Combinatorics of the two-species TASEP on a ring

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Abstract

The inhomogeneous two-species TASEP on a ring is an exclusion process that describes particles of different species hopping clockwise on a ring with parameters giving the hopping rates for different species. We introduce a combinatorial object that we call *toric rhombic alternative tableaux*, which are certain fillings of tableaux on a triangular lattice tiled with rhombi, and are in bijection with the well-studied *multiline queues* of Ferrari and Martin. Using the tableaux, we obtain a formula for the stationary probabilities of this TASEP, which specializes to results of Ayyer and Linusson. We obtain, in addition, an explicit determinantal formula for these probabilities. Furthermore, we extend our results to an inhomogeneous two-species ASEP with open boundaries by putting appropriate weights on the rhombic alternative tableaux, which are certain fillings of tableaux tiled by rhombi that give probabilities for the homogeneous two-species ASEP.

1 Introduction

The two-species totally asymmetric simple exclusion process (2-TASEP) on a ring is a model that describes the dynamics of particles of types 0, 1, and 2 hopping clockwise around a ring of n sites. Adjacent particles can swap places if the one on the left is of larger type. In the *homogeneous 2-TASEP*, the swapping rates are all equal, such as in Figure 1. We call a 2-TASEP in which each possible swap has a different rate the *inhomogeneous 2-TASEP*. In this paper, we study combinatorial solutions for the stationary probabilities of the 2-TASEP on a ring and related models.

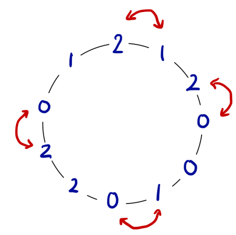

Figure 1: A TASEP on a ring of size $(4, 3, 4)$. The arrows indicate possible swaps at adjacent sites.

Our interest in the 2-TASEP on a ring stems from two directions. On one hand, the 2-TASEP is a specialization of the two-species asymmetric simple exclusion process (2-ASEP), in which adjacent particles swap places with rate 1 if the one on the left is of larger type, and with rate q otherwise for some parameter $0 \leq q \leq 1$. The 2-ASEP has recently been found to have a remarkable connection to moments of Macdonald polynomials [4]. Thus in our study of combinatorics of the 2-TASEP, we hope to gain insight on the more complex 2-ASEP for which combinatorics are not yet well understood. On the other hand, the 2-TASEP is a special case of the k-TASEP with k different types of particles, which has been studied extensively. The k-TASEP has a beautiful combinatorial solution in terms of *multiline queues* (MLQs) discovered by Ferrari and Martin in 2005 [8]. Our approach to solve the 2-TASEP uses tableaux, which are convenient in many ways. The tableaux are closely related to the well-studied *alternative tableaux*, which solve the 2-ASEP with open boundaries. Furthermore, the tableaux admit a natural addition of parameters which provide a solution for the inhomogeneous 2-TASEP.

Interest in the inhomogeneous k-TASEP arose from work of Lam and Williams [9], who studied a Markov Chain on the symmetric group, and conjectured that probabilities of this related model have a combinatorial solution consisting of polynomials with positive integer coefficients. The conjecture was proved for the 2-TASEP by A. Ayyer and S. Linusson [3] using multiline queues, and algebraically for the k-TASEP by C. Arita and K. Mallick [2]. In this paper we provide a tableaux proof which specializes to the latter.

This paper is organized as follows. In Section 2, we define the 2-TASEP on a ring is defined and give the solution in terms of multiline queues. We then describe the tableaux approach to get an equivalent solution. As a corollary, we obtain a determinantal formula for probabilities of states of the 2-TASEP on a ring. In Section 3, we give two bijections between MLQs and our tableaux. In Section 4, we obtain a solution to an inhomogeneous 2-TASEP that specializes to the solution in [3]. Finally in Section 5, we extend our solution to an inhomogeneous 2-ASEP with open boundaries.

2 The 2-TASEP and toric rhombic alternative tableaux

The 2-TASEP is a Markov chain describing particles of types 0, 1, and 2 hopping on a ring, with the larger particle types having "priority" over the smaller ones. The ring has n sites numbered 1 through n with each site occupied by one of the particle types, which we represent as a 1D periodic lattice $\mathbb{Z}/n\mathbb{Z}$. A state is represented by a word $X = X_1 \dots X_n$ where $X_i \in \{2, 1, 0\}$. The periodicity implies $X_1 X_2 \dots X_n$ and $X_2 \dots X_n X_1$ represent the same state. We say X is a state of the TASEP of size (k, r, ℓ) if it has k 2's, r 1's, and ℓ 0's. We denote by TASEP(k, r, ℓ) the set of states of size (k, r, ℓ). For example, Figure 1 shows a state of TASEP$(4, 3, 4)$.

The possible transitions of the 2-TASEP chain are the following: both 2 and 1 can swap with adjacent 0's to their right. Additionally, 2 can swap with an adjacent 1 to its right:

$$X\ 2\ 0\ Y \to X\ 0\ 2\ Y, \qquad X\ 2\ 1\ Y \to X\ 1\ 2\ Y, \qquad X\ 1\ 0\ Y \to X\ 0\ 1\ Y,$$

where X and Y are words in $\{2, 1, 0\}$. In the homogeneous 2-TASEP, all transitions occur with the same rate.

The inhomogeneous multispecies TASEP has also been studied; in this model, parameters represent different hopping rates for different particle types. We will discuss the inhomogeneous 2-TASEP in Section 4.

A Matrix Ansatz due to Derrida, Evans, Hakim, and Pasquier gives an explicit formula for the stationary probabilities of the states of the two-species TASEP on a ring [6].

Definition 2.1. Let $X = X_1 \dots X_n$ be a state of the 2-TASEP. For some set of matrices D, A, E define $\mathrm{dae}(X) = \mathrm{dae}(X_1) \dots \mathrm{dae}(X_n)$ to be the matrix product given by the map $\mathrm{dae}(2) \mapsto D$, $\mathrm{dae}(1) \mapsto A$, and $\mathrm{dae}(0) \mapsto E$. For example, $\mathrm{dae}(221021) = DDAEDA$.

(♣ check Zn for the ansatz ♣)

Theorem 2.1. Let D, A, E be matrices that satisfy the following relations:

$$DE = D + E$$
$$DA = A$$
$$AE = A$$

Then the stationary probability of a state X of the two-species TASEP on a ring of size (k, r, ℓ) is given by

$$\Pr(X) = \frac{1}{\binom{n}{k}\binom{n}{\ell}} \operatorname{tr}(\mathrm{dae}(X)),$$

where $\mathrm{dae}(X)$ is given by Definition 2.1.

Matrices D, A, E satisfying the Ansatz relations are not unique. One possible choice is:

$$D = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \\ & \vdots & & \ddots \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 & \\ 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \\ & \vdots & & \ddots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & \\ 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & \\ & \vdots & & \ddots \end{pmatrix}$$

Example 2.1. For the state $X = 12011020$, $\Pr(X) = \frac{8}{\binom{8}{2}\binom{8}{3}} \operatorname{tr}(ADEAAEDE) = \frac{32}{\binom{8}{2}\binom{8}{3}} = \frac{1}{49}$.

Remark. The fact that the partition function (i.e. normalizing factor) for the probabilities of a 2-TASEP of size (k, r, ℓ) is $\frac{1}{n}\binom{n}{k}\binom{n}{\ell}$ is well-known, and we will not prove it here. One way to see this is through enumeration of *multiline queues*, which we discuss in the following subsection.

2.1 Multline queues

Multiline queues (MLQs), first introduced by Ferrari and Martin, give an elegant combinatorial formula for the stationary probabilities of the k-TASEP on a ring [8]. The formula holds for any k, but for our purposes we define only MLQs that correspond to the 2-TASEP.

Let $k+r+\ell=n$. An MLQ of size (k,r,l) is a stack of two rows of balls and vacancies with the bottom row having $r+\ell$ balls and the top row having ℓ balls, all within a box of size $2\times n$. Locations are labeled from left to right with $1,\ldots,n$. We identify the left and right edges of the box, making it a cylinder; thus location 1 is to the right of and adjacent to location n.

Each MLQ corresponds to a state of the TASEP, which is determined by a *ball drop algorithm*, consisting of balls from the top row dropping to occupy balls in the bottom row. In this algorithm, top row balls drop to the bottom row and occupy the first unoccupied bottom row ball weakly to the right. Once all the top row balls have been dropped, each occupied bottom row ball is marked as a 0-ball, and each unoccupied bottom row ball is marked as a 1-ball. A state of the TASEP is read off the bottom row by associating 0-balls, 1-balls, and vacancies to type 0, 1, and 2 particles respectively. See Figure 2 for an example. We call this state the *type* of the MLQ. For an MLQ M of type $X=X_1\ldots X_n$, we denote by $M(j)$ the particle X_j.

Remark. The state read off the MLQ is independent of the order in which top row balls are dropped. However, in Section 3.2, we will require that balls are dropped from right to left, for the purpose of our bijections.

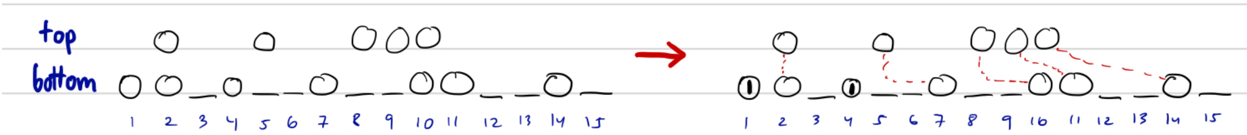

Figure 2: The MLQ of size $(8,2,5)$ on the left has type $X=102122022002202$, as seen from executing the ball dropping algorithm, where top row balls are dropped sequentially. The bottom row balls marked with a 1 are the 1-balls, and the unmarked balls are the 0-balls.

Definition 2.2. Let MLQ(X) be the set of *distinct* MLQs of type X. By distinct, we mean that no two are cyclic shifts of each other. Figure 3 shows the set MLQ(12020).

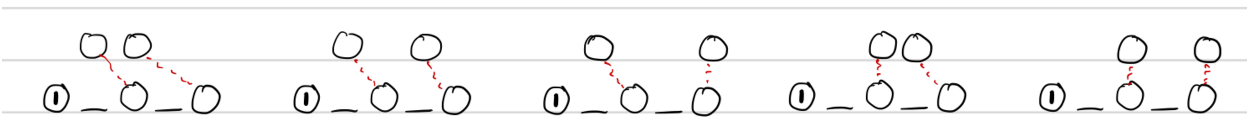

Figure 3: The set MLQ(X) for $X=12020$.

For a formal definition, let $B=\{x_1,\ldots,x_{r+\ell}\}$ be the locations of the bottom row balls. The balls that are occupied by a dropping top row ball are in the set of locations

$$H=\{x_i\ :\ \exists j \text{ such that there are } \geq j \text{ top row balls in the interval } [x_{i-j}+1,x_i]\}.$$

Then the balls in set H are the 0-balls, and the balls in set $B\backslash H$ are the 1-balls, which are mapped to type 0 and type 1 particles respectively.

Since the ℓ balls in the top row and the $r+\ell$ balls in the bottom row can be chosen independently, there is a total of $\binom{n}{\ell}\binom{n}{k}$ MLQs of size (k,r,ℓ) (where all cyclic shifts are counted as distinct objects). (♣ fix this. in fact, in all my formulas multiply |MLQ| by the order if the class of cyclic shifts. ♣)

The following theorem gives an expression for probabilities of the 2-TASEP in terms of the MLQs. We remark that this theorem also holds for the k-TASEP on a ring with a more general definition of MLQs.

Theorem 2.2 (Ferrari and Martin [8])**.** Let X be a state of the two-species TASEP on a ring of size (k, r, ℓ) with $n = k + r + \ell$. Then

$$\Pr(X) = \frac{o(X)}{\binom{n}{k}\binom{n}{\ell}} |\operatorname{MLQ}(X)|,$$

where $o(X)$ is the number of elements in the class of cyclic shifts of X.

Example 2.2. From Figure 3, we obtain that for $X = 12020$, $f(X) = 5$, and so $\Pr(X) = \frac{25}{\binom{5}{2}\binom{5}{2}} = \frac{1}{4}$.

2.2 Toric rhombic alternative tableaux

In this section we introduce tableaux that we call *toric rhombic alternative tableaux* (TRAT), which give a combinatorial formula for the stationary probabilities of the 2-TASEP. The TRAT are closely related to the *rhombic alternative tableaux* (RAT), which were defined by the author and Viennot in [13] as a solution for the 2-ASEP with open boundaries (see Section 5).

The TRAT are fillings with arrows of a tiling of a closed shape whose boundary is composed of south, southwest, and west edges on a triangular lattice. The tiles are three types of rhombic tiles which we call 20-tiles, 10-tiles, and 21-tiles. Each tile can contain an arrow that points either towards its left vertical edge or the top horizontal edge; we call them *left-arrows* and *up-arrows* correspondingly. The rules of the filling are that any tile that is "pointed to" by an arrow must be empty. We give a precise definition below.

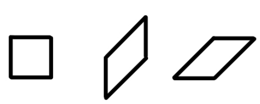

Figure 4: A 20-tile, a 21-tile, and a 10-tile.

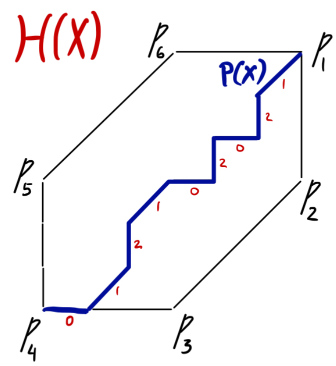

Figure 5: Toric hexagon $\mathcal{H}(X)$ with endpoints $p_1, \dots, p_6$, and the path $P(X)$.

Let $X = X_1 \dots X_n$ with $X_i \in \{2, 1, 0\}$ be a state of size (k, r, ℓ) of the two-species ASEP on a ring. Define a lattice path $P(X)$ as follows: reading X from left to right, draw a south edge for a 2, a southwest edge for a 1, and a west edge for a 0. Let $p_1 = (\ell + r, k + r)$ and $p_4 = (0, 0)$ be the coordinates of the vertices of $P(X)$. Define $p_2 = (\ell+r, r)$, $p_3 = (\ell, 0)$, $p_5 = (0, k)$, $p_6 = (r, k+r)$ and let $\{p_1, \dots, p_6\}$ be the endpoints of the hexagon $\mathcal{H}$ of size (k, r, ℓ) that contains $P(X)$. See Figure 5.

Definition 2.3. We call $\mathcal{H}(X)$ the hexagon $\mathcal{H}$ together with path $P(X)$. We say $\mathcal{H}(X)$ is a *toric diagram* of *type* X.

Definition 2.4. A *20-tile* is a rhombus with south and west edges. A *10-tile* is a rhombus with west and southwest edges. A *21-tile* is a rhombus with south and southwest edges. See Figure 4.

Now choose a tiling $\mathcal{T}$ with the 20-tiles, 21-tiles, and 10-tiles on the region of $\mathcal{H}(X)$ northwest of $P(X)$ and the region of $\mathcal{H}(X)$ southwest of P. For the remainder of the definition of the tableaux, this tiling is fixed. We call the tiled $\mathcal{H}(X)$ a *tiled toric hexagon.* Figure 6 shows an example of a toric hexagon $\mathcal{H}(X)$ of type $X = 120201210$.

Definition 2.5. A *north-strip* is a connected strip composed of adjacent 20- and 10-tiles. A *west-strip* is a connected strip composed of adjacent 20- and 21-tiles. The 20-tile and the 21-tile can contain a *left-arrow*, which is an arrow pointing to the left vertical edge of the tile, and is also pointing to every tile to its left in its west-strip. The 20-tile and the 10-tile can contain an *up-arrow*, which is an arrow pointing to the top horizontal edge of the tile, and is also pointing to every tile above it in its north strip. See Figure 6.

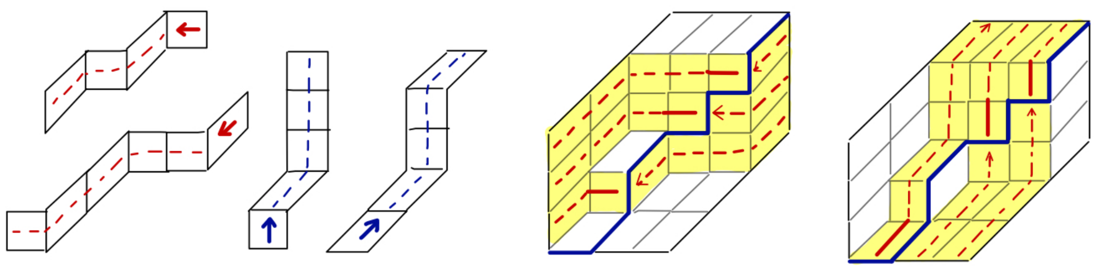

Figure 6: *Left*: west-strips containing a left-arrow which is pointing at the tiles to its left, as well as north-strips containing an up-arrow which is pointing at the tiles above it. *Middle*: a tiled toric hexagon with all its west-strips highlighted. *Right*: a tiled toric hexagon with all its north-strips highlighted. All strips begin at the tile to the northwest and adjacent to $P(X)$ and terminate at the tile to the southeast and adjacent to $P(X)$.

Identify the edges of $\mathcal{H}(X)$ between p_1 and p_6 with the edges between p_3 and p_4. Identify the edges of $\mathcal{H}(X)$ between p_1 and p_2 with the edges between p_5 and p_4. This makes $\mathcal{H}(X)$ a torus with one boundary component. If the edges of two tiles are identified, we say the tiles are adjacent. Following these identifications, north-strips and west-strips wrap around the hexagon. Each north-strip starts at the tile directly north $P(X)$ and ends at the tile directly south of $P(X)$. Similarly, each west-strip starts at the tile directly west of $P(X)$ and ends at the tile directly east of $P(X)$, as in Figure 6.

Definition 2.6. A tile is *pointed at* by an arrow if it is in the same west-strip to the left of a left-arrow or if it is in the same north-strip above an up-arrow. Conversely, a tile is *free* if it is not pointed at by any arrow.

