Interaction potential between coplanar uniformly charged disk and ring

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We consider a system made up of a uniformly charged disk and a uniformly charged ring that are coplanar with each other. The disk and ring have arbitrary radii and contain arbitrary amounts of net charge that is spread uniformly over area and length, respectively. This study is seen as the first necessary step towards a classical model treatment of a two-dimensional system of electrons in a weak perpendicular magnetic field. In this scenario, the uniformly charged disk represents the neutralizing background in a jellium approximation while a uniformly charged ring is viewed as a classical approximation to the cyclotron motion of the electron in presence of a uniform perpendicular magnetic field. In this work, we obtain an exact integral expression for the interaction potential energy of the system as a function of the separation distance between the centers of the two bodies and discuss various interesting emerging features of the model.

Keywords: Potential energy, Uniformly charged disk, Uniformly charged ring, Integer quantum Hall effect, Fractional quantum Hall effect.

I. INTRODUCTION

There have been many important developments and discoveries in the field of low-dimensional systems in the last decade [1–4]. A solid-state system in which the spatial dimension is less than that of the more common three-dimensional (3D) bulk counterpart manifests novel properties that have drawn the interest of many studies [5–16]. In practice, a two-dimensional (2D) system is a layer or a thin film [17] and a one-dimensional (1D) system is a thin wire [18]. Clusters of particles and very small crystals can be considered as zero-dimensional systems. In particular, the two-dimensional electron gas (2DEG) formed in a GaAs-AlGaAs heterostructure is a very useful system for investigating interaction and quantum effects. In such experimental systems, electrons are confined to a very narrow layer which, in theoretical studies, is considered ideally 2D. At low temperatures and with a high quality material, electrons can travel relatively long distances without much scattering. The most remarkable thing about such 2D systems is that, in a strong perpendicular magnetic field, they manifest novel quantum-mechanical phenomena such as the integer [19, 20] or the fractional quantum Hall effect [21–24].

A common treatment for interacting electrons in a solid is the jellium model where the positive neutralizing charges are assumed to be uniformly distributed in space. The neutralizing background charge interacts electrostatically with itself and the electrons. The most common geometry of the neutralizing background in quantum Hall studies of 2D systems of electrons in a perpendicular magnetic feld is the disk geometry [25–27]. Therefore, a typical 2D jellium model for N electrons in a perpendicular magnetic field sees them moving in 2D space in presence of a positively charged background disk with uniform (constant) charge density.

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A quantum treatment is required to understand quantum Hall systems that typically occur at very high magnetic fields. On the other hand, a classical treatment may work well for a weak magnetic field. Within the framework of classical theory, a particle with charge q, with mass m, moving with some velocity $\vec{v}$, may undergo cyclotron motion, if a uniform magnetic field is applied perpendicular to the plane of motion. Both the radius, R and the center of the circle of rotation of the charged particle can be easily calculated [28]. Based on this description, the simplest way to model a rotating charged particle is to see it as a uniformly charged ring with a constant linear charge density, $\lambda = q/(2\,\pi\,R)$. Therefore, a necessary first step in a classical treatment of a 2D system of electrons in a weak perpendicular magnetic field (with a disk jellium background) is the calculation of the interaction potential energy between a uniformly charged disk and a uniformly charged ring that is coplanar with it.

In this work, we calculate the interaction potential energy between a uniformly charged disk and a uniformly charged ring situated at an arbitrary distance relative to each other, but coplanar. The uniformly charged disk and ring contain arbitrary amounts of charge and have arbitrary radii. We obtain a closed-form analytical expression, in integral form, for the interaction potential energy as a function of the separation distance between the centers of the two bodies as well as their sizes. Some possible applications of this model are also discussed from the perspective of the consideration of a classical model for 2D systems of electrons in a vanishingly small (very weak) perpendicular magnetic magnetic field.

