A DISCRETE ELASTICITY COMPLEX ON THREE-DIMENSIONAL ALFELD SPLITS

SNORRE H. CHRISTIANSEN, JAY GOPALAKRISHNAN, JOHNNY GUZMÁN, AND KAIBO HU

Abstract. We construct conforming finite element elasticity complexes on the Alfeld splits of tetrahedra. The complex consists of vector fields and symmetric tensor fields, interlinked via the linearized deformation operator, the linearized curvature operator, and the divergence operator, respectively. The construction is based on an algebraic machinery that derives the elasticity complex from de Rham complexes, and smoother finite element differential forms.

1. Introduction

Differential complexes have become a powerful tool in the construction and analysis of numerical methods in the framework of finite element exterior calculus [6, 8]. The example of the de Rham complex together with its various finite element applications, especially in computational electromagnetism, is now very well known. The elasticity complex is another example with important applications in continuum mechanics and geometry.

In three space dimensions (3D), the elasticity complex reads as follows.

$$0 \longrightarrow \mathcal{R} \xrightarrow{\subset} C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0, \tag{1.1}$$

where $\mathbb{V} = \mathbb{R}^3$, $\mathcal{R} = \{a + b \times x : a, b \in \mathbb{R}^3\}$, $\mathbb{S}$ denotes the set of symmetric 3×3 matrices, def denotes the deformation operator equaling $\operatorname{sym}\operatorname{grad}$ which we shall also often write simply as ε (also known as the linearized strain), $\operatorname{inc} = \operatorname{curl} \circ \mathrm{T} \circ \operatorname{curl}$ gives the incompatibility operator, which in 3D is equivalent to the linearized Einstein tensor or the linearized Riemannian curvature. Here curl and div denote the curl and divergence operators applied row by row on a matrix field, respectively. The notation T in the definition of inc denotes the operation that maps a matrix to its transpose, which we also often denote simply by $'$ (prime). In mechanics, (1.1) bears the name of the *Kröner complex* [23, 31], due to Kröner's pioneering work on modeling defects of the continuum by the violation of Saint-Venant's compatibility condition, $\operatorname{inc} \circ \operatorname{def} = 0$. In the context of elasticity, the spaces after $\mathcal{R}$ in (1.1) correspond to the displacement, strain, stress (incompatibility), and the load, respectively. In geometry, the sequence (1.1) is referred to as the linearized *Calabi complex* [3, 12] and the spaces correspond to the embedding, the metric, and the curvature, respectively.

The following Sobolev space version of the complex (1.1)

$$0 \longrightarrow \mathcal{R} \xrightarrow{\subset} H^2 \otimes \mathbb{V} \xrightarrow{\varepsilon} H^1(\text{inc}, \mathbb{S}) \xrightarrow{\text{inc}} H(\text{div}, \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \longrightarrow 0, \tag{1.2}$$

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where $H^1(\operatorname{inc},\mathbb{S}) := \{u \in H^1 \otimes \mathbb{S} : \operatorname{inc} u \in L^2 \otimes \mathbb{S}\}$ and $H(\operatorname{div},\mathbb{S}) := \{u \in L^2 \otimes \mathbb{S} : \operatorname{div} u \in L^2 \otimes \mathbb{V}\}$, is relevant to variational principles in elasticity such as the Hellinger-Reissner principle. A proof of exactness of (1.2) is described in more detail in [24, p. 38–40].

Therefore a discrete version of (1.1) should be useful to understand the behavior of structure-preserving numerical methods. In this paper, we shall construct a discrete finite element subcomplex of (1.2). To the best of our knowledge, this is the first known finite element subcomplex of the elasticity complex, complete with conforming subspaces of all the spaces in the sequence and accompanying cochain projectors.

The Hellinger-Reissner principle involves the last two spaces, i.e., $H(\operatorname{div},\mathbb{S})$ and $L^2 \otimes \mathbb{V}$, in (1.2). The symmetry of the tensors makes it a challenging problem to construct conforming finite element discretization for these spaces. In two space dimensions (2D), Johnson and Mercier [30] constructed a stable finite element elasticity pair on the Clough-Tocher split. Later, Arnold and Winther [11] constructed the first finite element elasticity pair on triangular meshes with polynomial shape functions. This work was extended to 3D in [4] and further refined to reduce the number of degrees of freedom (dofs) in [29].

Despite the above-mentioned significant progress in the construction of finite elements for the last (stress-displacement) part of the elasticity complex (1.2), the question of how to construct an entire finite element subcomplex of (1.2) seems to have been left largely unanswered yet. The question is entirely natural from the viewpoint of completing the mathematical structure. Besides satisfying a mathematical curiosity, there are many other utilitarian and numerical reasons to tackle the question of constructing a discrete subcomplex. For example, a discrete complex contains an explicit characterization of the kernel of differential operators. This is crucial in designing robust solvers and preconditioners [21] in the framework of kernel-capturing subspace correction methods [32, 37]. Another reason is that the elasticity complex (1.1) is not only important for elasticity, but also for various applications where other parts of the complex are involved, e.g., the intrinsic elasticity [17] (involving compatible strain tensors), continuum modeling of defects [2] (involving the inc operator), and relativity [14, 33] (involving the metric and curvature). One needs no stretch of imagination to see progress in these areas being enabled by a discrete version of (1.1). Having said that, let us also note that this paper does not give any numerical method; the paper's sole focus is to reveal a mathematical structure analogous to (1.1) inherent in certain discrete spaces.

Specifically, we construct conforming finite element spaces that form a subcomplex of (1.2), with accompanying cochain projectors (defined on smoother subspaces), also referred to as "commuting projections." We use the Bernstein-Gelfand-Gelfand (BGG) construction [8, 10] as a tool to guide our construction. The BGG construction is an algebraic machinery that originated in Lie theory of geometry [13]. Later, it was introduced into numerical analysis as a way to derive differential complexes, such as the elasticity complex, from de Rham sequences [7, 10, 19]. The idea of the BGG construction (see (2.4) below) is to derive the elasticity complex from two copies of vector-valued de Rham complexes. To match the two complexes diagonally, the spaces of the same form degree in the two complexes should have different regularity. This was already noted by Arnold, Falk and Winther in their use of the BGG construction applied to the Hellinger-Reissner principle to derive finite element methods with weakly imposed symmetry [5]. To match the two de Rham sequences, they chose finite element spaces that satisfy certain algebraic conditions. When the degrees of certain spaces in the two sequences match exactly, one can see that the scheme with weakly imposed symmetry actually leads to strong symmetry. This was first observed in [25] where a provably stable set of spaces for a method imposing weak symmetry was shown to yield exactly symmetric stress approximations, by establishing connections between Stokes and elasticity systems, which can now be understood from the BGG viewpoint.

The BGG machinery was used to reinterpret the 2D Arnold-Winther element in [7]. Another elasticity pair by Hu and Zhang [28] was also explained in this way in [15], where the two de Rham sequences start with the Argyris and the Hermite elements, respectively. In 2D, there is another elasticity strain complex connecting the displacement, strain (metric) and incompatibility (curvature). Using the BGG diagrams, Christiansen and Hu [16] constructed conforming discrete strain complexes with applications in discrete geometry. These works helped put the pieces of the puzzle into place in 2D. Yet, several challenges remained to get to the 3D elasticity complex, due to the complexity of the differential structures in (1.2) and the difficulties in constructing smooth 3D discrete de Rham sequences. Thanks to recent progress on smooth finite element de Rham complexes [22, 26], a way out of the impasse finally emerged, at least on meshes of Alfeld splits of tetrahedra [1]. In this paper, we are thus finally able to construct a discrete elasticity complex on meshes of Alfeld splits.

Some parallel tracks of investigation by other groups of authors are related and interesting. Approaches to a discrete elasticity complex from a discrete geometric perspective can be found in [14, 27, 33]. The Regge calculus was originally proposed by Regge [36] and has several applications in quantum and numerical gravity. Due to its very weak continuity, establishing convergence might need further innovations. Christiansen [14] put the Regge calculus into a finite element context and fitted it into a discrete elasticity complex and Regge interpolation was used for shells recently in [35]. From this perspective, one of the results in this paper can be seen as providing a smoother analogue of the Regge elements, with $H^1(\text{inc})$-conformity (and additionally C^0 continuity). Our smoother spaces, while mathematically pleasing, do come at the price of increased number of dofs, so we emphasize again that this paper's goal is not to construct competitive numerical methods, but rather to reveal previously unknown mathematical structures.

The rest of the paper is organized as follows. In Section 2, we quickly present the essentials for the remainder of the paper, including results on spaces on Alfeld splits and the BGG resolution. In Sections 3 and 4, we present the two finite element de Rham complexes that will be used in the BGG construction. Section 5, the centerpiece of this paper, presents finite elements on Alfeld splits for each member of the discrete elasticity complex. Section 6 remarks on how the corresponding global finite element spaces may be constructed. A standalone appendix (Appendix A) gives an elementary argument for establishing supersmoothness results on three-dimensional Alfeld splits.

2. Preliminaries

To build an elasticity complex we shall employ two de Rham complexes of discrete spaces with extra smoothness (in comparison with the standard finite element spaces). We shall construct these spaces in the next two sections using the results of [22], which we recall in this section.

We work on Alfeld simplicial complexes and start by establishing notation associated to an Alfeld split. Starting with a tetrahedron $T = [x_0, \cdots, x_3]$, let T^{A} be an Alfeld triangulation of T, i.e., we choose an interior point z of T and we let $T_0 = [z, x_1, x_2, x_3]$, $T_1 = [z, x_0, x_2, x_3]$, $T_2 = [z, x_0, x_1, x_3]$, $T_3 = [z, x_0, x_1, x_2]$ and set $T^{\mathrm{A}} = \{T_0, T_1, T_2, T_3\}$. Let $\Delta_i(T)$ be the set of all i-dimensional subsimplexes of T.

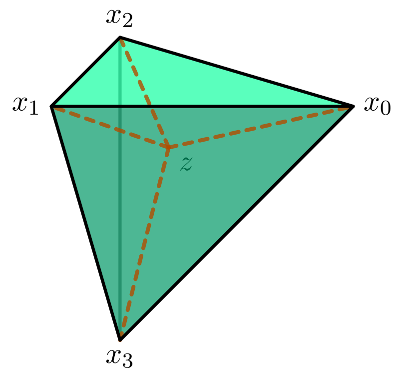

FIGURE 1. Alfeld split

The following spaces are well-known finite element spaces:

$$W_r^0(T^{\mathrm{A}}) = \{\omega \in H^1(T) : \omega|_K \in \mathcal{P}_r(K) \text{ for all } K \in T^{\mathrm{A}}\},$$
$$W_r^1(T^{\mathrm{A}}) = \{\omega \in H(\mathrm{curl}, T) : \omega|_K \in [\mathcal{P}_r(K)]^3 \text{ for all } K \in T^{\mathrm{A}}\},$$
$$W_r^2(T^{\mathrm{A}}) = \{\omega \in H(\mathrm{div}, T) : \omega|_K \in [\mathcal{P}_r(K)]^3 \text{ for all } K \in T^{\mathrm{A}}\},$$
$$W_r^3(T^{\mathrm{A}}) = \{\omega \in L^2(T) : \omega|_K \in \mathcal{P}_r(K) \text{ for all } K \in T^{\mathrm{A}}\}.$$

Their analogues with boundary conditions are

$$\mathring{W}_r^0(T^{\mathrm{A}}) = \{\omega \in W_r^0(T^{\mathrm{A}}) : \omega = 0 \text{ on } \partial T\},$$
$$\mathring{W}_r^1(T^{\mathrm{A}}) = \{\omega \in W_r^1(T^{\mathrm{A}}) : \omega \times n = 0 \text{ on } \partial T\},$$
$$\mathring{W}_r^2(T^{\mathrm{A}}) = \{\omega \in W_r^2(T^{\mathrm{A}}) : \omega \cdot n = 0 \text{ on } \partial T\},$$
$$\mathring{W}_r^3(T^{\mathrm{A}}) = \{\omega \in W_r^3(T^{\mathrm{A}}) : \int_T \omega = 0\}.$$

We define the following Lagrange spaces

$$L_r^0(T^{\mathrm{A}}) = W_r^0(T^{\mathrm{A}}),\ L_r^1(T^{\mathrm{A}}) = [W_r^0(T^{\mathrm{A}})]^3,\ L_r^2(T^{\mathrm{A}}) = L_r^1(T^{\mathrm{A}}),\ L_r^3(T^{\mathrm{A}}) = L_r^0(T^{\mathrm{A}}),$$

and their analogues with boundary conditions:

$$\mathring{L}_r^0(T^{\mathrm{A}}) = \mathring{W}_r^0(T^{\mathrm{A}}),\ \mathring{L}_r^1(T^{\mathrm{A}}) = [\mathring{W}_r^0(T^{\mathrm{A}})]^3,\ \mathring{L}_r^2(T^{\mathrm{A}}) = \mathring{L}_r^1(T^{\mathrm{A}}),\ \mathring{L}_r^3(T^{\mathrm{A}}) = L_r^3(T^{\mathrm{A}}) \cap \mathring{W}_r^3(T^{\mathrm{A}}).$$

Apart from the above standard spaces, we also need the following "smoother" spaces:

$$S_r^0(T^{\mathrm{A}}) = \{\omega \in L_r^0(T^{\mathrm{A}}) : \mathrm{grad}\,\omega \in L_{r-1}^1(T^{\mathrm{A}})\}, \quad \mathring{S}_r^0(T^{\mathrm{A}}) = \{\omega \in \mathring{L}_r^0(T^{\mathrm{A}}) : \mathrm{grad}\,\omega \in \mathring{L}_{r-1}^1(T^{\mathrm{A}})\},$$
$$S_r^1(T^{\mathrm{A}}) = \{\omega \in L_r^1(T^{\mathrm{A}}) : \mathrm{curl}\,\omega \in L_{r-1}^2(T^{\mathrm{A}})\}, \quad \mathring{S}_r^1(T^{\mathrm{A}}) = \{\omega \in \mathring{L}_r^1(T^{\mathrm{A}}) : \mathrm{curl}\,\omega \in \mathring{L}_{r-1}^2(T^{\mathrm{A}})\},$$
$$S_r^2(T^{\mathrm{A}}) = \{\omega \in L_r^2(T^{\mathrm{A}}) : \mathrm{div}\,\omega \in L_{r-1}^3(T^{\mathrm{A}})\}, \quad \mathring{S}_r^2(T^{\mathrm{A}}) = \{\omega \in \mathring{L}_r^2(T^{\mathrm{A}}) : \mathrm{div}\,\omega \in \mathring{L}_{r-1}^3(T^{\mathrm{A}})\},$$
$$S_r^3(T^{\mathrm{A}}) = L_r^3(T^{\mathrm{A}}), \quad \mathring{S}_r^3(T^{\mathrm{A}}) = \mathring{L}_r^3(T^{\mathrm{A}}).$$

When $r \le 3$, the space $S_r^0(T^{\mathrm{A}})$ coincides with $\mathcal{P}_r(T)$. More generally, the S-spaces have "extra" smoothness at the vertices as given in the next proposition.

Proposition 2.1.

(1) *Every function in* $S_r^0(T^A)$ *is* C^2 *at the vertices of* T.
(2) *Every function in* $\mathring{S}_r^0(T^A)$ *has vanishing second derivatives at the vertices of* T.
(3) *Every function in* $S_r^1(T^A)$ *is* C^1 *at the vertices of* T.
(4) *Every function in* $\mathring{S}_r^1(T^A)$ *has vanishing first derivatives at the vertices of* T.

The first two items follow from [1] and the remainder can be proved using a dimension counting argument found in [22]. Nonetheless, all the statements of the proposition follow from elementary arguments (without counting dimensions) detailed in Appendix A.

Consider the following sequences:

$$\mathbb{R} \longrightarrow W_r^0(T^{\mathrm{A}}) \xrightarrow{\text{grad}} W_{r-1}^1(T^{\mathrm{A}}) \xrightarrow{\text{curl}} W_{r-2}^2(T^{\mathrm{A}}) \xrightarrow{\text{div}} W_{r-3}^3(T^{\mathrm{A}}) \longrightarrow 0 \tag{2.1a}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{A}}) \xrightarrow{\text{grad}} L_{r-1}^1(T^{\mathrm{A}}) \xrightarrow{\text{curl}} W_{r-2}^2(T^{\mathrm{A}}) \xrightarrow{\text{div}} W_{r-3}^3(T^{\mathrm{A}}) \longrightarrow 0 \tag{2.1b}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{A}}) \xrightarrow{\text{grad}} S_{r-1}^1(T^{\mathrm{A}}) \xrightarrow{\text{curl}} L_{r-2}^2(T^{\mathrm{A}}) \xrightarrow{\text{div}} W_{r-3}^3(T^{\mathrm{A}}) \longrightarrow 0 \tag{2.1c}$$

$$\mathbb{R} \longrightarrow S_r^0(T^{\mathrm{A}}) \xrightarrow{\text{grad}} S_{r-1}^1(T^{\mathrm{A}}) \xrightarrow{\text{curl}} S_{r-2}^2(T^{\mathrm{A}}) \xrightarrow{\text{div}} S_{r-3}^3(T^{\mathrm{A}}) \longrightarrow 0. \tag{2.1d}$$

The first sequence is well known to be exact. The last three were shown to be exact in [22] for $r \geq 1$ and so were the corresponding sequences with boundary conditions (see for example (4.2) in [22] for the case with boundary conditions). Throughout, when a subscript indicating the degree is negative, the space is considered 0-dimensional. In fact, in the course of proving the exactness, the following representation of potentials was established in [22, proof of Theorem 3.1]. Here and throughout, we let $\mu \in C^0(T)$ denote the piecewise linear function on T^{A} such that $\mu(z) = 1$ and $\mu(x_i) = 0$ for $0 \leq i \leq 3$ (i.e., μ is a bubble function on T^{A}).

Proposition 2.2. *Let* $r \geq 0$. *For any* $w \in \mathring{W}_r^2(T^A)$ *with* $\operatorname{div} w = 0$, *there exists* $\gamma_j \in \mathcal{P}_j(T)^3$, $j = 0, 1, \cdots, r$, *such that*

$$u = \mu \sum_{\ell=0}^{r} \mu^\ell \gamma_{r-\ell} \tag{2.2}$$

satisfies $\operatorname{curl} u = w$. *Similarly, any* $w \in \mathring{W}_r^3(T^A)$ *also has (possibly different)* $\gamma_j \in \mathcal{P}_j(T)^3$, *which when combined to make the function* u *as in* (2.2), *satisfies* $\operatorname{div} u = w$.

We now collect the dimensions of the above-introduced spaces for any degree $r \geq 1$. A detailed discussion of first two counts below can be found, e.g., in [22]. The others are standard.

$$\dim \mathring{S}_r^0(T^{\mathrm{A}}) = \max\left(\frac{2}{3}(r-4)(r-3)(r-2),\ 0\right), \tag{2.3a}$$

$$\dim S_r^0(T^{\mathrm{A}}) = \binom{r+3}{3} + \frac{1}{2}(r-3)(r-2)(r-1) \tag{2.3b}$$

$$\dim \mathring{L}_r^0(T^{\mathrm{A}}) = 1 + 4(r-1) + 6\frac{(r-2)(r-1)}{2} + 4\frac{(r-3)(r-2)(r-1)}{6}, \tag{2.3c}$$

$$\dim L_r^0(T^{\mathrm{A}}) = 5 + 10(r-1) + 10\frac{(r-2)(r-1)}{2} + 4\frac{(r-3)(r-2)(r-1)}{6}, \tag{2.3d}$$

$$\dim \mathring{W}_r^1(T^{\mathrm{A}}) = 4(r+1) + 6(r-1)(r+1) + 4\frac{(r-2)(r-1)(r+1)}{2}, \tag{2.3e}$$

$$\text{(2.3f)} \qquad \dim \mathring{W}^2_r(T^{\mathrm{A}}) = 6\frac{(r+1)(r+2)}{2} + 4\frac{(r-1)(r+1)(r+2)}{2},$$

$$\text{(2.3g)} \qquad \dim \mathring{W}^3_r(T^{\mathrm{A}}) = 4\frac{(r+1)(r+2)(r+3)}{6} - 1,$$

$$\text{(2.3h)} \qquad \dim W^1_r(T^{\mathrm{A}}) = 10(r+1) + 10(r-1)(r+1) + 4\frac{(r-2)(r-1)(r+1)}{2},$$

$$\text{(2.3i)} \qquad \dim W^2_r(T^{\mathrm{A}}) = 10\frac{(r+1)(r+2)}{2} + 4\frac{(r-1)(r+1)(r+2)}{2},$$

$$\text{(2.3j)} \qquad \dim W^3_r(T^{\mathrm{A}}) = 4\frac{(r+1)(r+2)(r+3)}{6}.$$

We conclude this section by outlining the basic approach we shall adopt for constructing the elasticity complex on Alfeld splits. The approach is the same as what others [9] have pursued, known under the previously noted name, the BGG resolution. This theme is developed further in another recent work [10]. For our purposes here, it is sufficient to have the following simple result. Suppose $\mathcal{Z}_i, \mathcal{V}_i$ are Banach spaces, $r_i : \mathcal{Z}_i \to \mathcal{Z}_{i+1}$, $t_i : \mathcal{V}_i \to \mathcal{V}_{i+1}$, and $s_i : \mathcal{V}_i \to \mathcal{Z}_{i+1}$ are bounded linear operators such that the following diagram commutes:

$$\text{(2.4)} \qquad \begin{array}{ccccccc} \mathcal{Z}_0 & \xrightarrow{r_0} & \mathcal{Z}_1 & \xrightarrow{r_1} & \mathcal{Z}_2 & \xrightarrow{r_2} & \mathcal{Z}_3 \\ & \overset{s_0}{\nearrow} & & \overset{s_1}{\nearrow} & & \overset{s_2}{\nearrow} & \\ \mathcal{V}_0 & \xrightarrow{t_0} & \mathcal{V}_1 & \xrightarrow{t_1} & \mathcal{V}_2 & \xrightarrow{t_2} & \mathcal{V}_3 \end{array}$$

i.e., $r_{i+1}s_i = s_{i+1}t_i$ for $i = 0, 1$. We are interested in the situation where the top ($\mathcal{Z}$) sequence and the bottom ($\mathcal{V}$) sequence are complexes that form exact sequences. Then we have the following result that employs the Cartesian products $\mathcal{Z}_i \times \mathcal{V}_i$, which we write as $\left[\begin{smallmatrix}\mathcal{Z}_i\\ \mathcal{V}_i\end{smallmatrix}\right]$ so as to use the matrix multiplication pattern as a mnemonic. Specifically, $[r_0 \; s_0] : \left[\begin{smallmatrix}\mathcal{Z}_0\\ \mathcal{V}_0\end{smallmatrix}\right] \to \mathcal{Z}_1$ and $\left[\begin{smallmatrix}s_2\\ t_2\end{smallmatrix}\right] : \mathcal{V}_2 \to \left[\begin{smallmatrix}\mathcal{Z}_3\\ \mathcal{V}_3\end{smallmatrix}\right]$ are defined, respectively, by

$$\begin{bmatrix} r_0 & z_0 \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} = r_0 z + s_0 v, \qquad \begin{bmatrix} s_2 \\ t_2 \end{bmatrix} v = \begin{bmatrix} s_2 v \\ t_2 v \end{bmatrix}.$$

Proposition 2.3. *Suppose s_1 is a bijection.*

(1) *If $\mathcal{Z}_i$ and $\mathcal{V}_i$ are exact sequences and the diagram* (2.4) *commutes, then the following is an exact sequence:*

$$\begin{bmatrix} \mathcal{Z}_0 \\ \mathcal{V}_0 \end{bmatrix} \xrightarrow{[r_0 \; s_0]} \mathcal{Z}_1 \xrightarrow{t_1 \circ s_1^{-1} \circ r_1} \mathcal{V}_2 \xrightarrow{\left[\begin{smallmatrix}s_2\\ t_2\end{smallmatrix}\right]} \begin{bmatrix} \mathcal{Z}_3 \\ \mathcal{V}_3 \end{bmatrix}.$$

(2) *For the surjectivity of the last map $\left[\begin{smallmatrix}s_2\\ t_2\end{smallmatrix}\right]$, it is sufficient that r_2 and t_2 are surjective, $t_1 \circ t_2 = 0$, and $s_2 t_1 = r_2 s_1$.*

Proof. The range of $[r_0 \; s_0]$ is contained in the kernel of $t_1 \circ s_1^{-1} \circ r_1$ because for any $(z, v) \in \mathcal{Z}_0 \times \mathcal{V}_0$

$$t_1 s_1^{-1} r_1 (r_0 z + s_0 v) = t_1 s_1^{-1} r_1 s_0 v = t_1 s_1^{-1} s_1 t_0 v = t_1 t_0 v = 0,$$

where we have used the given assumptions that the top and bottom sequences in (2.4) are complexes and that the commutativity property, $r_1 s_0 = s_1 t_0$, holds. For the reverse inclusion, if $z \in \mathcal{Z}_1$ is in the kernel of $t_1 \circ s_1^{-1} \circ r_1$, then $s_1^{-1} r_1 z$ is in $\ker t_1 = \operatorname{range} t_0$, so there is a $v \in \mathcal{V}_0$ such that $s_1^{-1} r_1 z = t_0 v$, i.e., $0 = r_1 z - s_1 t_0 v = r_1 z - r_1 s_0 v$ using the commutativity property again. Hence, $z - s_0 v$ is in $\ker r_1 = \operatorname{range} r_0$, i.e., $z = s_0 v + r_0 z_0$ for some $z_0 \in \mathcal{Z}_0$, thus showing that z is in the range of $[r_0\ s_0]$ and completing the proof of range $[r_0\ s_0] = \ker(t_1 \circ s_1^{-1} \circ r_1)$.

