On Design of Robust Linear Quadratic Regulators

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Abstract

Closed-loop stability of uncertain linear systems is studied under the state feedback realized by a linear quadratic regulator (LQR). Sufficient conditions are presented that ensure the closed-loop stability in the presence of uncertainty, initially for the case of a non-robust LQR designed for a nominal model not reflecting the system uncertainty. Since these conditions are usually violated for a large uncertainty, a procedure is offered to redesign such a non-robust LQR into a robust one that ensures closed-loop stability under a predefined level of uncertainty. The analysis of this paper largely relies on the concept of inverse optimal control to construct suitable performance measures for uncertain linear systems, which are non-quadratic in structure but yield optimal controls in the form of LQR. The relationship between robust LQR and zero-sum linear quadratic dynamic games is established.

I. Introduction

Design of feedback control for dynamical systems heavily relies on mathematical models to describe their dynamics. These models are often approximate in nature and provide only partial knowledge of the system dynamics within certain level of uncertainty. Despite the inherent uncertainty in the system dynamics, a practical controller must be capable to robustly sustain its prescribed performance, and in particular, preserve its closed-loop stability. To address this essential requirement, major research effort has been directed toward robust control of uncertain systems, within both deterministic and stochastic frameworks. The present work concentrates on robust optimal control of uncertain linear systems within a deterministic framework formed around the notion of linear quadratic regulator (LQR).

Robust stabilization of uncertain linear systems has been extensively studied by several authors [1]–[15]. The resulting rich body of work directly relies on the notion of quadratic stabilizability and Lyapunov stability analysis, while relating to the LQR theory indirectly via the use of Riccati equation in construction of Lyapunov functions. The Riccati equation also plays a central role in the $\mathcal{H}_\infty$ approach to robust control of linear systems [16]–[19]. Prior work on robust control and stability analysis in the presence of uncertainty has been also extended to delayed [20], [21] and discrete-time [22] linear systems.

Prior work specifically addressing robust LQR is not too extensive. The authors in [23] propose a modified quadratic cost functional to improve the robustness of LQR against certain model variations. As discussed later, the result of [23] is a special case of the present work, even though the stability analysis in [23] relies on Lyapunov functions, different from the approach of this paper. In [24], again a Lyapunov function technique is employed to enhance the robustness of LQR by including a nonlinear component into its linear structure. The authors in [25], [26] adopt a dynamic programming approach within a discrete-time framework to develop a robust control law for linear systems. In their proposed scheme, the Bellman equation is solved at each instance of time under the worst case scenario of the model uncertainty.

This paper focuses on robust LQR by direct application of optimal control techniques, rather than the use of Lyapunov stability analysis. The core idea here is to construct a suitable cost functional as the performance index of a perturbed linear system, and evaluate the optimal closed-loop performance of this uncertain system by direct analysis of the cost functional. The constructed cost functional is not quadratic indeed, but inherently produces a valid LQR as the optimal control law for the perturbed linear system. The process of constructing a cost functional for a given optimal control law is a topic in *inverse optimal* control, which has been studied by several researchers [27]–[32].

In this paper, construction of cost functional and stability analysis of its associated optimal control law rely on our prior work on inverse optimal control [31], [32]. Some relevant results of this prior work are reproduced in Section III, which are then incorporated into a practical framework for optimal control of generally nonlinear perturbed systems. Later in Section IV, this framework is specifically applied to the case of uncertain linear systems to form a basis for robust LQR. Some results of Section IV appear in [31], [32] as individual examples, which in this paper, are consistently incorporated into a coherent framework for design of robust LQR control. A complete statement of the problem addressed in this paper is presented next in Section II.

II. Problem Statement

Consider the state-space equation

$$\dot{x}_t = \big(A + \delta A\,(x_t, u_t, t)\big)x_t + \big(B + \delta B\,(x_t, u_t, t)\big)u_t \tag{1}$$

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with the state vector $x_t \in \mathbb{R}^n$, the control vector $u_t \in \mathbb{R}^k$, and constant matrices A and B of proper dimensions, where the pair (A, B) is controllable. This nonlinear, time-varying equation is indeed used to describe the uncertain dynamics of a linear time-invariant system nominally represented by

$$\dot{x}_t = Ax_t + Bu_t. \tag{2}$$

The uncertainty in this linear system is reflected into the perturbed model (1) via the matrix-valued functions $\delta A(\cdot)$ and $\delta B(\cdot)$, which most generally can depend on all three of state, control, and time. As uncertain parameters, the explicit forms of these functions are not known; instead, their partial knowledge is provided in terms of bounds on their magnitude and assumptions on their structure, as discussed later.

