Optimal Control of a Large Population of Randomly Interrogated Interacting Agents

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Abstract

This article investigates a stochastic optimal control problem with linear Gaussian dynamics, quadratic performance measure, but non-Gaussian observations. The linear Gaussian dynamics characterizes a large number of interacting agents evolving under a centralized control and external disturbances. The aggregate state of the agents is only partially known to the centralized controller by means of the samples taken randomly in time and from anonymous randomly selected agents. Due to removal of the agent identity from the samples, the observation set has a non-Gaussian structure, and as a consequence, the optimal control law that minimizes a quadratic cost is essentially nonlinear and infinite-dimensional, for any finite number of agents. For infinitely many agents, however, this paper shows that the optimal control law is the solution to a reduced order, finite-dimensional linear quadratic Gaussian problem with Gaussian observations sampled only in time. For this problem, the separation principle holds and is used to develop an explicit optimal control law by combining a linear quadratic regulator with a separately designed finite-dimensional minimum mean square error state estimator. Conditions are presented under which this simple optimal control law can be adopted as a suboptimal control law for finitely many agents.

Index Terms

Linear quadratic Gaussian, non-Gaussian observation, separation principle, space-time sampling, stochastic control.

I. Introduction

THIS paper is mainly motivated by applications in control of magnetic fluids, in which a large number of interacting magnetic nanoparticles are collectively driven toward a desired target using an external magnetic field [1]–[8]. This magnetic field is controlled by a feedback loop that incorporates samples of the nanoparticle positions taken randomly in time and space by a high-resolution photodetector array. As a main component of a fluorescence imaging system [9], the photodetector array records intermittent flashes of light emitted randomly from the fluorescent coating of the magnetic nanoparticles excited by a laser source [10].

Beyond this motivating application, the mathematical model and problem formulation in this paper are general enough to fit into a broad class of applications in which the dynamics of a large number of randomly interrogated, interacting agents is manipulated by a centralized control identically applied to all these agents.

This formulation describes the dynamics of a large number of interacting agents by a set of coupled linear Gaussian state-space equations; each equation representing a single agent, and their coupling represents the interactions between the agents. The stochastic inputs included in the equations are intended to represent model uncertainty and external disturbances applied to the agents. A common control input applied to all equations characterizes an external manipulation identically affecting the agents. The aggregate state of the agents is partially observed by means of the samples taken randomly in time and in the set of agents (referred to as *space*). The distinctive feature of these samples is that the identify of the agents from which the samples are taken is not known to the measuring device. By removal of the agent identity from the measured samples, the observation set no longer remains linear Gaussian, as opposed to the more conventional sampling scheme that associates each sample to a specific agent.

The focus of this paper is on development of control laws for optimal regulation of the agents around the origin of their state space. This control task is formulated as minimization of an expected quadratic cost functional involving the aggregate state of the agents and the control applied to them collectively. This formulation would define a conventional linear quadratic Gaussian (LQG) problem, if each observed sample was tagged by the identity of an agent. In that case, the separation principle would hold, under which, the design of an optimal control law is decomposed into the convenient design of a linear quadratic regulator (LQR), and separately, a finite-dimensional minimum mean square error (MMSE) state estimator [11]–[19]. For the sampling scheme of this paper, however, the optimal control problem cannot be simply treated as an LQG problem, despite the linear Gaussian dynamics of the agents and the quadratic form of the adopted cost functional. In fact, for a finite number of agents, the solution to this problem is inherently nonlinear and infinite-dimensional.

For infinitely many agents, however, it is shown as the major contribution of this paper, that the formulated optimal control problem reduces to a finite-dimensional LQG problem holding the separation principle. The solution to this reduced problem introduces a finite-dimensional control law for infinitely many agents, although the optimal control law for any finite number of

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agents is essentially infinite-dimensional. This finding adds the case of this paper to the short list of few linear quadratic control problems with non-Gaussian measurements known for holding the separation principle [20]–[22]. This principle is exploited then to obtain the explicit form of the optimal control law by simply combining a LQR state feedback with a finite-dimensional MMSE state estimator.

The stochastic optimal control problem studied in this paper resembles the class of *mean field game* (MFG) problems, in the sense that both problems concern the dynamical behavior and design of optimal control for a large population of interacting agents [23]–[33]. Specifically, the linear Gaussian model and the quadratic cost functional of this paper apparently resembles the subclass of LQG mean field games [23]–[28]. Yet, the two problems fundamentally differ in the nature and objectives of control, and the structure of the observation set.

In this paper, control is essentially centralized, and is applied collectively to all agents by a single controller, while in MFG problems, each agent is individually controlled by its own local controller. Further, the control in this paper is generated from the partial observations of the aggregate state of all agents, and in this sense, the observation set is centralized. In MFG problems, however, the observation set is decentralized, in the sense that each agent only has access to the complete or partial knowledge of its own state, and possibly, the state of a single major player, not having a counterpart in this paper [24]–[32]. Finally, the control in this paper optimizes a single measure of the collective performance of all agents, in opposition to MFG problems in which each agent independently optimizes its own performance measure (and therefore, they are called games). A mean field optimal control problem studied in [34], similar to this paper, adopts a single measure of collective performance, but unlike this paper, shares the same decentralized structure of control and observation set with MFG problems.

The stochastic dynamical model in this paper with infinitely many agents is primarily intended to describe the dynamics of magnetic fluids controlled under magnetic fields. In [1]–[4], we represented this dynamics by a partial differential equation (PDE), considering the magnetic fluid as a continuum mass. It turns out that a PDE is an unnecessarily complex description for the dynamics of magnetic fluids, at least for the purpose of magnetic control design. In fact, a magnetic fluid is a discrete set consisting a large number of magnetic nanoparticles, rather than a continuum mass. Hence, the model of this paper seems a more natural framework to describe the dynamics of magnetic fluids. Furthermore, the PDE model of magnetic fluids adopted in [1]–[4] disregards the magnetic interactions between the nanoparticles comprising the magnetic fluid. This potentially important factor is appropriately accommodated by the model of this paper.

The dynamical behavior of the nanoparticles comprising a magnetic fluid is indeed nonlinear. The linear model proposed in this paper only describes the small deviations of the particles from a nominal trajectory generated by some coarse open-loop control. The objectives and design of this open-loop control are beyond the scope of this paper; the goal here is to develop a fine closed-loop control to maintain the particles close to the nominal trajectory despite the external disturbances and model uncertainty.

The remainder of this paper is organized into four sections followed by a conclusion and an appendix. In Section II, the dynamical model concerned in this paper is introduced and its associated optimal control problem is stated. The major results of this paper focusing on model reduction for infinitely many agents are presented in Section III. This section reduces the optimal control problem of Section II into a finite-dimensional LQG problem with measurements randomly sampled in time. The solution to this nonstandard LQG problem is presented in Section IV. The proofs of theorems and lemmas in Sections III and IV are presented in Appendix to improve the readability of the text. Finally, Section V investigates the possibility to adopt the simple optimal control law developed for infinitely many agents as a suboptimal control law for a large but finite number of agents.

II. Model and Problem Statement

A population of N interacting identical agents (or particles) is considered and their dynamics is represented by N coupled linear stochastic differential equations (SDE) of the form

$$dx_t^n = A_t x_t^n dt + \frac{1}{N}\sum_{i=1}^{N} F_t\left(x_t^n - x_t^i\right)dt + B_t u_t dt + D_t dw_t + G_t d\xi_t^n, \quad n = 1, 2, \ldots, N.$$

As a shorthand, these SDEs are expressed throughout the paper by the state-space equations

$$\dot{x}_t^n = A_t x_t^n + \frac{1}{N}\sum_{i=1}^{N} F_t\left(x_t^n - x_t^i\right) + B_t u_t + D_t \dot{w}_t + G_t \dot{\xi}_t^n, \tag{1}$$

where $x_t^n \in \mathbb{R}^{n_x}$, $n = 1, 2, \ldots, N$ denotes the state of agent n at time $t \geqslant 0$ and $u_t \in \mathbb{R}^{n_u}$ is a common control vector applied to all N agents identically. The stochastic process $\{w_t\}$ and the collection of N stochastic processes $\{\xi_t^n\}$, $n = 1, 2, \ldots, N$ are statistically independent standard Wiener processes of the dimensions n_w and n_ξ, respectively. The formal derivative of these processes are white noises denoted by $\{\dot{w}_t\}$ and $\left\{\dot{\xi}_t^n\right\}$. The time-varying matrices A_t, F_t, B_t, D_t, and G_t are assumed bounded, measurable, and of appropriate dimensions.

According to the set of state-space equations (1), the agents are coupled through three different mechanisms: the sum on the right-hand side describes the pairwise interactions between the agents, the control u_t identically manipulates all N agents, and $\dot{w}_t$ is a common disturbance or modelling error perturbing all agents in a similar manner. Beyond the coupling between the agents, each agent is individually perturbed by a Brownian motion represented by $\dot{\xi}_t^n$.

The initial states $x_0^1, x_0^2, \ldots, x_0^N$ of the stochastic state-space equations (1) are statistically independent of $\{w_t\}$ and $\{\xi_t^n\}$, $n = 1, 2, \ldots, N$, and are modeled as identically distributed Gaussian random vectors with the expected value $\mathrm{E}\left[x_0^n\right] = \bar{x}_0$ and the covariance and cross-covariance matrices

$$\mathrm{E}\left[\left(x_0^n - \bar{x}_0\right)\left(x_0^n - \bar{x}_0\right)^T\right] = S_z + S_e, \quad n = 1, 2, \ldots, N$$
$$\mathrm{E}\left[\left(x_0^n - \bar{x}_0\right)\left(x_0^m - \bar{x}_0\right)^T\right] = S_z, \quad n \neq m = 1, 2, \ldots, N,$$

where S_z and S_e are $n_x \times n_x$ positive semidefinite matrices. Hence, each initial state can be decomposed into the sum

$$x_0^n = z_0 + e_0^n, \quad n = 1, 2, \ldots, N \tag{2}$$

of two independent Gaussian random vectors z_0 and e_0^n of the expected values $\mathrm{E}\left[z_0\right] = \bar{x}_0$ and $\mathrm{E}\left[e_0^n\right] = 0$ and the covariance matrices S_z and S_e, respectively. The common term z_0 in this decomposition represents an identical shift of all agents from the origin, while $e_0^1, e_0^2, \ldots, e_0^N$ describe the random dispersion of the agents around the central point z_0.

The measurement model considered in this paper primarily intends to describe the output signal of a photodetector array constructing the space-time distribution of a large number of magnetic nanoparticles by the fluorescence imaging technique. This technique relies on the flashes of light emitted from the fluorescent coating of the nanoparticles excited by a coherent light source. These flashes of light are intrinsically emitted at random times from random nanoparticles by the nature of the fluorescence phenomenon [10]. Beyond this phenomenon, the measurement model of this paper is applicable to any scenario involving multiple agents that intermittently report their state to a supervisory center without a means for synchronization. This model is general enough to include the special case of periodic sampling in time (but not space) when the agents are synchronized by a common clock.

The observation set provided for closed-loop control of N agents is a discrete set of space-time points sampled randomly from them at the sampling times $0 < \tau_1 < \tau_2 < \tau_3 < \cdots$. The observation set generated during the time interval $[0, t]$ is denoted by $\mathcal{Y}_t^N$ and is expressed as

$$\mathcal{Y}_t^N = \emptyset, \quad t \in [0, \tau_1] \tag{3a}$$
$$\mathcal{Y}_t^N = \left\{\left(\tau_1, y_1\right), \left(\tau_2, y_2\right), \ldots, \left(\tau_k, y_k\right)\right\}, \quad t \in \left(\tau_k, \tau_{k+1}\right], \quad k = 1, 2, 3, \ldots. \tag{3b}$$

The spatial components $y_1, y_2, y_3, \ldots$ of the space-time points in this observation set are random vectors in $\mathbb{R}^{n_y}$ statistically depending on the state of the agents, as explained below.

Suppose C_t is an $n_y \times n_x$ bounded matrix and a continuous function of time. Assume that $\{\nu_1, \nu_2, \nu_3, \ldots\}$ is a sequence of independent integer random variables uniformly distributed on $\{1, 2, \ldots, N\}$, and that $\{v_1, v_2, v_3, \ldots\}$ is an independent identically distributed sequence of zero-mean Gaussian vectors in $\mathbb{R}^{n_y}$ with the covariance matrix V. Assume further that the sequences $\{\nu_k\}$, $\{v_k\}$, and $\{\tau_k\}$ are mutually independent, and independent of $\{w_t\}$, $\{\xi_t^n\}$, $n = 1, 2, \ldots, N$, and the initial states $x_0^1, x_0^2, \ldots, x_0^N$. Then, the spatial components y_k of the observation set are given by

$$y_k = C_{\tau_k} x_{\tau_k}^{\nu_k} + v_k, \quad k = 1, 2, 3, \ldots. \tag{4}$$

This expression constructs the observation set by randomly sampling the aggregate state $\left(x_t^1, x_t^2, \ldots, x_t^N\right)$ in the agent set (referred to as space) and in time. The $n_y \times n_x$ matrix C_t is incorporated into the model to extend its application to those measuring devices which can only observe an n_y-dimensional subspace of the entire state space of each agent ($n_y < n_x$). In addition, the random sequence $v_1, v_2, v_3, \ldots$ is introduced to represent the measurement noise.