We use the other commutativity property, $r_2 s_1 = s_2 t_1$, to prove $\operatorname{range}(t_1 \circ s_1^{-1} \circ r_1) = \ker \left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]$. Consider a v_2 in the latter kernel, i.e., $s_2 v_2 = 0$ and $t_2 v_2 = 0$. Since $\ker(t_2) = \operatorname{range}(t_1)$, there is a $v_1 \in \mathcal{V}_1$ such that $v_2 = t_1(v_1)$, so $0 = s_2 v_2 = s_2 t_1 v_1 = r_2 s_1 v_1$. Since $s_1 v_1 \in \ker(r_2) = \operatorname{range}(r_1)$, there is a $z_1 \in \mathcal{Z}_1$ such that $s_1 v_1 = r_1 z_1$, i.e., $v_1 = s_1^{-1} r_1 z_1$. Thus $v_2 = t_1 v_1 = t_1 s_1^{-1} r_1 z_1$, i.e., $\operatorname{range}(t_1 \circ s_1^{-1} \circ r_1) \supseteq \ker \left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]$. The reverse inclusion is easy.

To prove the final statement, consider a $z_3 \in \mathcal{Z}_3$ and $v_3 \in \mathcal{V}_3$. If t_2 is surjective, then there is a $\tilde{v}_2 \in \mathcal{V}_2$ such that $t_2 \tilde{v}_2 = v_3$. If r_2 is also surjective, then we can find a z_2 and $\tilde{z}_2$ in $\mathcal{Z}_2$ such that $r_2 z_2 = z_3$ and $r_2 \tilde{z}_2 = s_2 \tilde{v}_2$. It may now be easily verified that $v_2 = t_1 s_1^{-1} z_2 + \tilde{v}_2 - t_1 s_1^{-1} \tilde{z}_2$ in $\mathcal{V}_2$ satisfies $\left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right] v_2 = \left[\begin{smallmatrix} z_3 \\ v_3 \end{smallmatrix}\right]$. □

In the next two sections, we shall construct specific instances of the $\mathcal{Z}$ and $\mathcal{V}$ sequences in such a way that Proposition 2.3 may then be applied to produce an elasticity complex. For another example of application of this proposition, see the proof of exactness of (1.2) in [24, p. 38–40].

3. The first exact sequence of spaces

In this section, we develop one of the above-mentioned two sequences of spaces. This sequence is comprised of the following spaces:

$$
\begin{aligned}
V_r^0(T^{\mathrm{A}}) &= S_r^0(T^{\mathrm{A}}), \\
V_r^1(T^{\mathrm{A}}) &= \{\omega \in L_r^1(T^{\mathrm{A}}) :\ \omega \text{ is } C^1 \text{ at vertices of } T\}, \\
V_r^2(T^{\mathrm{A}}) &= \{\omega \in W_r^2(T^{\mathrm{A}}) :\ \omega \text{ is } C^0 \text{ at vertices of } T\}, \\
V_r^3(T^{\mathrm{A}}) &= W_r^3(T^{\mathrm{A}}).
\end{aligned}
$$

Each element of this V-sequence of spaces can be considered as a subspace of a corresponding space in (2.1b) obtained by increasing vertex smoothness of the middle two spaces. Note that we do not impose additional continuity at the interior vertex z, making these spaces slightly different from a sequence of similar spaces considered in [22]. The corresponding spaces with boundary conditions are given as follows:

$$
\begin{aligned}
\mathring{V}_r^0(T^{\mathrm{A}}) &= \mathring{S}_r^0(T^{\mathrm{A}}), \\
\mathring{V}_r^1(T^{\mathrm{A}}) &= \{\omega \in \mathring{L}_r^1(T^{\mathrm{A}}) :\ \operatorname{grad} \omega = 0 \text{ at vertices of } T\}, \\
\mathring{V}_r^2(T^{\mathrm{A}}) &= \{\omega \in \mathring{W}_r^2(T^{\mathrm{A}}) :\ \omega = 0 \text{ at vertices of } T\}, \\
\mathring{V}_r^3(T^{\mathrm{A}}) &= \mathring{W}_r^3(T^{\mathrm{A}}).
\end{aligned}
$$

Note an $\omega \in \mathring{L}_r^1(T^{\mathrm{A}})$ generally has a multivalued $\operatorname{grad} \omega$ at the vertices of T, so the statement "$\operatorname{grad} \omega = 0$ at vertices of T" above should be understood as follows: $\operatorname{grad} \omega$ exists at x_i (i.e., the multiple limiting values coincide) and equals 0. The statement "$\omega = 0$ at vertices of T" for $\omega \in \mathring{W}_r^2(T^{\mathrm{A}})$ above, and similar such statements later in the paper, carry the same tacit understanding.

3.1. **Characterizations and dimensions of the V spaces.** We shall now provide some characterizations of the $\mathring{V}$ spaces which makes their dimensions obvious. Let F^z denote the set of interior facets (2-subsimplices) of the mesh T^{A}. Each $f \in F^z$ has z as a vertex. The subcollection of three facets in F^z having x_i as a vertex is denoted by F_i^z. Let $w_{n,f}$ denote the normal component of w on an $f \in F^z$, i.e., $w_{n,f} = w \cdot n|_f$ where n is a unit normal to f of arbitrarily fixed orientation.

Lemma 3.1. *The following equalities hold:*

$$\mathring{V}_r^1(T^A) = \{\mu\, p :\ p \in L_{r-1}^1(T^A) \text{ satisfying } p(x_i) = 0, \text{ for } i = 0, 1, 2, 3\}, \tag{3.1}$$

$$\mathring{V}_r^2(T^A) = \{w \in \mathring{W}_r^2(T^A) :\ w_{n,f}(x_i) = 0 \text{ for all } f \in F_i^z, i = 0, 1, 2, 3\}. \tag{3.2}$$

Proof. Let $v \in \mathring{V}_r^1(T^{\mathrm{A}})$. On each T_i, since v vanishes on the facet where $\mu = 0$, we may factor it uniquely as $v = \mu p$ for some $p \in \mathcal{P}_{r-1}(T_i)^3$. Since v is continuous on T, we conclude that $p \in L_{r-1}^1(T^{\mathrm{A}})$. Moreover, since $\operatorname{grad} v(x_i) = (\operatorname{grad} \mu p)(x_i)$ and $\mu(x_i)$ are zero, $p(x_i)\operatorname{grad}\mu(x_i) = 0$. Hence $p(x_i) = 0$, so v is in the set on the right hand side of (3.1). Since the reverse inclusion "$\supseteq$" is easy to see, the set equality of (3.1) follows.

For (3.2), since the "$\subseteq$"-part is easy, let us focus on proving the reverse. Let v be in the set on the right hand side of (3.2). Consider a vertex, say x_1. Three facets of $T_1 = [z, x_1, x_2, x_3]$, namely $f_1 = [x_1, x_2, x_3], f_2 = [z, x_1, x_2], f_3 = [z, x_1, x_3]$, meet at x_1. Letting n_i denote the outward unit normal on f_i, observe that $\{n_1, n_2, n_3\}$ is a linearly independent set since T_1 has positive volume. The given conditions on v imply that $v_{n_1,f_1} \equiv 0$ and $v_{n_2,f_2}(x_1) = v_{n_3,f_3}(x_1) = 0$, i.e., three independent components of $v|_{T_1}(x_1) \in \mathbb{R}^3$ vanish, so $v|_{T_1}(x_1) = 0$. Repeating this argument at other x_i and T_j, we conclude that all limiting values of v at every vertex x_i vanish. Therefore $v \in \mathring{V}_r^2(T^{\mathrm{A}})$. □

Let $W_r^k(T) = \mathcal{P}_r\Lambda^k(T)$, not to be confused with $W_r^k(T^{\mathrm{A}})$. The well-known canonical degrees of freedom of this space provides the direct decomposition [8]

$$W_r^k(T) = \mathring{W}_r^k(T) \oplus W_r^{\partial,k}(T),$$

where $\mathring{W}_r^k(T)$ is the span of all interior shape functions of $\mathcal{P}_r\Lambda^k(T)$ and $W_r^{\partial,k}(T)$ is the span of the remaining shape functions. Let $L_r^{\partial,1}(T) = [W_r^{\partial,0}(T)]^3$.

Lemma 3.2. *The following equalities hold:*

$$\mathring{V}_r^1(T^A) = V_r^1(T^A) \cap \mathring{L}_r^1(T^A), \qquad \mathring{V}_r^2(T^A) = V_r^2(T^A) \cap \mathring{W}_r^2(T^A), \tag{3.3}$$

$$V_r^1(T^A) = \mathring{V}_r^1(T^A) \oplus L_r^{\partial,1}(T), \qquad V_r^2(T^A) = \mathring{V}_r^2(T^A) \oplus W_r^{\partial,2}(T). \tag{3.4}$$

Proof. Let $v \in V_r^1(T^{\mathrm{A}}) \cap \mathring{L}_r^1(T^{\mathrm{A}})$. Then v is C^1 at x_i, so $\operatorname{grad} v(x_i)$ is well-defined. Since v is zero along the three edges of T connected to x_i, three linearly independent components of the vector $\operatorname{grad} v_j(x_i)$ are zero for each $1 \le j \le 3$, so $\operatorname{grad} v(x_i) = 0$. Hence $\mathring{V}_r^1(T^{\mathrm{A}}) \supseteq V_r^1(T^{\mathrm{A}}) \cap \mathring{L}_r^1(T^{\mathrm{A}})$. Together with the obvious reverse inclusion, the first equality of (3.3) follows. The proof of the second is similar: indeed, if $w \in V_r^2(T^{\mathrm{A}}) \cap \mathring{W}_r^2(T^{\mathrm{A}})$, then w is C^0 at x_i, so $w(x_i)$ is a single-valued vector whose three independent components $w(x_i) \cdot n_j$ (for $j \ne i$) vanish, where n_j is the unit normal to the facet opposite to x_j. Hence $w \in \mathring{V}_r^2(T^{\mathrm{A}})$.

To prove the first decomposition of (3.4), first observe that the sets $\{u|_{\partial T} : u \in L_r^0(T^{\mathrm{A}})\}$ and $\{u|_{\partial T} : u \in \mathcal{P}_r(T)\}$ coincide. Consequently, the trace of any $v \in V_r^1(T^{\mathrm{A}}) \subseteq L_r^1(T^{\mathrm{A}})$ has a unique extension in $L_r^{\partial,1}(T)$, which we shall call v_L. Put $v_0 = v - v_L$. We claim that

$$v = v_0 + v_L$$

is the required decomposition. Indeed, since v_L is a polynomial on T (and hence smooth), the function $v_0 = v - v_L$ is in $V_r^1(T^{\mathrm{A}})$. Moreover, since the trace of v_0 is zero, $v_0 \in V_r^1(T^{\mathrm{A}}) \cap \mathring{L}_r^1(T^{\mathrm{A}})$, so by (3.3), $v_0 \in \mathring{V}_r^1(T^{\mathrm{A}})$. (The directness of the decomposition follows easily by examining the boundary values of the component spaces.)

The second decomposition in (3.4) is proved similarly. $\square$

Lemma 3.3. *The dimensions of the $\mathring{V}$ and V spaces for any $r \geq 1$ are as follows.*

$$\dim \mathring{V}_r^0(T^A) = \dim \mathring{S}_r^0(T^A), \qquad \dim V_r^0(T^A) = \dim S_r^0(T^A),$$
$$\dim \mathring{V}_r^1(T^A) = \max(2r^3 - 3r^2 + 7r - 15,\ 0), \qquad \dim V_r^1(T^A) = 6(r^2+1) + \dim \mathring{V}_r^1(T^A),$$
$$\dim \mathring{V}_r^2(T^A) = 2r^3 + 7r^2 + 7r - 10, \qquad \dim V_r^2(T^A) = 2r^3 + 9r^2 + 13r - 6,$$
$$\dim \mathring{V}_r^3(T^A) = \frac{2}{3}(r+1)(r+2)(r+3) - 1, \qquad \dim V_r^3(T^A) = \frac{2}{3}(r+1)(r+2)(r+3).$$

Proof. The counts in the first and last rows are obvious from the definition and (2.3). For the remainder, we first claim that

$$\dim \mathring{V}_r^1(T^{\mathrm{A}}) = \dim L_{r-1}^1(T^{\mathrm{A}}) - 12, \qquad \dim \mathring{V}_r^2(T^{\mathrm{A}}) = \dim \mathring{W}_r^2(T^{\mathrm{A}}) - 12. \tag{3.5}$$

Indeed, by virtue of (3.1) of Lemma 3.1, the dimension of $\mathring{V}_r^1(T^{\mathrm{A}})$ must equal that of $L_{r-1}^1(T^{\mathrm{A}})$ minus the number of independent constraints imposed by the condition "$p(x_i) = 0$" there, which amounts to three linearly independent constraints (one for each component) per vertex x_i. This yields the first count in (3.5). The second count in (3.5) follows from (3.2) of Lemma 3.1, where at each vertex x_i, there are three independent constraints, one for each interior facet connected to x_i. The lemma's expressions of $\dim \mathring{V}_r^1(T^{\mathrm{A}})$ and $\dim \mathring{V}_r^2(T^{\mathrm{A}})$ are now easily obtained by substituting (2.3) in (3.5), and simplifying, noting that $\mathring{V}_r^1(T^{\mathrm{A}})$ is trivial for $r = 1$.

It only remains to count $\dim V_r^1(T^{\mathrm{A}})$ and $\dim V_r^2(T^{\mathrm{A}})$. From (3.4) of Lemma 3.2,

$$\begin{aligned} \dim V_r^1(T^{\mathrm{A}}) &= \dim \mathring{V}_r^1(T^{\mathrm{A}}) + \dim L_r^{\partial,1}(T), \\ \dim V_r^2(T^{\mathrm{A}}) &= \dim \mathring{V}_r^2(T^{\mathrm{A}}) + \dim W_r^{\partial,2}(T). \end{aligned} \tag{3.6}$$

It is easy to see from the canonical set of degrees of freedom of $\mathcal{P}_r\Lambda^k(T)$ that

$$\dim L_r^{\partial,1}(T) = 3\left(4 + 6(r-1) + 4\frac{(r-1)(r-2)}{2}\right), \qquad \dim W_r^{\partial,2}(T) = 4\frac{(r+1)(r+2)}{2}.$$

Using this in (3.6) as well as previously computed dimensions of $\mathring{V}$ spaces, we obtain the stated expressions for $\dim V_r^1(T^{\mathrm{A}})$ and $\dim V_r^2(T^{\mathrm{A}})$. $\square$

3.2. **Exactness.** We now proceed to show that the following local sequences are exact:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} V_r^0(T^{\mathrm{A}}) \xrightarrow{\text{grad}} V_{r-1}^1(T^{\mathrm{A}}) \xrightarrow{\text{curl}} V_{r-2}^2(T^{\mathrm{A}}) \xrightarrow{\text{div}} V_{r-3}^3(T^{\mathrm{A}}) \longrightarrow 0, \tag{3.7}$$

$$0 \longrightarrow \mathring{V}_r^0(T^{\mathrm{A}}) \xrightarrow{\text{grad}} \mathring{V}_{r-1}^1(T^{\mathrm{A}}) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^{\mathrm{A}}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{\mathrm{A}}) \longrightarrow 0. \tag{3.8}$$

In the sequel, to indicate the null space of a differential operator $\mathscr{D}$ in relation to its domain Y, we use $\ker(\mathscr{D}, Y) := \{w \in Y : \mathscr{D}w = 0\}$. Note that $\mathring{V}_r^0(T^{\mathrm{A}}) = \mathring{S}_r^0(T^{\mathrm{A}})$ is nontrivial only for $r \geq 5$ (see (2.3)).

Lemma 3.4. *The sequence* (3.8) *is exact for any* $r \geq 5$. *For* $r = 3$ *and* 4, *the subsequence of* (3.8) *starting from* $\mathring{V}^1_{r-1}$ *is exact.*

Proof. By Proposition 2.1, any $w \in \mathring{V}^0_r(T^{\mathrm{A}})$ is C^2 at x_i, so $\operatorname{grad} w$ vanishes at x_i. Therefore, $\operatorname{grad} \mathring{V}^0_r(T^{\mathrm{A}}) \subseteq \mathring{V}^1_{r-1}(T^{\mathrm{A}})$. Of course, due to the boundary condition, grad : $\mathring{V}^0_r(T^{\mathrm{A}}) \to \mathring{V}^1_{r-1}(T^{\mathrm{A}})$ is also injective.

Proceeding to the next operator, it's easy to see that $\operatorname{curl} \mathring{V}^1_{r-1}(T^{\mathrm{A}}) \subseteq \ker(\operatorname{div}, \mathring{V}^2_{r-2}(T^{\mathrm{A}}))$. To prove the reverse inclusion, consider a $w \in \mathring{V}^2_{r-2}(T^{\mathrm{A}})$ with $\operatorname{div} w = 0$. Then, by Proposition 2.2, there is a $u = \mu v$ such that $\operatorname{curl} u = w$, where $v = \sum_{\ell=0}^{r-2} \mu^\ell \gamma_{r-2-\ell}$ and $\gamma_\ell \in \mathcal{P}_\ell(T)^3$. This implies that

$$\operatorname{grad} \mu \times v = w - \mu \operatorname{curl} v.$$

Since $w \in \mathring{V}^2_{r-2}(T^{\mathrm{A}})$, the right hand side above vanishes at all the vertices of T, and so does v. Hence $u = \mu v$ has vanishing $\operatorname{grad} u$ at the vertices of T, which implies $u \in \mathring{V}^1_{r-1}(T^{\mathrm{A}})$. Thus $\operatorname{curl} \mathring{V}^1_{r-1}(T^{\mathrm{A}}) = \ker(\operatorname{div}, \mathring{V}^2_{r-2}(T^{\mathrm{A}}))$.

Finally, consider the divergence operator. Since $\mathring{V}^2_{r-2}(T^{\mathrm{A}}) \subseteq \mathring{W}^2_{r-2}(T^{\mathrm{A}})$, and since the standard de Rham complex—the version of (2.1a) with boundary conditions—implies that $\operatorname{div} \mathring{W}^2_{r-2}(T^{\mathrm{A}}) = \mathring{W}^3_{r-3}(T^{\mathrm{A}})$, we have $\operatorname{div} \mathring{V}^2_{r-2}(T^{\mathrm{A}}) \subseteq \mathring{V}^3_{r-3}(T^{\mathrm{A}})$. To improve this inclusion to equality, recall that by the exactness of the version of (2.1c) with boundary conditions proved in [22], $\mathring{W}^3_{r-3}(T^{\mathrm{A}}) = \operatorname{div} \mathring{L}^2_{r-2}(T^{\mathrm{A}})$. Since $\mathring{V}^2_{r-2}(T^{\mathrm{A}}) \supseteq \mathring{L}^2_{r-2}(T^{\mathrm{A}})$, we conclude that $\operatorname{div} \mathring{V}^2_{r-2}(T^{\mathrm{A}}) = \mathring{V}^3_{r-3}(T^{\mathrm{A}})$. ☐

Lemma 3.5. *The sequence* (3.7) *is exact for any* $r \geq 3$.

Proof. The exactness of $\mathbb{R} \longrightarrow V^0_r(T^{\mathrm{A}}) \xrightarrow{\text{grad}} V^1_{r-1}(T^{\mathrm{A}})$ and $V^2_{r-2}(T^{\mathrm{A}}) \xrightarrow{\text{div}} V^3_{r-3}(T^{\mathrm{A}}) \longrightarrow 0$ follow easily using standard exactness results, so we shall only consider the curl case. Since it is obvious that $\operatorname{curl} V^1_{r-1}(T^{\mathrm{A}}) \subseteq \ker(\operatorname{div}, V^2_{r-2}(T^{\mathrm{A}}))$, let us prove the reverse inclusion. Consider a $\rho \in V^2_{r-2}(T^{\mathrm{A}})$ with $\operatorname{div} \rho = 0$. Let $\Pi\rho \in [\mathcal{P}_{r-2}(T)]^3$ be the canonical interpolant of ρ per the standard Nédélec (second type) degrees of freedom [34], defined for $r - 2 \geq 1$. Let $\psi = \rho - \Pi\rho$. By the well-known properties of Π, $\operatorname{div} \psi = 0$, and $\psi \cdot n = 0$ on ∂T. Since ρ is C^0 at x_i, ψ must vanish at the vertices of T. Thus ψ is in $\ker(\operatorname{div}, \mathring{V}^2_{r-2}(T^{\mathrm{A}}))$. By Lemma 3.4, there is an $\omega \in \mathring{V}^1_{r-1}(T^{\mathrm{A}})$ such that $\operatorname{curl} \omega_1 = \psi$. By a standard exactness result, we also know there is an $\omega_2 \in [\mathcal{P}_{r-1}(T)]^3$ such that $\operatorname{curl} \omega_2 = \Pi\rho$. Hence, $\operatorname{curl} \omega = \rho$ where $\omega = \omega_1 + \omega_2 \in V^1_{r-1}(T^{\mathrm{A}})$. ☐

3.3. **Degrees of freedom of the V spaces.** The degrees of freedom (dofs) of the V spaces are almost the same as the ones used in [22], the only difference being that some of our bubble spaces are less smooth at the interior point z. The unisolvency proofs are substantially similar to those in [22], so we do not write them out here. We only state the degrees of freedom.

Let $r \geq 5$. Then, a function $\omega \in V^0_r(T^{\mathrm{A}})$ is uniquely determined by the following dofs (see [22, Lemma 4.8]) :

(3.9a) $\quad D^\alpha \omega(a), \quad |\alpha| \leq 2, \quad a \in \Delta_0(T) \quad$ (40 dofs),

(3.9b) $\quad \int_e \omega\, \sigma, \quad \sigma \in \mathcal{P}_{r-6}(e), \quad e \in \Delta_1(T) \quad$ (6(r − 5) dofs),

$$(3.9c)\qquad \int_e \frac{\partial \omega}{\partial n_e^{\pm}}\,\sigma\,, \qquad \sigma \in \mathcal{P}_{r-5}(e), \quad e \in \Delta_1(T) \qquad (12(r-4) \text{ dofs}),$$

$$(3.9d)\qquad \int_F \omega\,\sigma\,, \qquad \sigma \in \mathcal{P}_{r-6}(F), \quad F \in \Delta_2(T) \qquad (4\frac{(r-5)(r-4)}{2} \text{ dofs}),$$

$$(3.9e)\qquad \int_F \frac{\partial \omega}{\partial n^F}\,\sigma\,, \qquad \sigma \in \mathcal{P}_{r-4}(F), \quad F \in \Delta_2(T) \qquad (4\frac{(r-3)(r-2)}{2} \text{ dofs}),$$

$$(3.9f)\qquad \int_T \operatorname{grad}\omega \cdot \operatorname{grad}\sigma\,, \qquad \sigma \in \mathring{V}_r^0(T^{\mathrm{A}}), \qquad (2\frac{(r-4)(r-3)(r-2)}{3} \text{ dofs}).$$

Here, $\{n_{e_+}, n_{e_-}\}$ is an orthonormal set spanning the plane orthogonal to the edge e, n^F denotes the outward unit normal of face F, and $n^F \cdot \operatorname{grad}\omega$ is abbreviated to $\partial\omega/\partial n^F$. In the case $r = 5$, the sets listed in (3.9b) and (3.9d) are omitted. It is easy to see that the sum of dofs above equal $\dim V_r^0(T^{\mathrm{A}})$ given in Lemma 3.3.