Definition 2.7. A TRAT of type X is a filling of the tiles of a tiled toric hexagon $\mathcal{H}(X)$ with left-arrows and up-arrows according to the following rules:

i. A tile pointed to by an arrow in the same strip must be empty.

ii. An empty tile must be pointed to by some up-arrow or some left-arrow.

Figure 7 shows an example of all possible fillings of $\mathcal{H}(X)$ for $X = 120201210$.

Definition 2.8. We define the *weight* of $\mathcal{H}(X)$ with tiling $\mathcal{T}$ to be the number of possible fillings with up-arrows and left-arrows of this tiling, and we denote it by $\mathrm{wt}_{\mathcal{T}}(X)$.

The following lemma is obtained by following the proof of the same result for rhombic alternative tableaux in Proposition 2.8 of [13].

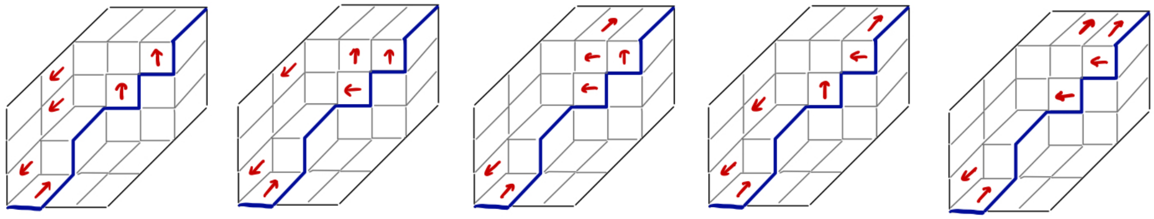

Figure 7: All possible fillings of $\mathcal{H}(X)$ for $X = 120201210$.

Lemma 2.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be two different tilings on $\mathcal{H}(X)$. Then

$$\mathrm{wt}_{\mathcal{T}}(X) = \mathrm{wt}_{\mathcal{T}'}(X).$$

As a consequence of the preceding lemma, we are able to define the weight of a state X.

Definition 2.9. Choose any tiling $\mathcal{T}$ on $\mathcal{H}(X)$ of size (k, r, ℓ). Define

$$\mathrm{weight}(X) = \mathrm{wt}_{\mathcal{T}}(X).$$

Definition 2.10. Let $X \in \mathrm{TASEP}(k, r, \ell)$. We denote by $o(X)$ the *order* of X, which is the number of elements in the class of cyclic shifts of X.

Our main result is the following.

Theorem 2.4. Let X be a state of the two-species TASEP on a ring of size (k, r, ℓ) with $n = k+r+\ell$. Then

$$\Pr(X) = \frac{o(X)}{\binom{n}{k}\binom{n}{\ell}} \mathrm{weight}(X).$$

Example 2.3. From Figure 3, we obtain that for $X = 120201210$, $f(X) = 5$, and so $\Pr(X) = \frac{9}{\binom{9}{3}\binom{9}{3}} \cdot 5 = \frac{5}{748}$ since there is a total of $\binom{9}{3}\binom{9}{3}$ MLQs of size $(3, 3, 3)$ and $o(X) = 9$. On the other hand, for $Y = 201201201$, $f(Y) = 8$, $o(Y) = 3$, and so $\Pr(Y) = \frac{1}{294}$.

We will first give a canonical Matrix Ansatz proof below, and in Section 3.2, we will show the TRAT is in bijection with the MLQs, from which our theorem follows due to Theorem 2.2.

We prove Theorem 2.4 by showing by induction on $|X|$ that $\mathrm{weight}(X)$ satisfies the same recurrences as the Matrix Ansatz of Theorem 2.1.

Proof. When X contains zero type 1 particles, an exceptional case occurs, since the trace of the matrix product of matrices D and E is no longer finite, so we cannot use the standard Matrix Ansatz proof. For X of size $(k, 0, \ell)$, $\mathcal{H}(X)$ is a $k \times \ell$ rectangle, and $\mathrm{weight}(X) = \binom{k+\ell}{k}$. (This can be derived with a standard lattice path bijection in the flavor of the Catalan tableaux that appear in [14], which we will not expand upon here.) It is also easy to check that the TASEP on a ring with fewer than two species of particles has uniform stationary distribution (each state has the same number of outgoing transitions as it has incoming transitions, so detailed balance holds). There is a total of $\binom{k+\ell}{k}^2$ MLQs and $\binom{k+\ell}{k}$ total states counting all cyclic shifts, and so $\Pr(X) = \frac{o(X)}{\binom{k+\ell}{k}}$ since X is counted $o(X)$ times. Consequently, Theorem 2.4 trivially holds in this case.

Let $f(X) = \text{tr}(\text{dae}(X))$, as defined in Theorem 2.1. When X has at least one type 1 particle, we will show that $\text{weight}(X) = f(X)$. Our proof is by induction on $|X|$.

For the base cases, when X has size $(k, r, 0)$ or $(0, r, \ell)$, $\mathcal{H}(X)$ consists of only 21-tiles or 10-tiles respectively, and so in each case, there is a unique filling of $\mathcal{H}(\mathcal{X})$. Thus since $DA = AE = A$, we trivially obtain $1 = \text{weight}(X) = f(X)$. Now, let n be such that for any $|W| < n$, it holds that $\text{weight}(W) = f(W)$.

Let X have size (k, r, ℓ) with $k, r, \ell > 0$ and $k + r + \ell = n$. Then there must be some cyclic shift of X such that one of the following occurs:

Case 1. $X = Y21$, or

Case 2. $X = Y20$

for some word Y of length $n - 2$. We fix the maximal tiling $\mathcal{T}_{max}$ on $\mathcal{H}(X)$ and consider each of these cases. (Recall that in the maximal tiling, every 2,1-corner has a 21-tile adjacent to it.)

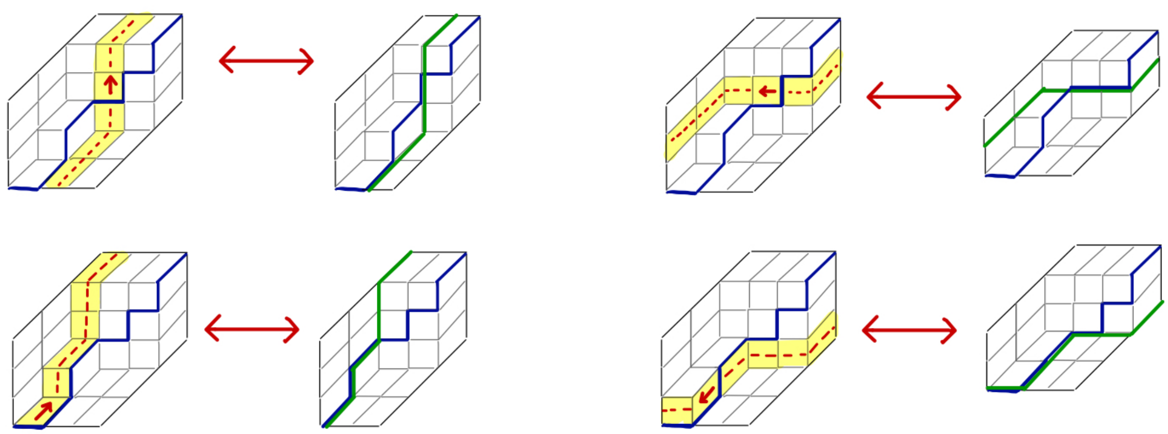

Figure 8: *Top left*: 20-tile t containing an up-arrow. *Top right*: 20-tile t containing a left-arrow. *Bottom left*: 10-tile t containing an up-arrow. *Bottom right*: 21-tile t containing a left-arrow. For all tableaux, the fillings are in bijection with fillings of tableaux with the highlighted strip containing t removed (the green path in the smaller tableaux indicates the location of the removed strip).

Case 1: $X = Y21$. The 21-tile adjacent to the 2, 1 pair of edges of $P(X)$ necessarily contains a left-arrow. Thus the remaining tiles of the west-strip $\mathfrak{s}$ originating at the 2-edge must be empty. Hence the fillings of $\mathcal{H}(X)$ are in bijection with $\mathcal{H}(X) \backslash \mathfrak{s}$ which is a tiled rhombic diagram of shape $Y1$. Consequently,

$$\text{weight}(Y21) = \text{weight}(Y1) = f(Y1) = f(Y21)$$

by the inductive hypothesis since $|Y1| < n$. Thus $\text{weight}(X) = f(X)$, as desired.

Case 2: $X = Y20$. The 20-tile adjacent to the 2, 0 pair of edges of $P(X)$ necessarily contains either a left-arrow or an up-arrow. In the left-arrow case, the remaining tiles of the west-strip $\mathfrak{s}$ originating at the 2-edge must be empty. Then the fillings of $\mathcal{H}(X)$ are in bijection with $\mathcal{H}(X) \backslash \mathfrak{s}$ which is a tiled rhombic diagram of shape $Y0$. In the up-arrow case, the remaining tiles of the north-strip $\mathfrak{s}$ originating at the 0-edge must be empty. Then the fillings of $\mathcal{H}(X)$ are in bijection with $\mathcal{H}(X) \backslash \mathfrak{s}$ which is a tiled rhombic diagram of shape $Y2$.

Figure 8 illustrates both of these cases.

Consequently,

$$\text{weight}(Y20) = \text{weight}(Y2) + \text{weight}(Y0) = f(Y2) + f(Y0) = f(Y20)$$

by the inductive hypothesis, and hence we obtain the desired result. □

Therefore, the TRAT indeed provide combinatorial formulae for the probabilities of the two-species TASEP on a ring.

2.3 Determinantal formula for probabilities of the two-species TASEP on a ring

We use the results of [12] to compute weight(X) using a determinantal formula that arises from the non-crossing paths Lingström-Gessel-Viennot Lemma.

Call an interval of of X consisting of 0 and 2 particles a *0,2-interval*. Partition X into r maximal 0,2-intervals $X^1, \ldots, X^r$.

Definition 2.11. Let $X \in \{0,2\}^{j+m}$ and let there be j 2's at locations $a_1, \ldots, a_j$. Define $\lambda(X)$ to be the partition associated to the Young diagram whose southeast boundary coincides with $P(X)$. Namely,

$$\lambda(X) = (m + 1 - a_1, m + 2 - a_2, \ldots, m + j - a_j).$$

For an example, see Figure 9.

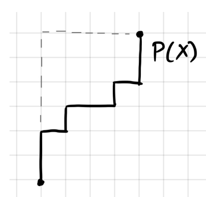

Figure 9: The Young diagram associated to 0,2-word $X = 2202002022$. Here $(a_1, \ldots, a_6) = (1, 2, 4, 7, 9, 10)$, and $\lambda(X) = (4, 4, 3, 1, 0, 0)$.

The following is derived in [12]. For a partition λ, define

$$A_\lambda = \left(\binom{\lambda_j + 1}{j - i + 1} \right)_{(i,j)}$$

Theorem 2.5. Let X be a state of the two-species ASEP on a ring. Partition X into 0,2-intervals $X_1, \ldots, X_{r+1}$. Then

$$\text{weight}(X) = \prod_{i=1}^{r+1} \det A_{\lambda(X_i)}.$$

3 Bijections

In this section, we describe two different bijections between MLQs and TRAT; one weight preserving and one not. The first bijection relies on a particular order for the ball drop algorithm on the MLQs which we discuss in the following subsection. In section 4.1, this weight-preserving bijection will permit us to define weighted multiline queues that give a combinatorial solution for the inhomogeneous TASEP. For the second bijection, the order of ball drops does not matter; we are still able to define weights on the MLQs, but the bijection with TRAT is no longer weight-preserving.

3.1 Refined multiline queue definition

Each multiline queue corresponds to a state of the circular ASEP, determined by the (order independent) *ball dropping algorithm* given in Section 1. We label the bottom row balls as 0-balls (balls occupied by a top row ball) and 1-balls (unoccupied balls).

To make our bijection well-defined, we first cyclically shift the MLQ to have a 1-ball at the left-most bottom row location. This implies no top row ball will wrap around the MLQ when it drops. Let the bottom row 0-balls be in locations $(x_1, \dots, x_\ell)$. Now, drop the top row balls from right to left. With each drop, the ball occupies the first unoccupied bottom row ball weakly to its right, while marking unmarked bottom row vacancies.

Let w_i be the number of unmarked vacancies that were marked by the dropping ball that occupied the 0-ball at location x_i. Set w_i to be the *weight* of x_i. Figure 10 shows an example with weights $(1, 1, 0, 0, 2, 0, 1)$.

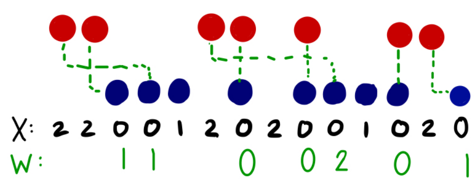

Figure 10: For $X = 22001202001020$, the hitting weights of the ball drops are $(1, 1, 0, 0, 2, 0, 1)$.

Lemma 3.1. At the end of the ball drop algorithm, the list of weights $(w_1, \dots, w_\ell)$ uniquely determines the initial configuration of top row balls.

The lemma is proved simply, by reversing the ball drop algorithm and "lifting" the bottom row 0-balls from right to left such that each ball at location x_i marks w_i unmarked vacancies. We call the reverse of a ball drop to a bottom row 0-ball at location x_i a *ball lift* from the 0-ball at location x_i, defined below.

Definition 3.1. Let $(x_1, \dots, x_\ell)$ be the locations of the 0-balls of an MLQ with corresponding weights $(w_1, \dots, w_\ell)$; all vacancies are initially unmarked. A *ball lift* from a 0-ball at location x_i with weight w_i is the following. A top row ball is placed directly above the w_i'th consecutive unmarked vacancy to the left of x_i, and each of those w_i vacancies becomes marked.

To show the ball lift is well-defined, i.e. that there are always w_i unmarked vacancies to the left of x_i with no 1-ball in between, we need the following lemma, the proof of which is obtained directly by following the ball-drop algorithm.

Lemma 3.2. Suppose M is an MLQ of size (k, r, ℓ), and in the bottom row, $x_1 < \dots < x_\ell$ are the locations of the 0-balls, and $b_1 \leq \dots \leq b_\ell$ are the locations of the *nearest* 1-balls, defined by

$$b_i = \max\{y < x_i \ : \ \text{1-ball at location } y\}.$$

Let $(w_1, \dots, w_\ell)$ be the weights on locations $(x_1, \dots, x_\ell)$ after the ball drops.

(i.) The conditions on $(w_1, \dots, w_\ell)$ are:

for each i,

$$\sum_{j:\ b_i < x_j \leq x_i} w_j + 1 \leq x_i - b_i.$$

In other words, there are enough vacancies to the left of x_i so that it can have weight w_i. We call such a list $(w_1, \dots, w_\ell)$ an X*-consistent* list.

(ii.)

Proof of Lemma 3.1. The lemma is equivalent to showing that M is the unique MLQ of type X with X-consistent weights $(w_1, \dots, w_\ell)$. We show this by reconstructing an MLQ of type X from an X-consistent list $(w_1, \dots, w_\ell)$. Let X have its 0 particles at locations $(x_1, \dots, x_\ell)$. Now perform ball lifts on the 0-balls from right to left (which is precisely the reverse of the ball-drop algorithm). For a ball lift with weight w_i to be possible, there must be at least w_i unmarked vacancies to the left of x_i with no 1-balls in between. This translates precisely to the requirement that

$$x_i - b_i - 1 - \sum_{j:\ b_i < x_j < x_i} w_j + 1 \geq w_i,$$

which we notice is the same as the condition placed on the w_i's in Lemma 3.2. Thus the ball drop algorithm has a well-defined inverse, and so the X-consistent list of weights $(w_1, \dots, w_\ell)$ corresponds to a unique MLQ of type X. □

3.2 Map from TRAT to MLQ

Let R be a TRAT of type $X \in \text{TASEP}(k, r, \ell)$. To describe a well-defined map R to a multiline queue, we first perform flips on the tiling of R to obtain the tiling $\mathcal{T}_X$, which we define below.

Definition 3.2. An X*-strip* is a north-strip obtained by reading X from left to right and placing a 20-tile for a 2 and a 10-tile for a 1 from top to bottom. Let $\mathcal{H}$ be a hexagon of size (k, r, ℓ). The tiling $\mathcal{T}_X$ is defined to be the top-justified placement of ℓ adjacent X-strips with $r\ell$ 21-tiles filling in the remaining space of $\mathcal{H}$. For an example, see Figure 11.

Note that, as in Figure 11, the edges of $P(X)$ are numbered from 1 to n from right to left, and so the X-strips have labels $x_1 < \cdots < x_\ell$ from right to left (which are also the locations of the 0's in X).

Lemma 3.3. The tiling $\mathcal{T}_X$ is a valid tiling of $\mathcal{H}$ with path $P(X)$.

The lemma is easily verified with a picture, such as in Figure 11, but we provide the proof below.

Proof. We want to show that all the edges of $P(X)$ coincide with edges of $\mathcal{T}_X$. We obtain $P(X)$ from $\mathcal{T}_X$ as follows.

Let $x_0 = 0$ and let $x_1, \dots, x_\ell$ be the locations of the ℓ 0's in X from left to right. Starting with $i = 1$, from the northeast corner of $\mathcal{H}$, draw a path P by following the east boundary of the i'th X-strip for $x_i - x_{i-1} - 1$ steps, and take a step west for the x_i'th step to switch to the $i+1$'st X-strip, up to $i = \ell$. After the x_ℓ'th step, follow the west boundary of the ℓ'th X-strip until the southwest corner of $\mathcal{H}$ is reached.

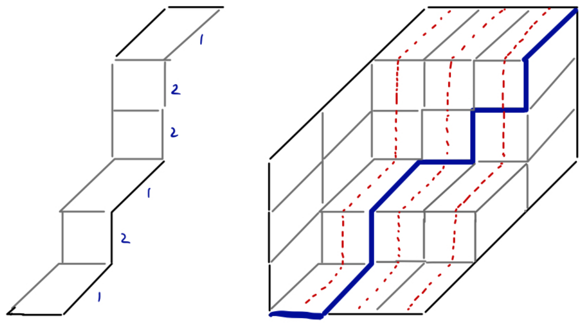

Figure 11: For $X = 120201210$, on the left is shown an X-strip, and on the right the tiling $\mathcal{T}_X$ on $\mathcal{H}(X)$ with path $P(X)$.

