Coulomb self-energy of a solid hemisphere with uniform volume charge density

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Calculation of the Coulomb self-energy of a solid hemisphere with uniform volume charge density represents a very challenging task. This system is an interesting example of a body that lacks spherical symmetry though it can be conveniently dealt with in spherical coordinates. In this work, we explain how to calculate the Coulomb self-energy of a solid hemisphere with uniform volume charge density by using a method that relies on the expansion of the Coulomb potential as an infinite series in terms of Legendre polynomials. The final result for the Coulomb self-energy of a uniformly charged solid hemisphere turns out to be quite simple.

Keywords: Electrostatic energy, Coulomb self-energy, Uniform charge density, Solid hemisphere.

I. INTRODUCTION

Knowing the Coulomb self-energy of a given charge distribution is very important since this quantity represents the electrostatic potential energy stored in the given system[1–6]. Generally speaking, the Coulomb self-energy of a charged body with an arbitrary shape cannot be calculated exactly in analytical form. For all such situations, one must resort to numerical methods. This implies that exact analytical results may be possible only for regular bodies with known charge distributions[7–9]. However, even in such cases, the calculation of the equilibrium charge distribution for conventional regular bodies such as a square, a cube or a cylinder is one of the most difficult problems in potential theory[10]. Trivial exceptions are systems such as a conducting spherical surface (which is a textbook example) or a conducting circular disk which illustrates very well how complicated this problem is[11].

This means that working on such a well-established field remains relevant to this day and continues to be a rewarding pursuit given that there are many challenges to overcome when solving problems of this nature. A remarkable example worth mentioning is the case of a charged straight wire. Although the theory has been developed for over a century, we still have ambiguous answers to the very simple question of what is the equilibrium charge distribution which makes the body of the straight charged wire an equipotential[12–15]. Since finding the equilibrium charge distribution on a given charged body is a very difficult problem to solve, one typically assumes a uniform charge distribution over the length, surface or volume of any given one-dimensional, two-dimensional or three-dimensional body, respectively. This is done in the hope of simplifying a little bit the mathematics of the problem.

Past work has shown that one can use a variety of specialized methods to successfully obtain analytical results for the Coulomb self-energy of bodies such as a uniformly charged solid cylinder[16], solid cube[17] or square plate[18]. On the other hand, a uniformly charged solid sphere or a uniformly charged spherical surface are typical textbook examples where various results are wellknown. For instance, a quick description of some mathematical approaches as well as a pedagogical implementation of the Fourier transform method to calculate the Coulomb self-energy of a solid sphere with uniform volume charge density is readily available[19]. At this juncture, it is worth noting that, while a uniformly charged square plate or a solid cube is a unique system as far as the symmetry is concerned, all the other bodies (solid cylinders, cylindrical shells, solid spheres, etc.) represent systems that possess cylindrical symmetry and/or spherical symmetry.

As already mentioned, the Coulomb self-energy of a solid sphere with constant volume charge density is easy to calculate[19]. Obviously, a solid sphere is made up of two solid hemispheres. Therefore, an interesting problem that arises is that of the calculation of the Coulomb self-energy of a single solid hemisphere with uniform volume charge density. Within the realm of various regular bodies[20–28], a uniformly charged solid hemisphere has axial symmetry, but it is most easily treated in spherical coordinates. However, direct integration methods as well as the Fourier transform method that works well to calculate the Coulomb self-energy of a solid sphere with uniform volume charge density do not succeed for the case of a solid hemisphere. This means that one must be proactive and search for some other special approach to attempt to solve this problem.

In this work, we apply a solution method which allows us to obtain an exact analytic result for the Coulomb self-energy of a solid hemisphere with uniform volume charge density. The starting point of this mathematical method is the expansion of the Coulomb potential as an infinite series in terms of Legendre polynomials. Its success hinges on the ability to transform a mathematical expression involving challenging integrals that cannot be calculated via direct integration into an expression that involves an infinite sum of terms that can be calculated exactly. At the end of the process, the final result obtained for the Coulomb self-energy of a uniformly charged solid hemisphere turns out to be quite simple.

II. MODEL AND RESULTS

We consider a solid hemisphere with uniform volume charge density. The solid hemisphere has a radius, R and contains a total charge, Q. The resulting uniform volume charge density is:

$$\rho = \frac{3\,Q}{2\,\pi\,R^3}\ . \tag{1}$$

The Coulomb self-energy of the uniformly charged solid hemisphere can be written as:

$$U = \frac{k\,\rho^2}{2}\iiint\limits_D d^3r \iiint\limits_D d^3r'\ \frac{1}{|\vec{r}-\vec{r}\,'|}\ , \tag{2}$$

where k is Coulomb's electric constant and D is the volume domain of the solid hemisphere. We choose a "northern" solid hemisphere and adopt a spherical system of coordinates with origin at the center of the solid hemisphere and $x-y$ plane on the "equator". For this choice, the volume domain, D reads:

$$D\ :\ \left\{0 \le (r,r') \le R\,;\ 0 \le (\theta,\theta') \le \frac{\pi}{2}\,;\ 0 \le (\varphi,\varphi') < 2\,\pi\right\}, \tag{3}$$

where r (r') is the radial distance from the origin, θ (θ') is the polar angle and φ (φ') is the azimuthal (longitudinal) angle of a pair of elementary charges located at position vectors, $\vec{r}$ ($\vec{r}\,'$) belonging to the solid hemisphere volume domain.

