|  | 137 | \emptyset | \emptyset |
|  | 110 | \emptyset | \emptyset |
|  | 110 | \emptyset | \emptyset |
|  | 110 | \emptyset | \emptyset |
|  | 100 | \emptyset | \emptyset |
|  | 147 | \emptyset | \emptyset |

| | | | |
|--|-----|-------------|-------------|
| | 120 | \emptyset | \emptyset |
| | 120 | \emptyset | \emptyset |
| | 120 | \emptyset | \emptyset |
| | 119 | \emptyset | \emptyset |
| | 155 | \emptyset | \emptyset |
| | 192 | \emptyset | \emptyset |
| | 348 | \emptyset | \emptyset |
| | 100 | \emptyset | \emptyset |
| | 100 | \emptyset | \emptyset |
| | 111 | \emptyset | \emptyset |
| | 244 | \emptyset | \emptyset |
| | 244 | \emptyset | \emptyset |
| | 244 | \emptyset | \emptyset |
| | 109 | $\{2\}$ | $\{?\}$ |
| | 110 | \emptyset | \emptyset |
| | 110 | \emptyset | \emptyset |
| | 218 | \emptyset | \emptyset |
| | 196 | \emptyset | \emptyset |
| | 120 | \emptyset | \emptyset |
| | 120 | \emptyset | \emptyset |
| | 345 | \emptyset | \emptyset |
| | 345 | \emptyset | \emptyset |
| | 345 | \emptyset | \emptyset |
| | 337 | \emptyset | \emptyset |
| | 174 | \emptyset | \emptyset |
| | 137 | \emptyset | \emptyset |
| | 180 | \emptyset | \emptyset |
| | 244 | \emptyset | \emptyset |

| | | | |
|--|-----|-----|-----|
| | 205 | {} | {} |
| | 434 | {} | {} |
| | 120 | {2} | {?} |
| | 120 | {2} | {?} |
| | 337 | {} | {} |
| | 214 | {} | {} |
| | 109 | {} | {} |
| | 162 | {} | {} |
| | 147 | {0} | {e} |
| | 205 | {} | {} |
| | 124 | {0} | {c} |
| | 109 | {0} | {?} |
| | 109 | {0} | {?} |
| | 137 | {0} | {?} |
| | 109 | {0} | {?} |
| | 171 | {0} | {a} |
| | 171 | {0} | {a} |
| | 298 | {0} | {b} |
| | 144 | {0} | {c} |
| | 144 | {0} | {a} |
| | 144 | {0} | {a} |
| | 243 | {0} | {a} |
| | 234 | {0} | {a} |
| | 267 | {} | {} |
| | 275 | {} | {} |
| | 307 | {} | {} |
| | 165 | {0} | {e} |
| | 165 | {0} | {e} |

A.2 The fusion ring of \mathcal{X}

Below, we give the fusion matrices L_X for tensoring on the left with $X \in \{f^{(2)}, f^{(4)}, P, Q, R, g\}$ for the fusion ring $K_0(\frac{1}{2}\mathcal{X})$ from Section 6.5 in the ordered basis

$$B = (1, f^{(2)}, f^{(4)}, P, Q, R, gP, gf^{(4)}, gf^{(2)}, g).$$

This means the (i, j) -th entry of L_X is the coefficient of X_i in $X \otimes X_j$, where X_k denotes the k -th element of B . For each object X , the fusion matrix for gX is $L_{gX} = L_g L_X = L_X L_g$, which is obtained from L_X by permuting the columns with the permutation $(10, 9, 8, 7, 5, 6, 4, 3, 2, 1)$.