Since the X-strip is obtained simply from excising the 0's from X, $P = P(X)$ by our construction. □

Without loss of generality, let X begin with a 1. Note that any north-strip that does not have an up-arrow below a 10-tile will necessarily acquire an up-arrow at the 10-tile. Thus there can be no arrows in north-strips above any 10-tiles. In particular, this implies the TRAT R of type X has all of the left-arrows and up-arrows contained in its ℓ X-strips above the path $P(X)$ in $\mathcal{H}(X)$, and so there is no ambiguity about which strip to start with.

We build an MLQ mlq(R) from R as follows. Let the bottom row of mlq(R) have type X. Let R have its north-strips at locations $x_1 < \cdots < x_\ell$ (from right to left) with a_i left-arrows in strip x_i for each i. Perform ball-lifts (of Definition 3.1) sequentially for $x_1, \ldots, x_\ell$, with weights $a_1, \ldots, a_\ell$, to obtain a unique MLQ mlq(R) with those weights. Figure 12 shows an example, and the following lemma shows our bijection is well-defined.

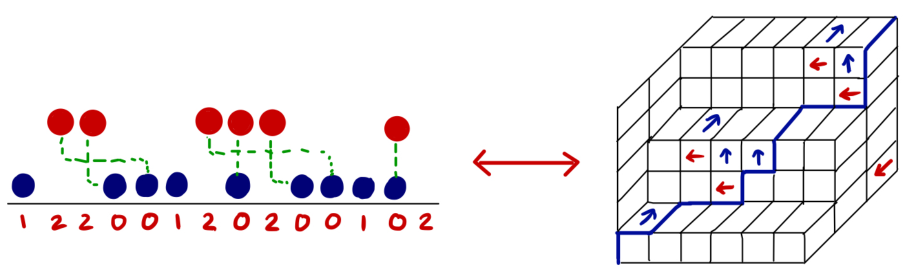

Figure 12: The weights of the ball drops of the MLQ on the left are $(1, 1, 0, 1, 1, 0)$, which is also the number of left-arrows in each north-strip from right to left in the corresponding TRAT on the right.

Lemma 3.4. Suppose the north-strips of R are at locations $x_1 < \cdots < x_\ell$, with strip x_i containing a_i left-arrows for each i. Then $(a_1, \ldots, a_\ell)$ is an X-consistent list, and thus there exists a unique MLQ of type X with weights $(a_1, \ldots, a_\ell)$.

Recall that a free 20-tile is one that does not have a left-arrow to its right in the same west-strip or an up-arrow below in the same north-strip.

Proof. Our proof will show that there is a natural map between the number of left-arrows in north-strip x_i in R and the weight w_i of the 0-ball at location x_i in $\mathrm{mlq}(R)$.

If a north-strip at location x_i contains a_i left-arrows, then it must have at least a_i free 20-tiles below its first 10-tile. Let b_i be the index of the diagonal strip containing the nearest 10-tile in strip x_i. Since diagonal strips cannot intersect, b_i is the index of the nearest diagonal edge to the right of x_i in $P(X)$. In other words, $b_i = \max\{y \ : \ y < x_i,\ X_y = 1\}$.[1] Then we have the following conditions on the a_i's. For each i,

$$\sum_{j:\ a_i < x_j \leq x_i} a_j + 1 \leq x_i - b_i.$$

Observe that the conditions on the list $(a_1, \ldots, a_\ell)$ make it an X-consistent list. Thus by Lemma 3.1, there exists a unique MLQ $M(R)$ of type X with weights $(a_1, \ldots, a_\ell)$, obtained by performing ball lifts sequentially for $x_1, \ldots, x_\ell$. This completes our proof. □

The inverse map from MLQ M to a TRAT $\mathrm{trat}(M)$ is obtained similarly. Let M have type X with the 0-balls at locations $x_1 < \cdots < x_\ell$, and with corresponding weights $(w_1, \ldots, w_\ell)$. Construct a TRAT with shape $\mathcal{H}(X)$ with tiling $\mathcal{T}_X$ such that strip x_i has w_i left-arrows for each i (strips are ordered from right to left). This construction is well-defined since the list $(w_1, \ldots, w_\ell)$ is X-consistent, which is a sufficient condition for a TRAT with such properties to exist. It is unique by construction: when filling the tableau from right to left, in each north-strip the left-arrows must be placed in consecutive free tiles from bottom to top. The maps $R \to \mathrm{mlq}(R)$ and $M \to \mathrm{trat}(M)$ are immediately inverses of each other.

3.3 Nested path map from MLQ to TRAT

Using nested lattice paths, we obtain a different bijection from MLQs to TRAT. In the following, we assume the MLQ has a 1-ball at its leftmost bottom row location.

A multiline queue naturally has an interpretation in terms of weighted lattice paths, where each row of the MLQ is mapped to a path, and each location in the row determines the type of edge that appears in the path. At $q = 0$, fillings of RAT are in bijection with weighted nested lattice paths in the usual 2-TASEP [12], and indeed this property is preserved in the case of the TRAT. Unfortunately, the pairs of lattice paths corresponding to the MLQs are not the same paths that are in bijection with the TRAT. That is, the pair of nested paths corresponding to the TRAT $\mathrm{trat}(M)$ is not the same as the pair of nested paths directly obtained from M. However, the set of paths of type X that is obtained from MLQs of type X is the same as the set of paths of type X obtained from TRAT of type X; see, for example, Figure 13.

Definition 3.3. A *2-TASEP compatible* pair of lattice paths is a pair of paths composed of south, west, and southwest edges that coincide at their endpoints, such that the space between the two paths can be completely tiled by squares.

We construct a pair of lattice paths from an MLQ as follows: the first path, P_1, is obtained by reading the bottom row of the MLQ and drawing a south edge for every vacancy, a west edge for every 0-ball, and a southwest edge for every 1-ball. The second path, P_2, is obtained by reading

[1] We note here that this definition of b_i is precisely the location of the first 10-tile only when the particular tiling $\mathcal{T}_X$ is used. That is because the order of the 2- and 1-edges in $P(X)$ matches the order of the 20- and 10-tiles in the x_i north-strip.

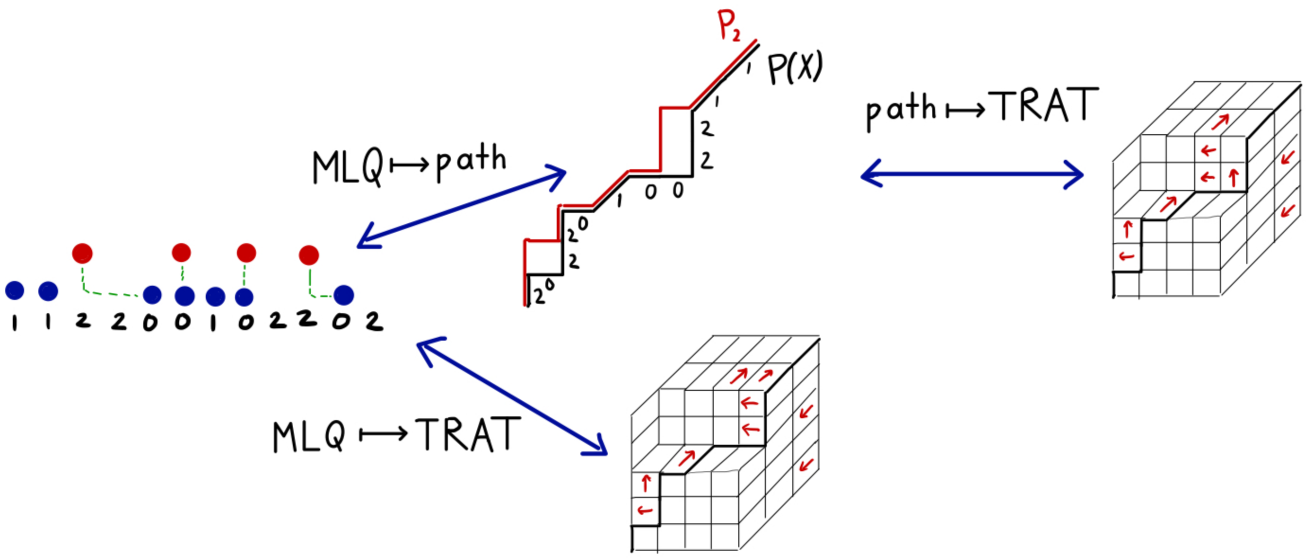

Figure 13: The map from a MLQ to a nested pair of lattice paths, which then maps to a TRAT via the canonical lattice path bijection. The second map is from the MLQ M to the TRAT trat(M). Notice that the two resulting TRAT are not the same.

the top row of the MLQ and drawing a southwest edge for a vacancy directly above a bottom row 1-ball, a south edge for a vacancy otherwise, and a west edge for a ball.

Lemma 3.5. By our construction, P_2 is weakly above P_1, and they coincide at every diagonal edge.

Proof. We consider an interval of vacancies and 0-balls between any two 1-balls at locations a and b in the bottom row of M. At each location $a < j \leq b$, there must be at least as many top row balls as there are bottom row 0-balls between locations a and j; otherwise, there will be an unoccupied 0-ball, which is a contradiction. Moreover, between a and b, there must be exactly the same number of top row balls as there are bottom row 0-balls. The latter implies P_2 and P_1 coincide at every diagonal edge. Hence for each $a < j \leq b$, P_1 takes at least as many steps south as does P_2 between edges a and j, and thus P_2 lies weakly above P_1. □

Lemma 3.6. P_1 and P_2 are 2-TASEP compatible paths.

Proof. By Lemma 3.5, the space between the two paths is always bounded by horizontal and vertical edges, and thus can be tiled completely by 20-tiles. □

It is easy to see that any pair of 2-TASEP compatible lattice paths corresponds to a unique multiline queue and vice versa. Building on the author's earlier paper [12], 2-TASEP compatible lattice paths are also in bijection with TRAT. This bijection arises from the canonical lattice path bijection of Catalan paths and Catalan tableaux in the well-studied case of the usual TASEP [14]. We describe the map briefly. A path weakly above the path $P(X)$ for a TRAT of shape $\mathcal{H}(X)$ is constructed as follows.

The path P_2 begins and ends at the endpoints of $P(X)$. It contains n edges, r of which are diagonal, k of which are vertical, and ℓ of which are horizontal; its edges are labeled from right to left. Suppose X has its type 1 particles at locations $b_1, \ldots, b_r$. Then P_2 has its diagonal edges at locations $b_1, \ldots, b_r$.

At each 0-edge of $P(X)$, the path P_2 takes j vertical steps down and one horizontal step left, where j is the total number of left-arrows in the 20-tiles of that 0-strip. Once P_2 has reached the left border of $\mathcal{H}(X)$, it takes vertical steps down until it reaches the left endpoint of $P(X)$. See an example in Figure 14.

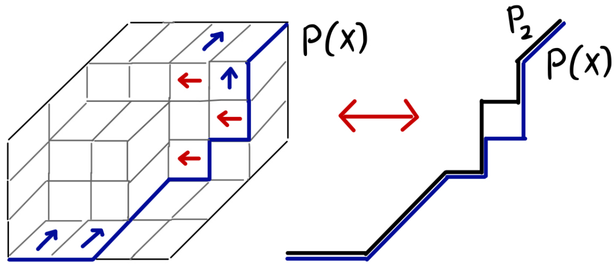

Figure 14: An example of the canonical map from a TRAT to a pair of nested lattice paths of type 1220201100.

The reverse map is as follows: begin with a pair of 2-ASEP compatible nested paths $P(X)$ and P_2, assuming X begins with a type 1 particle. Starting from right to left, let each 0-strip of the filling of $\mathcal{H}(X)$ contain j left-arrows in its 20-boxes, followed by an up-arrow, where j is the number of down-steps in P_2 preceding the horizontal step corresponding to the given 0-strip. There is a unique way of filling this 0-strip in such a way: the left-arrows must be in the lowest tiles possible, immediately followed by the up-arrow. See an example in Figure 13.

This bijection is well-defined due to the following lemma.

Lemma 3.7. 2-TASEP compatible paths of type X are in one to one correspondence with an X-consistent list.

Proof. Let $x_1 < \cdots < x_\ell$ be the indices of the 0 particles in X. Let a_i be the number of south steps in P_2 on the right of the x_i-column. Let b_i be the index of the nearest 1-particle to the left of x_i. P_2 is always weakly above P_1, and there can never be more south steps in P_2 than there are south steps in P_1 in the same interval. Thus

$$\sum_{j:\ x_i \geq x_j > b_i} a_j + 1 \leq x_i - b_i,$$

which is precisely the condition for $(a_1, \ldots, a_\ell)$ to be an X-consistent list.

On the other hand, if $(a_1, \ldots, a_\ell)$ satisfies the equation above, we have that at every x_i column, there are at least a_i possible south steps P_2 can take so that it is still weakly above P_1. Thus P_2 and P_1 with the given X-consistent list of south steps are indeed 2-TASEP compatible paths. □

4 Inhomogeneous 2-TASEP on a ring

We define the inhomogeneous 2-TASEP on a ring Markov chain as follows: let $X \in \text{TASEP}(k,r,\ell)$. The transitions on this Markov chain are:

$$X'20X'' \xrightarrow{\text{t}} X'02X''$$
$$X'21X'' \xrightarrow{\text{d}} X'12X''$$
$$X'10X'' \xrightarrow{\text{e}} X'01X''$$

where $0 \leq t,d,e \leq 1$ are parameters describing the hopping rates. When $t = d = e = 1$, we recover the usual 2-TASEP on a ring. When $t = e$, we recover the inhomogeneous TASEP studied by Ayyer and Linusson in [3], where they defined weights on MLQs to solve a conjecture of Lam and Williams [9] (our solution specializes to theirs after some manipulation). The advantage of our tableaux interpretation of 2-TASEP probabilities is that we can introduce additional weights to the TRAT which correspond to an inhomogeneous 2-TASEP. Define $\text{wt}(X)$ to be the unnormalized steady state probability of state X. We will show it is a polynomial in t,d,e with coefficients in $\mathbb{Z}^+$ by expressing it as a sum over the weighted tableaux.

The Matrix Ansatz of Theorem 2.1 naturally generalizes to the following inhomogeneous version.

Theorem 4.1. Let X be a state of the inhomogeneous 2-TASEP. Let D, A, and E be matrices satisfying:

$$\begin{aligned} tDE &= D+E \\ dDA &= A \\ eAE &= A \end{aligned} \tag{4.1}$$

then the stationary probability $\text{Prob}(X)$ is proportional to $\text{tr}(\text{dae}(X))$, where $\text{dae}(X)$ is given by Definition 2.1.

Define $\mathbf{d} = d^{-1}, \mathbf{e} = e^{-1}$, and $\mathbf{t} = t^{-1}$. A set of matrices that satisfy the conditions of the Ansatz are:

$$D^* = \begin{pmatrix} 0 & \mathbf{d} & 0 & 0 & \\ 0 & 0 & \mathbf{d} & 0 & \dots \\ 0 & 0 & 0 & \mathbf{d} & \\ 0 & 0 & 0 & 0 & \\ & \vdots & & & \ddots \end{pmatrix} \qquad A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ \mathbf{d} & 0 & 0 & 0 & \dots \\ \mathbf{d}^2 & 0 & 0 & 0 & \\ \mathbf{d}^3 & 0 & 0 & 0 & \\ & \vdots & & & \ddots \end{pmatrix} \qquad E^* = \begin{pmatrix} \mathbf{t} & 0 & 0 & 0 & \\ \mathbf{t}\,\mathbf{e} & \mathbf{t} & 0 & 0 & \dots \\ \mathbf{t}^2\,\mathbf{e} & \mathbf{t}^2 & \mathbf{t} & 0 & \\ \mathbf{t}^3\,\mathbf{e} & \mathbf{t}^3 & \mathbf{t}^2 & \mathbf{t} & \\ & \vdots & & & \ddots \end{pmatrix}$$

Example 4.1. For example, $\text{weight}(2201021) = \text{tr}(DDEAEDA) = \mathbf{d}\,\mathbf{e}^2\,\mathbf{t}^2 + \mathbf{d}^2\,\mathbf{e}\,\mathbf{t}^2 + \mathbf{d}^3\,\mathbf{e}\,\mathbf{t}$ where D, A, E satisfy Equations 4.1.

Definition 4.1. For $X \in \text{TASEP}(k,r,\ell)$, we call $\text{TRAT}(X)$ the set of TRAT on a toric diagram of type X with some fixed tiling $\mathcal{T}$.

We introduce a weight on the TRAT, which is a monomial in $\mathbf{t}, \mathbf{d}, \mathbf{e}$, and is denoted by $\text{wt}(R)$ for $R \in \text{TRAT}(X)$. We define $\text{wt}(X) = \sum_{R \in \text{TRAT}(X)} \text{wt}(R)$, which we will show satisfies the same recurrences as $\text{tr}(\text{dae}(X))$ in the Matrix Ansatz. Given the existence of D^*, A^*, E^* above, we will thus obtain that $\text{Prob}(X)$ is proportional to $\text{wt}(X)$.

Definition 4.2. Let R be a TRAT. Define the following:

- $|R|_{20}$ is the number of occupied 20-tiles in R.
- $|R|_{21}$ is the number of occupied 21-tiles in R.
- $|R|_{10}$ is the number of occupied 10-tiles in R.

Definition 4.3. Let $X \in \mathrm{TASEP}(k, r, \ell)$ with $R \in \mathrm{TRAT}(X)$. The *weight* of R denoted by $\mathrm{wt}(R)$, is given by

$$\mathrm{wt}(R) = \mathbf{t}^{|R|_{20}}\, \mathbf{d}^{|R|_{21}}\, \mathbf{e}^{|R|_{10}}.$$

For , define $\mathrm{wt}(X) = \sum_{R \in \mathrm{TRAT}(X)} \mathrm{wt}(R)$.

Example 4.2. For example, the TRAT in Figure 14 has weight $\mathbf{t}^4\,\mathbf{d}\,\mathbf{e}^2$ since it has 4, 1, and 2 occupied 20-tiles, 21-tiles, and 10-tiles, respectively.