Suppose a stabilizing linear state feedback

$$u_t = -\bar{K}x_t \tag{3}$$

with the $k \times n$ gain matrix $\bar{K}$ has been developed for the nominal model (2) via the LQR method. This control law is optimal in the sense that it minimizes the infinite horizon quadratic cost functional

$$J = \int_0^\infty \left(x_t^T Q x_t + u_t^T R u_t\right) dt \tag{4}$$

with the positive definite weight matrices Q and R of suitable dimensions (Q can be positive semidefinite provided that the pair (A, Q) is observable). These matrices have been chosen properly during the control design process, and are assumed known and fixed throughout this paper.

The first goal of this paper is to evaluate the closed-loop performance of the perturbed system (1) under the nominal control law (3). The specific task is to establish conditions on the functions $\delta A(\cdot)$ and $\delta B(\cdot)$ that preserve the closed-loop stability. Yet, the major objective of this paper is to refine the nominal control law (3) into a robust linear control law

$$u_t = -Kx_t \tag{5}$$

with a new LQR gain matrix K that ensures the closed-loop stability of the uncertain system (1) under given bounds on its unknown parameters $\delta A(\cdot)$ and $\delta B(\cdot)$. This is achieved by synthesizing a perturbed cost functional of the form

$$J_p = \int_0^\infty \left(x_t^T Q x_t + u_t^T R u_t + \theta(x_t, u_t, t)\right) dt \tag{6}$$

which results in a linear optimal control law of the form (5) with prescribed robustness. The perturbation $\theta(\cdot)$ in this cost functional is constructed in terms of $\delta A(\cdot)$ and $\delta B(\cdot)$, which is not necessarily a quadratic function.

III. Optimal Control of Perturbed Systems

The results of this paper are developed on the basis of two theorems from our prior work in [31], [32], reproduced here as Lemmas 1 and 2. These lemmas are modified then into Propositions 1 and 2 of this section to establish a basis for stability analysis of uncertain systems under optimal control.

Let $f(\cdot) : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ be a vector-valued function and consider the nonlinear state-space equation

$$\dot{x}_t = f(x_t, u_t) \tag{7}$$

with the state vector $x_t \in \mathbb{R}^n$, control vector u_t in the control set $\mathcal{U} \subset \mathbb{R}^k$, and an initial state x_0 given at $t = 0$. To measure the system performance, the infinite horizon cost functional

$$J = \int_0^\infty c(x_t, u_t)\, dt \tag{8}$$

is adopted with the Lagrangian $c(\cdot) : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$. Consider next the optimal control problem seeking a stationary control law $\mu(\cdot) : \mathbb{R}^n \to \mathcal{U}$ to minimize this cost functional under the state feedback control

$$u_t = \mu(x_t) \tag{9}$$

applied to the dynamical system (7).

The solution to this problem can be obtained by solving the Hamilton-Jacobi-Bellman (HJB) equation

$$\min_{u \in \mathcal{U}} \left\{\nabla V(x) \cdot f(x, u) + c(x, u)\right\} = 0 \tag{10}$$

for the value function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$. Here, ∇ and $\cdot$ denote the gradient and dot product operators, respectively. Under suitable regularity conditions, the optimal control law can be determined in terms of $V(\cdot)$ from the optimization problem

$$\mu(x) = \arg\min_{u \in \mathcal{U}} \left\{\nabla V(x) \cdot f(x, u) + c(x, u)\right\}. \tag{11}$$

The following lemma, replicated from [31, Prop. 1], provides a sufficient set of such regularity conditions.

Lemma 1: (Proposition 1 of [31]) Let the Lagrangian $c(\cdot)$ be continuous on $\mathbb{R}^n \times \mathcal{U}$ and define the cost functional (8) subject to the state-space equation (7). Assume that the HJB equation (10) admits a differentiable solution satisfying

$$0 \leqslant \varepsilon V(x) \leqslant c(x,u), \quad (x,u) \in \mathbb{R}^n \times \mathcal{U} \tag{12}$$

for some $\varepsilon > 0$. Then, a nonempty set of admissible controls exists under which the cost functional (8) remains bounded. Furthermore, over such set of admissible controls, this cost functional attains the minimum value

$$J^* = V(x_0)$$

under the stationary control law (11).