In the simplest form, the sampling times $\tau_1, \tau_2, \tau_3, \ldots$ can take deterministic values, for instance, multiples of a constant sampling period that implement a periodic sampling scheme. In the more general scenario of this paper, $\{\tau_1, \tau_2, \tau_3, \ldots\}$ is a point process consisting of the transition times of a counting process $\{\eta_t,\ t \geqslant 0\}$. This process is fairly general and only needs to satisfy two mild technical assumptions

$$\Pr\left\{\tau_k = \tau_{k+1}\right\} = 0, \quad k = 1, 2, 3, \ldots$$
$$\mathrm{E}\left[\alpha^{\eta_t}\right] < \infty, \quad |\alpha| < \infty, \quad 0 \leqslant t < \infty. \tag{5}$$

An example of $\{\eta_t\}$ which holds both these condition is a homogenous Poisson counter with the constant rate $\lambda > 0$. In addition, periodic sampling with a period h_s is represented by a point process $\{\tau_1, \tau_2, \tau_3, \ldots\}$ which assigns probability 1 to the single sample path $h_s, 2h_s, 3h_s \ldots$. Therefore, the results of this paper identically hold for a periodic sampling scheme.

In the context of the magnetic fluids controlled by magnetic fields, the original dynamics of the magnetic nanoparticles is highly nonlinear. However, this nonlinear dynamics can be linearized around a nominal trajectory to approximate it with the linear time-varying model (1). The nominal trajectory is generated by a coarse open-loop control designed to effectively drive the magnetic nanoparticles along a desired path, albeit in the absence of disturbances and modeling errors. Then, the linear model (1) is utilized for design of a fine feedback control that enhances the coarse open-loop control by suppressing disturbances and modeling errors.

The state and control vectors in the state-space equations (1) represent the deviations of the state and control of the original nonlinear system from their nominal values. Then, the goal of feedback control is to maintain x_t^n, $n = 1, 2, \ldots, N$ and u_t as close as possible to zero. This objective is formulated in this paper as minimizing the expected quadratic cost

$$J_N = \mathrm{E}\left[\int_0^T \left(\frac{1}{N}\sum_{n=1}^{N} \|x_t^n\|_{Q_t}^2 + \|u_t\|_{R_t}^2\right) dt + \frac{1}{N}\sum_{n=1}^{N} \|x_T^n\|_{Q_f}^2\right]. \tag{6}$$

Here, $T > 0$ is a fixed control time, Q_t, $t \geqslant 0$ and Q_f are $n_x \times n_x$ positive semidefinite matrices, R_t, $t \geqslant 0$ is an $n_u \times n_u$ positive definite matrix, and $\|x\|_Q$ denotes the weighted norm

$$\|x\|_Q = \left(x^T Q x\right)^{\frac{1}{2}}.$$

The problem in this paper is to develop an optimal control law that minimizes the cost functional (6) for infinitely many agents, i.e., for $N \to \infty$. This problem can be interpreted in different ways, for instance, the optimal control law can be viewed as the limit (at $N \to \infty$) of the sequence of optimal control laws developed for each $N = 1, 2, 3, \ldots$. Of course, this approach is barely tractable since it requires to obtain the optimal control law for each finite N, a problem inherently infinite-dimensional due to the non-Gaussian structure of the measurement model. A mathematically tractable alternative is established in Problem 1 below. Before stating this problem, it is necessary to provide a solid definition for a control law.

Definition 1: A *$\mathcal{Y}$-map* $\phi(\cdot)$ is a vector function assigning a vector $\phi\left(\bar{\mathcal{Y}}\right) \in \mathbb{R}^m$ to each instance

$$\bar{\mathcal{Y}} = \left\{(\bar{\tau}_1, \bar{y}_1), (\bar{\tau}_2, \bar{y}_2), \ldots, (\bar{\tau}_K, \bar{y}_K)\right\}$$

of the observation set $\mathcal{Y}_t^N$. A $\mathcal{Y}$-map is called *continuous* if it is continuous in $(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_K)$ for any fixed $(\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_K)$.

Definition 2: A *control law* $\mu(\cdot)$ is a time-dependent causal $\mathcal{Y}$-map (i.e., $\mu(\cdot, t)$ is a $\mathcal{Y}$-map for each fixed t) with values in $\mathbb{R}^{n_u}$. The causality of this map is defined by the property

$$\mu\left(\mathcal{Y}_s^N, t\right) = \mu\left(\mathcal{Y}_t^N, t\right), \quad s > t \geqslant 0.$$

The control law $\mu(\cdot)$ is *continuous* if $\mu(\cdot, t)$ is a continuous $\mathcal{Y}$-map for each fixed t.

The optimal control problem for infinitely many agents is formulated in this paper as follows.

Problem 1: For each fixed $N \in \mathbb{N}$, consider the stochastic state-space equations (1), the initial state (2), and the sampled observation set $\mathcal{Y}_t^N$ defined in (3). Let $\mu(\cdot)$ be a control law generating the $\mathcal{Y}_t^N$-measurable control u_t according to

$$u_t = \mu\left(\mathcal{Y}_t^N, t\right), \quad t \in [0, T]. \tag{7}$$

This control law is called admissible if it is continuous and for some $p > 1$, it satisfies the regularity condition

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathrm{E}\left[\|u_t\|^{2p}\right] < \infty. \tag{8}$$

For the control profile generated by (7) and its corresponding state trajectory $\left(x_t^1, x_t^2, \ldots, x_t^N\right)$, $t \in [0, T]$, construct the cost functional J_N in (6) and define the cost of each admissible control law $\mu(\cdot)$ as the limit

$$\mathcal{J}_\infty(\mu(\cdot)) = \lim_{N \to \infty} J_N. \tag{9}$$

Then, the goal is to minimize this cost functional on the set $\mathcal{C}$ of all admissible control laws, that is, to determine an optimal control law $\mu^*(\cdot) \in \mathcal{C}$ satisfying

$$\mathcal{J}_\infty(\mu^*(\cdot)) \leqslant \mathcal{J}_\infty(\mu(\cdot)), \quad \mu(\cdot) \in \mathcal{C}.$$

Remark 1: The set of admissible control laws includes at least a trivial control law $\mu(\cdot) = 0$, therefore it is nonempty.

In the remainder of this paper, a solution for Problem 1 is obtained in two steps. First, a reduced order finite-dimensional model is introduced in Section III that can adequately describe the dynamics of infinitely many agents to the extent necessary for solving Problem 1. This problem is next reformulated as a LQG problem for the reduced order system, which is a linear Gaussian state-space equation with a Gaussian observation set. For this new LQG problem, it is shown in Section IV that the separation principle holds, which makes it possible to develop an optimal control law via combining a LQR controller with a MMSE state estimator. Development of the optimal control law is discussed in details in Section IV.

III. MODEL REDUCTION FOR INFINITELY MANY AGENTS

As indicated by the stochastic state-space equation (1), the state of each agent is influenced by two categories of inputs: the common inputs u_t and $\dot{w}_t$ identically applied to all agents, and the individual inputs $\dot{\xi}_t^1, \dot{\xi}_t^2, \ldots, \dot{\xi}_t^N$ independently applied to each single agent. Similarly, each initial state x_0^n has two independent components: a common component z_0 shared by all agents, and an individual component e_0^n.

The linearity of the set of equations (1) allows to decompose the state x_t^n of each agent into a common component z_t and an individual component e_t^n such that

$$x_t^n = z_t + e_t^n, \quad n = 1, 2, \ldots, N. \tag{10}$$

Here, the common component z_t, $t \geqslant 0$ solves the SDE

$$\dot{z}_t = A_t z_t + B_t u_t + D_t \dot{w}_t \tag{11}$$

with the initial state z_0, and the individual components e_t^n are the solutions to the coupled SDEs

$$\dot{e}_t^n = A_t e_t^n + \frac{1}{N} \sum_{i=1}^{N} F_t \left(e_t^n - e_t^i \right) + G_t \dot{\xi}_t^n, \; n = 1, 2, \ldots, N \tag{12}$$

with the initial states e_0^n, $n = 1, 2, \ldots, N$. The initial states are Gaussian random vectors $z_0 \sim \mathcal{N}\left(\bar{x}_0, S_z\right)$ and $e_0^n \sim \mathcal{N}\left(0, S_e\right)$ independent of each other and independent of the stochastic processes $\{w_t\}$ and $\{\xi_t^n\}$, $n = 1, 2, \ldots, N$.

In terms of the states z_t and $e_t^1, e_t^2, \ldots, e_t^N$ of the state-space equations (11) and (12), the spatial components $y_1, y_2, y_3, \ldots$ of the observation set $\mathcal{Y}_t^N$ can be expressed as

$$y_k = C_{\tau_k} z_{\tau_k} + C_{\tau_k} e_{\tau_k}^{\nu_k} + v_k, \quad k = 1, 2, 3, \ldots \; . \tag{13}$$

The state vectors $e_t^1, e_t^2, \ldots, e_t^N$ in the dynamical system (12) are generated independent of the control input u_t. Therefore, they can be regarded as a colored noise process degrading the output vector (13).

In summary, the set of equations (11)-(13) can be interpreted as a dynamical system with the state z_t, the control input u_t, and the sampled output $\{y_1, y_2, y_3, \ldots\}$ which is corrupted by the colored measurement noise $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \ldots\right\}$ and the white measurement noise $\{v_1, v_2, v_3, \ldots\}$. The state of this system evolves in time according to the low-dimensional state-space equation (11), while the colored noise is generated by the high-dimensional state-space equations (12) and uniform spatial sampling of this high-dimensional state space.

It is clear from (12) that $\{e_t^n\}$, $n = 1, 2, \ldots, N$ are Gaussian stochastic processes. Therefore, conditioned on the sampling times $\{\tau_1, \tau_2, \tau_3, \ldots\}$, the random vectors $e_{\tau_k}^{\nu_k}$, $k = 1, 2, 3, \ldots$ are marginally Gaussian for any agent number N. However, by the following argument, they are not jointly Gaussian for a bounded N. Consider the random vector $\left(e_{t_1}^{\nu_1}, e_{t_2}^{\nu_2}, \ldots, e_{t_K}^{\nu_K}\right)$ for an arbitrary integer K and arbitrary but fixed sampling times $t_1, t_2, \ldots, t_K \in [0, T]$. Conditioned on $(\nu_1, \nu_2, \ldots, \nu_K)$, this random vector is Gaussian, its elements are conditionally dependent, and its conditional covariance matrix depends on the instances of $(\nu_1, \nu_2, \ldots, \nu_K)$. Therefore, its unconditional density function cannot remain Gaussian after averaging the conditional density function with respect to $(\nu_1, \nu_2, \ldots, \nu_K)$.

For the limiting case with infinite number of agents, it will be shown, however, that the random vector $\left(e_{t_1}^{\nu_1}, e_{t_2}^{\nu_2}, \ldots, e_{t_K}^{\nu_K}\right)$ converges in distribution to some Gaussian random vector with statistically independent elements. This key finding allows then to replace $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \ldots\right\}$ with a white Gaussian process, through which, the system of infinitely many agents is simply described by the finite-dimensional state-space equation (11) and a linear Gaussian measurement model. This key result is formally stated in Lemma 2. The proof of this lemma requires the technical background stated next in Lemma 1.

Lemma 1: Let A_t, F_t, and G_t be bounded and measurable functions and consider the set of coupled SDEs (12) driven by the independent standard Wiener processes $\left\{\xi_t^1, \xi_t^2, \ldots, \xi_t^N\right\}$. The initial states of these equations are independent zero-mean Gaussian random vectors of the bounded covariance matrix S_e. Then, the solution e_t^n to the SDE (12) can be decomposed into

$$e_t^n = \bar{e}_t^n + \frac{1}{\sqrt{N}} \beta_t, \quad n = 1, 2, \ldots, N, \quad t \in [0, T], \tag{14}$$

where $\bar{e}_t^n$ solves the SDE

$$\dot{\bar{e}}_t^n = \left(A_t + F_t\right) \bar{e}_t^n + G_t \dot{\xi}_t^n, \quad n = 1, 2, \ldots, N \tag{15}$$

with the same initial state as (12), and β_t is the solution to

$$\dot{\beta}_t = A_t \beta_t - \frac{1}{\sqrt{N}} \sum_{i=1}^{N} F_t \bar{e}_t^i \tag{16}$$

with the initial state $\beta_0 = 0$. Besides, $\{\bar{e}_t^n\}$ and $\{\beta_t\}$ maintain the following properties:

i. The set of stochastic processes $\{\bar{e}_t^n\}$, $n=1,2,\ldots,N$ are mutually independent and Gaussian, with zero mean and the covariance matrix Θ_t solving the Lyapunov matrix differential equation

$$\dot{\Theta}_t = (A_t + F_t)\,\Theta_t + \Theta_t\,(A_t + F_t)^T + G_t G_t^T \tag{17}$$

with the initial condition $\Theta_0 = S_e$.

ii. The stochastic process $\{\beta_t\}$ is zero-mean and Gaussian, and β_t has a uniformly bounded covariance matrix at each fixed $t \in [0,T]$.

Proof: See Appendix A. ■

An immediate consequence of Lemma 1 for infinitely many agents is that the Gaussian processes $\{e_t^n\}$, $n=1,2,\ldots,N$ tend to be statistically independent as $N \to \infty$. A heuristic argument for this claim maintains that $\{\bar{e}_t^n\}$, $n=1,2,\ldots,N$ are independent Gaussian processes according to (15), and that the random vector e_t^n converges to $\bar{e}_t^n$ as $N \to \infty$, by (14).

As the Gaussian processes $\{e_t^1\},\{e_t^2\},\{e_t^3\},\ldots$ become statistically independent for infinitely many agents (i.e., when $N \to \infty$), the samples $e_{t_1}^{\nu_1}, e_{t_2}^{\nu_2}, \ldots, e_{t_K}^{\nu_K}$ collected from these processes must be Gaussian and independent. This statement can be violated only by sampling the same process more than once. However, the occurrence probability of such event tends to 0 as $N \to \infty$. This heuristic argument is formalized in the following lemma.