The article is organized as follows. In Section II we explain the model and the theoretical approach. In Section III we display the key results and discuss their implications. In Section IV we briefly summarize the findings and provide some concluding remarks.

II. MODEL

The model consists of a perfectly 2D flat circular uniformly charged disk and a uniformly charged ring with the understanding that the two objects are coplanar. The disk has a radius, R_d and carries an arbitrary charge, Q that is spread uniformly over its surface. This gives rise to a constant surface density:

$$\sigma = \frac{Q}{\pi R_d^2} \ . \tag{1}$$

A cylindrical (2D polar) system of coordinates is chosen in such a way that the disk lies in the $x - y$ plane with its center at the origin.

On the other hand, the uniformly charged ring is situated at an arbitrary location in the $x - y$ plane where $\vec{r}_c$ represents the arbirary position of the center of the ring. We assume that a charge, q is spread uniformly over the length of the ring with radius, R resulting in a constant linear charge density:

$$\lambda = \frac{q}{2 \pi R} \ . \tag{2}$$

We assume Coulomb interaction between elementary charges, $dQ = \sigma\, d^2 r_1$ (object 1, disk) and $dq = \lambda\, dl_2$ (object 2, ring) where $d^2 r_1$ is an elementary surface area on the disk at position vector, $\vec{r}_1$ while dl_2 is an elementary length in the ring at a position vector, $\vec{r}_2 = \vec{r}_c + \vec{R}_2$ as shown in Fig. 1. In this geometry, $\vec{r}_c$ is the center-to-center separation vector and $\vec{R}_2$ is a 2D radius vector of magnitude, $|\vec{R}_2| = R$ relative to the center of the ring.

Based on symmetry arguments, we expect that the potential energy of interaction between the two charged objects will depend on the center-to-center distance, $r_c = |\vec{r}_c| \geq 0$ and respective radii, R_d and R. For this reason, we denote this quantity as $U_{12}(r_c, R_d, R)$. It is expected that this quantity will be Coulomb-like for $r_c \gg max(R_d, R)$ where $max(R_d, R)$ represents the largest of the two radii.

However, the outcome is not predictable for other different instances. As a result, we must carry out a full calculation of $U_{12}(r_c, R_d, R)$ in order to fully understand the features of this interaction potential. Such a calculation starts from the following integral expression:

$$\begin{aligned} U_{12}(r_c, R_d, R) &= k_e \iint_{Disk} dQ \oint_{Ring} dq \ \frac{1}{|\vec{r}_1 - \vec{r}_c - \vec{R}_2|} \\ &= k_e\, \sigma\, \lambda \iint_{Disk} d^2 r_1 \oint_{Ring} dl_2 \ \frac{1}{|\vec{r}_1 - \vec{r}_c - \vec{R}_2|} \end{aligned} \tag{3}$$

where k_e is Coulomb's electric constant, the 2D integral is over the disk domain denoted as "$Disk$" and the closed line integral is over the ring domain denoted as "$Ring$". The calculation of the quantity above is very challenging for direct integration techniques. However, we show in this work that there is an elegant way to obtain useful results by using the the method of Fourier transforms [29].

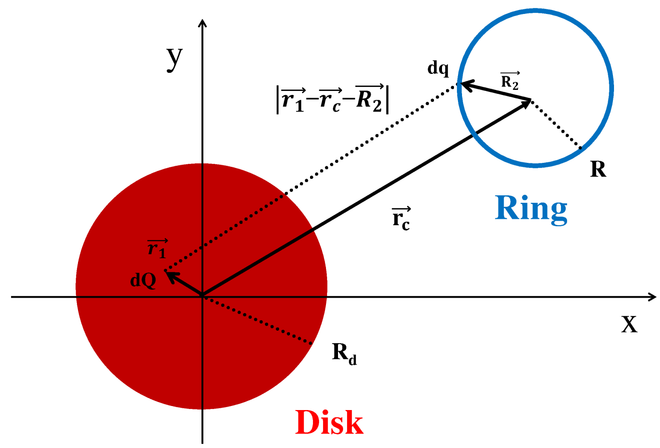

FIG. 1. Schematic view of a uniformy charged disk and a uniformly charged ring. The two objects are coplanar to each other. The radius of the disk is R_d while that of ring is R. The total charge of the disk is denoted Q while that of ring is q. The uniform surface charge density of the disk is $\sigma = Q/(\pi R_d^2)$ while the uniform linear charge density of the ring is $\lambda = q/(2 \pi R)$.