Proof: See the proof of Proposition 1 in [31]. ∎

Next, the closed-loop stability of the dynamical system (7) under the optimal control (11) is examined in Lemma 2. This lemma is a modification of [32, Thm. 2].

Lemma 2: (Theorem 2 of [32]) Suppose that $x = 0$ is an equilibrium point of the dynamical system (7) under $u_t = 0$, i.e., $f(0,0) = 0$. Assume that $c(\cdot)$ in the cost functional (8) is continuous and positive definite, i.e., $c(x,u) > 0$ for all $(x,u) \in \mathbb{R}^n \times \mathcal{U} - \{(0,0)\}$ and $c(0,0) = 0$. Assume further that $c(x,u) \to \infty$ as $\|x\| \to \infty$ for all $u \in \mathcal{U}$. Let $\mu(\cdot)$ be the optimal control law (11) under the assumptions of Lemma 1. Then, $x = 0$ is the globally asymptotically stable equilibrium of the dynamical system (7) under the state feedback (9).

Proof: See the proof of Theorem 2 in [32]. ∎

In the remainder of this section, Lemmas 1 and 2 are used to introduce a framework for robustness analysis of uncertain systems under optimal control. Suppose an uncertain system is described by the state-space equation

$$\dot{x}_t = f(x_t, u_t) + \delta f(x_t, u_t, t), \tag{13}$$

which is a perturbed form of the nominal dynamics (7). Here, the perturbation $\delta f(\cdot) : \mathbb{R}^n \times \mathcal{U} \times [0,\infty) \to \mathbb{R}^n$ is a function representing the model uncertainty, and not explicitly known. Since uncertainty is essentially time-varying, this function is allowed to explicitly depend on time, although the nominal model is time-invariant.

The goal here is to evaluate the closed-loop performance, mainly stability, of the uncertain system (13) under the same optimal control (11) developed for the nominal model (7). The evaluation method in this paper begins with constructing an auxiliary performance index for the perturbed system (13), which is identically optimized by the same nominal optimal control (11). The performance of the perturbed system (13) is then evaluated under the nominally optimal control law (11) by direct investigation of the auxiliary performance measure. Specifically, sufficient conditions for closed-loop stability of the uncertain system (13) are derived from Lemmas 1 and 2.

The auxiliary performance measure is constructed as

$$J_p = \int_0^\infty \big(c(x_t, u_t) - \nabla V(x_t) \cdot \delta f(x_t, u_t, t)\big) dt. \tag{14}$$

The Lagrangian in this cost functional is specifically chosen to yield an HJB equation with the same solution as (10), and as a result, to produce the same optimal control law (11). This HJB equation is expressed as

$$\min_{u \in \mathcal{U}} \big\{ \nabla V_p(x) \cdot \big(f(x,u) + \delta f(x,u,t)\big) + c(x,u) - \nabla V(x) \cdot \delta f(x,u,t) \big\} = 0$$

for the value function $V_p(\cdot)$, which is clearly solved by the same solution $V_p(\cdot) = V(\cdot)$ of (10) and the same minimizer as (11). Therefore, under suitable regularity conditions, (11) is an optimal control law for the perturbed system (13) with respect to the performance measure (14). These conditions are cautiously derived from Lemma 1, considering that both the uncertain dynamics (13) and the Lagrangian of (14) are time-varying, unlike the time-invariant setting of this lemma. The results are stated in the following proposition.

Proposition 1: Assume that $c(\cdot)$ and $\delta f(\cdot)$ are continuous functions and the HJB equation (10) admits a differentiable solution satisfying

$$0 \leqslant \varepsilon V(x) \leqslant c(x,u) - \nabla V(x) \cdot \delta f(x,u,t) \tag{15}$$

for some $\varepsilon > 0$ and all $(x,u,t) \in \mathbb{R}^n \times \mathcal{U} \times [0,\infty)$. Define the cost functional (14) subject to the state-space equation (13) with the initial state x_0. Then, a nonempty set of admissible controls exists under which the cost functional (14) remains bounded. Furthermore, over such set of admissible controls, this cost functional attains the minimum value

$$J_p^* = V(x_0)$$

under the stationary control law (11).

