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Sensor location for unknown input observers of second order infinite dimensional systems

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Abstract— **The problem of sensor placement for second order infinite dimensional systems is examined within the context of a disturbance-decoupling observer. Such an observer takes advantage of the knowledge of the spatial distribution of disturbances to ensure that the resulting estimation error dynamics are not affected by the temporal component of the disturbances. When such an observer is formulated in a second order setting, it results in a natural observer. Further, when the natural observer is combined with a disturbance decoupling observer, the necessary operator identities needed to ensure the well-posedness of the observer, are expressed in terms of the stiffness, damping, input and output operators. A further extension addresses the question of where to place sensors so that the resulting natural disturbance decoupling observer is optimal with respect to an appropriately selected performance measure. This paper proposes this performance measure which is linked to the mechanical energy of second order infinite dimensional systems. The proposed sensor optimization is demonstrated by a representative PDE in a second order setting.**

I. Introduction

This paper combines three different design aspects for the state estimation of infinite dimensional systems. The first one considers the observer design of infinite dimensional systems that have disturbances. A way to decouple the disturbances from the state estimation problem, thus removing completely the effects of the disturbances on the learning capability of the state estimator, is to use a *disturbance decoupling* observer. This approach, extensively applied in finite dimensional systems [1], aims at completely removing the effects of the disturbance signal by utilizing the knowledge of its distribution matrix. The enabling conditions are expressed in terms of a Sylvester equation and associated matrix identities reflecting the decoupling of the disturbance signal and which express observability conditions of the relevant system matrices. This *disturbance-decoupling*, or *unknown input*, observer design was extensively utilized in the fault detection community [2], [3] to decouple fault detection observers from the effects of disturbances with known distribution matrices.

The second design aspect deals with the design of *natural* observers for second order systems. Such systems represent mechanical systems and are formulated in a second order (in time) setting. They are called natural since they ensure that the derivative of the estimated position is equal to the estimated velocity. The estimator design is made in the second order setting in order to ensure this kinematic relationship of the estimated states. Such a link to the physics is lost when a second order system is placed in a first order state space form and then an observer is designed for it. This was first observed in [4] for single degree of freedom systems. The design of natural observers was subsequently extended to multi-degree of freedom systems (vector second order) [5], [6], [7] and second order infinite dimensional systems [8], [9]. The natural observer design was similarly applied for fault detection [10], [11], [12].

The design of a natural observer for second order systems to account for disturbance decoupling was considered in [13] for finite dimensional systems and was extended to the infinite dimensional case in [14]. However, an added design aspect not considered was the sensor location optimization. If more than a single sensor location ensures that the disturbance decoupling observer is feasible, then one must select the better location by appropriately selecting a performance measure. The best sensor location for *disturbance decoupling natural observers* (DDNO) is then selected from the set of feasible locations that ensure that the DDNO is feasible and which minimize the performance measure. This is in fact the contribution of this paper. Both finite and infinite dimensional second order systems are considered and the appropriate measure of the sensor location optimization is proposed to find the optimal sensor location amongst the candidate sensor locations that yield the "best" DDNO.

The sensor optimization for DDNO of second order finite dimensional systems is summarized in Section II and which uses the total energy of the error system. This translates to a location parameterized solution to an algebraic Lyapunov equation and the optimal sensor is found as the one that minimizes the trace of the location-parameterized Lyapunov matrix.

Migrating to the infinite dimensional case, care must be exercised when considering the relevant functional spaces. Accounting for unbounded stiffness and damping operators that also are defined in different spaces enables one to account for a wide range of damping in the class of second order infinite dimensional systems. This is formulated in the Gelfand quintuple (five space setting), [15] and is presented in Section III. The appropriate performance measure is similarly define energy of the system which gives rise to an operator Lyapunov equation parameterized by the candidate sensor locations. Assuming nuclearity of the Lyapunov operator allows one to represent the performance metric in terms of the location-parameterized Lyapunov operator.

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A numerical example involving the cable PDE in a single spatial dimension is included in Section IV to demonstrate the proposed sensor optimization results. Conclusions with future extensions follow in Section V.