A function $\omega \in V_{r-1}^1(T^{\mathrm{A}})$ is uniquely determined by the values (see [22, Lemma 4.15])

$$(3.10a)\qquad D^\alpha \omega(a), \qquad |\alpha| \le 1,\ a \in \Delta_0(T) \qquad (48 \text{ dofs})$$

$$(3.10b)\qquad \int_e \omega \cdot \kappa\,, \qquad \kappa \in [\mathcal{P}_{r-5}(e)]^3,\ e \in \Delta_1(T) \qquad (18(r-4) \text{ dofs})$$

$$(3.10c)\qquad \int_e (\operatorname{curl}\omega|_F \cdot n^F)\kappa\,, \qquad \kappa \in \mathcal{P}_{r-4}(e),\ e \in \Delta_1(F), F \in \Delta_2(T), \qquad (12(r-3) \text{ dofs})$$

$$(3.10d)\qquad \int_F (\omega \cdot n^F)\kappa\,, \qquad \kappa \in \mathcal{P}_{r-4}(F),\ F \in \Delta_2(T) \qquad (2(r-2)(r-3) \text{ dofs})$$

$$(3.10e)\qquad \int_F (n^F \times (\omega \times n^F)) \cdot \kappa\,, \qquad \kappa \in D_{r-5}(F),\ F \in \Delta_2(T) \qquad (4(r-3)(r-5) \text{ dofs})$$

$$(3.10f)\qquad \int_T \omega \cdot \kappa\,, \qquad \kappa \in \operatorname{grad}\mathring{V}_r^0(T^{\mathrm{A}}), \qquad (2\frac{(r-4)(r-3)(r-2)}{3} \text{ dofs})$$

$$(3.10g)\qquad \int_T \operatorname{curl}\omega \cdot \kappa\,, \qquad \kappa \in \operatorname{curl}\mathring{V}_{r-1}^1(T^{\mathrm{A}}), \qquad (\frac{4r^3 - 9r^2 + 5r - 33}{3} \text{ dofs})$$

where

$$(3.11)\qquad D_{r-5}(F) = [\mathcal{P}_{r-6}(F)]^2 + x\mathcal{P}_{r-6}(F)$$

is the local Raviart–Thomas space on the face F. The count of dofs in (3.10g) is obtained as a consequence of the exactness established in Lemma 3.4, i.e.,

$$\dim\left(\operatorname{curl}\mathring{V}_{r-1}^1(T^{\mathrm{A}})\right) = \dim \mathring{V}_{r-2}^2(T^{\mathrm{A}}) - \dim \mathring{V}_{r-3}^3(T^{\mathrm{A}}) = \frac{4r^3 - 9r^2 + 5r - 33}{3},$$

where we have also used the $\mathring{V}$-dimensions given in Lemma 3.3. The sum of the dofs in the last column above can be easily verified to equal the expression for $\dim V_{r-1}^1(T^{\mathrm{A}})$ given in Lemma 3.3.

Remark 3.6. Note that if a function $\omega \in V_{r-1}^1(T^{\mathrm{A}})$ has vanishing degrees of freedom (3.10a)–(3.10e) then $\omega|_{\partial T} = 0$ (see [22, Lemma 4.15]).

A function $\omega \in V^2_{r-2}(T^{\mathrm{A}})$ is uniquely determined by the values (see [22, Lemma 4.17])

(3.12a) $\omega(a)$, $\quad a \in \Delta_0(T)$ (12 dofs)

(3.12b) $\int_e (\omega \cdot n^F)\kappa\,,\quad \kappa \in \mathcal{P}_{r-4}(e),\ e \in \Delta_1(F),\ F \in \Delta_2(T)$ ($12(r-3)$ dofs)

(3.12c) $\int_F (\omega \cdot n^F)\kappa\,,\quad \kappa \in \mathcal{P}_{r-5}(F),\ F \in \Delta_2(T)$ ($2(r-3)(r-4)$ dofs)

(3.12d) $\int_T \omega \cdot \kappa\,,\quad \kappa \in \operatorname{curl} \mathring{V}^1_{r-1}(T^{\mathrm{A}})$ ($\frac{4r^3-9r^2+5r-33}{3}$ dofs)

(3.12e) $\int_T (\operatorname{div}\omega)\,(\operatorname{div}\kappa)\,,\quad \kappa \in \mathring{V}^2_{r-2}(T^{\mathrm{A}})$ ($\frac{2}{3}(r-2)(r-1)r-1$ dofs)

where we have used the exactness result of Lemma 3.4 again to count the number of dofs in (3.12e), $\dim(\operatorname{div} \mathring{V}^2_{r-2}(T^{\mathrm{A}})) = \dim \mathring{V}^3_{r-3}(T^{\mathrm{A}})$. The total number of dofs above can again be easily seen to equal $\dim V^2_{r-2}(T^{\mathrm{A}})$ in Lemma 3.3.

Finally, for $r \ge 5$, a function $\omega \in V^3_{r-3}(T^{\mathrm{A}})$ is uniquely determined by the following degrees of freedom (see [22, Lemma 4.18]):

(3.13a) $\int_T \omega\,,$ (1 dof)

(3.13b) $\int_T \omega\,\kappa\,,\quad \kappa \in \mathring{V}^3_{r-3}(T^{\mathrm{A}}),$ ($\frac{2}{3}r(r-1)(r-2)-1$ dofs).

With these dofs, the following result can be proved along the same lines as [22].

Theorem 3.7. *Let Π^V_i denote the canonical interpolation into $V^i_{r-i}(T^A)$ defined by the dofs of $V^i_{r-i}(T^A)$ set above. Then, for $r \ge 5$ the following diagram commutes*

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(\bar{T}) & \xrightarrow{\text{grad}} & \left[C^\infty(\bar{T})\right]^3 & \xrightarrow{\text{curl}} & \left[C^\infty(\bar{T})\right]^3 & \xrightarrow{\text{div}} & C^\infty(\bar{T}) & \longrightarrow & 0\\
 & & \downarrow \Pi^V_0 & & \downarrow \Pi^V_1 & & \downarrow \Pi^V_2 & & \downarrow \Pi^V_3 & & \\
\mathbb{R} & \longrightarrow & V^0_r(T^A) & \xrightarrow{\text{grad}} & V^1_{r-1}(T^A) & \xrightarrow{\text{curl}} & V^2_{r-2}(T^A) & \xrightarrow{\text{div}} & V^3_{r-3}(T^A) & \longrightarrow & 0.
\end{array}$$

4. The second exact sequence

In this section, we introduce the second exact sequence with smoother spaces required for the construction of the elasticity complex in the next section. Let

$$\begin{aligned}
Z^0_r(T^{\mathrm{A}}) &= S^0_r(T^{\mathrm{A}})\\
Z^1_r(T^{\mathrm{A}}) &= \{\omega \in S^1_r(T^{\mathrm{A}}) :\ \operatorname{curl}\omega \text{ is } C^1 \text{ at vertices of } T\},\\
Z^2_r(T^{\mathrm{A}}) &= \{\omega \in L^2_r(T^{\mathrm{A}}) :\ \omega \text{ is } C^1 \text{ at vertices of } T\},\\
Z^3_r(T^{\mathrm{A}}) &= \{\omega \in W^3_r(T^{\mathrm{A}}) :\ \omega \text{ is } C^0 \text{ at vertices of } T\}.
\end{aligned}$$

Note that each space in this sequence of Z-spaces is a subspace of a corresponding space in (2.1c) obtained by augmenting vertex smoothness to some of them. The corresponding spaces with boundary conditions are defined as follows:

$$\mathring{Z}^0_r(T^{\mathrm{A}}) = \mathring{S}^0_r(T^{\mathrm{A}}),$$
$$\mathring{Z}^1_r(T^{\mathrm{A}}) = \{\omega \in \mathring{S}^1_r(T^{\mathrm{A}}) : \ \operatorname{grad}\operatorname{curl}\omega = 0 \text{ at vertices of } T\},$$
$$\mathring{Z}^2_r(T^{\mathrm{A}}) = \{\omega \in \mathring{L}^2_r(T^{\mathrm{A}}) : \ \operatorname{grad}\omega = 0 \text{ at vertices of } T\},$$
$$\mathring{Z}^3_r(T^{\mathrm{A}}) = \{\omega \in \mathring{W}^3_r(T^{\mathrm{A}}) : \ \omega = 0 \text{ at vertices of } T\}.$$

It is easy to verify from these definitions that

$$\operatorname{grad} Z^0_r(T^{\mathrm{A}}) \subseteq Z^1_{r-1}(T^{\mathrm{A}}), \quad \operatorname{curl} Z^1_r(T^{\mathrm{A}}) \subseteq Z^2_{r-1}(T^{\mathrm{A}}), \quad \operatorname{div} Z^2_1(T^{\mathrm{A}}) \subseteq Z^3_{r-1}(T^{\mathrm{A}}), \tag{4.1}$$

and that similar inclusions hold for the $\mathring{Z}$ spaces with boundary conditions.

4.1. **Exactness and dimensions.** It is obvious from the above definitions that, we have, as equalities of sets, the identities

$$Z^2_r(T^{\mathrm{A}}) = V^1_r(T^{\mathrm{A}}) \quad \text{and} \quad \mathring{Z}^2_r(T^{\mathrm{A}}) = \mathring{V}^1_r(T^{\mathrm{A}}), \tag{4.2}$$

even though their form degrees do not match. The next relationships are also interesting.

Lemma 4.1. *The following identities hold:*

$$\mathring{Z}^1_r(T^A) = Z^1_r(T^A) \cap \mathring{S}^1_r(T^A), \tag{4.3}$$
$$\mathring{Z}^2_r(T^A) = Z^2_r(T^A) \cap \mathring{L}^2_r(T^A). \tag{4.4}$$

Proof. By (4.2), the equality of sets in(4.4) is immediate since it is exactly the left equality in (3.3). To show (4.3), it suffices to prove that $\mathring{S}^1_r(T^{\mathrm{A}}) \cap Z^1_r(T^{\mathrm{A}}) \subset \mathring{Z}^1_r(T^{\mathrm{A}})$. Let $\omega \in \mathring{S}^1_r(T^{\mathrm{A}}) \cap Z^1_r(T^{\mathrm{A}}) \subset \mathring{L}^1_r(T^{\mathrm{A}}) \cap V^1_r(T^{\mathrm{A}})$. Then $\operatorname{curl}\omega \in \mathring{L}^2_{r-1}(T^{\mathrm{A}}) \cap Z^2_{r-1}(T^{\mathrm{A}})$, so by (4.4), $\operatorname{curl}\omega \in \mathring{Z}^2_{r-1}(T^{\mathrm{A}})$. Hence $\operatorname{grad}\operatorname{curl}\omega = 0$ at the vertices of T and we conclude that $\omega \in \mathring{Z}^1_r(T^{\mathrm{A}})$. □

Now, we turn to the exactness properties of the sequences of Z and $\mathring{Z}$ spaces. Note that $\mathring{Z}^0_{r+1}(T^{\mathrm{A}}) = \mathring{S}^0_{r+1}(T^{\mathrm{A}})$ is nontrivial only for $r \geq 4$.

Lemma 4.2. *The following sequence is exact for any $r \geq 4$.*

$$0 \longrightarrow \mathring{Z}^0_{r+1}(T^A) \xrightarrow{\operatorname{grad}} \mathring{Z}^1_r(T^A) \xrightarrow{\operatorname{curl}} \mathring{Z}^2_{r-1}(T^A) \xrightarrow{\operatorname{div}} \mathring{Z}^3_{r-2}(T^A) \longrightarrow 0. \tag{4.5}$$

Proof. We shall only prove the exactness of $\mathring{Z}^2_{r-1}(T^{\mathrm{A}}) \xrightarrow{\operatorname{div}} \mathring{Z}^3_{r-2}(T^{\mathrm{A}}) \longrightarrow 0$, as all the other exactness properties can be shown easily from the known exactness of the $\mathring{S}$ sequence, i.e., the analogue of (2.1) with boundary conditions, and inclusions similar to (4.1) for the $\mathring{Z}$ sequence. To show that $\operatorname{div} : \mathring{Z}^2_{r-1}(T^{\mathrm{A}}) \to \mathring{Z}^3_{r-2}(T^{\mathrm{A}})$ is surjective, let $\rho \in \mathring{Z}^3_{r-2}(T^{\mathrm{A}}) \subset \mathring{W}^3_{r-2}(T^{\mathrm{A}})$. By Proposition 2.2, there is an ω of the form $\omega = \mu\psi$ where $\psi = \sum_{\ell=2}^{r-2} \mu^\ell \gamma_{r-2-\ell}$ and $\gamma_{r-2-\ell} \in [\mathcal{P}_{r-2-\ell}(T)]^3$, such that $\operatorname{div}\omega = \rho$. The last equality can be rewritten as

$$\operatorname{grad}\mu \cdot \psi = \rho - \mu \operatorname{div}\psi.$$

At the vertices of T, all the limit values of the right hand side above vanish, as ρ is in $\mathring{Z}^2_{r-2}(T^{\mathrm{A}})$. Since the three $\operatorname{grad}\mu$ vectors on the three faces of ∂T that meet at a vertex are linearly independent, we conclude that ψ vanishes at the vertices of T. Thus, by the product rule, $\operatorname{grad}\omega = \operatorname{grad}(\mu\psi)$ also vanishes at the vertices of T. Hence, $\omega \in \mathring{Z}^2_{r-1}(T^{\mathrm{A}})$. □

Let ℓ_i for each $i = 0, \ldots, 3$, denote a quadratic function in $\mathcal{P}_2(T)$ satisfying

$$\ell_i(x_j) = \delta_{ij}, \qquad \int_T \ell_i = 0. \tag{4.6}$$

E.g., one may set ℓ_i as the sum of a linear function and an edge bubble, $\ell_i = \lambda_i + c\lambda_j\lambda_k$ (using the barycentric coordinates λ_m of T) for some $j \neq k$ where the scalar c is chosen to make ℓ_i mean-free.

Lemma 4.3. *The following sequence is exact for $r \geq 4$.*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} Z^0_{r+1}(T^A) \xrightarrow{\operatorname{grad}} Z^1_r(T^A) \xrightarrow{\operatorname{curl}} Z^2_{r-1}(T^A) \xrightarrow{\operatorname{div}} Z^3_{r-2}(T^A) \longrightarrow 0. \tag{4.7}$$

Proof. All the exactness properties, except the last, follow immediately from the exactness of the S sequence. To prove the surjectivity of div : $Z^2_{r-1}(T^{\mathrm{A}}) \to Z^3_{r-2}(T^{\mathrm{A}})$, let $\rho \in Z^3_{r-2}(T^{\mathrm{A}})$. Using the ℓ_i in (4.6), set $\rho_2 = \sum_{i=0}^3 \ell_i\rho(x_i)$, which is in $Z^3_{r-2}(T^{\mathrm{A}})$ as $r-2 \geq 2$. Moreover, $\rho - \rho_2$ is in $\mathring{Z}^3_{r-2}(T^{\mathrm{A}})$. Hence, using Lemma 4.2, we can find $\omega_1 \in \mathring{Z}^2_{r-1}(T^{\mathrm{A}})$ such that $\operatorname{div}\omega_1 = \rho - \rho_2$. By standard exactness results, there is an $\omega_2 \in [\mathcal{P}_3(T)]^3$ such that $\operatorname{div}\omega_2 = \rho_2$. Hence, $\omega = \omega_1 + \omega_2 \in Z^2_{r-1}(T^{\mathrm{A}})$ satisfies $\operatorname{div}\omega = \rho$. □

Lemma 4.4. *The dimensions of the $\mathring{Z}$ and Z spaces are as follows.*

$$\dim \mathring{Z}^0_r(T^A) = \dim \mathring{S}^0_r(T^A), \qquad \dim Z^0_r(T^A) = \dim S^0_r(T^A), \qquad (r \geq 1), \tag{4.8}$$

$$\dim \mathring{Z}^1_r(T^A) = (2r-3)(r-3)(r-2), \quad \dim Z^1_r(T^A) = 2r^3 - 3r^2 + 13r - 4 \quad (r \geq 4). \tag{4.9}$$

The following formulas for the dimensions of $\mathring{Z}^2_r(T^A)$ and $\mathring{Z}^3_r(T^A)$ hold for $r \geq 2$, while those for $Z^2_r(T^A)$ and $Z^3_r(T^A)$ hold for $r \geq 1$:

$$\dim \mathring{Z}^2_r(T^A) = 2r^3 - 3r^2 + 7r - 15, \qquad \dim Z^2_r(T^A) = 2r^3 + 3r^2 + 7r - 9, \tag{4.10}$$

$$\dim \mathring{Z}^3_r(T^A) = \dim W^3_r(T^A) - 13, \qquad \dim Z^3_r(T^A) = \dim W^3_r(T^A) - 8. \tag{4.11}$$

Proof. Dimensions in (4.8) are obvious. Those in (4.10) are also obvious from (4.2) and Lemma 3.3. We proceed to prove (4.11) followed by (4.9). Let

$$\hat{Z}^3_r(T^{\mathrm{A}}) = \{w \in W^3_r(T^{\mathrm{A}}) : \; w(x_i) = 0\}. \tag{4.12}$$

We claim that

$$Z^3_r(T^{\mathrm{A}}) = \hat{Z}^3_r(T^{\mathrm{A}}) \oplus \mathcal{P}_1(T), \qquad \text{for } r \geq 1. \tag{4.13}$$

Indeed, given any $z \in Z^3_r(T^{\mathrm{A}})$, constructing a $\lambda = \sum_{i=0}^4 \lambda_i z(x_i)$, and putting $\hat{z} = z - \lambda$, we have the decomposition $z = \hat{z} + \lambda$, with $\hat{z} \in \hat{Z}^3_r(T^{\mathrm{A}})$ and $\lambda \in \mathcal{P}_1(T)$. This proves (4.13) since the reverse inclusion is obvious. It is easy to count the dimension of $\hat{Z}^3_r(T^{\mathrm{A}})$: the constraints $w(x_i) = 0$ form three linearly independent constraints at each x_i, so $\dim \hat{Z}^3_r(T^{\mathrm{A}}) = \dim W^3_r(T^{\mathrm{A}}) - 12$. Then, (4.13) yields

$$\dim Z^3_r(T^{\mathrm{A}}) = \dim \hat{Z}^3_r(T^{\mathrm{A}}) + 4 = \dim W^3_r(T^{\mathrm{A}}) - 8.$$

To count the dimension of $\mathring{Z}_r^3(T^{\mathrm{A}})$, let us start with $\tilde{Z}_r^3(T^{\mathrm{A}}) = \{w \in Z_r^3(T^{\mathrm{A}}) : \int_T w = 0\}$, whose dimension is obviously $\dim Z_3^r(T^{\mathrm{A}}) - 1$. We claim that, with $\mathcal{M}_2(T) = \mathrm{span}\{\ell_0, \ell_1, \ell_2, \ell_3\}$, where ℓ_i is as in (4.6), we have

$$\tilde{Z}_r^3(T^{\mathrm{A}}) = \mathring{Z}_r^3(T^{\mathrm{A}}) \oplus \mathcal{M}_2(T), \qquad \text{for } r \geq 2. \tag{4.14}$$

Indeed, given any $\tilde{z} \in \tilde{Z}_r^3(T^{\mathrm{A}})$, setting $\ell = \sum_{i=0}^4 \ell_i \tilde{z}(x_i)$ and $\mathring{z} = \tilde{z} - \ell$, we have the decomposition $\tilde{z} = \mathring{z} + \ell$ with $\mathring{z} \in \mathring{Z}_r^3(T^{\mathrm{A}})$ and $\ell \in \mathcal{M}_2(T)$. Combined with the obvious reverse inclusion, we obtain (4.14). Consequently, $\dim \mathring{Z}_r^3(T^{\mathrm{A}}) = \dim \tilde{Z}_3^r(T^{\mathrm{A}}) - \dim \mathcal{M}_2(T) = \dim Z_3^r(T^{\mathrm{A}}) - 5$. This finishes the proof of (4.11).

It only remains to prove (4.9), for which we use the already proved exactness. Restricting to $r \geq 4$ in order to apply Lemma 4.2, the rank-nullity theorem gives

$$\begin{aligned} \dim \mathring{Z}_r^1(T^{\mathrm{A}}) &= \dim \mathrm{curl}\,(\mathring{Z}_r^1(T^{\mathrm{A}})) + \dim \mathrm{grad}\,(\mathring{Z}_{r+1}^0(T^{\mathrm{A}})) \\ &= \dim \mathring{Z}_{r-1}^2(T^{\mathrm{A}}) - \dim \mathring{Z}_{r-2}^3(T^{\mathrm{A}}) + \dim \mathring{Z}_{r+1}^0(T^{\mathrm{A}}). \end{aligned}$$

This proves one of the equalities stated in (4.9). In the same way, Lemma 4.3 yields $\dim Z_r^1(T^{\mathrm{A}}) = \dim Z_{r-1}^2(T^{\mathrm{A}}) - \dim Z_{r-2}^3(T^{\mathrm{A}}) + \dim Z_{r+1}^0(T^{\mathrm{A}}) - 1$, whenever $r \geq 4$, thus proving the other equality. □

4.2. **Degrees of freedom and commuting projections for the Z spaces.** The degrees of freedom for $Z_{r+1}^0(T^{\mathrm{A}})$ are simply given by (3.9) with r replaced with $r+1$, so we start with those of $Z_r^1(T^{\mathrm{A}})$. We shall show that any $\omega \in Z_r^1(T^{\mathrm{A}})$, $r \geq 4$, is uniquely determined by the following dofs:

$$D^\alpha \omega(a),\ D^\beta \mathrm{curl}\,\omega(a), \qquad |\alpha| \leq 1,\ |\beta| = 1,\ a \in \Delta_0(T), \qquad \text{(80 dofs)} \tag{4.15a}$$

$$\int_e \omega \cdot \kappa\,, \qquad \kappa \in [\mathcal{P}_{r-4}(e)]^3,\ e \in \Delta_1(T), \qquad (18(r-3) \text{ dofs}) \tag{4.15b}$$

$$\int_e (\mathrm{curl}\,\omega) \cdot \kappa\,, \qquad \kappa \in [\mathcal{P}_{r-5}(e)]^3,\ e \in \Delta_1(T), \qquad (18(r-4) \text{ dofs}) \tag{4.15c}$$

$$\int_F (\omega \cdot n^F)\kappa\,, \qquad \kappa \in \mathcal{P}_{r-3}(F),\ F \in \Delta_2(T), \qquad (2(r-2)(r-1) \text{ dofs}) \tag{4.15d}$$

$$\int_F (n^F \times (\omega \times n^F)) \cdot \kappa\,, \qquad \kappa \in D_{r-4}(F),\ F \in \Delta_2(T), \qquad (4(r-2)(r-4) \text{ dofs}) \tag{4.15e}$$

$$\int_F (\mathrm{curl}\,\omega \times n^F) \cdot \kappa\,, \qquad \kappa \in [\mathcal{P}_{r-4}(F)]^2,\ F \in \Delta_2(T), \qquad (4(r-3)(r-2) \text{ dofs}) \tag{4.15f}$$

$$\int_T \omega \cdot \kappa\,, \qquad \kappa \in \mathrm{grad}\,\mathring{Z}_{r+1}^0(T^{\mathrm{A}}), \qquad (\tfrac{2}{3}(r-3)(r-2)(r-1) \text{ dofs}) \tag{4.15g}$$

$$\int_T \mathrm{curl}\,\omega \cdot \kappa\,, \qquad \kappa \in \mathrm{curl}\,\mathring{Z}_r^1(T^{\mathrm{A}}), \qquad (\tfrac{1}{3}(r-3)(r-2)(4r-7) \text{ dofs}). \tag{4.15h}$$

Here (4.15a) counts as 80 dofs, rather than 84, since at each vertex we have the identity $\mathrm{div}\,\mathrm{curl}\,\omega = 0$. In (4.15e), the space $D_{r-4}(F)$ is the Raviart-Thomas space (defined in (3.11)). In (4.15f), we have committed the usual abuse and written that κ is in $[\mathcal{P}_{r-4}(F)]^2$ instead of the (isomorphic) tangent plane of F. The count of dofs in (4.15h) follows from Lemmas 4.2 and 4.4.