Proof: The proof is a straightforward result of Lemma 1 applied to the auxiliary Lagrangian

$$c_p(x,u,t) = c(x,u) - \nabla V(x) \cdot \delta f(x,u,t).$$

For this Lagrangian and the dynamical system (13), the HJB equation remains same as (10), which results in the optimal control law (11). The regularity condition (12) in Lemma 1 is cautiously replaced with (15), noting that the Lagrangian is time-invariant in Lemma 1 but time-varying here. The proof of Lemma 1 (see the proof of [31, Prop. 1]) solely uses (12) to derive $\lim_{t\to\infty} V(x_t) = 0$ from $\lim_{t\to\infty} c(x_t, u_t) = 0$ for admissible controls. A similar conclusion can be drawn here from $\lim_{t\to\infty} c_p(x_t, u_t, t) = 0$ by enforcing (15). ■

The following proposition is the counterpart of Lemma 2 for uncertain systems, and analyzes the closed-loop stability of the perturbed system (13) under the control law (11).

Proposition 2: Suppose that $x = 0$ is an equilibrium point of the nominal system (7) under $u_t = 0$, i.e., $f(0, 0) = 0$, and that the perturbation $\delta f(\cdot)$ in (13) holds the property

$$\delta f(0, 0, t) = 0, \quad t \in [0, \infty).$$

Let $c(\cdot)$ be a continuous Lagrangian holding $c(0, 0) = 0$, and assume that the HJB equation (10) admits a differentiable solution $V(\cdot)$ satisfying (15). Assume further that $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$, and that $\lim_{\|x\|\to\infty} V(x) \geqslant \bar{V}$ for some $\bar{V} > 0$. Then, $x = 0$ is the globally asymptotically stable equilibrium of the uncertain system (13) under the optimal control law (11) applied as the state feedback (9).

Proof: The proof of [32, Thm. 2] shows that the optimal control law (11) holds the property $\mu(0) = 0$. Then, under the assumptions of this proposition, $x = 0$ is an equilibrium point of the uncertain system (13) under the optimal control law (11). Moreover, by an argument similar to [31, Prop. 1], the closed-loop state of (13) under the control law (11) and the assumption (15) holds $\lim_{t\to\infty} V(x_t) = 0$. This implies that $x = 0$ is the globally asymptotically stable equilibrium point of the closed-loop system, nothing that $V(x) = 0$ holds only at $x = 0$, by the assumptions of this proposition. ■

IV. Robust Linear Quadratic Regulators

This section exploits the general framework of Section III to especially study the robustness of uncertain linear systems under LQR control. First in Section IV-A, the closed-loop stability of such systems is examined for a non-robust LQR control naively designed for a nominal model of the system not reflecting its uncertainty. The resulting control may not be sufficiently robust to ensure its closed-loop stability in the presence of substantial uncertainty. Thus, a design procedure is introduced in Section IV-B to develop robust LQR controls capable of preserving their stability under a predefined level of uncertainty.

A. Stability of Uncertain Linear Systems Under LQR Control

Consider the LQR state feedback (3) designed to minimize the quadratic cost functional (4) subject to the nominal linear dynamics (2). It is well known that the gain matrix $\bar{K}$ of this linear state feedback is given by [33, p. 195]

$$\bar{K} = B^T R^{-1} \bar{P}, \tag{16}$$

where $\bar{P}$ is the positive definite matrix solving the algebraic Riccati equation

$$A^T \bar{P} + \bar{P} A - \bar{P} B R^{-1} B^T \bar{P} + Q = 0. \tag{17}$$

In addition, the minimum value of the cost functional (4) is given by $V(x_0)$ in terms of the quadratic value function

$$V(x) = x^T \bar{P} x. \tag{18}$$

Suppose the uncertain system (1) is controlled under the linear state feedback (3) with the gain matrix (16). To analyze the stability of the resulting closed-loop system, the method of Section III is adopted to construct a cost functional of the form (14) for the uncertain system (1) and the quadratic value function (18). The proposed non-quadratic cost functional is expressed as

$$J_p = \int_0^\infty \Big(x_t^T Q x_t + u_t^T R u_t - 2 x_t^T \bar{P} \big(\delta A(x_t, u_t, t) \, x_t + \delta B(x_t, u_t, t) \, u_t \big) \Big) dt. \tag{19}$$

Next, sufficient conditions are derived from Propositions 1 and 2 under which this cost functional is minimized by the linear state feedback (3) subject to the uncertain system (1). These conditions also imply closed-loop stability of (1) under the optimal control (3), as stated in the following proposition.