II. Problem formulation for second order finite dimensional systems

While n-degree of freedom systems (DOF), also known as vector second order systems, may not have the complexity of second order infinite dimensional system, they nonetheless share physical traits; the former can be viewed as the finite dimensional approximation of the latter. Such systems are governed by

$$\begin{aligned} &M\ddot{x}(t)+D\dot{x}(t)+Kx(t)=Bu(t)+Ew(t),\\ &y(t)=Cx(t). \end{aligned} \tag{1}$$

In the above, $M=M^T>0$ denotes the mass matrix, $D=D^T\geq 0$ denotes the damping matric, $K=K^T>0$ denotes the stiffness matrix and all are $n\times n$ matrices. The input matrix B is an $n\times n$ matrix and describes how the r-dimensional control signal u enters into the equations of motion. The position output vector y is obtained via the m position sensors via the $m\times n$ position output matrix C. The n-dimensional vector x is the position vector and $\dot{x}$ is the n-dimensional velocity vector. The term Ew denotes disturbance terms with w being the unknown input. The $n\times q$ matrix E denotes the manner in which the disturbance signal w is distributed in the states. The matrix E is assumed known.

The goal is to set up an observer that, despite the presence of the unknown disturbance signal w, can still provide an asymptotic estimate of the position and velocity vectors. This of course leads to a *disturbance-decoupling observer*, known also as an *unknown input observer*. If the above system is brought in a first order formulation, the link to the physical interpretation is lost. Thus, one must stay within the second order setting to design an observer; such an observer is termed a *natural observer*, because of its structure it ensures that the time derivative of the estimated position vector, denoted here by $\widehat{x}$ will be equal to the estimated velocity $\widehat{v}$ where $v=\dot{x}$ denotes the velocity. Such a relation is mathematically described by the identity

$$\frac{\mathrm{d}\widehat{x}}{\mathrm{d}t}=\widehat{v}=\widehat{\frac{\mathrm{d}x}{\mathrm{d}t}}, \tag{2}$$

or more compactly as $\dot{\widehat{x}}=\widehat{\dot{x}}$.

The natural observer for vector order systems was considered in [9] and its extension to unknown input observers was presented in [14]. However, the sensor optimization for such a disturbance decoupling natural observer (DDNO) has not been addressed. This optimization will be summarized here since it bears many similarities with its infinite dimensional counterpart.

The sensor location is reflected in the output matrix C. Thus we parameterized the output matric by the vector of candidate locations and henceforth use the notation $C(\theta)$. The goal is to select the sensor locations belonging to some parameter set $\theta\in\Theta$ that not only render the unknown-input natural observer feasible, but also minimize an appropriately selected performance measure; i.e., select the sensor location θ from the set of feasible locations that makes the resulting DDNO optimal.

The proposed θ-parameterized DDNO is given by

$$\begin{aligned} M\ddot{z}(t)+D_o(\theta)\dot{z}(t)+K_o(\theta)z(t)&=T(\theta)Bu(t)\\ &\quad+L(\theta)y(t)+N(\theta)\dot{y}(t)\\ \widehat{x}(t)&=z(t)+H(\theta)y(t). \end{aligned} \tag{3}$$

The estimate of the position state $x(t)$ is $\widehat{x}(t)$ and the state variables $z,\dot{z}$ are the position and velocity states of the observer. Due to its natural setting, the above disturbance decoupling observer ensures that the velocity estimate is simply given by the time derivative of $\widehat{x}(t)$.

For each $\theta\in\Theta$, the parameter-dependent matrices $D_o(\theta),K_o(\theta),T(\theta),L(\theta),N(\theta),H(\theta)$ satisfy the following conditions

$$\begin{aligned} &M\left(I-H(\theta)C(\theta)\right)M^{-1}E=0\\ &T(\theta)=M\left(I-H(\theta)C(\theta)\right)M^{-1}\\ &D_o(\theta)=T(\theta)D+N_1(\theta)C(\theta),\\ &K_o(\theta)=T(\theta)K+L_1(\theta)C(\theta),\\ &N_1(\theta)=-D_o(\theta)H(\theta),\\ &L_1(\theta)=-K_o(\theta)H(\theta),\\ &L(\theta)=L_1(\theta)+L_2(\theta),\\ &N(\theta)=N_1(\theta)+N_2(\theta), \end{aligned} \tag{4}$$

and which ensure that the DDNO is feasible for all $\theta\in\Theta$. Central to the stability and convergence of the above parameter-dependent DDNO is the estimation error equation $e=x-\widehat{x}$, expressed in a natural form

$$\begin{aligned} &M\ddot{e}(t)+D_o(\theta)\dot{e}(t)+K_o(\theta)e(t)=0,\\ &\varepsilon(t)=C(\theta)e(t). \end{aligned} \tag{5}$$