The quantity in Eq.(2) can be written more explicitly as a six-dimensional integral:

$$U = \frac{k\,\rho^2}{2}\int_0^R dr\,r^2\int_0^{\pi/2} d\theta\,\sin\theta\int_0^{2\,\pi} d\varphi\int_0^R dr'\,r'^{\,2}\int_0^{\pi/2} d\theta'\,\sin\theta'\int_0^{2\,\pi} d\varphi'\ \frac{1}{|\vec{r}-\vec{r}\,'|}\ . \tag{4}$$

One way to simplify the calculation of the integral above is to write the Coulomb potential factor, $1/|\vec{r}-\vec{r}\,'|$ as an infinite series expansion in terms of Legendre polynomials as explained in pg. 62 of Ref.(11):

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}P_l(\cos\gamma)\ , \tag{5}$$

where $l = 0, 1, \ldots$ is an index, $r_<$ ($r_>$) is the smaller (larger) of $r = |\vec{r}|$ and $r' = |\vec{r}\,'|$, γ is the angle included between vectors $\vec{r}$ and $\vec{r}\,'$ and $P_l(\cos\gamma)$ are Legendre polynomials[29]. Another important mathematical result that turns out to be very useful is the so-called addition theorem of Legendre polynomials. This theorem allows one to express a Legendre polynomial of order l in the angle γ in terms of products of the spherical harmonics, or equivalently, in terms of associated Legendre polynomials [See pg. 69 of Ref.(11) and/or pg. 599 of Ref.(29)] which allows one to write $P_l(\cos\gamma)$ as:

$$P_l(\cos\gamma) = P_l(\cos\,\theta)\,P_l(\cos\,\theta') + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}\,P_l^m(\cos\,\theta)\,P_l^m(\cos\,\theta')\,\cos\left[m\,(\varphi-\varphi')\right]\ , \tag{6}$$

where $P_l^m(\cos\theta)$ are the associated Legendre polynomials[30–32]. Integrals over angular variables are, sometimes, very complicated[33,34]. In this specific case, integration over the azimuthal angles, φ and φ' in Eq.(4) will produce:

$$\int_0^{2\,\pi} d\varphi\int_0^{2\,\pi} d\varphi'\ \frac{1}{|\vec{r}-\vec{r}\,'|} = (2\,\pi)^2\sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}\,P_l(\cos\,\theta)\,P_l(\cos\,\theta')\ . \tag{7}$$

The results above in Eq.(6) and Eq.(7) have been succesfullly used in earlier calculations to facilitate studies of systems with hemispherical geometry[35,36].

At this juncture, one can write the expression for the Coulomb self-energy as:

$$U = \frac{(2\,\pi)^2\,k\,\rho^2}{2}\sum_{l=0}^{\infty}\left[\int_0^{\pi/2} d\theta\ \sin\theta\,P_l(\cos\theta)\right]\left[\int_0^{\pi/2} d\theta'\,\sin\theta'\,P_l(\cos\theta')\right]\left[\int_0^R dr\,r^2\int_0^R dr'\,r'^{\,2}\,\frac{r_<^{\,l}}{r_>^{\,l+1}}\right]\Bigg\}\ . \tag{8}$$

One notes that the integrals over polar angles, θ (θ') in Eq.(8) are of the same type and may be coined in

the form of an integral of Legendre polynomials over half range[36] written as:

$$\int_0^{\pi/2} d\theta \, \sin\theta \, P_l(\cos\theta) = \int_0^1 dx \, P_l(x) \; . \tag{9}$$

To simplify the expressions, one can write the quantity in Eq.(8) by using a compact notation as:

$$U = \frac{(2\,\pi)^2 \, k \, \rho^2}{2} \sum_{l=0}^{\infty} a_l^2 \, c_l(R) \; , \tag{10}$$

where

$$a_l = \int_0^1 dx \, P_l(x) \; , \tag{11}$$

and

$$c_l(R) = \int_0^R dr \, r^2 \int_0^R dr' \, r'^2 \, \frac{r_<^l}{r_>^{l+1}} \; . \tag{12}$$

