



Scalar Conformal Primary Fields in the Brownian Loop Soup

Federico Camia^{1,2,3} , Valentino F. Foit² , Alberto Gandolfi² , Matthew Kleban⁴

¹ Science Division, New York University Abu Dhabi, Saadiyat Island, Abu Dhabi, United Arab Emirates.
E-mail: federico.camia@nyu.edu

² Department of Mathematics, Vrije Universiteit Amsterdam, De Boelelaan 1111, Amsterdam 1081 HV, The Netherlands. E-mail: foit@nyu.edu; albertogandol@nyu.edu

³ Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012, USA

⁴ Center for Cosmology and Particle Physics, New York University, 726 Broadway, New York, New York 10003, USA. E-mail: kleban@nyu.edu

Received: 18 October 2021 / Accepted: 6 December 2022

Published online: 19 December 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract: The Brownian loop soup is a conformally invariant statistical ensemble of random loops in two dimensions characterized by an intensity $\lambda > 0$, with central charge $c = 2\lambda$. Recent progress resulted in an analytic form for the four-point function of a class of scalar conformal primary “layering vertex operators” $\mathcal{O}_\beta$ with dimensions (Δ, Δ) , with $\Delta = \frac{\lambda}{10}(1 - \cos \beta)$, that compute certain statistical properties of the model. The Virasoro conformal block expansion of the four-point function revealed the existence of a new set of operators with dimensions $(\Delta + k/3, \Delta + k'/3)$, for all non-negative integers k, k' satisfying $|k - k'| \equiv 0 \pmod{3}$. In this paper we introduce the edge counting field $\mathcal{E}(z)$ that counts the number of loop boundaries that pass close to the point z . We rigorously prove that the n -point functions of $\mathcal{E}$ are well defined and behave as expected for a conformal primary field with dimensions $(1/3, 1/3)$. We analytically compute the four-point function $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle$ and analyze its conformal block expansion. The operator product expansions of $\mathcal{E} \times \mathcal{E}$ and $\mathcal{E} \times \mathcal{O}_\beta$ contain higher-order edge operators with “charge” β and dimensions $(\Delta + k/3, \Delta + k/3)$. Hence, we have explicitly identified all scalar primary operators among the new set mentioned above. We also re-compute the central charge by an independent method based on the operator product expansion and find agreement with previous methods.

1. Introduction

This article is concerned with a new family of conformal field theories that arise from the Brownian loop soup. In this section we provide some background, introduce some terminology, discuss the structure of the paper and present the main results.

1.1. The Brownian loop soup. The Brownian loop soup (BLS) [1] is an ideal gas of Brownian loops with a distribution chosen so that it is invariant under local conformal

transformations. The BLS is implicit in the work of Symanzik [2] on Euclidean quantum field theory, more precisely, in the representation of correlation functions of Euclidean fields in terms of random paths that are locally statistically equivalent to Brownian motion. This representation can be made precise for the Gaussian free field, in which case the random paths are independent of each other and can be generated as a Poisson process.

The BLS is closely related not only to Brownian motion and the Gaussian free field but also to the Schramm-Loewner Evolution (SLE) and Conformal Loop Ensembles (CLEs). It provides an interesting and useful link between Brownian motion, field theory, and statistical mechanics. Partly motivated by these connections, as well as by a potential application to cosmology in the form of a conformal field theory for eternal inflation [3], three of the present authors introduced a set of operators that compute properties of the BLS and discovered new families of conformal primary fields depending on a real parameter β [4]. One such family are the fields $\mathcal{O}_\beta$. These operators have scaling dimensions $\Delta(\beta) = \frac{\lambda}{10}(1 - \cos \beta)$ and are periodic under $\beta \rightarrow \beta + 2\pi$, with $\mathcal{O}_0 \equiv \mathcal{O}_{2\pi} = \mathbb{1}$ (the identity operator). Their n -point function $\langle \mathcal{O}_{\beta_1}(z_1) \dots \mathcal{O}_{\beta_n}(z_n) \rangle_{\mathbb{C}}$ in the full plane is identically zero unless $\sum_{j=1}^n \beta_j = 0 \pmod{2\pi}$, which is reminiscent of the “charge neutrality” or “charge conservation” condition that applies to vertex operators of the free boson [5]. The existence of the operators $\mathcal{O}_\beta$ as generalized random fields (i.e., random distributions in the sense of Schwartz) was proved in [6] when $\Delta(\beta) < 1/2$.

These operators were further studied in [7], where it is shown that the operator product expansion (OPE) $\mathcal{O}_{\beta_i} \times \mathcal{O}_{\beta_j}$ predicts the existence of operators of dimensions $(\Delta_{ij} + \frac{k}{3}, \Delta_{ij} + \frac{k'}{3})$ for all non-negative integers k, k' satisfying $|k - k'| = 0 \pmod{3}$, where $\Delta_{ij} = \frac{\lambda}{10}(1 - \cos(\beta_i + \beta_j))$. The simplest case is $k = k' = 1$ and $\beta_i + \beta_j = 0 \pmod{2\pi}$ so that $\Delta_{ij} = 0$ and the dimensions are $(1/3, 1/3)$. These results were derived by exploiting a connection between the BLS and the $O(n)$ model in the limit $n \rightarrow 0$. Further generalizations of the layering operators were explored in [8].

While the analysis in [7] demonstrated that new operators must exist and allowed us to compute their dimensions and three-point function coefficients with $\mathcal{O}_\beta$, it did not provide a clue as to how they are defined in terms of loops of the BLS loop ensemble. In this paper we introduce a new field $\mathcal{E}(z)$ that counts the number of outer boundaries of BLS loops that pass close to z and rigorously prove that its n -point functions are well defined and behave as expected for a primary field. We identify $\mathcal{E}$ with the operator of dimensions $(1/3, 1/3)$ discovered in [7], compute the four-point function $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}}$, and perform its Virasoro conformal block expansion. This provides further information about three-point function coefficients and the spectrum of primary operators. We further define higher order ($k = k' > 1$) and charged ($\beta \neq 0$) generalizations of this operator that can be identified with the operators of dimensions $(\Delta_{ij} + \frac{k}{3}, \Delta_{ij} + \frac{k}{3})$. In other words, we identify and explicitly define in terms of the loops all spin-zero primary fields emerging from the Virasoro conformal block expansion derived in [7].

This corpus of results establishes the BLS as a novel conformal field theory (CFT), or class of conformal field theories, with certain unique features (such as the periodicity of the operator dimensions in the charge β). Nevertheless, many aspects of this CFT remain mysterious — among other things, the nature of the operators with non-zero spin, $|k - k'| \neq 0$. The relation of this CFT to other better-known CFTs and its possible role as a model for physical phenomena also remains unclear.

1.2. Conformal field theory. Conformal field theory (CFT) is the study of a special class of Euclidean quantum field theories endowed with conformal symmetry. For decades, CFT has attracted a great deal of attention in both the physics and mathematics communities because of its central role in the description of critical phenomena (second-order phase transitions) and in string theory, and as a playground to study interacting quantum field theories. CFT has also had a big impact on various aspects of modern mathematics, for example with the introduction of the concept of vertex algebra by Borchers [9, 10].

In two dimensions, the conformal symmetry is so powerful that it allows to provide a general framework [11] that leads to very strong predictions. Indeed, two-dimensional conformal field theories represent a rare example of quantum field theories that can be exactly solved, and the physics literature contains a wealth of results on two-dimensional conformal field theories (see, for example, [5, 12–14] and Part II of [15]).

The mathematics literature on CFT is also vast. Rigorous approaches to CFT can be broadly divided in three different groups: a geometrical approach initiated by Segal [16]; an algebraic approach, initially due to Borchers [9, 10] and Frenkel, Lepowsky and Meurman [17], and developed further by Frenkel, Huang and Lepowsky [18] and Kac [19]; a functional analytic approach, pioneered by Wassermann [20] and Gabbiani and Fröhlich [21], in which techniques from algebraic quantum field theory are employed.

In this paper, we don't attempt to give a precise definition of a general CFT, instead we deal with a specific class of models derived from the BLS. Practically speaking, for us a CFT is essentially a collection of (limiting) correlation functions satisfying a conformal covariance property (see, e.g., Theorem 2.3 below.) These are obtained from the n -point correlation functions of local observable fields, defined with the help of ultraviolet cutoffs, when the cutoffs are sent to zero. Heuristically, these limiting functions are interpreted as the n -point correlation functions of Euclidean quantum fields. When their n -point functions satisfy conformal covariance, in the sense of Lemma 2.2 and Theorem 2.3, the fields are called *conformal primary fields*, *primary operators* or simply (*conformal*) *primaries*. These are the fundamental building blocks of any conformal field theory.

We point out that in this paper we do not deal with the question of the existence of the fields themselves, beyond their correlation functions. For some results in this direction, the interested reader is referred to [6] for results on the fields $\mathcal{O}_\beta$, to [22] for results on related fields, and to [23] for results on the Ising spin (magnetization) field.

An important tool in CFT, and one that will appear often in the rest of this paper, is the *operator product expansion* (OPE),

$$\mathcal{A}(z_1) \times \mathcal{B}(z_2) = \sum_k C_{\mathcal{A},\mathcal{B}}^{\mathcal{P}_k}(z_1, z_2) \mathcal{P}_k(z_2), \quad (1)$$

a formal expansion of the product of two fields, $\mathcal{A}$ and $\mathcal{B}$, at different points, z_1 and z_2 , as a (possibly infinite) sum of local fields $\mathcal{P}_i$. Since the fields involved in an OPE are typically not defined pointwise, the expression (1) is only defined in expectation, that is,

$$\langle \mathcal{A}(z_1) \mathcal{B}(z_2) \mathcal{G} \rangle = \sum_k C_{\mathcal{A},\mathcal{B}}^{\mathcal{P}_k}(z_1, z_2) \langle \mathcal{P}_k(z_2) \mathcal{G} \rangle, \quad (2)$$

where $\mathcal{G}$ is any product of local fields at points different from z_1, z_2 . This expansion is useful because it allows to probe the *spectrum* of the theory, i.e., the collection $\{\mathcal{P}_i\}_i$ of all local fields of the CFT.

The functional form of the coefficient $C_{\mathcal{A},\mathcal{B}}^{\mathcal{P}_k}$ of the OPE is determined by the requirement of conformal covariance up to certain constants, which are called the *structure*

constants of the theory. In a certain sense, “solving” a CFT is essentially equivalent to identifying its central charge, spectrum, and structure constants [5]. In this paper, using a combination of rigorous and theoretical-physics methods, we achieve progress in these directions for a particular family of CFTs obtained from the BLS.

1.3. Preliminary definitions. If A is a set of loops in a domain D , the partition function of the BLS restricted to loops from A can be written as

$$Z_A = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\mu_D^{\text{loop}}(A) \right)^n, \quad (3)$$

where $\lambda > 0$ is a constant and μ^{loop} is a measure on planar loops in D called *Brownian loop measure* and defined as

$$\mu_D^{\text{loop}} := \int_D \int_0^\infty \frac{1}{2\pi t^2} \mu_{D,z,t}^{\text{br}} dt d\mathbf{A}(z), \quad (4)$$

where $\mathbf{A}$ denotes area and $\mu_{D,z,t}^{\text{br}}$ is the restriction of the complex Brownian bridge measure with starting point z and duration t to loops that stay in D .¹ Z_A can be thought of as the grand canonical partition function of a system of loops with fugacity λ , and the BLS can be shown to be conformally invariant and to have central charge $c = 2\lambda$ (see [1, 4]).

In this paper we will only be concerned with the outer boundaries of Brownian loops. More precisely, given a planar loop γ in $\mathbb{C}$, its outer boundary or “edge” $\ell = \ell(\gamma)$ is the boundary of the unique infinite component of $\mathbb{C} \setminus \gamma$.² Note that, for any planar loop γ , $\ell(\gamma)$ is a simple closed curve, i.e., a closed loop without self-intersections, unless γ has cut points. Since the complex Brownian bridge assigns probability zero to loops γ with one or more cut points, in this paper, we will work with collections $\mathcal{L}$ of simple loops ℓ which are the outer boundaries of the loops from a BLS and for us, with a slight abuse of terminology, a BLS will be a collection of simple loops. With this understanding, the $\lambda \rightarrow 0$ limit (interpreted appropriately) reduces to the case of a single self-avoiding loop. There is a unique (up to an overall multiplicative constant) conformally invariant measure on such loops [25], which is also believed to describe the $n \rightarrow 0$ limit of the critical $O(n)$ model. Exploiting this conjectural connection allowed us to obtain exact results for certain correlation functions here and in our previous work [7].

Given a simple loop ℓ , let $\bar{\ell}$ denote its *interior*, i.e. the unique bounded simply connected component of $\mathbb{C} \setminus \ell$. In other words, a point z belongs to $\bar{\ell}$ if ℓ disconnects z from infinity, in which case we write $z \in \bar{\ell}$. In [4], the authors studied the correlation functions of the *layering operator* or *field*³ $V_\beta(z) = \exp(i\beta \sum_{\ell: z \in \bar{\ell}} \sigma_\ell(z))$, where σ_ℓ are independent, symmetric, (± 1) -valued Boolean variables associated to the loops. One difficulty arises immediately due to the scale invariance of the BLS, which implies that the sum at the exponent is infinite with probability one. This difficulty can be overcome by imposing a short-distance cutoff $\delta > 0$ on the diameter of loops (essentially removing

¹ We note that the Brownian loop measure should be interpreted as a measure on “unrooted” loops, that is, loops without a specified starting point. Unrooted loops are equivalence classes of rooted loops. The interested reader is referred to [1] for more details.

² Models that consider loops in their entirety are also interesting and are studied in [22, 24].

³ In this paper we use the terms *field* and *operator* interchangeably.

