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Adaptive RKHS-based functional estimation of structurally perturbed second order infinite dimensional systems

Michael A. Demetriou

Abstract— **This paper proposes a new approach for the adaptive functional estimation of second order infinite dimensional systems with structured perturbations. First, the proposed observer is formulated in the natural second order setting thus ensuring the time derivative of the estimated position is the estimated velocity, and therefore called natural adaptive observer. Assuming that the system does not yield a positive real system when placed in first order form, then the next step in deriving parameter adaptive laws is to assume a form of input-output collocation. Finally, to estimate structured perturbations taking the form of functions of the position and/or velocity outputs, the parameter space is not identified by a finite dimensional Euclidean space but instead is considered in a Reproducing Kernel Hilbert Space. Such a setting allows one not to be restricted by a priori assumptions on the dimension of the parameter spaces. Convergence of the position and velocity errors in their respective norms is established via the use of a parameter-dependent Lyapunov function, specifically formulated for second order infinite dimensional systems that include appropriately defined norms of the functional errors in the reproducing kernel Hilbert spaces. Boundedness of the functional estimates immediately follow and via an appropriate definition of a persistence of excitation condition for functional estimation, a functional convergence follows. When the system is governed by vector second order dynamics, all abstract spaces for the state evolution collapse to a Euclidean space and the natural adaptive observer results simplify. Numerical results of a second order PDE and a multi-degree of freedom finite dimensional mechanical system are presented.**

I. Introduction

The argument to support the examination of second order infinite dimensional systems in their natural second order setting relies on the preservation of physics. As was delineated in [1], the observer design for second order systems in the second order setting ensures that the derivative of the position estimate is indeed the velocity estimate. This cannot be guaranteed when a second order system is brought in a first order form. For finite dimensional systems, this physical relationship is attained asymptotically, but is not present in the transient stage. For infinite dimensional systems brought into a first order setting for the general case, the state vector components may not directly relate to the estimated position and estimated velocity, [2]. Additionally the strict positive realness of a first order infinite dimensional system is difficult to backtrack into the second order setting.

Thus, the adaptive estimation of structurally perturbed second order infinite dimensional systems cannot directly benefit from earlier works on adaptive techniques of positive real infinite dimensional systems in a first order setting [3]. The closest one can reach to the coupling of the input and output operators, as a condition guaranteed in positive real infinite dimensional systems, is through collocation whereby the output (observation) operator is the adjoint of the input operator, symbolically written in the generic form $\mathcal{C}=\mathcal{B}^*$. A case resembling the positive realness case and also the collocation was considered in [4] whereby a weighted multiple of the output operator was equal to the adjoint of the input operator, symbolically written as $M\mathcal{C}=\mathcal{B}^*$.

When unknown functions are to be estimated via adaptive techniques do not admit a series expansion parametrization, then one must consider a Hilbert space instead of a Euclidean space as the parameter space. The approach to use Reproducing Kernel Hilbert Spaces in the context of adaptive functional estimation was by spearheaded by Kurdila and co-workers [5], [6], [7], [8]. When one considers a finite dimensional system and utilizes adaptive techniques along with RKHS for functional estimation, the resulting dynamical system is rendered infinite dimensional.

Migrating to the adaptive functional estimation of infinite dimensional using only output information to generate the adaptive laws, was the effort in [9]. The infinite dimensional system assumed a positive realness condition and, following the fundamental work in [5], [6], [8], extended the RKHS-based adaptive functional estimation results.

Following the above arguments on the current state on the use of RKHS-based adaptive functional estimation, this paper considers a class of structurally perturbed second order infinite dimensional systems and presents a natural adaptive observer with a RKHS-based adaptive functional estimation.

II. Problem formulation

The class of 2nd order infinite dimensional systems, often representing structural PDEs such as beams, plates and cables, is described by an evolution equation defined over a five space setting and given by

$$\begin{aligned}&\ddot{\zeta}+\mathcal{D}\dot{\zeta}+\mathcal{K}\zeta=\mathcal{B}u+\mathcal{B}f_p(y_p)+\mathcal{B}f_v(y_v) \text{ in } V_1^*,\\ &\zeta(0)=\zeta_0\in V_1, \quad \dot{\zeta}(0)=\zeta_1\in V_2.\end{aligned} \tag{1}$$

It follows the abstract framework considered in [4]. In summary, one has the Gelfand quituple $V_1\hookrightarrow V_2\hookrightarrow H\hookrightarrow V_2^*\hookrightarrow V_1^*$ with H as the pivot space and duality pairings $\langle\cdot,\cdot\rangle_{V_i^*,V_i}$, $i=1,2$, [10]. The space V_1 is continuously and densely embedded in V_2 and V_2 densely and compactly embedded in H, and their conjugate dual spaces are denoted by V_1^* and V_2^*, respectively. Such a five space setting allows for a larger class of damping operators. Associated with the

M. A. Demetriou is with WPI, Aerospace Engineering Department, Worcester, MA 01609, USA, mdemetri@wpi.edu. The author acknowledges financial support from NSF-CMMI grant # 1825546.

above are the position and velocity measurements

$$y_p(t) = \mathcal{C}_p\zeta(t), \quad y_v(t) = \mathcal{C}_v\dot{\zeta}(t). \tag{2}$$

The damping operator $\mathcal{D} \in \mathcal{L}(V_2, V_2^*)$ is symmetric, bounded and $V_2 - H$ coercive, [10]. The stiffness operator $\mathcal{K} \in \mathcal{L}(V_1, V_1^*)$ is symmetric, bounded and V_1 coercive, the input operator $\mathcal{B} \in L(U, V_1^*)$, the position output operator $\mathcal{C}_p \in \mathcal{L}(V_1, Y_p)$ and the velocity output operator $\mathcal{C}_v \in \mathcal{L}(V_2, Y_v)$. The input space $U = \mathbb{R}^r$, the position output space $Y_p = \mathbb{R}^{n_p}$ and the velocity output space $Y_v = \mathbb{R}^{n_v}$.

