Exact analytical results for the electrostatic potential due to a uniformly charged finite rectangular plate

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We explain a general mathematical method that allows one to calculate the electrostatic potential created by a uniformly charged rectangular plate with arbitrary length and width at an arbitrary point in space. Exact analytical results for the electrostatic potential due to a uniformly charged finite rectangular plate are shown for special cases in order to illustrate the implementation of the formalism. Results of this nature are very important to various problems in physical sciences, applied mathematics and potential theory.

Keywords: Rectangular plate, Electrostatic potential, Potential theory, Uniform charge distribution, Coulomb potential.

I. INTRODUCTION

Calculation of the electrostatic potential created by charged body is one of the most important problems in potential theory and electrostatics [1–5]. However, in most cases, this problem for a charged body with arbitrary shape can be solved only numerically and not analytically. One reason for such an outcome has to do with the fact that the exact determination of the equilibrium charge distribution (that makes the body an equipotential) represents a very difficult mathematical problem to solve even for apparently simple cases such a straight finite wire [6–9]. Notable exceptions where the equilibrium charge distribution is analytically known are few trivial cases such as a conducting spherical surface or a conducting circular disk [10]. For these reasons, when dealing with those situations in which the distribution of charge cannot be calculated exactly, one resorts to approximations where the most common one is that of a uniform charge distribution. However, even for the case of a uniform charge distribution, an exact analytical calculation of the electrostatic potential is possible only for those bodies that have a regular shape and possess some symmetry. As typical examples where calculations are simple we may mention systems such as a spherical surface with uniform surface charge density, a solid sphere with uniform volume charge density [11] and few similar systems with either spherical or axial symmetry.

On the other hand, a uniformly charged rectangular plate represents a system without spherical or axial symmetry. For this reason, the calculation of its electrostatic potential at an arbitrary point in space represents a very challenging problem. Obtaining an analytic expression for such a case is of great importance since many electric and electronic devices contain charged, flat square/rectangular plates as their components. The most noteworthy example would be a parallel-plate capacitor consisting of two oppositely charged finite-sized square/rectangular [12] or circular plates [13, 14]. As well known, capacitors are key building blocks for any device that serve the main purpose of storing electric energy in circuits. Knowing the electrostatic potential creating by a plate gives one the opportunity to obtain the amount of the electrostatic energy stored in the plate as well as the electrostatic interaction energy between the two plates. In particular, the problem of the energy stored in a charged body, namely, calculation of its Coulomb self-energy [15, 16] is intrinsically connected to that of the electrostatic potential. For these reasons, in this work, we consider the problem of a uniformly charged finite rectangular plate and calculate in analytical closed form the electrostatic potential created by such a system at an arbitrary point in space. The solution method that we explain is general and rather elegant. The expressions obtained allow one to calculate the electrostatic potential at any arbitrary point where, in our opinion, direct integration methods may either not succeed, or are very difficult to apply.

II. MODEL AND RESULTS

The model under consideration is that of uniformly charged rectangular plate with arbitrary length and width, L_x and L_y, respectively. The constant surface charge density of the uniformly charged rectangular plate is written as:

$$\sigma = \frac{Q}{L_x\, L_y} \ , \tag{1}$$

where Q is the total charge uniformly spread on the surface of the rectangular plate. The Cartesian system of coordinates is chosen in such a way as to have its origin at the center (intersection of diagonals) of the rectangular plate. The x and y axes are taken parallel to the sides, L_x and L_y, respectively, with z perpendicular to the rectangular plate. A view of the system lying on the $z = 0$ plane is shown in Fig. 1.

With this choice of the coordinative system, one can write the expression for the Coulomb electrostatic poten-