III. RESULTS AND DISCUSSIONS

We define the 2D Fourier transform integrals as:

$$F(\vec{k}) = \iint d^2 r\, e^{i \vec{k} \vec{r}} f(\vec{r}) \tag{4}$$

and

$$f(\vec{r}) = \iint \frac{d^2 k}{(2\pi)^2} e^{-i \vec{k} \vec{r}} F(\vec{k}) \ , \tag{5}$$

where $\vec{k}$ and $\vec{r}$ are 2D vectors. The 2D Fourier transform of a Coulomb-like function, $1/|\vec{r}|$ is $2\pi/|\vec{k}|$. By thinking of $\vec{r}_1 - \vec{r}_c - \vec{R}_2$ as the vector $\vec{r}$ in Eq.(5) one has:

$$\frac{1}{|\vec{r}_1 - \vec{r}_c - \vec{R}_2|} = \iint \frac{d^2 k}{(2\pi)^2} e^{-i \vec{k} (\vec{r}_1 - \vec{r}_c - \vec{R}_2)} \frac{2\pi}{k} \ , \tag{6}$$

where $k = |\vec{k}| \geq 0$ is the magnitude of 2D vector $\vec{k}$.

After substituting the expression from Eq.(6) into Eq.(3) one has:

$$U_{12}(r_c,R_d,R) = k_e\,\sigma\,\lambda \iint \frac{d^2k}{(2\,\pi)^2}\, e^{i\,\vec{k}\,\vec{r}_c}\, \frac{2\,\pi}{k} \iint_{Disk} d^2r_1\, e^{-i\,\vec{k}\,\vec{r}_1} \oint_{Ring} dl_2\, e^{+i\,\vec{k}\,\vec{R}_2} \ . \tag{7}$$

The reader can verify that:

$$\iint_{Disk} d^2r\, e^{\pm i\,\vec{k}\,\vec{r}} = 2\,\pi\,R_d^2\, \frac{J_1(k\,R_d)}{(k\,R_d)} \ , \tag{8}$$

where R_d is the radius of the disk and $J_1(x)$ is a Bessel function of the first kind of order 1. Integrals over angular variables are, sometimes, very challenging [30]. However, in this case, one can easily check that:

$$\oint_{Ring} dl_2\, e^{\pm i\,\vec{k}\,\vec{R}_2} = 2\,\pi\,R\,J_0(k\,R) \ , \tag{9}$$

where $|\vec{R}_2| = R$ and $J_0(x)$ is a Bessel function of the first kind of order 0. The next step is to substitute the results from Eq.(8) and Eq.(9) into Eq.(7). After straightforward, but somehow lengthy, algebraic manipulations one obtains:

$$U_{12}(r_c,R_d,R) = 2\,k_e\,Q\,q \int_0^\infty dk\, J_0(k\,r_c)\, \frac{J_1(k\,R_d)}{(k\,R_d)}\, J_0(k\,R) \ . \tag{10}$$

The integral expression in the above form is very appealing because it allows one to obtain quite easily the expected results in the limit of the interacting objects becoming particles (point charges). To check such limiting results one must rely on the knowledge of the following mathematical formulae:

$$\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2} \ \ ; \ \ J_0(x=0) = 1 \ \ ; \ \ \int_0^\infty dx\, J_0(x) = 1 \ . \tag{11}$$

First of all, one can easily see from the results in Eq.(11) that:

$$U_{12}(r_c,R_d=0,R=0) = \frac{k_e\,Q\,q}{r_c} \ , \tag{12}$$

which represents correctly the Coulomb energy of two charged particles at a separation distance of r_c.