The perturbation terms $f_p(\cdot) : \mathbb{R}^{n_p} \to \mathbb{R}^r$ and $f_v(\cdot) : \mathbb{R}^{n_v} \to \mathbb{R}^r$ satisfy Lipschitz-type conditions in order to guarantee existence of solutions for (1), (2), see [4]. Following the conditions presented in [4] for the well-posedness of (1), (2), one requires an L_2-type boundedness of the form $B\left(u(\cdot) + f_p(y_p(\cdot)) + f_v(y_v(\cdot))\right) \in L_2(0, \infty; V_2^*)$.

We can now present the estimation and control objectives in increasing complexity:

1) Set-up an adaptive natural observer to estimate the structured perturbation terms $f_p(y_p)$ and $f_v(y_v)$.
2) Set up an adaptive natural observer to estimate the terms $f_p(y_p)$ and $f_v(y_v)$ and use these estimates in the control design to "cancel" their effects.
3) Set up an adaptive natural observer to estimate the structured perturbation terms $f_p(y_p)$ and $f_v(y_v)$ and the estimates of the now assumed unknown static position G_p and velocity G_v gains, and use the estimates of the terms in the control design to "cancel" their effects, and use the estimates of static position $\widehat{\Theta}_p$ and velocity $\widehat{\Theta}_v$ gains as a means to regulate the closed-loop system to zero with a prescribed rate.
4) Set up an adaptive natural observer to estimate the structured perturbation terms $f_p(y_p)$ and $f_v(y_v)$ and the position $\zeta(t)$ and velocity $\dot{\zeta}(t)$ states. Use the estimates of the structured perturbation terms in the control design to "cancel" their effects and use the position $\widehat{\zeta}(t)$ and velocity $\widehat{\dot{\zeta}}(t)$ state estimates in a state controller as a means to regulate the closed-loop system to zero with a prescribed rate.
5) Set up adaptive natural observer to ensure that the 2nd order system follows a reference model described by a 2nd order infinite dimensional system. Use the estimates of $f_p(y_p)$ and $f_v(y_v)$ in the control design to "cancel" their effects. Either use the estimates of G_p, G_v to ensure tracking is achieved, or use the state estimates in a feedback controller to achieve tracking.

In this paper, we will be concerned with objective 1. The adaptive estimation of the structured perturbation terms $f_p(y_p)$ and $f_v(y_v)$ will be examined in a Euclidean space $\mathbb{R}^m$ and which constitutes a modified case of the one presented in [4], and also on a Reproducing Kernel Hilbert Space (RKHS). The latter approach follows the earlier exposure by the author for addressing structured perturbations of strictly positive real infinite dimensional systems [9] in a first order form. Since the system under consideration is not placed in a first order setting and when in the first order setting it does not immediately yield a strictly positive real system, then one has to remain in the natural 2nd order setting. The enabling condition to extract adaptive laws for the estimates of either the structure perturbation terms or the estimates of the feedback gains is a modified collocation condition which relates the input and output operators.

Assumption 1: There exist matrices M_p and M_v of dimensions $r \times n_p$ and $r \times n_v$, respectively, satisfying

$$M_p\mathcal{C}_p = \mathcal{B}^* \quad \text{and} \quad M_v\mathcal{C}_v = \mathcal{B}^*, \tag{3}$$

and $\min(n_p, n_v) \geq r$.

The other enabling condition, which may be required to improve the coercivity of the damping operator and/or to improve the convergence properties of an associated error system has to do with the output injection terms.

Assumption 2 (Matching Condition): There exist static gains $G_p \in \mathcal{L}(Y_p, U)$ and $G_v \in \mathcal{L}(Y_v, U)$ such that the following matching conditions

$$\mathcal{D} + \mathcal{B}G_v\mathcal{C}_v = \mathcal{D}_o \quad \text{and} \quad \mathcal{K} + \mathcal{B}G_p\mathcal{C}_p = \mathcal{K}_o \tag{4}$$

are satisfied. The closed-loop damping $\mathcal{D}_o$ and stiffness $\mathcal{K}_o$ operators have desired coercivity bounds that improve the convergence of the estimation errors associated with a natural observer for the perturbation-free nominal system.
Following [2], it should be remarked that in the absence of structured perturbations in (1), the associated natural observer for (1), (2) with Assumption 2 will guarantee the asymptotic convergence of the state position and velocity estimation errors in the stronger norms.

III. Adaptive natural observers

The adaptive observer for (1), in a 2nd order setting, takes the form of another 2nd order infinite dimensional system with appropriate output injection terms

$$\begin{aligned}
&\ddot{\widehat{\zeta}}(t) + \mathcal{D}\dot{\widehat{\zeta}}(t) + \mathcal{K}\widehat{\zeta}(t) = \mathcal{B}u(t) \\
&+\mathcal{B}G_p\left(y_p(t) - \mathcal{C}_p\widehat{\zeta}(t)\right) + \mathcal{B}G_v\left(y_v(t) - \mathcal{C}_v\dot{\widehat{\zeta}}(t)\right) \\
&+\mathcal{B}\left(\widehat{f}_p(y_p(t)) + \widehat{f}_v(y_v(t))\right) \\
&\widehat{\zeta}(0) = \widehat{\zeta}_0 \in V_1, \quad \dot{\widehat{\zeta}}(0) = \widehat{\zeta}_1 \in V_2.
\end{aligned} \tag{5}$$

The estimated position is denoted by $\widehat{\zeta}(t)$ and the estimated velocity is $\dot{\widehat{\zeta}}(t)$. Due to the 2nd order structure of (5), the derivative of the estimated position $d\widehat{\zeta}(t)/dt$ *is* equal to the estimated velocity $\widehat{\dot{\zeta}}(t)$, i.e. $\dot{\widehat{\zeta}} = \widehat{\dot{\zeta}}$. The term $\widehat{f}_p(y_p)$ denotes the adaptive estimate of the position structured perturbation term $f_p(y_p)$, and $\widehat{f}_v(y_v)$ denotes the adaptive estimate of the velocity structured perturbation term $f_v(y_v)$.