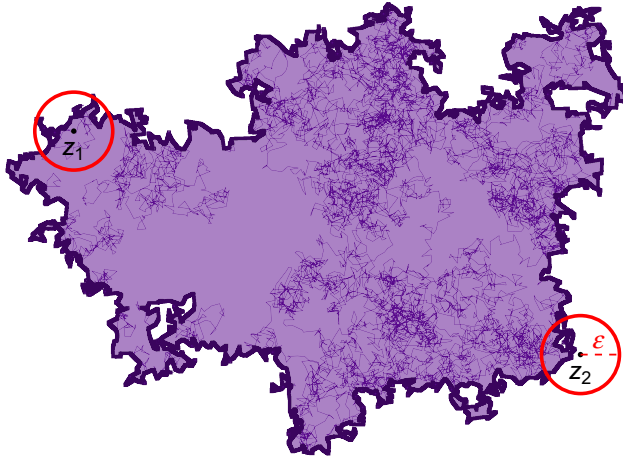


Fig. 1. A Brownian loop (thin NYU violet line) and its boundary (thick violet line; the interior is shaded). Such a loop would contribute to the two-point function of edge operators inserted at z_1 and z_2 because the loop comes within ε of both. It would contribute to a layering operator inserted at z_1 (but not z_2) because z_1 (but not z_2) is in the interior of the loop (that is, the loop separates z_1 from infinity, but not z_2)

from the loop soup all loops with diameter smaller than δ .⁴) As shown in [4], the cutoff δ can be removed by rescaling the cutoff version V_β^δ of V_β by $\delta^{-2\Delta(\beta)}$ and sending $\delta \rightarrow 0$. When $\delta \rightarrow 0$, the n -point correlation functions of $\delta^{-2\Delta(\beta)} V_\beta^\delta$ converge to conformally covariant quantities [4], showing that the limiting field is a scalar conformal primary field with real and positive scaling dimension varying continuously as a periodic function of β , namely as $\Delta(\beta) = \bar{\Delta}(\beta) = \frac{\lambda}{10}(1 - \cos \beta)$. This limiting field is further studied in [7], where its canonically normalized version is denoted by $\mathcal{O}_\beta$.⁵

The *edge field* $\mathcal{E}(z)$ studied in this paper counts the number of loops ℓ passing within a short-distance ε of the point z . The cutoff and renormalization procedure described in Sect. 2 shows that $\mathcal{E}$ has well defined n -point functions which are conformally covariant, and that it behaves like a scalar conformal primary with scaling dimension $(1/3, 1/3)$. This scaling dimension can be understood qualitatively as follows. It is known that the fractal dimension of the boundary of a Brownian loop is $4/3$ [26]. Fattening the loop's boundary into a strip⁶ of width ε , a fractal dimension of $4/3$ means that the area of the strip is proportional to $\varepsilon^{2/3}$. Hence the probability for a loop to come within ε of a given point scales as $\varepsilon^{2/3}$. Loops that contribute to the two-point function of the edge operator with itself must come close to both points (Fig. 1). Therefore the two-point function must be proportional to the ratio $|\varepsilon/z_{12}|^{4/3}$, where $\varepsilon^{4/3}$ is proportional to the square of the probability mentioned above and the power of $z_{12} := z_1 - z_2$ follows from invariance under an overall scale transformation $(\varepsilon, z) \rightarrow (\lambda\varepsilon, \lambda z)$. This dependence on $|z_{12}|$ is that of a scalar operator with dimension $(1/3, 1/3)$.

In Sect. 6.1 we identify additional scalar fields resulting from combinations of the edge field $\mathcal{E}$ with itself that we denote by $\mathcal{E}^{(k)}$ and call *higher-order* edge operators.

⁴ An additional infrared cutoff or a “charge neutrality” or “charge conservation” condition may be necessary in some circumstances — we refer the interested reader to [4] for more details.

⁵ By *canonically normalized* we mean that the full-plane two-point function $\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle_{\mathbb{C}} = |z - z'|^{-2\Delta(\beta)}$, where $\langle \cdot \rangle_{\mathbb{C}}$ denotes expectation with respect to the BLS on the full plane.

⁶ Recipes for Wiener sausages in Brownian soups are available on special request.

These fields have holomorphic and anti-holomorphic dimension $k/3$ for all non-negative integers k . In Sect. 6.2 we discuss “charged” versions of the (higher-order) edge operators resulting from combinations of the edge field with itself and with the layering field $\mathcal{O}_\beta$; we denote these by $\mathcal{E}_\beta^{(k)}$ and call them *charged* edge operators. These fields have holomorphic and anti-holomorphic dimension $\Delta(\beta) + \frac{k}{3}$, with non-negative integer k . The higher-order and charged edge operators complete the list of all scalar primary fields in the conformal block expansion derived in [7].

1.4. Structure of the paper. This paper contains both rigorous results and “physics-style” arguments and is written with a mixed audience of mathematicians and physicists in mind. The rigorous results are generally presented as lemmas or theorems in the text; they include explicit expressions for certain correlation functions and the proof that the n -point correlation functions of the edge operator $\mathcal{E}$ and of the higher-order edge operators $\mathcal{E}^{(k)}$ are conformally covariant. The proofs of some of the rigorous results are collected in the appendix to avoid breaking the flow of the paper.

More precisely, the main rigorous results are presented in Sects. 2, 3, 5 and 6.1 and, for the sake of clarity, they are stated as lemmas and theorems. Equations (6.19) of [7] and (52) of [27], used in Sects. 3 and 5, respectively, are not rigorous, as well as the identification in (95) of Sect. 5. The calculations in Sect. 4 are rigorous up to equation (74). The definitions and results of Sects. 6.2, 7 and 8 rely on various assumptions and physics-style theoretical arguments.

Objects denoted with script letters, such as $\mathcal{E}$ and $\mathcal{O}_\beta$ (which we call either *fields* or *operators*, following standard physics terminology), require a regularization and are not defined pointwise. They may exist as generalized functions (distributions in the sense of Schwartz), but we do not investigate this issue in this paper. (For some results in this direction, see [6, 22].) Expectation values of such objects represent the limits of the corresponding expectation values of the regularized objects, which are defined pointwise (almost everywhere). We stress that, while the fields/operators we introduce in this paper may only be defined formally, their n -point functions, which are the objects we work with, are well defined (pointwise) as limits of the n -point functions of the corresponding regularized fields.

The edge operator $\mathcal{E}$ is introduced in Sect. 2, where its correlation functions are discussed. Section 3 contains the computation of $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}}$, including the structure constant $C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}}$. Section 4 contains a derivation of the OPE of $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$ and the identification of the edge operator $\mathcal{E}$ with the primary operator of dimension $(1/3, 1/3)$ discovered in [7]. Section 5 contains the calculation of the full-plane four-point function $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}}$. Higher-order and charged edge operators are introduced in Sects. 6.1 and 6.2, respectively, where their correlation functions are discussed. The Virasoro conformal block expansion resulting from the four-point function $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}}$ is developed in Sect. 7.1, while Sect. 7.2 contains a direct derivation of the full-plane three-point function $\langle \mathcal{E}(z_1) \mathcal{E}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}}$, including the structure constant $C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}}$. Section 8 contains a new derivation of the fact that the central charge of the BLS with intensity λ is $c = 2\lambda$.

1.5. Summary of the main results. The domains D considered in this paper are the full (complex) plane $\mathbb{C}$, the upper-half plane $\mathbb{H}$ or any domain conformally equivalent to $\mathbb{H}$. In this section and in the rest of the paper, we use $\langle \cdot \rangle_D$ to denote expectation with respect

to the BLS in D . The domain will be explicitly present in our notation when we want to emphasize its role; if the domain is not denoted in a particular expression (for example, if we use $\langle \cdot \rangle$ instead of $\langle \cdot \rangle_D$ or μ^{loop} instead of μ_D^{loop}), it means that that expression is valid for any of the domains mentioned above.

The first group of main results concerns the Brownian loop measure μ_D^{loop} in a domain D , the n -point functions of the edge operator $\mathcal{E}$, which can be expressed in terms of μ_D^{loop} , and the relation between $\mathcal{E}$ and $\mathcal{O}_\beta$.⁷ To formulate the results, we let ϑ_ε denote the scaling limit of the probability that, in critical site percolation on the triangular lattice, there are one open and two closed paths crossing the annulus with inner radius ε and outer radius 1, known as a *three-arm event*. The existence of the limit is guaranteed by the existence of the full scaling limit of critical percolation [24], and it is known that $\vartheta_\varepsilon \sim \varepsilon^{2/3}$ (see Lemma A.2 for a precise statement).

- In Sect. 2 we prove that, for any collection of distinct points $z_1, \dots, z_k \in D$ with $k \geq 2$, letting $B_\varepsilon(z_j)$ denote the disk of radius ε centered at z_j , the following limit exists:

$$\alpha_D^{z_1, \dots, z_k} := \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-k} \mu_D^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j = 1, \dots, k). \quad (5)$$

Moreover, $\alpha_D^{z_1, \dots, z_k}$ is conformally covariant in the sense that, if D' is a domain conformally equivalent to D and $f : D \rightarrow D'$ is a conformal map, then

$$\alpha_{D'}^{f(z_1), \dots, f(z_k)} = \left(\prod_{j=1}^k |f'(z_j)|^{-2/3} \right) \alpha_D^{z_1, \dots, z_k}. \quad (6)$$

- The field $\mathcal{E}$ formally defined by

$$\mathcal{E}(z) := \frac{\hat{c}}{\sqrt{\lambda}} \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} (\mathcal{N}_\varepsilon(z) - \langle \mathcal{N}_\varepsilon(z) \rangle), \quad (7)$$

where $\mathcal{N}_\varepsilon(z)$ counts the number of loops ℓ that come to distance ε of z ,⁸ behaves like a conformal primary field with scaling dimension $2/3$. The constant $\hat{c}$ can be chosen so that $\mathcal{E}$ is canonically normalized, i.e.

$$\langle \mathcal{E}(z_1) \mathcal{E}(z_2) \rangle_{\mathbb{C}} = |z_1 - z_2|^{-4/3}. \quad (8)$$

- More precisely, we prove that, if D' is a domain conformally equivalent to D and $f : D \rightarrow D'$ is a conformal map, then

$$\langle \mathcal{E}(f(z_1)) \dots \mathcal{E}(f(z_n)) \rangle_{D'} = \left(\prod_{j=1}^n |f'(z_j)|^{-2/3} \right) \langle \mathcal{E}(z_1) \dots \mathcal{E}(z_n) \rangle_D. \quad (9)$$

- Letting $z_{jk} := z_j - z_k$, in Sect. 3 we prove that

$$\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}} = C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} \frac{1}{|z_{12}|^{4\Delta(\beta)}} \left| \frac{z_{12}}{z_{13} z_{23}} \right|^{2/3}. \quad (10)$$

⁷ The edge operator is properly defined in Sect. 2 below.

⁸ We note that $\mathcal{N}_\varepsilon(z)$ is infinite with probability one because of the scale invariance of the BLS, but its centered version $\mathcal{E}_\varepsilon(z) := \mathcal{N}_\varepsilon(z) - \langle \mathcal{N}_\varepsilon(z) \rangle$ has well defined n -point functions — see Lemma 2.1.

Using the non-rigorous equation (6.19) of [7], we obtain the *three-point structure constant*

$$C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} = -\sqrt{\lambda}(1 - \cos \beta) \frac{2^{7/6} \pi}{3^{1/4} \sqrt{5} \Gamma(1/6) \Gamma(4/3)}. \quad (11)$$

- In Sect. 4 we argue that the OPE of $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$ takes the form

$$\begin{aligned} \mathcal{O}_\beta(z) \times \mathcal{O}_{-\beta}(z') &= |z - z'|^{-4\Delta(\beta)} \left(\mathbb{1} + C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} |z - z'|^{2/3} \mathcal{E}(z) + C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}^{(2)}} |z - z'|^{4/3} \mathcal{E}^{(2)}(z) \right. \\ &\quad \left. + o(|z - z'|^{4/3}) \right), \end{aligned} \quad (12)$$

where $\mathbb{1}$ is the identity operator and

$$\left(C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}^{(2)}} \right)^2 = \frac{1}{2} \left(C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} \right)^4. \quad (13)$$

- In Sect. 5 we prove that the mixed full-plane four-point function $\left\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \right\rangle_{\mathbb{C}}$ exists and is conformally covariant. Using a non-rigorous result of Simmons and Cardy [27], we argue that it has the following explicit expression:

$$\begin{aligned} \left\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \right\rangle_{\mathbb{C}} &= |z_{12}|^{-4\Delta(\beta)} \\ &\cdot \left[\frac{1 + \cos \beta}{2} |z_{34}|^{-4/3} + \frac{1 - \cos \beta}{2} Z_{\text{twist}} + \lambda (1 - \cos \beta)^2 \hat{\alpha}_{z_1|z_2}^{z_3} \hat{\alpha}_{z_1|z_2}^{z_4} \right], \end{aligned} \quad (14)$$

where

$$\hat{\alpha}_{z_j|z_k; \mathbb{C}}^{z_l} = \frac{2^{7/6} \pi}{3^{1/4} \sqrt{5} \Gamma(1/6) \Gamma(4/3)} \left| \frac{z_{jk}}{z_{jl} z_{kl}} \right|^{2/3} \quad (15)$$

and

$$\begin{aligned} Z_{\text{twist}} &= \left| \frac{z_{13} z_{24}}{z_{34}^2 z_{23} z_{14}} \right|^{2/3} \left[\left| {}_2F_1 \left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}, \frac{z_{12} z_{34}}{z_{13} z_{24}} \right) \right|^2 \right. \\ &\quad \left. - \frac{4\Gamma(\frac{2}{3})^6}{\Gamma(\frac{4}{3})^2 \Gamma(\frac{1}{3})^4} \left| \frac{z_{12} z_{34}}{z_{13} z_{24}} \right|^{2/3} \left| {}_2F_1 \left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}, \frac{z_{12} z_{34}}{z_{13} z_{24}} \right) \right|^2 \right]. \end{aligned} \quad (16)$$

- In Sect. 6.1 we prove that the higher-order edge operators $\mathcal{E}^{(k)}$ behave like canonically normalized primary fields. More precisely, for each $k \in \mathbb{N}$,

$$\left\langle \mathcal{E}^{(k)}(z_1) \mathcal{E}^{(k)}(z_2) \right\rangle_{\mathbb{C}} = |z_1 - z_2|^{-4k/3}. \quad (17)$$

Moreover, if D' is a domain conformally equivalent to D and $f : D \rightarrow D'$ is a conformal map, then

$$\begin{aligned} &\left\langle \mathcal{E}^{(k_1)}(f(z_1)) \dots \mathcal{E}^{(k_n)}(f(z_n)) \right\rangle_{D'} \\ &= \left(\prod_{j=1}^n |f'(z_j)|^{-2k_j/3} \right) \left\langle \mathcal{E}^{(k_1)}(z_1) \dots \mathcal{E}^{(k_n)}(z_n) \right\rangle_D. \end{aligned} \quad (18)$$

- In Sect. 6.2 we further generalize the edge operators $\mathcal{E}^{(k)}$ mixing them with the layering operator V_β .
- In Sect. 7.1 we argue that the OPE of $\mathcal{O}_\beta \times \mathcal{E}$ takes the form

$$\mathcal{O}_\beta(z) \times \mathcal{E}(z') = C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{O}_\beta} |z - z'|^{-2/3} \mathcal{O}_\beta(z) + C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{E}_\beta} \mathcal{E}_\beta(z) + \mathcal{R} \quad (19)$$

where $C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{O}_\beta} = C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}}$,

$$\left(C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{E}_\beta}\right)^2 = \frac{1 + \cos \beta}{2} \quad (20)$$

and, here and below, $\mathcal{R}$ represents the remaining terms in the expansion.