In the $R \to 0$ limit (when the ring becomes a point with charge, q) one obtains:

$$U_{12}(r_c,R_d,R=0) = 2\,k_e\,Q\,q \int_0^\infty dk\, J_0(k\,r_c)\, \frac{J_1(k\,R_d)}{(k\,R_d)} = q\,V_{Disk}(r_c,R_d) \ , \tag{13}$$

where $V_{Disk}(r_c,R_d) = 2\,k_e\,Q \int_0^\infty dk\, J_0(k\,r_c)\, \frac{J_1(k\,R_d)}{(k\,R_d)}$ represents the electrostatic potential created by a uniformly charged disk with radius, R_d and total charge, Q at a distance r_c away from its center on the plane of the disk. Derivation of the electrostatic potential created by a uniformly charged disk at an arbitrary point in space and various mathematical expressions (with slight differences of notation from the current work) are available from an earlier study [31].

In the $R_d \to 0$ limit (when the disk becomes a point with charge, Q) one obtains:

$$U_{12}(r_c,R_d=0,R) = k_e\,Q\,q \int_0^\infty dk\, J_0(k\,r_c)\, J_0(k\,R) = Q\,V_{Ring}(r_c,R) \ , \tag{14}$$

where $V_{Ring}(r_c,R) = k_e\,q \int_0^\infty dk\, J_0(k\,r_c)\, J_0(k\,R)$ denotes the electrostatic potential created by a uniformly charged ring with radius, R and total charge, q at a distance r_c away from its center on the plane of the ring. A calculation of the electrostatic potential created by a uniformly charged ring at an arbitrary point in space and various mathematical expressions of this quantity (with slight differences of notation from the current work) are readily available in literature [32].

An exact analytic result is possible for $r_c = 0$ and arbitrary R_d and R. This setup corresponds to the situation in which the disk and the ring are concentric. For such a case, the integral is written as:

$$U_{12}(r_c=0,R_d,R) = \frac{2\,k_e\,Q\,q}{R_d} \int_0^\infty \frac{dk}{k}\, J_1(k\,R_d)\, J_0(k\,R) \ . \tag{15}$$

With help from integral formulas for Bessel functions [31] one eventually obtains:

$$U_{12}(r_c=0,R_d,R)=\frac{2\,k_e\,Q\,q}{R_d}\begin{cases}\frac{2}{\pi}\,E\left[\left(\frac{R}{R_d}\right)^2\right] & ; \quad 0\le\frac{R}{R_d}\le 1\\ \frac{2}{\pi}\left\{\left(\frac{R}{R_d}\right)\,E\left[\left(\frac{R_d}{R}\right)^2\right]+\frac{1-\left(\frac{R}{R_d}\right)^2}{\left(\frac{R}{R_d}\right)}K\left[\left(\frac{R_d}{R}\right)^2\right]\right\} & ; \quad 1<\frac{R}{R_d}<\infty\ ,\end{cases} \tag{16}$$

where

$$K(m)=\int_0^{\pi/2}\frac{d\,\theta}{\sqrt{1-m\,\sin^2(\theta)}}\ , \tag{17}$$

and

$$E(m)=\int_0^{\pi/2}d\,\theta\,\sqrt{1-m\,\sin^2(\theta)}\ , \tag{18}$$

are, respectively, the complete elliptic integral of the first and second kind defined, respectively, for $0\le m<1$ and $0\le m\le 1$ as in the book by Arfken and Weber [33] (see pages 355-356). One can see from the defining expressions of the complete elliptic integrals that $K(m=0)=\pi/2$, $K(m\to 1)\to\infty$, $E(m=0)=\pi/2$ and $E(m=1)=1$.