One can check combinatorially, for instance following the proof of Theorem (♣ what? ♣) of [13], that

$$\begin{aligned} t\,\mathrm{wt}(X'20X'') &= \mathrm{wt}(X'2X'') + \mathrm{wt}(X'0X''), \\ d\,\mathrm{wt}(X'21X'') &= \mathrm{wt}(X'1X''), \\ e\,\mathrm{wt}(X'10X'') &= \mathrm{wt}(X'1X'') \end{aligned}$$

for some 2-TASEP words X', X''. This is done by reducing a TRAT of type X to a tableau of smaller size by removing a north-strip or a west-strip. We will not reproduce this (fairly standard) proof, and instead we will further build on the connection between the TRAT and the multiline queues by defining a weighted version of the multiline queues using the bijection of Section 3.2, and then proving the recurrences are satisfied on the weighted MLQs.

4.1 Multiline queue associated to the inhomogeneous 2-TASEP on a ring

We introduce a *weighted multiline queue* (WMLQ) that generalizes the definition of the multiline queue of Section 3.1. The ball drop algorithm for the WLMQ is the same as for the usual MLQ, and the type of the WMLQ is also obtained in the same way.

Definition 4.4. A 0-ball is *unrestricted* if, immediately following its ball drop, there is an unmarked vacancy to its left with no 1-ball in between.

Example 4.3. In Figure 15, the 0-balls in locations 3 and 5 are unrestricted because at the time they are occupied, the vacancy at location 2 remains unmarked. However, when the 0-balls at locations 6 and 11 are occupied, there are no unmarked vacancies to their left before the nearest 1-ball, so those 0-balls are restricted.

Definition 4.5. A *weighted multiline queue* (WMLQ) is a usual multiline queue with weights $\mathbf{t}, \mathbf{d}, \mathbf{e}$ assigned to each entry in the bottom row, as follows:

- every 1-ball receives weight 1,
- every marked vacancy receives a weight of $\mathbf{t}$,
- if a 0-ball is unrestricted, that ball receives a weight of $\mathbf{t}$,
- at the end of all the ball drops, every unmarked vacancy receives a weight of $\mathbf{d}$,
- at the end of all the ball drops, every remaining bottom row ball receives a weight of $\mathbf{e}$.

Definition 4.6. The *weight* $\mathrm{wt}(M)$ of an MLQ M is the monomial obtained by taking the product of the weights assigned to the bottom row. In other words, if we define $\mathrm{rest}(M)$ to be the number of restricted 0-balls and $\mathrm{umv}(M)$ to be the number of unmarked vacancies, for $M \in \mathrm{MLQ}(k,r,\ell)$ we obtain

$$\mathrm{wt}(M) = \mathbf{t}^{k+\ell-\mathrm{rest}(M)-\mathrm{umv}(M)}\,\mathbf{d}^{\mathrm{umv}(M)}\,\mathbf{e}^{\mathrm{rest}(M)}\,.$$

Example 4.4. The MLQ in Figure 15 has type $X = 12200120200102$ and weight $\mathrm{wt}(M) = \mathbf{t}^8\,\mathbf{d}\,\mathbf{e}^2$. Observe that the 0-balls in locations 5, 8, 10, and 13 are unrestricted and have weight $\mathbf{t}$, and the 0-balls at locations 4 and 11 are restricted and have weight $\mathbf{e}$. All vacancies except for the one at location 14 are marked and have weight $\mathbf{t}$.

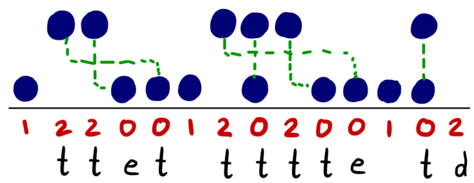

Figure 15: Weighted MLQ $M \in \mathrm{MLQ}(X)$ for $X = 12200120200102$ and weight $\mathrm{wt}(M) = \mathbf{t}^8\,\mathbf{d}\,\mathbf{e}^2$.

Proposition 4.2.

$$\mathrm{wt}(X) = \sum_{M \in WMLQ(X)} \mathrm{wt}(M).$$

Remark. The definition of the ball drops differs from the usual definition of bully paths on MLQ's, since the ball drops must occur in order from right to left, whereas for usual bully paths, the order of the ball drops is inconsequential. The reason for this in our algorithm is to determine which balls receive weight $\mathbf{t}$, and which receive weight $\mathbf{e}$. Recall that a bottom row ball receives weight $\mathbf{t}$ only if there is an unmarked vacancy to its left immediately following its ball drop.

When we set $t = e$ (and $\mathbf{t} = \mathbf{e}$), the weighted MLQ reduces to the following: let $u(M)$ be the number of unmarked vacancies of an MLQ of size (k,r,ℓ). Then $\mathrm{wt}(M) = \mathbf{t}^{\ell+k-u(M)}\,\mathbf{d}^{u(M)}$. From the formula in [3], the weight of M is computed to be $t^k\left(\frac{d}{t}\right)^{k-u(M)} = t^{u(M)}d^{k-u(M)}$. Since $\mathbf{t} = t^{-1}$ and $\mathbf{d} = d^{-1}$, we see these two weights are equivalent up to a constant factor of $t^{\ell+k}d^k$.

To show the weighted MLQ's indeed provide a formula for inhomogeneous 2-TASEP probabilities, we give a standard Matrix Ansatz proof.

Lemma 4.3. Let M be an MLQ whose entries are represented as $M = \binom{y_1\ \cdots\ y_n}{x_1\ \cdots\ x_n}$ for $x_i \in \{0,1,2\}$ and $y_i \in \{\mathbf{v},\mathbf{b}\}$, with $\mathbf{0},\mathbf{1},\mathbf{2}$ representing a bottom row 0-ball, 1-ball, or vacancy respectively, and with $\mathbf{v},\mathbf{b}$ representing a top row vacancy or ball, respectively. Suppose $x_i, x_{i+1} = \mathbf{2},\mathbf{0}$, and let $M' = \binom{y_1\ \cdots\ y_{i-1}}{x_1\ \cdots\ x_{i-1}}$ and $M'' = \binom{y_{i+2}\ \cdots\ y_n}{x_{i+2}\ \cdots\ x_n}$. Then

$$\mathrm{wt}(M) = \begin{cases} \mathbf{t}\,\mathrm{wt}(M'\binom{y_i}{x_i}M'') & \text{if } y_{i+1} = \mathbf{b} \\ \mathbf{t}\,\mathrm{wt}(M'\binom{y_i}{x_{i+1}}M'') & \text{if } y_{i+1} = \mathbf{v}. \end{cases} \tag{4.2}$$

Proof. If $y_{i+1} = \mathbf{v}$, then the 0-ball at x_{i+1} must be occupied by some top row ball that passes location i. Thus x_i is necessarily a marked vacancy after x_{i+1} is occupied, and hence has weight t. Removing the vacancy and shifting y_i to location $i+1$ has no effect on the rest of the MLQ.

If $y_{i+1} = \mathbf{b}$, then removing the entire column at location $i+1$ has no effect on the rest of the MLQ since the ball at y_{i+1} always drops directly on top of the 0-ball at x_{i+1}. Moreover, the 0-ball at x_{i+1} acquires weight t since at the time it is occupied, the vacancy at x_i is unmarked - since y_{i+1} is dropped before any top row balls to its left. Thus we obtain Equation (4.2). □

Theorem 4.4. Let $X \in \text{TASEP}(k, r, \ell)$ and let $f(X) = \text{tr}(\text{dae}(X))$ as defined in Theorem 4.1. Then

$$f(X) = \sum_{M \in \text{MLQ}(X)} \text{wt}(M). \tag{4.3}$$

Proof. Our proof is by induction on the size of X.

For our base case, we consider X which has no instance of 20. For such X, there is a unique MLQ, since no bottom row 0-ball has a vacancy to its left without a 1-ball in between, so every 0-ball must be occupied by a top row ball directly above it. This also implies there are no marked vacancies and every 0-ball is restricted. Thus the k vacancies contribute weight $\mathbf{d}^k$ and the ℓ 0-balls contribute weight $\mathbf{e}^\ell$, so $\sum_{M \in \text{MLQ}(X)} \text{wt}(M) = \mathbf{d}^k\,\mathbf{e}^\ell$. On the Matrix Ansatz side, we directly obtain $f(X) = \mathbf{d}^k\,\mathbf{e}^\ell$. In particular, this is true with $k = 0$ or $\ell = 0$.

Now, suppose we have K, L, such that for any $X \in \text{TASEP}(k, r, \ell)$ with $k \leq K$ and $\ell < L$, Equation (4.3) is satisfied. We will show that Equation (4.3) is also satisfied for $Y \in \text{TASEP}(k, r, \ell)$ where $k, \ell = K, L$. By our base case, if Y has no instance of 20, we are done. Otherwise, let $Y = Y'20Y''$ with $2, 0$ in positions $i, i+1$.

We partition the set of MLQs of type Y into two depending on the contents of column $i+1$:

$$\text{MLQ}(Y) = \left\{ M \in \text{MLQ}(Y) : M = M' \binom{y_i}{\mathbf{2}} \binom{\mathbf{v}}{\mathbf{0}} M'' \right\} \bigcup \left\{ M \in \text{MLQ}(Y) : M = M' \binom{y_i}{\mathbf{2}} \binom{\mathbf{b}}{\mathbf{0}} M'' \right\}.$$

We show the following are bijections:

$$\text{MLQ}(Y'0Y'') \Longleftrightarrow \left\{ M \in \text{MLQ}(Y) : M = M' \binom{y_i}{\mathbf{2}} \binom{\mathbf{v}}{\mathbf{0}} M'' \right\} \tag{4.4}$$

$$\text{MLQ}(Y'2Y'') \Longleftrightarrow \left\{ M \in \text{MLQ}(Y) : M = M' \binom{y_i}{\mathbf{2}} \binom{\mathbf{b}}{\mathbf{0}} M'' \right\} \tag{4.5}$$

For the first equation, let $M = M'\binom{y_i}{\mathbf{2}}\binom{\mathbf{v}}{\mathbf{0}}M'' \in \text{MLQ}(Y)$. Then $M'\binom{y_i}{\mathbf{0}}M'' \in \text{MLQ}(Y'0Y'')$ since the 0-ball is still occupied by the same ball in both M and the reduced MLQ. Moreover, given an MLQ $\hat{M} = M'\binom{y_i}{\mathbf{0}}M'' \in \text{MLQ}(Y'0Y'')$, it is immediate that inserting two vacancies to obtain $M'\binom{y_i}{\mathbf{2}}\binom{\mathbf{v}}{\mathbf{0}}M''$ gives back M, thus establishing the bijection.

Similarly, let $M = M'\binom{y_i}{\mathbf{2}}\binom{\mathbf{b}}{\mathbf{0}}M'' \in \text{MLQ}(Y)$. Then $M'\binom{y_i}{\mathbf{2}}M'' \in \text{MLQ}(Y'2Y'')$ since the column at location $i+1$ in M had no effect on the rest of the MLQ, keeping its type the same minus the 0 in location $i+1$. It's clear this is a bijection as well. Thus we obtain

$$\sum_{\substack{M \in \text{MLQ}(Y),\\ M = M'\mathbf{x}_i\mathbf{x}_{i+1}M''}} \text{wt}(M) = \sum_{\substack{M \in \text{MLQ}(Y),\\ M = M'\mathbf{x}_i\mathbf{x}_{i+1}M''}} \sum_{x} \text{wt}(M'\binom{x}{\mathbf{2}}\binom{\mathbf{b}}{\mathbf{0}}M'') + \text{wt}(M'\binom{x}{\mathbf{2}}\binom{\mathbf{v}}{\mathbf{0}}M'').$$

By Lemma 4.3, this equals

$$\sum_{\substack{M\in\mathrm{MLQ}(Y),\\ M=M'\mathbf{x}_i\mathbf{x}_{i+1}M''}} \sum_x \mathbf{t}\,\mathrm{wt}(M'\binom{x}{\mathbf{2}}M'') + \mathbf{t}\,\mathrm{wt}(M'\binom{x}{\mathbf{0}}M''),$$

which reduces to

$$\sum_{M\in\mathrm{MLQ}(Y'20Y'')} \mathrm{wt}(M) = \sum_{M\in\mathrm{MLQ}(Y'0Y'')} \mathrm{wt}(M) + \sum_{M\in\mathrm{MLQ}(Y'2Y'')} \mathrm{wt}(M)$$

by our arguments above. Consequently, by our induction assumption and Theorem 4.1, this equals

$$\sum_{M\in\mathrm{MLQ}(Y)} \mathrm{wt}(M) = \mathbf{t}\left(f(Y'2Y'') + f(Y'0Y'')\right) = f(Y),$$

thus completing the proof. □

Theorem 4.5. Let $T = TRAT(M)$ for weighted MLQ M. Then

$$\mathrm{wt}(T) = \mathrm{wt}(M).$$

Proof. Every north-strip in a TRAT has exactly one up-arrow, and every west-strip has exactly one left-arrow. If an up-arrow (resp. left-arrow) is not in a 20-tile, it is in a 10-tile (resp. 21-tile). Following the MLQ-TRAT bijection in Section 3.2, by construction the left-arrows in the 20-tiles are precisely those that correspond to marked vacancies in M, and hence $|T|_{21} = \mathrm{umv}(M)$. Unmarked vacancies contribute $\mathbf{d}$ to $\mathrm{wt}(M)$, so the power of $\mathbf{d}$ is the same for $\mathrm{wt}(T)$ and $\mathrm{wt}(M)$.

Recall that we call a free tile in a TRAT one that is not pointed to by (or already contains) a left-arrow. In each north-strip, the up-arrow is placed in the bottom-most free tile. Let M have a 0-ball at location j with dropping weight w. When this 0-ball is unrestricted, there is an unmarked gap to its left at the time it is occupied. Balls are dropped from right to left and north-strips of T are filled from right to left: thus an unmarked gap to the left of a 0-ball implies there is a free 20-tile in north-strip j above the w 20-tiles containing the left-arrows. Consequently, the up-arrow is contained in a 20-tile in strip j, contributing a weight of t. On the other hand if the 0-ball is restricted, the opposite occurs, and there are no free 20-tiles in north-strip j. Thus the up-arrow is contained in a 10-tile, and so $|T|_{10} = \mathrm{rest}(M)$ with the latter contributing a weight of $\mathbf{e}$ to $\mathrm{wt}(M)$. Thus the power of $\mathbf{e}$ is the same for $\mathrm{wt}(T)$ and $\mathrm{wt}(M)$, from which we can deduce that $\mathrm{wt}(T) = \mathrm{wt}(M)$. □

Example 4.5. In Figure **??** we show all multiline queues in the set $\mathrm{MLQ}(2,1,1)$ and their corresponding TRAT, along with the weights. From Theorem 4.4 we conclude that

$$\begin{aligned}
\Pr(2210) &= \frac{1}{\mathcal{Z}_{2,1,1}}\,\mathbf{d}^2\,\mathbf{e}\\
\Pr(2021) &= \frac{1}{\mathcal{Z}_{2,1,1}}(\mathbf{t}\,\mathbf{d}\,\mathbf{e} + \mathbf{t}\,\mathbf{d}^2)\\
\Pr(2201) &= \frac{1}{\mathcal{Z}_{2,1,1}}(\mathbf{t}^2\,\mathbf{e} + \mathbf{t}^2\,\mathbf{d} + \mathbf{t}\,\mathbf{d}^2),
\end{aligned}$$

where $\mathcal{Z}_{2,1,1} = \mathbf{d}^2\,\mathbf{e} + \mathbf{t}\,\mathbf{d}\,\mathbf{e} + \mathbf{t}\,\mathbf{d}^2 + \mathbf{t}^2\,\mathbf{e} + \mathbf{t}^2\,\mathbf{d} + \mathbf{t}\,\mathbf{d}^2$.

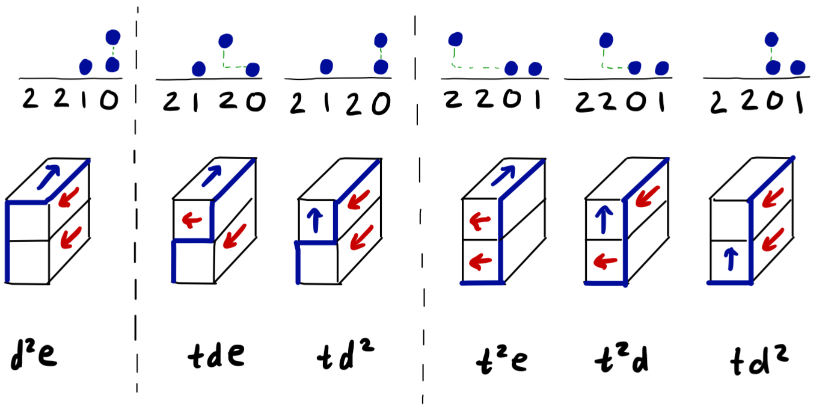

Figure 16: All 6 elements of MLQ$(2,1,0)$ and the corresponding elements of TRAT$(2,1,0)$ along with their respective weights.

Corollary 4.6. The TRAT Markov chain projects to the inhomogeneous TASEP when the stationary probability of a TRAT is its weight. Thus for X a state of the inhomogeneous TASEP,

$$\text{Prob}(X) \propto \sum_{T \in TRAT(X)} \text{wt}(T).$$

Remark. We obtain a solution to the inhomogeneous TASEP studied in [8] by setting $\mathbf{e} = \mathbf{t} = x_1$, $\mathbf{d} = x_2$, where x_1 is the rate of the transition $20 \to 02$, $10 \to 01$, and x_2 is the rate of the transition $21 \to 12$.

5 Inhomogeneous two-species ASEP with open boundaries and enhanced rhombic alternative tableaux

The two-species asymmetric simple exclusion process with open boundaries (2-ASEP) is a Markov chain whose states are configurations of particles of type 0, 1, and 2 on a finite lattice with open boundaries. The states are represented by words $X = X_1 \dots X_n$ with $X_i \in \{2,1,0\}$ and the possible transitions are:

- two adjacent particles $X_i X_{i+1}$ can swap with rate 1 if $X_i > X_{i+1}$ or rate q if $X_i < X_{i+1}$,
- at X_1, particle 0 can be replaced by particle 2 with rate α, and
- at X_n, particle 2 can be replaced by particle 0 with rate β.