The construction of the estimates $\widehat{f}_p(y_p), \widehat{f}_v(y_v)$ in the appropriate spaces constitutes the contribution of this work. First, the estimates will be considered over a Euclidean space $\mathbb{R}^m$ of fixed dimension m and subsequently will be defined over a RKHS (infinite dimensional) which provides a natural parameter space for functional estimation.

A. Functional estimation over $\mathbb{R}^m$

To realize the adaptive estimates $\widehat{f}_p(y_p), \widehat{f}_v(y_v)$, one must provide the appropriate assumptions for their parametrization and subsequently extract the relevant adaptive laws via a Lyapunov-redesign method [11]. Towards that, one makes the following parametrization assumption.

Assumption 3 (Euclidean parametrization): The structured perturbation terms are assumed to admit the following parametrizations

$$f_p(y_p) = g_p(y_p)\Theta_p \quad \text{and} \quad f_v(y_v) = g_v(y_v)\Theta_v, \tag{6}$$

where the parameters Θ_p, Θ_v are m-dimensional unknown constant vectors and $g_p(\cdot) : \mathbb{R}^+ \times \mathbb{R}^{n_p} \to \mathbb{R}^{r\times m}$, $g_v(\cdot) : \mathbb{R}^+ \times \mathbb{R}^{n_v} \to \mathbb{R}^{r\times m}$, are known matrix functions.

Using Assumption 3, the adaptive observer with the parameter estimates now takes the form

$$\begin{aligned}
&\ddot{\widehat{\zeta}}(t) + \mathcal{D}\dot{\widehat{\zeta}}(t) + \mathcal{K}\widehat{\zeta}(t) = \mathcal{B}u(t) \\
&+\mathcal{B}G_p\left(y_p(t) - C_p\widehat{\zeta}(t)\right) + \mathcal{B}G_v\left(y_v(t) - C_v\dot{\widehat{\zeta}}(t)\right) \\
&+\mathcal{B}\left(g_p(y_p(t))\widehat{\Theta}_p(t) + g_v(y_v(t))\widehat{\Theta}_v(t)\right), \\
&\widehat{\zeta}(0) = \widehat{\zeta}_0 \in V_1, \quad \dot{\widehat{\zeta}}(0) = \widehat{\zeta}_1 \in V_2,
\end{aligned} \tag{7}$$

where $\widehat{\Theta}_p(t)$ denotes the *adaptive estimate* of the unknown Θ_p and $\widehat{\Theta}_v(t)$ is the *adaptive estimate* of the unknown Θ_v.

To extract the appropriate adaptive laws for $\widehat{\Theta}_p(t), \widehat{\Theta}_v(t)$ one considers the position error $e(t) = \zeta(t) - \widehat{\zeta}(t)$. Due to the structure of the proposed natural observer, the velocity error is precisely $\dot{e}(t) = \dot{\zeta}(t) - \dot{\widehat{\zeta}}(t)$. Subtracting the above from (1), one arrives at the error system

$$\begin{aligned}
&\ddot{e}(t) + \mathcal{D}\dot{e}(t) + \mathcal{K}e(t) = -\mathcal{B}G_pC_pe(t) - \mathcal{B}G_vC_v\dot{e}(t) \\
&+\mathcal{B}g_p(y_p)\left(\Theta_p - \widehat{\Theta}_p(t)\right) + \mathcal{B}g_v(y_v)\left(\Theta_v - \widehat{\Theta}_v(t)\right), \\
&e(0) = \zeta_0 - \widehat{\zeta}_0 \in V_1, \quad \dot{e}(0) = \zeta_1 - \widehat{\zeta}_1 \in V_2,
\end{aligned} \tag{8}$$

The error system (8) is key to extracting the update laws for the unknown $\widehat{\Theta}_p(t), \widehat{\Theta}_v(t)$. Using the following parameter-dependent Lyapunov function

$$\begin{aligned}
V(e,\dot{e}) \;=\; &\frac{\gamma}{2}\Big(\langle \dot{e}(t), \dot{e}(t)\rangle + \langle \mathcal{K}_o e(t), e(t)\rangle\Big) \\
&+\frac{1}{2}\langle \mathcal{D}_o e(t), e(t)\rangle + \langle e(t), \dot{e}(t)\rangle \\
&+\widetilde{\Theta}_p^T(t)\Gamma_p^{-1}\widetilde{\Theta}_p(t) + \widetilde{\Theta}_v^T(t)\Gamma_v^{-1}\widetilde{\Theta}_v(t),
\end{aligned} \tag{9}$$

with $\widetilde{\Theta}_p(t) \triangleq \Theta_p - \widehat{\Theta}_p(t)$ and $\widetilde{\Theta}_v(t) \triangleq \Theta_v - \widehat{\Theta}_v(t)$ denoting the position and velocity *parameter errors* [11], the desired adaptations are given by

$$\begin{aligned}
\dot{\widehat{\Theta}}_p(t) &= -\Gamma_p g_p(y_p)\Big[M_pCe(t) + \gamma M_vC\dot{e}(t)\Big], \\
\dot{\widehat{\Theta}}_v(t) &= -\Gamma_v g_v(y_v)\Big[M_pCe(t) + \gamma M_vC\dot{e}(t)\Big].
\end{aligned} \tag{10}$$