For $\omega \in Z^2_{r-1}(T^{\mathrm{A}})$, $r \geq 4$, we define the following dofs:

$$
\begin{aligned}
&(4.16a) \qquad && D^\alpha \omega(a), && |\alpha| \le 1 && a \in \Delta_0(T), && \text{(48 dofs)}\\
&(4.16b) && \int_e \omega \cdot \kappa\,, && \kappa \in [\mathcal{P}_{r-5}(e)]^3 && e \in \Delta_1(T), && (18(r-4) \text{ dofs})\\
&(4.16c) && \int_F \omega \cdot \kappa\,, && \kappa \in [\mathcal{P}_{r-4}(F)]^3 && F \in \Delta_2(T), && (6(r-3)(r-2) \text{ dofs})\\
&(4.16d) && \int_T \omega \cdot \kappa\,, && \kappa \in \operatorname{curl} \mathring{Z}^1_r(T^{\mathrm{A}}) && && (\tfrac{1}{3}(r-3)(r-2)(4r-7) \text{ dofs})\\
&(4.16e) && \int_T (\operatorname{div} \omega)\, \kappa\,, && \kappa \in \operatorname{div} \mathring{Z}^2_{r-1}(T^{\mathrm{A}}) && && (\tfrac{2}{3}(r+1)r(r-1) - 13 \text{ dofs}).
\end{aligned}
$$

The counts in (4.16d) and (4.16e) follow from Lemmas 4.2 and 4.4.

For any $r \geq 4$, a function $\omega \in Z^3_{r-2}(T^{\mathrm{A}})$ is uniquely determined by the following dofs:

$$
\begin{aligned}
&(4.17a) \qquad && \omega(a), && a \in \Delta_0(T) && \text{(4 dofs)}\\
&(4.17b) && \int_T \omega\,, && && \text{(1 dof)}\\
&(4.17c) && \int_T \omega\, \kappa\,, && \kappa \in \mathring{Z}^3_{r-2}(T^{\mathrm{A}}), && (\tfrac{2}{3}(r+1)r(r-1) - 13 \text{ dofs}).
\end{aligned}
$$

The unisolvency of (4.17) is obvious from our definition of $\mathring{Z}^3_{r-2}(T^{\mathrm{A}})$, so we shall now focus on proving the unisolvency of the dofs in (4.16) and (4.15).

Lemma 4.5. *For any $r \geq 4$, the dofs* (4.16) *uniquely determine a function in $Z^2_{r-1}(T^A)$.*

Proof. The total number of dofs in (4.16) is exactly the dimension of the space $Z^2_{r-1}(T^{\mathrm{A}})$. Hence, we only need to show that if $\omega \in Z^2_{r-1}(T^{\mathrm{A}})$ and the dofs in (4.16) vanish, then $\omega \equiv 0$. The dofs in (4.16a) and (4.16b) make ω vanish on all the edges of T. Then the dofs in (4.16c) make ω vanish on all the faces of T. This, together with the zero first derivatives of ω at vertices (due to (4.16a)) shows that $\omega \in \mathring{Z}^2_{r-1}(T^{\mathrm{A}})$. Then (4.16e) implies $\operatorname{div} \omega = 0$. Using the exactness of the sequence (4.5) and the dofs (4.16d) we conclude that $\omega \equiv 0$. ☐

Lemma 4.6. *For any $r \geq 4$, the dofs* (4.15) *uniquely determine a function in $Z^1_r(T^A)$.*

Proof. We first note that $\dim Z^1_r(T^{\mathrm{A}})$ is equal to the total number of dofs in (4.15). To prove unisolvency, consider an $\omega \in Z^1_r(T^{\mathrm{A}})$ for which all the dofs in (4.15) vanish. Using the dofs (4.15a)–(4.15e), the inclusion $Z^1_r(T^{\mathrm{A}}) \subset V^1_r(T^{\mathrm{A}})$, and Remark 3.6, we conclude that ω vanishes on ∂T. From (4.15a) and (4.15c) we have that $\operatorname{curl} \omega$ vanishes on all the edges of T. Then using (4.15f) we conclude that $\operatorname{curl} \omega \times n$ vanishes on ∂T. Also, $\operatorname{curl} \omega \cdot n$ vanishes on ∂T, since we have already established that ω vanishes on ∂T. Thus $\operatorname{curl} \omega$ vanishes on ∂T. Using the vertex dofs again, we conclude that $\omega \in \mathring{Z}^1_r(T^{\mathrm{A}})$. Now, the dofs (4.15h) show that $\operatorname{curl} \omega = 0$ on T. By the exactness property (4.5) and the dofs (4.15g) we conclude that $\omega \equiv 0$. ☐

Theorem 4.7. *Let Π^Z_k denote the canonical interpolant defined using the dofs of $Z^k_{r+1-k}(T^A)$. Then, for $r \geq 5$, the following diagram commutes*

$$\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & C^\infty(T) & \xrightarrow{\text{grad}} & [C^\infty(T)]^3 & \xrightarrow{\text{curl}} & [C^\infty(T)]^3 & \xrightarrow{\text{div}} & C^\infty(T) & \longrightarrow & 0 \\
 & & \downarrow \Pi^Z_0 & & \downarrow \Pi^Z_1 & & \downarrow \Pi^Z_2 & & \downarrow \Pi^Z_3 & & \\
\mathbb{R} & \longrightarrow & Z^0_{r+1}(T^A) & \xrightarrow{\text{grad}} & Z^1_r(T^A) & \xrightarrow{\text{curl}} & Z^2_{r-1}(T^A) & \xrightarrow{\text{div}} & Z^3_{r-2}(T^A) & \longrightarrow & 0.
\end{array}$$

Proof. First, we show that for any $\rho \in C^\infty(T)$, $\operatorname{grad} \Pi^Z_0 \rho = \Pi^Z_1 \operatorname{grad} \rho$. We do this by showing that all dofs of (4.15) vanish when applied to $u = \operatorname{grad} \Pi^Z_0 \rho - \Pi^Z_1 \operatorname{grad} \rho \in Z^1_r(T^{\mathrm{A}})$ and applying Lemma 4.6.

Using the vertex dofs of $Z^0_{r+1}(T^{\mathrm{A}})$—see (3.9a)—and (4.15a), together with $\operatorname{curl} \operatorname{grad} = 0$, we see that the dofs of (4.15a) vanish on u. To show that the dofs of (4.15b) applied to u on edge e, namely $\int_e u \cdot \kappa$, also vanish, we split $\kappa \in \mathcal{P}_{r-4}(e)^3$ into tangential and normal components $\kappa = \kappa_e t_e + \kappa_+ n^+_e + \kappa_- n^-_e$ and proceed. By (3.9) with $r+1$ in place of r,

$$\int_e \operatorname{grad} (\Pi^Z_0 \rho) \cdot n^\pm_e \, \kappa_\pm = \int_e \operatorname{grad} \rho \cdot n^\pm_e \, \kappa_\pm \,, \qquad \text{by (3.9c)},$$

$$\int_e \operatorname{grad} (\Pi^Z_0 \rho - \rho) \cdot t_e \; \kappa_e = -\int_e (\Pi^Z_0 \rho - \rho) \; \partial_{t_e} \kappa_e = 0, \qquad \text{by (3.9a)–(3.9b)}.$$

Together with (4.15b), we conclude that $\int_e u \cdot \kappa = \int_e (\operatorname{grad} \Pi^Z_0 \rho - \Pi^Z_1 \operatorname{grad} \rho) \cdot \kappa = 0$. Proceeding to the dofs of (4.15c) applied to u, for any $\kappa \in [\mathcal{P}_{r-5}(e)]^3$,

$$\int_e (\operatorname{curl} u) \cdot \kappa = -\int_e (\operatorname{curl} \Pi^Z_1 \operatorname{grad} \rho) \cdot \kappa = -\int_e (\operatorname{curl} \operatorname{grad} \rho) \cdot \kappa = 0.$$

Thus all vertex and edge dofs vanish when applied to u.

Next, consider the face and inner dofs. Let $w_F = n^F \times (w \times n^F)$ denote the tangential component of a vector field w on F, let $\operatorname{grad}_F, \operatorname{curl}_F, \operatorname{rot}_F$, and div_F denote the standard surface differential operators on F, and let ν^F denote the outward unit normal on ∂F lying in the tangent plane of F. It is easy to see that the dofs of (4.15d) vanish on u using (3.9e) with $r+1$ in place of r. For the dofs in (4.15e), let $\kappa \in D_{r-4}(F)$ and let ν^F denote the outward unit normal on ∂F lying in the tangent plane of F. By the properties of the Raviart-Thomas space, $\nu^F \cdot \kappa|_e$ is in $\mathcal{P}_{r-5}(e)$. Applying (4.15e) and then integrating by parts,

$$\begin{aligned}
\int_F u_F \cdot \kappa &= \int_F \operatorname{grad}_F (\Pi^Z_0 \rho - \rho) \cdot \kappa \\
&= -\int_F (\Pi^Z_0 \rho - \rho) \cdot \operatorname{div}_F \kappa + \int_{\partial F} (\Pi^Z_0 \rho - \rho) \nu^F \cdot \kappa \,.
\end{aligned}$$

Applying (3.9) with $r+1$ in place of r, the last term above vanishes due to (3.9b) and the penultimate term vanishes due to (3.9d). Hence the dofs of (4.15e) applied to u vanish. The dofs of (4.15f) and (4.15h) applied to u are, of course, zero simply because $\operatorname{curl} \operatorname{grad}$ vanishes. Finally, the dofs of (4.15g) applied to u yield zero due to (3.9f).

Let us proceed to show the second commuting diagram property, namely $\operatorname{curl} \Pi^Z_1 \rho = \Pi^Z_2 \operatorname{curl} \rho$ for all $\rho \in [C^\infty(T)]^3$. Putting $u = \operatorname{curl} \Pi^Z_1 \rho - \Pi^Z_2 \operatorname{curl} \rho \in Z^2_{r-1}(T^{\mathrm{A}})$, we now show that all dofs in (4.16) vanish on u. (Then the result follows from Lemma 4.5.) It is easy to see that the vertex dofs applied to u are zero due to (4.15a) and (4.16a), and that the edge dofs applied to u are zero due to (4.15c) and (4.16b). For the face dofs applied to u, namely $\int_F u\cdot\kappa$, we proceed by splitting κ into its normal component $\kappa_n n^F$ and the remaining tangential component $\kappa_t = n \times (\kappa \times n)$. The latter gives

$$\int_F u \cdot \kappa_t = \int_F u \times n^F \cdot \kappa \times n^F = \int_F (\operatorname{curl} \Pi_1^Z \rho) \times n^F \cdot \kappa \times n^F - \int_F (\Pi_2^Z \operatorname{curl} \rho) \times n^F \cdot \kappa \times n^F.$$

Applying (4.16c) to the last term and (4.15f) to the penultimate term, the result is zero. To show that the face dofs with the normal component are also zero, we use (4.16c) and integrate by parts:

$$\int_F u \cdot \kappa_n n^F = \int_F \operatorname{curl} (\Pi_1^Z \rho - \rho) \cdot n^F \kappa_n = \int_F \operatorname{curl}_F (\Pi_1^Z \rho - \rho)_F\, \kappa_n = \int_F (\Pi_1^Z \rho - \rho)_F \operatorname{rot}_F \kappa_n,$$

where the boundary terms arising from the integration by parts were zeroed out due to (4.15b). The last term above vanishes by (4.15e). Hence all the face dofs $\int_F u\cdot\kappa$ vanish. It is easy to see that the inner dofs in (4.16d) and (4.16e) applied to u also yield zero. Hence $u = 0$.

The final commuting diagram property stated in the theorem, $\operatorname{div} \Pi_2^Z \rho = \Pi_3^Z \operatorname{div} \rho$, for all $\rho \in [C^\infty(T)]^3$, is also proved using similar arguments using the previously established lemmas. □

5. Local elasticity complexes

5.1. **Derived exact sequences.** The two exact sequences of spaces, (3.7) and (4.7), that we have developed in the previous sections can now be put together to deduce an elasticity sequence. We do so by applying Proposition 2.3. To this end, we need connecting operators between the sequences as in (2.4). Let $\mathbb{M}$ denote the space of 3×3 matrices. Define $\Xi : \mathbb{M} \to \mathbb{M}$ by $\Xi M = M' - \operatorname{tr}(M)\mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix and $(\cdot)'$ denotes the transpose. We note that this operator is invertible and

$$\Xi^{-1} M = M' - \frac{1}{2}\operatorname{tr}(M)\mathbb{I}. \tag{5.1}$$

As usual, we let $\operatorname{sym} M = \frac{1}{2}(M + M')$, $\operatorname{skw} M = \frac{1}{2}(M - M')$, and put $\mathbb{K} = \operatorname{skw}(\mathbb{M})$ and $\mathbb{S} = \operatorname{sym}(\mathbb{M})$. We define $\operatorname{mskw} : \mathbb{V} \to \mathbb{K}$ by

$$\operatorname{mskw} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

and set $\operatorname{vskw} = \operatorname{mskw}^{-1} \circ \operatorname{skw}$. It is easy to check that the following two identities hold:

$$\operatorname{div} \Xi = 2\operatorname{vskw}\ \operatorname{curl}\,, \tag{5.2}$$

$$\Xi\operatorname{grad} = -\operatorname{curl}\ \operatorname{mskw}. \tag{5.3}$$

These identities imply that the following diagram commutes:

$$\begin{array}{ccccccc} Z^0_{r+1}(T^{\mathrm{A}}) \otimes \mathbb{V} & \xrightarrow{\operatorname{grad}} & Z^1_r(T^{\mathrm{A}}) \otimes \mathbb{V} & \xrightarrow{\operatorname{curl}} & Z^2_{r-1}(T^{\mathrm{A}}) \otimes \mathbb{V} & \xrightarrow{\operatorname{div}} & Z^3_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V} \\ & \nearrow^{-\operatorname{mskw}} & & \nearrow^{\Xi} & & \nearrow^{2\operatorname{vskw}} & \\ V^0_r(T^{\mathrm{A}}) \otimes \mathbb{V} & \xrightarrow{\operatorname{grad}} & V^1_{r-1}(T^{\mathrm{A}}) \otimes \mathbb{V} & \xrightarrow{\operatorname{curl}} & V^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V} & \xrightarrow{\operatorname{div}} & V^3_{r-3}(T^{\mathrm{A}}) \otimes \mathbb{V}. \end{array} \tag{5.4}$$

Lemma 5.1. *Both* $\operatorname{vskw} : \mathring{V}^2_{r-2}(T^A) \otimes \mathbb{V} \to \hat{Z}^3_{r-2}(T^A) \otimes \mathbb{V}$ *and* $\operatorname{vskw} : V^2_{r-2}(T^A) \otimes \mathbb{V} \to Z^3_{r-2}(T^A) \otimes \mathbb{V}$ *are surjective operators.*

Proof. Given any $z \in \hat{\mathring{Z}}^3_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$, we must find some $v \in \mathring{V}^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$ such that $2\,\mathrm{vskw}\, v = z$. First, we take any $\tilde{v} \in \mathring{V}^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$ satisfying $\int_T 2\,\mathrm{vskw}\, \tilde{v} = \int_T z$. Then consider $z - 2\,\mathrm{vskw}\, \tilde{v} \in \mathring{Z}^3_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$. Due to the exactness of the $\mathring{Z}$ sequence (4.5), there exists $w \in \mathring{Z}^2_{r-1}(T^{\mathrm{A}}) \otimes \mathbb{V}$ such that $\mathrm{div}\, w = z - 2\,\mathrm{vskw}\, \tilde{v}$. Then $v = \mathrm{curl}\, \Xi^{-1} w + \tilde{v}$ is in $\mathring{V}^2_{r-1} \otimes \mathbb{V}$ and $2\,\mathrm{vskw}(v) = z$ by (5.2). The proof of the other surjectivity is similar and easier. □

Theorem 5.2. *The sequence*

$$\begin{bmatrix} Z^0_{r+1}(T^A)\otimes\mathbb{V} \\ V^0_r(T^A)\otimes\mathbb{V} \end{bmatrix} \xrightarrow{[\mathrm{grad},\, -\mathrm{mskw}]} Z^1_r(T^A)\otimes\mathbb{V} \xrightarrow{\mathrm{curl}\,\Xi^{-1}\mathrm{curl}} V^2_{r-2}(T^A)\otimes\mathbb{V} \xrightarrow{\begin{bmatrix} 2\mathrm{vskw} \\ \mathrm{div} \end{bmatrix}} \begin{bmatrix} Z^3_{r-2}(T^A)\otimes\mathbb{V} \\ V^3_{r-3}(T^A)\otimes\mathbb{V} \end{bmatrix}$$

is exact for $r \geq 4$. Moreover, the last operator is surjective.

Proof. Identities (5.1) and (4.2) imply that $\Xi : V^1_{r-1}(T^{\mathrm{A}}) \otimes \mathbb{V} \to Z^2_{r-1}(T^{\mathrm{A}}) \otimes \mathbb{V}$ is a bijection. The exactness of the top and bottom sequences in (5.4) were established in Lemmas 4.3 and 3.5, for $r \geq 4$ and $r \geq 3$, respectively. Hence the proof of the stated exactness reduces to an application of Proposition 2.3. The statement on surjectivity of the last map also follows from Proposition 2.3 after using Lemma 5.1. □

Note that the kernel of $[\mathrm{grad}\,, -\,\mathrm{mskw}]$ is $\{(a - b \wedge x, -b) : a, b \in \mathbb{V}\}$. Indeed, let $(u, v) \in \ker[\mathrm{grad}\,, -\,\mathrm{mskw}]$. Then $\mathrm{grad}\, u - \mathrm{mskw}\, v = 0$ implies that $\varepsilon(u) := \mathrm{sym}\,\mathrm{grad}\, u = 0$. Therefore $u|_T = a_T + b_T \wedge x$ is an infinitesimal rigid body motion for some constant vectors a_T and b_T on each $T \in T^{\mathrm{A}}$. For any face $F = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in T^{\mathrm{A}}$, if we choose the origin of the Euler vector field x on F, then $a_{T_1} = a_{T_2}$, $b_{T_1} = b_{T_2}$ due to the C^0 continuity of u. This implies that $u|_{T_1}$ and $u|_{T_2}$ are the restriction of the same affine vector field to T_1 and T_2, respectively. Therefore globally $u = a + b \wedge x$, for some constant vectors $a, b \in \mathbb{V}$. Then we obtain $v = -b$. The conclusion also follows from a general algebraic construction (c.f., [10]). With this algebraic machinery, one can show that the cohomology of the elasticity complex of Theorem 5.2 is isomorphic to the product of the cohomology of the Z-complex and the V-complex. In particular, one obtains the kernel of $[\mathrm{grad}\,, -\,\mathrm{mskw}]$ to be isomorphic to $\mathbb{V} \times \mathbb{V}$ since the kernel of $\mathrm{grad} : Z^0_{r+1}(T^{\mathrm{A}}) \otimes \mathbb{V} \to Z^1_r(T^{\mathrm{A}}) \otimes \mathbb{V}$ and the kernel of $\mathrm{grad} : V^0_r(T^{\mathrm{A}}) \otimes \mathbb{V} \to V^1_{r-1}(T^{\mathrm{A}}) \otimes \mathbb{V}$ are both equal to $\mathbb{V}$.

An analogous result holds for the spaces with boundary conditions. To see this, first note that instead of (5.4), we now have the following commuting diagram:

$$\begin{array}{ccccccccc}
\mathring{Z}^0_{r+1}(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\mathrm{grad}} & \mathring{Z}^1_r(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\mathrm{curl}} & \mathring{Z}^2_{r-1}(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\mathrm{div}} & \hat{\mathring{Z}}^3_{r-2}(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\int} & \mathbb{V} \\
& \overset{-\mathrm{mskw}}{\nearrow} & & \overset{\Xi}{\nearrow} & & \overset{2\mathrm{vskw}}{\nearrow} & & & \\
\mathring{V}^0_r(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\mathrm{grad}} & \mathring{V}^1_{r-1}(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\mathrm{curl}} & \mathring{V}^2_{r-2}(T^{\mathrm{A}})\otimes\mathbb{V} & \xrightarrow{\mathrm{div}} & \mathring{V}^3_{r-3}(T^{\mathrm{A}})\otimes\mathbb{V}. & &
\end{array}$$

Here we have modified the $\mathring{Z}$ sequence slightly from (4.2). We have used the space $\hat{\mathring{Z}}^3_r(T^{\mathrm{A}})$ defined in (4.12) since $\mathrm{vskw}(\mathring{V}^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V})$ is generally contained only in $\hat{\mathring{Z}}^3_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$, not $\mathring{Z}^3_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$. The last operator in the top sequence is just $z \mapsto \int_T z\, dx$ and the sequence is exact even with this modification. Hence the proof of the next result is completely analogous to that of Theorem 5.2. (Note that surjectivity of the last map is not claimed in the theorem.)

Theorem 5.3. *The following sequence is exact for $r \geq 4$:*

$$\begin{bmatrix} \mathring{Z}^0_{r+1}(T^A)\otimes\mathbb{V} \\ \mathring{V}^0_r(T^A)\otimes\mathbb{V} \end{bmatrix} \xrightarrow{[\mathrm{grad},\, -\mathrm{mskw}]} \mathring{Z}^1_r(T^A)\otimes\mathbb{V} \xrightarrow{\mathrm{curl}\,\Xi^{-1}\mathrm{curl}} \mathring{V}^2_{r-2}(T^A)\otimes\mathbb{V} \xrightarrow{\begin{bmatrix} 2\mathrm{vskw} \\ \mathrm{div} \end{bmatrix}} \begin{bmatrix} \hat{\mathring{Z}}^3_{r-2}(T^A)\otimes\mathbb{V} \\ \mathring{V}^3_{r-3}(T^A)\otimes\mathbb{V} \end{bmatrix}.$$

5.2. **Discrete elasticity complex.** We now proceed to define an elasticity complex useful for designing mixed methods with strongly imposed symmetry. To do this, we first note that $\mathrm{curl}\,\Xi^{-1}\mathrm{curl}$ maps skew symmetric matrices to 0. Indeed,

$$\mathrm{curl}\,\Xi^{-1}\mathrm{curl}\,\mathrm{mskw} = -\mathrm{curl}\,\Xi^{-1}\,\Xi\mathrm{grad} = -\mathrm{curl}\,\mathrm{grad} = 0. \tag{5.5}$$

Also, note that $\mathrm{curl}\,\Xi^{-1}\mathrm{curl}$ maps all matrix fields to symmetric matrix fields because

$$\mathrm{vskw}\,\mathrm{curl}\,\Xi^{-1}\mathrm{curl} = \frac{1}{2}\mathrm{div}\,\Xi\,\Xi^{-1}\mathrm{curl} = \frac{1}{2}\mathrm{div}\,\mathrm{curl} = 0. \tag{5.6}$$

Our elasticity complexes will be formed using the following spaces:

$U^0_{r+1}(T^{\mathrm{A}}) = Z^0_{r+1}(T^{\mathrm{A}})\otimes\mathbb{V}$, $\quad \mathring{U}^0_{r+1}(T^{\mathrm{A}}) = \mathring{Z}^0_{r+1}(T^{\mathrm{A}})\otimes\mathbb{V}$,

$U^1_r(T^{\mathrm{A}}) = \{\mathrm{sym}(u) : u \in Z^1_r(T^{\mathrm{A}})\otimes\mathbb{V}\}$, $\quad \mathring{U}^1_r(T^{\mathrm{A}}) = \{\mathrm{sym}(u) : u \in \mathring{Z}^1_r(T^{\mathrm{A}})\otimes\mathbb{V}\}$,

$U^2_{r-2}(T^{\mathrm{A}}) = \{\omega \in V^2_{r-2}(T^{\mathrm{A}})\otimes\mathbb{V} : \mathrm{skw}\ \omega = 0\}$, $\quad \mathring{U}^2_{r-2}(T^{\mathrm{A}}) = \{\omega \in \mathring{V}^2_{r-2}(T^{\mathrm{A}})\otimes\mathbb{V} : \mathrm{skw}\ \omega = 0\}$,

$U^3_{r-3}(T^{\mathrm{A}}) = V^3_{r-3}(T^{\mathrm{A}})\otimes\mathbb{V}$, $\quad \mathring{U}^3_{r-3}(T^{\mathrm{A}}) = \{u \in V^3_{r-3}(T^{\mathrm{A}})\otimes\mathbb{V} : u \perp \mathcal{R}\}$.