The 2-ASEP has been studied by many including [15, **?**, 1, 7], and a tableaux solution was recently discovered by the author and X. Viennot in [13]. In Section 5.1 we generalize the 2-ASEP to an inhomogeneous process similar to the inhomogeneous TASEP on a ring of Section 4, and likewise generalize the tableaux to solve the inhomogeneous model.

In the (homogeneous) 2-ASEP the transitions are governed by parameters α, β, and q as follows.

Let X' and X'' be 2-ASEP words:

$$X'20X'' \underset{q}{\overset{1}{\rightleftarrows}} X'02X'' \qquad\qquad 0X' \overset{\alpha}{\to} 2X'$$

$$X'21X'' \underset{q}{\overset{1}{\rightleftarrows}} X'12X'' \qquad\qquad X'2 \overset{\beta}{\to} X'0$$

$$X'10X'' \underset{q}{\overset{1}{\rightleftarrows}} X'01X''$$

The number of type 1 particles is conserved. Thus we define 2-ASEP(n, r) to be the set of 2-ASEP words of length n with exactly r particles of type 1.

Remark. Classically, the 2-ASEP is described as a model describing the dynamics of two species of particles, *heavy* and *light*, hopping left and right on a a lattice of n sites, such that heavy particles can replace a vacancy at the first location, and can be replaced by a vacancy at the n'th location. In the bulk, a particle can swap places with a vacancy or two adjacent particles of different species can swap, with rates that favor the particles hopping to the right with priority given to the heavy particle. In our case, the heavy particles, light particles and vacancies are represented by particles of type 2, 1, and 0 respectively.

A tableaux formula for the stationary probabilities of the 2-ASEP is obtained using rhombic alternative tableaux (RAT) [13]. The RAT are precisely the tableaux obtained by taking the northwest portion of a TRAT, except with additional filling rules. We denote the region of $\mathcal{H}(X)$ northwest of $P(X)$ in the TRAT of type X by $\Gamma(X)$, which we call a *rhombic diagram*. We carry over the definition of a tiled rhombic diagram, north-strips, west-strips, up-arrows, and left-arrows from previous sections. Recall that when we say a tile is *pointed at* by an up-arrow (resp. left-arrow), that means there is an up-arrow below the tile in the same north-strip (resp. left-arrow to the right of the tile in the same west-strip).

Definition 5.1. A *rhombic alternative tableau* (RAT) of type $X \in$ 2-ASEP(n, r) is a rhombic diagram $\Gamma(X)$ with some tiling $\mathcal{T}_X$ that is filled with up-arrows, left-arrows, and q's according to the following filling rules:

- a tile must be empty if it is pointed at by an up-arrow or a left-arrow, and
- if a tile is not pointed at by an arrow, it must contain an up-arrow, a left-arrow, or a q.

For $X \in$ 2-ASEP(k, r, ℓ), if a RAT R is of type X, we say R has size (k, r, ℓ).

Definition 5.2. The *weight* of a RAT R of size (k, r, ℓ) is given by

$$\text{wt}(R) = \alpha^{k+\#\{\text{up-arrows}\}}\beta^{\ell+\#\{\text{left-arrows}\}}q^{\#\{q\text{'s}\}}.$$

For $X \in$ 2-ASEP(n, r), we define $RAT_{\mathcal{T}}(X)$ to be the set of RAT of type X with some fixed tiling $\mathcal{T}$. We denote the set of all RAT of size (n, r) by $RAT(n, r)$. More precisely,

$$RAT(n, r) = \bigcup_{Y \in \text{2-ASEP}(n,r)} RAT_{\mathcal{T}_Y}(Y)$$

where $\{\mathcal{T}_Y\}_Y$ is some set of tilings of rhombic diagrams $\{\Gamma(Y)\}_Y$ where Y ranges over all possible states.

Theorem 5.1 ([13])**.** The steady state probability of state $X \in$ 2-ASEP(n,r) is

$$\text{Prob}(X) = \frac{1}{\mathcal{Z}_{n,r}} \sum_{R \in RAT_{\mathcal{T}_X}(X)} \text{wt}(R),$$

where $\mathcal{T}_X$ is a fixed tiling of $\Gamma(X)$ and $\mathcal{Z}_{n,r} = \sum_{R \in RAT(n,r)} \text{wt}(R)$ is the partition function.

5.1 Inhomogeneous 2-ASEP

In the inhomogeneous 2-ASEP, swaps of different species of particles occur at different rates which depend on both particles involved in the swap, analogous to the inhomogeneous generalization in Section 4. Let $X \in$ 2-ASEP(n,r) be an ASEP word. The transitions on the inhomogeneous 2-ASEP Markov chain are:

$$X'20X'' \underset{\text{q}}{\overset{\text{t}}{\rightleftarrows}} X'02X'' \qquad\qquad 0X' \overset{\alpha}{\to} 2X'$$
$$X'21X'' \underset{\text{q}}{\overset{\text{d}}{\rightleftarrows}} X'12X'' \qquad\qquad X'2 \overset{\beta}{\to} X'0$$
$$X'10X'' \underset{\text{q}}{\overset{\text{e}}{\rightleftarrows}} X'01X''$$

where $0 \le t, d, e, q, \alpha, \beta \le 1$ are parameters describing the hopping rates. (When $t = d = e = 1$, we recover the usual 2-ASEP.)

The following Matrix Ansatz is a modification of an inhomogeneous version of the canonical Derrida-Evans-Hakim-Pasquier Matrix Ansatz [6] and the two-species Matrix Ansatz of Uchiyama [15].

Theorem 5.2 ([6])**.** Let D, A, E be matrices and $\langle w|, |v\rangle$ vectors satisfying:

$$tDE = qED + D + E \qquad\qquad \langle w|E = \frac{1}{\alpha}\langle w| \tag{5.1}$$
$$dDA = qAD + A \qquad\qquad D|v\rangle = \frac{1}{\beta}|v\rangle$$
$$eAE = qEA + A$$

then the stationary probability of state $X \in$ 2-ASEP(n,r) is proportional to $\langle w|\,\text{dae}(X)|v\rangle$, where dae$(X)$ is given by Definition 2.1.

and where $[y^k]P(y)$ denotes the coefficient of y^k in polynomial $P(y)$.

Remark. In the TASEP case (when $q = 0$), the Matrix Ansatz equations in the bulk become

$$tDE = D + E, \qquad dDA = A, \qquad eAE = A. \tag{5.2}$$

5.2 Rhombic alternative tableaux and the inhomogeneous 2-ASEP with open boundaries

Remarkably, there is a natural extension of our definition of the weighted TRAT to a weighted RAT, from which we obtain a formula for probabilities of the inhomogeneous 2-ASEP with open boundaries for general q.

Recall that for a RAT $R \in \mathrm{RAT}(k, r, \ell)$, $\mathrm{wt}(R)$ is a monomial in α, β, q given by

$$\mathrm{wt}(R) = \alpha^{\#\{\text{up-arrows}\}} \beta^{\ell + \#\{\text{left-arrows}\}} q^{\$\{q\text{'s}\}}.$$

Also recall that we denote by $|R|_{20}$ (resp. $|R|_{21}$, $|R|_{10}$) the number of occupied 20-tiles (resp. 21-tiles, 10-tiles) in R.

Definition 5.3. The *enhanced weight* of a RAT R is defined as follows:

$$\mathrm{wt}_e(R) = \mathrm{wt}(R)\, \mathbf{t}^{|R|_{20}}\, \mathbf{d}^{|R|_{21}}\, \mathbf{e}^{|R|_{10}}.$$

For example, in Figure 17, we obtain $\mathrm{wt}(R) = \alpha^{13}\beta^9 q^{12}$ and $\mathrm{wt}_e(R) = \mathrm{wt}(R)\, \mathbf{t}^{13}\, \mathbf{d}^2\, \mathbf{e}^6$.

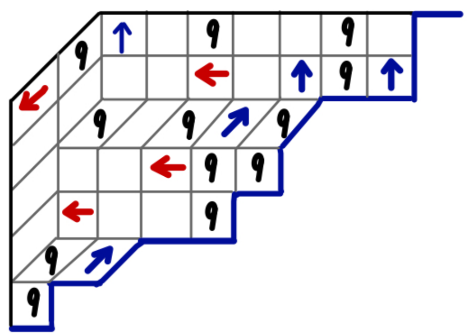

Figure 17: For this RAT of type $X = 0221202001020 \in$ 2-ASEP$(8, 2, 5)$, $|T|_2 0$=13, $|T|_{21} = 2$, and $|T|_{10} = 6$, so $\mathrm{wt}_e(R) = \alpha^{13}\beta^9 q^{12}\, \mathbf{t}^{13}\, \mathbf{d}^2\, \mathbf{e}^6$.

Definition 5.4. We define the weight of a state $X \in$ 2-ASEP(n, r) to be the weight generating function of all RAT of type X with some fixed tiling $\mathcal{T}$:

$$\mathrm{weight}(X) = \sum_{R \in \mathrm{RAT}_{\mathcal{T}}(X)} \mathrm{wt}_e(R).$$

Theorem 5.3. Let $X \in$ 2-ASEP(n, r) be a state of the inhomogeneous two-species ASEP with open boundaries and parameters $\alpha, \beta, q, t, d, e$ governing the transition rates. The stationary probability of state X is

$$\mathrm{Prob}(X) = \frac{1}{\mathcal{Z}_{n,r}} \sum_{R \in \mathrm{RAT}_{\mathcal{T}}(X)} \mathrm{wt}_e(R).$$

for some fixed tiling $\mathcal{T}$, and where $\mathcal{Z}_{n,r} = \sum_{R \in \mathrm{RAT}(n,r)}$ is the partition function.

Proof. Once again we give the standard Matrix Ansatz proof by showing that the enhanced weight wt_e on the RAT satisfies the recurrences of the inhomogeneous Matrix Ansatz of Theorem 5.2. Showing this along with a base case would imply $\mathrm{weight}(X) = f(X)$, where we define

$$f(X) = (\alpha\beta)^{n-r} \langle w|\, \mathrm{dae}(X) |v\rangle,$$

with $\langle w|$, $|v\rangle$, and $\mathrm{dae}(X)$ defined as in Theorem 5.2. The proof builds upon the inductive proof of Theorem 2.4, but uses enhanced weight $\mathrm{wt}_e(R)$ for usual RAT instead of the usual $\mathrm{wt}(R)$ for TRAT. Recall that we call $\Gamma(X)$ the shape of a rhombic diagram of type X.

Our proof is by induction on the area of $\Gamma(X)$, which we denote by $|\Gamma(X)|$. The base cases where $|\Gamma(X)| = 0$ is when our state X has type $X_{min} = 0^\ell 1^r 2^k$ for some k, r, ℓ. In this case, there is a single RAT of that type with volume 0, so trivially weight$(X_{min}) = \alpha^k \beta^\ell$. On the other hand, $f(X_{min}) = (\alpha\beta)^{k+\ell} \langle w|E^\ell A^r D^k|v\rangle = \alpha^k\beta^\ell = \text{weight}(X_{min})$.

Now, suppose that for any Y such that $|\Gamma(Y)| < N$, weight$(Y) = f(Y)$.

Let $X \in$ 2-ASEP(n, r) with $|\Gamma(X)| = N > 0$. X must have an adjacent pair of locations containing the particles 21, 10, or 20. We examine all three of these cases. For some Y, Z:

I. $X = Y21Z$,

II. $X = Y10Z$,

III. $X = Y20Z$.

Case I: $X = Y21Z$. The 21-tile adjacent to the 2, 1 pair of edges of $P(X)$ contains either a left-arrow that contributes weight $\beta\,\mathbf{d}$, or a q, which contributes weight $q\,\mathbf{d}$. In the former case, the remaining tiles of the west-strip $\mathfrak{s}$ originating at the 2-edge must be empty. Hence those fillings are in bijection with $\Gamma(X)\backslash\mathfrak{s}$ which are all possible fillings of $\Gamma(Y1Z)$. In the latter case, the 21-tile containing the q can be removed with the rest of the tiling of $\Gamma(X)$ unaffected. Thus those fillings are in bijection with fillings of $\Gamma(Y12Z)$. Thus combinatorially we obtain

$$\text{weight}(X) = \text{weight}(Y21Z) = \alpha\beta\,\mathbf{d}\,\text{weight}(Y1Z) + q\,\mathbf{d}\,\text{weight}(Y12Z),$$

where the α in the first term comes from removing a 2-edge with weight α to obtain the smaller shape $\Gamma(Y1Z)$. Since $|\Gamma(Y1Z)| < N$ and $|\Gamma(Y12Z)| < N$, by the inductive hypothesis, this equals

$$\begin{aligned}\text{weight}(X) =& \alpha\beta\,\mathbf{d}\,f(Y1Z) + q\,\mathbf{d}\,f(Y12Z)\\ =& \frac{\alpha\beta}{d}(\alpha\beta)^{n-r-1}\langle w|\,\text{dae}(Y1Z)|v\rangle + \frac{q}{d}(\alpha\beta)^{n-r}\langle w|\,\text{dae}(Y12Z)|v\rangle\\ =& (\alpha\beta)^{n-r}\langle w|\,\text{dae}(Y21Z)|v\rangle\\ =& f(X).\end{aligned}$$

Case II: $X = Y10Z$. The 10-tile adjacent to the 1, 0 pair of edges of $P(X)$ contains either an up-arrow that contributes weight $\alpha\,\mathbf{e}$, or a q, which contributes weight $q\,\mathbf{e}$. In the former case, the remaining tiles of the north-strip $\mathfrak{s}$ originating at the 0-edge must be empty. Hence those fillings are in bijection with $\Gamma(X)\backslash\mathfrak{s}$ which are all possible fillings of $\Gamma(Y1Z)$. In the latter case, the 10-tile containing the q can be removed with the rest of the tiling of $\Gamma(X)$ unaffected. Thus those fillings are in bijection with fillings of $\Gamma(Y01Z)$. Thus combinatorially we obtain

$$\text{weight}(X) = \text{weight}(Y10Z) = \alpha\beta\,\mathbf{e}\,\text{weight}(Y1Z) + q\,\mathbf{e}\,\text{weight}(Y01Z),$$

and by the inductive hypothesis, this equals

$$\begin{aligned}\text{weight}(X) =& \alpha\beta\,\mathbf{e}\,f(Y1Z) + q\,\mathbf{e}\,f(Y12Z)\\ =& \frac{\alpha\beta}{e}(\alpha\beta)^{n-r-1}\langle w|\,\text{dae}(Y1Z)|v\rangle + \frac{q}{e}(\alpha\beta)^{n-r}\langle w|\,\text{dae}(Y01Z)|v\rangle\\ =& (\alpha\beta)^{n-r}\langle w|\,\text{dae}(Y10Z)|v\rangle\\ =& f(X).\end{aligned}$$

Case III: $X = Y20Z$. The 20-tile adjacent to the 2, 0 pair of edges of $P(X)$ contains either a left-arrow that contributes weight $\beta\,\mathbf{t}$, an up-arrow that contributes weight $\alpha\,\mathbf{t}$, or a q, which contributes weight $q\,\mathbf{t}$. In the first case, the remaining tiles of the west-strip $\mathfrak{s}$ originating at the 2-edge must be empty. Hence those fillings are in bijection with $\Gamma(X)\backslash\mathfrak{s}$ which are all possible fillings of $\Gamma(Y0Z)$. In the second case, the remaining tiles of the north-strip $\mathfrak{s}$ originating at the 0-edge must be empty. Hence those fillings are in bijection with $\Gamma(X)\backslash\mathfrak{s}$ which are all possible fillings of $\Gamma(Y2Z)$. In the third case, the 20-tile containing the q can be removed with the rest of the tiling of $\Gamma(X)$ unaffected. Thus those fillings are in bijection with fillings of $\Gamma(Y02Z)$. Thus combinatorially we obtain

$$\text{weight}(Y21Z) = \alpha\beta\,\mathbf{t}\,\text{weight}(Y0Z) + \alpha\beta\,\mathbf{t}\,\text{weight}(Y2Z) + q\,\mathbf{t}\,\text{weight}(Y02Z),$$

and by the inductive hypothesis, this equals

$$\begin{aligned}\text{weight}(X) =&\alpha\beta\,\mathbf{t}\,f(Y0Z) + \alpha\beta\,\mathbf{t}\,f(Y2Z) + q\,\mathbf{t}\,f(Y02Z)\\ =&\frac{\alpha\beta}{t}(\alpha\beta)^{n-r-1}\left(\langle w|\,\text{dae}(Y0Z)|v\rangle + \langle w|\,\text{dae}(Y2Z)|v\rangle\right) + \frac{q}{t}(\alpha\beta)^{n-r}\langle w|\,\text{dae}(Y02Z)|v\rangle\\ =&(\alpha\beta)^{n-r}\langle w|\,\text{dae}(Y20Z)|v\rangle\\ =&f(X).\end{aligned}$$

Thus $\text{weight}(X) = f(X)$, as desired. □

Therefore, the RAT with enhanced weights indeed provide formulae for the probabilities of the inhomogeneous two-species ASEP.

5.3 Acyclic multiline queues

The connection between rhombic tableaux and multiline queues extends to the open boundary case of the 2-TASEP. Following the idea of the TRAT-MLQ bijection of Section 3.2, we define *acyclic multiline queues*, which we also call AMLQs.

Definition 5.5. An *acyclic MLQ* (AMLQ) of type $X \in$ 2-ASEP(n, r) is a configuration of two rows of balls on a lattice of size $2 \times n$ with open boundaries. There are $0 \le \ell \le n - r$ balls in the top row and $\ell + r$ balls in the bottom row, with the following restriction: for each $1 \le i \le \ell + r$, the i'th bottom row ball (from the left) has at least i top row balls weakly to its left. We denote the set of AMLQs of type X by AMLQ(X), and we denote by AMLQ(n, r) the set of acyclic MLQs of size (n, r):

$$\text{AMLQ}(n, r) = \bigcup_{X\in \text{2-ASEP}(n,r)} \text{AMLQ}(X).$$

In other words, an acyclic MLQ is an MLQ with open boundaries (which is not invariant under cyclic shifts), and in which every top row ball occupies a bottom row ball to its right. Figure 18 shows an example of a configuration which is and one which isn't an acyclic MLQ.