The symmetric positive definite matrices Γ_p, Γ_v are the adaptive gain matrices [11]. Essential to the Lyapunov-based adaptations was the matching condition in Assumption 2 that enables one to use static output injection terms, as opposed to full observer gain terms, to arrive at an error system that had the appropriate stability properties. The stability and convergence results of the proposed adaptive observer are summarized in the following lemma. Its proof can easily be established by following the approach for the similar problem of model reference adaptive control of 2nd order infinite dimensional systems in [4].

Lemma 1: Consider the 2nd order infinite dimensional system (1) with measurements (2). Assume that the collocation condition in Assumption 1 and the matching condition in Assumption 2 are satisfied and that (1), (2) constitutes an *admissible plant* in the sense that L_2 control signals result in L_2 output signals for the Lipschitz-type nonlinearities $f_p(\cdot)$ and $f_v(\cdot)$. Then the proposed natural adaptive observer (7) that assumes the parameterizations (6) along with the adaptations (10) result in a well-posed system with

$$\lim_{t\to\infty}\|e(t)\|_{V_1} = 0 \quad \text{and} \quad \lim_{t\to\infty}|\dot{e}(t)|_H = 0, \tag{11}$$

with the parameter estimates $\widehat{\Theta}_p(\cdot), \widehat{\Theta}_v(\cdot) \in L_\infty(0,\infty;\mathbb{R}^m)$. Additionally, parameter convergence in the sense of

$$\lim_{t\to\infty}|\widetilde{\Theta}_p(t)|_{\mathbb{R}^m} = 0 \quad \text{and} \quad \lim_{t\to\infty}|\widetilde{\Theta}_v(t)|_{\mathbb{R}^m} = 0, \tag{12}$$

can be established when a *persistence of excitation* condition is satisfied. This takes a special form for 2nd order infinite dimensional systems [12] and which requires the existence of T_0, δ_0 and ε_0 such that for each admissible parameter pair $(\Theta_p, \Theta_v) \in \mathbb{R}^m \times \mathbb{R}^m$ with unity Euclidean norm and each sufficiently large $t > 0$, there exists $\bar{t} \in [t, t+T_0]$ such that

$$\left\|\int_{\bar{t}}^{\bar{t}+T_0} \mathcal{B}\Big(g_p(y_p(\tau))\Theta_p + g_v(y_v(\tau))\Theta_v\Big)\,\mathrm{d}\tau\right\|_{V_2^*} \geq \varepsilon_0. \tag{13}$$

B. Functional estimation over RKHS

When the expansion (6) which yields parameterizations in the Euclidean space $\mathbb{R}^m \times \mathbb{R}^m$ is no longer guaranteed, one must resort to Hilbert spaces for parameterizations. Following the earlier work on positive real infinite dimensional [9], we extend the parameter spaces for the current class of 2nd order infinite dimensional systems. We denote by Q_p and Q_v the Hilbert spaces of functions defined on the respective output spaces Y_p and Y_v, with $f_p : Y_p \to Q_p$ and $f_v : Y_v \to Q_v$ with the evaluation functionals over Q_p and Q_v which evaluate each of the functions at the points $y_p \in Y_p$ and $y_v \in Y_v$ by

$$\begin{aligned}
\lambda_{y_p} &: f_p \to f_p(y_p), \qquad \forall y_p \in Y_p, \\
\lambda_{y_v} &: f_v \to f_v(y_v), \qquad \forall y_v \in Y_v.
\end{aligned} \tag{14}$$

This is interpreted as

$$f_p(y_p) = \lambda_{y_p}(f_p) \quad \text{and} \quad f_v(y_v) = \lambda_{y_v}(f_v).$$

One would like to have the evaluation functionals above to be bounded. Indeed with the appropriate construction of their associated kernels, one has that the evaluation functionals $\lambda_{y_p}, \lambda_{y_v}$ are bounded and the associated Hilbert space Q_p and Q_v, taking the role of the parameter spaces, are RKHS. Via the use of the Riesz representation theorem we have that for all position outputs $y_p \in Y_p$ there is an element (kernel)

$$\kappa_p : Y_p \times Y_p \to \mathbb{R}^m$$

with $\kappa_{y_p} = \kappa_p(y_p, \cdot)$ that enjoys the reproducing property

$$f_p(y_p) = \lambda_{y_p}(f_p) = \langle f_p, \kappa_p(y_p,\cdot)\rangle_{Q_p} = \langle f_p, \kappa_{y_p}\rangle_{Q_p}, \tag{15}$$

for all $f_p \in Q_p$, and for all $y_p \in Y_p$. Similarly, we have that for all velocity outputs $y_v \in Y_v$ there is an element

$$\kappa_v : Y_v \times Y_v \to \mathbb{R}^m$$

with $\kappa_{y_v} = \kappa_v(y_v, \cdot)$ that enjoys the reproducing property

$$f_v(y_v) = \lambda_{y_v}(f_v) = \langle f_v, \kappa_v(y_v, \cdot)\rangle_{Q_v} = \langle f_v, \kappa_{y_v}\rangle_{Q_v}, \qquad (16)$$