- In Sect. 7.2 we argue that the OPE of $\mathcal{E} \times \mathcal{E}$ contains the terms

$$\begin{aligned} \mathcal{E}(z) \times \mathcal{E}(z') \\ = |z - z'|^{-4/3} \left(\mathbb{1} + C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} |z - z'|^{2/3} \mathcal{E}(z) + C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}^{(2)}} |z - z'|^{4/3} \mathcal{E}^{(2)}(z) + \mathcal{R} \right), \end{aligned} \quad (21)$$

where the *three-point structure constants* are

$$C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} = \frac{1}{\sqrt{\lambda}} \frac{2^{13/6} 3^{1/4} \sqrt{5} \pi^{3/2} \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})^3 \Gamma(\frac{7}{6})} \quad (22)$$

$$C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}^{(2)}} = \sqrt{2}. \quad (23)$$

- In Sect. 8 we show that the central charge of the BLS can be independently re-derived to be $c = 2\lambda$ by computing the two-point function of the stress-tensor,

$$\langle T(z_1) T(z_2) \rangle_{\mathbb{C}} = \frac{c/2}{z_{12}^4}, \quad (24)$$

from (14) by applying the OPEs of $\mathcal{E} \times \mathcal{E}$ and $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$.

2. The Edge Counting Operator

For a domain $D \subseteq \mathbb{C}$, a point $z \in \mathbb{C}$, a real number $\varepsilon > 0$, and a collection $\mathcal{L}$ of simple loops in D , let $n_z^\varepsilon(\mathcal{L})$ denote the number of loops $\ell \in \mathcal{L}$ such that $\ell \cap B_\varepsilon(z) \neq \emptyset$, where $B_\varepsilon(z)$ denotes the disk of radius ε centered at z . We define *formally* the “random variable” $\mathcal{N}_\varepsilon(z) = n_z^\varepsilon(\mathcal{L})$ where $\mathcal{L}$ is distributed like the collection of outer boundaries $\ell = \ell(\gamma)$ of the loops γ of a Brownian loop soup in D with intensity λ (see Sect. 1.3).

$\mathcal{N}_\varepsilon(z)$ counts the number of loops γ of a Brownian loop soup whose “edge” ℓ (the outer boundary) comes ε -close to z ; it is only formally defined because it is infinite with probability one. Nevertheless, we will be interested in the fluctuations of $\mathcal{N}_\varepsilon(z)$ around its infinite mean, which can be formally written as

$$\begin{aligned} \mathcal{E}_\varepsilon(z) &:= \mathcal{N}_\varepsilon(z) - \langle \mathcal{N}_\varepsilon(z) \rangle_D \\ &= \mathcal{N}_\varepsilon(z) - \lambda \mu_D^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset), \end{aligned} \quad (25)$$

where $\langle \cdot \rangle_D$ denotes expectation with respect to the Brownian loop soup in D (of fixed intensity λ) and μ_D^{loop} is the Brownian loop measure restricted to D , i.e. the unique (up to a multiplicative constant) conformally invariant measure on simple planar loops [25].

To make precise sense of (25), we define $N_\varepsilon^\delta(z) := n_z^\varepsilon(\mathcal{L}^\delta)$ and $E_\varepsilon^\delta(z) := N_\varepsilon^\delta(z) - \langle N_\varepsilon^\delta(z) \rangle$, where $\mathcal{L}^\delta$ is a Brownian loop soup with cutoff $\delta > 0$, obtained by taking the usual Brownian loop soup and removing all loops with diameter⁹ smaller than δ . The random variables $N_\varepsilon^\delta(z)$ and $E_\varepsilon^\delta(z)$ are well defined because of the cutoffs ε and δ and due to the fact that the BLS is *thin* in the sense of [28]. In Lemma A.1 of the appendix we show that, while $\mathcal{E}_\varepsilon(z)$ is only formally defined, its n -point functions, defined as limits of the n -point functions of the corresponding regularized quantities, i.e.,

$$\langle \mathcal{E}_\varepsilon(z_1) \dots \mathcal{E}_\varepsilon(z_n) \rangle_D := \lim_{\delta \rightarrow 0} \langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D, \quad (26)$$

exist for all collections of points $z_1, \dots, z_n$ at distance greater than 2ε from each other, with $n \geq 2$. In fact,

$$\langle \mathcal{E}_\varepsilon(z_1) \dots \mathcal{E}_\varepsilon(z_n) \rangle_D = \langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D \quad (27)$$

for all $\varepsilon, \delta > 0$ sufficiently small (depending on the points $z_1, \dots, z_n$), so that the limit in (26) can be dropped.¹⁰

There is a closed-form expression for the n -point functions defined in (26) in terms of the Brownian loop measure μ_D^{loop} , as stated in the following lemma, whose proof is presented in the appendix.

Lemma 2.1. *For any $\varepsilon > 0$ and any collection of distinct points $z_1, \dots, z_n \in D$ at distance greater than 2ε from each other, with $n \geq 2$, let Π denote the set of all partitions of $\{1, \dots, n\}$ such that each element I_l of $\{I_1, \dots, I_r\} \in \Pi$ has cardinality $|I_l| \geq 2$; then*

$$\langle \mathcal{E}_\varepsilon(z_1) \dots \mathcal{E}_\varepsilon(z_n) \rangle_D = \sum_{\{I_1, \dots, I_r\} \in \Pi} \lambda^r \prod_{l=1}^r \mu_D^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I_l). \quad (28)$$

We remind the reader that ϑ_ε denotes the scaling limit of the probability that, in critical site percolation on the triangular lattice, there are one open and two closed paths crossing the annulus with inner radius ε and outer radius 1, known as a three-arm event, and that $\vartheta_\varepsilon \sim \varepsilon^{2/3}$. A central result of this paper is the fact that the field formally defined by

$$\mathcal{E}(z) := \frac{\hat{c}}{\sqrt{\lambda}} \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \mathcal{E}_\varepsilon(z) \quad (29)$$

behaves like a conformal primary, where the constant $\hat{c}$ is chosen to ensure that $\mathcal{E}$ is canonically normalized, i.e.,

$$\langle \mathcal{E}(z_1) \mathcal{E}(z_2) \rangle_{\mathbb{C}} = |z_1 - z_2|^{-4/3}. \quad (30)$$

As we mentioned in Sect. 1.4, $\mathcal{E}(z)$ is actually defined in terms of limits of correlation functions, i.e.,

$$\langle \mathcal{E}(z_1) \dots \mathcal{E}(z_n) \rangle_D := \left(\frac{\hat{c}}{\sqrt{\lambda}} \right)^n \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-n} \langle \mathcal{E}_\varepsilon(z_1) \dots \mathcal{E}_\varepsilon(z_n) \rangle_D. \quad (31)$$

⁹ The diameter of a loop is defined as the largest distance between any two points on the loop.

¹⁰ This happens because, when ε and δ are small, in the calculation of $\langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D$, due to the centering of $E_\varepsilon^\delta(z_j)$, the only terms that do not vanish become independent of δ (see the proof of Lemma A.1 in the appendix).

The existence of the limit in the right hand side of (31) relies crucially on the following lemma, which is interesting in its own right, and whose proof is given at the end of this section.

Lemma 2.2. *Let $D \subseteq \mathbb{C}$ be either the complex plane $\mathbb{C}$ or the upper-half plane $\mathbb{H}$ or any domain conformally equivalent to $\mathbb{H}$. For any collection of distinct points $z_1, \dots, z_k \in D$ with $k \geq 2$, the following limit exists:*

$$\alpha_D^{z_1, \dots, z_k} := \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-k} \mu_D^{loop}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j = 1, \dots, k). \quad (32)$$

Moreover, $\alpha_D^{z_1, \dots, z_k}$ is conformally covariant in the sense that, if D' is a domain conformally equivalent to D and $f : D \rightarrow D'$ is a conformal map, then

$$\alpha_{D'}^{f(z_1), \dots, f(z_k)} = \left(\prod_{j=1}^k |f'(z_j)|^{-2/3} \right) \alpha_D^{z_1, \dots, z_k}. \quad (33)$$

For any collection of points $z_1, \dots, z_n \in D$ and any subset $S = \{z_{j_1}, \dots, z_{j_k}\}$ of $\{z_1, \dots, z_n\}$, let $\alpha_D^S := \alpha_D^{z_{j_1}, \dots, z_{j_k}}$. The existence and the conformal covariance of the limit in (31) is the content of the next theorem, which is one of the main results of this paper.

Theorem 2.3. *Let $D \subseteq \mathbb{C}$ be either the complex plane $\mathbb{C}$ or the upper-half plane $\mathbb{H}$ or any domain conformally equivalent to $\mathbb{H}$. For any collection of distinct points $z_1, \dots, z_n \in D$ with $n \geq 2$, the following limit exists:*

$$g_D(z_1, \dots, z_n) := \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-n} \langle E_\varepsilon(z_1) \dots E_\varepsilon(z_n) \rangle_D. \quad (34)$$

Moreover, if $\mathcal{S} = \mathcal{S}(z_1, \dots, z_n)$ denotes the set of all partitions of $\{z_1, \dots, z_n\}$ such that each element S_l of $(S_1, \dots, S_r) \in \mathcal{S}$ has cardinality $|S_l| \geq 2$, then

$$g_D(z_1, \dots, z_n) = \sum_{(S_1, \dots, S_r) \in \mathcal{S}} \lambda^r \alpha_D^{S_1} \dots \alpha_D^{S_r}. \quad (35)$$

Furthermore, $g_D(z_1, \dots, z_n)$ is conformally covariant in the sense that, if D' is a domain conformally equivalent to D and $f : D \rightarrow D'$ is a conformal map, then

$$g_{D'}(f(z_1), \dots, f(z_n)) = \left(\prod_{k=1}^n |f'(z_k)|^{-2/3} \right) g_D(z_1, \dots, z_n). \quad (36)$$

Proof. The existence of the limit in (34) follows from (28) combined with the existence of the limit in (32). The expression in (35) follows directly from (28) and the definition of $\alpha^{z_1, \dots, z_k}(D)$ in (32). The conformal covariance expressed in (36) is an immediate consequence of (35) and (33). $\square$

Using the notation introduced in (29), we will write

$$\langle \mathcal{E}(z_1) \dots \mathcal{E}(z_n) \rangle_D := \frac{\hat{c}^n}{\lambda^{n/2}} g_D(z_1, \dots, z_n), \quad (37)$$

despite the fact that $\mathcal{E}$ is only formally defined. To simplify the notation, we define

$$\hat{\alpha}_D^{z_1, \dots, z_k} := \hat{c}^k \alpha_D^{z_1, \dots, z_k}. \quad (38)$$

In particular, using this notation, the two-, three- and four-point functions are

$$\langle \mathcal{E}(z_1) \mathcal{E}(z_2) \rangle_D = \hat{\alpha}_D^{z_1, z_2} \quad (39a)$$

$$\langle \mathcal{E}(z_1) \mathcal{E}(z_2) \mathcal{E}(z_3) \rangle_D = \frac{1}{\sqrt{\lambda}} \hat{\alpha}_D^{z_1, z_2, z_3} \quad (39b)$$

$$\begin{aligned} \langle \mathcal{E}(z_1) \mathcal{E}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_D &= \frac{1}{\lambda} \hat{\alpha}_D^{z_1, z_2, z_3, z_4} \\ &+ \hat{\alpha}_D^{z_1, z_2} \hat{\alpha}_D^{z_3, z_4} + \hat{\alpha}_D^{z_1, z_3} \hat{\alpha}_D^{z_2, z_4} + \hat{\alpha}_D^{z_1, z_4} \hat{\alpha}_D^{z_2, z_3}. \end{aligned} \quad (39c)$$

We note that, combining (31) with (26), one can see that the definition of $\langle \mathcal{E}(z_1) \dots \mathcal{E}(z_n) \rangle_D$ requires a double limit. In our presentation, we have first taken the limit $\delta \rightarrow 0$ and then the limit $\varepsilon \rightarrow 0$. However, the validity of (27) for all ε , δ sufficiently small shows that the order of the limits is immaterial.

We conclude this section with the proof of Lemma 2.2.

Proof of Lemma 2.2. Consider the full scaling limit of critical percolation in D constructed in [24] and denote it by $\mathcal{F}_D$. $\mathcal{F}_D$ is a collection of non-simple, non-crossing loops distributed like CLE_6 in D [29]. As explained in Section 8 of [25], the “outer perimeters” of loops from $\mathcal{F}_D$ are (almost surely) simple loops distributed like the outer boundaries of Brownian loops. Hence, there is a close connection between the Brownian loop measure μ_D^{loop} and the collection of loops constructed in [24].

More precisely, let $\mathbb{P}$ denote the distribution of $\mathcal{F}_D$ and $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Since $\mathcal{F}_D$ is conformally invariant, if A is a measurable set of self-avoiding loops and $\mathcal{N}_A$ is the number of loops Γ from $\mathcal{F}_D$ such that their outer perimeters $\ell(\Gamma)$ are in A , $\mathbb{E}(\mathcal{N}_A)$ defines a conformally invariant measure on self-avoiding loops. Moreover, since the measure μ_D^{loop} is unique, up to a multiplicative constant, we must have

$$\mu_D^{\text{loop}}(A) = \Theta \mathbb{E}(\mathcal{N}_A), \quad (40)$$

where $0 < \Theta < \infty$ is a constant.

Now consider the set of simple loops $S_\varepsilon = \{\ell \in D : \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j = 1, \dots, k\}$. Thanks to the scale invariance of μ_D^{loop} and $\mathcal{F}_D$, we can assume without loss of generality that the points $z_1, \dots, z_k$ are at distance much larger than 1 from each other and from ∂D . We write $\mathcal{F}_D \in S_\varepsilon$ to indicate the event that a configuration from $\mathcal{F}_D$ contains at least one loop Γ such that $\ell(\Gamma) \in S_\varepsilon$.