Lemma 5.4. AMLQ(X) is in bijection with MLQ$(1X1)$.

Proof. Let $A \in$ AMLQ(X). Let A_M be an MLQ obtained by appending a column to the left and right of A, containing a vacancy in the top row and a 1-ball in the bottom row. The leftmost column of A_M trivially contains a 1-ball, and since every top row ball in A occupies some ball weakly to its right without wrapping around, the bottom row ball at the rightmost location of the A_M must

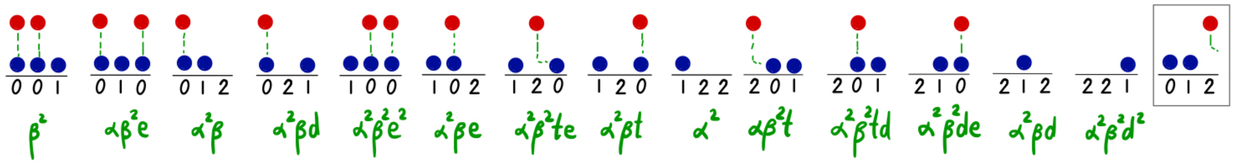

Figure 18: All the acyclic MLQs of size $(3,1)$ are shown. The rightmost (boxed) configuration is not an AMLQ since the top row ball must wrap around to occupy the bottom row.

remain unoccupied; thus $A_M \in \mathrm{MLQ}(1X1)$. On the other hand, let $A_M \in \mathrm{MLQ}(1X1)$ be cyclically shifted so that its type read from left to right is $1X1$. The right-most 1-ball must remain unoccupied, so all top row balls must occupy bottom row balls without wrapping around. By chopping off the leftmost and rightmost columns of A_M, we get back $A \in \mathrm{AMLQ}(X)$. □

We fix some definitions to simplify notation.

Definition 5.6. Let A be an AMLQ.

- $\mathrm{rest}(A)$ is the number of restricted 0-balls in A.
- $\mathrm{umv}(A)$ is the number of unmarked vacancies in A.
- $\text{u-free}(A)$ is be the number of restricted 0-balls to the left of the leftmost 1-ball in A.
- $\text{l-free}(A)$ is be the number of unmarked vacancies to the right of the rightmost 1-ball in A.

Definition 5.7. The *weight* of an acyclic MLQ $A \in AMLQ(n,r)$ is

$$\mathrm{wt}(A) = \alpha^{n-r-\text{u-free}(A)}\beta^{n-r-\text{l-free}(M)}.$$

Theorem 5.5. Let X be a state of the two-species TASEP of size (n,r). Then

$$\Pr(X) = \frac{1}{\mathcal{Z}_{n,r}} \sum_{A \in \mathrm{AMLQ}(X)} \mathrm{wt}(A),$$

where $\mathcal{Z}_{n,r} = \sum_{A \in \mathrm{AMLQ}(n,r)} \mathrm{wt}(A)$ is the partition function.

In Figure 18, the α, β weights of all AMLQs of size $(3,1)$ are given (after setting the other variables to equal 1).

The proof of our theorem is through a weight-preserving bijection of acyclic MLQs with RAT at $q=0$. Let us denote the set of $\mathrm{RAT}_{\mathcal{T}}(X)$ at $q=0$ by $\mathrm{RAT}_{\mathcal{T}}^{q=0}(X)$.

Proof. Let $A \in \mathrm{AMLQ}(X)$ for $X \in 2\text{-ASEP}(n,r)$, and let $X' = 1X1 \in 2\text{-ASEP}(n+2, r+2)$. Let $A_M \in \mathrm{MLQ}(1X1)$ be the MLQ obtained by appending a column containing a 1-ball in the bottom row to the left and right of A. Let $T = TRAT(A_M)$ be the TRAT obtained by applying the ball drop algorithm to A_M. Now apply flips to the tiling of T until the rightmost (resp. leftmost) diagonal strip consists of a row of ℓ adjacent 10-tiles (resp. k adjacent 21-tiles), obtaining tiling $\mathcal{T}$ in which the leftmost and rightmost diagonal strips are aligned with the north and west boundaries of T.

We define T_U to be the rhombic tableau obtained by taking the region of T northwest of $P(X)$. T_U satisfies the rules of Definition 5.1, and so $T_U \in \mathrm{RAT}_{\mathcal{T}}^{q=0}(1X1)$. We claim that the region of T southeast of the path $P(X)$ contains no arrows. Suppose a north-strip of T has its top-most tile,

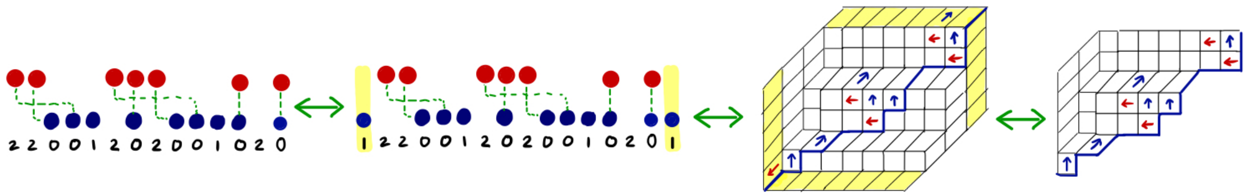

Figure 19: Let $X = 22001202001020$. From left to right, we have: $A \in \text{AMLQ}(X) \longleftrightarrow A_M \in \text{MLQ}(1X1) \longleftrightarrow \text{trat}(A_M) \in \text{TRAT}(1X1) \longleftrightarrow \text{rat}(A) \in \text{RAT}(X)$. The highlighted columns in A_M which are appended to A correspond to the highlighted diagonal strips in $\text{trat}(A_M)$ which are subsequently removed to obtain $\text{rat}(A)$.

which is a 10-tile contained in the rightmost diagonal strip, free. Then that 10-tile will necessarily contain an up-arrow. Similarly, suppose a west-strip of T has its left-most tile, which is a 21-tile contained in the leftmost diagonal strip, free. Then that 21-tile will necessarily contain a left-arrow. Thus every arrow in T is contained in T_U, and so a TRAT $T \in TRAT(1X1)$ can be uniquely recreated from a rhombic tableau $T_U \in RAT^{q=0}(1X1)$. Consequently, this map is a bijection.

Now define R to be the tableau obtained by chopping off the rightmost and leftmost diagonal strips of T_U: recall that with the tiling $\mathcal{T}$, these strips are bordering the northwest boundary of T_U, so chopping them off results in a proper rhombic diagram of type X with a filling with up-arrows and left-arrows that still satisfy Definition 5.1 (for $q = 0$); thus $R \in \text{RAT}^{q=0}_{\mathcal{T}}(X)$. Moreover, T_U can be recreated from R by re-attaching the external diagonal strips and placing a up-arrow in the topmost tile of every north-strip that is free of an up-arrow, and a left-arrow in the leftmost tile of every west-strip that is free of a left-arrow. We call $\text{rat}(A) = R$. Thus $\text{rat} : \text{AMLQ}(X) \to \text{RAT}^{q=0}_{\mathcal{T}}(X)$ is a bijection.

Define u-free(R) to be the number of north-strips that are free of up-arrows and l-free(R) to be the number of west-strips that are free of left-arrows. Then

$$\begin{aligned} \text{wt}(R) &= \alpha^k \beta^\ell \alpha^{\#\text{ up-arrows}} \beta^{\#\text{ left-arrows}} \\ &= (\alpha\beta)^{n-r} \alpha^{-\text{ u-free}(R)} \beta^{-\text{ l-free}(R)}. \end{aligned}$$

By our construction, u-free(R) (resp. l-free(R)) is the number of tiles containing an up-arrow in the rightmost diagonal strip (resp. left-arrow in the leftmost diagonal strip) of R. By following the ball drop algorithm, we see that an up-arrow in the rightmost diagonal strip precisely corresponds to the unmarked vacancies left of the leftmost 1-ball in A, and a left-arrow in the leftmost diagonal strip precisely corresponds to the restricted 0-balls right of the rightmost 1-ball in A. Thus u-free(R) = u-free(A) and l-free(R) = l-free(A) from Definition 5.7.

For $X \in$ 2-ASEP(n, r), let $A \in \text{AMLQ}(X)$ and $R = \text{rat}(A) \in \text{RAT}^{q=0}(X)$. By the above $\text{wt}(R) = \text{wt}(A)$, and so with Theorem 5.1, we obtain

$$\Pr(X) = \frac{1}{\mathcal{Z}_{n,r}} \sum_{R \in \text{RAT}^{q=0}_{\mathcal{T}}(X)} \text{wt}(R) = \frac{1}{\mathcal{Z}_{n,r}} \sum_{A \in \text{AMLQ}(X)} \text{wt}(A),$$

as desired, where $\mathcal{Z}_{n,r} = \sum_{A \in \text{AMLQ}(n,r)} \text{wt}(A)$. □

5.3.1 Weighted acyclic multiline queues

We can define weighted acyclic MLQs in the same way that we define weighted MLQs, but with an additional condition: restricted 0-balls left of the leftmost 1-ball and unmarked vacancies right of the rightmost 1-ball receive weight 1. Following our bijection with $RAT^{q=0}$, we obtain an enhanced weight wt_e which is a monomial in $\alpha, \beta, \mathbf{t}, \mathbf{d}, \mathbf{e}$ and which gives us a formula for stationary probabilities of the inhomogeneous 2-TASEP with open boundaries.

Definition 5.8. Let $A \in \mathrm{AMLQ}(n, r)$, and let $\mathrm{wt}(A)$ be the α, β weight of A from Definition 5.7. Define the enhanced weight of A to be

$$\mathrm{wt}_e(A) = \mathrm{wt}(A)\, \mathbf{t}^{n-r} \left(\frac{\mathbf{e}}{\mathbf{t}}\right)^{\mathrm{rest}(M)} \left(\frac{\mathbf{d}}{\mathbf{t}}\right)^{\mathrm{umv}(M)} \mathbf{e}^{-\,\text{u-free}(M)}\, \mathbf{d}^{-\,\text{l-free}(M)} \tag{5.3}$$

$$= (\alpha\beta\, \mathbf{t})^{n-r} \left(\frac{\mathbf{e}}{\mathbf{t}}\right)^{\mathrm{rest}(M)} \left(\frac{\mathbf{d}}{\mathbf{t}}\right)^{\mathrm{umv}(M)} (\alpha\, \mathbf{e})^{-\,\text{u-free}(M)} (\beta\, \mathbf{d})^{-\,\text{l-free}(M)}. \tag{5.4}$$

Example 5.1. In Figure 19, we have $A \in \mathrm{AMLQ}(X)$ on the left and $\mathrm{rat}(A) \in \mathrm{RAT}(X)$ on the right for $X = 22001202001020 \in$ 2-ASEP$(14, 2)$. For both, $\text{u-free}(A) = \text{u-free}(\mathrm{rat}(A)) = 1$ and $\text{l-free}(A) = \text{l-free}(\mathrm{rat}(A)) = 1$; there are 8, 2, and 0 occupied 20-tiles, 10-tiles, and 21-tiles respectively in R, and so both objects have $\mathrm{wt}(A) = \mathrm{wt}(\mathrm{rat}(A)) = \alpha^{14-1}\beta^{14-1} = (\alpha\beta)^{13}$ and enhanced weight $\mathrm{wt}_e(A) = \mathrm{wt}_e(\mathrm{rat}(A)) = (\alpha\beta)^{13}\, \mathbf{t}^8\, \mathbf{e}^2$.

For another example, see Figure 18, in which the weights of all elements of $\mathrm{AMLQ}(3, 1)$ are given.

In the following theorem, we show that the enhanced weight $\mathrm{wt}_e(A)$ satisfies the Matrix Ansatz for the inhomogeneous 2-TASEP in Theorem 5.2, and consequently gives a formula for the stationary probabilities of the inhomogeneous 2-TASEP.

Our bijection between $RAT^{q=0}(n, r)$ and $AMLQ(n, r)$ is weight preserving as stated by the following lemma, with an example shown in Figure **??**.

Lemma 5.6. Let $A \in \mathrm{AMLQ}(n, r)$ and let $R = RAT^{q=0}(A)$. Then $\mathrm{wt}_e(A) = \mathrm{wt}_e(R)$.

Proof. We want to show that

i. $|R|_{20} = n - r - \mathrm{rest}(M) - \mathrm{umv}(M)$,

ii. $|R|_{21} = \mathrm{umv}(M) - \text{l-free}(M)$, and

iii. $|R|_{10} = \mathrm{rest}(M) - \text{u-free}(M)$.

From our proof of Theorem 4.5, we immediately get (i.). Now, observe that the contents of M to the left of its leftmost 1-ball and to the right of its rightmost 1-ball correspond to the northeast and southwest regions of R that lie outside of the diagonal strips. Thus those regions consist of only 20-tiles, and hence the unmarked vacancies that would normally correspond to a left-arrow in a 21-tile (counted by $\text{l-free}(M)$), and the restricted 0-balls that would normally correspond to an up-arrow in a 10-tile (counted by $\text{u-free}(M)$), no longer contribute to the weight of the AMLQ. Thus we obtain (ii.) and (iii.), completing the proof of the lemma. □

Theorem 5.3 combined with Lemma 5.6 gives the following result.

Corollary 5.7. Let $X \in$ 2-ASEP(n, r) be a state of the inhomogeneous TASEP with open boundaries and parameters α, β, t, d, e governing the rates of transitions. The stationary probability of state X is

$$\text{Prob}(X) = \frac{1}{\mathcal{Z}_{n,r}} \sum_{A \in \text{AMLQ}(X)} \text{wt}_e(A),$$

where $\mathcal{Z}_{n,r} = \sum_{A \in \text{AMLQ}(n,r)} \text{wt}_e(A)$ is the partition function.

6 Markov chains on tableaux that project to the 2-ASEP

The structure of the toric rhombic tableaux yields a natural Markov chain on these tableaux that projects to the two-species TASEP on a ring, in the flavor of the Markov chain on the rhombic alternative tableaux projecting to the two-species ASEP in [11], which in turn generalized the Markov chain on the alternative tableaux in [5]. As a result, we also obtain a Markov chain on the two-species multiline queues by following the bijection of Section 3.2, which we describe in Section 6.2. We are then able to extend this Markov chain to the inhomogeneous weighted multiline queue.

By projection of Markov chains, we mean the following definition, which is precisely Definition 3.20 from [5].

Definition 6.1. Let M and N be Markov chains on finite sets X and Y, and let f be a surjective map from X to Y. We say that M *projects* to N if the following properties hold:

- If $x_1, x_2 \in X$ with $Prob_M(x_1 \to x_2) > 0$, then $Prob_M(x_1 \to x_2) = Prob_N(f(x_1) \to f(x_2))$.
- If y_1 and y_2 are in Y and $Prob_N(y_1 \to y_2) > 0$, then for each $x_1 \in X$ such that $f(x_1) = y_1$, there is a unique $x_2 \in X$ such that $f(x_2) = y_2$ and $Prob_M(x_1 \to x_2) > 0$; moreover, $Prob_M(x_1 \to x_2) = Prob_N(y_1 \to y_2)$.

Furthermore, we have the following Proposition 6.1, which implies Corollary 6.2 below.

Let $\text{Prob}_M(x_0 \to x; t)$ denote the probability that if we start at state x_0 at time 0, then we are in state x at time t. From the following proposition of [5], we obtain that if M projects to N, then a walk on the state diagram of M is indistinguishable from a walk on the state diagram of N.

Proposition 6.1. Suppose that M projects to N. Let $x_0 \in X$ and $y_0, y_1 \in Y$ such that $f(x_0) = y_0$. Then

$$\text{Prob}_N(y_0 \to y_1) = \sum_{x' \text{ s.t. } f(x')=y_1} \text{Prob}_M(x_0 \to x_1)$$

Call the Markov chain on the TRAT the TRAT chain, and call the Markov chain on the two-species ASEP on a ring the ASEP chain.

Corollary 6.2. Suppose the TRAT chain projects to the ASEP chain via the map type : $TRAT \to ASEP$. Let X be a state of the the ASEP chain and let $TRAT(X) = \{R \in TRAT \ : \ \text{type}(R) = X\}$. Then the steady state probability of state X is equal to the sum of the steady state probabilities that the TRAT chain is in any of the states $R \in TRAT(X)$.

6.1 Definition of the TRAT Markov chain

In this section, we no longer require fixing a particular tiling on $\mathcal{H}(X)$.

Definition 6.2. A *corner* of a tableau is a consecutive pair of 2 and 0, 2 and 1, or 1 and 0 edges (we call these 20-, 21-, and 10-corners, respectively).

Let R be a tableau with a 20-corners, b 21-corners, and c 10-corners. Then there are $a+b+c$ possible transitions out of R. The Markov transitions are based on the following: each corner corresponds, up to tiling equivalence, to either a north-strip with an up-arrow in its bottom-most box (the *up-arrow case*), or a west-strip with a left-arrow in its right-most box (the *left-arrow case*). We define insertion of strips precisely below.

Definition 6.3. We introduce the notion of *positive length* of a strip $\mathfrak{s}$ to mean the number of tiles of that strip that are contained northwest of $P(X)$, and we denote it by $L_+(\mathfrak{s})$. (Note that all north strips and all west strips have the same total length of $r+k$ and $r+\ell$, respectively.)

Definition 6.4. Let $\mathcal{H}(X)$ have tiling $\mathcal{T}$. The *insertion of a north-strip* $s_N(x)$ to the left of a point x on $P(X)$ is defined as follows. Take the path $p_N(x)$ that begins and ends at x by traveling north up the vertical or diagonal edges of $\mathcal{T}$. It is important that $p_N(x)$ is as far to the right as possible, meaning that, if at some point the path has a choice between taking a vertical edge or a diagonal edge, it always chooses the diagonal one. Now replace each vertical edge of $p_N(x)$ with a 20-tile and each diagonal edge of $p_N(x)$ with a 10-tile. The newly inserted tiles form the north-strip $s_N(x)$. The horizontal edge adjacent to x becomes a new 0-edge in $P(X)$. See Figure 20.