Proposition 3: Suppose $\delta A(\cdot)$ and $\delta B(\cdot)$ in the uncertain system (1) are confined to the uncertainty sets $\mathscr{U}_A$ and $\mathscr{U}_B$, respectively, that is

$$\delta A(x, u, t) \in \mathscr{U}_A, \quad \delta B(x, u, t) \in \mathscr{U}_B \tag{20}$$

for every $(x, u, t) \in \mathbb{R}^n \times \mathbb{R}^k \times [0, \infty)$. Let $\bar{P}$ be the positive definite solution to the algebraic Riccati equation (17) with a controllable (A, B) and positive definite Q and R. Assume the matrix inequality

$$Q - \delta A^T \bar{P} - \bar{P} \delta A - \bar{P} \delta B R^{-1} \delta B^T \bar{P} \geqslant \varepsilon \bar{P} \tag{21}$$

holds for some $\varepsilon > 0$ and all $\delta A \in \mathscr{U}_A$ and $\delta B \in \mathscr{U}_B$. Then, under the linear state feedback (3) with the gain matrix (16), the uncertain system (1) has a globally asymptotically stable equilibrium at $x = 0$, and for the initial state x_0 produces the minimum value $J_p^* = x_0^T \bar{P} x_0$ of the cost functional (19).

Proof: This is a special case of Propositions 1 and 2 for which the regularity condition (15) is explicitly given by

$$x^T Q x + u^T R u - 2x^T \bar{P}\big(\delta A\,(x,u,t)\,x + \delta B\,(x,u,t)\,u\big) \geqslant \varepsilon x^T \bar{P} x, \quad (x,u,t) \in \mathbb{R}^n \times \mathbb{R}^k \times [0,\infty)\,.$$

This condition is implied by the matrix inequality

$$Q - \delta A^T\,(x,u,t)\,\bar{P} - \bar{P}\delta A\,(x,u,t) - \bar{P}\delta B\,(x,u,t)\,R^{-1}\delta B^T\,(x,u,t)\,\bar{P} \geqslant \varepsilon \bar{P}, \tag{22}$$

which is straightforwardly concluded from

$$\begin{aligned} u^T R u - 2x^T \bar{P}\delta B\,(x,u,t)\,u &\geqslant \min_{v\in\mathbb{R}^k}\big(v^T R v - 2x^T \bar{P}\delta B\,(x,u,t)\,v\big) \\ &= -x^T \bar{P}\delta B\,(x,u,t)\,R^{-1}\delta B^T\,(x,u,t)\,\bar{P}x. \end{aligned}$$

However, (22) together with (20) leads to (21). ■

B. Design of Robust LQR for Uncertain Linear Systems

The matrix inequality (21) provides a sufficient condition for closed-loop stability of an uncertain linear system when it is controlled by an LQR designed for its nominal model. For a large uncertainty, however, this condition can be violated, and as a result, the closed-loop stability cannot be ensured. In that case, the nominal LQR (3) is redesigned into the robust LQR (5) with a new gain matrix K given by

$$K = B^T R^{-1} P. \tag{23}$$

Here, P is the solution to a new Riccati equation developed in this section.