Recall that $\mathrm{inc}\,u = \mathrm{curl}\,(\mathrm{curl}\,u)'$ for a symmetric matrix field u. In fact, when u is symmetric, $(\mathrm{curl}\,u)' = \Xi^{-1}\mathrm{curl}\,u$ as

$$\mathrm{tr}(\mathrm{curl}\,u) = 0, \tag{5.7}$$

so $\mathrm{inc}\,u = \mathrm{curl}\,\Xi^{-1}\mathrm{curl}\,u$. The elasticity complexes with the newly defined U spaces are as follows.

$$0 \longrightarrow \mathcal{R} \xrightarrow{\subset} U^0_{r+1}(T^{\mathrm{A}}) \xrightarrow{\varepsilon} U^1_r(T^{\mathrm{A}}) \xrightarrow{\mathrm{inc}} U^2_{r-2}(T^{\mathrm{A}}) \xrightarrow{\mathrm{div}} U^3_{r-3}(T^{\mathrm{A}}) \longrightarrow 0, \tag{5.8}$$

Here is the analogue with boundary conditions:

$$0 \longrightarrow \mathring{U}^0_{r+1}(T^{\mathrm{A}}) \xrightarrow{\varepsilon} \mathring{U}^1_r(T^{\mathrm{A}}) \xrightarrow{\mathrm{inc}} \mathring{U}^2_{r-2}(T^{\mathrm{A}}) \xrightarrow{\mathrm{div}} \mathring{U}^3_{r-3}(T^{\mathrm{A}}) \longrightarrow 0. \tag{5.9}$$

Theorem 5.4. *The sequences* (5.8) *and* (5.9) *are exact sequences for $r \geq 4$.*

Proof. First we must show that (5.8) is a complex. By (5.5), $\mathrm{curl}\,\Xi^{-1}\mathrm{curl} \circ \varepsilon = \mathrm{curl}\,\Xi^{-1}\mathrm{curl} \circ \mathrm{grad} = 0$. Also, it's obvious that $\mathrm{div} \circ \mathrm{curl}\,\Xi^{-1}\mathrm{curl} = 0$. Hence it suffices to verify that the operators map into the right spaces. Let $w \in U^0_{r+1}(T^{\mathrm{A}})$. Then we have $\mathrm{grad}\,w \in Z^1_r(T^{\mathrm{A}}) \otimes \mathbb{V}$ and therefore $\varepsilon(w) \in U^1_r(T^{\mathrm{A}})$. Next, consider a $u \in U^1_r(T^{\mathrm{A}})$. Then $\omega = \mathrm{curl}\,\Xi^{-1}\mathrm{curl}\,u$ has zero skew symmetric part due to (5.6), so is in $U^2_{r-2}(T^{\mathrm{A}})$. Finally, if $v \in U^2_{r-2}(T^{\mathrm{A}}) \subseteq V^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$, then $\mathrm{div}\,v \in V^3_{r-3}(T^{\mathrm{A}}) \otimes \mathbb{V} = U^3_{r-3}(T^{\mathrm{A}})$.

Now we prove exactness. Let $u \in U^3_{r-3}(T^{\mathrm{A}})$. We use the surjectivity of the last map in the exact sequence of Theorem 5.2. Accordingly, for $(0, u) \in [Z^3_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}] \times [V^3_{r-3}(T^{\mathrm{A}}) \otimes \mathbb{V}]$, there is a $w \in V^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$ such that $\mathrm{div}\,w = u$ and $2\mathrm{vskw}\ w = 0$. Thus, $w \in U_{r-2}(T^{\mathrm{A}})$ and $\mathrm{div}\,w = u$, establishing the surjectivity of div in (5.8).

Next, let $u \in U^2_{r-2}(T^{\mathrm{A}})$ with $\mathrm{div}\,u = 0$. Then, u is in the kernel of the last operator in the exact sequence of Theorem 5.2. Hence, there is a $v \in Z^1_r(T^{\mathrm{A}})\otimes\mathbb{V}$ such that $\mathrm{curl}\,\Xi^{-1}\mathrm{curl}\,v = u$.

But, by (5.5), $\operatorname{curl}\Xi^{-1}\operatorname{curl}(\operatorname{sym} v) = \operatorname{curl}\Xi^{-1}\operatorname{curl} v = u$. Thus we have found a function $w = \operatorname{sym} v$ in $U_r^1(T^{\mathrm{A}})$ satisfying $\operatorname{curl}\Xi^{-1}\operatorname{curl} w = u$.

Finally, let $u \in U_r^1(T^{\mathrm{A}})$ with $\operatorname{curl}\Xi^{-1}\operatorname{curl} u = 0$. Then $u = \operatorname{sym}(z)$ for some $z \in Z_r^1(T^{\mathrm{A}}) \otimes \mathbb{V}$ and $\operatorname{curl}\Xi^{-1}\operatorname{curl} z = 0$ by (5.5). By Theorem 5.2, $z = \operatorname{grad} w - \operatorname{mskw} s$ for some $w \in Z_{r+1}^0(T^{\mathrm{A}}) \otimes \mathbb{V} = U_{r+1}^0(T^{\mathrm{A}})$ and $s \in V_r^0(T^{\mathrm{A}}) \otimes \mathbb{V}$. Then $u = \operatorname{sym} z = \varepsilon(w) - \operatorname{sym}(\operatorname{mskw} s) = \varepsilon(w)$.

The proof of exactness of (5.9) proceeds similarly using Theorem 5.3 in place of Theorem 5.2, except where it concerns the surjectivity of the last map: to prove that div in (5.9) is onto, consider a $u \in \mathring{U}_{r-3}^3(T^{\mathrm{A}})$. By the exactness of the sequence (3.8) we have a $v \in \mathring{V}_{r-2}^2(T^{\mathrm{A}}) \times \mathbb{V}$ such that $\operatorname{div} v = u$. For any constant vector $c \in \mathbb{R}^3$, since $u \perp \mathcal{R}$,

$$\int_T \operatorname{vskw}\, v \cdot c = \int_T v : \operatorname{mskw}\, c = \int_T v : \operatorname{grad}(c \times x) = -\int_T \operatorname{div} v \cdot (c \times x) = -\int_T u \cdot (c \times x) = 0.$$

Therefore, $2 \operatorname{vskw}\, v \in \mathring{Z}_{r-2}^3(T^{\mathrm{A}})$. By the exactness of the sequence (4.5), there exists an $m \in \mathring{Z}_{r-1}^2(T^{\mathrm{A}})$ such that $\operatorname{div} m = 2 \operatorname{vskw}\, v$. Hence, by (5.2) we get $2 \operatorname{vskw} \operatorname{curl}(\Xi^{-1} m) = 2 \operatorname{vskw}\, v$ and so $\operatorname{div} w = u$ where $w = v - \operatorname{curl}(\Xi^{-1} m) \in \mathring{U}_{r-2}^2(T^{\mathrm{A}})$. □

Lemma 5.5. *When $r \geq 4$,*

$$\dim U_{r+1}^0(T^A) = 2r^3 + 16r + 12, \qquad \dim \mathring{U}_{r+1}^0(T^A) = 2(r-3)(r-2)(r-1), \tag{5.10}$$

$$\dim U_r^1(T^A) = 4r^3 - 3r^2 + 17r - 6, \qquad \dim \mathring{U}_r^1(T^A) = 4r^3 - 21r^2 + 29r - 6, \tag{5.11}$$

$$\dim U_{r-2}^2(T^A) = 4r^3 - 9r^2 + 5r - 12, \qquad \dim \mathring{U}_{r-2}^2(T^A) = r(r-1)(4r-11), \tag{5.12}$$

$$\dim U_{r-3}^3(T^A) = 2r(r-1)(r-2), \qquad \dim \mathring{U}_{r-3}^3(T^A) = 2r^3 - 6r^2 + 4r - 6. \tag{5.13}$$

Proof. By Lemma 5.1,

$$\dim U_{r-2}^2(T^{\mathrm{A}}) = \dim V_{r-2}^2(T^{\mathrm{A}}) \otimes \mathbb{V} - \dim Z_{r-2}^3(T^{\mathrm{A}}) \otimes \mathbb{V} = 4r^3 - 9r^2 + 5r - 12,$$
$$\dim \mathring{U}_{r-2}^2(T^{\mathrm{A}}) = \dim \mathring{V}_{r-2}^2(T^{\mathrm{A}}) \otimes \mathbb{V} - \dim \mathring{Z}_{r-2}^3(T^{\mathrm{A}}) \otimes \mathbb{V} = r(r-1)(4r-11).$$

The dimensions of the spaces with form degrees 0 and 3 easily follow from the previously established dimensions. Finally, $\dim U_r^1(T^{\mathrm{A}})$ and $\dim \mathring{U}_r^1(T^{\mathrm{A}})$ are computed using the exactness results of Theorem 5.4. □

5.3. **An $H^1(\mathrm{inc})$-conforming finite element.** The next result gives more insight into the structure of $U_r^1(T^{\mathrm{A}})$, and in particular, shows that $U_r^1(T^{\mathrm{A}})$ is a conforming subspace of $H^1(\mathrm{inc}, T) = \{s \in H^1(T, \mathbb{S}) : \operatorname{inc} s \in L^2(T, \mathbb{S})\}$ on the Alfeld split. After proving it, we shall develop dofs that are designed to help enforce global conformity in $H^1(\mathrm{inc})$. Let $\mathcal{P}_r(T^{\mathrm{A}}; \mathbb{S})$ denote the space of symmetric matrices whose entries are polynomials of degree r on each tetrahedron of the Alfeld split T^{A}, and let $H^1(\Omega; \mathbb{S})$ denote the space of symmetric matrix fields with each entry in $H^1(\Omega)$.

Theorem 5.6. *We have the following characterizations of* $U_r^1(T^A)$ *and* $\mathring{U}_r^1(T^A)$.

$$U_r^1(T^A) = \{u \in H^1(T;\mathbb{S}) : u \in \mathcal{P}_r(T^A;\mathbb{S}),\ (\operatorname{curl} u)' \in W_{r-1}^1(T^A)\otimes\mathbb{V},$$
$$u \text{ is } C^1 \text{ at vertices of } T \text{ and } \operatorname{inc} u \text{ is } C^0 \text{ at vertices of } T\}.$$
$$\mathring{U}_r^1(T^A) = \{u \in \mathring{H}^1(T;\mathbb{S}) : u \in \mathcal{P}_r(T^A;\mathbb{S}),\ (\operatorname{curl} u)' \in \mathring{W}_{r-1}^1(T^A)\otimes\mathbb{V},\ \text{all first order}$$
$$\text{derivatives of } u \text{ and } \operatorname{inc} u \text{ vanishes at the vertices of } T\}.$$

Proof. Let $M_r^1(T^{\mathrm{A}})$ denote the space on the right hand side of the first equality. We claim that

$$U_r^1(T^{\mathrm{A}}) \subseteq M_r^1(T^{\mathrm{A}}). \tag{5.14}$$

Indeed, if $u \in U_r^1(T^{\mathrm{A}})$, then $u = \operatorname{sym} z$ for some $z \in Z_r^1(T^{\mathrm{A}})\otimes\mathbb{V}$, so by (5.3),

$$\Xi^{-1}\operatorname{curl} z = \Xi^{-1}\operatorname{curl} u + \Xi^{-1}\operatorname{curl}\operatorname{skw} z = \Xi^{-1}\operatorname{curl} u - \operatorname{grad}\operatorname{vskw} z.$$

Since the last term is in $W_{r-1}^1(T^{\mathrm{A}})\otimes\mathbb{V}$ and $\Xi^{-1}\operatorname{curl} z$ is in $L_{r-1}^1(T^{\mathrm{A}})\otimes\mathbb{V}$, we conclude that $\Xi^{-1}\operatorname{curl} u$ is in $W_{r-1}^1(T^{\mathrm{A}})\otimes\mathbb{V}$. By Proposition 2.1(3), z is C^1 at the vertices, hence so is $u = \operatorname{sym} z$. Moreover, since $\operatorname{curl} z$ is C^1 at the vertices, by (5.5), $\operatorname{curl}\Xi^{-1}\operatorname{curl} u = \operatorname{curl}\Xi^{-1}\operatorname{curl} z$ is C^0 at the vertices. This proves (5.14).

To prove the reverse inclusion, let $m \in M_r^1(T^{\mathrm{A}})$. Put $\sigma = \operatorname{curl}\Xi^{-1}\operatorname{curl} m$. By the definition of $M_r^1(T^{\mathrm{A}})$, we know that σ is C^0 at the vertices, and moreover, $\Xi^{-1}\operatorname{curl} m \in W_{r-1}^1(T^{\mathrm{A}})\otimes\mathbb{V}$ so that σ is in $W_{r-2}^2(T^{\mathrm{A}})\otimes\mathbb{V}$. Hence σ is in $V_{r-2}^2(T^{\mathrm{A}})\otimes\mathbb{V}$. In fact, σ is in the kernel of the last operator in the exact sequence of Theorem 5.2 since $\operatorname{div}\sigma = 0$ and $\operatorname{vskw}(\sigma) = 0$ due to (5.6). Hence there is a $z \in Z_r^1(T^{\mathrm{A}})\otimes\mathbb{V}$ such that $\sigma = \operatorname{curl}\Xi^{-1}\operatorname{curl} z$.

Now consider $q = \Xi^{-1}\operatorname{curl}(m - z)$. Clearly, $\operatorname{curl} q = 0$. In addition, the definitions of $M_r^1(T^{\mathrm{A}})$ and $Z_r^1(T^{\mathrm{A}})$ imply that q is in $W_{r-1}^1(T^{\mathrm{A}})\otimes\mathbb{V}$. Hence the exactness of the W-sequence yields a v in $W_r^0(T^{\mathrm{A}})\otimes\mathbb{V}$ such that $\operatorname{grad} v = q$. By Proposition 2.1(3), z is C^1 at the vertices, so q is C^0 at the vertices, which in turn implies that v is C^1 at the vertices of T.

To finish the proof, put $\theta = m + \operatorname{mskw}(v)$. Then θ is C^1 at the vertices of T, and by (5.3),

$$\operatorname{curl}\theta = \operatorname{curl} m + \operatorname{curl}\operatorname{mskw} v = \operatorname{curl} m - \Xi\operatorname{grad} v = \operatorname{curl} z.$$

Hence θ is in $Z_r^1(T^{\mathrm{A}})\otimes\mathbb{V}$. Since $m = \operatorname{sym}(\theta)$, we conclude that $m \in U_r^1(T^{\mathrm{A}})$.

The proof of the characterization of $\mathring{U}_r^1(T^{\mathrm{A}})$ is similar. □

For further study of the complex, we collect some identities in the next lemma, several of which involve surface operators we now discuss. Let $F \in \Delta_2(T)$ and let n be its unit normal vector pointing out of T. Fix two tangent vectors t_1, t_2 in $n^\perp$, such that the ordered set $(b_1, b_2, b_3) = (t_1, t_2, n)$ is a orthonormal right-handed basis for $\mathbb{R}^3$. Any matrix field $u : T \to \mathbb{R}^{3\times3}$ can be written as $\sum_{i,j=1}^3 u_{ij} b_i b_j'$ with scalar components $u_{ij} : T \to \mathbb{R}$. Let $u_{nn} = n'un$ and $\operatorname{tr}_F u = \sum_{i=1}^2 t_i' u t_i$. With $s = t_1, t_2, n$, or any linear combination thereof, let

$$u_{FF} = \sum_{i,j=1}^2 u_{ij} t_i t_j', \qquad u_{Fs} = \sum_{i=1}^2 (s'ut_i)t_i', \qquad u_{sF} = \sum_{i=1}^2 (t_i'us)t_i, \tag{5.15}$$

Equivalently, $u_{FF} = QuQ$, $u_{Fs} = s'uQ$, and $u_{sF} = Qus$, where $P = nn'$ and $Q = I - P$. Next, considering scalar-valued (component) functions ϕ, w_i, q_i and u_{ij}, we rewrite the standard surface operators we have used before (in the proof of Theorem 4.7) on the left, while defining additional operations needed on the right using the left definitions:

$$\operatorname{grad}_F \phi = (\partial_{t_1}\phi)t_1 + (\partial_{t_2}\phi)t_2, \qquad \operatorname{grad}_F (w_1 t_1 + w_2 t_2) = t_1(\operatorname{grad}_F w_1)' + t_2(\operatorname{grad}_F w_2)',$$

$$\operatorname{rot}_F \phi = (\partial_{t_2}\phi)t_1 - (\partial_{t_1}\phi)t_2, \qquad \operatorname{rot}_F (q_1 t_1' + q_2 t_2') = t_1(\operatorname{rot}_F q_1)' + t_2(\operatorname{rot}_F q_2)',$$

$$\operatorname{curl}_F (w_1 t_1 + w_2 t_2) = \partial_{t_1} w_2 - \partial_{t_2} w_1, \qquad \operatorname{curl}_F u_{FF} = t_1' \operatorname{curl}_F (u_{Ft_1})' + t_2' \operatorname{curl}_F (u_{Ft_2})'.$$

For vector functions v, let $v_F = Qv = n \times (v \times n)$. It is easy to see that

$$n \cdot \operatorname{curl} v = \operatorname{curl}_F v_F, \qquad (\operatorname{grad} v)_{FF} = \operatorname{grad}_F v_F. \qquad n \times \operatorname{rot}_F \phi = \operatorname{grad}_F \phi. \tag{5.16}$$

Lemma 5.7. *For a (smooth enough) matrix-valued function u,*

$$s' (\operatorname{curl} u)\, n = \operatorname{curl}_F (u_{Fs})', \text{ for any } s \in \mathbb{R}^3, \tag{5.17a}$$

$$\big[(\operatorname{curl} u)'\big]_{Fn} = \operatorname{curl}_F u_{FF}. \tag{5.17b}$$

If in addition u is symmetric, then

$$(\operatorname{inc} u)_{nn} = \operatorname{curl}_F (\operatorname{curl}_F u_{FF})', \tag{5.17c}$$

$$(\operatorname{inc} u)_{Fn} = \operatorname{curl}_F \big[(\operatorname{curl} u)'\big]_{FF}, \tag{5.17d}$$

$$\operatorname{tr}_F \operatorname{curl} u = -\operatorname{curl}_F (u_{Fn})'. \tag{5.17e}$$

For a (smooth enough) vector-valued function v,

$$2(\operatorname{curl} \varepsilon(v))' = \operatorname{grad} \operatorname{curl} v, \tag{5.17f}$$

$$2\big[(\operatorname{curl} \varepsilon(v))'\big]_{FF} = \operatorname{grad}_F (\operatorname{curl} v)_F, \tag{5.17g}$$

$$\operatorname{curl} v = n(\operatorname{curl}_F v_F) + \operatorname{rot}_F (v \cdot n) + n \times \partial_n v, \tag{5.17h}$$

$$2[\varepsilon(v)]_{nF} = 2[\varepsilon(v)_{Fn}]' = \operatorname{grad}_F (v \cdot n) + \partial_n v_F, \tag{5.17i}$$

$$\operatorname{tr}_F (\operatorname{rot}_F v_F') = \operatorname{curl}_F v_F. \tag{5.17j}$$

Proof. The first identity (5.17a) follows from (5.16). The second follows from the first:

$$\begin{aligned}
\big[(\operatorname{curl} u)'\big]_{Fn} &= \sum_{i=1}^{2} \big[n'(\operatorname{curl} u)' t_i\big]\; t_i' && \text{by (5.15)}\\
&= \sum_{i=1}^{2} t_i' \operatorname{curl}_F (u_{Ft_i})' && \text{by (5.17a)},
\end{aligned}$$

which equals $\operatorname{curl}_F u_{FF}$ per our definition. To prove (5.17c),

$$\begin{aligned}
(\operatorname{inc} u)_{nn} = n'(\operatorname{curl}(\operatorname{curl} u)')n &= \operatorname{curl}_F \Big(\big[(\operatorname{curl} u)'\big]_{Fn}\Big)' && \text{by (5.17a)}\\
&= \operatorname{curl}_F (\operatorname{curl}_F u_{FF})' && \text{by (5.17b)}.
\end{aligned}$$

To prove (5.17d), we use (5.17b) to obtain $[(\operatorname{inc} u)']_{Fn} = \operatorname{curl}_F [(\operatorname{curl} u)']_{FF}$ and use the symmetry of $\operatorname{inc} u$. To prove (5.17e), we use (5.7), noting that trace is invariant under a basis change, to obtain $0 = \operatorname{tr}(\operatorname{curl} u) = \operatorname{tr}_F (\operatorname{curl} u) + n'(\operatorname{curl} u)n$. The last term, by (5.17a), equals $\operatorname{curl}_F (u_{Fn})'$, thus proving (5.17e). To prove (5.17f),

$$\begin{aligned}
2\operatorname{curl} \varepsilon(v) &= -2\operatorname{curl}(\operatorname{skw} \operatorname{grad} v) = -2\operatorname{curl}(\operatorname{mskw}\ \operatorname{vskw}\ \operatorname{grad} v)\\
&= \Xi\operatorname{grad}(2\operatorname{vskw}\ \operatorname{grad} v) = \Xi\operatorname{grad}(\operatorname{curl} v)
\end{aligned}$$

which equals the right hand side of (5.17f) since $\operatorname{tr}(\operatorname{grad} \operatorname{curl} v) = 0$. Equation (5.17g) follows from (5.17f) and (5.16). Proving the remaining identities involves mere definition chasing. □

The identities in the next lemma are obtained by integration by parts on a face. Stokes theorem gives, for $q = q_1 t_1' + q_2 t_2'$,

$$\int_F \operatorname{curl}_F u_{FF} \cdot q = \int_F u_{FF} : \operatorname{rot}_F q + \int_{\partial F} u_{FF} t \cdot q' . \tag{5.18}$$

Here and in the sequel, t denotes a unit tangent vector along ∂F (not to be confused with t_i) oriented with respect to n to satisfy the right hand rule. We use "$\cdot$" for inner products between row-vectors (as in the first term of (5.18)) as well between column vectors (as in the last term of (5.18)), and we use ":" to denote the Frobenius inner product between matrices. Using $\mathbb{I}_F = t_1 t_1' + t_2 t_2'$, we define $\operatorname{dev}_F u_{FF} = u_{FF} - \mathbb{I}_F (\operatorname{tr}_F u_{FF})/2$, which is used below.