The complexity of the calculation for $r_c=0$ as seen from the result in Eq.(16) seems to hint that explicit exact analytical results are not feasible for arbitrary r_c. The only other result that we can guess for finite arbitrary R_d and R would involve the $r_c\to\infty$ limit where one expects:

$$U_{12}(r_c\to\infty,R_d,R)=\frac{k_e\,Q\,q}{r_c}\ . \tag{19}$$

From this point on, we assume that both $R_d>0$ and $R>0$. This means that we can introduce a dimensionless variable either as r_c/R_d or r_c/R in order to simplify the calculation of the integral in Eq.(10). A quick look at Eq.(10) suggests that the simplest choice is to express the energy in units of $k_e\,Q\,q/R_d$ and, thus, write the integral in Eq.(10) as:

$$U_{12}(r_c,R_d,R)=\frac{2\,k_eQ\,q}{R_d}\int_0^{\infty}\frac{du}{u}J_0\left(\frac{r_c}{R_d}u\right)\,J_1(u)\,J_0\left(\frac{R}{R_d}u\right)\ , \tag{20}$$

where $u=k\,R_d$ is an auxiliary variable. As one can see, the resulting integral depends on two parameters, r_c/R_d and R/R_d. From a mathematical point of view, the expression in Eq.(20) involves a variant of the integral of the product of three Bessel functions divided by a power function of the form:

$$\int_0^{\infty}\frac{dx}{x}\,J_0\,(a\,x)\;J_1(x)\,J_0(b\,x)\ , \tag{21}$$

where a,b are real parameters. As far as we know, there are no compact analytical expressions available for such integrals at arbitrary values of a and b. Therefore, given

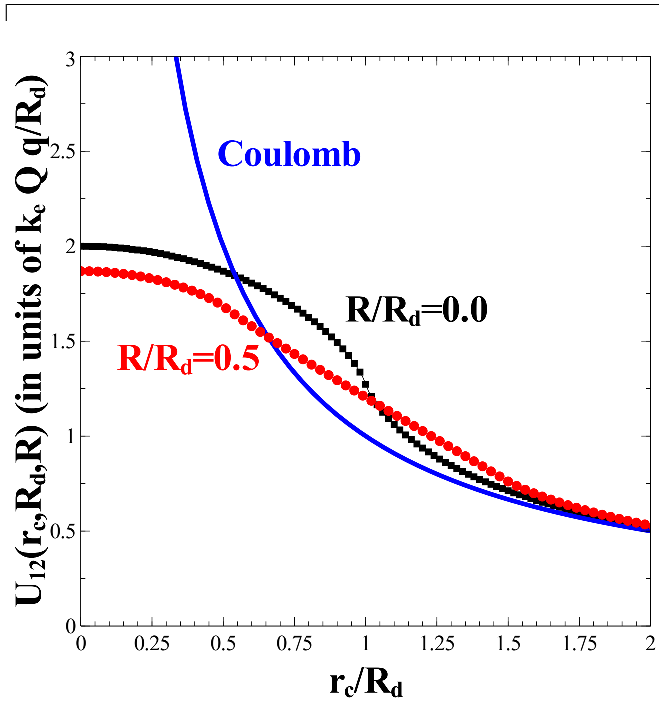

FIG. 2. Electrostatic interaction energy between a uniformly charged disk with radius, R_d and total charge, Q and a uniformly charged ring with radius, R and total charge, q. The case of a ring with radius smaller than that of a disk is considered, $R/R_d=0.5$. The quantity is calculated as a function of r_c/R_d where r_c is the center-to-center distance between the the disk and ring (filled circles). The result is compared to the case $R/R_d=0.0$ which would represent the interaction energy of a disk with a point charge (filled squares) and a standard Coulomb interaction potential, $k_e\,Q\,q/r_c$ (solid line). The energy is expressed in units of $k_e\,Q\,q/R_d$.

that a complete exact calculation cannot be done analytically, the integral presentation in Eq.(20) is of fundamental importance because it is compact, simple and easy to implement numerically.