If for each $\theta\in\Theta$ the conditions (4) are satisfied, then the error equation (5) is a stable system with both the position and velocity errors converging to zero asymptotically, despite the presence of the unknown disturbance input in (1). The conditions to ensure (4), are given in [11] and summarized in [14]. They are stated here for the case of θ-dependent matrices

- The first condition $T(\theta)E=0$ for each $\theta\in\Theta$ with the transformation matrix $T(\theta)$ given by $T(\theta)=I-MH(\theta)C(\theta)M^{-1}$, has a solution if and only if $\mathrm{rank}(C(\theta)M^{-1}E)=\mathrm{rank}(M^{-1}E)$ for each $\theta\in\Theta$.
- The solution to $T(\theta)E=0$ is given via the pseudoinverse of the matrix $C(\theta)M^{-1}E$ via

$$\begin{aligned} &E=MH(\theta)C(\theta)M^{-1}E\Rightarrow M^{-1}E=H(\theta)C(\theta)M^{-1}E\Rightarrow\\ &H(\theta)=M^{-1}E\left(C(\theta)M^{-1}E\right)^{\dagger}=\\ &M^{-1}E\left[\left(C(\theta)M^{-1}E\right)^T\left(C(\theta)M^{-1}E\right)\right]^{-1}\left(C(\theta)M^{-1}E\right)^T \end{aligned}$$

  This is feasible since $C(\theta)M^{-1}E$ is a full column rank matrix.

The other conditions that are imposed are $(C(\theta), T(\theta)K)$ and $(C(\theta), T(\theta)D)$ be detectable uniformly for each $\theta \in \Theta$.

To find the optimal $\theta \in \Theta$, one considers the error equation (5). When viewed in a first order form, it becomes

$$\begin{bmatrix} K_o(\theta) & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} 0 & K_o(\theta) \\ K_o(\theta) & D_o(\theta) \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The appropriate metric to optimize the DDNO is to consider the energy of the above error system and minimize this energy with respect to $\theta \in \Theta$. This takes the form of a θ-dependent Lyapunov equation.

The above system can be compactly written as

$$\dot{E}(t) = \mathbb{A}(\theta)E(t), \tag{6}$$

where

$$E = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \ \mathbb{A}(\theta) = -\begin{bmatrix} K_o(\theta) & 0 \\ 0 & M \end{bmatrix}^{-1} \begin{bmatrix} 0 & K_o(\theta) \\ K_o(\theta) & D_o(\theta) \end{bmatrix}.$$

For this θ-parameterized dynamical system, the natural optimization measure is the cost function which is based on the mechanical energy

$$J(\theta) = \int_0^\infty E^T(\tau)\mathbb{Q}(\theta)E(\tau)\,\mathrm{d}\tau, \tag{7}$$

where the weight matrix $\mathbb{Q}(\theta)$ is

$$\mathbb{Q}(\theta) = \begin{bmatrix} K_o(\theta) & \mathbf{I} \\ \mathbf{0} & M \end{bmatrix}, \quad \theta \in \Theta.$$

The weighted inner product in the optimization measure represents the kinetic ($e^T K_o e$) and potential ($\dot{e}^T M \dot{e}$) energies of the closed-loop error system (5) and constitutes the natural selection for the optimization measure. The solution to this minimization problem is

$$\min_{\theta \in \Theta} J(\theta) = \min_{\theta \in \Theta} E^T(0)\Pi(\theta)E(0), \tag{8}$$

where $\Pi(\theta)$ is the solution to the θ-parameterized Lyapunov equation

$$\mathbb{A}^T(\theta)\Pi(\theta) + \Pi(\theta)\mathbb{A}(\theta) + \mathbb{Q}(\theta) = \mathbf{0}, \ \ \forall \theta \in \Theta. \tag{9}$$

A further simplification removes the dependence on the initial data $E(0)$ by assuming that $E(0)$ is a Gaussian random vector in $\mathbb{R}^{2n}$ with a zero mean and unit covariance. In such a case, the above optimization simplifies and hence the optimal value of the parameter is

$$\theta^{opt} = \arg \min_{\theta \in \Theta} \mathrm{tr}\ (\Pi(\theta)). \tag{10}$$

The construction of the optimal disturbance decoupling observers is summarized in Algorithm 1.