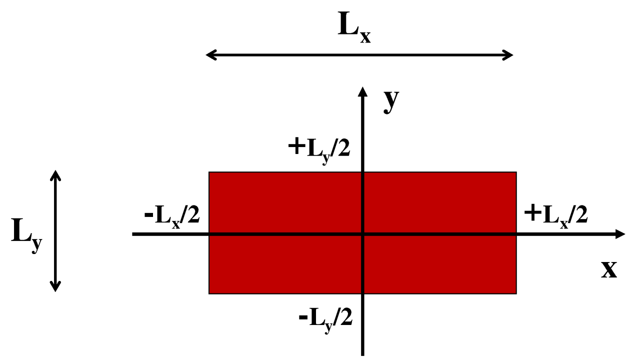

FIG. 1: Schematic view of a rectangular plate with uniform surface charge density lying in the $z = 0$ plane of a Cartesian system of coordinates (view from above). The z-axis (not shown) is perpendicular to the plate pointing towards the reader. The rectangular domain containing charge lies between the points $-L_{x,y}/2$ and $+L_{x,y}/2$ in the respective x and y directions. The rectangular plate has arbitrary length and width, respectively, L_x and L_y.

tial due to the rectangular plate as:

$$V(x,y,z,L_x,L_y) = k_e\,\sigma \int_{-L_x/2}^{+L_x/2} dx' \int_{-L_y/2}^{+L_y/2} dy'\ \frac{1}{|\vec{r}-\vec{r}\,'|}\ , \tag{2}$$

where k_e denotes the Coulomb's electric constant, $\vec{r} = (x,y,z)$ is an arbitrary point in space for the calculation of the potential, $\vec{r}\,' = (x\,',y\,',z\,' = 0)$ is the position vector of an elementary surface, $dx'dy'$ and the rectangular domain containing the charge has the form: $\Big\{ -L_x/2 \le x\,' \le +L_x/2\ ;\ -L_y/2 \le y\,' \le +L_y/2 \Big\}$. The electrostatic potential function, $V(x,y,z,L_x,L_y)$ shows L_x and L_y as explicit arguments in addition to the position coordinates, x, y and z. This is done on purpose with the intent of drawing close attention to the fact that we are dealing with a rectangular plate where lengths L_x and L_y are arbitrary. Based on this notation, the quantity $V(x,y,z,L,L)$ would represent the electrostatic potential created at an arbitrary point in space due to a uniformly charged square plate with length, L.

A useful Laplace-type transformation allows one to, quite generally, express $1/|\vec{r}-\vec{r}\,'|$ as:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{2}{\sqrt{\pi}} \int_0^\infty du\, e^{-u^2\,(\vec{r}-\vec{r}\,')^2}\ . \tag{3}$$

In order to see how the quantity in the right-hand-side (RHS) of Eq.(3) can be expressed as a Laplace transform, we introduce a new auxiliary variable, $t = u^2$ which allows us to write:

$$\frac{2}{\sqrt{\pi}} \int_0^\infty du\, e^{-u^2\,(\vec{r}-\vec{r}\,')^2} = \int_0^\infty dt\, e^{-t\,(\vec{r}-\vec{r}\,')^2}\, \frac{1}{\sqrt{\pi\, t}}\ . \tag{4}$$

We remind the reader that the Laplace transform of a function $f(t)$ is defined by

$$F(s) = \int_0^\infty dt\, e^{-s\,t}\, f(t)\ , \tag{5}$$

where s is a parameter and $F(s)$ is another function, for instance, see pg. 28 in Ref. [17]. Clearly, the quantity in the RHS of Eq.(4) can be seen as a Laplace transform of function $f(t) = 1/\sqrt{\pi\, t}$ with parameter, $s = (\vec{r}-\vec{r}\,')^2$. Though the integral is not very difficult, one may rely on standard tables of Laplace transforms, for instance, see pg. 32 in Ref. [17] to verify that the Laplace transform of $f(t) = 1/\sqrt{\pi\, t}$ is $F(s) = 1/\sqrt{s}$. This value is in full agreement with the original result in Eq.(3) since $s = (\vec{r}-\vec{r}\,')^2$ and, thus, $F(s) = 1/\sqrt{s} = 1/|\vec{r}-\vec{r}\,'|$. At this juncture, it is worth noting that the transformation in Eq.(3) was first introduced in a brief Letter [18] that dealt with the calculation of the electric potential of a uniformly charged square plate on its plane. It was pointed out in that Letter that several exact results previously presented in a paper by Aghamohammadi [19] could be obtained via a very different approach.