For each $j = 1, \dots, k$, consider the annulus $A_{\varepsilon,1}(z_j) := B_1(z_j) \setminus B_\varepsilon(z_j)$ centered at z_j with outer radius 1 and inner radius ε . Because of our assumption on the distances between the points z_j , $j = 1, \dots, k$, the annuli do not overlap. The configurations from $\mathcal{F}_D$ for which $\mathcal{N}_{S_\varepsilon} > 0$ (i.e., such that $\mathcal{F}_D \in S_\varepsilon$) are those that contain at least one loop Γ whose outer perimeter $\ell(\Gamma)$ intersects $B_\varepsilon(z_j)$ for each $j = 1, \dots, k$. They can be split in two groups as described below, where a *three-arm event* inside $A_{\varepsilon,1}(z_j)$ refers to the presence of a loop Γ such that the annulus $A_{\varepsilon,1}(z_j)$ is crossed from the inside of $B_\varepsilon(z_j)$ to the outside of $B_1(z_j)$ by two disjoint outer perimeter paths belonging to $\ell(\Gamma)$ and by one path within the complement of the unique unbounded component of $\mathbb{C} \setminus \Gamma$.

- (i) Configurations that induce a three-arm event inside $A_{\varepsilon,1}(z_j)$ for each $j = 1, \dots, k$, for which $\mathcal{N}_{S_\varepsilon} = 1$.
- (ii) Configuration that induce more than three arms in $A_{\varepsilon,1}(z_j)$ for at least one $j = 1, \dots, k$, for which $\mathcal{N}_{S_\varepsilon} \geq 1$.

The probability of a three-arm event in $A_{\varepsilon,1}(z_j)$ is $\vartheta_\varepsilon \sim \varepsilon^{2/3}$ as $\varepsilon \rightarrow 0$, while the probability to have four or more arms in $A_{\varepsilon,1}(z_j)$ is $o(\vartheta_\varepsilon)$ as $\varepsilon \rightarrow 0$ [30]; therefore

$$\vartheta_\varepsilon^{-k} \mathbb{E}(\mathcal{N}_{S_\varepsilon}) = \vartheta_\varepsilon^{-k} \mathbb{P}(\mathcal{F}_D \in S_\varepsilon \text{ and there is a three-arm event in each } A_{\varepsilon,1}(z_j)) + O(\varepsilon). \quad (41)$$

It follows from the construction of $\mathcal{F}_D$ in [24], which uses the locality of SLE₆, that a configuration in group (i) can be constructed by first generating independent configurations inside $B_1(z_j)$ for each $j = 1, \dots, k$, requiring that each induces a three-arm event in $A_{\varepsilon,1}(z_j)$, and then generating a “matching” configuration in $D \setminus \bigcup_{j=1}^k B_1(z_j)$. A configuration inside $B_1(z_j)$ contains loops and arcs starting and ending on $\partial B_1(z_j)$. Moreover, since $A_{\varepsilon,1}(z_j)$ contains a three-arm event, exactly one outer perimeter arc starting and ending on $\partial B_1(z_j)$ intersects $B_\varepsilon(z_j)$. Each arc in $B_1(z_j)$ has a pair of endpoints on $\partial B_1(z_j)$. We let $\mathcal{I}_j$ denote the collection of endpoints on $\partial B_1(z_j)$, together with the information regarding which endpoints are connected to each other, and we denote by ν_j^ε the distribution of $\mathcal{I}_j$, conditioned on the occurrence of a three-arm event. An important observation is that, conditioned on $\mathcal{I}_j$ for each $j = 1, \dots, k$, the configuration in $D \setminus \bigcup_{j=1}^k B_1(z_j)$ is independent of the configurations inside $B_1(z_j)$ for $j = 1, \dots, k$. If we let G denote the event that endpoints on $\partial B_1(z_j)$ are connected in $D \setminus \bigcup_{j=1}^k B_1(z_j)$ in such a way that overall the resulting configuration in D is in S_ε , this observation allows us to write

$$\begin{aligned} & \mathbb{P}(\mathcal{F}_D \in S_\varepsilon \text{ and there is a three-arm event in } A_{\varepsilon,1}(z_j) \forall j = 1, \dots, k) \\ &= \mathbb{P}(\mathcal{F}_D \in S_\varepsilon \mid \text{there is a three-arm event in } A_{\varepsilon,1}(z_j) \forall j = 1, \dots, k) \\ & \quad \mathbb{P}(\text{there is a three-arm event in } A_{\varepsilon,1}(z_j) \forall j = 1, \dots, k) \\ &= \vartheta_\varepsilon^k \int \mathbb{P}(G \mid \mathcal{I}_1, \dots, \mathcal{I}_k) \prod_{j=1}^k d\nu_j^\varepsilon(\mathcal{I}_j). \end{aligned} \quad (42)$$

Combining this with (41), we obtain

$$\lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-k} \mathbb{E}(\mathcal{N}_{S_\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \int \mathbb{P}(G \mid \mathcal{I}_1, \dots, \mathcal{I}_k) \prod_{j=1}^k d\nu_j^\varepsilon(\mathcal{I}_j), \quad (43)$$

where $\mathbb{P}(G \mid \mathcal{I}_1, \dots, \mathcal{I}_k)$ does not depend on ε and ν_j^ε is the distribution of endpoints on $\partial B_1(z_j)$ conditioned on the occurrence of a three-arm event in $A_{\varepsilon,1}(z_j)$, or equivalently on the existence of a single outer perimeter arc starting and ending on $\partial B_1(z_j)$ and intersecting $B_\varepsilon(z_j)$.

Now observe that requiring the existence of a single outer perimeter arc that intersects $B_\varepsilon(z_j)$ and sending $\varepsilon \rightarrow 0$ is equivalent to centering the disk $B_1(z_j)$ at a *typical* point¹¹ z_j on the outer perimeter of a loop from $\mathcal{F}_D$ which exits $B_1(z_j)$ and therefore has diameter greater than 1. Therefore, the weak limit $\lim_{\varepsilon \rightarrow 0} \nu_j^\varepsilon$ exists: it is given by the distribution of endpoints of arcs for a disk of radius 1 centered at a typical point on the outer perimeter of a loop from $\mathcal{F}_D$ of diameter larger than 1. Equivalently, by scale invariance, it is the distribution of endpoints of arcs on $\partial B_r(z)$ for a disk $B_r(z)$ centered

¹¹ Here typical means that it is not a *pivotal* point, i.e., a point on the outer perimeter of two loops. Pivotal points have a lower fractal dimension.

at a typical point z on the outer perimeter of a loop from $\mathcal{F}_D$, with diameter r smaller than the diameter of the loop. Therefore, if we call ν this distribution, from (40) and (43) we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \vartheta_{\varepsilon}^{-k} \mu_D^{\text{loop}}(S_{\varepsilon}) &= \Theta \lim_{\varepsilon \rightarrow 0} \vartheta_{\varepsilon}^{-k} \mathbb{E}(\mathcal{N}_{S_{\varepsilon}}) \\ &= \Theta \int \mathbb{P}(G|\mathcal{I}_1, \dots, \mathcal{I}_k) \prod_{j=1}^k d\nu(\mathcal{I}_j), \end{aligned} \quad (44)$$

proving the existence of the limit in (32).

In order to prove (33), consider a domain D' conformally equivalent to D and a conformal map $f: D \rightarrow D'$, and let $z'_j = f(z_j)$, $s_j = |f'(z_j)|$ for each $j = 1, \dots, k$, and $S'_{\varepsilon} = \{\ell \in D' : \ell \cap B_{\varepsilon}(z'_j) \neq \emptyset \forall j = 1, \dots, k\}$. We are interested in the behavior of

$$\alpha_{D'}^{z'_1, \dots, z'_k} = \lim_{\varepsilon \rightarrow 0} \vartheta_{\varepsilon}^{-k} \mu_{D'}(S'_{\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \vartheta_{\varepsilon}^{-k} \mu_D(\ell \cap f^{-1}(B_{\varepsilon}(z'_j)) \neq \emptyset \forall j = 1, \dots, k). \quad (45)$$

To evaluate this limit, we will use the fact that

$$\begin{aligned} \vartheta_{\varepsilon}^{-k} [\mu_D(\ell \cap f^{-1}(B_{\varepsilon}(z'_j)) \neq \emptyset \forall j = 1, \dots, k) - \mu_D(\ell \cap B_{\varepsilon/s_j}(z_j) \neq \emptyset \forall j = 1, \dots, k)] \\ = o(1) \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (46)$$

To see this, let $A_{r_j, R_j}(z_j) = B_{R_j}(z_j) \setminus B_{r_j}(z_j)$ denote the thinnest annulus centered at z_j containing the symmetric difference of $f^{-1}(B_{\varepsilon}(z'_j))$ and $B_{\varepsilon/s_j}(z_j)$ and note that

$$\begin{aligned} |\mu_D(\ell \cap f^{-1}(B_{\varepsilon}(z'_j)) \neq \emptyset \forall j = 1, \dots, k) - \mu_D(\ell \cap B_{\varepsilon/s_j}(z_j) \neq \emptyset \forall j = 1, \dots, k)| \\ \leq \mu_D(\ell \cap B_{R_j}(z_j) \neq \emptyset \forall j = 1, \dots, k \text{ and } \ell \cap B_{r_l}(z_l) = \emptyset \text{ for some } l = 1, \dots, k). \end{aligned} \quad (47)$$

Since f^{-1} is analytic and $(f^{-1}(z'_j))' = 1/s_j$, for every $w \in \partial B_{\varepsilon}(z'_j)$, $|z_j - f^{-1}(w)| = |f^{-1}(z'_j) - f^{-1}(w)| = \varepsilon/s_j + O(\varepsilon^2)$, which implies that $R_j - r_j = O(\varepsilon^2)$ and $R_j = O(\varepsilon)$. The second line of (47) can be bounded above by a constant times $\vartheta_{\varepsilon}^k \times o(1)$, as we now explain. The factor $\vartheta_{\varepsilon}^k$ comes from the requirement that ℓ intersect $B_{R_{z_j}}(z_j)$ for each $j = 1, \dots, k$ and the factor $o(1)$ comes from the requirement that ℓ intersect $\partial B_{R_l}(z_l)$ but not $\partial B_{r_l}(z_l)$ for at least one $l = 1, \dots, k$. More precisely, one can consider disks D_j of radius $N\varepsilon$ centered at $z_1, \dots, z_k$, for some N large but fixed, and first explore the region outside these disks. Using percolation arguments similar to those in the first part of the proof, one gets a factor $\vartheta_{N\varepsilon}^k = O(\vartheta_{\varepsilon}^k)$ from the requirement that ℓ intersect each D_j . Inside each disk $\{D_j\}_{j=1, \dots, k}$, one has a Brownian excursion of linear size $N\varepsilon$ that gets to distance $O(\varepsilon^2)$ of $\partial B_{r_j}(z_j)$ without intersecting it. The μ_D^{loop} -measure of loops producing such excursions can be shown to be of order $o(1)$, as $\varepsilon \rightarrow 0$, by arguments similar to those in the proof of Lemma 6.5 of [31], which provides an upper bound for the probability that a Brownian loop gets close to a deterministic loop without touching it. The upper bound implies that the probability in question goes to zero when the ratio between the linear size of the deterministic loop and the minimal distance between the

loops diverges, provided that the Brownian loop has linear size comparable to that of the deterministic loop. In the present case, that ratio is of order $1/\varepsilon$ and the Brownian excursion has diameter of order $N\varepsilon$, comparable to the diameter of $\partial B_{r_j}(z_j)$.

Hence, from (45), (46) and (32), using Lemma A.2 from the appendix, we obtain

$$\begin{aligned}
 & \alpha_{D'}^{z'_1, \dots, z'_k} \\
 &= \lim_{\varepsilon \rightarrow 0} (\vartheta_\varepsilon^{-k} \mu_D(\ell \cap B_{\varepsilon/s_j}(z_j)) \neq \emptyset \forall j = 1, \dots, k) + o(1) \\
 &= \left(\prod_{j=1}^k s_j^{-2/3} \right) \lim_{\varepsilon \rightarrow 0} \left(\prod_{j=1}^k \left(\frac{\vartheta_\varepsilon / \vartheta_{\varepsilon/s_j}}{s_j^{2/3}} \right)^{-1} \vartheta_{\varepsilon/s_j}^{-1} \right) \\
 & \quad \mu_D(\ell \cap B_{\varepsilon/s_j}(z_j)) \neq \emptyset \forall j = 1, \dots, k \\
 &= \left(\prod_{j=1}^k s_j^{-2/3} \right) \alpha_D^{z_1, \dots, z_k}, \tag{48}
 \end{aligned}$$

which concludes the proof. $\square$

3. Correlation Functions with a “Twist”

In this section we present a simple method to compute certain types of correlation functions involving two vertex layering operators. Later, as an application, we will use this method to show how the edge operator $\mathcal{E}$ emerges from the OPE of $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$. From now on, we will drop the subscript D from the expectation $\langle \cdot \rangle_D$, μ_D^{loop} , $\alpha_D^{z_1, \dots, z_k}$ and similar expressions when D can be any domain.

To explain the method mentioned above, in the next paragraph we use $\{\cdot\}$ to denote an unnormalized sum, where $\{\cdot\}$ is formally defined by the relation

$$\langle \cdot \rangle := \frac{\{\cdot\}}{Z} \tag{49}$$

and $Z := \{1\}$ denotes the partition function.¹² If we define

$$\{\cdot\}_{z_1, z_2}^* \equiv \{\cdot\}_{z_1, z_2; \beta}^* := \{\cdot \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2)\} \tag{50}$$

and

$$\langle \cdot \rangle_{z_1, z_2}^* \equiv \langle \cdot \rangle_{z_1, z_2; \beta}^* := \frac{\{\cdot\}_{z_1, z_2}^*}{\{1\}_{z_1, z_2}^*}, \tag{51}$$

then we can write

$$\begin{aligned}
 \langle \mathcal{A} \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle &= \frac{\{\mathcal{A} \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2)\}}{\{1\}} \\
 &= \frac{\{1\}_{z_1, z_2}^*}{\{1\}} \frac{\{\mathcal{A}\}_{z_1, z_2}^*}{\{1\}_{z_1, z_2}^*}
 \end{aligned}$$

¹² We note that, while $\langle \cdot \rangle$ is a well defined expectation (with respect to the BLS), $\{\cdot\}$ and Z do not in general exist separately.

$$= \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle \langle \mathcal{A} \rangle_{z_1, z_2}^*, \quad (52)$$

where $\mathcal{A}$ can be any combination of edge operators.