Lemma 2: For each fixed integer N, define the Gaussian stochastic processes $\{e_t^1\},\{e_t^2\},\ldots,\{e_t^N\}$ as the solutions to the coupled SDEs (12) with independent zero-mean Gaussian initial states of the covariance matrix S_e. Let $\{\nu_1,\nu_2,\nu_3,\ldots\}$ be a sequence of independent integer random variables with uniform distribution on $\{1,2,\ldots,N\}$. For any integer K and any choice of distinct sampling times $t_1,t_2,\ldots,t_K \in [0,T]$, define the random vector

$$\psi_N = \left(e_{t_1}^{\nu_1}, e_{t_2}^{\nu_2}, \ldots, e_{t_K}^{\nu_K}\right). \tag{18}$$

Consider the $n_x \times n_x$ positive definite matrix Θ_t solving the Lyapunov equation (17) with the initial state S_e, and let M_t be any $n_x \times n_x$ matrix decomposing Θ_t into

$$M_t M_t^T = \Theta_t, \quad t \in [0,T]. \tag{19}$$

Let $\{\zeta_1,\zeta_2,\zeta_3,\ldots\}$ be a sequence of independent Gaussian random vectors in $\mathbb{R}^{n_x}$ with zero mean and identity covariance matrix, and define the Gaussian random vector

$$\psi = \left(M_{t_1}\zeta_1, M_{t_2}\zeta_2, \ldots, M_{t_K}\zeta_K\right). \tag{20}$$

Then, the sequence of random vector $\psi_1,\psi_2,\psi_3,\ldots$ converges in distribution to the Gaussian random vector ψ as $N \to \infty$, i.e., the limit

$$\lim_{N\to\infty} \Pr\{\psi_N \in \mathscr{C}\} = \Pr\{\psi \in \mathscr{C}\} \tag{21}$$

holds for any continuity set $\mathscr{C} \subset \mathbb{R}^{Kn_x}$ of the probability measure on the right-hand side.

Proof: See Appendix B. ■

Remark 2: An alternative and equivalent definition for the convergence of random vectors in distribution is given in terms of expected value [35, p. 253]. This new definition replaces the limit (21) in Lemma 2 with the limit

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\psi_N\right)\right] = \mathrm{E}\left[\phi\left(\psi\right)\right]$$

which holds for any bounded continuous scalar function $\phi(\cdot)$.

A direct implication of Lemma 2 is that the non-Gaussian colored noise $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \ldots\right\}$ in the observation model (13) can be replaced for infinitely many agents by a white Gaussian process $\{M_{\tau_1}\zeta_1, M_{\tau_2}\zeta_2, M_{\tau_3}\zeta_3, \ldots\}$. This replacement results in a reduced order finite-dimensional system representing the dynamics of infinitely many agents. This reduced order system maps the control input u_t in the SDE (11) into the output

$$\mathcal{Y}_t = \emptyset, \quad t \in [0,\tau_1] \tag{22a}$$

$$\mathcal{Y}_t = \left\{(\tau_1,y_1),(\tau_2,y_2),\ldots,(\tau_k,y_k)\right\}, \quad t \in (\tau_k,\tau_{k+1}], \quad k=1,2,3,\ldots \tag{22b}$$

by solving this SDE with an initial state $z_0 \sim \mathcal{N}(\bar{x}_0, S_z)$, and then, taking its state z_t to generate the spatial components

$$y_k = C_{\tau_k} z_{\tau_k} + C_{\tau_k} M_{\tau_k} \zeta_k + v_k, \quad k=1,2,3,\ldots\,. \tag{23}$$

Here, $\tau_1,\tau_2,\tau_3,\ldots$ denote the transition times of the counting process $\{\eta_t\}$, the matrix M_t solves (19), and $\{\zeta_1,\zeta_2,\zeta_3,\ldots\}$ and $\{v_1,v_2,v_3,\ldots\}$ are zero-mean white Gaussian processes with the covariance matrices $I_{n_x \times n_x}$ (identity matrix) and V, respectively. These white processes, the counting process $\{\eta_t\}$, the initial state z_0, and the Wiener process $\{w_t\}$ in (11) are mutually independent.

Remark 3: For infinitely many agents, Lemma 1 implies

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e_t^n = 0, \quad t \geqslant 0,$$

where the convergence is understood in the mean square sense. This limit in turn results in

$$z_t = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_t^n, \quad t \geqslant 0.$$

By this expression, the state z_t of the state-space equation (11) is the ensemble average of the state of agents. This quantity represents the center of mass of the agents in case the state of each agent represents its position in the 3D Euclidean space. Therefore, the reduced order system simplifies control of a population of infinitely many agents to control of their center of mass, while the deviations from the center of mass appear as an additional source of measurement noise.

For future reference, an N-agent system $\mathscr{S}_N$ as well as the reduced order system $\mathscr{S}$ are formally defined as follows.

Definition 3: For any integer N, the *N-agent system $\mathscr{S}_N$* is a stochastic rule that generates the observation set $\mathcal{Y}_t^N$ from the input process $\{u_s,\ 0 \leqslant s \leqslant t\}$ according to (3)-(4) and by solving the coupled SDEs (1) with the initial states (2).

Definition 4: The *reduced order system $\mathscr{S}$* is a stochastic rule that generates the observation set $\mathcal{Y}_t$ in terms of the input process $\{u_s,\ 0 \leqslant s \leqslant t\}$ according to (22)-(23) and by solving the SDE (11) with the initial state $z_0 \sim \mathcal{N}(\bar{x}_0, S_z)$.

It must be shown next that the sequence $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \ldots$ of N-agent systems converges to the reduced order system $\mathscr{S}$ in some reasonable sense. A possible approach followed in this paper is to show that the sequence $\mathcal{Y}_t^1, \mathcal{Y}_t^2, \mathcal{Y}_t^3, \ldots$ converges in distribution to $\mathcal{Y}_t$ when the systems $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \ldots$ and $\mathscr{S}$ are under the same control law. This approach is formalized in the following lemma.

Lemma 3: Consider the stochastic systems $\mathscr{S}$ and $\mathscr{S}_N$ as defined in Definitions 4 and 3. Assume that these systems are controlled under the same control law $\mu(\cdot)$, i.e., the controls applied to $\mathscr{S}$ and $\mathscr{S}_N$ are generated by $u_t = \mu(\mathcal{Y}_t, t)$ and $u_t = \mu\left(\mathcal{Y}_t^N, t\right)$, respectively. Then, for any continuous control law $\mu(\cdot)$, the sequence of outputs $\mathcal{Y}_t^1, \mathcal{Y}_t^2, \mathcal{Y}_t^3, \ldots$ converges in distribution to $\mathcal{Y}_t$, i.e, the limit

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] = \mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)\right], \quad t \in [0, T] \tag{24}$$

holds for every bounded and continuous scalar $\mathcal{Y}$-map $\phi(\cdot)$ [see Definition 1]. Furthermore, at each fixed $t \in [0, T]$, this limit holds (and $\mathrm{E}[\phi(\mathcal{Y}_t)]$ is bounded) for every nonnegative continuous $\phi(\cdot)$ for which there exists $p > 1$ such that

$$\sup_{N\in\mathbb{N}} \mathrm{E}\left[\phi^p\left(\mathcal{Y}_t^N\right)\right] < \infty. \tag{25}$$

Proof: See Appendix C. ■

Remark 4: A more heuristic notion of convergence can be defined for a sequence of systems $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \ldots$ in terms of the convergence of conditional probability measures. In this notion, the sequence $\mathcal{Y}_t^1, \mathcal{Y}_t^2, \mathcal{Y}_t^3, \ldots$ converges in distribution to $\mathcal{Y}_t$, while the systems $\mathscr{S}_1, \mathscr{S}_2, \mathscr{S}_3, \ldots$ and $\mathscr{S}$ are excited by the same control input. In particular, the limit

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right) | \mathscr{U}_t\right] = \mathrm{E}\left[\phi\left(\mathcal{Y}_t\right) | \mathscr{U}_t\right], \quad t \in [0, T]$$

must hold for all bounded, continuous $\mathcal{Y}$-maps $\phi(\cdot)$ and every deterministic bounded control profile $\mathscr{U}_t = \{u_s,\ 0 \leqslant s \leqslant t\}$. Convergence of stochastic systems in this sense is implied by the convergence in the sense of Lemma 3, taking the control law $\mu(\cdot)$ as a deterministic function not depending on $\mathcal{Y}_t^N$.

The main results of this section are presented in Theorem 1 below. This theorem states that the optimal control Problem 1 is well defined, and reformulates it as another optimal control problem subject to the reduced order system $\mathscr{S}$. This new problem admits an explicit solution presented in Section IV.

Theorem 1: Let $\mu(\cdot) \in \mathcal{C}$ be any admissible control law and $\mathcal{Y}_t$ be the observation set (22) generated by the reduced order system $\mathscr{S}$ in Definition 4. Apply the feedback control

$$u_t = \mu(\mathcal{Y}_t, t), \quad t \in [0, T]$$

to the stochastic state-space equation (11) with the Gaussian initial state $z_0 \sim \mathcal{N}(\bar{x}_0, S_z)$. In terms of this control and the state z_t of the state-space equation (11) define the functional

$$\mathcal{J}(\mu(\cdot)) = \mathrm{E}\left[\int_0^T \left(\|z_t\|_{Q_t}^2 + \|u_t\|_{R_t}^2\right) dt + \|z_T\|_{Q_f}^2\right] + J_e, \tag{26}$$

where the constant J_e is given by

$$J_e = \int_0^T \mathrm{tr}\left\{Q_t \Theta_t\right\} dt + \mathrm{tr}\left\{Q_f \Theta_T\right\} \tag{27}$$

in terms of the solution Θ_t to the Lyapunov matrix differential equation (17) with the initial state S_e. Then, the limit (9) in Problem 1 exists and the functional $\mathcal{J}_\infty(\mu(\cdot))$ in this problem is equal to $\mathcal{J}(\mu(\cdot))$ in the sense that

$$\mathcal{J}_\infty(\mu(\cdot)) = \mathcal{J}(\mu(\cdot)), \quad \mu(\cdot) \in \mathcal{C}.$$

Proof: See Appendix D. ■

IV. Optimal Control

This section presents a solution to Problem 1 based on the reduced order system $\mathscr{S}$ and Theorem 1. By this theorem, the infinite-dimensional optimal control Problem 1 reduces to the finite-dimensional problem of minimizing the quadratic cost functional (26) subject to the reduced order system $\mathscr{S}$. In fact, this new formulation represents a nonstandard LQG problem with the linear Gaussian dynamics (11), the linear Gaussian measurements (23), and the quadratic cost functional (26). The problem is nonstandard as the observation set includes samples randomly taken in time, rather than a continuous-time signal or samples taken uniformly in time. Yet, the separation principle identically holds for this nonstandard LQG problem, as shown in Section IV-A. Based on this principle, the solution to Problem 1 is decomposed into an estimation and a control problem discussed in Sections IV-B and IV-C, respectively.

A. Separation Principle

Consider the reduced order system consisting of the linear Gaussian dynamics (11) and the linear Gaussian observations sampled randomly in time according to (23). The history of the samples up to time t is collected in the observation set $\mathcal{Y}_t$, as defined in (22), and the problem is to obtain and admissible control law to minimize the quadratic cost (26). This section verifies that the separation principle holds for this problem, in the sense that it can be solved in two separate steps: designing an optimal control law while pretending the actual state of (11) is available for feedback, and then, replacing this state with its MMSE estimate, pretending that the control applied to (11) is a solely deterministic function of time. To verify this property, it must be shown that the information provided for estimation does not depend on the choice of control law.

In this paper, the separation principle is verified by adopting a *stochastic open loop* approach from [16], [19]. For a detailed discussion of this approach, the reader is referred to [16], [19], while its implication to the problem of this paper is discussed here. Let $z_t^{u=0}$ be the solution to the state-space equation (11) under the zero control $u_t = 0$, and $\mathcal{Y}_t^{u=0}$ denote its associated observation set generated as (22) with the measurements

$$y_k^{u=0} = C_{\tau_k} z_{\tau_k}^{u=0} + C_{\tau_k} M_{\tau_k} \zeta_k + v_k. \tag{28}$$

Suppose that $\mu(\cdot)$ is an admissible control law and construct the stochastic process $\left\{z_t^{w=0}\right\}$ as the solution to

$$\dot{z}_t^{w=0} = A_t z_t^{w=0} + B_t \mu\left(\mathcal{Y}_t^{u=0}, t\right) \tag{29}$$

with the initial state $z_0^{w=0} = 0$. In terms of $z_t^{u=0}$ and $z_t^{w=0}$, construct the observation set $\mathcal{Y}_t$ as (22) with the measurements

$$y_k = C_{\tau_k}\left(z_{\tau_k}^{u=0} + z_{\tau_k}^{w=0}\right) + C_{\tau_k} M_{\tau_k} \zeta_k + v_k. \tag{30}$$

Then, it must be shown that

$$\sigma\left(\mathcal{Y}_t\right) = \sigma\left(\mathcal{Y}_t^{u=0}\right), \quad t \in [0, T], \tag{31}$$

where $\sigma\left(\mathcal{Y}_t\right)$ and $\sigma\left(\mathcal{Y}_t^{u=0}\right)$ denote the σ-algebras generated by $\mathcal{Y}_t$ and $\mathcal{Y}_t^{u=0}$, respectively.