Based on the result from Eq.(3), the electrostatic potential under consideration can be written as:

$$V(x,y,z,L_x,L_y) = k_e\,\sigma\, \frac{2}{\sqrt{\pi}} \int_0^\infty du\, e^{-u^2\,z^2} \int_{-L_x/2}^{+L_x/2} dx'\, e^{-u^2\,(x-x')^2} \int_{-L_y/2}^{+L_y/2} dy'\, e^{-u^2\,(y-y')^2}\ . \tag{6}$$

Up to this point the treatment is straightforward and quite general. At this juncture, we introduce the following two auxiliary functions which are defined as:

$$f(u,x,L_x) = \int_{-L_x/2}^{+L_x/2} dx'\, e^{-u^2\,(x-x')^2}\ , \tag{7}$$

and

$$f(u,y,L_y) = \int_{-L_y/2}^{+L_y/2} dy'\, e^{-u^2\,(y-y')^2} \ . \qquad (8)$$

Aside differences in notation, these functions are similar in spirit to the one used to solve the problem of the electrostatic potential due to a uniformly charged straight wire [20]. One can calculate explicitly that:

$$f(u,x,L_x) = \frac{\sqrt{\pi}}{2\,u} \left\{ erf\left[u \left(\frac{L_x}{2} - x \right) \right] + erf\left[u \left(\frac{L_x}{2} + x \right) \right] \right\} \qquad (9)$$

where $erf(x) = (2/\sqrt{\pi}) \int_0^x dt\, e^{-t^2}$ is an error function. Obviously, one attains the function $f(u,y,L_y)$ by replacing x with y and L_x with L_y in Eq.(9).

This means that one can write the expression for the electrostatic potential in Eq.(6) in a very compact way as:

$$V(x,y,z,L_x,L_y) = k_e\,\sigma\, \frac{2}{\sqrt{\pi}} \int_0^{\infty} du\, e^{-u^2\,z^2}\, f(u,x,L_x)\, f(u,y,L_y) \ . \qquad (10)$$

The one-dimensional integral expression in Eq.(10) represents a concise and a convenient general result which allows one to calculate the electrostatic potential due to a uniformly charged finite rectangular plate at any arbitrary point in space. The only remaining task left at this point is to calculate the resulting integrals which are either available in the mathematical literature or can be calculated analytically.

In order to illustrate this point, let us show some exact analytical results for two selected cases, $V(0,0,0,L_x,L_y)$ and $V(0,0,z,L_x,L_y)$. With little mathematical effort, one can see that:

$$V(0,0,0,L_x,L_y) = k_e\,\sigma\, 2\sqrt{\pi} \int_0^{\infty} du\, \frac{erf(u\,\frac{L_x}{2})\, erf(u\,\frac{L_y}{2})}{u^2} \ . \qquad (11)$$

The integral appearing in Eq.(11) has the form:

$$I(a,b) = \int_0^{\infty} dx\, \frac{erf(a\,x)\, erf(b\,x)}{x^2} = \frac{2}{\sqrt{\pi}} \left[a\, \sinh^{-1}\left(\frac{b}{|a|}\right) + b\, \sinh^{-1}\left(\frac{a}{|b|}\right) \right] , \qquad (12)$$

where $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$ is an inverse hyperbolic sine function and a, b are considered to be real constants (one must be careful to write $\sqrt{a^2} = |a|$ and $\sqrt{b^2} = |b|$ for arbitrary real a and b). Note that both the error function and the inverse hyperbolic sine function are odd functions. Thus, one can easily check that:

$$I(a,b) = -I(-a,b) \ \ , \ \ I(a,b) = -I(a,-b) \ . \qquad (13)$$

The use of the formula in Eq.(12) immediately leads to the expression:

$$V(0,0,0,L_x,L_y) = k_e\,\sigma \left[2\,L_x\, \sinh^{-1}\left(\frac{L_y}{L_x}\right) + 2\,L_y\, \sinh^{-1}\left(\frac{L_x}{L_y}\right) \right] . \qquad (14)$$

The process is clear in case one wants to express the final result in terms of the total charge, Q instead of the constant surface charge, σ. For such a situation, one uses the fact that $\sigma = Q/(L_x\,L_y)$, and from here one proceeds to obtain:

$$V(0,0,0,L_x,L_y) = k_e\,Q\left[\frac{2}{L_y}\,\sinh^{-1}\left(\frac{L_y}{L_x}\right) + \frac{2}{L_x}\,\sinh^{-1}\left(\frac{L_x}{L_y}\right)\right]\,. \tag{15}$$