for all $f_v \in Q_v$, and for all $y_v \in Y_v$. The extraction of the adaptive laws in the parameter spaces (RKHS's) via Lyapunov redesigned methods relies on the use of the *adjoints* of the evaluation functionals. Define the adjoint of the position evaluation functional $\lambda^*_{y_p} : Y_p \to Q_p$ via

$$\langle C_p\phi, \lambda_{y_p}(f_p)\rangle_{Y_p} = \langle C_p\phi\kappa_{y_p}, f_p\rangle_{Q_p} = \langle \lambda^*_{y_p}(C_p\phi), f_p\rangle_{Q_p}, \quad (17)$$

for $\phi \in V_1$. Similarly, define the adjoint of the velocity evaluation functional $\lambda^*_{y_v} : Y_v \to Q_v$ via

$$\langle C_v\phi, \lambda_{y_v}(f_v)\rangle_{Y_v} = \langle C_v\phi\kappa_{y_v}, f_v\rangle_{Q_v} = \langle \lambda^*_{y_v}(C_v\phi), f_v\rangle_{Q_v}, \quad (18)$$

for $\phi \in V_2$. Using (14), the 2[nd] order system (1) is written as

$$\begin{aligned} &\ddot{\zeta} + \mathcal{D}\dot{\zeta} + \mathcal{K}\zeta = \mathcal{B}u + \mathcal{B}\lambda_{y_p(t)}(f_p) + \mathcal{B}\lambda_{y_v(t)}(f_p) \text{ in } V_1^*, \\ &\zeta(0) = \zeta_0 \in V_1, \quad \dot{\zeta}(0) = \zeta_1 \in V_2. \end{aligned} \qquad (19)$$

The natural adaptive observer (7) is now redefined as

$$\begin{aligned} \ddot{\widehat{\zeta}}(t) + \mathcal{D}_o\dot{\widehat{\zeta}}(t) + \mathcal{K}_o\widehat{\zeta}(t) &= \mathcal{B}u(t) + \mathcal{B}G_py_p \\ &\quad + \mathcal{B}G_vy_v + \mathcal{B}\lambda_{y_p(t)}(\widehat{f}_p) + \mathcal{B}\lambda_{y_v(t)}(\widehat{f}_v), \qquad (20) \\ \widehat{\zeta}(0) = \widehat{\zeta}_0 \in V_1, \quad \dot{\widehat{\zeta}}(0) &= \widehat{\zeta}_1 \in V_2. \end{aligned}$$

Using (4), the error system resulting from (19), (20) is

$$\begin{aligned} &\ddot{e}(t) + \mathcal{D}_o\dot{e}(t) + \mathcal{K}_oe(t) = \mathcal{B}\lambda_{y_p(t)}(\widetilde{f}_p) + \mathcal{B}\lambda_{y_v(t)}(\widetilde{f}_v) \\ &e(0) = \zeta_0 - \widehat{\zeta}_0 \in V_1, \quad \dot{e}(0) = \zeta_1 - \widehat{\zeta}_1 \in V_2. \end{aligned} \qquad (21)$$

The adaptive laws for the functional estimates are made possible via a different Lyapunov function, given by (*cf.* (9))

$$\begin{aligned} V(e,\dot{e}) \quad = \quad & \frac{\gamma}{2}\Big(\langle \dot{e}(t), \dot{e}(t)\rangle + \langle \mathcal{K}_oe(t), e(t)\rangle\Big) \\ &+ \frac{1}{2}\langle \mathcal{D}_oe(t), e(t)\rangle + \langle e(t), \dot{e}(t)\rangle \qquad (22) \\ &+ \langle \mathcal{G}_p^{-1}\widetilde{f}_p, \widetilde{f}_p\rangle_{Q_p} + \langle \mathcal{G}_v^{-1}\widetilde{f}_v, \widetilde{f}_v\rangle_{Q_v}, \end{aligned}$$

where now the positive self-adjoint linear operators $\mathcal{G}_p \in \mathcal{L}(Q_p, Q_p)$ and $\mathcal{G}_v \in \mathcal{L}(Q_v, Q_v)$ acquire the role of the adaptive gain matrices Γ_p, Γ_v in the adaptations (10). With the above Lyapunov function, the adaptive laws are extracted via the following identities

$$\begin{aligned} \langle \varphi, \mathcal{B}\lambda_{y_p}(f_p)\rangle_{V_1,V_1^*} &= \langle \mathcal{B}^*\varphi, \lambda_{y_p}(f_p)\rangle_{Y_p} \\ &= \langle \lambda^*_{y_p}(\mathcal{B}^*\varphi), f_p\rangle_{Q_p}, \;\; \varphi \in V_1, f_p \in Q_p, \end{aligned} \qquad (23)$$

$$\begin{aligned} \langle \varphi, \mathcal{B}\lambda_{y_v}(f_v)\rangle_{V_2,V_2^*} &= \langle \mathcal{B}^*\varphi, \lambda_{y_v}(f_v)\rangle_{Y_v} \\ &= \langle \lambda^*_{y_v}(\mathcal{B}^*\varphi), f_v\rangle_{Q_2}, \;\; \varphi \in V_2, f_v \in Q_v. \end{aligned} \qquad (24)$$

For short, by defining the position and velocity output errors

$$\varepsilon_p(t) = C_pe(t), \qquad \varepsilon_v(t) = C_v\dot{e}(t),$$

and using (23), (24) we have

$$\langle e(t), \mathcal{B}\lambda_{y_p(t)}(\widetilde{f}_p)\rangle_{V_1,V_1^*} = \langle \lambda^*_{y_p(t)}(M_p\varepsilon_p(t)), \widetilde{f}_p\rangle_{Q_p},$$

$$\langle \dot{e}(t), \mathcal{B}\lambda_{y_p(t)}(\widetilde{f}_v)\rangle_{V_2,V_2^*} = \langle \lambda^*_{y_v(t)}(M_v\varepsilon_v(t)), \widetilde{f}_v\rangle_{Q_v}.$$