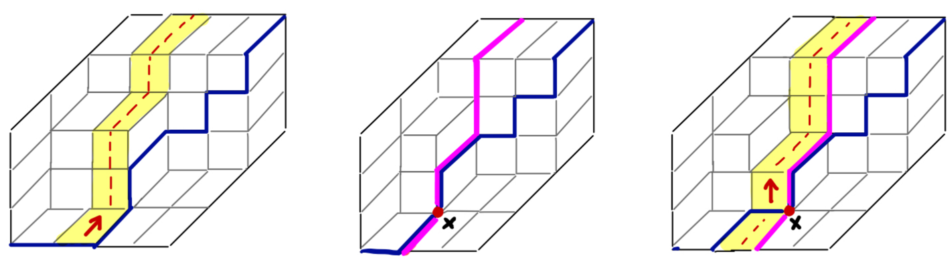

Figure 20: In this figure is shown a transition from a TRAT R of type 12020121**10**0 to a TRAT R' of type 1202012**01**0. On the left is TRAT R with a corner tile containing an up-arrow in north-strip $\mathfrak{s}$, and $L_+(\mathfrak{s}) = 6$. In the middle is the TRAT with the north-strip $\mathfrak{s}$ removed. Location x is chosen since $L_+(s_N(x)) = 5$; $p_N(x)$ is marked by the red path. On the right, $s_N(x)$ is inserted to build the TRAT R'.

Definition 6.5. Let $\mathcal{H}(X)$ have tiling $\mathcal{T}$. The *insertion of a west-strip* $s_W(x)$ above a point x on $P(X)$ is defined as follows. Take the path $p_W(x)$ that begins and ends at x by traveling west along the horizontal or diagonal edges of $\mathcal{T}$. It is important that $p_W(x)$ is as far to the south as possible, meaning that, if at some point the path has a choice between taking a horizontal edge or a diagonal edge, it always chooses the diagonal one. Now replace each horizontal edge of $p_W(x)$ with a 20-tile and each diagonal edge of $p_W(x)$ with a 21-tile. The newly inserted tiles form the west-strip $s_W(x)$. The vertical edge adjacent to x becomes a new 2-edge in $P(X)$. See Figure 21.

Definition 6.6. The transitions of the TRAT chain are defined as follows.

- For the **up-arrow transition**, see Figure 20. Let the up-arrow be contained in north-strip $\mathfrak{s}$. Remove $\mathfrak{s}$, and let x be the right-most location of $\mathcal{H}(X)$ such that $L_+(s_N(x)) = L_+(\mathfrak{s}) - 1$. Insert the north-strip $s_N(x)$ at x and place an up-arrow in its bottom-most box.
- For the **left-arrow transition**, see Figure 21. Let the left-arrow be contained in west-strip $\mathfrak{s}$. Remove $\mathfrak{s}$, and let x be the right-most location of $\mathcal{H}(X)$ such that $L_+(s_W(x)) = L_+(\mathfrak{s}) - 1$. Insert the west-strip $s_W(x)$ at x and place a left-arrow in its right-most box.

Remark. There is a subtlety arising from the choice of a tiling on $\mathcal{H}(X)$: there may be no 21-tile

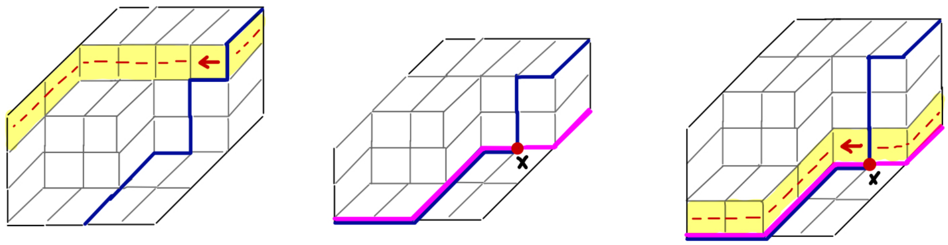

Figure 21: In this figure is shown a transition from a TRAT R of type 1**20**2201100 to a TRAT R' of type 1**02**2201100. On the left is TRAT R with a corner tile containing a left-arrow in west-strip $\mathfrak{s}$, and $L_+(\mathfrak{s}) = 6$. In the middle is the TRAT with the west-strip $\mathfrak{s}$ removed. Location x is chosen since $L_+(S_W(x)) = 5$; $p_W(x)$ is marked by the red path. On the right, $s_W(x)$ is inserted to build the TRAT R'.

adjacent to a 21 corner, or there may be no 10-tile adjacent to a 10 corner. However, it is well known that any two tilings can be obtained from one another via some series of flips, where a flip is a switch of configurations in Figure 22. A *filling-preserving flip* is a weight-preserving map from the filling of a tiling $\mathcal{T}$ to a filling of a tiling $\mathcal{T}'$, where $\mathcal{T}$ and $\mathcal{T}'$ differ by a single flip. Figure 22 shows the four possible cases of a filling-preserving flip. A full proof of this property of rhombic tableaux is given in [13].

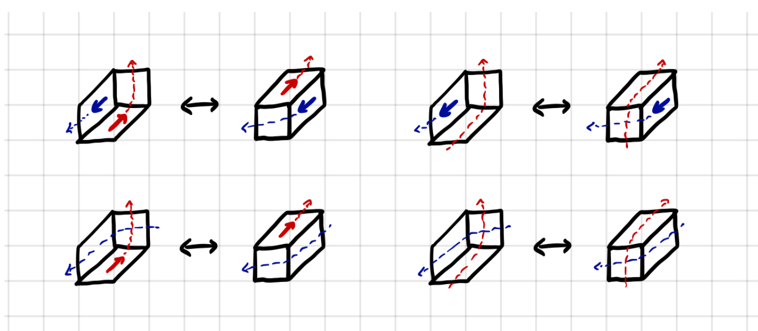

Figure 22: A *flip* is the switch from one hexagonal configuration to another in a tiling. In this figure we see the four cases of possible fillings of a hexagonal configuration of tiles in a TRAT. The dashed blue lines through the west-strips (resp. red lines through the north strips) represent the presence of left-arrows (resp. up-arrows) in those strips.

Transition at a 20 corner.

There is necessarily a 20-tile containing either an up-arrow or a left-arrow adjacent to that corner. In the former case, the up-arrow transition is performed. In the latter case, the left-arrow transition is performed.

Transition at a 21 corner.

If there is a 21-tile adjacent to that corner, it must necessarily contain a left-arrow. If there is no 21-tile adjacent to that corner, perform flips on $\mathcal{H}(X)$ until a 21-tile appears in the desired location. This tile must necessarily contain a left-arrow, and we perform the 21 transition on this new tiling. In both of these cases, the left-arrow transition is performed.

Transition at a 10 corner.

If there is a 10-tile adjacent to that corner, it must necessarily contain an up-arrow. If there is no 10-tile adjacent to that corner, perform flips on $\mathcal{H}$ until such a tile appears in the desired location. This tile must necessarily contain an up-arrow, and we perform the 10 transition on this new tiling. In both of these cases, the up-arrow transition is performed.

Remark. A notable case occurs when $X = 1X'2$, and the transition is the 21 corner at the ends of $P(X)$. The Markov transition is the usual 21 corner left-arrow transition, but that may be hard to visualize without first rotating the TRAT one edge to the left so that its type becomes $21X'$.

In Figure 23, we show an example of the Markov chain on all the states of $TRAT(2, 1, 2)$.

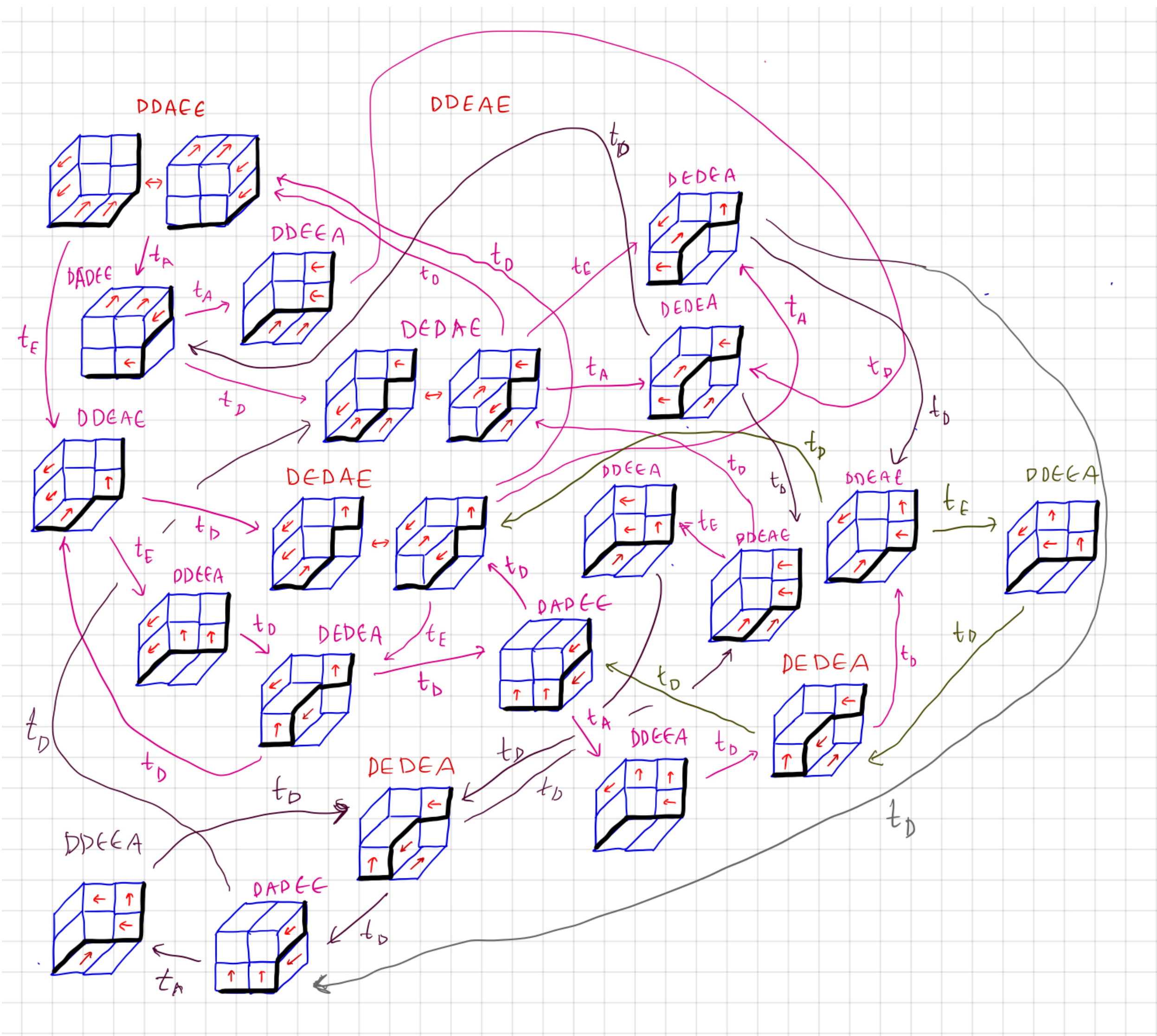

Figure 23: The Markov chain on states of size $(2, 1, 2)$. In this figure, $t_D = t$, $t_A = d$, and $t_E = e$.

Our Markov chain is a projection onto the 2-TASEP on a ring if the following lemmas hold.

Lemma 6.3. If there is a transition $X \to Y$ on the ASEP chain, then for any TRAT R_X of type X, there exists exactly one tableau R_Y such that there is a transition $R_X \to R_Y$ on the TRAT chain.

Proof. A pair of consecutive edges on $P(X)$ of a TRAT R of type X is a *corner* if and only if that pair corresponds to one of the pairs $\{20, 21, 10\}$. Let X be a state of the ASEP chain, and let R_X be a TRAT of type X. Since every corner of a TRAT R_X has exactly one transition coming out of it, each possible transition $X \to Y$ on the ASEP chain corresponds to precisely one transition $R_X \to R'$ on the TRAT chain. We show that type$(R') = Y$.

The consequence of a transition at a corner $\mathfrak{c}$ on the TRAT chain is that the (north- or west-) strip at that corner is cut out, shifted by one box to the northwest, and replaced in a location where the segment of $P(X)$ on that shifted strip lines up with the rest of $P(X)$ on the tableau. This results in $P(X)$ undergoing a move in which the order of edges flips at corner $\mathfrak{c}$. Thus $P(X)$ becomes $P(Y)$, and so the shape of R' is a toric hexagon with path $P(Y)$ and hence type$(R') = Y$. □

Lemma 6.4. There is a uniform stationary distribution on the TRAT chain.

Proof. We want to show that detailed balance holds for a uniform distribution on the tableaux when each transition has probability 1. This is true if each tableau has an equal number of transitions going into and out of it.

By our definition of the TRAT chain, each corner of a tableau corresponds to precisely one transition coming out of it. By observing the image of the TRAT chain, we see that each corner also corresponds to precisely one transition going into the tableau (we omit the sufficiently straightforward definition of the reverse chain, which is simply the reverse of our definition of the forward chain). Thus the above claim holds true, which proves the lemma. □

6.2 Markov Chain on multiline queues's that projects to the inhomogeneous 2-TASEP on a ring

From the Markov chain on the TRAT and following the bijection of Section 3.2, we construct a minimal Markov chain on the weighted MLQs, which we call Ω^{MLQ}, that is different from both Markov chains in [8] and in [3], that projects to the inhomogeneous 2-TASEP on a ring. The Markov chain is minimal in the sense that every nontrivial transition in Ω^{MLQ} corresponds to a nontrivial transition in the TASEP.

Let M be an MLQ. We call $\Omega_i^{MLQ}(M)$ the MLQ obtained from a transition on M at location i. We denote by $M(i)$ the TASEP particle corresponding to location i in M. Recall that if the bottom row contains a vacancy at location i, then $M(i) = 2$; if the bottom row contains a 0-ball (i.e. one that is hit by a dropping top row ball), $M(i) = 0$; and if the bottom row contains a 1-ball (i.e. that is not hit by a dropping top row ball), $M(i) = 1$.

Definition 6.7. We call a ball in the bottom row *occupied* if there is a ball directly above it. Otherwise if there is a vacancy above it, we call it *vacant.*

Note that 1-balls are necessarily vacant. Moreover, no path from a top row ball to the 0-ball it occupies can pass through a 1-ball.

Definition 6.8. The transition at location i, denoted by Ω_i^{MLQ}, is given by the following rules.

- *Occupied jump:* if a transition occurs at an occupied ball at location i, let $j < i - 1$ be the nearest index left of $i - 1$ such that $M(j) \neq 0$, that is, $j = \max\{j < i - 1 : M(j) \neq 0\}$. A ball is inserted in the top row at location $j + 1$, shifting all top row contents at locations $j + 1, \ldots, i - 1$ one spot to the right.

- *Vacant jump:* if a transition occurs at a vacant ball at location i, let $j > i$ be the nearest index right of i such that $M(j) \neq 2$, that is, $j = \min\{j > i : M(j) \neq 2\}$. A vacancy is inserted in the top row at location j, shifting all top row contents at locations $i+1, \ldots, j-1$ one spot to the left.

In both cases, the bottom row contents of locations $i-1, i$ are swapped.

Figure 24 shows examples of each of the occupied and vacant jumps.

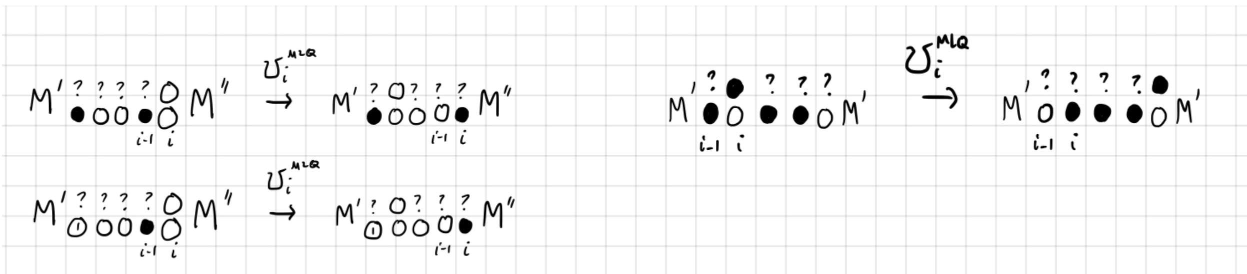

Figure 24: On the left are the occupied jumps; the top example is of a 20 jump and the bottom one is of a 10 jump. On the right is a vacant jump. In this case, the bottom row ball can be either a 0-ball or a 1-ball for a 20 or 21 jump, respectively.

Remark. Though Ω^{MLQ} has some similarities with the minimal Markov chain described in Section 5 of [3], out transitions are different. In particular, our Markov chain is equivalent to the TRAT Markov chain through the MLQ-TRAT bijection, which is addressed in the following lemma.

Lemma 6.5. Let M be an MLQ. The MLQ-TRAT bijection gives the following correspondences, which are illustrated by Figures 27 and 28.

- An occupied jump on M at location i corresponds to an up-arrow transition in the TRAT chain at edges $(i-1, i)$, which is a 20 transition (resp. 10 transition) at locations $(i-1, i)$ on the ASEP chain when $M(i-1)$ is a vacancy (resp. 1-ball).
- A vacant jump on M at location i corresponds to a left-arrow transition in the TRAT chain at edges $(i-1, i)$, which is a 20 transition (resp. 21 transition) at locations $(i-1, i)$ on the ASEP chain when $M(i)$ is a 0-ball (resp. 1-ball).

Proof. Let M be an MLQ and $T = TRAT(M)$, and let $M' = \Omega_i^{MLQ}(M)$ and $T' = TRAT(M')$.

If a ball at location i is occupied, $M(i) = 0$, and furthermore by the definition of our bijection there are no left-arrows in column i of T, and hence an up-arrow is contained in the bottom-most possible tile. Thus if the edges $(i-1, i)$ are at a corner of T, that corner tile contains an up-arrow. Now, since $j < i-1$ is the largest index such that $M(j) \neq 0$, $M(j+1) = 0$. Thus our definition of the occupied jump is equivalent to removing the column i from M, and inserting it to the right of column j. On the other hand, in T' that means removing column i with the up-arrow in its bottom-most tile and re-inserting it to the left of edge j, where j is the closest non-horizontal edge to the right of $i-1$. The new column has an up-arrow in its bottom-most box since M' has an occupied ball at location j. The rest of M' is left unchanged from M, and hence the rest of T' is left unchanged from T. This is precisely the definition of the TRAT transition at corner $i-1, i$ with an up-arrow in that corner.