The condition (31) is verified by induction as follows. First, for any two successive sampling times τ_k and τ_{k+1}, it is shown that assuming the equality in (31) holds on $t \in [0, \tau_k]$, it will necessarily hold on $t \in [0, \tau_{k+1}]$. To that end, it is observed from (28) and (30) that

$$y_k = y_k^{u=0} + C_{\tau_k} z_{\tau_k}^{w=0}. \tag{32}$$

Moreover, (29) implies that $z_{\tau_k}^{w=0}$ is a sole function of $\mathcal{Y}_{\tau_k}^{u=0}$, and therefore, measurable with respect to both $\sigma\left(\mathcal{Y}_{\tau_k}^{u=0}\right)$, and by the induction assumption, $\sigma\left(\mathcal{Y}_{\tau_k}\right)$. It is observed from (22) that $\mathcal{Y}_t = \left\{\mathcal{Y}_{\tau_k}, (\tau_k, y_k)\right\}$ and $\mathcal{Y}_t^{u=0} = \left\{\mathcal{Y}_{\tau_k}^{u=0}, \left(\tau_k, y_k^{u=0}\right)\right\}$ hold on $t \in (\tau_k, \tau_{k+1}]$. Then, substituting (32) into the former and comparing the expressions

$$\begin{aligned} \mathcal{Y}_t &= \left\{\mathcal{Y}_{\tau_k}, \left(\tau_k, y_k^{u=0} + C_{\tau_k} z_{\tau_k}^{w=0}\right)\right\} \\ \mathcal{Y}_t^{u=0} &= \left\{\mathcal{Y}_{\tau_k}^{u=0}, \left(\tau_k, y_k^{u=0}\right)\right\} \end{aligned}$$

while $\sigma\left(\mathcal{Y}_{\tau_k}\right) = \sigma\left(\mathcal{Y}_{\tau_k}^{u=0}\right)$, the equality $\sigma\left(\mathcal{Y}_t\right) = \sigma\left(\mathcal{Y}_t^{u=0}\right)$ is concluded on $t \in (\tau_k, \tau_{k+1}]$, and therefore, on $t \in [0, \tau_{k+1}]$. Since $\mathcal{Y}_t = \mathcal{Y}_t^{u=0} = \emptyset$, $t \in [0, \tau_1]$, the equality in (31) trivially holds on $[0, \tau_1]$. Then, by induction, $\sigma\left(\mathcal{Y}_t\right) = \sigma\left(\mathcal{Y}_t^{u=0}\right)$ holds on $t \in [0, \tau_{last}]$, where τ_{last} is the last (largest) sampling time in the interval $[0, T]$. Finally, the condition (31) is established as $\sigma\left(\mathcal{Y}_t\right)$ and $\sigma\left(\mathcal{Y}_t^{u=0}\right)$ remain unchanged over $t \in [\tau_{last}, T]$.

B. Minimum Mean Square Error Estimator

Once applicability of the separation principle is verified, the MMSE state estimator for the dynamical system $\mathscr{S}$ is reduced to a continuous-discrete Kalman filter [36], [37]. Specifically, consider the state-space equation (11) with the state vector z_t, the control vector u_t, the initial state $z_0 \sim \mathcal{N}\left(\bar{x}_0, S_z\right)$, and the observation set $\mathcal{Y}_t$ generated according to (22)-(23). Denote the conditional mean and conditional covariance matrix of z_t given $\mathcal{Y}_t$ by $\hat{z}_t = \mathrm{E}\left[z_t | \mathcal{Y}_t\right]$ and $\Sigma_t = \mathrm{cov}\left(z_t | \mathcal{Y}_t\right)$, respectively.

Then, under any $\mathcal{Y}_t$-measurable control u_t, the estimators $\hat{z}_t$ and Σ_t can be determined from a continuous-discrete Kalman filter adopted from [36], [37]. Using the notation of this paper, this filter is expressed by the stochastic state-space equations

$$\dot{\hat{z}}_t = A_t \hat{z}_t + B_t u_t + L_t \left(y_{\eta_t} - C_t \hat{z}_t\right) \dot{\eta}_t \tag{33a}$$

$$\dot{\Sigma}_t = A_t \Sigma_t + \Sigma_t A_t^T + D_t D_t^T - L_t C_t \Sigma_t \dot{\eta}_t \tag{33b}$$

$$\dot{\Theta}_t = \left(A_t + F_t\right) \Theta_t + \Theta_t \left(A_t + F_t\right)^T + G_t G_t^T \tag{33c}$$

$$L_t = \Sigma_t C_t^T \left(C_t \left(\Sigma_t + \Theta_t\right) C_t^T + V\right)^{-1} \tag{33d}$$

with the state vector $(\hat{z}_t, \Sigma_t, \Theta_t)$ and the initial state

$$(\hat{z}_0, \Sigma_0, \Theta_0) = (\bar{x}_0, S_z, S_e)\,. \tag{34}$$

Among these equations, the first two are SDEs driven by the counting process $\{\eta_t\}$ (with the formal derivative $\{\dot{\eta}_t\}$), the third is an ordinary differential equation (ODE), and the last is a shorthand for expressing the gain matrix L_t in terms of the state vector $(\hat{z}_t, \Sigma_t, \Theta_t)$. The interpretation of the SDEs (33a) and (33b) is as follows. During the time interval $t \in (\tau_{k-1}, \tau_k]$ between successive sampling times τ_{k-1} and τ_k, the counting process η_t does not change and $\dot{\eta}_t = 0$ holds. Therefore, the SDEs (33a) and (33b) reduce [36], [37, p. 194] to the ODEs

$$\dot{\hat{z}}_t = A_t \hat{z}_t + B_t u_t \tag{35a}$$

$$\dot{\Sigma}_t = A_t \Sigma_t + \Sigma_t A_t^T + D_t D_t^T\,. \tag{35b}$$

These equations resemble the Kalman-Bucy filter without any measurements over $t \in (\tau_{k-1}, \tau_k]$.

At each sampling time τ_k, a new sample y_k is observed and is made available to the Kalman filter (33). This new sample is then incorporated into the state estimation via a discontinuity in the states of (33a) and (33b) represented by [37, p. 194]

$$\hat{z}_{\tau_k^+} = \hat{z}_{\tau_k} + L_{\tau_k} \left(y_k - C_{\tau_k} \hat{z}_{\tau_k}\right) \tag{36a}$$

$$\Sigma_{\tau_k^+} = \Sigma_{\tau_k} - L_{\tau_k} C_{\tau_k} \Sigma_{\tau_k}\,. \tag{36b}$$

These equations resemble the measurement update phase in a conventional Kalman filter.

It is worth mentioning that in opposition to the conventional Kalman and Kalman-Bucy filters which essentially work with deterministic conditional covariance matrices, for the Kalman filter (33), this matrix is a stochastic process depending on the observation. Due to this dependence on the observation, the overall system (33) is technically nonlinear, as opposed to the conventional Kalman or Kalman-Bucy filters which are solely linear systems. Moreover, the conditional covariance matrix in (33) cannot be precomputed like a conventional Kalman or Kalman-Bucy filter, rather, it must be generated in real time.

Remark 5: The conditional covariance matrix Σ_t generated by the continuous-discrete Kalman filter (33) has a feature vital to the development of an optimal control law in Section IV-C, namely, it does not depend on the choice of the control law that maps $\mathcal{Y}_t$ into u_t. According to (33b), $\{\Sigma_t\}$ is a stochastic process solely generated by the counting process $\{\eta_t\}$, which indeed is constructed independent of the choice of control law.

C. Optimal Control Law

Under the assumption of perfect state knowledge, it has been shown in [38, Ch. 8] that the linear state feedback

$$u_t = -K_t z_t \tag{37}$$

minimizes the quadratic cost functional (26) subject to the linear Gaussian dynamics (11). The optimal feedback gain K_t is an $n_u \times n_x$ matrix expressed as

$$K_t = R_t^{-1} B_t^T P_t \tag{38}$$

based on the $n_x \times n_x$ matrix P_t solving the Riccati differential equation

$$\dot{P}_t = -P_t A_t - A_t^T P_t + P_t B_t R_t^{-1} B_t^T P_t - Q_t \tag{39}$$

backward in time with the terminal condition

$$P_T = Q_f\,. \tag{40}$$

In the absence of the complete state knowledge, the state z_t in the control law (37) is replaced with its MMSE estimate $\hat{z}_t$ generated by the Kalman filter (33), leading to the control law

$$u_t = -K_t \hat{z}_t\,. \tag{41}$$

The optimality of this control law is verified in Theorem 2 of this section. For future reference, the control law (41) that combines the estimator (33) with the linear state feedback (37) is denoted by $\mu^*\left(\cdot\right)$ and is formally defined as follows.

Let K_t be the gain matrix (38) and consider the system of differential equations consisting of

$$\dot{\hat{z}}_t = \left(A_t - B_t K_t\right)\hat{z}_t + L_t\left(y_{\eta_t} - C_t \hat{z}_t\right)\dot{\eta}_t$$

and (33b)-(33d), with the state vector $(\hat{z}_t, \Sigma_t, \Theta_t)$. Suppose $\bar{\mathcal{Y}}_t$ is an instance of the observation set $\mathcal{Y}_t$ in (22) [or $\mathcal{Y}_t^N$ in (3)], and determine its associated instance $\left(\bar{\hat{z}}_t, \bar{\Sigma}_t, \Theta_t\right)$ of the state vector by solving this system of equations over $[0, t]$ with the initial state (34). The control law $\mu^*(\cdot)$ is defined then as a deterministic mapping that maps the instance $\bar{\mathcal{Y}}_t$ into

$$\mu^*\left(\bar{\mathcal{Y}}_t, t\right) = -K_t \bar{\hat{z}}_t, \quad t \in [0, T]. \tag{42}$$

Lemma 4: The control law $\mu^*(\cdot)$ defined according to (42) is admissible for the optimal control Problem 1.

Proof: See Appendix E. ■

The following theorem presents the main result of this paper by offering a solution to the optimal control Problem 1.

Theorem 2: The control law $\mu^*(\cdot)$ in (42) minimizes the cost functional $\mathcal{J}_\infty(\cdot)$ in Problem 1 on the set of all admissible control laws $\mathcal{C}$. Moreover, the minimum value of this cost functional is given by

$$\mathcal{J}_\infty\left(\mu^*(\cdot)\right) = J_z + J_e$$

in terms of the constants J_e in (27) and J_z defined as

$$J_z = \bar{x}_0^T P_0 \bar{x}_0 + \operatorname{tr}\left\{P_0 S_z\right\} + \int_0^T \operatorname{tr}\left\{D_t^T P_t D_t\right\} dt + \operatorname{E}\left[\int_0^T \operatorname{tr}\left\{P_t B_t R_t^{-1} B_t^T P_t \Sigma_t\right\} dt\right]. \tag{43}$$

Here, $\bar{x}_0$ and S_z are the initial state (34) of the estimator (33), the matrices B_t, D_t, and R_t are parameters of Problem 1, P_t is the solution to the Riccati differential equation (39) with the terminal condition (40), and Σ_t is the solution to the stochastic differential equation (33b) with the initial state S_z.

Proof: Consider the reduced order system $\mathscr{S}$ as defined in Definition 4 and the quadratic cost functional $\mathcal{J}(\mu(\cdot))$ as given by (26). This system is represented by the state vector z_t, the control u_t, and the observation set $\mathcal{Y}_t$. For this system, the conditional mean $\hat{z}_t = \operatorname{E}\left[z_t|\mathcal{Y}_t\right]$ and the conditional covariance matrix $\Sigma_t = \operatorname{cov}\left(z_t|\mathcal{Y}_t\right)$ solve the state-space equations (33) with the initial state (34).

It is shown in [38, p. 289] that the cost functional $\mathcal{J}(\mu(\cdot))$ can be rewritten as

$$\mathcal{J}(\mu(\cdot)) = \operatorname{E}\left[\int_0^T \left\|u_t + K_t \hat{z}_t\right\|_{R_t}^2 dt\right] + J_z + J_e. \tag{44}$$

As discussed in Remark 5, $\{\Sigma_t\}$ does not depend on the choice of control law that generates a $\mathcal{Y}_t$-measurable control. Hence, the constant J_z as defined in (43) cannot depend on the choice of the control law either. This statement trivially holds for J_e as well. Noting that the integrand on the right-hand side of (44) is nonnegative and that $J_z + J_e$ is a constant not depending on this integrand, the minimum of $\mathcal{J}(\mu(\cdot))$ is attained when this integrand vanishes, i.e., $u_t = -K_t \hat{z}_t$ must hold on $t \in [0, T]$, almost everywhere. This choice is in fact the control law $\mu^*(\cdot)$ and results in a minimum value of $J_z + J_e$.

By Lemma 4, the control law $\mu^*(\cdot)$ is admissible, and by Theorem 1, the cost functional $\mathcal{J}_\infty(\mu(\cdot))$ is equal to $\mathcal{J}(\mu(\cdot))$ for all admissible control laws. It is concluded that $\mu^*(\cdot)$ also minimizes $\mathcal{J}_\infty(\cdot)$ over the set of all admissible control laws, and that $J_z + J_e$ is the minimum value of $\mathcal{J}_\infty(\cdot)$. ■

V. Suboptimal Control for Finitely Many Agents

The reduced order system introduced in Definition 4 offers a finite-dimensional description for the dynamics of infinitely many agents, at least to the extent needed for the input-output characterization and design of an optimal control law. Based on this finite-dimensional system, a control law was developed in Section IV to be optimal for infinite number of agents. This control law is finite-dimensional and easy to implement by a set of state-space equations.