This simple formula will be very useful in the rest of the paper thanks to the observation that $\langle \cdot \rangle_{z_1, z_2}^*$ is the expectation with respect to the measure $\mu_{z_1, z_2; \beta}^* \equiv \mu_{z_1, z_2}^*$ defined by¹³

$$\mu_{z_1, z_2}^*(\ell) := \begin{cases} \mu^{\text{loop}}(\ell) & \text{if } \ell \text{ does not separate } z_1, z_2 \\ e^{i\beta\sigma_\ell} \mu^{\text{loop}}(\ell) & \text{if } z_1 \in \bar{\ell}, z_2 \notin \bar{\ell} \\ e^{-i\beta\sigma_\ell} \mu^{\text{loop}}(\ell) & \text{if } z_1 \notin \bar{\ell}, z_2 \in \bar{\ell}, \end{cases} \quad (53)$$

where $\sigma_\ell = \pm 1$ is a symmetric Boolean variable assigned to ℓ . In other words, $\langle \cdot \rangle_{z_1, z_2}^*$ is the expectation with respect to the measure μ_{z_1, z_2}^* , whose Radon-Nikodym derivative with respect to μ^{loop} is given by

$$\frac{d\mu_{z_1, z_2}^*}{d\mu^{\text{loop}}}(\ell) = \frac{\mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2)}{\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle}(\ell) := \begin{cases} 1 & \text{if } \ell \text{ does not separate } z_1, z_2 \\ e^{i\beta\sigma_\ell} & \text{if } z_1 \in \bar{\ell}, z_2 \notin \bar{\ell} \\ e^{-i\beta\sigma_\ell} & \text{if } z_1 \notin \bar{\ell}, z_2 \in \bar{\ell}. \end{cases} \quad (54)$$

As a first example, we use the method in the proof of the following theorem.

Theorem 3.1. *Using the notation introduced in (29) and (38), we have that*

$$\begin{aligned} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle &:= \frac{\hat{c}}{\sqrt{\lambda}} \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}_\varepsilon(z_3) \rangle \\ &= -\sqrt{\lambda}(1 - \cos \beta) \hat{\alpha}_{z_1|z_2}^{z_3} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle, \end{aligned} \quad (55)$$

where

$$\hat{\alpha}_{z_1|z_2}^{z_3} := \hat{c} \alpha_{z_1|z_2}^{z_3} \quad (56)$$

and

$$\alpha_{z_1|z_2}^{z_3} \equiv \alpha_{z_2|z_1}^{z_3} := \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_3) \neq \emptyset, \ell \text{ separates } z_1, z_2). \quad (57)$$

$\alpha_{z_1|z_2}^{z_3}$ and consequently $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle$ are conformally covariant in the sense of Lemma 2.2 and Theorem 2.3.

Proof. Consider the regularized fields $N_\varepsilon^\delta(z) := n_z^\varepsilon(\mathcal{L}^\delta)$ and $E_\varepsilon^\delta(z) := N_\varepsilon^\delta(z) - \langle N_\varepsilon^\delta(z) \rangle$, introduced earlier, as well as the regularized layering field $O_\beta^\delta(z)$ obtained from $\mathcal{L}^\delta$ (i.e., the “canonically normalized” version of the layering field $V_\beta^\delta(z)$ of [4]). With this notation and using (52), we have

$$\begin{aligned} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}_\varepsilon(z_3) \rangle &:= \lim_{\delta \rightarrow 0} \delta^{-4\Delta(\beta)} \langle O_\beta^\delta(z_1) O_{-\beta}^\delta(z_2) E_\varepsilon^\delta(z_3) \rangle \\ &= \lim_{\delta \rightarrow 0} \delta^{-4\Delta(\beta)} \langle O_\beta^\delta(z_1) O_{-\beta}^\delta(z_2) \rangle \langle \mathcal{E}_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^*. \end{aligned} \quad (58)$$

Now note that, according to (53)–(54), the contributions to $\langle N_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^*$ and $\langle N_\varepsilon^\delta(z_3) \rangle$ from loops that do not separate z_1 and z_2 are the same, while the contribution to

¹³ This is analogous to what is discussed in Sect. 2 of [32].

$\langle N_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^*$ from loops that do separate z_1 and z_2 comes with a factor $\cos \beta$ because of the averaging over $\sigma_\ell = \pm 1$ (recall that $\{\sigma_\ell\}_{\ell \in \mathcal{L}}$ is distributed like a collection of independent, (± 1) -valued, symmetric random variables). Therefore, we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} \langle E_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^* &= \lim_{\delta \rightarrow 0} \left[\langle N_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^* - \langle N_\varepsilon^\delta(z_3) \rangle \right] \\ &= \lim_{\delta \rightarrow 0} \left[(\cos \beta - 1) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_3) \neq \emptyset, \ell \text{ separates } z_1, z_2) \right] \\ &= -\lambda(1 - \cos \beta) \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_3) \neq \emptyset, \ell \text{ separates } z_1, z_2). \end{aligned} \quad (59)$$

Combining (59) with (58), and using the convergence of the two-point function of the layering operator from [4], gives

$$\begin{aligned} &\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle \\ &= -\lambda(1 - \cos \beta) \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_3) \neq \emptyset, \ell \text{ separates } z_1, z_2) \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle. \end{aligned} \quad (60)$$

To conclude the proof, it suffices to show the existence and conformal covariance of

$$\hat{\alpha}_{z_1|z_2}^{z_3} \equiv \hat{\alpha}_{z_2|z_1}^{z_3} := \hat{c} \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_3) \neq \emptyset, \ell \text{ separates } z_1, z_2). \quad (61)$$

These follow from the proof of Lemma 2.2 applied to the ensemble of loops that separate z_1 and z_2 . $\square$

So far our discussion has been completely general and independent of the domain D . If we now specify that $D = \mathbb{C}$ and note that the operators $\mathcal{O}_\beta, \mathcal{O}_{-\beta}$ are normalized so that

$$\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle_{\mathbb{C}} = |z_1 - z_2|^{-4\Delta(\beta)}, \quad (62)$$

we get, from (55),

$$\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}} = -\sqrt{\lambda}(1 - \cos \beta) \hat{\alpha}_{z_1|z_2; \mathbb{C}}^{z_3} |z_1 - z_2|^{-4\Delta(\beta)}. \quad (63)$$

The conformal covariance of (63), obtained in Theorem 3.1, implies that its form is fixed up to a multiplicative constant (see, for example, the proof of Theorem 4.5 of [4]). Therefore, letting $z_{jk} := z_j - z_k$, we have

$$\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}} = C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} \frac{1}{|z_{12}|^{4\Delta(\beta)}} \left| \frac{z_{12}}{z_{13} z_{23}} \right|^{2/3}, \quad (64)$$

where $C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}}$ is a numerical coefficient.

Until now, our discussion has been rigorous; we are now going to use a nonrigorous ingredient, namely, the determination of the coefficient $C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}}$ for $\beta_1 = \beta_2 = \pi$, obtained using nonrigorous methods in [7], where it is called $C^{(1,1)}$. Comparing (63) with (64) and using the expression for $C^{(1,1)}$ from Eq. (6.19) of [7] gives

$$\hat{\alpha}_{z_1|z_2; \mathbb{C}}^{z_3} = \frac{2^{7/6} \pi}{3^{1/4} \sqrt{5} \Gamma(1/6) \Gamma(4/3)} \left| \frac{z_{12}}{z_{13} z_{23}} \right|^{2/3}. \quad (65)$$

Together with (63), this leads to the following expression for the three-point function coefficient for general values of β :

$$C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} = -\sqrt{\lambda}(1 - \cos \beta) \frac{2^{7/6} \pi}{3^{1/4} \sqrt{5} \Gamma(1/6) \Gamma(4/3)}. \quad (66)$$

4. OPE and the Edge Operator

In this section, applying the method presented in the previous section, we show how the edge operator $\mathcal{E}$ emerges from the Operator Product Expansion (OPE) of $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$. It is shown in [7] that the OPE of the product of two vertex operators, $\mathcal{O}_{\beta_i} \times \mathcal{O}_{\beta_j}$, contains operators of dimensions $(\Delta_{ij} + \frac{k}{3}, \Delta_{ij} + \frac{k'}{3})$ for non-negative integers k, k' , where $\Delta_{ij} = \frac{\lambda}{10}(1 - \cos(\beta_i + \beta_j))$. In what follows, we identify the operator of dimensions $(1/3, 1/3)$ with the edge operator $\mathcal{E}$.

The calculations we present in this section are rigorous until equation (74). To proceed beyond that, we need an assumption, expressed by (75), which we believe to be correct, as we argue below.

If $N^\delta(z)$ denotes the number of loops of diameter larger than δ that contain z in their interior, it was shown in [4] that the two-point function

$$\begin{aligned} \langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle &\propto \lim_{\delta \rightarrow 0} \delta^{-2\Delta(\beta)} \left\langle e^{i\beta N^\delta(z)} e^{-i\beta N^\delta(z')} \right\rangle \\ &= \lim_{\delta \rightarrow 0} \delta^{-2\Delta(\beta)} \exp(-\lambda(1 - \cos \beta) \mu^{\text{loop}}(\ell \text{ separates } z, z', \text{diam}(\ell) > \delta)) \end{aligned} \quad (67)$$

exists.

We are interested in the sub-leading behavior of $\mathcal{O}_\beta(z) \times \mathcal{O}_{-\beta}(z')$ when $z' \rightarrow z$. The two-point function $\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle$ diverges when $z' \rightarrow z$ (see (62)), so we normalize $\mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z')$ by its expectation. Taking two distinct points $z_1, z_2 \neq z, z'$ and using (52), we can write

$$\begin{aligned} &\left\langle \frac{\mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z')}{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle} \mathcal{O}_{\beta'}(z_1) \mathcal{O}_{-\beta'}(z_2) \right\rangle \\ &= \langle \mathcal{O}_{\beta'}(z_1) \mathcal{O}_{-\beta'}(z_2) \rangle \frac{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle_{z_1, z_2; \beta'}^*}{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle}. \end{aligned} \quad (68)$$

To compute the right hand side of the equation above, we note that the loops that do not separate z_1 and z_2 contribute equally to $\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle_{z_1, z_2; \beta'}^*$ and $\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle$, so their contributions cancel out in the ratio on the right-hand side. The loops that do separate z_1, z_2 contribute differently, as we have already seen in the computation leading to (55). An analogous computation using (67) gives

$$\begin{aligned} &\frac{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle_{z_1, z_2; \beta'}^*}{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle} \\ &= \exp \left[(1 - \cos \beta') \lambda (1 - \cos \beta) \mu^{\text{loop}}(\ell \text{ separates } z \text{ from } z' \text{ and } z_1 \text{ from } z_2) \right] \end{aligned}$$

$$\begin{aligned}
 &= 1 + (1 - \cos \beta') \lambda (1 - \cos \beta) \mu^{\text{loop}}(\ell \text{ separates } z \text{ from } z' \text{ and } z_1 \text{ from } z_2) \\
 &\quad + O(\mu^{\text{loop}}(\ell \text{ separates } z \text{ from } z' \text{ and } z_1 \text{ from } z_2)^2),
 \end{aligned} \tag{69}$$

as $|z - z'| \rightarrow 0$. We now let $\varepsilon = |z - z'|$ and observe that

$$\begin{aligned}
 &\mu^{\text{loop}}(\ell \text{ separates } z, z' \text{ and } z_1, z_2) = \mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset \text{ and } \ell \text{ separates } z_1, z_2) \\
 &\quad - \mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset, \ell \text{ does not separate } z, z' \text{ and } \ell \text{ separates } z_1, z_2) \\
 &= \mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset \text{ and } \ell \text{ separates } z_1, z_2) \\
 &\quad \cdot \left[1 - \frac{\mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset, \ell \text{ does not separate } z, z' \text{ and } \ell \text{ separates } z_1, z_2)}{\mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset \text{ and } \ell \text{ separates } z_1, z_2)} \right],
 \end{aligned} \tag{70}$$

where

$$\mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset \text{ and } \ell \text{ separates } z_1, z_2) = O(\vartheta_\varepsilon) \text{ as } \varepsilon \rightarrow 0, \tag{71}$$

which follows from the proof of Lemma 2.2. Letting

$$\begin{aligned}
 \tilde{c}_\varepsilon &\equiv \tilde{c}_\varepsilon(z, z'; z_1, z_2) \\
 &:= 1 - \frac{\mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset, \ell \text{ does not separate } z, z' \text{ and } \ell \text{ separates } z_1, z_2)}{\mu^{\text{loop}}(\ell \cap B_\varepsilon(z) \neq \emptyset \text{ and } \ell \text{ separates } z_1, z_2)}
 \end{aligned} \tag{72}$$

and using (69)–(72), (59), and the fact that $\vartheta_\varepsilon \sim \varepsilon^{2/3}$, we can write

$$\frac{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle_{z_1, z_2; \beta'}^*}{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle} = 1 - (1 - \cos \beta) \tilde{c}_\varepsilon \langle E_\varepsilon(z) \rangle_{z_1, z_2; \beta'}^* + o(\varepsilon^{2/3}) \text{ as } \varepsilon \rightarrow 0. \tag{73}$$

Combining this with (68), we obtain

$$\begin{aligned}
 &\left\langle \frac{\mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z')}{\langle \mathcal{O}_\beta(z) \mathcal{O}_{-\beta}(z') \rangle} \mathcal{O}_{\beta'}(z_1) \mathcal{O}_{-\beta'}(z_2) \right\rangle \\
 &= \langle \mathcal{O}_{\beta'}(z_1) \mathcal{O}_{-\beta'}(z_2) \rangle - (1 - \cos \beta) \tilde{c}_\varepsilon \langle \mathcal{O}_{\beta'}(z_1) \mathcal{O}_{-\beta'}(z_2) E_\varepsilon(z) \rangle + o(\varepsilon^{2/3})
 \end{aligned} \tag{74}$$

as $\varepsilon \rightarrow 0$.