In fact, we were able to calculate the integral expression in Eq.(20) numerically, with very high precision, for arbitrary values of r_c/R and R/R_d by using standard integration packages [34]. A sample of such results is displayed in Fig. 2 where we show the resulting interaction potential energy, $U_{12}(r_c,R_d,R)$ in units of $k_e\,Q\,q/R_d$ as a function of r_c/R_d for the case of $R/R_d=0.5$ (filled circles). This result is compared to the case $R/R_d=0.0$ which would represent the interaction energy of a disk with a point charge (filled squares) and a standard Coulomb interaction potential (solid line) of the form:

$$U_C(r_c) = \frac{1}{r_c/R_d}\,\frac{k_e\,Q\,q}{R_d}\ . \qquad (22)$$

As can be clearly seen, the differences between the functions become visible for distances in the range, $r_c/R_d < 2$. Interestingly, the standard Coulomb interaction potential is always weaker than the disk-ring potential at large distances. As the separation distance decreases, the Coulomb potential grows faster and eventually intersects with disk-ring and disk-point potential energy curves at some small separation distance. For distances shorter than this value, the Coulomb potential climbs much faster while the disk-ring and disk-point potentials slowly saturate to their respective finite values at $r_c = 0$.

IV. CONCLUSIONS

To summarize, we calculated the potential energy of interaction between a uniformly charged disk and a coplanar uniformly charged ring as a function of the distance separating them. This calculation is a necessary first step in a classical treatment of a 2D electron system subject to a weak perpendicular magnetic field where the disk represents the jellium background while the ring approximates the rotating electron in cyclotron motion. We derived a closed-form formula in integral form for this quantity in terms of the arbitrary center-to-center separation distance between the two objects and their respective geometry, namely, the arbitrary radii of disk and ring. We checked that the general expression for the potential energy reduces to the appropriate forms in the limiting cases of (i) particle-particle; (ii) disk-particle; and (iii) ring-particle interactions in which the particle (a point charge) is coplanar to the disk and ring for cases (ii) and (iii), respectively. We note that the interaction potential is always finite at $r_c = 0$ except for the particle-particle special case. At this juncture, we also remark that we looked more closely to the situation where the radius of the ring is smaller than that of the disk since we are typically thinking of a model in which many rotating electrons, namely, uniformly charged rings are contained inside the disk. For such a case, the results for the interaction energy are similar to those for a pair of identical coplanar uniformly charged nanodisks [35] since the potential created by the large disk as felt by the smaller ring (assuming $R_d > R$) is the dominant factor.

The model studied in this work is an important component for the implementation of a classical model to study 2D electron systems in a weak magnetic field wherever the picture of a rotating electron is seen as a reasonable starting point. It is known that 2D systems of electrons in quantum Hall effect studies [36–40] typically happen in very high magnetic field and require a full quantum treatment. However, it is possible that an effective classical treatment may work reasonaby well in the limit of a weak magnetic field. By adopting this logic, one may argue that such a treatment can give some useful information even for Fermi liquid states in half-filled Landau levels at large filling factors [41]. Such states can be viewed as representing effective charged particles called composite fermions (electrons coupled to an even number of magnetic flux quanta) in a vanishingly small (very weak) effective magnetic field based on the present Chern-Simons theories for even-denominator filled quantum Hall states [42, 43].