The simplicity of this control law motivates its deployment as a suboptimal control for finitely many agents. This idea is explained mathematically as follows. Let $\mathcal{J}_N(\mu(\cdot))$ denote the value of the cost functional (6) for the dynamical system (1) with N agents under the control law $\mu(\cdot) \in \mathcal{C}$. By Theorem 1, this value can be expressed in terms of $\mathcal{J}(\mu(\cdot))$ in (26) as

$$\mathcal{J}_N(\mu(\cdot)) = \mathcal{J}(\mu(\cdot)) + \mathcal{E}_N(\mu(\cdot)),$$

where $\mathcal{E}_N(\cdot)$ is a functional holding $\lim_{N\to\infty} \mathcal{E}_N(\mu(\cdot)) = 0$ for any fixed $\mu(\cdot) \in \mathcal{C}$. Hence, $\mathcal{J}_N(\cdot) \simeq \mathcal{J}(\cdot)$ is a reasonable approximation for a sufficiently large N, and as a result, $\mathcal{J}(\cdot)$ can be minimized instead of $\mathcal{J}_N(\cdot)$ to determine a suboptimal control law. In this case, the approximation error is bounded within the interval

$$0 \leqslant \mathcal{J}_N\left(\mu^*(\cdot)\right) - \inf_{\mu(\cdot)\in\mathcal{C}} \mathcal{J}_N(\mu(\cdot)) \leqslant \varepsilon_N, \tag{45}$$

where the nonnegative constant ε_N is given by

$$\varepsilon_N = \sup_{\mu(\cdot)\in\mathcal{C}} 2\left|\mathcal{E}_N\left(\mu\left(\cdot\right)\right)\right|. \tag{46}$$

As a result, $\mu^*\left(\cdot\right)$ is ε_N-optimal for $\mathcal{J}_N\left(\cdot\right)$ in the sense that

$$\mathcal{J}_N\left(\mu\left(\cdot\right)\right) \geqslant \mathcal{J}_N\left(\mu^*\left(\cdot\right)\right) - \varepsilon_N, \quad \mu\left(\cdot\right) \in \mathcal{C}.$$

Following a quite sophisticated process, Theorem 1 together with Lemmas 1 through 3 prove the pointwise convergence of the functional $\mathcal{J}_N\left(\cdot\right)$ to $\mathcal{J}\left(\cdot\right)$ for each fixed $\mu\left(\cdot\right)\in\mathcal{C}$, albeit without explicitly providing an expression for $\left|\mathcal{E}_N\left(\cdot\right)\right|$ or some upper bound on its value. Yet, the proofs of these theorem and lemmas present strong indications to conjecture that there must be a constant $c > 0$ to upper bound ε_N in (46) by

$$\varepsilon_N \leqslant \frac{c}{\sqrt{N}}. \tag{47}$$

These indications include the upper bounds (54) and (66) (see the proofs of Lemma 2 and Theorem 1), which predict a rate of convergence $1/\sqrt{N}$ for $\mathcal{J}_N\left(\cdot\right)$, as well as the boundedness condition (8) which can extend the convergence of $\mathcal{J}_N\left(\cdot\right)$ from pointwise to uniform sense. This extension allows to establish a bound on $\left|\mathcal{E}_N\left(\mu\left(\cdot\right)\right)\right|$ independent of $\mu\left(\cdot\right)$. A complete proof of (47), although seems possible, will be far from concise and needs several more pages beyond the limitation of this paper (and possibly additional technical assumptions). Therefore, the formal proof of the upper bound (47) is skipped in this paper.

Even after proving this upper bound, it only implies that the approximation error (45) will decay with a rate $1/\sqrt{N}$, without being able to provide a tight, numerically tractable value of c. Without the knowledge of c, (47) will be barely an effective numeric tool to evaluate the suboptimal performable of $\mu^*\left(\cdot\right)$. For practical applications, this section proposes an alternative approach to establish lower bounds on the number N of the agents under which $\mu^*\left(\cdot\right)$ is near-optimal. This approach relies on the fact that $\mathscr{S}$ is derived from $\mathscr{S}_N$ based on two crucial properties: first, the stochastic processes $\left\{e_t^1\right\},\ldots,\left\{e_t^N\right\}$ tend to become mutually independent as $N\to\infty$, and second, the probability of sampling a single process $\left\{e_t^n\right\}$ more than once tends to 0 as $N\to\infty$. For approximating $\mathscr{S}_N$ with $\mathscr{S}$, the value of N must be then chosen large enough to hold a close approximation of these properties.

A measure of mutual independence for the pair of Gaussian processes $\left\{e_t^n\right\}$ and $\left\{e_t^m\right\}$ is their cross-correlation function

$$\gamma_N\left(t_1,t_2\right) = \frac{\mathrm{E}\left[\left(e_{t_1}^n\right)^T e_{t_2}^m\right]}{\left(\mathrm{E}\left[\left\|e_{t_1}^n\right\|^2\right]\mathrm{E}\left[\left\|e_{t_2}^m\right\|^2\right]\right)^{1/2}}, \quad n\neq m.$$

For infinitely many agents, this function takes the exact value of 0. To attain a close approximation of this vlaue, N must be chosen sufficiently large to maintain the absolute value of this function below a small threshold $0<\delta_1\ll 1$ over the entire control period $t\in[0,T]$. This condition explicitly requires the number of agents N to satisfy the inequality

$$\sup_{t_1,t_2\in[0,T]} \left|\gamma_N\left(t_1,t_2\right)\right| \leqslant \delta_1. \tag{48}$$

It can be shown using Lemma 1 that the left-hand side of this inequality converges to 0 with the rate $1/N$ as $N\to\infty$. Thus, there must exist some value of N to satisfy the inequality. An explicit expression for the cross-correlation function $\gamma_N\left(\cdot\right)$ can be derived from Lemma 1 via a straightforward but lengthy procedure, which is skipped here.

For an infinite number of agents, the probability of multiple sampling of a single process is identically zero over the entire control time $[0,T]$. For a close approximation, this probability must be kept below a small threshold $0<\delta_2\ll 1$. To be less conservative, this condition is enforced only over a coherence time of the stochastic process $\left\{e_t^n\right\}$ instead of the entire control period. The coherence time is typically much shorter than the control time and is defined as the time interval between two samples of the stochastic process $\left\{e_t^n\right\}$ that are approximately independent. The coherence time is defined here as

$$T_c\left(N\right) = \inf\left\{s\,\middle|\,\sup_{t\in[0,T]}\left|\rho_N\left(t,t+s\right)\right|\leqslant\delta_1\right\}$$

in terms of the autocorrelation function

$$\rho_N\left(t_1,t_2\right) = \frac{\mathrm{E}\left[\left(e_{t_1}^n\right)^T e_{t_2}^n\right]}{\left(\mathrm{E}\left[\left\|e_{t_1}^n\right\|^2\right]\mathrm{E}\left[\left\|e_{t_2}^n\right\|^2\right]\right)^{1/2}}. \tag{49}$$

Using the law of total probability [35, p. 25], the probability of multiple sampling in the interval $\left[t,t+T_c\left(N\right)\right]$ is given by

$$1-\sum_{k=0}^{N}\frac{1}{N^k}\cdot\frac{N!}{(N-k)!}\,\Pr\left\{\eta_{t+T_c(N)}-\eta_t=k\right\},$$

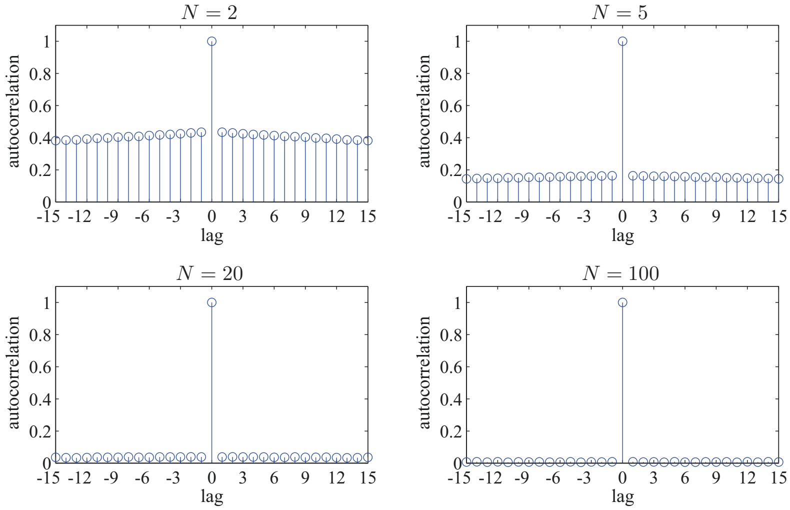

Fig. 1. Autocorrelation of the random sequence $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\right\}$ estimated from sample paths of length 10^6 for $N = 2, 5, 20, 100$.

where $\{\eta_t\}$ is the counting process used for sampling in time. Hence, the number of agents N must be large enough to keep this probability below δ_2 for every $t \in [0, T]$. This requirement constrains the smallest value of N to satisfy the inequality

$$\inf_{t\in[0,T]} \sum_{k=0}^{N} \frac{1}{N^k} \cdot \frac{N!}{(N-k)!} \Pr\left\{\eta_{t+T_c(N)} - \eta_t = k\right\} \geqslant 1 - \delta_2. \tag{50}$$

The analysis above demonstrates the essence of the problem that leads to a lower bound on N. Yet, the key question of how to select the numerical values of δ_1 and δ_2 is not addressed. A plausible answer to this question requires a difficult trade-off between two extremes: smaller values of δ_1 and δ_2 result in closer approximations for the dynamics of the agents, but also in excessively conservative lower bounds for N. In addition to the difficulty in choosing δ_1 and δ_2, solving the inequality conditions (48) and (50) is not straightforward analytically.

A more practical approach to establish a lower bound on N is the use of Monte Carlo methods and a whiteness test. This approach is in particular easy to implement for a time-invariant system (1) with matrices A_t, F_t, and G_t independent of time. It is reminded that the random sequence $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\right\}$ is not white for any finite N, but it tends to a white sequence as $N \to \infty$, as shown in Lemma 2. This fact is the core property used for model reduction in Section III, which eventually led to the optimal control law of Section IV. Hence, for any finite value of N for which $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\right\}$ is effectively white, this control law must perform near optimal.

To evaluate the whiteness of $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\right\}$ for a given number of agents, a long sample path of this random sequence is generated by numerically solving the coupled SDEs (12), and then, applying the random sampling procedure introduced in Section II. Next, the whiteness of the sequence is evaluated, for example, by estimating its autocorrelation function [defined similar to (49)], and comparing it against the delta function.[1]

This procedure is demonstrated for the numerical parameter values $A_t = -1$, $F_t = 0.2$, and $G_t = 0.8$. With these values, the state of each agent is a scalar, but this scalar represents only one element of the 3-dimensional position vector of a magnetic particle moving in a homogeneous and isotropic environment. The sampling is performed by a homogenous Poisson process with the constant rate $\lambda = 100$. In Fig. 1, the autocorrelation function of $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\right\}$ is illustrated for different values of $N = 2, 5, 20, 100$. It is observed that this function tends to the delta function as N increases. As a measure of whiteness, the maximum distance (∞-norm) between the autocorrelation function and the delta function is illustrated versus N in Fig. 2. By this figure, the autocorrelation function tends to the delta function asymptotically, which implies that $\left\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\right\}$ becomes in effect white for a large enough N, e.g., $N \geqslant 50$.

A. Simulation Results

Section IV-C indicates that the control law $\mu^*(\cdot)$ defined in (42) is optimal for infinite number of agents. This section examines the performance of this control law for finite number of agents using computer simulations. A MATLAB simulator simultaneously solves the stochastic state-space equations (1), generates the observation set $\mathcal{Y}_t^N$ according to (3), solves the equations of the Kalman filter (33), and applies the feedback control (41) to the dynamical system (1). The numerical values of the simulation parameters are presented in TABLE I.

[1]This indeed is only a simple practical test of whiteness; strictly speaking, whiteness is not implied by uncorrelatedness for non-Gaussian processes. For test of whiteness, more advanced techniques exist [39], for example, by using the auto-mutual information function [40].

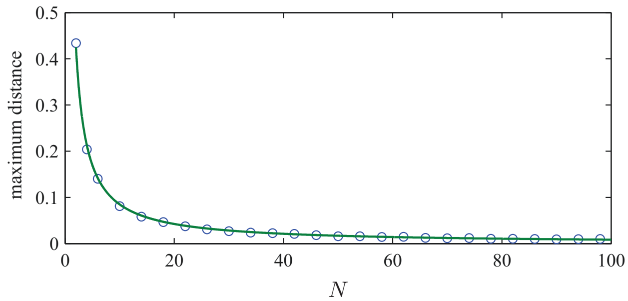

Fig. 2. Measure of whiteness of $\{e_{\tau_1}^{\nu_1}, e_{\tau_2}^{\nu_2}, e_{\tau_3}^{\nu_3}, \dots\}$ versus N. The vertical axis shows the maximum distance of the autocorrelation function from the delta function, as a measure of whiteness. As $N \to \infty$, this distance clearly tends to 0, which implies that the autocorrelation function tends to the delta function asymptotically.

N	A_t	F_t	B_t	D_t	G_t	C_t	λ	V	S_z	S_e	Q_t	R_t	Q_f	T
20	-1	0.2	1	2.5	0.8	1	100	0.04	0.16	0.4	1	0.001	0.031	10

TABLE I
NUMERICAL VALUES OF THE SIMULATION PARAMETERS.