At this point we make the natural conjecture that, as long as the points z, z_1, z_2 are distinct, the limit

$$\tilde{c} := \lim_{z' \rightarrow z} \tilde{c}_\varepsilon \equiv \lim_{z' \rightarrow z} \tilde{c}_\varepsilon(z, z'; z_1, z_2) \tag{75}$$

exists and is independent of the domain and of z, z_1, z_2 . To see why this conjecture is justified, one can use arguments analogous to those in the proof of Lemma 2.2. Thinking in terms of the full scaling limit of critical percolation, as described in the proof of Lemma 2.2, one can split the loops separating z_1, z_2 and intersecting $B_\varepsilon(z)$ into excursions from $\partial B_\varepsilon(z)$ either inside or outside the disk. As explained in the proof of Lemma 2.2, the excursions inside and outside $B_\varepsilon(z)$ are independent of each other, conditioned on the location on $\partial B_\varepsilon(z)$ of their starting and ending points. Since the limit

in (75) is determined only by the behavior of the excursions inside $B_\varepsilon(z)$, it should not depend on the domain and on z_1, z_2 .

Using the conjecture expressed by (75) and the formal definition (29) of the edge operator, we are lead to conjecture the following behavior:

$$\frac{\mathcal{O}_\beta(z)\mathcal{O}_{-\beta}(z')}{\langle \mathcal{O}_\beta(z)\mathcal{O}_{-\beta}(z') \rangle} = \mathbb{1} - (1 - \cos \beta) \frac{\tilde{c}}{\hat{c}} \sqrt{\lambda} |z - z'|^{2/3} \mathcal{E}(z) + \mathcal{R} \quad (76)$$

as $z' \rightarrow z$, where $\mathbb{1}$ denotes the identity operator and $\mathcal{R}$ represent additional terms in the expansion responsible for the term $o(\varepsilon^{2/3})$ in (74). For z away from any boundary and in the limit $z' \rightarrow z$, using (62), we can assume that

$$\begin{aligned} \mathcal{O}_\beta(z) \times \mathcal{O}_{-\beta}(z') \\ = |z - z'|^{-4\Delta(\beta)} \left(\mathbb{1} - \sqrt{\lambda}(1 - \cos \beta) \frac{\tilde{c}}{\hat{c}} |z - z'|^{2/3} \mathcal{E}(z) + o(|z - z'|^{2/3}) \right), \end{aligned} \quad (77)$$

which shows how the edge operator emerges from the OPE of two layering vertex operators.

In order to check for internal consistency, we determine $\tilde{c}/\hat{c}$. To do this we insert the OPE (77) in the three-point function

$$\begin{aligned} \langle \mathcal{O}_\beta(z_1)\mathcal{O}_{-\beta}(z_2)\mathcal{E}(z_3) \rangle_{\mathbb{C}} \\ = |z_{12}|^{-4\Delta(\beta)} \left(-\sqrt{\lambda}(1 - \cos \beta) \frac{\tilde{c}}{\hat{c}} \langle \mathcal{E}(z_1)\mathcal{E}(z_3) \rangle_{\mathbb{C}} |z_{12}|^{2/3} + o(|z_{12}|^{2/3}) \right). \end{aligned} \quad (78)$$

Comparing this with (63), using (65) and the fact that $\mathcal{E}$ is assumed to be canonically normalized, so that

$$\langle \mathcal{E}(z_1)\mathcal{E}(z_3) \rangle_{\mathbb{C}} = |z_{13}|^{-4/3}, \quad (79)$$

we get

$$\begin{aligned} \frac{\tilde{c}}{\hat{c}} |z_{13}|^{-4/3} |z_{12}|^{2/3} + o(|z_{12}|^{2/3}) &= \hat{\alpha}_{z_1|z_2;\mathbb{C}}^{z_3} \\ &= \frac{2^{7/6}\pi}{3^{1/4}\sqrt{5}\Gamma(1/6)\Gamma(4/3)} \left| \frac{z_{12}}{z_{13}z_{23}} \right|^{2/3}. \end{aligned} \quad (80)$$

Dividing both sides of the equation above by $|z_{12}|^{2/3}$ and letting $z_2 \rightarrow z_1$ gives

$$\frac{\tilde{c}}{\hat{c}} = \frac{2^{7/6}\pi}{3^{1/4}\sqrt{5}\Gamma(1/6)\Gamma(4/3)}. \quad (81)$$

Based on general principles and on the conformal block expansion performed in [7], the OPE of $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$ should read

$$\begin{aligned} \mathcal{O}_\beta(z) \times \mathcal{O}_{-\beta}(z') \\ = |z - z'|^{-4\Delta(\beta)} \left(\mathbb{1} + C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\phi_{1/3,1/3}} |z - z'|^{2/3} \phi_{1/3,1/3}(z) + o(|z - z'|^{2/3}) \right), \end{aligned} \quad (82)$$

where $\phi_{1/3,1/3}$ is an operator of dimension $(1/3, 1/3)$. In order to identify $\phi_{1/3,1/3}$ with the edge operator $\mathcal{E}$, we need to identify $C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\phi_{1/3,1/3}}$ with the coefficient $C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^\mathcal{E}$ given in (66). Comparing (82) with (77), and using (81), this gives

$$C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\phi_{1/3,1/3}} = -\sqrt{\lambda}(1 - \cos \beta) \frac{2^{7/6} \pi}{3^{1/4} \sqrt{5} \Gamma(1/6) \Gamma(4/3)}, \quad (83)$$

which indeed coincides with (66).

5. A Mixed Four-Point Function

The method introduced in Sect. 3 can be used to calculate the mixed four-point function

$$\begin{aligned} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle &= \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle \langle \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{z_1, z_2}^* \\ &= \lambda^{-1} \hat{c}^2 \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-2} \langle \mathcal{E}_\varepsilon(z_3) \mathcal{E}_\varepsilon(z_4) \rangle_{z_1, z_2}^*. \end{aligned} \quad (84)$$

The result is given in the following theorem.¹⁴

Theorem 5.1. *We have that*

$$\begin{aligned} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle \\ = \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle \left[\hat{\alpha}^{z_3, z_4} - (1 - \cos \beta) \hat{\alpha}_{z_1|z_2}^{z_3, z_4} + \lambda(1 - \cos \beta)^2 \hat{\alpha}_{z_1|z_2}^{z_3} \hat{\alpha}_{z_1|z_2}^{z_4} \right], \end{aligned} \quad (85)$$

where

$$\hat{\alpha}_{z_1|z_2}^{z_3, z_4} := \hat{c}^2 \alpha_{z_1|z_2}^{z_3, z_4} \quad (86)$$

with

$$\alpha_{z_1|z_2}^{z_3, z_4} \equiv \alpha_{z_2|z_1}^{z_3, z_4} := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-4/3} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4; \ell \text{ separates } z_1, z_2). \quad (87)$$

$\alpha_{z_1|z_2}^{z_3, z_4}$ and consequently $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle$ are conformally covariant in the sense of Lemma 2.2 and Theorem 2.3.

Proof. Using the random variables defined in the paragraph above (26), a bit of algebra shows that

$$\begin{aligned} \langle E_\varepsilon(z_3) E_\varepsilon(z_4) \rangle_{z_1, z_2}^* &= \lim_{\delta \rightarrow 0} \langle E_\varepsilon^\delta(z_3) E_\varepsilon^\delta(z_4) \rangle_{z_1, z_2}^* \\ &= \lim_{\delta \rightarrow 0} \left\langle \left[N_\varepsilon^\delta(z_3) - \langle N_\varepsilon^\delta(z_3) \rangle \right] \left[N_\varepsilon^\delta(z_4) - \langle N_\varepsilon^\delta(z_4) \rangle \right] \right\rangle_{z_1, z_2}^* \\ &= \lim_{\delta \rightarrow 0} \left\langle \left[N_\varepsilon^\delta(z_3) - \langle N_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^* \right] \left[N_\varepsilon^\delta(z_4) - \langle N_\varepsilon^\delta(z_4) \rangle_{z_1, z_2}^* \right] \right\rangle_{z_1, z_2}^* \\ &\quad + \langle E_\varepsilon(z_3) \rangle_{z_1, z_2}^* \langle E_\varepsilon(z_4) \rangle_{z_1, z_2}^*. \end{aligned} \quad (88)$$

¹⁴ The terms $\hat{\alpha}^{z_3, z_4}$ and $\hat{\alpha}_{z_k, z_l}^{z_j}$ are defined in (32), (38) and (56), (57), respectively, and $\hat{c}$ is introduced in (29) and chosen so that $\mathcal{E}$ is canonically normalized (see (30)).

Now note that

$$\lim_{\delta \rightarrow 0} \left\langle \left[N_\varepsilon^\delta(z_3) - \langle N_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^* \right] \left[N_\varepsilon^\delta(z_4) - \langle N_\varepsilon^\delta(z_4) \rangle_{z_1, z_2}^* \right] \right\rangle_{z_1, z_2}^* \quad (89)$$

is exactly analogous to $\langle E_\varepsilon(z_3) E_\varepsilon(z_3) \rangle$, with the measure μ^{loop} replaced by μ_{z_1, z_2}^* . Therefore, combining Lemma 2.1 with (53), we have that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left\langle \left[N_\varepsilon^\delta(z_3) - \langle N_\varepsilon^\delta(z_3) \rangle_{z_1, z_2}^* \right] \left[N_\varepsilon^\delta(z_4) - \langle N_\varepsilon^\delta(z_4) \rangle_{z_1, z_2}^* \right] \right\rangle_{z_1, z_2}^* \\ &= \lambda \mu_{z_1, z_2}^* (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4) \\ &= \lambda \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4; \ell \text{ does not separate } z_1, z_2) \\ &\quad + \lambda \cos \beta \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4; \ell \text{ separates } z_1, z_2) \\ &= \lambda \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4) \\ &\quad - \lambda (1 - \cos \beta) \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4; \ell \text{ separates } z_1, z_2). \end{aligned} \quad (90)$$

Using this and (59), we obtain

$$\begin{aligned} & \langle E_\varepsilon(z_3) E_\varepsilon(z_4) \rangle_{z_1, z_2}^* = \lambda \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4) \\ &\quad - \lambda (1 - \cos \beta) \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4; \ell \text{ separates } z_1, z_2) \\ &\quad + \lambda^2 (1 - \cos \beta)^2 \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_3) \neq \emptyset, \ell \text{ separates } z_1, z_2) \\ &\quad \cdot \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_4) \neq \emptyset, \ell \text{ separates } z_1, z_2). \end{aligned} \quad (91)$$

Inserting this expression in (84) gives (85) with

$$\hat{\alpha}_{z_1|z_2}^{z_3, z_4} := \hat{c}^2 \alpha_{z_1|z_2}^{z_3, z_4} \quad (92)$$

and

$$\alpha_{z_1|z_2}^{z_3, z_4} \equiv \alpha_{z_2|z_1}^{z_3, z_4} := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-4/3} \mu^{\text{loop}} (\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ for } j = 3, 4; \ell \text{ separates } z_1, z_2), \quad (93)$$

where the existence of the limit its conformal covariance follow from the proof of Lemma 2.2 applied to the ensemble of loops that separate z_1 and z_2 . $\square$

We note that $\alpha_{z_1|z_2}^{z_3, z_4} \equiv \alpha_{z_1|z_2; D}^{z_3, z_4}$ depends on the domain D . When $D = \mathbb{C}$, non-rigorous arguments allow us to relate $\alpha_{z_1|z_2}^{z_3, z_4}$ to the quantity

$$\begin{aligned} Z_{\text{twist}} &:= \left| \frac{z_{13} z_{24}}{z_{34}^2 z_{23} z_{14}} \right|^{2/3} \\ &\cdot \left[\left| {}_2F_1 \left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}, x \right) \right|^2 - \frac{4\Gamma(\frac{2}{3})^6}{\Gamma(\frac{4}{3})^2 \Gamma(\frac{1}{3})^4} |x|^{2/3} \left| {}_2F_1 \left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}, x \right) \right|^2 \right] \end{aligned} \quad (94)$$

corresponding to equation (52) of [27], where $x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$.

In the language of [27], Z_{twist} is the four-point function of a pair of “2-leg” operators $\phi_{0,1}$ with a pair of “twist” operators $\phi_{2,1}$ ¹⁵ in the $O(n)$ model in the limit $n \rightarrow 0$. The

¹⁵ The subscripts label the positions of the operators in the Kac table.

“2-leg” operator $\phi_{0,1}(z)$ forces a self-avoiding loop of the $O(n)$ model to go through z , while a pair of “twist” operators $\phi_{2,1}(z_1)\phi_{2,1}(z_2)$ acts like $\mathcal{O}_\pi(z_1)\mathcal{O}_{-\pi}(z_2)$ in the sense that the weight of each loop that separates z_1 and z_2 is multiplied by -1 . Simmons and Cardy [27] compute this four-point function for the $O(n)$ model for $-2 < n < 2$, which in the case of $n = 0$ leads to (94). The $n = 0$ case of the $O(n)$ model corresponds to a self-avoiding loop whose properties are described by μ^{loop} , as we will now explain.

Strictly speaking, when $n = 0$ all loops are suppressed, but the inclusion of a pair of 2-leg operators guarantees the presence of at least one loop. Sending $n \rightarrow 0$ then singles out the “one loop sector” described by μ^{loop} , since all other “sectors” produce a contribution of higher order in n (see the discussion preceding Eq. (49) of [27]).

Something analogous happens in the case of the four-point function (85). As explained above, the pair of operators $\mathcal{O}_\pi(z_1)\mathcal{O}_{-\pi}(z_2)$ acts like $\phi_{2,1}(z_1)\phi_{2,1}(z_2)$, while the presence of a pair of edge operators guarantees the existence of at least one loop. Since the loop soup can be thought of as a gas of loops in the grand canonical ensemble with fugacity λ , the four-point function can be written as a sum of contributions from various “sectors” characterized by the number of loops. Because of the normalization of the edge operator, which includes a factor of $\lambda^{-1/2}$, the contribution of the “one loop sector” is of order $O(1)$, while all other contributions are of order $O(\lambda)$, as one can clearly see from (85). As a result, sending $\lambda \rightarrow 0$ in (85) singles out the “one loop sector” just like sending $n \rightarrow 0$ in the case of the $O(n)$ four-point function calculated by Simmons and Cardy [27]. The two limits can be directly compared because all operators involved are canonically normalized. We can therefore conjecture that

$$\begin{aligned} Z_{\text{twist}} &= \lim_{\lambda \rightarrow 0} \langle \mathcal{O}_\pi(z_1)\mathcal{O}_{-\pi}(z_2)\mathcal{E}(z_3)\mathcal{E}(z_4) \rangle_{\mathbb{C}} \\ &= \hat{\alpha}_{\mathbb{C}}^{z_3, z_4} - 2\hat{\alpha}_{z_1|z_2; \mathbb{C}}^{z_3, z_4}, \end{aligned} \quad (95)$$

where we used (62) and (85) to compute the limit.