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[1] M. A. Kastner, Phys. Today **46**, 24 (June 1993).
[2] D. Heitmann and J. P. Kotthaus, Phys. Today **46**, 56 (June 1993).
[3] R. C. Ashoori, Nature **379**, 413 (1996).
[4] O. Ciftja, Nanomaterials **13**, 364 (2023).
[5] O. Ciftja, B. Sutton, and A. Way, AIP Adv. **3**, 052110 (2013).
[6] O. Ciftja, Physica B **458**, 92 (2015).
[7] O. Ciftja, Sci. Rep. **11**, 3181 (2021).
[8] U. Merkt, J. Huser, and M. Wagner, Phys. Rev. B **43**, 7320 (1991).
[9] S. Bednarek, B. Szafran, and J. Adamowski, Phys. Rev. B. **59**, 13036 (1999).
[10] J. Kainz, S. A. Mikhailov, A. Wensauer, and U. Rössler, Phys. Rev. B **65**, 115305 (2002).
[11] H. Saarikoski, M. J. Puska, and R. M. Nieminen, Int. J. Quantum Chem. **91**, 490 (2003).
[12] M. B. Tavernier, E. Anisimovas, F. M. Peeters, B. Szafran, J. Adamowski, and S. Bednarek, Phys. Rev. B **68**, 205305 (2003).
[13] O. Ciftja, J. Phys.: Condens. Matter **19**, 046220 (2007).
[14] O. Ciftja, Mod. Phys. Lett. B **23**, 3055 (2009).
[15] N. Q. Huong and N. H. Hong, J. Magn. Magn. Mater. **534**, 167944 (2021).
[16] E. Kawakami, P. Scarlino, L. R. Schreiber, J. R. Prance, D. E. Savage, M. G. Lagally, M. A. Eriksson, and L. M. K. Vandersypen, Appl. Phys. Lett. **103**, 132410 (2013).
[17] D. Elam, E. Ortega, A. Nemashkalo, Y. Strzhemechny, A. Ayon, A. Ponce, and A. A. Chabanov, Appl. Phys. Lett. **119**, 142101 (2021).
[18] N. N. Dat and N. T. T. Hien, Eur. Phys. J. B **95**, 31 (2022).
[19] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[20] O. Ciftja, J. Phys. Chem. Solids **156**, 110131 (2021).
[21] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[22] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[23] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[24] O. Ciftja and C. Wexler, Solid State Commun. **122**, 401 (2002).
[25] R. Morf and B.I. Halperin, Phys. Rev. B **33**, 2221 (1986).
[26] O. Ciftja and C. Wexler, Phys. Rev. B **67**, 075304 (2003).
[27] G. Dev and J. K. Jain, Phys. Rev. B **45**, 1223 (1992).
[28] O. Ciftja, Cent. Eur. J. Phys. **11**(2), 173 (2013).
[29] O. Ciftja, Eur. J. Phys. **42**, 025204 (2021).
[30] O. Ciftja, J. Phys.: Conf. Ser. **200**, 022002 (2010).
[31] O. Ciftja and I. Hysi, Appl. Math. Lett. **24**, 1919 (2011).
[32] O. Ciftja, A. Babineaux, and N. Hafeez, Eur. J. Phys. **30**, 623 (2009).
[33] G. B. Arfken and H. J. Weber, Mathematical Methods For Physicists, Fifth Edition, Harcourt/Academic Press, Burlington, Massachusetts, USA (2001).
[34] Wolfram Research, Inc., Mathematica, Version 9.0, Champaign, Illinois, USA (2012).
[35] O. Ciftja and I. Berry, AIP Adv. **8**, 035209 (2018).
[36] E. Rezayi, and N. Read, Phys. Rev. Lett. **72**, 900 (1994).
[37] O. Ciftja, Europhys. Lett. **74**, 486 (2006).
[38] J. Xia, J. Math. Phys. **40**, 150 (1999).
[39] O. Ciftja, Int. J. Mod. Phys. B **24**, 3489 (2010).
[40] O. Ciftja, Sci. Rep. **12**, 2383 (2022).
[41] J. Quintanilla and O. Ciftja, Sci. Rep. **13**, 1400 (2023).
[42] B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
[43] E. Rezayi and N. Read, Phys. Rev. Lett. **72**, 900 (1994).