The simulation results are presented in Fig. 3. A comparison between the state value z_t and its estimate $\hat{z}_t$ is illustrated in Fig. 3(a), and the estimation error $z_t - \hat{z}_t$ is shown versus time in Fig. 3(b). The variance of the estimation error is numerically computed as 0.148. This value is compared with the expected value $\mathrm{E}\left[\Sigma_t\right]$ estimated from Fig. 3(c) as 0.135. The small gap between these numbers indicates that the actual variance of the estimation error is close to what is expected of the Kalman filter. A stronger supporting evidence for near-optimal performance of the Kalman filter is the whiteness of its innovations process. Specifically, the distance between the autocorrelation function of this process and the delta function is computed numerically as small as 0.02.

The feedback control $u_t = -K_t \hat{z}_t$ is illustrated versus time in Fig. 3(d). The performance of this closed-loop control is compared with the open-loop control $u_t = 0$ in Fig. 3(e), when both controls are applied to the state-space equation (11), and in Fig. 3(f) when the controls are applied to (1). In both cases, the control objective is to maintain the states z_t and x_t^n as close as possible to 0. Evidently, the closed-loop control is far more effective in achieving this goal for both z_t and x_t^n. The average power of z_t is estimated as 0.26 and 3.2 for closed-loop and open-loop controls, respectively. The corresponding numbers for x_t^n are 0.63 and 3.6.

VI. Conclusion

Optimal control of a large number of interacting agents was considered under a centralized control and a common external disturbance acting on them collectively, as well as independent disturbances applied to them individually. These agents imitate the magnetic nanoparticles comprising a magnetic fluid, which are driven toward a target by a controlled magnetic field, while being perturbed by a common disturbance collectively, and by Brownian motion individually. The dynamics of the agents was described by linear stochastic state-space equations driven by white Gaussian processes and a common input regarded as the centralized control. Subject to this linear Gaussian dynamics, an optimal control problem was defined aimed at minimizing a quadratic cost functional with an imperfect knowledge of the state of the agents. This knowledge was provided by samples taken randomly in time from the state of anonymous randomly chosen agents.

Due to the non-Gaussian structure of the observation set, the optimal control law for this problem is nonlinear and infinite-dimensional in essence. Yet, it was shown that for infinitely many agents, the problem can be systematically reduced into a finite-dimensional but nonstandard LQG problem holding the separation principal. Using this principal, the optimal control law was determined explicitly by combining a linear quadratic regulator with a separately designed finite-dimensional MMSE state estimator. The possibility of adopting this explicit control law to approximate the optimal control for finitely many agents was investigated, and conditions on the number of agents for a close approximation were presented.

Appendix

Proof of Lemmas and Theorem 1

A. Proof of Lemma 1

Assuming that $\bar{e}_t^n$ and β_t are the solutions to (15) and (16) with the initial states $\bar{e}_0^n = e_0^n$ and $\beta_0 = 0$, respectively, it is straightforward to verify by direct inspection that (14) solves the SDE (12). The proof of statements i and ii are given below.

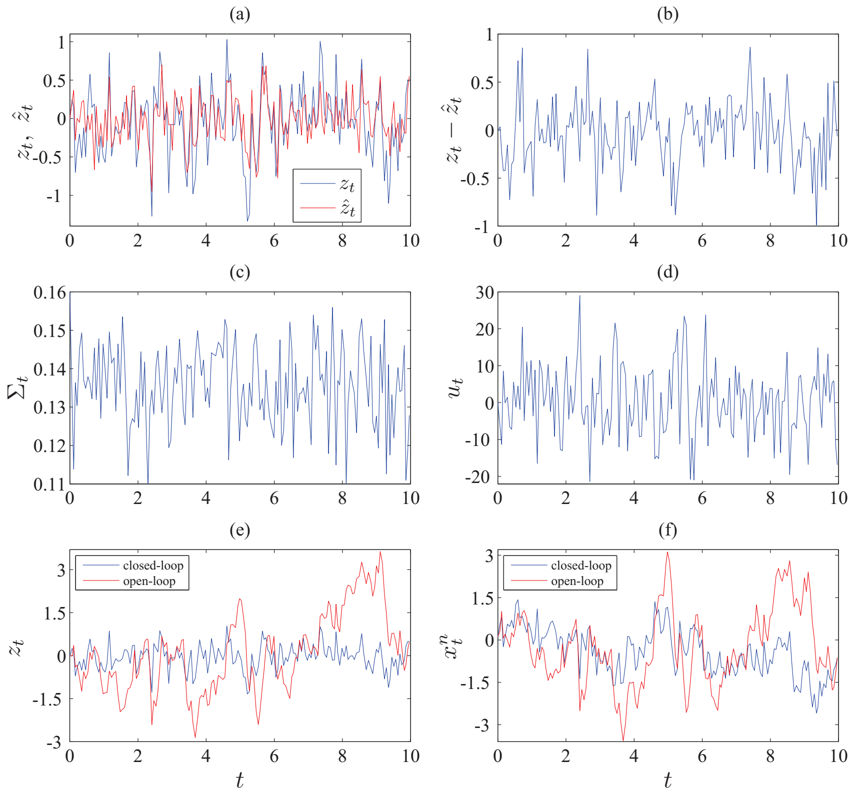

Fig. 3. Simulation results: (a) the state z_t of the dynamical system (11) and its estimate $\hat{z}_t$ versus time; (b) the estimation error $z_t - \hat{z}_t$; (c) conditional covariance Σ_t; (d) suboptimal control $u_t = -K_t\hat{z}_t$; (e) comparison of z_t under the closed-loop control $u_t = -K_t\hat{z}_t$ and the open-loop control $u_t = 0$; (f) same comparison for the state x_t^n of the dynamical system (1), where $n = 2$ is randomly chosen.

Proof of i: The stochastic processes $\{\bar{e}_t^n\}$, $n = 1, 2, \dots, N$ are generated by linear filtering of jointly independent Wiener processes $\{\xi_t^n\}$, $n = 1, 2, \dots, N$, therefore, they are Gaussian, zero-mean, and jointly independent. It is shown in [38, p. 66] that the covariance matrix of $\{\bar{e}_t^n\}$ evolves in time according to the matrix differential equation (17).

Proof of ii: The stochastic process $\{\beta_t\}$ is generated by the linear state-space equation

$$\dot{\beta}_t = A_t\beta_t - F_t\bar{e}_t$$

with zero initial state and the stochastic input

$$\bar{e}_t = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \bar{e}_t^i. \tag{51}$$

Since $\{\bar{e}_t\}$ is zero-mean and Gaussian, $\{\beta_t\}$ will be zero-mean and Gaussian as well. Moreover, $\bar{e}_t$ has a bounded covariance matrix Θ_t not depending on N, implying that β_t must have a uniformly bounded covariance matrix.

B. Proof of Lemma 2

By Lévy's continuity theorem [35, p. 322], convergence of the random vectors $\psi_1, \psi_2, \psi_3, \dots$ in distribution is concluded from pointwise convergence of their characteristic functions:

$$\lim_{N\to\infty} \mathrm{E}\left[\exp\left(j\omega^T\psi_N\right)\right] = \mathrm{E}\left[\exp\left(j\omega^T\psi\right)\right], \quad \omega \in \mathbb{R}^{Kn_x}.$$

By an additional condition of this theorem, the characteristic function of ψ must be continuous at $\omega = 0$, which trivially holds for the Gaussian random vector ψ. Hence, the pointwise convergence of the characteristic functions is verified in the remaining of this proof.

By Lemma 1, the random vector ψ_N can be written as

$$\psi_N = \bar{\psi}_N + \frac{1}{\sqrt{N}}\chi_N,$$

where $\bar{\psi}_N$ and χ_N are $Kn_x \times 1$ random vectors defined as

$$\bar{\psi}_N = \left(\bar{e}_{t_1}^{\nu_1}, \bar{e}_{t_2}^{\nu_2}, \dots, \bar{e}_{t_K}^{\nu_K}\right)$$
$$\chi_N = \left(\beta_{t_1}, \beta_{t_2}, \dots, \beta_{t_K}\right).$$

The characteristic function of ψ_N is then expressed as

$$\mathrm{E}\left[\exp\left(j\omega^T\psi_N\right)\right] = \mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)\right] + R_N\left(\omega\right), \tag{52}$$

where the complex function $R_N\left(\omega\right)$ is defined as

$$R_N\left(\omega\right) = \mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)\left(\exp\left(\frac{j\omega^T\chi_N}{\sqrt{N}}\right) - 1\right)\right].$$

It is shown next that this function converges to 0 as $N \to \infty$. To that end, the identity $e^{j2\theta} - 1 = j2e^{j\theta}\sin\theta$ is employed to rewrite $R_N\left(\omega\right)$ as

$$R_N\left(\omega\right) = \mathrm{E}\left[j2\exp\left(j\omega^T\bar{\psi}_N + \frac{j\omega^T\chi_N}{2\sqrt{N}}\right)\sin\left(\frac{\omega^T\chi_N}{2\sqrt{N}}\right)\right].$$

Taking the absolute value of both sides, applying the inequality

$$\left|\mathrm{E}\left[X\right]\right| \leqslant \mathrm{E}\left[\left|X\right|\right], \tag{53}$$

noting that $\left|e^{j\theta}\right| = 1$, and using the inequality $\left|\sin\theta\right| \leqslant \left|\theta\right|$, it is straightforward to show

$$\begin{aligned}\left|R_N\left(\omega\right)\right| &\leqslant 2\mathrm{E}\left[\left|\sin\left(\frac{\omega^T\chi_N}{2\sqrt{N}}\right)\right|\right] \\ &\leqslant \frac{1}{\sqrt{N}}\,\mathrm{E}\left[\left|\omega^T\chi_N\right|\right].\end{aligned} \tag{54}$$

By Lemma 1, $\omega^T\chi_N$ is a Gaussian random variable with zero mean and bounded variance, implying that

$$\lim_{N\to\infty} R_N\left(\omega\right) = 0, \quad \omega \in \mathbb{R}^{Kn_x}.$$

Let $\mathscr{D}$ be the event that the random variables $\nu_1, \nu_2, \ldots, \nu_K$ take distinct values, and denote its complement by $\mathscr{D}'$. Using the statistical independence of $\nu_1, \nu_2, \ldots, \nu_K$, the probability of $\mathscr{D}$ is computed as

$$q_N = \frac{N\left(N-1\right)\left(N-2\right)\cdots\left(N-K+1\right)}{N^K}.$$

Clearly, q_N tends to 1 as $N \to \infty$.

The law of total expectation [35, p. 79] implies that

$$\begin{aligned}\mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)\right] &= \mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\mathscr{D}\right]q_N + \mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\mathscr{D}'\right]\left(1-q_N\right) \\ &= \mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\mathscr{D}\right] + \tilde{R}_N\left(\omega\right),\end{aligned} \tag{55}$$

where $\tilde{R}_N\left(\omega\right)$ is defined as

$$\tilde{R}_N\left(\omega\right) = \left(1-q_N\right)\left(\mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\mathscr{D}'\right] - \mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\mathscr{D}\right]\right).$$

Taking the absolute value of both sides, applying the triangle inequality and (53), and noting that $\left|e^{j\theta}\right| = 1$, it is shown that

$$\left|\tilde{R}_N\left(\omega\right)\right| \leqslant 2\left(1-q_N\right).$$

This inequality implies that

$$\lim_{N\to\infty} \tilde{R}_N\left(\omega\right) = 0, \quad \omega \in \mathbb{R}^{Kn_x}.$$

Conditioned on $\nu_1, \nu_2, \ldots, \nu_K$ and the event $\mathscr{D}$, the random vectors $\bar{e}_{t_1}^{\nu_1}, \bar{e}_{t_2}^{\nu_2}, \ldots, \bar{e}_{t_K}^{\nu_K}$ are zero-mean Gaussian and jointly independent with the covariance matrices $\Theta_{t_1}, \Theta_{t_1}, \ldots, \Theta_{t_K}$. Therefore, their conditional characteristic function is the same as the characteristic function of ψ. This fact and the law of total expectation result in

$$\begin{aligned}\mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\mathscr{D}\right] &= \mathrm{E}\left[\mathrm{E}\left[\exp\left(j\omega^T\bar{\psi}_N\right)|\nu_1, \nu_2, \ldots, \nu_K, \mathscr{D}\right]|\mathscr{D}\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\exp\left(j\omega^T\psi\right)\right]|\mathscr{D}\right] \\ &= \mathrm{E}\left[\exp\left(j\omega^T\psi\right)\right].\end{aligned}$$

Combining this result with (52) and (55), the characteristic functions of ψ_N is expressed as

$$\mathrm{E}\left[\exp\left(j\omega^T\psi_N\right)\right] = \mathrm{E}\left[\exp\left(j\omega^T\psi\right)\right] + R_N\left(\omega\right) + \tilde{R}_N\left(\omega\right).$$

Then, the proof is completed by taking the limit of both sides as $N \to \infty$, and noting that the last two terms on the right-hand side vanish as $N \to \infty$.