If a ball at location i is vacant and $M(i) = 0$ (resp. $M(i) = 1$), there is a left-arrow in the corner 20-tile (resp. 21-tile) at edges $(i-1, i)$ of T. (Note that If $M(i) = 1$, we assume the tiling of T has a 21-tile at the $(i-1, i)$ corner. If the tiling does not have such a tile, we perform filling-preserving flips until it does.) Since $j > i$ is the smallest index such that $M(j) \neq 2$, $M(j-1) = 2$. Recall that in a vacant jump, every top row entry from column j to $i-1$ is shifted one location to the right, and a vacancy is placed in the top row of column j. In particular, if $M(j) = 0$, the hitting weight of the ball at location j increases by 1, while keeping all others unchanged. On the other hand, in T' that means removing row i with the left-arrow in its right-most tile and re-inserting this row to the right of edge j, where j is the closest non-vertical edge to the left of $i-1$. (We assume the row is inserted such that the tiling of T' is standard.) Since the hitting weight of the ball at location j increased by 1 while keeping all others unchanged, T' has an extra left-arrow in column j, which corresponds precisely to inserting a row with a left-arrow in its right-most box to the right of edge j. The latter is the definition of the TRAT transition at corner $(i-1, i)$ with a left-arrow in that corner. □

6.3 Markov chain on acyclic multiline queues that projects to the inhomogeneous 2-TASEP with open boundaries

There is a Markov Chain on the acyclic MLQs that projects to the 2-TASEP with open boundaries, that has the same bulk transitions as the usual MLQ chain from Section 6.2. This Markov Chain is obtained directly by pushing the Markov Chain on $RAT^{q=0}$ from [11] through the $\mathrm{RAT}^{q=0} \to \mathrm{AMLQ}$ bijection. We call this Markov chain $\Omega^{\mathrm{AMLQ}}_{MC}$.

Definition 6.9. Let $A \in \mathrm{AMLQ}(n, r)$. We define the left and right boundary transitions on Ω^{AMLQ} as follows, with examples shown in Fiture 25.

- **Left boundary transition Ω^{AMLQ}_0:** If A has a 0-ball at its left boundary, let $j > 1$ be the nearest index that does not contain a bottom row vacancy. The leftmost 0-ball is necessarily occupied by a ball above it. Replace the leftmost bottom row 0-ball by a vacancy, remove the leftmost top row ball, and shift all top row contents left of location $j+1$ one location to the left, and insert a vacancy in the top row of location j.
- **Right boundary transition Ω^{AMLQ}_n:** If A has a vacancy at its right boundary, let $j < n$ be the nearest index that does not contain a 0-ball. There is necessarily a top row vacancy above the rightmost bottom row vacancy. Remove the rightmost column and insert a column consisting of a 0-ball occupied ball a ball above it at location $j+1$.

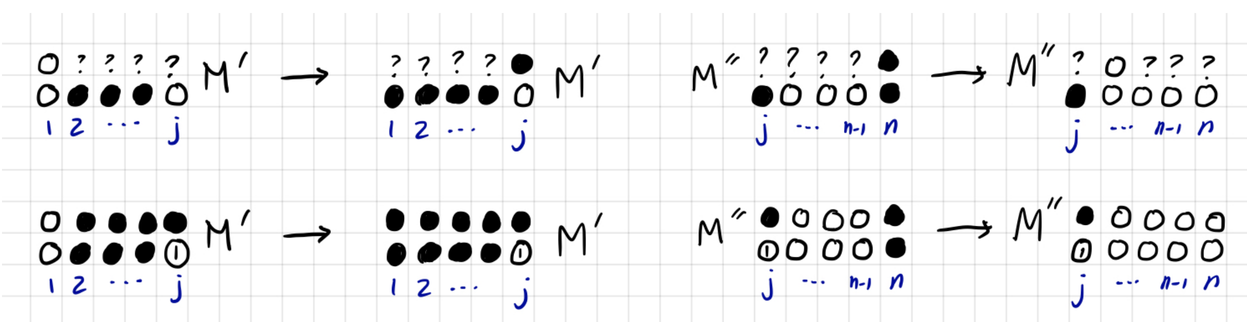

Figure 25: On top, we see a left boundary transition on the left and a right boundary transition on the right. On the bottom we see special cases of each when $M(j) = 1$.

To prove Ω^{AMLQ} indeed projects to the 2-TASEP with open boundaries, we recall the Markov Chain on $\mathrm{RAT}^{q=0}$ which it is in bijection with. We present it briefly below.

Theorem 6.6 ([11])**.** There is a Markov chain ζ^{RAT} on $\mathrm{RAT}^{q=0}(n,r)$, where each $R \in \mathrm{RAT}^{q=0}(n,r)$ has weight $\mathrm{wt}(R)$, that projects to the 2-ASEP(n,r).

We briefly describe the transitions of ζ^{RAT}.

Definition 6.10. The transitions ζ_i^{RAT} for $0 \le i \le n$ of ζ^{RAT} are as follows.

For $1 \le i < n$, let $\zeta_i^{RAT} : RAT(n,r) \to RAT(n,r)$ be a transition occurring at edges $(i, i+1)$: in the 2-ASEP word $X = X_1 \dots X_n$, this corresponds to the transition $X_i X_{i+1} \to X_{i+1} X_i$. For $i = 0$, this transition represents the boundary transition at the first edge of the RAT; in the 2-ASEP this corresponds to the transition $0X' \to 2X'$. Tor $i = n$, the transition represents the boundary transition at the last edge of the RAT; in the 2-ASEP this corresponds to the transition $X'2 \to X'0$.

For $1 \le i < n$, there are three cases. If $X_i = X_{i+1}$, then ζ_i^{RAT} is the identity. If $X_i > X_{i+1}$, assume the tiling $\mathcal{T}_X$ has an $X_i X_{i+1}$-tile adjacent to the corresponding corner. We have two possible cases for the contents of that tile.

- If the tile contains an up-arrow, $\zeta_i^{\mathrm{RAT}}(R)$ is a RAT obtained by removing the north-strip beginning at the corner, shortening it by tile, and re-inserting it in the rightmost possible location with an up-arrow still in its bottom-most tile. Then $\mathrm{wt}(\zeta_i^{\mathrm{RAT}}(R)) = \mathrm{wt}(R)$. (If the strip had length 1 to start, it is reinserted as a single horizontal edge at the right corner of R, in which case $\alpha\, \mathrm{wt}(\zeta_i^{\mathrm{RAT}}(R)) = \mathrm{wt}(R)$.)
- If the tile contains a left-arrow, $\zeta_i^{\mathrm{RAT}}(R)$ is a RAT obtained by removing the west-strip beginning at the corner, shortening it by tile, and re-inserting it in the bottom-most possible location with a left-arrow still in its right-most tile. Then $\mathrm{wt}(\zeta_i^{\mathrm{RAT}}(R)) = \mathrm{wt}(R)$. (If the strip had length 1 to start, it is reinserted as a single vertical edge at the left corner of R, in which case $\beta\, \mathrm{wt}(\zeta_i^{\mathrm{RAT}}(R)) = \mathrm{wt}(R)$.)

For $i = 0$, the rightmost boundary edge of the RAT must be horizontal. This edge is removed, and instead a west-strip of greatest possible length is inserted, while preserving the semiperimiter of the RAT. The strip is inserted in the lowest possible location and a left-arrow is placed in its rightmost tile. In this case $\mathrm{wt}(\zeta_i^{\mathrm{RAT}}(R)) = \alpha\, \mathrm{wt}(R)$.

For $i = n$, the leftmost boundary edge of the RAT must be vertical. This edge is removed, and instead a north-strip of greatest possible length is inserted, while preserving the semiperimiter of the RAT. The strip is inserted in the lowest possible location and a left-arrow is placed in its rightmost tile. In this case $\mathrm{wt}(\zeta_i^{\mathrm{RAT}}(R)) = \alpha\, \mathrm{wt}(R)$.

Figure 26 shows examples of each of these transitions. For proofs and technical details, see [11].

Lemma 6.7. Let $A \in \mathrm{AMLQ}$, and let $R = \mathrm{RAT}^{q=0}(A)$. Then $\Omega_i^{\mathrm{AMLQ}}(A) = \mathrm{RAT}^{MC}(R)$.

Proof. In the bulk, the proof carries over from the MLQ and TRAT case. It remains to show the equivalence for Ω_0^{AMLQ} and Ω_n^{AMLQ}. We show the following:

i. Ω_0^{AMLQ} corresponds to the right boundary transition ζ_0^{RAT} on a RAT, from Definition 6.10.

ii. Ω_n^{AMLQ} corresponds to the left boundary transition ζ_n^{RAT} on a RAT, from Definition 6.10.

In both cases, it is immediate that $\mathrm{type}(\Omega_0^{\mathrm{AMLQ}}(M)) = \mathrm{type}(\zeta_0^{\mathrm{RAT}})(R)$ (resp. $\mathrm{type}(\Omega_n^{\mathrm{AMLQ}}(M)) = \mathrm{type}(\zeta^{\mathrm{RAT}})_n(R))$ since only the first (resp. last) location of M and R changes its type.

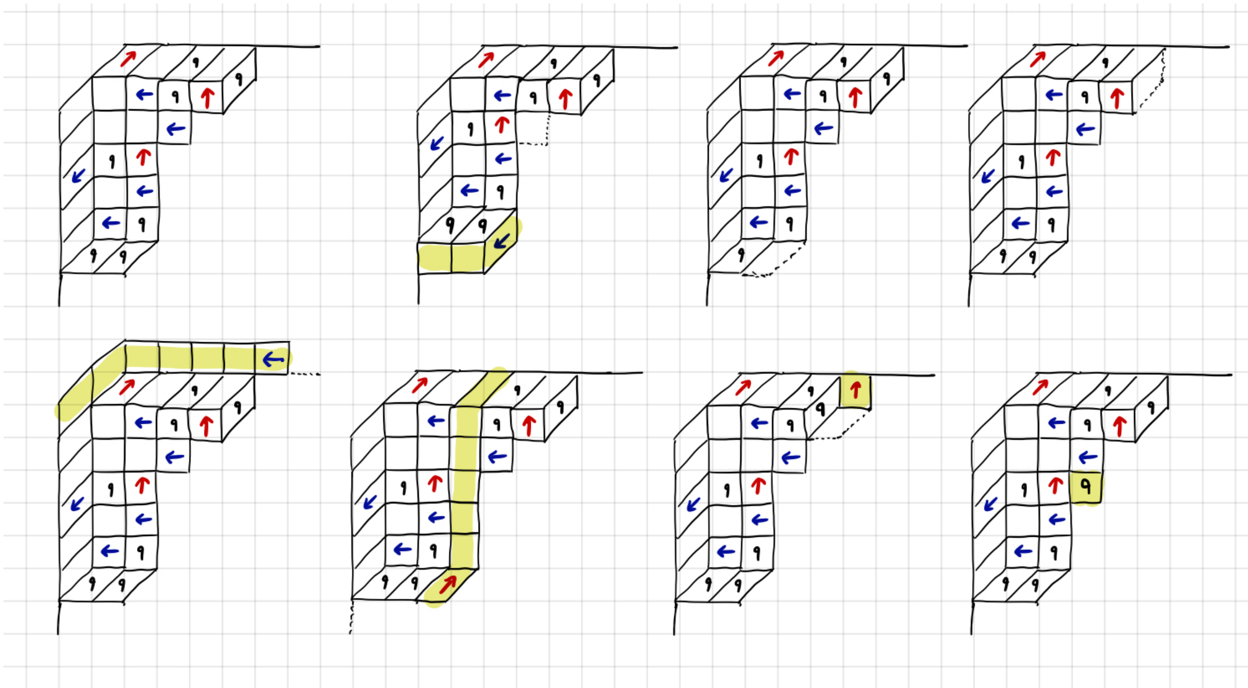

Figure 26: Examples of some of the possible transitions on the RAT Markov chain ζ^{RAT} for the state 00210212221002. The highlighted strips are those which are inserted while performing a transition at the edges indicated by dashed lines.

For (i.), we let $j = \min\{j > 1 : \; M(j) \neq 2\}$. Inserting a vacancy above the ball at location j in $\Omega_0^{\mathrm{AMLQ}}(M)$ results in a left-arrow being added to the north-strip j in $\zeta_n^{\mathrm{RAT}}(R)$. Since $(j-1, j)$ is a corner, and since we have replaced the first 0-ball with a vacancy, this is equivalent to adding a west-strip with a left-arrow in its rightmost tile at the $(j-1, j)$ corner, which is precisely the definition of ζ_0^{RAT}.

For (ii.), we let $j = \max\{j < n : \; M(j) \neq 0\}$. A column of an MLQ consisting of an occupied 0-ball corresponds to a north-strip with an up-arrow in the bottommost free location. Since $(j, j+1)$ is a corner of $\zeta^{\mathrm{RAT}})_n(R)$, a column in $\Omega_n^{\mathrm{AMLQ}}(M)$ consisting of an occupied 0-ball at location $j+1$ corresponds to a north-strip with an up-arrow in its bottommost tile adjacent to that corner. This is precisely the definition of ζ_n^{RAT}. $\square$

7 Concluding remarks

The tableaux method in this paper has a few advantages. First, it allows us to solve a more general, symmetric version of the inhomogeneous TASEP on a ring with three parameters for the hopping rates. Second, it establishes a connection between the well-studied multiline queue method of Ferrari and Martin solving the multispecies TASEP on a ring, and the alternative tableaux method originally introduced by Corteel and Williams. The multiline queues give combinatorics for the multispecies TASEP on a ring for any number of species but only for $q = 0$. On the other hand, thus far the tableaux method has only been useful the the 2-ASEP with open boundaries, albeit for general q. Our bijection makes us hopeful to find tableaux combinatorics for the two-species ASEP on a ring

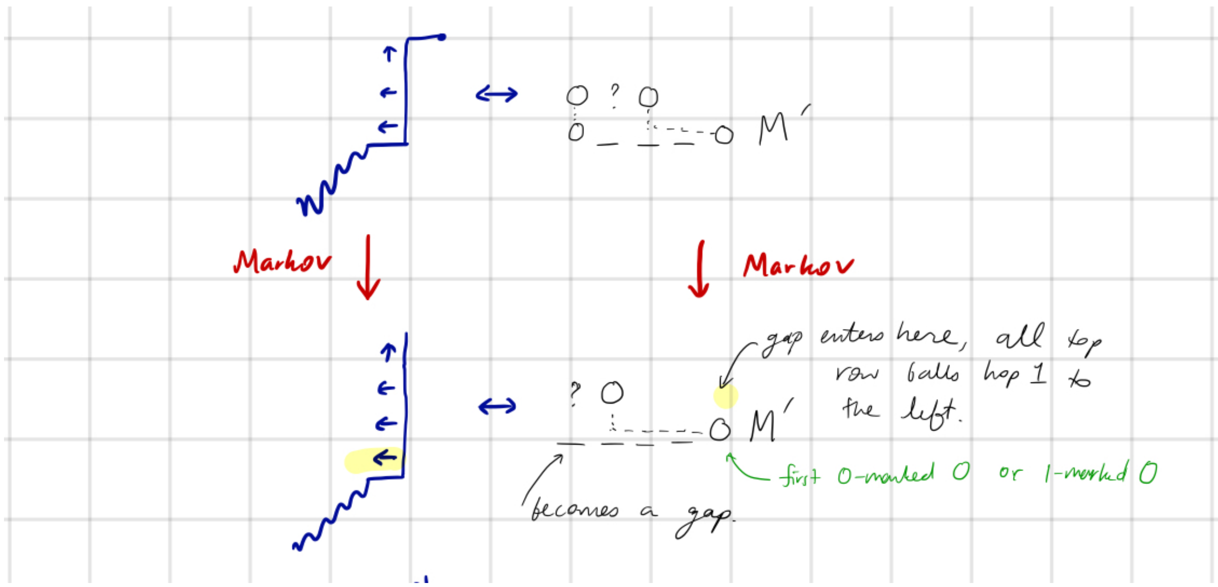

Figure 27: Left boundary Markov transition.

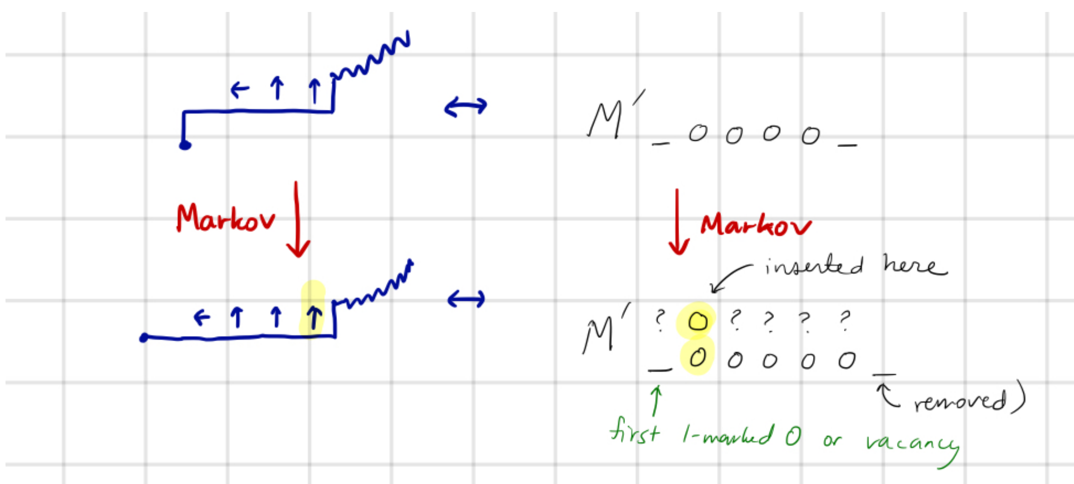

Figure 28: Right boundary Markov transition.

with general q. Furthermore, if would be interesting to put weights on the multiline queues or the acyclic MLQs to incorporate the q parameter.

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