The strategy in this paper is to replace the weight matrix Q in the quadratic cost functional (4) with $Q+\Theta_a\,(P)+\Theta_b\,(P)$, where $\Theta_a\,(P)$ and $\Theta_b\,(P)$ are positive definite (or positive semidefinite) matrices depending on the solution P to the Riccati equation which must be developed. Then, the robust LQR (5) must minimize the quadratic cost functional

$$J = \int_0^\infty \Big(x_t^T\big(Q + \Theta_a\,(P) + \Theta_b\,(P)\big)x_t + u_t^T R u_t\Big)dt$$

subject to the nominal linear system (2). To realize this goal, the gain K of this robust LQR is obtained from (23) with P solving the algebraic matrix equation

$$A^T P + PA - PBR^{-1}B^T P + \Theta_a\,(P) + \Theta_b\,(P) + Q = 0. \tag{24}$$

To ensure the robustness of this LQR using Proposition 3, the matrix inequality

$$Q + \Theta_a\,(P) + \Theta_b\,(P) - \delta A^T P - P\delta A - P\delta B R^{-1}\delta B^T P \geqslant \varepsilon P \tag{25}$$

is enforced for every $\delta A \in \mathscr{U}_A$ and $\delta B \in \mathscr{U}_B$ with $\varepsilon > 0$. To that end, $\Theta_a\,(\cdot)$ and $\Theta_b\,(\cdot)$ are constructed with the property

$$\Theta_a\,(P) - \delta A^T P - P\delta A \geqslant \varepsilon P, \quad \delta A \in \mathscr{U}_A \tag{26a}$$

$$\Theta_b\,(P) - P\delta B R^{-1}\delta B^T P \geqslant 0, \quad \delta B \in \mathscr{U}_B, \tag{26b}$$

which clearly implies (25) for any positive definite Q (or a positive semidefinite Q such that (A,Q) is observable). In addition, $\Theta_a\,(\cdot)$ and $\Theta_b\,(\cdot)$ must be chosen in such a manner that (24) remains a valid Riccati equation admitting a unique positive definite solution. The construction process of $\Theta_a\,(\cdot)$ and $\Theta_b\,(\cdot)$ is discussed in the remainder of this section.

In this paper, the uncertainty set $\mathscr{U}_A$ includes all instances of δA with the largest singular value not exceeding a given constant $\alpha > 0$, that is

$$\mathscr{U}_A = \big\{\delta A \in \mathbb{R}^{n\times n}|\ \|\delta A\| \leqslant \alpha\big\}\,, \tag{27}$$

where $\|\cdot\|$ denotes the induced 2-norm of matrices. For this uncertainty set, the following lemma proposes an appropriate structure for $\Theta_a\,(\cdot)$.

Lemma 3: Suppose P is a positive definite matrix and S is any square matrix satisfying $\|S\| \leqslant 1$, where $\|\cdot\|$ denotes the induced 2-norm of matrices. Then, the matrix inequality

$$\frac{1}{2}\left(S^T P + PS\right) \leqslant \eta P$$

holds for $\eta > 0$ defined as

$$\eta = \sqrt{\max_{\|w\|=1}\,(w^T P^{-1} w)\,(w^T P w)}\,. \tag{28}$$

Proof: See Appendix. ∎

Using Lemma 3, a suitable form for $\Theta_a(\cdot)$ is proposed as

$$\Theta_a(P) = (2\lambda + \varepsilon) P \tag{29}$$

with a constant parameter $\lambda > 0$. By this lemma, the matrix inequality (26a) identically holds for the uncertainty set (27) under any $0 < \alpha \leqslant \lambda/\eta$, where η is given by (28).

Remark 1: The authors of [23] adopt a Lyapunov stability analysis technique and the solution P to the Riccati equation

$$(\lambda I + A)^T P + P(\lambda I + A) - PBR^{-1}B^T P + Q = 0 \tag{30}$$

with $\lambda > 0$ (I denotes the identity matrix) to derive a robust LQR gain from (23). This equation is indeed the same as (24) with $\Theta_a(P) = 2\lambda P$ and $\Theta_b(P) = 0$, in the notation of this paper. It is shown in [23] that for some $\lambda > 0$, the robust gain derived from (23) and (30) stabilizes the uncertain system (1) under the linear control (5) for $\delta A(\cdot) \in \mathscr{U}_A$ and $\delta B(\cdot) = 0$. Similar results are presented in [31], [32] following the same analysis method of this paper. Yet, the upper bound $\alpha \leqslant \lambda/\eta$ established here using Lemma 3 is tighter than its counterpart in [23], [31], [32].