One may find convenient to express the value of the electrostatic potential in terms of, what we call, the "symmetric" unit, $k_e\,Q/\sqrt{L_x\,L_y}$ resulting in the following expression:

$$V(0,0,0,L_x,L_y) = \frac{k_e\,Q}{\sqrt{L_x\,L_y}}\left[2\sqrt{\frac{L_x}{L_y}}\,\sinh^{-1}\left(\frac{L_y}{L_x}\right) + 2\sqrt{\frac{L_y}{L_x}}\,\sinh^{-1}\left(\frac{L_x}{L_y}\right)\right]\,. \tag{16}$$

The result for a square plate ($L_x = L_y = L$) readily follows:

$$V(0,0,0,L,L) = \frac{k_e\,Q}{L}\,4\,\sinh^{-1}(1) \approx 3.52549\,\frac{k_e\,Q}{L}\,. \tag{17}$$

Another advantage of using the "symmetric" unit is that one can easily compare the electrostatic potential of a square plate to that of a rectangular plate with the same surface area. For instance, let us assume that we have a rectangular plate with length $L_x = 2\,L$ and width $L_y = L/2$. Note that this rectangular plate has an area, $L_x\,L_y = L^2$ identical to that of a square plate with length, L. By using Eq.(16) for $L_x/L_y = 4$ one easily sees that:

$$V(0,0,0,2\,L,\frac{L}{2}) \approx 3.08458\,\frac{k_e\,Q}{L}\,. \tag{18}$$

The fact that a uniformly charged rectangular plate creates a smaller potential (in this case at the center) than its square counterpart with the same surface area is easy to understand. Obviously, given the same amount of total charge and the same surface charge density (since the square plate and the rectangular plate have the same area), there is more charge of the rectangular plate situated away from the center as the case of its square counterpart. This leads to a somewhat smaller electrostatic potential created by the rectangular plate at the center as shown by the the result in Eq.(18).

The calculation of the electrostatic potential along the z-axis leads to a slightly more complicated integral as can be seen below:

$$V(0,0,z,L_x,L_y) = k_e\,\sigma\,2\,\sqrt{\pi}\int_0^\infty du\,e^{-u^2\,z^2}\,\frac{erf(u\,\frac{L_x}{2})\,erf(u\,\frac{L_y}{2})}{u^2}\,. \tag{19}$$

The integral appearing in the above expression has the following mathematical form that has been solved analytically:

$$F(a,b,c) = \int_0^\infty dx\,e^{-a^2x^2}\,\frac{erf(b\,x)\,erf(c\,x)}{x^2} = \frac{2}{\sqrt{\pi}}\left[b\,\sinh^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) + c\,\sinh^{-1}\left(\frac{b}{\sqrt{a^2+c^2}}\right) - a\,\tan^{-1}\left(\frac{b\,c}{a\,\Delta}\right)\right]\,, \tag{20}$$

where $\tan^{-1}(x)$ is an inverse tangent function, a, b, c are considered to be real constants and

$$\Delta = \sqrt{a^2+b^2+c^2}\,, \tag{21}$$

is an auxiliary parameter. Details of the calculation of the above integral can be found in the Appendix B of an earlier work [21].

By applying the formula from Eq.(20) to the integral expression appearing in Eq.(19), one eventually obtains the result:

$$V(0,0,z,L_x,L_y) = k_e\,\sigma\left[2\,L_x\,\sinh^{-1}\left(\frac{L_y}{\sqrt{4\,z^2+L_x^2}}\right)+2\,L_y\,\sinh^{-1}\left(\frac{L_x}{\sqrt{4\,z^2+L_y^2}}\right)-4\,z\,\tan^{-1}\left(\frac{L_x\,L_y}{2\,z\,\sqrt{4\,z^2+L_x^2+L_y^2}}\right)\right]. \tag{22}$$