Lemma 5.8. *Let u be a symmetric matrix-valued function and let q_i and ϕ be scalar-valued functions. For smooth enough u, $q = q_1 t_1' + q_2 t_2'$, and ϕ, we have*

$$\begin{aligned} \int_F (\operatorname{inc} u)_{nn} \phi = & \int_F u_{FF} : \operatorname{rot}_F (\operatorname{rot}_F \phi)' \\ & + \int_{\partial F} (\operatorname{curl}_F u_{FF}) t \, \phi \, ds + \int_{\partial F} u_{FF} t \cdot (\operatorname{rot}_F \phi)' , \end{aligned} \tag{5.19}$$

$$\begin{aligned} \int_F (\operatorname{inc} u)_{Fn} \cdot q = & \int_F \left[(\operatorname{curl} u)' \right]_{FF} : \operatorname{dev}_F \operatorname{rot}_F q - \frac{1}{2} \int_F u_{nF} \cdot \operatorname{rot}_F \operatorname{curl}_F q' \\ & + \int_{\partial F} \left[(\operatorname{curl} u)' \right]_{FF} t \cdot q' - \frac{1}{2} \int_{\partial F} (u_{nF} \cdot t) \operatorname{curl}_F q' . \end{aligned} \tag{5.20}$$

Proof. By Stokes theorem and Lemma 5.7's (5.17c),

$$\int_F (\operatorname{inc} u)_{nn} \phi = \int_{\partial F} (\operatorname{curl}_F u_{FF})' \cdot t \phi + \int_F (\operatorname{curl}_F u_{FF})' \cdot \operatorname{rot}_F \phi .$$

In the last term, writing $(\operatorname{curl}_F u_{FF})' \cdot \operatorname{rot}_F \phi$ as $\operatorname{curl}_F u_{FF} \cdot (\operatorname{rot}_F \phi)'$ and integrating by parts again using (5.18), we prove (5.19). To prove (5.20), we start by using (5.17d) and (5.18):

$$\int_F (\operatorname{inc} u)_{Fn} \cdot q = \int_{\partial F} \left[(\operatorname{curl} u)' \right]_{FF} t \cdot q' + \int_F \left[(\operatorname{curl} u)' \right]_{FF} : \operatorname{rot}_F q . \tag{5.21}$$

Note that $[(\operatorname{curl} u)']_{FF} : \operatorname{rot}_F q = [(\operatorname{curl} u)']_{FF} : \operatorname{dev}_F \operatorname{rot}_F q + \frac{1}{2} (\operatorname{tr}_F [(\operatorname{curl} u)']_{FF})(\operatorname{tr}_F \operatorname{rot}_F q)$. Also, by (5.17e), $\operatorname{tr}_F [(\operatorname{curl} u)']_{FF} = \operatorname{tr}_F (\operatorname{curl} u) = -\operatorname{curl}_F (u_{Fn})'$, and by (5.17j), $\operatorname{tr}_F \operatorname{rot}_F q = \operatorname{curl}_F q'$. Hence the last term of (5.21), after a further integration by parts, becomes

$$\int_F \left[(\operatorname{curl} u)' \right]_{FF} : \operatorname{rot}_F q = -\frac{1}{2} \int_F u_{Fn}' \cdot \operatorname{rot}_F \operatorname{curl}_F q' - \frac{1}{2} \int_{\partial F} u_{Fn}' \cdot t \operatorname{curl}_F q' .$$

Thus (5.20) follows after using $u_{Fn}' = u_{nF}$. □

The next lemma is an exactness result in 2D, which we state on the face F for our purposes. Let $\varepsilon_F(v) = \operatorname{sym} \operatorname{grad}_F v$ for $v \in t_1 H^1(F) + t_2 H^1(F)$. Spaces like the latter will be abbreviated to $[H^1(F)]^2$ (e.g., $[\mathcal{P}_{r-5}(F)]^2$ in the next lemma). Let b_F denote the face bubble, i.e., the product of the three barycentric coordinates of the vertices of F.

Lemma 5.9. *Let u_{FF} be as in (5.15) with u_{ij} in $\mathcal{P}_r(F)$ and $u_{FF}' = u_{FF}$. If $\operatorname{curl}_F (\operatorname{curl}_F u_{FF})' = 0$ and both $u_{FF}|_{\partial F} = 0$ and $(t \cdot \operatorname{curl}_F u_{FF})|_{\partial F} = 0$, then $u_{FF} = \varepsilon_F(b_F^2 \phi)$ where $\phi \in [\mathcal{P}_{r-5}(F)]^2$.*

Proof. Since $\operatorname{curl}_F (\operatorname{curl}_F u_{FF})' = 0$ and the tangential component of $(\operatorname{curl}_F u_{FF})'$ vanishes on ∂F, we have $(\operatorname{curl}_F u_{FF})' = \operatorname{grad}_F (b_F \psi)$ for some $\psi \in \mathcal{P}_{r-2}(F)$. Put $g = b_F \psi (t_1 t_2' - t_2 t_1')$. Observe that $\operatorname{grad}_F (b_F \psi) = (\operatorname{curl}_F g)'$ and $\operatorname{sym}(g) = 0$. Thus, $\operatorname{curl}_F (u_{FF} - g) = 0$ and $u_{FF} - g$

vanishes on ∂F. Hence, there exists $\phi \in [\mathcal{P}_{r-5}(F)]^2$ such that $\operatorname{grad}_F(b_F^2\phi) = u_{FF} - g$. We conclude by noting that $u_{FF} = \operatorname{sym} u_{FF} = \varepsilon_F(b_F^2\phi)$. □

With these preparations, we proceed to develop degrees of freedom for $U_r^1(T^{\mathrm{A}})$. Instead of directly using the definition of $U_r^1(T^{\mathrm{A}})$ as the symmetric part of another space, we use its alternate characterization in Theorem 5.6 to design its dofs. Let t_e denote a unit tangent vector (of arbitrarily fixed orientation) along an edge e. We will need the space of rigid displacements within a face F, namely $\mathcal{R}(F) := \{d_1t_1' + d_2t_2' + c((x\cdot t_1)t_2' - (x\cdot t_2)t_1') : c, d_i \in \mathbb{R}\}$.

Lemma 5.10. *For any $r \geq 4$, the functionals*

(5.22a)	$D^\alpha u(a)$,	$\lvert\alpha\rvert \leq 1, a \in \Delta_0(T)$,	(96)
(5.22b)	$\operatorname{inc} u(a)$,	$a \in \Delta_0(T)$,	(24)
(5.22c)	$\int_e u : \kappa$,	$\kappa \in \operatorname{sym}[\mathcal{P}_{r-4}(e)]^{3\times 3}, e \in \Delta_1(T)$,	$(6\cdot 6(r-3))$
(5.22d)	$\int_e (\operatorname{curl} u)' t_e \cdot \kappa$,	$\kappa \in [\mathcal{P}_{r-3}(e)]^3, e \in \Delta_1(T)$	$(6\cdot 3\cdot (r-2))$
(5.22e)	$\int_e (\operatorname{inc} u) n^F \cdot \kappa$,	$\kappa \in [\mathcal{P}_{r-4}(e)]^3, e \in \Delta_1(F), F \in \Delta_2(T)$	$(4\cdot 3\cdot 3(r-3))$
(5.22f)	$\int_F (\operatorname{inc} u)_{nn}\, \kappa$,	$\kappa \in \mathcal{P}_{r-5}(F)/\mathcal{P}_1(F), F \in \Delta_2(T)$	$\left(4\cdot\left[\frac{1}{2}(r-4)(r-3)-3\right]\right)$
(5.22g)	$\int_F (\operatorname{inc} u)_{Fn} \cdot \kappa$,	$\kappa \in [\mathcal{P}_{r-5}(F)]^2/\mathcal{R}(F), F \in \Delta_2(T)$,	$\left(4\cdot\left[2\frac{1}{2}(r-4)(r-3)-3\right]\right)$
(5.22h)	$\int_F u_{FF} : \kappa$,	$\kappa \in \varepsilon_F\left(b_F^2\,[\mathcal{P}_{r-5}(F)]^2\right), F \in \Delta_2(T)$	$(4\cdot 2\cdot \frac{1}{2}(r-4)(r-3))$
(5.22i)	$\int_F [(\operatorname{curl} u)']_{FF} : \kappa$,	$\kappa \in \operatorname{grad}_F(b_F[\mathcal{P}_{r-3}(F)]^2), F \in \Delta_2(T)$	$(4\cdot 2\cdot \frac{1}{2}(r-2)(r-1))$
(5.22j)	$\int_F u_{Fn} \cdot \kappa$,	$\kappa \in \operatorname{grad}_F(b_F^2\mathcal{P}_{r-5}(F)), \quad F \in \Delta_2(T)$,	$(4\cdot \frac{1}{2}(r-4)(r-3))$
(5.22k)	$\int_F u_{nn}\, \kappa$,	$\kappa \in \mathcal{P}_{r-3}(F), \qquad F \in \Delta_2(T)$	$(4\cdot \frac{1}{2}(r-2)(r-1))$
(5.22l)	$\int_T \operatorname{inc} u : \operatorname{inc} \kappa$,	$\kappa \in \mathring{U}_r^1(T^A)$,	$(2r^3 - 9r^2 + 7r + 6)$
(5.22m)	$\int_T u : \varepsilon(\kappa)$,	$\kappa \in \mathring{U}_{r+1}^0(T^A)$,	$(2(r-3)(r-2)(r-1))$

form a unisolvent set of degrees of freedom for $U_r^1(T^A)$.

Proof. The number of dofs add up to the dimension of $U_r^1(T^{\mathrm{A}})$ given in Lemma 5.5. Suppose that all dofs of (5.22) vanish for a $u \in U_r^1(T^{\mathrm{A}})$. We must show that $u \equiv 0$. The following conclusions are immediate from (5.22a, 5.22c), (5.22b, 5.22e), and (5.22a, 5.22d), respectively:

$$(5.23)\qquad u|_e = 0, \qquad (\operatorname{inc} u)n^F|_e = 0, \qquad (\operatorname{curl} u)'t|_e = 0, \qquad \text{for } e \in \Delta_1(T),\ F \in \Delta_2(T).$$

In particular, the last equality, in conjunction with (5.17b) of Lemma 5.7 shows that $0 = n'(\operatorname{curl} u)'t = (\operatorname{curl}_F u_{FF})'t$ on ∂F. Hence all terms on the right hand side of (5.19) vanish when $\phi \in \mathcal{P}_1(F)$. Thus Lemma 5.8, combined with (5.22f) and (5.23), yields $(\operatorname{inc} u)_{nn} = 0$ on all $F \in \Delta_2(T)$. Next, before proceeding to use (5.22g), observe that any $q = c((x\cdot t_1)t_2' - (x\cdot t_2)t_1')$, with

$c \in \mathbb{R}$, has $\operatorname{rot}_F q = c(t_1 t_1' + t_2 t_2')$. Hence $\operatorname{dev}_F \operatorname{rot}_F \mathcal{R}(F) = 0$. Similarly, $\operatorname{rot}_F \operatorname{curl}_F [\mathcal{R}(F)]' = 0$. Hence all terms on the right hand side of (5.20) vanish for $q \in \mathcal{R}(F)$. Therefore, Lemma 5.8 combined with (5.22g), (5.23), and the observation that $(\operatorname{inc} u)_{nF} = [(\operatorname{inc} u)_{Fn}]'$ leads us to conclude that $(\operatorname{inc} u)_{nF} = 0$ on all faces of T. Thus, $(\operatorname{inc} u)n|_{\partial T} = 0$.

In particular, due to (5.17c), $\operatorname{curl}_F (\operatorname{curl}_F u_{FF})' = 0$ on any $F \in \Delta_2(T)$. This implies, by virtue of (5.23) and Lemma 5.9, that $u_{FF} = \varepsilon_F(b_F^2 \phi)$ for some $\phi \in [\mathcal{P}_{r-5}(F)]^2$, and hence by (5.22h), we have u_{FF} vanishes on F. This implies, by (5.17b), that $[(\operatorname{curl} u)']_{Fn}$ vanishes on F, i.e., $n'(\operatorname{curl} u)'Q|_{\partial T} = 0$. In fact, all of $(\operatorname{curl} u)'Q$ vanishes on ∂T, as we shall now show.

On a face F, by (5.17d), $\operatorname{curl}_F [(\operatorname{curl} u)']_{FF} = 0$, and by (5.23), $[(\operatorname{curl} u)']_{FF} t|_{\partial F} = 0$. Hence, $(\operatorname{curl} u)'_{FF} = \operatorname{grad}_F (b_F \phi)$ for a $\phi \in [\mathcal{P}_{r-3}(F)]^2$. The dofs of (5.22i) then show that $[(\operatorname{curl} u)']_{FF}$ vanishes on F. We have already shown that $[(\operatorname{curl} u)']_{Fn}$ also vanishes. Hence $(\operatorname{curl} u)'Q|_{\partial T} = 0$. Since we know from the first characterization of Theorem 5.6 that $(\operatorname{curl} u)' \in W_{r-1}^1(T^{\mathrm{A}}) \otimes \mathbb{V}$, we have just shown that $(\operatorname{curl} u)' \in \mathring{W}_{r-1}^1(T^{\mathrm{A}}) \otimes \mathbb{V}$.

Next, we proceed to show that $u|_{\partial T} = 0$. On a face F, since $[(\operatorname{curl} u)']_{FF} = 0$, by (5.17e), $\operatorname{tr}_F [(\operatorname{curl} u)']_{FF} = \operatorname{tr}_F (\operatorname{curl} u)' = \operatorname{curl}_F u_{Fn} = 0$. Moreover, by (5.23), $u_{Fn}|_{\partial F} = 0$, so $u_{Fn} = \operatorname{grad}_F (b_F^2 \phi)$ for $\phi \in \mathcal{P}_{r-5}(F)$. Thus, by (5.22j) we conclude that u_{Fn} vanishes on F. Of course, we also have $u_{nn}|_{\partial T} = 0$ due to (5.22k). Thus $u|_{\partial T} = 0$.

We are now in a position to apply the second characterization of Theorem 5.6, to conclude that $u \in \mathring{U}_r^1(T^{\mathrm{A}})$. Hence (5.22l) implies $\operatorname{inc} u \equiv 0$ on T. Using the exactness of the sequence (5.8) given by Theorem 5.4 and the dofs of (5.22m), we see that $u \equiv 0$ on T. □

5.4. **Degrees of freedom of the remaining elements.** In this subsection, we give unisolvent dofs for the other spaces in the complex, $U_{r+1}^0(T^{\mathrm{A}})$, $U_{r-2}^2(T^{\mathrm{A}})$ and $U_{r-3}^3(T^{\mathrm{A}})$. We begin by proposing the following dofs for $U_{r+1}^0(T^{\mathrm{A}})$ and proving them to be unisolvent.

Lemma 5.11. *For any $r \geq 4$, the functionals*

(5.24a)	$D^\alpha \omega(a)$,	$\lvert\alpha\rvert \leq 2,\ a \in \Delta_0(T)$,	(120)
(5.24b)	$\displaystyle\int_e \omega \cdot \kappa$,	$\kappa \in [\mathcal{P}_{r-5}(e)]^3,\ e \in \Delta_1(T)$,	$(18(r-4))$
(5.24c)	$\displaystyle\int_e \frac{\partial \omega}{\partial n_e^\pm} \cdot \kappa$,	$\kappa \in [\mathcal{P}_{r-4}(e)]^3, e \in \Delta_1(T)$,	$(36(r-3))$
(5.24d)	$\displaystyle\int_F [\varepsilon(\omega)]_{Fn} \cdot \kappa$,	$\kappa \in \operatorname{grad}_F(b_F^2 \mathcal{P}_{r-5}(F)), F \in \Delta_2(T)$,	$(2(r-4)(r-3))$
(5.24e)	$\displaystyle\int_F \varepsilon_F(\omega_F) : \varepsilon_F(b_F^2 \kappa)$	$\kappa \in [\mathcal{P}_{r-5}(F)]^2, F \in \Delta_2(T)$,	$(4(r-4)(r-3))$
(5.24f)	$\displaystyle\int_F \frac{\partial(\omega \cdot n)}{\partial n} \kappa$,	$\kappa \in \mathcal{P}_{r-3}(F), F \in \Delta_2(T)$,	$(2(r-2)(r-1))$
(5.24g)	$\displaystyle\int_F \left[(\operatorname{curl} \varepsilon(\omega))'\right]_{FF} : \kappa$,	$\kappa \in \operatorname{grad}_F(b_F[\mathcal{P}_{r-3}(F)]^2), F \in \Delta_2(T)$,	$(4(r-2)(r-1))$
(5.24h)	$\displaystyle\int_T \varepsilon(\omega) : \varepsilon(\kappa)$,	$\kappa \in \mathring{U}_{r+1}^0(T^A)$,	$(2(r-3)(r-2)(r-1))$

form a unisolvent set of degrees of freedom for $U_{r+1}^0(T^A)$.

Proof. It is easily verified that the total number of dofs equals the dimension of $U_{r+1}^0(T^{\mathrm{A}})$ given in Lemma 5.5. Consider an $\omega \in U_{r+1}^0(T^{\mathrm{A}})$ on which the dofs (5.24) vanish. By standard arguments, dofs (5.24a), (5.24b) and (5.24c) imply that

$$\omega|_e = 0, \qquad (\operatorname{grad}\omega)|_e = 0, \qquad \text{for } e \in \Delta_1(T). \tag{5.25}$$

Hence, on any face $F \in \Delta_2(T)$, we have $\omega_F \in b_F^2[P_{r-5}(F)]^2$, so (5.24e) implies that $\omega_F = 0$. Also note that (5.25) implies $(\operatorname{curl}\omega)_F \in b_F[P_{r-3}(F)]^2$, so the dofs of (5.24g), in view of the identity (5.17g), imply that $(\operatorname{curl}\omega)_F = 0$. This in turn implies, after taking the cross product with n on both sides of (5.17h) and using (5.16), that

$$\partial_n \omega_F = \operatorname{grad}_F(\omega\cdot n). \tag{5.26}$$

Hence (5.17i) yields $2[\varepsilon(\omega)]_{nF} = \operatorname{grad}_F(\omega\cdot n) + \partial_n\omega_F = 2\operatorname{grad}_F(\omega\cdot n)$. The latter is in $b_F^2[P_{r-5}(F)]^2$ due to (5.25), so the dofs of (5.24d) give $\omega\cdot n = 0$ on F. Combining with the already shown $\omega_F \equiv 0$, we summarize: all components of ω vanish on ∂T.

Next, we will show that all first derivatives of ω also vanish on ∂T. Consider an $F \in \Delta_2(T)$ and let K denote one of the subtetrahedra T_i which has F as a face. Then, since $\omega\cdot n$ vanishes on F, there must exist a $p \in \mathcal{P}_r(K)$ such that $\omega\cdot n = \mu p$. Since $\partial_n(\omega\cdot n)|_F = (\partial_n\mu)p|_F$ vanishes on ∂F by (5.25), there exists a $\psi \in \mathcal{P}_{r-3}(F)$ such that $p = b_F\psi$, i.e., $\partial_n(\omega\cdot n)|_F = (\partial_n\mu)b_F\psi$. Hence (5.24f) yields $\partial_n(\omega\cdot n) = 0$. By (5.26), $\partial_n\omega_F$ also vanishes. Since $\omega|_{\partial T} \equiv 0$, all the tangential derivatives of ω also vanish, so we conclude that $(\operatorname{grad}\omega)|_{\partial T} \equiv 0$. Thus $\omega \in \mathring{U}^0_{r+1}(T^{\mathrm{A}})$. Therefore, (5.24h) shows that ω vanishes. □

Lemma 5.12 (Dofs of the stress space)**.** *For any $r \geq 4$, the functionals*

(5.27a)	$\sigma(a)$,	$a \in \Delta_0(T)$,	(6×4 dofs)
(5.27b)	$\int_e \sigma n^F \cdot \kappa\,,$	$\kappa \in [\mathcal{P}_{r-4}(e)]^3,\ e \in \Delta_1(F),\ F \in \Delta_2(T),$	($3 \times 12(r-3)$ dofs)
(5.27c)	$\int_F \sigma n^F \cdot \kappa\,,$	$\kappa \in [\mathcal{P}_{r-5}(F)]^3,\ f \in \Delta_2(T),$	($3 \times 2(r-3)(r-4)$ dofs)
(5.27d)	$\int_T \sigma : \kappa\,,$	$\kappa \in \operatorname{inc}\mathring{U}^1_{r-1}(T^A),$	($2r^3 - 9r^2 + 7r + 6$ dofs)
(5.27e)	$\int_T \operatorname{div}\sigma\cdot\kappa\,,$	$\kappa \in \mathring{U}^3_{r-3}(T^A),$	($2r^3 - 6r^2 + 4r - 6$ dofs)

form a unisolvent set of degrees of freedom for $U^2_{r-2}(T^A)$.

Proof. The stated counts of the dofs (obtained using the exactness given in Theorem 5.4 and standard dimensions) sum up to the expression for $\dim U^2_{r-2}(T^{\mathrm{A}})$ in Lemma 5.5. To prove unisolvency, let all dofs of (5.27) vanish for a $\sigma \in U^2_{r-2}(T^{\mathrm{A}}) \subset V^2_{r-2}(T^{\mathrm{A}}) \otimes \mathbb{V}$. Then, (5.27a)–(5.27c), together with $\operatorname{skw}\sigma(x_i) = 0$ yield $\sigma n|_{\partial T} = 0$ (as in the proof of Lemma 4.5). Thus, by Lemma 3.2, σ is in $\mathring{V}^2_{r-2} \otimes \mathbb{V}$, and since $\operatorname{skw}\sigma \equiv 0$, we conclude that $\sigma \in \mathring{U}^2_{r-2}(T^{\mathrm{A}})$. By Theorem 5.4, $\mathring{U}^3_{r-3}(T^{\mathrm{A}}) = \operatorname{div}\mathring{U}^2_{r-2}(T^{\mathrm{A}})$, so the dofs of (5.27e) yield $\operatorname{div}\sigma = 0$. Using the exactness result of Theorem 5.4 again, we conclude that there is a $u \in \mathring{U}^1_{r-1}(T^{\mathrm{A}})$ such that $\sigma = \operatorname{inc} u$. Hence, using (5.27d), we conclude that $\sigma = 0$. □

Finally, the dofs of $U^3_{r-3}(T^{\mathrm{A}})$ are just "three copies" of the dofs of $V^3_{r-1}(T^{\mathrm{A}})$. It is immediate that any $w \in U^3_{r-3}(T^{\mathrm{A}})$ is uniquely defined by the following functionals:

$$\int_T w \cdot \kappa\,, \qquad \kappa \in \mathcal{R}, \qquad \text{(6 dofs)}, \tag{5.28a}$$

$$\int_T w \cdot v\,, \qquad v \in \mathring{U}^3_{r-3}(T^{\mathrm{A}}), \qquad (2r^3 - 6r^2 + 4r - 6 \text{ dofs}). \tag{5.28b}$$

5.5. **Commuting projections.** In this subsection, we study the accompanying cochain projectors of our elasticity complex (5.8). These projections are simply the canonical finite element interpolants, denoted by Π^U_j, $j = 0, 1, 2, 3$, defined by the already given degrees of freedom of $U^0_{r+1}(T^{\mathrm{A}}), U^1_r(T^{\mathrm{A}}), U^2_{r-2}(T^{\mathrm{A}})$, and $U^3_{r-3}(T^{\mathrm{A}})$, respectively. Note that Π^U_3 is just the L^2 projection into $U^3_{r-3}(T^{\mathrm{A}})$ since $\mathring{U}^3_{r-3}(T^{\mathrm{A}}) \oplus \mathcal{R} = U^3_{r-3}(T^{\mathrm{A}})$.

Theorem 5.13. *The following diagram commutes for the indicated degrees r:*

$$\begin{array}{ccccccccccc} \mathcal{R} & \xrightarrow{\subset} & C^\infty(\bar{T}) \otimes \mathbb{V} & \xrightarrow{\varepsilon} & C^\infty(\bar{T}) \otimes \mathbb{S} & \xrightarrow{\text{inc}} & C^\infty(\bar{T}) \otimes \mathbb{S} & \xrightarrow{\text{div}} & C^\infty(\bar{T}) \otimes \mathbb{V} & \longrightarrow & 0 \\ & & \Big\downarrow \Pi^U_0 & (r\geq 4) & \Big\downarrow \Pi^U_1 & (r\geq 4) & \Big\downarrow \Pi^U_2 & (r\geq 6) & \Big\downarrow \Pi^U_3 & & \\ \mathcal{R} & \xrightarrow{\subset} & U^0_{r+1}(T^A) & \xrightarrow{\varepsilon} & U^1_r(T^A) & \xrightarrow{\text{inc}} & U^2_{r-2}(T^A) & \xrightarrow{\text{div}} & U^3_{r-3}(T^A) & \longrightarrow & 0. \end{array} \tag{5.29}$$

Proof. First, we show that

$$\operatorname{div} \Pi^U_2 \sigma = \Pi^U_3 \operatorname{div} \sigma \qquad \text{for } r \geq 6. \tag{5.30}$$

Since $w = \operatorname{div} \Pi^U_2 \sigma - \Pi^U_3 \operatorname{div} \sigma$ is in $U^3_{r-3}(T^{\mathrm{A}})$, it suffices to prove that the dofs of (5.28) applied to w vanish. It is obvious from (5.27e) that the dofs of (5.28b) applied to w vanish. The dofs of (5.28a) also vanish because, for $\kappa \in \mathcal{R}$

$$\begin{aligned} \int_T w \cdot \kappa &= \int_T \operatorname{div} \Pi^U_2 \sigma \cdot \kappa - \int_T \operatorname{div} \sigma \cdot \kappa && \text{by (5.28a)} \\ &= \int_{\partial T} (\Pi^U_2 \sigma)\, n \cdot \kappa - \int_{\partial T} \sigma\, n \cdot \kappa = 0 && \text{by (5.27c)} \end{aligned}$$

for $r \geq 6$, so (5.30) follows.