C. Proof of Lemma 3

The boundedness property (5) of η_t implies (e.g., by using Markov's inequality)

$$\lim_{K\to\infty} \Pr\{\eta_t > K\} = 0. \tag{56}$$

Application of the law of total expectation [35, p. 79] yields

$$\mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] = \sum_{k=0}^{K} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k\right]\Pr\{\eta_t = k\} + \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t > K\right]\Pr\{\eta_t > K\}$$

which in turn leads to the inequality

$$\begin{aligned}\left|\mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] - \sum_{k=0}^{K} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k\right]\Pr\{\eta_t = k\}\right| &= \left|\mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t > K\right]\right|\Pr\{\eta_t > K\} \\ &\leqslant \bar{\phi}\Pr\{\eta_t > K\}.\end{aligned}$$

Here, $0 < \bar{\phi} < \infty$ is an upper bound of $|\phi(\cdot)|$ and its existence is assumed by the lemma. By letting first $N \to \infty$ and next $K \to \infty$ in both sides of this inequality and using (56), it is concluded that

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] = \sum_{k=0}^{\infty} \Pr\{\eta_t = k\} \lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k\right]. \tag{57}$$

The law of total expectation also implies

$$\mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)\right] = \sum_{k=0}^{\infty} \Pr\{\eta_t = k\}\, \mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)|\eta_t = k\right]. \tag{58}$$

It is shown that the left-hand sides of (57) and (58) are equal by verifying for all $k = 0, 1, 2, 3, \ldots$ that

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k\right] = \mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)|\eta_t = k\right]. \tag{59}$$

This equality trivially holds for $k = 0$, therefore it is proven for $k = 1, 2, 3, \ldots$ as follows.

Let ψ_N and ψ be the random vectors (18) and (20) defined in Lemma 2 (replace $t_1, t_2, \ldots, t_K$ with $\tau_1, \tau_2, \ldots, \tau_k$). Define the scalar function $\tilde{\phi}(\cdot)$ as the conditional expected value

$$\tilde{\phi}\left(\psi_N\right) = \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k, \psi_N\right]. \tag{60}$$

The dynamical systems $\mathscr{S}_N$ and $\mathscr{S}$ have similar structures and are controlled under the same feedback control law; they differ only in the probability distributions of ψ_N and ψ. Hence, the same function $\tilde{\phi}(\cdot)$ represents the conditional expectation

$$\mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k, \psi\right] = \tilde{\phi}\left(\psi\right). \tag{61}$$

Since by assumption, $\phi(\cdot)$ is a bounded $\mathcal{Y}$-map, $\tilde{\phi}(\cdot)$ is a bounded function. On the other hand, $\tilde{\phi}(\cdot)$ is generated via a sequence of operations performed on the continuous control law $\mu(\cdot)$ and continuous $\mathcal{Y}$-map $\phi(\cdot)$. These operations are integration, function decomposition, and expected value, all preserving continuity. Therefore, $\tilde{\phi}(\cdot)$ is a continuous function. Then, application of Lemma 2 and Remark 2 to $\tilde{\phi}(\cdot)$ yields

$$\lim_{N\to\infty} \mathrm{E}\left[\tilde{\phi}\left(\psi_N\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k\right] = \mathrm{E}\left[\tilde{\phi}\left(\psi\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k\right].$$

Substituting (60) and (61) into this equality and using the smoothing property of conditional expectation result in

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k\right] = \mathrm{E}\big[\phi\left(\mathcal{Y}_t\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k\big].$$

Conditioned on $\eta_t = k$, taking the expected value of both sides of this equality leads to

$$\mathrm{E}\Big[\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k\right]|\eta_t = k\Big] = \mathrm{E}\big[\,\mathrm{E}\big[\phi\left(\mathcal{Y}_t\right)|\eta_t = k, \tau_1, \tau_2, \ldots, \tau_k\big]|\eta_t = k\big].$$

Since $\phi(\cdot)$ is bounded by assumption, Lebesgue's dominated convergence theorem [35, p. 187] allows to change the order of limit and outer expected value in the left-hand side. Then, the smoothing property of conditional expectation leads to (59).

To prove the second statement, define the scalar function

$$\tilde{\phi}_\alpha(\cdot) = \min\{\phi(\cdot), \alpha\}$$

indexed by the positive scalar α. Then, the expected value on the left-hand side of (24) can be expressed as

$$\mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] = \mathrm{E}\left[\tilde{\phi}_\alpha\left(\mathcal{Y}_t^N\right)\right] + \delta_N(\alpha), \tag{62}$$

where $\delta_N(\alpha)$ is defined as

$$\delta_N(\alpha) = \mathrm{E}\left[\left(\phi\left(\mathcal{Y}_t^N\right) - \alpha\right) I\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\}\right]$$

in terms of the indicator function

$$I\{\phi > \alpha\} = \begin{cases} 1, & \phi > \alpha \\ 0 & \phi \leqslant \alpha. \end{cases}$$

Since $\delta_N(\alpha)$ in (62) is nonnegative, Lyapunov's inequality [35, p. 193] and the assumption (25) result in

$$\begin{aligned} \mathrm{E}\left[\tilde{\phi}_\alpha\left(\mathcal{Y}_t^N\right)\right] &\leqslant \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] \\ &\leqslant \left(\mathrm{E}\left[\phi^p\left(\mathcal{Y}_t^N\right)\right]\right)^{\frac{1}{p}} \\ &\leqslant \sup_{N\in\mathbb{N}} \left(\mathrm{E}\left[\phi^p\left(\mathcal{Y}_t^N\right)\right]\right)^{\frac{1}{p}} \\ &< \infty. \end{aligned}$$

Noting that $\tilde{\phi}_\alpha(\cdot)$ is a bounded and continuous function, the first statement of this lemma implies

$$\mathrm{E}\left[\tilde{\phi}_\alpha\left(\mathcal{Y}_t\right)\right] = \lim_{N\to\infty} mathrmE\left[\tilde{\phi}_\alpha\left(\mathcal{Y}_t^N\right)\right] < \infty. \tag{63}$$

In addition to being bounded, the left-hand side of this equality is an increasing function of α, leading to the existence and boundedness of the limit

$$\lim_{\alpha\to\infty} \mathrm{E}\left[\tilde{\phi}_\alpha\left(\mathcal{Y}_t\right)\right] = \mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)\right] < \infty. \tag{64}$$

By taking the limit of (62) first as $N \to \infty$ and next $\alpha \to \infty$ and making use of (63) and (64), it is concluded that

$$\lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right)\right] = \mathrm{E}\left[\phi\left(\mathcal{Y}_t\right)\right] + \lim_{\alpha\to\infty}\lim_{N\to\infty} \delta_N(\alpha).$$

This equality proves (24) by showing next that the double limit on the right-hand side vanishes.

The definitions of $\delta_N(\alpha)$ and the indicator function together with Hölder's inequality [35, p. 193] imply

$$\begin{aligned} \delta_N(\alpha) &\leqslant \mathrm{E}\left[\phi\left(\mathcal{Y}_t^N\right) I\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\}\right] \\ &\leqslant \left(\mathrm{E}\left[\phi^p\left(\mathcal{Y}_t^N\right)\right]\right)^{\frac{1}{p}} \left(\mathrm{E}\left[I\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\}^q\right]\right)^{\frac{1}{q}} \\ &= \left(\mathrm{E}\left[\phi^p\left(\mathcal{Y}_t^N\right)\right]\right)^{\frac{1}{p}} \Pr\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\}^{\frac{1}{q}} \\ &\leqslant \sup_{N\in\mathbb{N}} \left(\mathrm{E}\left[\phi^p\left(\mathcal{Y}_t^N\right)\right]\right)^{\frac{1}{p}} \sup_{N\in\mathbb{N}} \Pr\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\}^{\frac{1}{q}}, \end{aligned} \tag{65}$$

where q is a constant satisfying $p^{-1} + q^{-1} = 1$. However,

$$f_N(\alpha) \triangleq \Pr\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\}$$

is a decreasing function of α and $\lim_{\alpha\to\infty} f_N(\alpha) = 0$ holds for all $N \in \mathbb{N}$. Thus, for any $\epsilon > 0$, there exists $\bar{\alpha}_N$ such that

$$0 \leqslant f_N(\alpha) < \epsilon, \quad \alpha > \bar{\alpha}_N.$$

Setting $\bar{\alpha} = \sup_{N\in\mathbb{N}} \bar{\alpha}_N$ and noting that $f_N(\alpha)$ is decreasing in α, it is concluded that

$$0 \leqslant \sup_{N\in\mathbb{N}} f_N(\alpha) < \epsilon, \quad \alpha > \bar{\alpha}$$

which is equivalent to

$$\lim_{\alpha\to\infty} \sup_{N\in\mathbb{N}} \Pr\left\{\phi\left(\mathcal{Y}_t^N\right) > \alpha\right\} = 0.$$

This limit together with (65) implies

$$\lim_{\alpha\to\infty}\lim_{N\to\infty} \delta_N(\alpha) = 0.$$

D. Proof of Theorem 1

Let $\mu(\cdot)$ be an admissible control law and in terms of the observation set $\mathcal{Y}_t^N$ of the dynamical system $\mathscr{S}_N$ apply the feedback control $u_t = \mu\left(\mathcal{Y}_t^N, t\right)$ to the set of stochastic state-space equations (1). By Lemma 1 and (10), the resulting state vectors $x_t^1, x_t^2, \ldots, x_t^N$ can be equivalently generated through

$$x_t^n = z_t + \bar{e}_t^n + \frac{1}{\sqrt{N}}\,\beta_t, \quad n = 1, 2, \ldots, N,$$

where z_t is the state of (11) under the same control, and $\bar{e}_t^n$ and β_t are the solutions to (15) and (16), respectively. Using this expression, the quadratic cost functional (6) is rewritten as

$$J_N = \mathrm{E}\left[\int_0^T \left(\|z_t\|_{Q_t}^2 + \|u_t\|_{R_t}^2\right) dt + \|z_T\|_{Q_f}^2\right] + J_e + \delta_N,$$

where δ_N is defined in terms of $\bar{e}_t$ in (51) as

$$\delta_N = \frac{2}{\sqrt{N}}\left(\int_0^T \mathrm{E}\left[z_t^T Q_t \bar{e}_t\right] dt + \mathrm{E}\left[z_T^T Q_f \bar{e}_T\right]\right) + \frac{2}{\sqrt{N}}\left(\int_0^T \mathrm{E}\left[z_t^T Q_t \beta_t\right] dt + \mathrm{E}\left[z_T^T Q_f \beta_T\right]\right)$$
$$+ \frac{2}{N}\left(\int_0^T \mathrm{E}\left[\beta_t^T Q_t \bar{e}_t\right] dt + \mathrm{E}\left[\beta_T^T Q_f \bar{e}_T\right]\right) + \frac{1}{N}\left(\int_0^T \mathrm{E}\left[\beta_t^T Q_t \beta_t\right] dt + \mathrm{E}\left[\beta_T^T Q_f \beta_T\right]\right). \tag{66}$$

By Lemma 1, $\|\bar{e}_t\|$ and $\|\beta_t\|$ both have uniformly bounded second moments. Further, it is shown later in this proof that for a control u_t generated by an admissible control law, $\|z_t\|$ has a uniformly bounded second moment for all $t \in [0, T]$. Since both Q_t and Q_f are bounded matrices, it can be concluded that the expressions in parentheses on the right-hand side of (66) are bounded (easily shown by the Cauchy-Schwarz [35, p. 192] inequality). The boundedness of these expressions implies

$$\lim_{N\to\infty} \delta_N = 0.$$

Let $\Phi(\cdot)$ denote the state transition matrix of A_t. Then, the solution to the SDE (11) is given by

$$z_t = \int_0^t \Phi(t, s)\, B_s u_s ds + g_t,$$

where g_t is a Gaussian random vector defined as

$$g_t = \Phi(t, 0)\, z_0 + \int_0^t \Phi(t, s)\, D_s dw_s.$$

Using the triangle inequality and properties of induced matrix norm, it is concluded that

$$\|z_t\| \leqslant \int_0^t \|\Phi(t, s)\, B_s\| \cdot \|u_s\|\, ds + \|g_t\|. \tag{67}$$

Application of Minkowski's inequality [35, p. 194] (for its integral version see [41]) to the right-hand side of (67) yields

$$\begin{aligned}
\mathrm{E}\left[\|z_t\|^{2p}\right]^{\frac{1}{2p}} &\leqslant \mathrm{E}\left[\left|\int_0^t \|\Phi(t, s)\, B_s\| \cdot \|u_s\|\, ds + \|g_t\|\right|^{2p}\right]^{\frac{1}{2p}} \\
&\leqslant \int_0^t \|\Phi(t, s)\, B_s\| \cdot \mathrm{E}\left[\|u_s\|^{2p}\right]^{\frac{1}{2p}} ds + \mathrm{E}\left[\|g_t\|^{2p}\right]^{\frac{1}{2p}} \\
&\leqslant \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathrm{E}\left[\|u_s\|^{2p}\right]^{\frac{1}{2p}} \sup_{t\in[0,T]} \int_0^t \|\Phi(t, s)\, B_s\|\, ds + \sup_{t\in[0,T]} \mathrm{E}\left[\|g_t\|^{2p}\right]^{\frac{1}{2p}}.
\end{aligned} \tag{68}$$

In the right-hand side of this inequality, the first multiplicative term is bounded by assumption, the second multiplicative term is bounded since A_t and B_t are bounded matrices, and the additive term is bounded since g_t is a Gaussian random vector of bounded covariance matrix. These facts imply that

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathrm{E}\left[\|z_t\|^{2p}\right] < \infty, \quad t \in [0, T].$$

Two conclusions are drawn from this result. First, applying Lyapunov's inequality [35, p. 193] to the random variable $\|z_t\|$ implies that it has a bounded second moment. Second, from Minkowski's inequality and the regularity condition (8) it is concluded that

$$\sup_{N\in\mathbb{N}} \mathrm{E}\left[\left(\int_0^T \left(\|z_t\|_{Q_t}^2 + \|u_t\|_{R_t}^2\right) dt + \|z_T\|_{Q_f}^2\right)^p\right] < \infty. \tag{69}$$