By recalling that Coulomb's electric constant can be written as $k_e = 1/(4\,\pi\,\epsilon_0)$, one may rewrite the result in Eq.(22) in the following form:

$$V(0,0,z,L_x,L_y) = \frac{\sigma}{\pi\,\epsilon_0}\left[\frac{L_x}{2}\,\sinh^{-1}\left(\frac{\frac{L_y}{2}}{\sqrt{z^2+\left(\frac{L_x}{2}\right)^2}}\right)+\frac{L_y}{2}\,\sinh^{-1}\left(\frac{\frac{L_x}{2}}{\sqrt{z^2+\left(\frac{L_y}{2}\right)^2}}\right)-z\tan^{-1}\left(\frac{\frac{L_x}{2}\,\frac{L_y}{2}}{z\,\sqrt{z^2+\left(\frac{L_x}{2}\right)^2+\left(\frac{L_y}{2}\right)^2}}\right)\right] \tag{23}$$

This expression, with the proper substitutions of $L_x = 2\,a$ and $L_y = 2\,b$, is in agreement with the formula found in Eq.(3) of a recent work by Fagundes [22] which uses a different method of calculation.

The prior result was obtained via direct integration techniques through careful integration by parts, trigonometric substitution and partial fraction decomposition with the details of such a calculation covering around four pages in Appendix A of that work [22]. In contrast, the present approach avoids a direct integration of the defining expression of the electrostatic potential in Eq.(2). Instead, we choose to transform such an expression in a way as to obtain a more convenient one-dimensional integral presentation as provided by the quantity in Eq.(10). There is only one type of integral appearing in the quantity in Eq.(10). This integral is of the general form given by Eq.(20). This means that we have the discretion to use only one integral formula to obtain the result in Eq.(22) for $V(0,0,z,L_x,L_y)$ as well as calculate $V(x,y,z,L_x,L_y)$, which is going to be very long, with the added benefit that such an integral formula is already available from the literature as shown in Eq.(20).

If one wants to express the result in Eq.(22) in terms of the total charge, Q one substitutes $\sigma = Q/(L_x\,L_y)$ to obtain:

$$V(0,0,z,L_x,L_y) = k_e\,Q\left[\frac{2}{L_y}\,\sinh^{-1}\left(\frac{L_y}{\sqrt{4\,z^2+L_x^2}}\right)+\frac{2}{L_x}\,\sinh^{-1}\left(\frac{L_x}{\sqrt{4\,z^2+L_y^2}}\right)-\frac{4\,z}{L_x\,L_y}\,\tan^{-1}\left(\frac{L_x\,L_y}{2\,z\,\sqrt{4\,z^2+L_x^2+L_y^2}}\right)\right]. \tag{24}$$

As illustrated earlier for the case of $V(0,0,0,L_x,L_y)$ in Eq.(16), one can express the electrostatic potential, $V(0,0,z,L_x,L_y)$ in terms of the "symmetric" unit of $k_e\,Q/\sqrt{L_x\,L_y}$. In Fig. 2, we show the dependence of the electrostatic potential, $V(0,0,z,L_x,L_y)$ (in units of $k_e\,Q/\sqrt{L_x\,L_y}$) as a function of the dimensionless distance, $|z|/\sqrt{L_x\,L_y}$ for the cases of a square plate with $L_x = L$ and $L_y = L$ (solid line) and a rectangular plate with $L_x = 2\,L$ and $L_y = L/2$ (dotted line). Note that the values of L_x and L_y above are chosen in such a way that both the square and the rectangular plate have the same surface area. As can be already deduced from the results in Eq.(17) and Eq.(18) one can conclude that, for equal charge and equal surface area, the electrostatic potential of the uniformly charged rectangular plate along the z direction is always smaller than that of the square counterpart. Obviously, the electrostic potential becomes Coulomb-like in the $|z| \to \infty$ limit (for fixed finite L_x and L_y).

The result for a square plate ($L_x = L_y = L$) reads:

$$V(0,0,z,L,L) = \frac{k_e\,Q}{L}\left[4\,\sinh^{-1}\left(\frac{L}{\sqrt{4\,z^2+L^2}}\right) - \frac{4\,z}{L}\,\tan^{-1}\left(\frac{L^2}{2\,z\,\sqrt{4\,z^2+2\,L^2}}\right)\right]. \tag{25}$$

Note the expression in Eq.(25) is equivalent and can be written as:

$$V(0,0,z,L,L) = \frac{k_e\,Q}{L}\left[4\,\sinh^{-1}\left(\frac{1}{\sqrt{4\,Z^2+1}}\right) - 4\,Z\,\tan^{-1}\left(\frac{1}{2\,Z\,\sqrt{4\,Z^2+2}}\right)\right], \tag{26}$$