Algorithm 1 Optimal natural DDO-finite dimensional case

initialize: Determine the set of admissible sensor locations Θ ensure the solvability of the natural disturbance decoupled observer in (4). Implicitly embedded in the solvability conditions are the observability conditions for the pairs $(C(\theta), T(\theta)L)$ and $(C(\theta), T(\theta)D)$.
iterate: For each $\theta \in \Theta$, set up the $2n \times 2n$ matrix $A(\theta)$
solve: For each $\theta \in \Theta$, solve the Lyapunov equation (9)
select: Select the optimal sensor location using

$$\theta^{opt} = \arg \min_{\theta \in \Theta} \mathrm{tr}\ (\Pi(\theta)).$$

implement: Implement natural DDO (3) with $\theta = \theta^{opt}$

III. Problem formulation for second order infinite dimensional systems

The analysis of second order systems involves a five-space setting in order to allow the damping operators defined over different spaces than the stiffness operators. The second order infinite dimensional systems are described by the evolution equation

$$\begin{aligned} &\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t) + Ew(t), \quad \text{in } V_1^*, \\ &x(0) = x_0 \in V_1, \quad \dot{x}(0) = v_0 \in H, \\ &y(t) = Cx(t), \end{aligned} \tag{11}$$

where the five-space setting $V_1 \hookrightarrow V_2 \hookrightarrow H \hookrightarrow V_2^* \hookrightarrow V_1^*$ allows one to appropriately define the damping and stiffness operators. The Hilbert space H serves as the pivot space and the embeddings are compact with the following norm bounds: $|\phi|_H \le c_1\|\phi\|_{V_1}$, $|\phi|_H \le c_2\|\phi\|_{V_2}$ and $|\phi|_{V_2} \le c_2\|\phi\|_{V_1}$. Using the above, we have that the damping operator $D \in \mathcal{L}(V_2, V_2^*)$ is symmetric and bounded with $|\langle D\phi, \psi\rangle_{V_2^*,V_2}| \le \delta_u\|\phi\|_{V_2}\|\psi\|_{V_2}$, and V_2-H coercive with $\langle D\phi, \phi\rangle_{V_2^*,V_2} + \lambda_\ell|\phi|_H^2 \ge \delta_\ell\|\phi\|_{V_2}^2$, for $\phi, \psi \in V_2$. Similar conditions are satisfied by the stiffness operators in the larger space V_1 with a stronger coercivity condition: symmetric, bounded with $|\langle K\phi, \psi\rangle_{V_1^*,V_1}| \le \kappa_u\|\phi\|_{V_1}\|\psi\|_{V_1}$, and V_1-coercive with $\langle K\phi, \phi\rangle_{V_1^*,V_1} \ge \kappa_\ell\|\phi\|_{V_1}^2$, for $\phi, \psi \in V_1$. The output operator $C \in \mathcal{L}(V_1, Y)$ where Y is the finite dimensional output space, the control input operator $B \in \mathcal{L}(U, V_1^*)$ where U is the finite dimensional control space and the disturbance input operator $E \in \mathcal{L}(W, V_1^*)$ where W is the finite dimensional disturbance space. Finally, with $Bu + Ew \in L_2(0,t;V_2^*)$ one can claim well-posedness meaning that $x \in L_2(0,t;V_1)$, $\dot{x} \in L_2(0,t;V_2)$ and $\ddot{x} \in L_2(0,t;H)$, see [15], [16].

Unlike the finite dimensional counterpart (3), the infinite dimensional case of a disturbance-decoupling observer requires the following assumption.

Assumption 1 (output differentiability): The output operator is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = C\dot{x}(t) \tag{12}$$

Additionally, it is assumed that the counterpart of the DDNO in (3) can be realized for more than a single sensor location. Thus, it is assumed that the set of candidate sensor locations Θ consists of all points that render the system (11) exponentially detectable. For each sensor location $\theta \in \Theta$, the corresponding DDNO for (11) is given by

$$\begin{aligned}
&\ddot{z}(t)+D_o(\theta)\dot{z}(t)+K_o(\theta)z(t)=T(\theta)Bu(t)\\
&\qquad\qquad +L(\theta)y(t)+N(\theta)\dot{y}(t), \quad \text{in } V_1^*,\\
&\widehat{x}(t)=z(t)+H(\theta)y(t),\\
&V_1^*\ni z(0)\neq x_0-HCx_0,\\
&H\ni \dot{z}(0)\neq v_0-HCv_0.
\end{aligned} \tag{13}$$