For the uncertainty set $\mathscr{U}_B$, two models are considered at different generality levels. The first model, which is easier to analyze, restricts the columns of δB to be in the column space of B, that is, a $k \times k$ matrix Δ exists to hold $\delta B = B\Delta$. Then, $\mathscr{U}_B$ is defined as

$$\mathscr{U}_B = \left\{ \delta B = B\Delta \in \mathbb{R}^{n\times k} \middle| \left\| R^{1/2}\Delta R^{-1/2} \right\| \leqslant \gamma \right\} \tag{31}$$

for some $\gamma > 0$. Here, $R^{1/2}$ is the square root of R and $\|\cdot\|$ denotes the induced 2-norm of matrices. It is straightforward to verify that

$$\Theta_b(P) = \gamma^2 PBR^{-1}B^T P \tag{32}$$

satisfies (26b) under the uncertainty set (31).

For the choices of $\Theta_a(\cdot)$ defined in (29) and $\Theta_b(\cdot)$ in (32), the algebraic equation (24) is the valid Riccati equation

$$\Big(\left(\lambda + \tfrac{1}{2}\varepsilon\right) I + A\Big)^T P + P\Big(\left(\lambda + \tfrac{1}{2}\varepsilon\right) I + A\Big) - PB\left(\frac{R}{1-\gamma^2}\right)^{-1} B^T P + Q = 0, \tag{33}$$

provided that $0 < \gamma < 1$. The solution to this equation yields a robust LQR, as stated in the following proposition.

Proposition 4: Take any $\lambda > 0$ and $0 < \gamma < 1$ and assume that P is the unique positive definite solution to the Riccati equation (33) with a controllable (A, B), observable (A, Q), and $\varepsilon > 0$. In terms of P, determine the LQR gain matrix K from (23) and the constant η from (28). Take $0 < \alpha \leqslant \lambda/\eta$ and define the uncertainty sets $\mathscr{U}_A$ and $\mathscr{U}_B$ by (27) and (31), respectively. Then, under the linear state feedback (5), the uncertain system (1) has a globally asymptotically stable equilibrium at $x = 0$ for any $\delta A(\cdot) \in \mathscr{U}_A$ and $\delta B(\cdot) \in \mathscr{U}_B$. Moreover, this control minimizes the cost functional (6) for

$$\theta(x, u, t) = 2x^T P\left(\left(\lambda + \tfrac{1}{2}\varepsilon\right) I - \delta A(x, u, t)\right) x + 2x^T P\left(\tfrac{1}{2}\gamma^2 BKx - \delta B(x, u, t)\, u\right).$$

Proof: As (A, B) is controllable, $\Big(\left(\lambda + \tfrac{1}{2}\varepsilon\right) I + A, B\Big)$ is also controllable. Then, with an observable pair (A, Q), the Riccati equation (33) has a unique positive definite solution. In terms of this solution P, determine $\Theta_a(P)$ and $\Theta_b(P)$ from (29) and (32), respectively, and replace Q in the cost functional (19) with $Q + \Theta_a(P) + \Theta_b(P)$. The statements of this proposition are then concluded from Proposition 3 by verifying (21) with Q replaced by $Q + \Theta_a(P) + \Theta_b(P)$, i.e.,

$$Q + (2\lambda + \varepsilon) P + \gamma^2 PBR^{-1}B^T P - \delta A^T P - P\delta A - P\delta BR^{-1}\delta B^T P \geqslant \varepsilon P.$$

∎

A more general form of the uncertainty set $\mathscr{U}_B$ considered in this paper consists of all matrices δB with a bound on their weighted 2-norm according to

$$\mathscr{U}_B = \left\{ \delta B \in \mathbb{R}^{n\times k} \middle| \left\| \delta BR^{-1/2} \right\| \leqslant \beta \right\}, \tag{34}$$

where $\beta > 0$ is given constant. It is straightforward to verify for this uncertainty set that

$$\Theta_b(P) = \beta^2 P^2$$

holds the inequality condition (26b). For this choice of $\Theta_b(\cdot)$, the algebraic equation (24) is explicitly expressed as

$$\Big(\left(\lambda + \tfrac{1}{2}\varepsilon\right) I + A\Big)^T P + P\Big(\left(\lambda + \tfrac{1}{2}\varepsilon\right) I + A\Big) - PBR^{-1}B^T P + \beta^2 P^2 + Q = 0, \tag{35}$$

which indeed is not a conventional Riccati equation. Rather, it is a generalized Riccati equation playing a central role in the zero-sum linear quadratic (LQ) dynamic games [34].