The adaptive laws, expressed in weak form, are

$$\begin{aligned} \langle \dot{\widetilde{f}}_p, p\rangle_{Q_p} &= -\langle \mathcal{G}_p\lambda^*_{y_p(t)}[M_p\varepsilon_p + \gamma M_v\varepsilon_v], p\rangle_{Q_p} \\ &= -\langle [M_p\varepsilon_p + \gamma M_v\varepsilon_v], \lambda_{y_p(t)}(\mathcal{G}_pp)\rangle_{Q_p} \end{aligned} \qquad (25)$$

for all test functions $p \in Q_p$, and

$$\begin{aligned} \langle \dot{\widetilde{f}}_v, v\rangle_{Q_v} &= -\langle \mathcal{G}_v\lambda^*_{y_v(t)}[M_p\varepsilon_p + \gamma M_v\varepsilon_v], v\rangle_{Q_v} \\ &= -\langle [M_p\varepsilon_p + \gamma M_v\varepsilon_v], \lambda_{y_v(t)}(\mathcal{G}_vv)\rangle_{Q_v} \end{aligned} \qquad (26)$$

for all test functions $v \in Q_v$. In strong form they are

$$\dot{\widetilde{f}}_p = -\mathcal{G}_p\lambda^*_{y_p(t)}\Big[M_p\varepsilon_p(t) + \gamma M_v\varepsilon_v(t)\Big] \qquad (27)$$

$$\dot{\widetilde{f}}_v = -\mathcal{G}_v\lambda^*_{y_v(t)}\Big[M_p\varepsilon_p(t) + \gamma M_v\varepsilon_v(t)\Big]. \qquad (28)$$

The well-posedness of the natural adaptive observer (20) with adaptations (25), (26) can be established by constructing arguments similar to those presented in [4] for natural observers of structurally perturbed 2[nd] order infinite dimensional systems with adaptation in $\mathbb{R}^m$, and in [9] for adaptive observers of structurally perturbed positive real infinite dimensional systems with adaptations in RKHS.

The stability and convergence, in a similar fashion to Lemma 1 for estimation in $\mathbb{R}^m \times \mathbb{R}^m$, is given below.

Lemma 2: Consider the 2[nd] order infinite dimensional system (19) with measurements obtained by (2). Assume that the collocation-like condition in Assumption 1 and the matching condition in Assumption 2 are satisfied and that (19), (2) constitutes an admissible plant in the sense that L_2 control signals result in L_2 output signals for the Lipschitz-type nonlinearities $f_p(\cdot)$ and $f_v(\cdot)$. Then the proposed natural adaptive observer (20) along with the adaptations (27), (28) result in a well-posed system with

$$\lim_{t\to\infty} \|e(t)\|_{V_1} = 0 \quad \text{and} \quad \lim_{t\to\infty} |\dot{e}(t)|_H = 0, \qquad (29)$$

with the functional estimates $\widehat{f}_p(\cdot) \in L_\infty(0,\infty; Q_p)$, $\widehat{f}_v(\cdot) \in L_\infty(0,\infty; Q_v)$. Parameter convergence in the sense of

$$\lim_{t\to\infty} \|\widehat{f}_p(t) - f_p\|_{Q_p} = 0 \quad \text{and} \quad \lim_{t\to\infty} |\widehat{f}_v(t) - f_v|_{Q_v} = 0, \quad (30)$$

can be established when a persistence of excitation condition is satisfied [13]. This takes a special form for 2[nd] order infinite dimensional systems [12] and which requires the existence of T_0, δ_0 and ε_0 such that for each admissible pair $p \in Q_p, v \in Q_v$ with $|p|_{Q_p} = 1$, $|v|_{Q_v} = 1$ and each sufficiently large $t > 0$, there exists $\bar{t} \in [t, t+T_0]$ such that

$$\left\| \int_{\bar{t}}^{\bar{t}+T_0} \mathcal{B}\left(\lambda_{y_p(\tau)}(p) + \lambda_{y_v(\tau)}(v)\right) \mathrm{d}\tau \right\|_{V_2^*} \geq \varepsilon_0. \qquad (31)$$

Remark 1: Central to the implementation of the scheme in Lemma 1 is the parametrization in (6) and for the scheme in Lemma 2 is the property in (16).

IV. Adaptive RKHS-based functional estimation of structurally perturbed vector 2[nd] order systems

The case of a 2[nd] order evolution system over $\mathbb{R}^n \times \mathbb{R}^n$ with structured perturbations constitutes a special case of the natural adaptive observer (20). In this case all five spaces in (19) collapse to a single Euclidean space with $V_1 = V_2 = H =$

$V_2^* = V_1^* = \mathbb{R}^n$. While the resulting natural adaptive observer will be finite dimensional, the adaptive estimation will be infinite dimensional since it is defined over the RKHS Q_p and Q_v. All operators in (19), (2) and (27), (28) are replaced with their matrix equivalents.