The constraints on the initial conditions are imposed to avoid the trivial case. The state $\widehat{x}(t)$ is the estimated state position and by its construction, the DDNO (13) ensures that the estimated velocity is exactly equal to $\dot{\widehat{x}}(t)$. The state $z(t)$ is the state of the DDNO. The location parameterized operators D_o, K_o, T, L, N, H must satisfy the following conditions for each $\theta\in\Theta$ in order to achieve the sought after disturbance decoupling. The counterparts of (4) are given by

$$\begin{aligned}
&(I-H(\theta)C(\theta))E=0 \text{ in } V_1^*, && H(\cdot)\in\mathcal{L}(Y,V_2^*)\\
&T(\theta)=(I-H(\theta)C(\theta)), && T(\cdot)\in\mathcal{L}(V_1^*,V_2^*)\\
&D_o(\theta)=T(\theta)D+N_1(\theta)C(\theta), && N_1(\cdot)\in\mathcal{L}(Y,V_2^*)\\
&K_o(\theta)=T(\theta)K+L_1(\theta)C(\theta), && L_1(\cdot)\in\mathcal{L}(Y,V_1^*)\\
&N_2(\theta)=-(T(\theta)D+N_1(\theta)C(\theta))H(\theta) && \\
&\qquad=-D_o(\theta)H(\theta), && N_2(\cdot)\in\mathcal{L}(Y,V_2^*)\\
&L_2(\theta)=-(T(\theta)K+L_1(\theta)C)H(\theta) && \\
&\qquad=-K_o(\theta)H(\theta), && L_2(\cdot)\in\mathcal{L}(Y,V_1^*),\\
&N(\theta)=N_1(\theta)+N_2(\theta), && N(\cdot)\in\mathcal{L}(Y,V_2^*)\\
&L(\theta)=L_1(\theta)+L_2(\theta), && L(\cdot)\in\mathcal{L}(Y,V_1^*).
\end{aligned} \tag{14}$$

Following the case of a fixed sensor in [14], for each $\theta\in\Theta$, the pairs $(C(\theta),T(\theta)K)$ and $(C(\theta),T(\theta)D)$ are exponentially detectable, uniformly for $\theta\in\Theta$. This ensures that for each $\theta\in\Theta$ the operators $D_o(\theta)$ and $K_o(\theta)$ are V_2 and V_1 coercive, respectively. Defining the position estimation error $e=x-\widehat{x}$, the combination of (11), (13), (14) yields

$$\ddot{e}(t)+D_o(\theta)\dot{e}(t)+K_o(\theta)e(t)=0, \text{ in } V_1, \ \theta\in\Theta. \tag{15}$$

The well-posedness of (13) and convergence of the error (15) easily follow from [14]. They are stated here for each $\theta\in\Theta$.

Lemma 1: The proposed DDNO (13) is well-posed if the θ-parameterized pairs $(C(\theta),T(\theta)K)$ and $(C(\theta),T(\theta)D)$ are exponentially detectable for each $\theta\in\Theta$ and all the conditions in (14) are satisfied.

Lemma 2: The estimation error (15) is a stable volution system for each $\theta\in\Theta$ and

$$\lim_{t\to\infty}\|e(t)\|_{V_1}=0, \quad \lim_{t\to\infty}|\dot{e}(t)|=0,$$

thus rendering the θ-parameterized DDNO in (13) a natural disturbance decoupling observer for the perturbed second order system (11).

Following the finite dimensional case for the optimal sensor location, the appropriate performance metric for the location-parameterized state error (15) is the infinite horizon energy given by

$$J(\theta)=\int_0^\infty \|e(\tau)\|_{V_1}^2+|\dot{e}(\tau)|_H^2\,\mathrm{d}\tau. \tag{16}$$

To arrive at a first order formulation, one first considers the appropriate state space, given by $\mathbb{X}=V_1\times H$ and equipped by the inner product and norm

$$\langle\Phi,\Psi\rangle_{\mathbb{X}}=\langle\Phi_1,\Psi_1\rangle_{V_1}+\langle\Phi_2,\Psi_2\rangle_H$$

$$|\Phi|_{\mathbb{X}}^2=\|\Phi_1\|_{V_1}^2+|\Phi_2|_H^2,$$

for $\Phi=(\Phi_1,\Phi_2)$, $\Psi=(\Psi_1,\Psi_2)$. Next, consider $\mathbb{Y}=V_1\times V_2$ endowed with the norm

$$|\Phi|_{\mathbb{Y}}^2=\|\Phi_1\|_{V_1}^2+\|\Phi_2\|_{V_2}^2.$$

Then we have that $\mathbb{Y}\hookrightarrow\mathbb{X}\hookrightarrow\mathbb{Y}^*$ and thus can define the *theta*-parameterized state operator associated with (15) via

$$\mathcal{A}(\theta)\Phi=\begin{bmatrix}0 & I\\ -K_o(\theta) & -D_o(\theta)\end{bmatrix}\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}, \ \Phi=(\Phi_1,\Phi_2)\in\mathbb{Y}.$$