Next, we show that

$$\operatorname{inc} \Pi^U_1 u = \Pi^U_2 \operatorname{inc} u \qquad \text{for } r \geq 4. \tag{5.31}$$

Let $\sigma = \operatorname{inc} \Pi^U_1 u - \Pi^U_2 \operatorname{inc} u$. To show that the dofs (5.27) vanish on σ, we begin by noting that (5.22b) and (5.27a) imply that the dofs (5.27a) vanish for σ. Similarly, the dofs (5.27b) applied to σ also vanish due to (5.22e) and (5.27b). To show that the dofs of (5.27c) also vanish on σ, we split them into normal and tangential parts after using (5.27c) on $\operatorname{inc} u$:

$$\int_F \sigma n^F \cdot \kappa = g_n + g_F, \quad g_n = \int_F \left[\operatorname{inc}(\Pi^U_1 u - u)\right]_{nn} \kappa_n, \quad g_F = \int_F \left[\operatorname{inc}(\Pi^U_1 u - u)\right]_{nF} \kappa_F. \tag{5.32}$$

Note that g_n vanishes for any $\kappa_n \in \mathcal{P}_{r-5}(F)/\mathcal{P}_1(F)$ due to (5.22f). In fact, g_n vanishes for any $\kappa_n \in \mathcal{P}_{r-5}(F)$, as we now show. Observe that by (5.19) of Lemma 5.8, for a $p_1 \in \mathcal{P}_1(F)$,

$$\int_F \sigma_{nn}\, p_1 = \int_{\partial F} \left[\operatorname{curl}_F (\Pi^U_1 u - u)_{FF}\right] t\, p_1\, ds + \int_{\partial F} (\Pi^U_1 u - u)_{FF} t \cdot (\operatorname{rot}_F p_1)'\,. \tag{5.33}$$

By (5.17b), $\operatorname{curl}_F(\Pi_1^U u - u)_{FF} t\, p_1 = [\operatorname{curl}(\Pi_1^U u - u)']_{Fn} t\, p_1 = \operatorname{curl}(\Pi_1^U u - u)' : p_1 n t'$, so the first term on the right hand side of (5.33) vanishes by (5.22d). The last term of (5.33) also vanishes because $(\Pi_1^U u - u)_{FF} t \cdot (\operatorname{rot}_F p_1)' = Q(\Pi_1^U u - u) Q t \cdot (\operatorname{rot}_F p_1)' = (\Pi_1^U u - u) Q t \cdot Q(\operatorname{rot}_F p_1)' = (\Pi_1^U u - u) : Q(\operatorname{rot}_F p_1)' t$ thus allowing us to apply (5.22c) whenever $r - 4 \geq 0$. Thus from (5.33) we conclude that $g_n = 0$ for all $\kappa_n \in \mathcal{P}_{r-5}(F) = \mathcal{P}_1(F) \oplus \mathcal{P}_{r-5}(F)/\mathcal{P}_1(F)$. Next consider g_F. Obviously, (5.22g) shows that $g_F = 0$ whenever $\kappa_F \in [\mathcal{P}_{r-5}(F)]^2/\mathcal{R}(F)$. However, for $\kappa_F \in \mathcal{R}(F)$, we may conduct a similar argument as above but now using (5.20) of Lemma 5.8, to conclude that $g_F = 0$ for all $\kappa_F \in [\mathcal{P}_{r-5}(F)]^2$. Thus, returning to (5.32), we have $\int_F \sigma n^F \cdot \kappa = 0$ for all $\kappa \in [\mathcal{P}_{r-5}(F)]^3$, i.e., all dofs of (5.27c) applied to σ vanish. It is easy to see that the remaining dofs of (5.27d) and (5.27e) applied to σ also vanish, thus finishing the proof of (5.31).

Finally, we will prove that

$$\varepsilon(\Pi_0^U \omega) = \Pi_1^U \varepsilon(\omega), \qquad \text{for } r \geq 4. \tag{5.34}$$

Letting $u = \varepsilon(\Pi_0^U \omega) - \Pi_1^U \varepsilon(\omega)$, it is enough to show that the dofs of (5.22) applied to u vanish. Let us dispose off the obvious implications first: *(i)* $\operatorname{inc} \circ \varepsilon = 0$ implies that the dofs of (5.22b), (5.22e), (5.22f), (5.22g) and (5.22l) applied to u vanish; *(ii)* (5.24a), (5.24d), (5.24e), (5.24f), (5.24g) applied to ω, taken together with (5.22a), (5.22j), (5.22h), (5.22k), (5.22i) applied to $\varepsilon(\omega)$, each respectively imply that (5.22a), (5.22j), (5.22h), (5.22k), (5.22i) applied to u vanish. It only remains to prove that the dofs of (5.22c) and (5.22d) applied to u vanish. To this end, it is useful to employ the edge-based orthonormal basis $\{n_e^+, n_e^-, t_e\}$ and write $\kappa \in \operatorname{sym}[\mathcal{P}_{r-4}(e)]^{3\times 3}$ as $\kappa = \kappa_{11} n_e^+ (n_e^+)' + \kappa_{12}\big(n_e^+ (n_e^-)' + n_e^- (n_e^+)'\big) + \kappa_{13}\big(n_e^+ t_e' + t_e (n_e^+)'\big) + +\kappa_{22} n_e^- (n_e^-)' + \kappa_{23}\big(n_e^+ (n_e^-)' + n_e^- (n_e^+)'\big) + \kappa_{33} t_e t_e'$ where $\kappa_{ij} \in \mathcal{P}_{r-4}(e)$. Then,

$$\begin{aligned}
\int_e u : \kappa &= \int_e [\varepsilon(\Pi_0^U \omega) - \Pi_1^U \varepsilon(\omega)] : \kappa = \int_e \varepsilon(\Pi_0^U \omega - \omega) : \kappa && \text{by (5.22c)} \\
&= \int_e \operatorname{grad}(\Pi_0^U \omega - \omega) : \kappa = \int_e \operatorname{grad}(\Pi_0^U \omega - \omega) : (\kappa_{13} n_e^+ t_e' + \kappa_{33} t_e t_e') && \text{by (5.24c)} \\
&= \int_e \operatorname{grad}(\Pi_0^U \omega - \omega) t_e \cdot (\kappa_{13} n_e^+ + \kappa_{33} t_e).
\end{aligned}$$

Now that the integrand contains a tangential derivative, we may integrate by parts, to see that the integral vanishes after an application of (5.24a) and (5.24b). Thus the dofs of (5.22c) applied to u vanish. To examine the dofs (5.22d), letting $\kappa \in [\mathcal{P}_{r-3}(e)]^3$, we note that

$$\begin{aligned}
\int_e (\operatorname{curl} u)' t_e \cdot \kappa &= \int_e \big[\operatorname{curl} \varepsilon(\Pi_0^U \omega - \omega)\big]' t_e \cdot \kappa && \text{by (5.22d)} \\
&= \frac{1}{2} \int_e \big[\operatorname{grad} \operatorname{curl}(\Pi_0^U \omega - \omega)\big] t_e \cdot \kappa && \text{by (5.17f)} \\
&= -\frac{1}{2} \int_e \operatorname{curl}(\Pi_0^U \omega - \omega) \cdot \partial_t \kappa && \text{by (5.24a)}
\end{aligned}$$

where in the last step, we have integrated by parts, and put $\partial_t \kappa = (\operatorname{grad} \kappa) t_e$. The curl in the integrand above can be decomposed into terms involving $\partial_t(\Pi_0^U \omega - \omega)$ and those involving $\partial_{n_e^\pm}(\Pi_0^U \omega - \omega)$. The former terms can be integrated by parts yet again, which after using (5.24a) and (5.24b), vanish. The latter terms also vanish by (5.24c) which we may apply as $\partial_t \kappa$ is of degree at most $r - 4$. $\square$

6. Global complexes

We have developed a number of new finite elements on Alfeld splits in the previous sections. In this section, we briefly discuss how the elements on Alfeld splits may be put together to construct global finite element spaces. Throughout this section, Ω denotes a contractible polyhedral domain in $\mathbb{R}^3$, subdivided by $\mathcal{T}_h$, a conforming tetrahedral mesh (and h denotes maximal element diameter). Let $\mathcal{T}_h^{\mathrm{A}}$ be the refinement obtained performing an Alfeld split to each mesh tetrahedron $T \in \mathcal{T}_h$, e.g., by connecting the barycenter of T with its vertices. We consider finite element spaces on $\mathcal{T}_h^{\mathrm{A}}$ built using the previously discussed elements. Every local dof we defined previously was associated to a subsimplex, so it is standard to go from the local dofs to the global dofs associated to the simplicial complex $\mathcal{T}_h$. We use $\#_k$ to denote the number of k-dimensional simplexes in $\mathcal{T}_h$. For example, $\#_0$ and $\#_1$ are the numbers of vertices and edges, respectively.

6.1. **The global V complex.** We begin with the standard finite element sequence. Let $W_r^0(\mathcal{T}_h^{\mathrm{A}})$, $W_r^1(\mathcal{T}_h^{\mathrm{A}})$, $W_r^2(\mathcal{T}_h^{\mathrm{A}})$, and $W_r^3(\mathcal{T}_h^{\mathrm{A}})$, denote the standard conforming finite element subspaces of $H^1(\Omega), H(\mathrm{curl}\,,\Omega), H(\mathrm{div},\Omega)$, and $L^2(\Omega)$ whose elements when restricted to a mesh element $T \in \mathcal{T}_h$ are in $W_r^0(T^{\mathrm{A}})$, $W_r^1(T^{\mathrm{A}})$, $W_r^2(T^{\mathrm{A}})$, and $W_r^3(T^{\mathrm{A}})$, respectively. Let

$$
\begin{aligned}
&V_r^0(\mathcal{T}_h^{\mathrm{A}}) = \{\omega \in C^1(\Omega) : \omega \text{ is } C^2 \text{ at vertices of } \mathcal{T}_h,\ \omega|_T \in V_r^0(T^{\mathrm{A}}) \text{ for all } T \in \mathcal{T}_h\},\\
&V_r^1(\mathcal{T}_h^{\mathrm{A}}) = \{\omega \in [C^0(\Omega)]^3 :\ \omega \text{ is } C^1 \text{ at vertices of } \mathcal{T}_h,\ \omega|_T \in V_r^1(T^{\mathrm{A}}) \text{ for all } T \in \mathcal{T}_h\},\\
&V_r^2(\mathcal{T}_h^{\mathrm{A}}) = \{\omega \in H(\mathrm{div},\Omega) :\ \omega \text{ is } C^0 \text{ at vertices of } \mathcal{T}_h,\ \omega|_T \in V_r^2(T^{\mathrm{A}}) \text{ for all } T \in \mathcal{T}_h\},\\
&V_r^3(\mathcal{T}_h^{\mathrm{A}}) = W_r^3(\mathcal{T}_h^{\mathrm{A}}).
\end{aligned}
$$

Equivalently, we could have defined our global spaces using the dofs of the local V spaces by making them single-valued when they are shared by multiple tetrahedra. The fact that this will lead to equivalent spaces can be established using the unisolvency proofs of the local V spaces (this is already known for the space V_r^0). Below we also use global continuity to define the global Z and U, however, they equivalently can be defined using dofs. After inheriting the global V dofs from the prior local V dofs, we may define global interpolation operators into these finite element spaces in the canonical way. Then the following global analogue of Theorem 3.7 can be easily proved.

Theorem 6.1. *Let $\Pi_{i,h}^V$ denote the canonical global finite element interpolant onto $V_{r-i}^i(\mathcal{T}_h^A)$. Then for $r \geq 5$ the following diagram commutes:*

$$
\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\mathrm{grad}} & [C^\infty(\Omega)]^3 & \xrightarrow{\mathrm{curl}} & [C^\infty(\Omega)]^3 & \xrightarrow{\mathrm{div}} & C^\infty(\Omega)\\
\big\downarrow{\scriptstyle \Pi_{0,h}^V} & & \big\downarrow{\scriptstyle \Pi_{1,h}^V} & & \big\downarrow{\scriptstyle \Pi_{2,h}^V} & & \big\downarrow{\scriptstyle \Pi_{3,h}^V}\\
V_r^0(\mathcal{T}_h^A) & \xrightarrow{\mathrm{grad}} & V_{r-1}^1(\mathcal{T}_h^A) & \xrightarrow{\mathrm{curl}} & V_{r-2}^2(\mathcal{T}_h^A) & \xrightarrow{\mathrm{div}} & V_{r-3}^3(\mathcal{T}_h^A).
\end{array}
$$

An exactness result analogous to Lemma 3.4 also holds for these global V spaces. In order to prove it, we are not able to use the projections $\Pi_{i,h}^V$ directly, since the functions we will apply them to are not sufficiently regular. This technical problem is overcome in the proof below by zeroing out the degrees of freedom requiring higher regularity and using the well-known existence of a regular potential (see e.g, [18]):

$$
\forall u \in L^2(\Omega),\ \exists v \in [H^1(\Omega)]^3 \text{ such that } \operatorname{div} v = u. \tag{6.1}
$$

With the above-mentioned modified interpolant and (6.1), the global exactness for $r \geq 5$ follows easily as seen below. The $r = 4$ case is also interesting, but since no local dofs for gluing $V_r^0(T^A)$ are known for this case, the same proof does not work. Yet, we are able to prove the partial exactness result that $\operatorname{div} : V_2^2(\mathcal{T}_h^{\mathrm{A}}) \to V_1^3(\mathcal{T}_h^{\mathrm{A}})$ is onto when $r = 4$, using a technique inspired by Stenberg [38, Theorem 1], who showed how dofs in standard mixed methods (for the Poisson problem) can be reduced by imposing vertex continuity. Our $V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}})$ space has similar continuity restrictions at the vertices of $\mathcal{T}_h$.

Theorem 6.2. *The sequence*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} V_r^0(\mathcal{T}_h^A) \xrightarrow{\text{grad}} V_{r-1}^1(\mathcal{T}_h^A) \xrightarrow{\text{curl}} V_{r-2}^2(\mathcal{T}_h^A) \xrightarrow{\text{div}} V_{r-3}^3(\mathcal{T}_h^A) \longrightarrow 0, \tag{6.2}$$

is exact for $r \geq 5$. When $r = 4$, the divergence operator remains surjective.

Proof. To show that $\operatorname{div} : V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}}) \to V_{r-3}^3(\mathcal{T}_h^{\mathrm{A}})$ is onto, let $v \in V_{r-3}^3(\mathcal{T}_h^{\mathrm{A}})$. By (6.1), there exists an $\omega \in [H^1(\Omega)]^3$ such that $\operatorname{div} \omega = v$. We now proceed to modify $\Pi_{2,h}^V$ and apply it to ω. The first modification involves zeroing out vertex dofs to avoid taking values of ω at the vertices. The second modification involves a rearrangement of $r-3$ face dofs that helps prove the theorem's assertion for the $r = 4$ case. These modifications result in the $\tilde{\Pi}_{2,h}^V$ given next. For every $F \in \Delta_2(T)$, arbitrarily choose an edge of F and denote it by e_F. Then define $\tilde{\Pi}_{2,h}^V \omega \in V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}})$ such that on an element $T \in \mathcal{T}_h$, the function $\omega_T = (\tilde{\Pi}_{2,h}^V \omega)|_T$ is given by the equations

$$\omega_T(a) = 0, \qquad a \in \Delta_0(T) \tag{6.3a}$$

$$\int_e (\omega \cdot n^F)\kappa, \qquad \kappa \in \mathcal{P}_{r-4}(e),\ e \in \Delta_1(F),\ e \neq e_F,\ F \in \Delta_2(T) \tag{6.3b}$$

$$\int_F (\omega \cdot n^F)\kappa, \qquad \kappa \in \mathcal{P}_{r-4}(F),\ F \in \Delta_2(T) \tag{6.3c}$$

$$\int_T \omega_T \cdot \kappa = \int_T \omega \cdot \kappa, \qquad \kappa \in \operatorname{curl} \mathring{V}_{r-1}^1(T^{\mathrm{A}}) \tag{6.3d}$$

$$\int_T (\operatorname{div} \omega_T)\, \kappa = \int_T (\operatorname{div} \omega)\, \kappa, \qquad \kappa \in \mathring{V}_{r-3}^3(T^{\mathrm{A}}) \tag{6.3e}$$

These equations, being a minor modification of previously given unisolvent dofs (3.12), can easily be shown to uniquely define a $\omega_T \in V_{r-2}^2(T^{\mathrm{A}})$, so $\tilde{\Pi}_{3,h}^V \omega$ is a well defined function in $V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}})$ for any ω in $H^1(\Omega)^3$ (e.g., the integral on the right hand side of (6.3c) is bounded for any ω in $H^1(\Omega)^3$ by a trace theorem). Now, when $r \geq 4$, for any constant κ, we have $\int_T \operatorname{div}(\omega_T - \omega)\kappa = \int_{\partial T} (\omega_T - \omega) \cdot n \kappa = 0$ by (6.3c). Hence (6.3e) yields $\operatorname{div} \tilde{\Pi}_{2,h}^V \omega = \Pi_{3,h}^V \operatorname{div} \omega = \Pi_{3,h}^V v = v$. This proves the stated surjectivity of divergence for $r = 4$ as well as for $r \geq 5$.

Continuing, restricting to the $r \geq 5$ case, for any $u \in \ker(\operatorname{curl}, V_{r-1}^1(\mathcal{T}_h^{\mathrm{A}}))$, there exists $v \in W_r^0(\mathcal{T}_h^{\mathrm{A}})$ such that $u = \operatorname{grad} v$ by the exactness of the standard finite element de Rham complex (the W sequence). Since u is C^1 at the vertices of $\mathcal{T}_h$, v is C^2 at the vertices, so $v \in V_r^0(\mathcal{T}_h^{\mathrm{A}})$.

Finally, to show that $\operatorname{curl} V_{r-1}^1(\mathcal{T}_h^{\mathrm{A}}) = \ker(\operatorname{div}, V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}}))$, it suffices to prove that their dimensions are equal. To this end, we note from (3.9), (3.10), (3.12), and (3.13) that the following dimension count holds:

$$
\begin{aligned}
\dim(V_r^0(\mathcal{T}_h^{\mathrm{A}})) &= 10\#_0 + (3r-13)\#_1 + (r^2-7r+13)\#_2 + \frac{2}{3}(r-4)(r-3)(r-2)\#_3,\\
\dim(V_{r-1}^1(\mathcal{T}_h^{\mathrm{A}})) &= 12\#_0 + 3(r-4)\#_1 + \frac{3}{2}(r-2)(r-3)\#_2 + (2r^3-9r^2+19r-27)\#_3,\\
\dim(V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}})) &= 3\#_0 + \frac{1}{2}(r+2)(r-3)\#_2 + (2r^3-5r^2+3r-12)\#_3,\\
\dim(V_{r-3}^3(\mathcal{T}_h^{\mathrm{A}})) &= \frac{2}{3}r(r-1)(r-2)\#_3.
\end{aligned}
$$

By the exactness properties we have already proven, $\dim \operatorname{curl} V_{r-1}^1(\mathcal{T}_h^{\mathrm{A}}) = \dim V_{r-1}^1(\mathcal{T}_h^{\mathrm{A}}) - \dim V_r^0(\mathcal{T}_h^{\mathrm{A}}) + 1$ and $\dim \ker(\operatorname{div}, V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}})) = \dim V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}}) - \dim V_{r-3}^2(\mathcal{T}_h^{\mathrm{A}})$. These numbers are equal because the Euler formula, together with the dimensions given above, yields $\dim(V_r^0(\mathcal{T}_h^{\mathrm{A}})) - \dim(V_{r-1}^1(\mathcal{T}_h^{\mathrm{A}})) + \dim(V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}})) - \dim(V_{r-3}^3(\mathcal{T}_h^{\mathrm{A}})) = 1$. $\square$

6.2. **The global Z complex.** Let $Z_r^0(\mathcal{T}_h^{\mathrm{A}}) = V_r^0(\mathcal{T}_h^{\mathrm{A}})$, and

$$
\begin{aligned}
Z_r^1(\mathcal{T}_h^{\mathrm{A}}) &= \{\omega \in [C^0(\Omega)]^3 : \operatorname{curl}\omega \in [C^0(\Omega)]^3,\ \omega \text{ and } \operatorname{curl}\omega \text{ are } C^1 \text{ at vertices of } \mathcal{T}_h,\\
&\qquad\qquad \text{and } \omega|_T \in Z_r^1(T^{\mathrm{A}}) \text{ for all } T \in \mathcal{T}_h\},\\
Z_r^2(\mathcal{T}_h^{\mathrm{A}}) &= \{\omega \in [C^0(\Omega)]^3 :\ \omega \text{ is } C^1 \text{ at vertices of } \mathcal{T}_h,\ \omega|_T \in Z_r^2(T^{\mathrm{A}}) \text{ for all } T \in \mathcal{T}_h\},\\
Z_r^3(\mathcal{T}_h^{\mathrm{A}}) &= \{\omega \in L^2(\Omega) :\ \ \omega \text{ is } C^0 \text{ at vertices of } \mathcal{T}_h,\ \omega|_T \in Z_r^3(T^{\mathrm{A}}) \text{ for all } T \in \mathcal{T}_h,\}.
\end{aligned}
$$

These spaces inherit global dofs from the previously given local Z dofs. The following global analogue of Theorem 4.7 can be easily proved.

Theorem 6.3. *Let $\Pi_{i,h}^Z$ denote the canonical global finite element interpolant onto $Z_{r-i+1}^i(\mathcal{T}_h^A)$. Then for $r \geq 4$ the following diagram commutes:*

$$
\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\text{grad}} & [C^\infty(\Omega)]^3 & \xrightarrow{\text{curl}} & [C^\infty(\Omega)]^3 & \xrightarrow{\text{div}} & C^\infty(\Omega)\\
\downarrow \Pi_0^Z & & \downarrow \Pi_1^Z & & \downarrow \Pi_2^Z & & \downarrow \Pi_3^Z\\
Z_{r+1}^0(\mathcal{T}_h^A) & \xrightarrow{\text{grad}} & Z_r^1(\mathcal{T}_h^A) & \xrightarrow{\text{curl}} & Z_{r-1}^2(\mathcal{T}_h^A) & \xrightarrow{\text{div}} & Z_{r-2}^3(\mathcal{T}_h^A).
\end{array}
$$

Theorem 6.4. *The sequence*

$$
0 \longrightarrow \mathbb{R} \xrightarrow{\subset} Z_{r+1}^0(\mathcal{T}_h^A) \xrightarrow{\text{grad}} Z_r^1(\mathcal{T}_h^A) \xrightarrow{\text{curl}} Z_{r-1}^2(\mathcal{T}_h^A) \xrightarrow{\text{div}} Z_{r-2}^3(\mathcal{T}_h^A) \longrightarrow 0 \tag{6.4}
$$

is exact for $r \geq 4$.