This leads to

$$\hat{\alpha}_{z_1|z_2; \mathbb{C}}^{z_3, z_4} = \frac{\hat{\alpha}_{\mathbb{C}}^{z_3, z_4} - Z_{\text{twist}}}{2}, \quad (96)$$

with

$$\hat{\alpha}_{\mathbb{C}}^{z_3, z_4} = \frac{1}{|z_3 - z_4|^{4/3}}, \quad (97)$$

from (39a), (30). Combining (96) and (97) with (94), (85) and (65) provides an explicit expression for the full-plane mixed four-point function $\langle \mathcal{O}_\beta(z_1)\mathcal{O}_{-\beta}(z_2)\mathcal{E}(z_3)\mathcal{E}(z_4) \rangle_{\mathbb{C}}$.

6. Higher-Order and Charged Edge Operators

We will now extend the analysis of the edge operator $\mathcal{E}$ to all spin-zero operators that have non-zero fusion with the vertex operators. We will show that they have holomorphic and anti-holomorphic conformal dimensions

$$(\Delta(\beta) + k/3, \Delta(\beta) + k/3), \quad (98)$$

with $\Delta(\beta) = \frac{\lambda}{10}(1 - \cos \beta)$, for any non-negative integer k . They correspond to the operators indicated on the diagonal of Fig. 2b. We will first define the operators with $\beta = 0$ and dimensions $(k/3, k/3)$ for $k \geq 2$, which will be denoted $\mathcal{E}^{(k)}$ and will be called

higher-order edge operators. We will then see that the operators $\mathcal{E}_\beta^{(k)}$ with dimensions $(\Delta(\beta) + k/3, \Delta(\beta) + k/3)$ with $\beta \neq 0$ are a product of $\mathcal{O}_\beta$ with a modified version of $\mathcal{E}^{(k)}$. These will be called *charged edge operators*.

6.1. Higher-order edge operators. Searching for new primary operators, we are guided by their conformal dimensions. For the operators with dimensions $(k/3, k/3)$, it is natural to consider powers of edge operators. However, these are not well defined. Indeed, even if we keep both ε and δ cutoffs, it is clear that $(E_\varepsilon^\delta(z))^k$ is not the correct starting point because its mean is not zero. A better choice, inspired by

$$\begin{aligned} E_\varepsilon^{(1);\delta}(z) &:= E_\varepsilon^\delta(z) = N_\varepsilon^\delta(z) - \lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \\ &= \left(\frac{\partial}{\partial x} - \lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \right) x^{N_\varepsilon^\delta(z)} \Big|_{x=1}, \end{aligned} \quad (99)$$

is given, for each integer $k \geq 2$, by

$$\begin{aligned} E_\varepsilon^{(k);\delta}(z) &:= \left(\frac{\partial}{\partial x} - \lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \right)^k x^{N_\varepsilon^\delta(z)} \Big|_{x=1} \\ &= \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} N_\varepsilon^\delta(z) \dots (N_\varepsilon^\delta(z) - (k-j) + 1) \\ &\quad \cdot \left(\lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \right)^j \\ &\quad + (-1)^k \left(\lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \right)^k. \end{aligned} \quad (100)$$

This definition is valid in any domain D . Since $N_\varepsilon^\delta(z) = n_z^\varepsilon(\mathcal{L}^\delta)$ (see Sect. 3 above (59) and Appendix A) is a Poisson random variable with parameter $\lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset)$, we have that

$$\begin{aligned} &\langle N_\varepsilon^\delta(z)(N_\varepsilon^\delta(z) - 1) \dots (N_\varepsilon^\delta(z) - (k-j) + 1) \rangle \\ &= \left(\lambda\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \right)^{k-j}, \end{aligned} \quad (101)$$

which implies that $\langle E_\varepsilon^{(k);\delta}(z) \rangle_{\mathbb{C}} = 0$ for every $\delta > 0$.

With this notation, for each $k \geq 1$, we formally define the *order k edge operator*

$$\mathcal{E}^{(k)}(z) := \frac{\hat{c}^k}{\sqrt{k!}\lambda^{k/2}} \lim_{\delta, \varepsilon \rightarrow 0} \vartheta_\varepsilon^{-k} E_\varepsilon^{(k);\delta}(z). \quad (102)$$

As we will see at the end of this section, the constant in front of the limit is chosen in such a way that $\mathcal{E}^{(k)}$ is canonically normalized, i.e.,

$$\langle \mathcal{E}^{(k)}(z_1) \mathcal{E}^{(k)}(z_2) \rangle_{\mathbb{C}} = |z_1 - z_2|^{-4k/3}. \quad (103)$$

For $k = 1$, we recover the edge operator, i.e., $\mathcal{E}^{(1)} \equiv \mathcal{E}$.

Definition (102) is formal in the sense that $\mathcal{E}^{(k)}(z)$ is only well defined within n -point correlation functions. In order to show that $\mathcal{E}^{(k)}$ has well-defined n -point functions, we start with an intermediate result, for which we need the following notation. Given a collection of points $z_1, \dots, z_n$ and a vector $\mathbf{k} = (k_1, \dots, k_n)$, $k_j \in \mathbb{N}$, we denote by $\mathcal{M} \equiv \mathcal{M}(z_1, \dots, z_n; k_1, \dots, k_n)$ the collection of all multisets¹⁶ M such that

- (1) the elements S of M are sets contained in $\{z_1, \dots, z_n\}$ with $|S| > 1$,
- (2) the multiplicities $m_M(S)$ are such that $\sum_{S \in M} m_M(S) \mathbf{I}(z_j \in S) = k_j$ for each $j = 1, \dots, n$ and each $M \in \mathcal{M}$.

Condition (2) on the multiplicities essentially says that each point z_j has multiplicity exactly k_j in each multiset M . Note that $\mathcal{M}$ can be empty since conditions (1) and (2) cannot necessarily be satisfied simultaneously for generic choices of the vector $\mathbf{k}$.

For a set S , let I_S denote the set of indices such that $j \in I_S$ if and only if $z_j \in S$. Then we have the following lemma, proved in the appendix.

Lemma 6.1. *For any $n \geq 2$ and $\delta, \varepsilon > 0$, for any collection of points $z_1, \dots, z_n$ at distance greater than 2ε from each other, with the notation introduced above, we have that*

$$\begin{aligned} \left\langle \prod_{j=1}^n E_\varepsilon^{(k_j)}(z_j) \right\rangle &:= \lim_{\delta \rightarrow 0} \left\langle \prod_{j=1}^n E_\varepsilon^{(k_j); \delta}(z_j) \right\rangle \\ &= \left(\prod_{j=1}^n k_j! \right) \sum_{M \in \mathcal{M}} \prod_{S \in M} \frac{1}{m_M(S)!} \left(\lambda \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \forall z_j \in S) \right)^{m_M(S)} \mathbf{I}(\mathcal{M} \neq \emptyset), \end{aligned} \quad (104)$$

where $\mathbf{I}(\mathcal{M} \neq \emptyset)$ denotes the indicator function of the event that $\mathcal{M}$ is not empty.

The next theorem shows that it is also possible to remove the ε cutoff and demonstrates that the operators $\mathcal{E}^{(k)}$ are primaries with dimensions $(k/3, k/3)$ for all non-negative integer k .

Theorem 6.2. *Let $D \subseteq \mathbb{C}$ be either the complex plane $\mathbb{C}$ or the upper-half plane $\mathbb{H}$ or any domain conformally equivalent to $\mathbb{H}$. With the notation of the previous lemma, for any collection of distinct points $z_1, \dots, z_n \in D$ with $n \geq 2$ and any vector $\mathbf{k} = (k_1, \dots, k_n)$ with $k_j \in \mathbb{N}$ such that $\mathcal{M}$ is not empty, we have that*

$$\begin{aligned} \mathcal{G}_D(z_1, \dots, z_n; k_1, \dots, k_n) &:= \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-\sum_{j=1}^n k_j} \left\langle E_\varepsilon^{(k_1)}(z_1) \dots E_\varepsilon^{(k_n)}(z_n) \right\rangle_D \\ &= \left(\prod_{j=1}^n k_j! \right) \sum_{M \in \mathcal{M}} \prod_{S \in M} \frac{1}{m_M(S)!} (\lambda \alpha^S)^{m_M(S)}. \end{aligned} \quad (105)$$

Moreover, $\mathcal{G}_D(z_1, \dots, z_n; k_1, \dots, k_n)$ is conformally invariant in the sense that, if D' is a domain conformally equivalent to D and $f : D \rightarrow D'$ is a conformal map, then

$$\begin{aligned} \mathcal{G}_{D'}(f(z_1), \dots, f(z_n); k_1, \dots, k_n) \\ = \left(\prod_{j=1}^n |f'(z_j)|^{-2k_j/3} \right) \mathcal{G}_D(z_1, \dots, z_n; k_1, \dots, k_n). \end{aligned} \quad (106)$$

¹⁶ A multiset is a set whose elements have multiplicity ≥ 1 .

Proof. From the expression for the n -point function in Lemma 6.1, using the fact that $\sum_{S \in \mathcal{M}} m_M(S) \mathbf{I}(z_j \in S) = k_j$, for each $j = 1, \dots, n$ and each $M \in \mathcal{M}$, we see that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-\sum_{j=1}^n k_j} \left\langle \prod_{j=1}^n E_{\beta_j; \varepsilon}^{(k_j)}(z_j) \right\rangle \\ &= \left(\prod_{j=1}^n k_j! \right) \sum_{M \in \mathcal{M}} \prod_{S \in M} \frac{1}{m_M(S)!} \left(\lambda \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-|S|} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \forall z_j \in S) \right)^{m_M(S)} \\ &= \left(\prod_{j=1}^n k_j! \right) \sum_{M \in \mathcal{M}} \prod_{S \in M} \frac{1}{m_M(S)!} (\lambda \alpha^S)^{m_M(S)}, \end{aligned} \quad (107)$$

where the last equality follows from Lemma 2.2. Equation (106) now follows immediately from the last expression and Lemma 2.2. $\square$

Using (105) and the definition of order k edge operator (102), we can now write the correlation of n higher-order edge operators as

$$\begin{aligned} & \left\langle \mathcal{E}^{(k_1)}(z_1) \dots \mathcal{E}^{(k_n)}(z_n) \right\rangle_D \\ &= \left(\prod_{j=1}^n \frac{\hat{c}^{k_j}}{k_j! \lambda^{k_j/2}} \right) \mathcal{G}_D(z_1, \dots, z_n; k_1, \dots, k_n) \\ &= \left(\prod_{j=1}^n \lambda^{-k_j/2} \right) \sum_{M \in \mathcal{M}} \prod_{S \in M} \frac{1}{m_M(S)!} (\lambda \hat{\alpha}^S)^{m_M(S)}. \end{aligned} \quad (108)$$

In view of (106), these n -point functions are manifestly conformally covariant, showing that the higher-order edge operators are conformal primaries.

If $n = 2$ and $k_1 = k_2 = k$, it is easy to see that the set $\mathcal{M}$ contains a single multiset with only one element $S = \{z_1, z_2\}$ with multiplicity k . Therefore, specializing (108) to this case with $D = \mathbb{C}$ gives

$$\left\langle \mathcal{E}^{(k)}(z_1) \mathcal{E}^{(k)}(z_2) \right\rangle_{\mathbb{C}} = (\hat{\alpha}^{z_1, z_2})^k = (\langle \mathcal{E}(z_1) \mathcal{E}(z_2) \rangle_{\mathbb{C}})^k = |z_1 - z_2|^{-4k/3}, \quad (109)$$

which shows that $\mathcal{E}^{(k)}$ is canonically normalized.

6.2. Charged edge operators. We now apply a “twist” to the (higher-order) edge operators and introduce a new set of operators. A *charged edge operator* is essentially an edge operator “seen from” the perspective of a measure $\mu_{z; \beta}^* \equiv \mu_z^*$ defined by

$$\mu_z^*(\ell) := \begin{cases} \mu^{\text{loop}}(\ell) & \text{if } z \notin \bar{\ell} \\ e^{i\beta\sigma_\ell} \mu^{\text{loop}}(\ell) & \text{if } z \in \bar{\ell} \end{cases} \quad (110)$$

where $\sigma_\ell = \pm 1$ is a symmetric Boolean variable assigned to ℓ . The measure μ_z^* is constructed from μ^{loop} by assigning an additional phase $e^{i\beta\sigma_\ell}$ to each loop covering z . The construction is similar to that of μ_{z_1, z_2}^* , introduced in Sect. 3, to which we refer the reader.

We note that, when taking expectations, one sums over the two possible values of σ_ℓ with equal probability, so that loops ℓ that do not cover z get weight $\mu^{\text{loop}}(\ell)$, while loops ℓ that cover z get weight $\cos \beta \mu^{\text{loop}}(\ell)$.

With this in mind, for any $\beta \in [0, 2\pi)$, the simplest charged edge operator with cutoffs $\delta, \varepsilon > 0$, corresponding to the “twisted” or “charged” version of (99), is defined as

$$\begin{aligned} E_{\beta;\varepsilon}^{(1);\delta}(z) &\equiv E_{\beta;\varepsilon}^\delta(z) \\ &:= V_\beta^\delta(z) \left(N_\varepsilon^\delta(z) - \lambda(\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \notin \bar{\ell}) \right. \\ &\quad \left. + \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \in \bar{\ell}) \cos \beta \right), \end{aligned} \quad (111)$$

where

$$V_\beta^\delta(z) := \exp \left(i\beta \sum_{\substack{\ell \in \mathcal{L}^\delta \\ z \in \bar{\ell}}} \sigma_\ell \right), \quad (112)$$

the layering operator with cutoff $\delta > 0$ introduced in [4], induces a phase $e^{i\beta\sigma_\ell}$ for each loop ℓ such that $z \in \bar{\ell}$, and

$$\begin{aligned} &\lambda(\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \notin \bar{\ell}) \\ &\quad + \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \in \bar{\ell}) \cos \beta) \end{aligned} \quad (113)$$

is the expectation of $N_\varepsilon^\delta(z)$ under the measure μ_z^* .