The error equation (15) written as a first order dynamical system is

$$\dot{E}(t)=\mathcal{A}(\theta)E(t).$$

The value of the cost (16) is given by

$$J^{opt}(\theta)=\langle\mathbf{\Pi}(\theta)E(0),E(0)\rangle_{\mathbb{X}}, \quad \theta\in\Theta,$$

where for each $\theta\in\Theta$, $\mathbf{\Pi}(\theta)$ is the solution to the operator Lyapunov equation

$$\mathcal{A}^*(\theta)\mathbf{\Pi}(\theta)+\mathbf{\Pi}(\theta)\mathcal{A}(\theta)+\mathcal{Q}=0 \tag{17}$$

where $\mathcal{Q}$ is an appropriately selected coercive operator and which then prompts to optimal sensor location via

$$\theta^{opt}=\arg\min_{\theta\in Theta}\operatorname{trace}\mathbf{\Pi}(\theta). \tag{18}$$

The above is enabled via the assumption that the initial data $E(0)=(e(0),\dot{e}(0))$ is a Gaussian random vector in $\mathbb{X}$ with zero mean and unit covariance. The existence of a optimizer (minimiser) to (18) depends on the Lipschitz continuity of operator $\mathbf{\Pi}(\theta)$ with respect to the sensor location $\theta\in\Theta$. Such a condition can be satisfied if the θ-parameterized state operator $\mathcal{A}(\theta)$ generates an exponentially stable C_0-semigroup on $\mathbb{X}$ and $\mathcal{A}(\theta)$ has compact resolvent. These can be guaranteed via the conditions of the exponential detectability and the relation between the operators $K_0(\theta)$ and $D_0(\theta)$ (see [17, Theorem 3B.1 in Appendix 3B]).

IV. Numerical Examples

To demonstrate the sensor optimization for the DDNO, we consider the wave PDE equation in one spatial dimension given by

$$\begin{aligned}
&\frac{\partial^2 x}{\partial t^2}(t,\xi)-a_2\frac{\partial^2\partial x}{\partial\xi^2\partial t}(t,\xi)-a_1\frac{\partial^2 x}{\partial\xi^2}(t,\xi)\\
&\qquad=b(\xi)u(t)+e(\xi)w(t),
\end{aligned} \tag{19}$$

in the spatial domain $\Omega=[0,\ell]=[0,1]$, furnished with Dirichlet boundary conditions $x(t,0)=0=x(t,1)$ and initial condition $x(0,\xi)=5\sin(2\pi\xi)$, $x_t(0,\xi)=0$. This is the same example considered in the earlier work [14]. The constant parameters were selected as $a_1 = 0.2$, $a_2 = 0.01$. One immediately identifies $V_1(0,\ell) = V_2(0,\ell) = H_0^1(0,\ell)$ and $H = L_2(0,\ell)$.

The spatial distribution of the actuator and of the disturbance inputs were selected as

$$b(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.765, 0.815] \\ 0 & \text{otherwise} \end{cases},$$

$$e(\xi) = \begin{cases} 1 & \text{if } \xi \in [0.50, 0.70] \\ 0 & \text{otherwise} \end{cases},$$

The stiffness and damping operators for (17) are given by

$$\langle K\phi, \psi \rangle_{V_1^*, V_1} = -a_1 \int_0^\ell \phi''(\xi)\psi''(\xi)\,\mathrm{d}\xi,$$

$$\langle D\phi, \psi \rangle_{V_2^*, V_2} = -a_2 \int_0^\ell \phi''(\xi)\psi''(\xi)\,\mathrm{d}\xi,$$

with the control and disturbance input operators given by

$$\langle Bu(t), \phi \rangle = \int_{0.765}^{0.815} \phi(\xi)\,\mathrm{d}\xi\, u(t),$$

$$\langle Ew(t), \phi \rangle = \int_{0.50}^{0.70} \phi(\xi)\,\mathrm{d}\xi\, w(t).$$