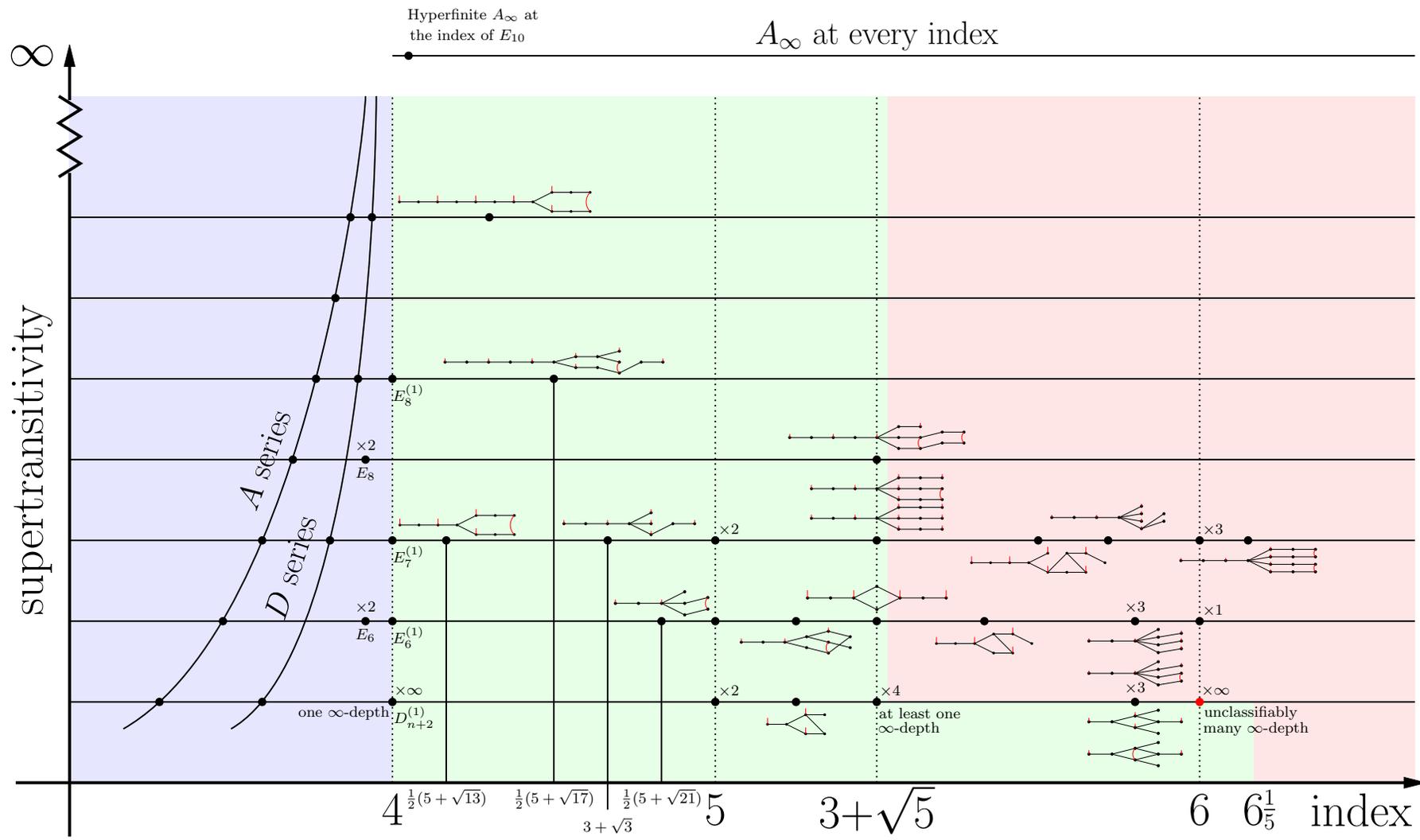
$$\begin{aligned}
 L_{f^{(2)}} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} &
 L_{f^{(4)}} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 3 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} &
 L_P &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 1 & 3 & 2 & 0 & 0 \\ 1 & 1 & 3 & 4 & 4 & 2 & 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 & 5 & 2 & 4 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 4 & 2 & 4 & 3 & 1 & 1 \\ 0 & 0 & 2 & 3 & 3 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 L_Q &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 4 & 5 & 2 & 4 & 3 & 1 & 0 \\ 1 & 1 & 3 & 5 & 4 & 2 & 5 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 3 & 4 & 5 & 2 & 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 & 5 & 2 & 4 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 & 3 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} &
 L_R &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} &
 L_g &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

A.3 Subfactors with index in $(4, 5\frac{1}{4}]$

Combining the results of this paper with the previous results on the classification of small index subfactors, we obtain the following complete list. The non Temperley-Lieb-Jones irreducible subfactor planar algebras with index in $(4, 5\frac{1}{4}]$ are:

| index | principal graph | name | # of subfactors | citations |
|--|-----------------|---|-----------------|---|
| $\frac{1}{2}(5 + \sqrt{13})$ ~ 4.37720 | | Haagerup | 2d | [AH99] |
| | | extended Haagerup | 2d | [BMPS12] |
| $\frac{1}{2}(5 + \sqrt{17})$ | | Asaeda-Haagerup | 2d | [AH99] |
| $3 + \sqrt{3}$ | | 3311 | 2d | existence [GHJ89], classification [Kaw95a; IJMS12] |
| $\frac{1}{2}(5 + \sqrt{21})$ | | 2221 | 2c | existence [Izu01], classification [Han10] |
| 5 | | $\mathbb{Z}/5\mathbb{Z}$ | 1 | classification [Izu97] |
| 5 | | $\mathbb{Z}/2\mathbb{Z} \subset D_{10}$ | 1 | — — |
| 5 | | $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ | 1 | — — |
| 5 | | $S_4 \subset S_5$ | 2d | classification [IMPPS15] |
| 5 | | $A_4 \subset A_5$ | 2d | — — |
| ~ 5.04892 | | $\mathfrak{su}(2)_5$ | 1 | existence [Wen88], classification [MP14c] |
| ~ 5.04892 | | $\mathfrak{su}(3)_4$ | 1 | — — |
| $3 + \sqrt{5}$ | | $A_3 \otimes A_4 = (A_3 * A_4)/\sim_1$ | 1 | classification [Liu15; IMP13] |
| $3 + \sqrt{5}$ | | $(A_3 * A_4)/\sim_2$ | 2d | existence [BH94], classification [Liu15; IMP13] |
| $3 + \sqrt{5}$ | | $(A_3 * A_4)/\sim_3$ | 2d | existence [IMP13] (due to Izumi), classification [Liu15; IMP13] |
| $3 + \sqrt{5}$ | | $A_3 * A_4$ | 2d | [BJ97] |
| $3 + \sqrt{5}$ | | 2D2 | 2d | existence [Izu16; MP14b], classification [MP14b] |
| $3 + \sqrt{5}$ | | $3\mathbb{Z}/4\mathbb{Z}$ | 2d | existence [Izu16; PP15], classification [Izu16] |
| $3 + \sqrt{5}$ | | $3\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | 1 | existence [Izu16; MP15], classification [Izu16] |
| $3 + \sqrt{5}$ | | 4442 | 1 | existence [MP15; Izu16], classification [MP14b] |

A.4 The map of subfactors



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