Proof. Let $\mathcal{A}$ denote the set of all (constant) trace-free 3×3 matrices. To show that div is onto, let $v \in Z_{r-2}^3(\mathcal{T}_h^{\mathrm{A}})$. By (6.1), there exists an $\omega \in [H^1(\Omega)]^3$ such that $\operatorname{div}\omega = v$. Let $\tilde{\Pi}_{2,h}^Z\omega \in Z_{r-1}^2(\mathcal{T}_h^{\mathrm{A}})$ be such that on each $T \in \mathcal{T}_h^{\mathrm{A}}$, its element restriction $\omega_T = (\tilde{\Pi}_{2,h}^Z\omega)|_T$

satisfies

$$\begin{aligned}
&\omega_T(a) = 0, && && a \in \Delta_0(T),\\
&(\operatorname{grad}\omega_T)(a) : \kappa = 0, && \kappa \in \mathcal{A}, && a \in \Delta_0(T),\\
&\operatorname{tr}(\operatorname{grad}\omega_T)(a) = \operatorname{tr}(\operatorname{grad}\omega)(a), && && a \in \Delta_0(T),\\
&\int_e \omega_T \cdot \kappa \, = 0, && \kappa \in [\mathcal{P}_{r-5}(e)]^3, && e \in \Delta_1(T),\\
&\int_F \omega_T \cdot \kappa \, = \int_F \omega \cdot \kappa\,, && \kappa \in [\mathcal{P}_{r-4}(F)]^3, && F \in \Delta_2(T),\\
&\int_T \omega_T \cdot \kappa \, = \int_T \omega \cdot \kappa\,, && \kappa \in \operatorname{curl} \mathring{Z}_r^1(T^{\mathrm{A}}), &&\\
&\int_T \operatorname{div}\omega_T \cdot \kappa \, = \int_T \operatorname{div}\omega \cdot \kappa\,, && \kappa \in \operatorname{div} \mathring{Z}_{r-1}^2(T^{\mathrm{A}}). &&
\end{aligned}$$

Here we have used the same technique of zeroing out certain dofs that we used in the proof of Theorem 6.2. The right hand sides of the equations above are bounded since ω is in $[H^1(\Omega)]^3$ and since $\operatorname{tr}(\operatorname{grad}\omega) = \operatorname{div}\omega = v \in Z_{r-2}^3(T^{\mathrm{A}})$. These equations uniquely determine $\tilde{\Pi}_{2,h}^Z \omega \in Z_{r-1}^2(\mathcal{T}_h^{\mathrm{A}})$ due to the unisolvency of (4.16) proved in Lemma 4.5. An argument analogous to the one we used to prove that $\operatorname{div}\Pi_2^Z = \Pi_3^Z \operatorname{div}$ in the proof of Theorem 4.7 now yields $\operatorname{div}\tilde{\Pi}_{2,h}^Z\omega = \operatorname{div}\omega = v$. It is easy to prove that $\operatorname{grad} : Z_{r+1}^0(\mathcal{T}_h^{\mathrm{A}}) \to \ker(\operatorname{curl}, Z_r^1(\mathcal{T}_h^{\mathrm{A}}))$ is onto (see proof of Theorem 6.2).

Finally, we perform a dimension count of the global degrees of freedom to show that $\operatorname{curl} : Z_r^1(\mathcal{T}_h^{\mathrm{A}}) \to \ker(\operatorname{div}, Z_{r-1}^2(\mathcal{T}_h^{\mathrm{A}}))$ is onto. To this end, we note from (3.9), (4.15), (4.16), and (4.17) that the following dimension count holds:

$$\begin{aligned}
\dim(Z_{r+1}^0(\mathcal{T}_h^{\mathrm{A}})) &= 10\#_0 + (3r-10)\#_1 + (r^2-5r+7)\#_2 + \frac{2}{3}(r-3)(r-2)(r-1)\#_3,\\
\dim(Z_r^1(\mathcal{T}_h^{\mathrm{A}})) &= 20\#_0 + 3(2r-7)\#_1 + \frac{5}{2}(r-2)(r-3)\#_2\\
&\quad + \left(\frac{2}{3}(r-3)(r-2)(r-1) + \frac{1}{3}(r-3)(r-2)(4r-7)\right)\#_3,\\
\dim(Z_{r-1}^2(\mathcal{T}_h^{\mathrm{A}})) &= 12\#_0 + 3(r-4)\#_1 + \frac{3}{2}(r-2)(r-3)\#_2\\
&\quad + \left(\frac{1}{3}(r-3)(r-2)(4r-7) + \frac{2}{3}(r+1)r(r-1) - 13\right)\#_3,\\
\dim(Z_{r-2}^3(\mathcal{T}_h^{\mathrm{A}})) &= \#_0 + \left(\frac{2}{3}(r+1)r(r-1) - 12\right)\#_3.
\end{aligned}$$

By the Euler formula, we have

$$\dim(Z_{r+1}^0(\mathcal{T}_h^{\mathrm{A}})) - \dim(Z_r^1(\mathcal{T}_h^{\mathrm{A}})) + \dim(Z_{r-1}^2(\mathcal{T}_h^{\mathrm{A}})) - \dim(Z_{r-2}^3(\mathcal{T}_h^{\mathrm{A}})) = 1,$$

which shows that $\dim\operatorname{curl} Z_r^1(\mathcal{T}_h^{\mathrm{A}}) = \dim\ker(\operatorname{div}, Z_{r-1}^2(\mathcal{T}_h^{\mathrm{A}}))$. □

6.3. **The global U complex.** The global elasticity complex consists of

$$U_{r+1}^0(\mathcal{T}_h^{\mathrm{A}}) = Z_{r+1}^0(\mathcal{T}_h^{\mathrm{A}}) \otimes \mathbb{V}, \qquad U_r^1(\mathcal{T}_h^{\mathrm{A}}) = \left\{\operatorname{sym}(u) : u \in Z_r^1(\mathcal{T}_h^{\mathrm{A}}) \otimes \mathbb{V}\right\},$$
$$U_{r-2}^2(\mathcal{T}_h^{\mathrm{A}}) = \{\omega \in V_{r-2}^2(\mathcal{T}_h^{\mathrm{A}}) \otimes \mathbb{V} : \operatorname{skw}\omega = 0\}, \quad U_{r-3}^3(\mathcal{T}_h^{\mathrm{A}}) = Z_{r-3}^3(\mathcal{T}_h^{\mathrm{A}}).$$

To show that these spaces form an exact global complex, we follow the same procedure as for the local complex, starting with a global analogue of Theorem 5.2. Note that like in the local case, the global space $V^1_{r-1}(\mathcal{T}_h^{\mathrm{A}}) \otimes \mathbb{V}$ is in bijective correspondence with $Z^2_{r-1}(\mathcal{T}_h^{\mathrm{A}}) \otimes \mathbb{V}$ via Ξ. Also, $\mathrm{vskw} : V^2_{r-2}(\mathcal{T}_h^{\mathrm{A}}) \otimes \mathbb{V} \mapsto Z^3_{r-2}(\mathcal{T}_h^{\mathrm{A}} \otimes \mathbb{V})$ is easily seen to be surjective.

Theorem 6.5. *For $r \geq 5$, the sequence*

$$\begin{bmatrix} Z^0_{r+1}(\mathcal{T}_h^A)\otimes\mathbb{V} \\ V^0_r(\mathcal{T}_h^A)\otimes\mathbb{V} \end{bmatrix} \xrightarrow{[\mathrm{grad},\, -\mathrm{mskw}]} Z^1_r(\mathcal{T}_h^A)\otimes\mathbb{V} \xrightarrow{\mathrm{curl}\,\Xi^{-1}\mathrm{curl}} V^2_{r-2}(\mathcal{T}_h^A)\otimes\mathbb{V} \xrightarrow{\begin{bmatrix} 2\mathrm{vskw} \\ \mathrm{div} \end{bmatrix}} \begin{bmatrix} Z^3_{r-2}(\mathcal{T}_h^A)\otimes\mathbb{V} \\ V^3_{r-3}(\mathcal{T}_h^A)\otimes\mathbb{V} \end{bmatrix}$$

is exact and the kernel of the first operator above is isomorphic to $\mathcal{R}$. When $r = 4$, the last operator remains surjective.

Proof. The case $r \geq 5$ follows by the $r \geq 5$ case of Theorem 6.2 and Proposition 2.3's item (1). The statement for $r = 4$ follows from the surjectivity of the divergence asserted by Theorem 6.2 in the $r = 4$ case and Proposition 2.3's item (2). ☐

Theorem 6.6. $U^1_r(\mathcal{T}_h^A) = \{u \in H^1(\Omega;\mathbb{S}) : (\mathrm{curl}\, u)' \in W^1_{r-1}(\mathcal{T}_h^A) \otimes \mathbb{V},$ *u is C^1 at the mesh vertices of $\mathcal{T}_h$,* $\mathrm{inc}\, u$ *is C^0 at the mesh vertices of $\mathcal{T}_h$, and $u|_T \in U^1_r(T^A)$ for all mesh elements $T \in \mathcal{T}_h\}$.*

Proof. This can be proved along the lines of the proof of Theorem 6.6 using Theorem 6.5. ☐

Theorem 6.7. *The following sequence of global finite element spaces*

$$0 \longrightarrow \mathcal{R} \xrightarrow{\subset} U^0_{r+1}(\mathcal{T}_h^A) \xrightarrow{\varepsilon} U^1_r(\mathcal{T}_h^A) \xrightarrow{\mathrm{inc}} U^2_{r-2}(\mathcal{T}_h^A) \xrightarrow{\mathrm{div}} U^3_{r-3}(\mathcal{T}_h^A) \longrightarrow 0. \tag{6.6}$$

is a complex and is exact (on contractible domains) for $r \geq 4$.

Proof. For $r \geq 5$, the proof is along the lines of the proof of Theorem 5.4 using Theorem 6.5.

For $r = 4$, first note that the surjectivity of $\mathrm{div} : U^2_{r-2}(\mathcal{T}_h^{\mathrm{A}}) \to U^3_{r-3}(\mathcal{T}_h^{\mathrm{A}})$ follows from Theorem 6.5. Next, we show that $\varepsilon(U^0_{r+1}(\mathcal{T}_h^{\mathrm{A}})) = \ker(\mathrm{inc}, U^1_r(\mathcal{T}_h^{\mathrm{A}}))$ for $r = 4$. Any $u \in U^1_r(\mathcal{T}_h^{\mathrm{A}})$ with $\mathrm{inc}\, u = 0$ may be written as $\varepsilon(v)$ for some $v \in H^2(\Omega)$ by the exactness of (1.2). Now, on each mesh element $T \in \mathcal{T}_h$, split into an Alfeld split T^{A}, the local exactness result of Theorem 5.4, applied with $r = 4$, shows that there is a $w_T \in U^0_{r+1}(T^{\mathrm{A}})$ satisfying $\varepsilon(w_T) = u|_T$. In other words, $\varepsilon(w_T - u)|_T = 0$, which implies that on each T, the function v must equal a polynomial of the form $v|_T = w_T + r_T$ for some $r_T \in \mathcal{R}(T) \subset [\mathcal{P}_1(T)]^3$. Thus $u = \varepsilon(v)$ and $v \in H^2(\Omega) \cap \mathcal{P}_5(\mathcal{T}_h^{\mathrm{A}}) \subseteq U^0_5(\mathcal{T}_h^{\mathrm{A}})$. To complete the proof of exactness, we now only need to show that $\mathrm{curl} : U^1_r(\mathcal{T}_h^{\mathrm{A}}) \to \ker(\mathrm{div}, U^2_{r-2}(\mathcal{T}_h^{\mathrm{A}}))$ is onto. To this end, we note from (3.9), (5.22), (5.27), and (5.28) that the following dimension count holds:

$$\begin{aligned}
\dim(U^0_{r+1}(\mathcal{T}_h^{\mathrm{A}})) &= 10\#_0 + (3r-10)\#_1 + (r^2 - 5r + 7)\#_2 + N_0\#_3, \\
\dim(U^1_r(\mathcal{T}_h^{\mathrm{A}})) &= 30\#_0 + 3(3r-8)\#_1 + 3/2(3r^2 - 11r + 4)\#_2 + N_1\#_3, \\
\dim(U^2_{r-2}(\mathcal{T}_h^{\mathrm{A}})) &= 6\#_0 + 3/2(r-3)(r+2)\#_2 + N_2\#_3, \\
\dim(U^3_{r-3}(\mathcal{T}_h^{\mathrm{A}})) &= N_3\#_3.
\end{aligned}$$

By the definition of interior degrees of freedom, we have $N_0 + N_2 = N_1 + N_3 - 6$. By the Euler formula, we have

$$\dim(U^0_{r+1}(\mathcal{T}_h^{\mathrm{A}})) - \dim(U^1_r(\mathcal{T}_h^{\mathrm{A}})) + \dim(U^2_{r-2}(\mathcal{T}_h^{\mathrm{A}})) - \dim(U^3_{r-3}(\mathcal{T}_h^{\mathrm{A}})) = 6,$$

which shows that $\dim \mathrm{curl}\, U^1_r(\mathcal{T}_h^{\mathrm{A}}) = \dim \ker(\mathrm{div}, U^2_{r-2}(\mathcal{T}_h^{\mathrm{A}}))$. ☐

Appendix A. Supersmoothness

Consider a tetrahedron T and its Alfeld split $\{T_i\}$ as in the rest of the paper. Proposition 2.1's items (1) and (2) are a consequence of the following fact proved in [1]: if $v \in C^1(T)$ and $v|_{T_i}$ is in $C^\infty(T_i)$, then v is C^2 at the vertices of T. Such serendipitous "supersmoothness" at some points was observed on triangles earlier [20]. In Theorem A.1 below, we establish a supersmoothness result in the same spirit for 1-forms. In fact, the earlier result of Alfeld follows from the theorem, as noted in Corollary A.2. Items (3) and (4) of Proposition 2.1 follow from the arguments below. (The proof will show that the assumption that $v|_{T_i}$ is infinitely smooth can be relaxed, but this generalization is not important for our purposes.)

Theorem A.1. *Suppose v is in $C^0(T)^3$, $v_i = v|_{T_i} \in C^\infty(T_i)^3$, and* $\operatorname{curl} v$ *is C^0 at the vertices x_i of T. Then v is C^1 at x_i.*

Proof. Let $F_{ij} = \partial T_i \cap \partial T_j$ and let TF_{ij} denote the tangent plane of F_{ij}. Let $c \in \mathbb{V}$ and $\tau \in TF_{ij}$. The first observation needed for this proof is that

$$c \cdot (\operatorname{grad} v_i)\tau = c \cdot (\operatorname{grad} v_j)\tau \quad \text{on } F_{ij}. \tag{A.1}$$

This is because the continuity of v requires $(v_i - v_j) \cdot c$ to vanish on F_{ij} for any $c \in \mathbb{V}$, so its tangential derivatives also vanish on F_{ij}.

We claim that at a vertex of T on F_{ij}, we also have

$$\tau \cdot (\operatorname{grad} v_i)c = \tau \cdot (\operatorname{grad} v_j)c. \tag{A.2}$$

To show this, consider x_1, a common vertex of T and F_{23}. Then, since the scalar $\tau \cdot (\operatorname{grad} v_i)(x_1)\, c$ equals its transpose, we have

$$\begin{aligned}
\tau \cdot (\operatorname{grad} v_2)\, c &= c \cdot (\operatorname{grad} v_2)'\tau && \text{at } x_1,\\
&= c \cdot \big((\operatorname{grad} v_2)' - (\operatorname{grad} v_2)\big)\,\tau + c \cdot (\operatorname{grad} v_2)\tau \\
&= c \cdot \big((\operatorname{grad} v_2)' - (\operatorname{grad} v_2)\big)\,\tau + c \cdot (\operatorname{grad} v_3)\tau && \text{by (A.1)},\\
&= c \cdot \big((\operatorname{grad} v_3)' - (\operatorname{grad} v_3)\big)\,\tau + c \cdot (\operatorname{grad} v_3)\tau && \text{as } \operatorname{curl} v \text{ is } C^0 \text{ at } x_1\\
&= \tau \cdot (\operatorname{grad} v_3)\, c.
\end{aligned}$$

This argument can be repeated at other vertices to finish the proof of (A.2).

Now we are ready to show that v is C^1 at x_i. Let $\tau_i = (x_i - z)/\|x_i - z\|$ and $\tau_{ij} = (x_i - x_j)/\|x_i - x_j\|$. Without loss of generality, we focus on one vertex, say x_1. At x_1,

$$c \cdot (\operatorname{grad} v_2)\tau_0 = c \cdot (\operatorname{grad} v_3)\tau_0 \qquad \tau_0(\operatorname{grad} v_2)c = \tau_0(\operatorname{grad} v_3)c \tag{A.3a}$$

$$c \cdot (\operatorname{grad} v_2)\tau_{10} = c \cdot (\operatorname{grad} v_3)\tau_{10} \qquad \tau_{10}(\operatorname{grad} v_2)c = \tau_{10}(\operatorname{grad} v_3)c. \tag{A.3b}$$

The left equalities follow from (A.1) and the right ones from (A.2). Furthermore, at x_1 we have

$$\begin{aligned}
\tau_{12} \cdot (\operatorname{grad} v_2)\tau_{13} &= \tau_{12} \cdot (\operatorname{grad} v_0)\tau_{13} && \text{by (A.1) applied to } F_{20}\\
&= \tau_{12} \cdot (\operatorname{grad} v_3)\tau_{13} && \text{by (A.2) applied to } F_{03}.
\end{aligned}$$

Therefore, $\tau_{12} \cdot (\operatorname{grad} v_2)\tau_{13} = \tau_{12} \cdot (\operatorname{grad} v_3)\tau_{13}$ at x_1. Writing τ_{12} as a linear combination of τ_0, τ_{10} and τ_{13}, and using the equalities of (A.3) in the right panel, we conclude that

$$\tau_{13} \cdot (\operatorname{grad} v_2)\tau_{13} = \tau_{13} \cdot (\operatorname{grad} v_3)\tau_{13} \qquad \text{at } x_1. \tag{A.4}$$

The identities of (A.4) and (A.3) together yield the equality of $\operatorname{grad} v_2$ and $\operatorname{grad} v_3$ at x_1. Repeating this argument for every pair of v_i meeting at a vertex, the proof is finished. □

Corollary A.2. *If $w \in C^1(T)$ and $w|_{T_i}$ is in $C^\infty(T_i)$, then w is C^2 at the vertices of T.*

Proof. This follows by applying Theorem A.1 with $v = \operatorname{grad} w$. □

References

[1] P. Alfeld. A trivariate Clough–Tocher scheme for tetrahedral data. *Computer Aided Geometric Design*, 1(2):169–181, 1984.

[2] S. Amstutz and N. Van Goethem. The incompatibility operator: from Riemann's intrinsic view of geometry to a new model of elasto-plasticity. In *Topics in Applied Analysis and Optimisation*, pages 33–70. Springer, 2019.

[3] A. Angoshtari and A. Yavari. Differential complexes in continuum mechanics. *Archive for Rational Mechanics and Analysis*, 216(1):193–220, 2015.

[4] D. Arnold, G. Awanou, and R. Winther. Finite elements for symmetric tensors in three dimensions. *Mathematics of Computation*, 77(263):1229–1251, 2008.

[5] D. Arnold, R. Falk, and R. Winther. Mixed finite element methods for linear elasticity with weakly imposed symmetry. *Mathematics of Computation*, 76(260):1699–1723, 2007.

[6] D. N. Arnold. *Finite element exterior calculus*. SIAM, 2018.

[7] D. N. Arnold, R. S. Falk, and R. Winther. Differential complexes and stability of finite element methods II: The elasticity complex. In *Compatible spatial discretizations*, pages 47–67. Springer, 2006.

[8] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta numerica*, 15:1–155, 2006.

[9] D. N. Arnold, R. S. Falk, and R. Winther. Mixed finite element methods for linear elasticity with weakly imposed symmetry. *Math. Comp.*, 76(260):1699–1723 (electronic), 2007.

[10] D. N. Arnold and K. Hu. Complexes from complexes. *arXiv preprint arXiv:2005.12437*, 2020.

[11] D. N. Arnold and R. Winther. Mixed finite elements for elasticity. *Numerische Mathematik*, 92(3):401–419, 2002.

[12] E. Calabi. On compact, Riemannian manifolds with constant curvature I. *Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1961*, pages 155–180, 1961.

[13] A. Čap, J. Slovák, and V. Souček. Bernstein-Gelfand-Gelfand sequences. *Annals of Mathematics*, 154(1):97–113, 2001.

[14] S. H. Christiansen. On the linearization of Regge calculus. *Numerische Mathematik*, 119(4):613–640, 2011.

[15] S. H. Christiansen, J. Hu, and K. Hu. Nodal finite element de Rham complexes. *Numerische Mathematik*, 139(2):411–446, 2018.

[16] S. H. Christiansen and K. Hu. Finite element systems for vector bundles: elasticity and curvature. *arXiv preprint arXiv:1906.09128*, 2019.

[17] P. G. Ciarlet, L. Gratie, and C. Mardare. Intrinsic methods in elasticity: a mathematical survey. *Discrete & Continuous Dynamical Systems-A*, 23(1&2):133, 2009.

[18] M. Costabel and A. McIntosh. On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Mathematische Zeitschrift*, 265(2):297–320, 2010.

[19] M. Eastwood. A complex from linear elasticity. In *Proceedings of the 19th Winter School" Geometry and Physics"*, pages 23–29. Circolo Matematico di Palermo, 2000.

[20] G. Farin. Bézier polynomials over triangles. Technical Report TR/91, Department of Mathematics, Brunel University, Uxbridge, UK, 1980.

[21] P. E. Farrell, L. Mitchell, L. R. Scott, and F. Wechsung. A Reynolds-robust preconditioner for the Reynolds-robust Scott-Vogelius discretization of the stationary incompressible Navier-Stokes equations. *arXiv preprint arXiv:2004.09398*, 2020.

[22] G. Fu, J. Guzman, and M. Neilan. Exact smooth piecewise polynomial sequences on Alfeld splits. *Mathematics of Computation*, 2020.

[23] N. V. Goethem. The non-Riemannian dislocated crystal: A tribute to Ekkehart Kröner (1919-2000). *Journal of Geometric Mechanics*, 2:303, 2010.

[24] J. Gopalakrishnan. *Oberwolfach Reports: MFO Workshop 2225*. MFO, 2022. Workshop on "Hilbert Complexes: Analysis, Applications, and Discretizations" held 19 Jun – 25 Jun 2022. https://doi.org/10.14760/OWR-2022-29.

[25] J. Gopalakrishnan and J. Guzmán. A second elasticity element using the matrix bubble. *IMA Journal of Numerical Analysis*, 32(1):352–372, 2012.

[26] J. Guzmán and M. Neilan. Inf-sup stable finite elements on barycentric refinements producing divergence–free approximations in arbitrary dimensions. *SIAM Journal on Numerical Analysis*, 56(5):2826–2844, 2018.

[27] P. Hauret, E. Kuhl, and M. Ortiz. Diamond elements: a finite element/discrete-mechanics approximation scheme with guaranteed optimal convergence in incompressible elasticity. *International Journal for Numerical Methods in Engineering*, 72(3):253–294, 2007.

[28] J. Hu and S. Zhang. A family of conforming mixed finite elements for linear elasticity on triangular grids. *arXiv preprint arXiv:1406.7457*, 2014.

[29] J. Hu and S. Zhang. A family of symmetric mixed finite elements for linear elasticity on tetrahedral grids. *Science China Mathematics*, 58(2):297–307, 2015.

[30] C. Johnson and B. Mercier. Some equilibrium finite element methods for two-dimensional elasticity problems. *Numerische Mathematik*, 30(1):103–116, 1978.

[31] E. Kröner. Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. *Arch. Rational Mech. Anal.*, 4:273–334 (1960), 1960.

[32] Y.-J. Lee, J. Wu, J. Xu, and L. Zikatanov. Robust subspace correction methods for nearly singular systems. *Mathematical Models and Methods in Applied Sciences*, 17(11):1937–1963, 2007.

[33] L. Li. *Regge finite elements with applications in solid mechanics and relativity.* PhD thesis, University of Minnesota, 2018.

[34] J.-C. Nédélec. A new family of mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 50(1):57–81, 1986.

[35] M. Neunteufel and J. Schöberl. Avoiding Membrane Locking with Regge Interpolation. *Preprint: arXiv:1907.06232*, 2020.

[36] T. Regge. General relativity without coordinates. *Il Nuovo Cimento (1955-1965)*, 19(3):558–571, 1961.

[37] J. Schöberl. *Robust multigrid methods for parameter dependent problems.* PhD thesis, Johannes Kepler Universität Linz, 1999.

[38] R. Stenberg. A nonstandard mixed finite element family. *Numerische Mathematik*, 115(1):131–139, 2010.

Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway

Email address: snorrec@math.uio.no

Portland State University (MTH), PO Box 751, Portland, OR 97207, USA

Email address: gjay@pdx.edu

Division of Applied Mathematics, Brown University, Box F, 182 George Street, Providence, RI 02912, USA

Email address: johnny_guzman@brown.edu

School of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis, MN 55455-0488, USA

Email address: khu@umn.edu