V. Numerical Examples

2nd order PDE: We first consider a 2nd order PDE which is representative of 2nd order infinite dimensional system, and given by the wave equation

$$w_{tt}(t,\xi) + 0.001 w_t(t,\xi) - 0.05 w_{t\xi\xi}(t,\xi) - w_{\xi\xi}(t,\xi) \\ + b(\xi)\big(u(t) + f_p(y_p)\big), \qquad 0 < \xi < \ell.$$

Dirichlet boundary conditions are assumed with $w(t,0) = w(t,\ell) = 0$ and initial conditions $w(0,\xi) = 0.5\sin(\pi\xi) + \exp(-100(\xi - \ell/3)^2)$, $w_t(0,\xi) = 0.5(1 - \cos(2\pi\xi/\ell)) + \exp(-100(\xi - 0.65\ell)^2)$. The spatial domain was $[0,\ell] = [0,1]$. The relevant spaces here are $H = L^2(0,\ell)$, $V_1 = V_2 = H_0^1(0,\ell)$ with $V_1^* = V_2^* = H^{-1}(0,\ell)$.

The spatial distribution of the input is given by $b(\xi) = \delta(\xi - 0.5\ell)$ with both the position and velocity measurements collocated to the input and given by $y_p(t) = w(t, 0.5\ell)$, $y_v(t) = w_t(t, 0.5\ell)$. In this case, we have $r = n_p = n_v = 1$ with M_p, M_v in (3) given by $M_p = M_v = 1$. The static gains in the matching condition (4) were selected as $G_p = 1$, $G_v = 2$. The unknown nonlinearities were selected as $f_p(y) = -y^3$ and $f_v(y_v) = 0$.

A finite element based approximation scheme was used to approximate the wave PDE, with a total of 50 linear elements. The resulting finite dimensional vector order system was integrated in the interval $[0,600]$s using the Matlab `ode45` solver. The initial conditions for the state estimates were selected as $\widehat{w}(0,\xi) = 0 = \dot{\widehat{w}}(0,\xi)$. For the functional estimation, radial basis functions (gaussian functions) were used with kernel $\kappa_{y_p}(p) = \exp(-\frac{|y_p - p|^2}{2\sigma_p^2})$ and $\sigma_p = \frac{1}{2\sqrt{\log(2)}}$ with the means p evenly distributed in the "spatial" interval $[-7,7]$. To approximate $\widehat{f}_p(y_p)$ using a finite dimensional subspace $Q_p^N \subset Q_p$, a total of $N = 121$ radial basis functions were used with the kernel approximation

$$\widehat{f}_p(y_p) = \sum_{i=1}^{N} \widehat{\theta}_i(t) \kappa_{y_{pi}}(\cdot).$$

Using the initial guess $\widehat{f}_p(y_p(0)) = 0$ yielding $\widehat{\theta}_i(0) = 0$, $i = 1,\ldots,N$, the adaptation (27) was implemented with $\mathcal{G}_p = 0.1$.

The evolution of the L_2 state error norm $\sqrt{\|e(t)\|_{V_1}^2 + |\dot{e}(t)|_H^2}$ is depicted in Figure 1 where it is observed that it asymptotically converges to zero. The time evolution of the unknown term $f_p(y_p(t))$ and its adaptive functional estimate $\widehat{f}_p(y_p(t)$ are depicted in Figure 2a where it is observed that the adaptive functional estimate converges to the true function. However, this does not reveal how well the adaptive functional estimate learns the unknown function. A way to assess the amount of learning is via the normalized functional error given by

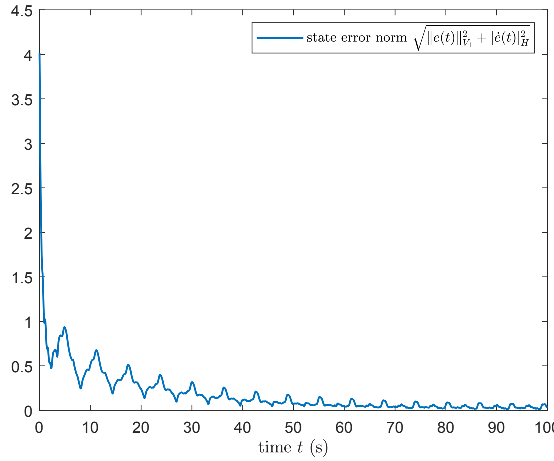

Fig. 1. Evolution of L_2 state norm $\sqrt{\|e(t)\|_{V_1}^2 + |\dot{e}(t)|_H^2}$.

$$\|f_p(y_p(t)) - \widehat{f}(y_p(t))\|_{Q_p^N}^2 = \frac{\int_{-7}^{7}\Big(f_p(y) - \widehat{f}_p(y)\Big)^2 \mathrm{d}y}{\int_{-7}^{7} f_p^2(y)\,\mathrm{d}y} = \frac{\int_{-7}^{7}\Big(-y^3 - \widehat{f}_p(y)\Big)^2 \mathrm{d}y}{\int_{-7}^{7} y^6\,\mathrm{d}y},$$

and which is depicted in Figure 2b. This functional error starts at a value of 100%, since $\widehat{f}_p(y) = 0$ and converges to a small value. At the final time $t = 600$s this error is 5.14%. The true nonlinearity $f_p(y)$ along with the adaptive estimate $\widehat{f}_p(y_p)$ at $t = 100$s are plotted against the "spatial" variable y and are depicted in Figure 3. It is observed that the adaptive estimate identifies the function $-y^3$.