Define the scalar $\mathcal{Y}$-map $\phi(\cdot)$ by the conditional expectation

$$\phi\left(\mathcal{Y}_T^N\right) = \mathrm{E}\left[\int_0^T \left(\|z_t\|_{Q_t}^2 + \|u_t\|_{R_t}^2\right) dt + \|z_T\|_{Q_f}^2 \,\middle|\, \mathcal{Y}_T^N\right].$$

Applying Lyapunov's inequality to the conditional expectation on the right-hand side establishes the upper bound

$$\mathrm{E}\left[\left(\int_0^T \left(\|z_t\|_{Q_t}^2 + \|u_t\|_{R_t}^2\right) dt + \|z_T\|_{Q_f}^2\right)^p \,\middle|\, \mathcal{Y}_T^N\right]$$

on $\phi^p\left(\mathcal{Y}_T^N\right)$. By taking the expected value of this upper bound, applying the smoothing property of conditional expectation, and using (69), it can be shown that

$$\sup_{N\in\mathbb{N}} \mathrm{E}\left[\phi^p\left(\mathcal{Y}_T^N\right)\right] < \infty.$$

Following an argument similar to the proof of Lemma 3, the $\mathcal{Y}$-map $\phi(\cdot)$ is continuous for any admissible control law (which is continuous by definition). Furthermore, this $\mathcal{Y}$-map is inherently nonnegative. Therefore, the second statement of Lemma 3 is applicable to $\phi(\cdot)$, which leads to

$$\begin{aligned}
\mathcal{J}_\infty(\mu(\cdot)) &= \lim_{N\to\infty} J_N \\
&= \lim_{N\to\infty} \left(\mathrm{E}\left[\phi\left(\mathcal{Y}_T^N\right)\right] + J_e + \delta_N\right) \\
&= \lim_{N\to\infty} \mathrm{E}\left[\phi\left(\mathcal{Y}_T^N\right)\right] + J_e + \lim_{N\to\infty} \delta_N \\
&= \mathrm{E}\left[\phi\left(\mathcal{Y}_T\right)\right] + J_e \\
&= \mathcal{J}(\mu(\cdot)).
\end{aligned}$$

E. Proof of Lemma 4

Before presenting the proof, the following lemma is stated for future use. Suppose that $X_k \in \mathbb{R}^{n_k}$, $k = 1, 2, \ldots, K$ are random vectors and M_k, $k = 1, 2, \ldots, K$ are $n \times n_k$ bounded matrices. Then, the inequality

$$\mathrm{E}\left[\left\|\sum_{k=1}^K M_k X_k\right\|^{2p}\right]^{\frac{1}{2p}} \leqslant \sum_{k=1}^K \|M_k\| \cdot \mathrm{E}\left[\|X_k\|^{2p}\right]^{\frac{1}{2p}} \tag{70}$$

holds for every $p > 1$, where $\|M_k\|$ denotes the induced matrix 2-norm of M_k. The proof is a straightforward application of the triangle inequality and Minkowski's inequality [35, p. 194].

The control law $\mu^*(\cdot)$ given by (42) is trivially a continuous and causal $\mathcal{Y}$-map. Therefore, only the regularity condition (8) must be verified for this control law. According to (42), the control u_t in (8) is generated via the following process. The observation set $\mathcal{Y}_t^N$ of the system $\mathscr{S}_N$ is applied to the MMSE estimator (33) to generate $\hat{z}_t$, then the control $u_t = -K_t\hat{z}_t$ is constructed in terms of $\hat{z}_t$. This control is applied to the set of coupled SDEs (1) to generate the state vectors $x_t^1, x_t^2, \ldots, x_t^N$, which in turn, generate $\mathcal{Y}_t^N$ via (3).

It is shown first that the gain matrix L_t in (33d) is uniformly bounded, i.e., there exists a constant $c_1 > 0$ independent of N and $\mathcal{Y}_t^N$ such that

$$\|L_t\| < c_1 < \infty, \quad t \in [0, T], \tag{71}$$

where $\|L_t\|$ is interpreted as the induced matrix 2-norm of L_t. The proof of this fact is based on the inequality

$$\begin{aligned}
\|L_t\| &= \left\|\Sigma_t C_t^T \left(C_t\left(\Sigma_t + \Theta_t\right) C_t^T + V\right)^{-1}\right\| \\
&\leqslant \|\Sigma_t\| \cdot \left\|C_t^T\right\| \cdot \left\|\left(C_t\left(\Sigma_t + \Theta_t\right) C_t^T + V\right)^{-1}\right\| \\
&\leqslant \|\Sigma_t\| \cdot \left\|C_t^T\right\| \cdot \left\|\left(C_t\Theta_t C_t^T + V\right)^{-1}\right\|
\end{aligned}$$

which is concluded from $\|M_1 M_2\| \leqslant \|M_1\| \cdot \|M_2\|$ property of the induced norms and positive definiteness of Σ_t. Consider the Lyapunov matrix differential equation

$$\dot{\bar{\Sigma}}_t = A_t\bar{\Sigma}_t + \bar{\Sigma}_t A_t^T + D_t D_t^T$$

with the same initial condition S_z as (33b). The solution $\bar{\Sigma}_t$ to this equation is an upper bound on Σ_t, therefore $\|\Sigma_t\| \leqslant \left\|\bar{\Sigma}_t\right\|$. Hence, the bounded constant c_1 chosen as

$$c_1 = \sup_{t\in[0,T]} \left\|\bar{\Sigma}_t\right\| \cdot \left\|C_t^T\right\| \cdot \left\|\left(C_t\Theta_t C_t^T + V\right)^{-1}\right\|$$

satisfies (71).

Let z_t and $\hat{z}_t$ be the solutions to (11) and (33a) under the same control input, and define the error vector

$$\delta_t = z_t - \hat{z}_t. \tag{72}$$

Suppose that $\tau_1, \tau_2, \ldots, \tau_{\eta_T}$ are the sampling times of the observation set $\mathcal{Y}_T^N$. Subtracting both sides of (35a) from (11), the dynamics of δ_t over each interval $t \in (\tau_{k-1}, \tau_k]$ between the successive sampling times τ_{k-1} and τ_k is obtained as

$$\dot{\delta}_t = A_t \delta_t + D_t \dot{w}_t.$$

The solution to this equation for the interval $t \in (\tau_{k-1}, \tau_k]$ is given by

$$\delta_t = \Phi(t, \tau_{k-1})\, \delta_{\tau_{k-1}^+} + \int_{\tau_{k-1}}^{t} \Phi(t, s)\, D_s dw_s, \tag{73}$$

where $\Phi(\cdot)$ is the state transition matrix of A_t. This expression identically holds for $t \in [0, \tau_1]$ with $\delta_{\tau_0} \triangleq \delta_0 = z_0 - \bar{x}_0$, and for $t \in (\tau_{\eta_T}, T]$ by replacing $\delta_{\tau_{k-1}}$ with δ_{η_T}.

Define the scalar $\bar{\delta}_t$ as the conditional expected value

$$\bar{\delta}_t = \mathrm{E}\left[\|\delta_t\|^{2p} \,|\tau_1, \tau_2, \ldots, \tau_{\eta_T} \right]^{\frac{1}{2p}}, \quad t \in [0, T].$$

Applying (70) to (73) with $\mathrm{E}[\cdot|\tau_1, \tau_2, \ldots, \tau_{\eta_T}]$ replacing $\mathrm{E}[\cdot]$, and noting that $\{w_t\}$ is independent of $\tau_1, \tau_2, \ldots, \tau_{\eta_T}$ yield

$$\bar{\delta}_t \leqslant \|\Phi(t, \tau_{k-1})\|\, \bar{\delta}_{\tau_{k-1}^+} + \mathrm{E}\left[\left\| \int_{\tau_{k-1}}^{t} \Phi(t, s)\, D_s dw_s \right\|^{2p} |\tau_{k-1} \right]^{\frac{1}{2p}}.$$

This inequality leads to

$$\bar{\delta}_t \leqslant c_2 \bar{\delta}_{\tau_{k-1}^+} + c_3, \quad t \in (\tau_{k-1}, \tau_k], \tag{74}$$

where the bounded constants $c_2 \geqslant 1$ and $c_3 > 0$ are given by

$$c_2 = \sup_{t_1, t_2 \in [0,T]} \|\Phi(t_1, t_2)\|$$

$$c_3 = \sup_{t_1, t_2 \in [0,T]} \mathrm{E}\left[\left\| \int_{t_1}^{t_2} \Phi(t, s)\, D_s dw_s \right\|^{2p} \right]^{\frac{1}{2p}}.$$

The boundedness of c_3 is simply concluded from the fact that the integral inside the norm is a zero-mean Gaussian vector with a bounded covariance matrix.

Since $\{z_t\}$ is a continuous process, $z_{\tau_k^+} = z_{\tau_k}$ holds at each transition time $t = \tau_k$. Then, by subtracting both sides of (36a) from z_{τ_k} and using (13) and (72), it is concluded that

$$\delta_{\tau_k^+} = \delta_{\tau_k} - L_{\tau_k} C_{\tau_k} \delta_{\tau_k} - L_{\tau_k} \left(C_{\tau_k} e_{\tau_k}^{\nu_k} + v_k \right).$$

Application of (70) to the right-hand side of this equation leads to the inequality

$$\bar{\delta}_{\tau_k^+} \leqslant (1 + c_1 c_4)\, \bar{\delta}_{\tau_k} + c_1 c_5, \tag{75}$$

where the bounded positive constants c_4 and c_5 are defined as

$$c_4 = \sup_{t \in [0,T]} \|C_t\|$$

$$c_5 = \sup_{t \in [0,T]} \mathrm{E}\left[\|C_t e_t^n + v_k\|^{2p} \right]^{\frac{1}{2p}}.$$

Substituting (74) with $t = \tau_k$ into the right-hand side of (75) results in the recursive inequality

$$\bar{\delta}_{\tau_k^+} \leqslant c_6 \bar{\delta}_{\tau_{k-1}^+} + c_7, \tag{76}$$

where $c_6 = (1 + c_1 c_4)\, c_2 > 1$ and $c_7 = (1 + c_1 c_4)\, c_3 + c_1 c_5$.

Starting from the bounded initial value

$$\bar{\delta}_{\tau_0^+} \triangleq c_8 = \mathrm{E}\left[\|z_0 - \bar{x}_0\|^{2p} \right]^{\frac{1}{2p}} < \infty,$$

the recursive application of (76) for $k = 1, 2, \ldots, K$ yields

$$\bar{\delta}_{\tau_K^+} \leqslant c_8 c_6^K + c_7 \sum_{k=0}^{K-1} c_6^k \leqslant \max\{c_7, c_8\} \frac{c_6^{K+1} - 1}{c_6 - 1}.$$

Since $c_6 > 1$, the right-hand side of the second inequality is increasing in K which implies that

$$\bar{\delta}_{\tau_0^+}, \bar{\delta}_{\tau_1^+}, \bar{\delta}_{\tau_2^+}, \ldots, \bar{\delta}_{\tau_{\eta_T}^+} \leqslant \max\left\{c_7, c_8\right\} \frac{c_6^{\eta_T+1}-1}{c_6-1}.$$

Combining this result with (74) leads to the inequality

$$\bar{\delta}_t \leqslant c_2 \max\left\{c_7, c_8\right\} \frac{c_6^{\eta_T+1}-1}{c_6-1} + c_3, \quad t \in [0, T]$$

which is equivalently written as

$$\mathrm{E}\left[\left\|\delta_t\right\|^{2p} \middle| \tau_1, \tau_2, \ldots, \tau_{\eta_T}\right] \leqslant \left(c_2 \max\left\{c_7, c_8\right\} \frac{c_6^{\eta_T+1}-1}{c_6-1} + c_3\right)^{2p}, \quad t \in [0, T].$$

By taking the unconditional expected value of both sides of this inequality, applying Minkowski's inequality, and using the boundedness property (5) of η_T, it is concluded that

$$\mathrm{E}\left[\left\|\delta_t\right\|^{2p}\right] \leqslant \Delta < \infty, \quad t \in [0, T], \tag{77}$$

where the bounded constant Δ is defined as

$$\Delta = c_2 \max\left\{c_7, c_8\right\} \frac{\mathrm{E}\left[c_6^{2p(\eta_T+1)}\right]^{\frac{1}{2p}} - 1}{c_6-1} + c_3.$$

By substituting $u_t = K_t\left(z_t - \delta_t\right)$ into the linear state-space equation (11), this equation is written as

$$\dot{z}_t = \left(A_t - B_t K_t\right) z_t - B_t K_t \delta_t + D_t \dot{w}_t.$$

With δ_t satisfying (77), an argument similar to (68) shows that there exists a constant $0 < Z < \infty$ such that

$$\mathrm{E}\left[\left\|z_t\right\|^{2p}\right] \leqslant Z < \infty, \quad t \in [0, T].$$

Finally, for any $N \in \mathbb{N}$, application of (70) implies

$$\begin{aligned} \mathrm{E}\left[\left\|u_t\right\|^{2p}\right] &= \mathrm{E}\left[\left\|K_t\left(z_t - \delta_t\right)\right\|^{2p}\right] \\ &\leqslant \left\|K_t\right\|^{2p}\left(\mathrm{E}\left[\left\|z_t\right\|^{2p}\right]^{\frac{1}{2p}} + \mathrm{E}\left[\left\|\delta_t\right\|^{2p}\right]^{\frac{1}{2p}}\right)^{2p} \\ &\leqslant \left(Z^{\frac{1}{2p}} + \Delta^{\frac{1}{2p}}\right)^{2p} \sup_{t\in[0,T]} \left\|K_t\right\|^{2p}, \quad t \in [0, T]. \end{aligned}$$

Since K_t is a bounded matrix, this inequality verifies (8).

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