Two sensor measurements were considered, with one fixed at an a priori location in the spatial domain Ω and the other one parameterized by the location $\theta \in \Omega$. Thus one has

$$y(t) = \begin{bmatrix} \int_0^1 c_1(\xi)x(t,\xi)\,d\xi \\ \int_0^1 c_2(\xi;\theta)x(t,\xi)\,d\xi \end{bmatrix} = \begin{bmatrix} C_1 x(t) \\ C_2(\theta)x(t) \end{bmatrix},$$

with the spatial distribution of the sensors given by the boxcar function

$$C_1\phi = \langle c_1, \phi \rangle = \int_0^1 c_1(\xi)\phi(\xi)\,\mathrm{d}\xi = \int_{0.573}^{0.593} \phi(\xi)\,\mathrm{d}\xi,$$

$$C_2(\theta)\phi = \langle c_2(\theta), \phi \rangle = \int_{\theta-\delta\xi}^{\theta+\delta\xi} \phi(\xi)\,\mathrm{d}\xi,$$

where the half-length of the sensor support is $\delta\xi = \ell/100$. The set of candidate sensor locations consist of all locations in the interval $[\delta\xi, \ell - \delta\xi]$ that render the resulting second order infinite dimensional system exponentially detectable.

To simulate the system with its DDNO and perform the sensor optimization, a Galerkin-based finite dimensional approximation using linear elements was used. The sensor optimization computed the finite dimensional approximation of the operator Lyapunov equation and provided the θ-parameterized cost. As presented in Figure 1, the normalized cost J is plotted for all candidate sensor locations. As observed, it predicts the optimal location of the second sensor to be at $\theta = 0.583$, which coincides with the fixed sensor. Thus, the optimal sensor location was selected as the one that provided the second smallest value of the normalized cost J and which placed the second sensor at the optimal location $\theta^{opt} = 0.6986$.

Using as the disturbance input w the function $w(t) = 320 + 40\sin(2\pi\omega_1 t)$, where $\omega_1 = \sqrt{\lambda_{min}(K)}$ denotes the minimum of the square root of the eigenvalues of the stiffness matrix, designed as a means to represent the worst possible disturbance, the proposed DDNO was simulated in the interval $[0, 10]$s. An H^∞ filter, designed for the first order formulation of the infinite dimensional system and which used the same sensors was simulated in order to provide a direct comparison of the filters to provide state estimates despite the presence of a spatially distributed disturbance.

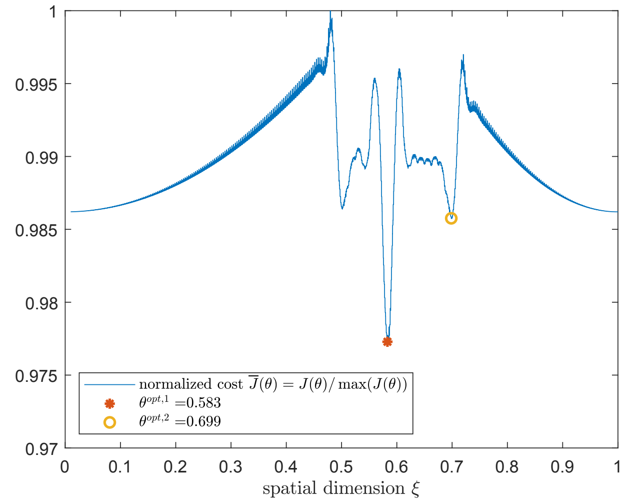

Fig. 1. Optimal cost J against the sensor locations.

The spatial distribution of the state position error $x(t,\xi) - \widehat{x}(t,\xi)$ at the final time $t = 10$s for both the proposed optimal DDNO and the optimal H^∞ filters is depicted in Figure 2. Similarly, the spatial distribution of the state velocity error $x_t(t,\xi) - \widehat{x}_t(t,\xi)$ at the final time $t = 10$s is depicted in Figure 3. Both figures provide similar results on the successful performance of the proposed DDNO to completely decouple the effects of the disturbance signals in reconstructing the process state. Such a performance cannot be duplicated by the H^∞ filter which cannot minimize the effects of the disturbance $w(t)$ on the successful reconstruction of the position and velocity states of (19).

Finally, the time evolution of the state error norm given by

$$\sqrt{\|e(t)\|_{V_1}^2 + |\dot{e}(t)|_H^2} = \sqrt{\int_0^\ell \left(a_1 \frac{\partial^2 e}{\partial \xi^2}(t,\xi)\right)^2 d\xi + \int_0^\ell \left(\frac{\partial e}{\partial t}(t,\xi)\right)^2 d\xi}$$

for both filters is presented in Figure 4. The performance of the proposed DDNO is far better than that of the robust filter provided by the H^∞ design.