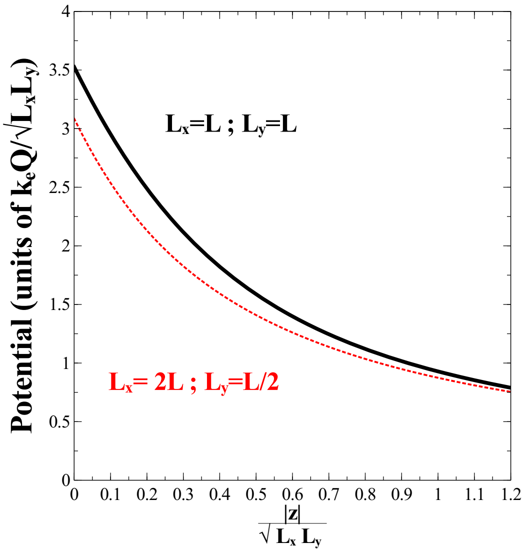

FIG. 2: Plot of $V(0,0,z,L_x,L_y)$ in units of $k_e\,Q/\sqrt{L_x\,L_y}$ as a function of dimensionless distance, $|z|/\sqrt{L_x\,L_y}$ for two cases: (i) Square plate, $L_x = L; L_y = L$ (solid line) and (ii) Rectangular plate, $L_x = 2\,L; L_y = L/2$ (dotted line). Note that the square and the rectangular plate have the same surface area.

where we introduce the dimensionless variable, $Z = z/L$. The result in Eq.(26) is the same as the one reported in Eq.(12) of an earlier work [23] that dealt solely with a uniformly charged square plate, thus, serving as a good check of accuracy and consistency.

III. CONCLUSIONS

In this work we explain the application of a general mathematical method that allows one to calculate the electrostatic potential at an arbitrary point in space due to a uniformly charged rectangular plate with arbitrary length and width. Calculation of the electrostatic potential due to a uniformly charged rectangular plate is a very difficult task for standard integration techniques that must involve Cartesian coordinates. For most of its part, the method presented here represents an elegant way to calculate the electrostatic potential by avoiding to calculate the initial cumbersome integrals in the formula that defines it. Case in point, the expression in Eq.(10) stands out as a very useful general result that allows one to obtain the electrostatic potential at any arbitrary point in space if one is inclined to use simple numerical methods that can calculate one-dimensional integrals. By proceeding further, we also note that explicit exact analytic results are also possible once one completes the calculation of the resulting one-dimensional integrals. To illustrate this point, we show several exact analytical results for the electrostatic potential due to a uniformly charged finite rectangular plate for special cases.

We believe that the present mathematical method that leads to the result in Eq.(10) is the simplest possible way to lead to a useful general expression of the electrostatic potential created by a uniformly charged finite rectangular plate at an arbitrary point in space. In fact, we point out that a fully analytic expression for $V(x,y,z,L_x,L_y)$ in Eq.(10) can be derived for the general scenario where all quantities x, y, z, L_x, L_y are arbitrary since the resulting integrals can be done analytically. However, the resulting mathematical expression for such a case is expected to be very long and we leave it out of this work. In our opinion, a calculation of $V(x,y,z,L_x,L_y)$ by using direct integration techniques from the start will lead to formidable mathematical difficulties. As already seen for the case of $V(0,0,z,L_x,L_y)$, calculations of electrostatic potentials by direct integration of charge densities can be cumbersome and lead to long analytical solutions [22]. At this juncture, we also point out that one can obtain the electrostatic potential at an arbitrary point in space, namely $V(x,y,z,L_x,L_y)$ in Eq.(10), by avoiding an explicit analytic calculation of the resulting integrals in Eq.(10) and do them numerically using appropriate software [24]. Overall, the reported results are related to important problems in electrostatics and applied mathematics that deal with potential theory. From this point of view, this work may be of interest to both specialized researchers and broad audiences [25].

CRediT authorship contribution statement

Orion Ciftja: Conceptualization, Methodology, Validation, Writing - original draft , Writing - review and editing, Supervision. Brent Ciftja: Methodology, Validation, Formal analysis, Investigation, Writing - reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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