The specific generalized Riccati equation (35) is associated with a dynamic game involving two players u_t and w_t which independently control the linear dynamics

$$\dot{x}_t = Ax_t + Bu_t + w_t.$$

The goal of players in this game is to, respectively, minimize and maximize the quadratic functional

$$J_{\text{game}} = \int_0^{\infty} e^{(2\lambda+\varepsilon)t} \left(x_t^T Q x_t + u_t^T R u_t - \beta^{-2} w_t^T w_t \right) dt.$$

Then, existence of a unique positive definite solution for (35) is equivalent to the existence of a saddle-point equilibrium for this LQ dynamic game. The class of LQ dynamic games and conditions for the existence of saddle-point equilibria for them have been extensively studied beyond the scope of this paper. These conditions are not repeated here; instead, the interested reader is referred to [34] for details.

The existence of a positive definite solution for (35) clearly depends on the size of parameter β. For $\beta = 0$, (35) trivially reduces to a conventional Riccati equation admitting a unique positive definite solution. By increasing β from $\beta = 0$ toward positive values, this solution continues to exist up to some maximum value β_m. This maximum value defines the largest uncertainty in $\delta B(\cdot)$ which can be successfully handled by the robust LQR (5) with the gain matrix (23) derived from the solution P to (35). Under this LQR, the uncertain system (1) minimizes the cost functional (6) with

$$\theta(x,u,t) = 2x^T P \left(\left(\lambda + \tfrac{1}{2}\varepsilon \right) I - \delta A(x,u,t) \right) x + 2x^T P \left(\tfrac{1}{2}\beta^2 P x - \delta B(x,u,t)\, u \right),$$

and remains stable for every $\delta A(\cdot) \in \mathscr{U}_A$ and $\delta B(\cdot) \in \mathscr{U}_B$, where $\mathscr{U}_A$ and $\mathscr{U}_B$ are given by (27) and (34), respectively.

V. Conclusion

Adopting an inverse optimal control approach, closed-loop stability of uncertain linear systems under an LQR feedback control was studied. Sufficient conditions were established to ensure the feedback stability in the presence of uncertainty. Moreover, a procedure for design of robust LQR control was presented to enforce these conditions for a predefined level of uncertainty. The role of linear quadratic dynamic games in the proposed design procedure was discussed.

Appendix
Proof of Lemma 3

The statement of lemma is proven by showing that

$$\frac{x^T P S x}{x^T P x} \leqslant \eta \tag{36}$$

holds for all $x \neq 0$ if $\|S\| \leqslant 1$ and $P > 0$. By singular value decomposition, S is expressed as $S = UDV^T$, where D is a diagonal matrix with diagonal elements in $[0,1]$ and U and V are unitary matrices. Substituting this decomposition into the left-hand side of (36) and rearranging terms result in

$$\frac{x^T P S x}{x^T P x} = \frac{\left(U^T x\right)^T \left(U^T P U\right) \left(D V^T U\right) \left(U^T x\right)}{\left(U^T x\right)^T \left(U^T P U\right) \left(U^T x\right)} = \frac{y^T \Pi z}{y^T \Pi y},$$

where $\Pi = U^T P U$ is a positive definite matrix, $y = U^T x$, and $z = \left(D V^T U\right) y$ holds the constraint $\|z\| \leqslant \|y\|$. Then, it is concluded that

$$\begin{aligned} \frac{x^T P S x}{x^T P x} &\leqslant \max_{\substack{\|z\| \leqslant \|y\| \\ y \neq 0}} \frac{y^T \Pi z}{y^T \Pi y} = \max_{y \neq 0} \left(\frac{y^T \Pi}{y^T \Pi y} \right) \left(\frac{\Pi y}{\|\Pi y\|} \|y\| \right) \\ &= \max_{y \neq 0} \frac{\|\Pi y\| \cdot \|y\|}{\left\|\Pi^{1/2} y\right\|^2} = \max_{w \neq 0} \frac{\left\|\Pi^{-1/2} w\right\| \cdot \left\|\Pi^{1/2} w\right\|}{\|w\|^2} \\ &= \sqrt{\max_{\|w\|=1} \left(w^T \Pi^{-1} w\right) \left(w^T \Pi w\right)} = \eta. \end{aligned}$$

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