One can calculate both quantities, a_l (that depends on l) and $c_l(R)$ (that depends on l and R) in analytical form. The result for $c_l(R)$ reads:

$$c_l(R) = \frac{2}{5} \, \frac{R^5}{(l+3)} \; . \tag{13}$$

Hence,

$$U = \frac{(2\,\pi)^2}{5} \, k \, \rho^2 \, R^5 \sum_{l=0}^{\infty} \frac{a_l^2}{(l+3)} \; . \tag{14}$$

The calculation of a_l is challenging but has been done in an earlier work[36] with the final results being: $a_0 = 1$, $a_2 = a_4 = a_6 = \ldots = 0$ and $a_1 \neq a_3 \neq a_5 \neq \ldots \neq 0$. One can write the results in a succinct mathematical way by substituting the dummy index, l with a new index, n so that, $l = 2\,n$ $(n = 1, 2, \ldots)$ gives $l = 2, 4, \ldots$ while $l = 2\,n + 1$ $(n = 0, 1, 2, \ldots)$ gives the odd values, $l = 1, 3, \ldots$ as below:

$$a_0 = \int_0^1 dx \, P_0(x) = 1 \; , \tag{15}$$

$$a_{2\,n} = \int_0^1 dx \, P_{2\,n}(x) = 0 \quad ; \quad n = 1, 2, \ldots \; , \tag{16}$$

and

$$a_{2\,n+1} = \int_0^1 dx \, P_{2\,n+1}(x) = \frac{(-1)^n}{2^{2\,n+1}\,(n+1)} \, \frac{(2\,n)!}{(n!)^2} \; ; \; n = 0, 1, \ldots \tag{17}$$

Based on the above results, one can write the infinite sum in Eq.(14) as:

$$\sum_{l=0}^{\infty} \frac{a_l^2}{(l+3)} = \frac{a_0^2}{3} + \sum_{n=0}^{\infty} \frac{a_{2\,n+1}^2}{(2\,n+4)} = \frac{a_0^2}{3} + \frac{a_1^2}{4} + \frac{a_3^2}{6} + \ldots \; , \tag{18}$$

where $a_0 = 1$ and a_1, a_3, ... are given from Eq.(17). It follows that:

$$\sum_{l=0}^{\infty} \frac{a_l^2}{(l+3)} = \frac{(32 - 3\,\pi)}{18\,\pi} \; . \tag{19}$$

The formula above was obtained by using symbolic computation software[37] and the result was checked numerically to a very high degree of accuracy. One substitutes the result from Eq.(19) into Eq.(14) to obtain:

$$U = (4\,\pi) \, \frac{(32 - 3\,\pi)}{90} \, k \, \rho^2 \, R^5 \; . \tag{20}$$

One can easily verify that:

$$k \, \rho^2 \, R^5 = \left(\frac{3}{2\,\pi} \right)^2 \, \frac{k \, Q^2}{R} \; . \tag{21}$$

Therefore, based on Eq.(21), one can write the quantity in Eq.(20) as:

$$U = \frac{(32 - 3\,\pi)}{10\,\pi} \, \frac{k \, Q^2}{R} \; , \tag{22}$$

where Q represents the total charge spread uniformly throughout the volume of a solid hemisphe with radius R. The analytical exact result in Eq.(22) is, unexpectedly, quite simple in view of the many challenges faced while solving the problem.

III. CONCLUSIONS

We calculated exactly the Coulomb self-energy of a solid hemisphere with uniform volume charge density. Calculation of this quantity which represents the electrostatic energy stored in such a system poses a very difficult mathematical problem partly because the body lacks spherical symmetry. Nevertheless, it is shown in this work that an exact analytic result is possible by using a method that relies on an infinite series expansion of the standard Coulomb potential in terms of Legendre polynomials. The method leads to new integrals, that while challenging, can be calculated analytically. Ultimately, one ends up with an expression for the Coulomb self-energy given as an infinite sum that can be calculated exactly. The final result for the Coulomb self-energy of a solid hemisphere with uniform volume charge density turns out to be amazingly simple.

Obtaining such a result is quite rewarding given the many challenges which one must face when solving problems of this nature. The outcome is also important since the solution method and treatment of the problem can be useful to various different disciplines[38–43]. For example, individuals working in the field of applied mathematics or applications of computational methods in sciences may use the present result to gauge the accuracy

of various numerical methods and/or standard computational software. After all, the calculation of the Coulomb self-energy of a solid hemisphere with uniform volume charge density is not a simple problem to solve. As far as we know, its calculation cannot be done analytically via straightforward standard integration methods. As a result, the quantity in Eq.(4) represents a challenging six-dimensional integral that is difficult to calculate even numerically and, there will always be some resulting numerical errors when such a calculation is done via standard numerical methods or tools[44]. Therefore, it is pointed out that the work reported here can be of interest to a broader readership that extends beyond that associated with the field of electrostatics.

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