V. Conclusions

The design of a disturbance decoupling, or unknown input, observer for second order systems was combined with sensor optimization for a class of second order finite and infinite dimensional systems. The candidate sensor locations were selected from the available locations that provided a form of observability and ensure that all the conditions for the

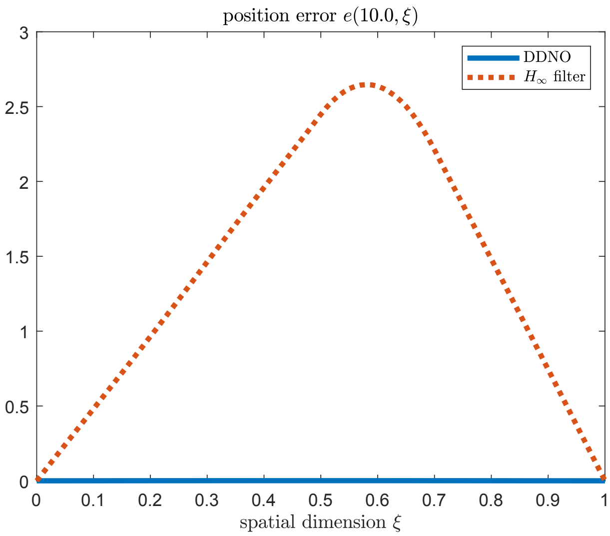

Fig. 2. Spatial evolution of the position error.

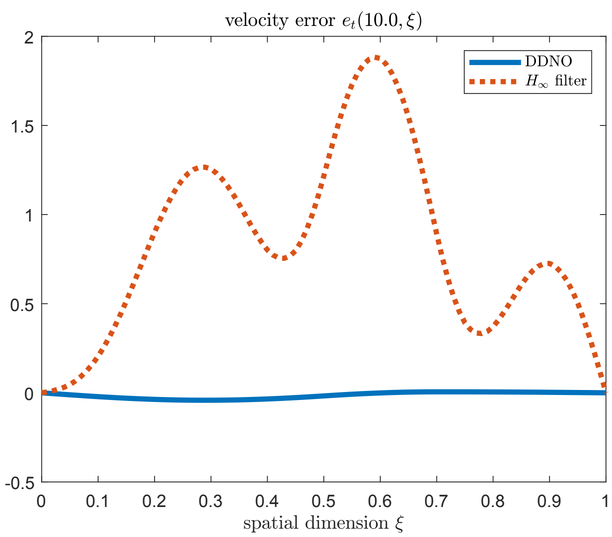

Fig. 3. Spatial evolution of the velocity error.

unknown input observer were feasible. Then the optimization provided the optimization measure to be the energy of the resulting estimation error. The proposed optimization scheme was demonstrated with a cable PDE in one spatial dimension.

A natural extension for the DDNO is to combine the observer design with the controller design and thus arrive at a closed-loop performance measure to optimize both the sensors and the actuators of the second order dynamical systems. Another direction is to consider singularly perturbed vector second order systems as presented in [18], or semilinear vector second order systems and include the disturbance-decoupling feature along with the sensor optimization. These aspects are currently being examined by the authors and will appear in a forthcoming publication.

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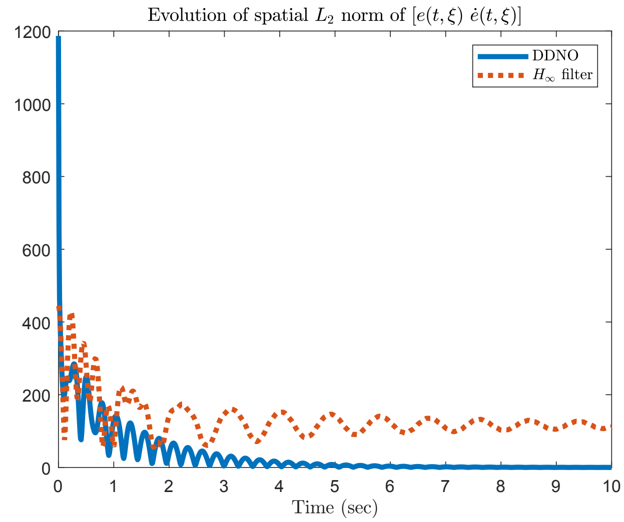

Fig. 4. Time evolution of error norm.

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