2nd order ODE: Here, a 3DOF system is considered

$$M\ddot{w}(t) + D\dot{w}(t) + Kw(t) = B\Big(u(t) + f_p(y_p(t))\Big)$$

with $M = \mathrm{diag}\ (1,2,1)$, $B = [0\ 0\ 1]^T$, $C_p = C_v = B^T$

$$K = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3.6 & -2.1 & 0 \\ -2.1 & 3.6 & -1.5 \\ 0 & -1.5 & 1.5 \end{bmatrix}.$$

A more general version of the matching condition is considered for the finite dimensional case and which takes the form of $D + G_vC_v = D_o$ and $K + G_pC_p = K_o$, with $G_p = [6\ 9\ 9]^T$, $G_v = [17.0432\ 21.56\ 12.3]^T$, and $\gamma = 3.0182$, $\mathcal{G}_p = 15$. The initial condition for the plant was $w(0) = \begin{bmatrix} 0.50 & 0.25 & 0.15 \end{bmatrix}^T$, $\dot{w}(0) = \begin{bmatrix} 0.1 & 0.2 & 0.3 \end{bmatrix}^T$, and for the natural observer was $\widehat{w}(0) = \mathbf{0}_{3\times 1} = \dot{\widehat{w}}(0)$. The nonlinear function was $f_p(y) = -\alpha y^3$ with $\alpha = 0.5$.

The evolution of the energy norm of the error system is depicted in Figure 4a and the output position error is depicted in Figure 4b. Both plots show the convergence of the errors to zero. The nonlinearity $f_p(y)$ along with the estimate $\widehat{f}_p(y_p)$ at $t = 60$s are plotted against the "spatial" variable y and depicted in Figure 5. It is observed that the adaptive estimate identifies the function $-0.5y^3$.

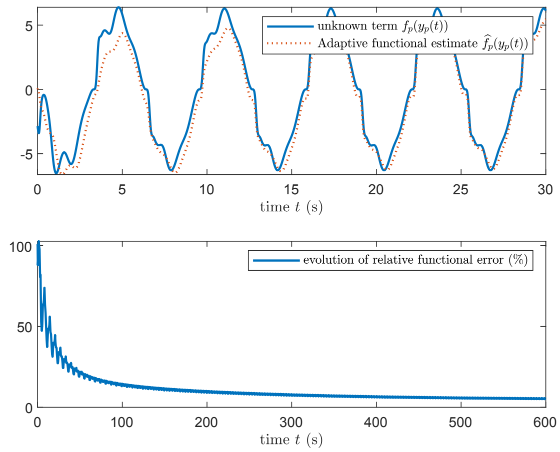

Fig. 2. (a) Time evolution of the unknown term $f_p(y_p(t))$ and its adaptive functional estimate $\widehat{f}_p(y_p(t)$; (b) time evolution of the functional error.

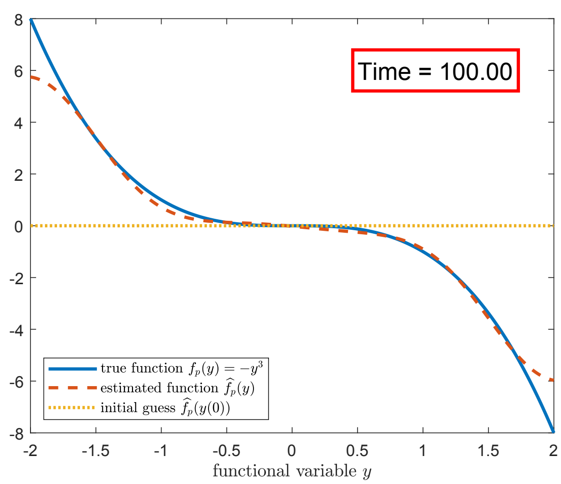

Fig. 3. Adaptive functional estimate $\widehat{f}_p(y_p(t)$ at final time $t = 100$s and the actual $f_p(y) = -y^3$ versus the functional variable ("spatial variable") y.

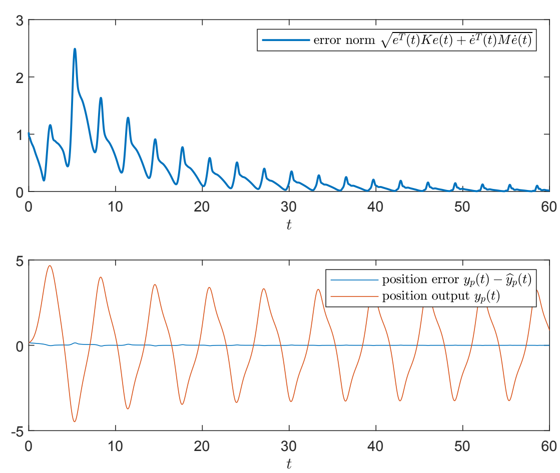

Fig. 4. Evolution of L_2 state norm.

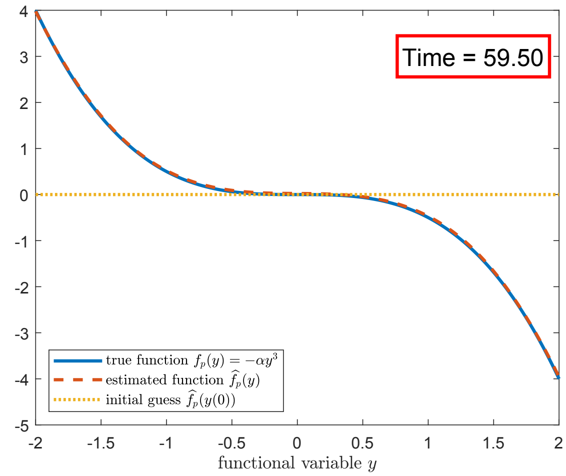

Fig. 5. Adaptive functional estimate $\widehat{f}_p(y_p(t))$ at final time $t = 60$s and the actual $f_p(y) = -0.5y^3$ versus the functional variable ("spatial variable") y.

VI. Conclusions

An adaptive functional estimation scheme for structurally perturbed 2^{nd} order infinite dimensional systems was presented and which utilized a RKHS-based adaptive functional estimation. The natural adaptive observer did not require a positive realness condition but instead imposed a collocation condition as a means to extract the adaptive laws for the RKHS-based functional estimation.

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