Generalizing this to any $k \in \mathbb{N}$, the “twisted” or “charged” version of (102) is given by

$$\begin{aligned} E_{\beta;\varepsilon}^{(k);\delta}(z) &:= V_\beta^\delta(z) \left[\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} N_\varepsilon^\delta(z) \dots (N_\varepsilon^\delta(z) - (k-j) + 1) \right. \\ &\quad \cdot \left(\lambda(\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \notin \bar{\ell}) \right. \\ &\quad \left. + \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \in \bar{\ell}) \cos \beta \right)^j \\ &\quad \left. + (-1)^k \left(\lambda(\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \notin \bar{\ell}) \right. \right. \\ &\quad \left. \left. + \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \in \bar{\ell}) \cos \beta \right)^k \right]. \end{aligned} \quad (114)$$

We now formally define the *charged (order k) edge operator*

$$\mathcal{E}_\beta^{(k)}(z) := \lim_{\delta, \varepsilon \rightarrow 0} (c'\delta)^{-2\Delta(\beta)} \frac{\hat{c}^k}{k! \lambda^{k/2}} \vartheta_\varepsilon^{-k} E_{\beta;\varepsilon}^{(k);\delta}(z), \quad (115)$$

where c' is a normalization constant needed to obtain the canonically normalized operator $\mathcal{O}_\beta$ from V_β^δ , which depends on the domain (see [7]). For completeness, we also define $\mathcal{E}_\beta^{(0)} \equiv \mathcal{O}_\beta$. Unlike their uncharged counterparts, the charged operators $\mathcal{E}_\beta^{(k)}$ are not canonically normalized for general $\beta \neq 0$.

As an example, we compute the two-point function of the simplest charged edge operators, with charge conservation. To that end, we write $E_{\beta;\varepsilon}^\delta(z)$ as

$$\begin{aligned} E_{\beta;\varepsilon}^\delta(z) &= V_\beta^\delta(z) \left(N_\varepsilon^\delta(z) - \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset) \right. \\ &\quad \left. + (1 - \cos \beta) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \in \bar{\ell}) \right) \\ &= V_\beta^\delta(z) E_\varepsilon^\delta(z) + (1 - \cos \beta) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z) \neq \emptyset, z \in \bar{\ell}) V_\beta^\delta(z). \end{aligned} \quad (116)$$

Using this expression and the method introduced in Sect. 3, we have

$$\begin{aligned} \left\langle E_{\beta;\varepsilon}^\delta(z_1) E_{-\beta;\varepsilon}^\delta(z_2) \right\rangle &= \left\langle V_\beta^\delta(z_1) V_{-\beta}^\delta(z_2) E_\varepsilon^\delta(z_1) E_\varepsilon^\delta(z_2) \right\rangle \\ &\quad + (1 - \cos \beta) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_2) \neq \emptyset, z_2 \in \bar{\ell}) \left\langle V_\beta^\delta(z_1) E_\varepsilon^\delta(z_1) V_{-\beta}^\delta(z_2) \right\rangle \\ &\quad + (1 - \cos \beta) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_1) \neq \emptyset, z_1 \in \bar{\ell}) \left\langle V_{-\beta}^\delta(z_2) E_\varepsilon^\delta(z_2) V_\beta^\delta(z_1) \right\rangle \\ &\quad + (1 - \cos \beta)^2 \lambda^2 \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_1) \neq \emptyset, z_1 \in \bar{\ell}) \\ &\quad \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_2) \neq \emptyset, z_2 \in \bar{\ell}) \left\langle V_\beta^\delta(z_1) V_{-\beta}^\delta(z_2) \right\rangle \\ &= \left\langle V_\beta^\delta(z_1) V_{-\beta}^\delta(z_2) \right\rangle \left[\left\langle E_\varepsilon^\delta(z_1) E_\varepsilon^\delta(z_2) \right\rangle_{z_1, z_2}^* \right. \\ &\quad + (1 - \cos \beta) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_2) \neq \emptyset, z_2 \in \bar{\ell}) \left\langle E_\varepsilon^\delta(z_1) \right\rangle_{z_1, z_2}^* \\ &\quad + (1 - \cos \beta) \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_1) \neq \emptyset, z_1 \in \bar{\ell}) \left\langle E_\varepsilon^\delta(z_2) \right\rangle_{z_1, z_2}^* \\ &\quad \left. + (1 - \cos \beta)^2 \lambda^2 \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_1) \neq \emptyset, z_1 \in \bar{\ell}) \right. \\ &\quad \left. \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_2) \neq \emptyset, z_2 \in \bar{\ell}) \right]. \end{aligned} \quad (117)$$

After identifying z_3 with z_1 and z_4 with z_2 , we can use (59) and (91) to simplify the above expression. A simple calculation shows that, for any $\delta < |z_1 - z_2|$,

$$\begin{aligned} \left\langle E_{\beta;\varepsilon}^\delta(z_1) E_{-\beta;\varepsilon}^\delta(z_2) \right\rangle &= \left\langle V_\beta^\delta(z_1) V_{-\beta}^\delta(z_2) \right\rangle \left[\lambda \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, j = 1, 2) \right. \\ &\quad - (1 - \cos \beta) \lambda \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, j = 1, 2; \ell \text{ separates } z_1, z_2) \\ &\quad + \lambda^2 (1 - \cos \beta)^2 \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_1) \neq \emptyset, z_2 \in \bar{\ell}, z_1 \notin \bar{\ell}) \\ &\quad \left. \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_2) \neq \emptyset, z_1 \in \bar{\ell}, z_2 \notin \bar{\ell}) \right]. \end{aligned} \quad (118)$$

Using definition (115), we obtain

$$\begin{aligned} \left\langle \mathcal{E}_\beta(z_1) \mathcal{E}_{-\beta}(z_2) \right\rangle &= \lim_{\delta \rightarrow 0} (\hat{c}' \delta)^{-4\Delta(\beta)} \left\langle V_\beta^\delta(z_1) V_{-\beta}^\delta(z_2) \right\rangle \\ &\quad \hat{c}^2 \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-2} \left[\mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, j = 1, 2) \right. \\ &\quad - (1 - \cos \beta) \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, j = 1, 2; \ell \text{ separates } z_1, z_2) \\ &\quad + \lambda (1 - \cos \beta)^2 \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_1) \neq \emptyset, z_2 \in \bar{\ell}, z_1 \notin \bar{\ell}) \\ &\quad \left. \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_2) \neq \emptyset, z_1 \in \bar{\ell}, z_2 \notin \bar{\ell}) \right] \end{aligned}$$

$$\begin{aligned}
 &= \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle \left[\hat{\alpha}_{z_1|z_2}^{z_1, z_2} - (1 - \cos \beta) \hat{\alpha}_{z_1|z_2}^{z_1, z_2} \right. \\
 &\quad \left. + \lambda (1 - \cos \beta)^2 \hat{c}^2 \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-2} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_1) \neq \emptyset, z_2 \in \bar{\ell}, z_1 \notin \bar{\ell}) \right. \\
 &\quad \left. \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_2) \neq \emptyset, z_1 \in \bar{\ell}, z_2 \notin \bar{\ell}) \right]. \quad (119)
 \end{aligned}$$

At this point, we should note that unfortunately the existence of the limits

$$\begin{aligned}
 \alpha_{z_1|z_2}^{z_1, z_2} &= \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-2} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, j = 1, 2; \ell \text{ separates } z_1, z_2), \\
 \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \mu^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, z_k \in \bar{\ell}, z_j \notin \bar{\ell}) \quad (120)
 \end{aligned}$$

does not follow from Lemma 2.2. It is, however, reasonable to assume that they exist. Indeed, in the case of the first limit, observing that

$$\lim_{\substack{z_3 \rightarrow z_1 \\ z_4 \rightarrow z_2}} Z_{\text{twist}} = 0 \quad (121)$$

and using (96) suggests that, in the full plane,

$$\hat{\alpha}_{z_1|z_2; \mathbb{C}}^{z_1, z_2} = \frac{1}{2} \hat{\alpha}_{\mathbb{C}}^{z_1, z_2}. \quad (122)$$

The second limit in (120) should also exist; moreover, if

$$\hat{\alpha}_{\mathbb{C}}^{z_j}(z_k; z_j) := \hat{c} \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \mu_{\mathbb{C}}^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, z_k \in \bar{\ell}, z_j \notin \bar{\ell}) \quad (123)$$

does exist, arguments like those used in the second part of the proof of Lemma 2.2 imply that, for any $s > 0$, $\hat{\alpha}_{\mathbb{C}}^{s, z}(0; z) = s^{-2/3} \hat{\alpha}_{\mathbb{C}}^{z, z}(0; z)$. Since $\hat{\alpha}_{\mathbb{C}}^{z_j}(z_k; z_j)$ only depends on $|z_j - z_k|$, this would in turn imply that $\hat{\alpha}_{\mathbb{C}}^{z_j}(z_k; z_j)$ must take the form $\text{const} \cdot |z_j - z_k|^{-2/3}$.

If the considerations above are correct, then it follows from (119) that $\langle \mathcal{E}_\beta(z_1) \mathcal{E}_{-\beta}(z_2) \rangle_{\mathbb{C}}$ behaves like the correlation function between two conformal primaries of scaling dimension $2\Delta(\beta) + 2/3$, as desired. Indeed, we conjecture that, similarly to (122), $\hat{\alpha}_{\mathbb{C}}^{z_j}(z_k; z_j) = \frac{1}{2} \hat{\alpha}_{z_k; \mathbb{C}}^{z_j}$, which would lead to

$$\begin{aligned}
 \langle \mathcal{E}_\beta(z_1) \mathcal{E}_{-\beta}(z_2) \rangle_{\mathbb{C}} &= \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle_{\mathbb{C}} \left(\frac{1}{2} (1 + \cos \beta) \hat{\alpha}_{\mathbb{C}}^{z_1, z_2} + \frac{\lambda}{4} \hat{\alpha}_{z_2; \mathbb{C}}^{z_1} \hat{\alpha}_{z_1; \mathbb{C}}^{z_2} \right) \\
 &\sim |z_1 - z_2|^{-4\Delta(\beta) - 4/3}, \quad (124)
 \end{aligned}$$

where the existence and the scaling behavior of

$$\hat{\alpha}_{z_k; \mathbb{C}}^{z_j} := \hat{c} \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon^{-1} \mu_{\mathbb{C}}^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, z_k \in \bar{\ell}) \quad (125)$$

follow from the proof of Lemma 2.2.

7. The Primary Operator Spectrum

The four-point function of a conformal field theory contains information about the three-point function coefficients, as well as the spectrum of primary operators. In the following two sections, we perform the Virasoro conformal block expansion of the new four-point function (85) in the full plane, and derive the three-point coefficient involving three edge operators through the OPE of the edge operator as an illustration of the conformal block expansion.

7.1. Virasoro conformal blocks. By a global conformal transformation, one can always map three of the four points of a four-point function $\langle \mathcal{A}_1(z_1)\mathcal{A}_2(z_2)\mathcal{A}_3(z_3)\mathcal{A}_4(z_4) \rangle_{\mathbb{C}}$ to fixed values, where $\mathcal{A}_j(z_j)$ here denotes a generic primary operator evaluated at z_j . The remaining dependence is only on the cross-ratio $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and its complex conjugate $\bar{x}$, which are invariant under global conformal transformations. The following discussion parallels Sect. 6 of [7]. Following the notation of Section 6.6.4 of [5], it is customary to set $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ and $z_4 = 0$. The resulting function

$$G_{34}^{21}(x) := \lim_{z_1 \rightarrow \infty} z_1^{2\Delta_1} \bar{z}_1^{2\bar{\Delta}_1} \langle \mathcal{A}_1(z_1)\mathcal{A}_2(1)\mathcal{A}_3(x)\mathcal{A}_4(0) \rangle_{\mathbb{C}} \quad (126)$$

can be decomposed into Virasoro conformal blocks according to

$$G_{34}^{21}(x) = \sum_{\mathcal{P}} C_{34}^{\mathcal{P}} C_{12}^{\mathcal{P}} \mathcal{F}_{34}^{21}(\mathcal{P}|x) \bar{\mathcal{F}}_{34}^{21}(\mathcal{P}|\bar{x}). \quad (127)$$

The sum over $\mathcal{P}$ runs over all primary operators in the theory, and the $C_{ij}^{\mathcal{P}}$ are the three-point function coefficients of the operators labeled by $i, j, \mathcal{P}$, that is,

$$\begin{aligned} \langle \mathcal{A}_l(z_1)\mathcal{A}_j(z_2)\mathcal{P}(z_3) \rangle_{\mathbb{C}} &= C_{lj}^{\mathcal{P}} z_{12}^{-(\Delta_l+\Delta_j-\Delta_{\mathcal{P}})} z_{13}^{-(\Delta_l+\Delta_{\mathcal{P}}-\Delta_j)} z_{23}^{-(\Delta_j+\Delta_{\mathcal{P}}-\Delta_l)} \\ &\quad \bar{z}_{12}^{-(\bar{\Delta}_l+\bar{\Delta}_j-\bar{\Delta}_{\mathcal{P}})} \bar{z}_{13}^{-(\bar{\Delta}_l+\bar{\Delta}_{\mathcal{P}}-\bar{\Delta}_j)} \bar{z}_{23}^{-(\bar{\Delta}_j+\bar{\Delta}_{\mathcal{P}}-\bar{\Delta}_l)}, \end{aligned} \quad (128)$$

where $\Delta_j, \bar{\Delta}_j$ are the scaling dimensions of the corresponding fields.

The functions $\mathcal{F}, \bar{\mathcal{F}}$ are called Virasoro conformal blocks and are fixed by conformal invariance. Each conformal block can be written as a power series

$$\mathcal{F}_{34}^{21}(\mathcal{P}|x) = x^{\Delta_{\mathcal{P}}-\Delta_3-\Delta_4} \sum_{K=0}^{\infty} \mathcal{F}_K x^K, \quad (129)$$

where coefficients $\mathcal{F}_K$ can be fully determined by the central charge c , and the conformal dimensions $\Delta_j, \Delta_{\mathcal{P}}$ of the five operators involved. $\bar{\mathcal{F}}$ is determined analogously.

In the case of (85), we obtain

$$\begin{aligned} G_{34}^{21}(x) &= \lim_{z_1 \rightarrow \infty} |z_1|^{4\Delta(\beta)} \langle \mathcal{O}_{\beta}(z_1)\mathcal{O}_{-\beta}(1)\mathcal{E}(x)\mathcal{E}(0) \rangle_{\mathbb{C}} \\ &= \lambda \frac{4 \cdot 2^{1/3} \pi^2}{5\sqrt{3}\Gamma(\frac{1}{6})^2 \Gamma(\frac{4}{3})^2} \frac{(1 - \cos \beta)^2}{|1 - x|^{2/3}} + \frac{1 + \cos \beta}{2|x|^{4/3}} + \frac{1 - \cos \beta}{2|x|^{4/3}|1 - x|^{2/3}} \\ &\quad \cdot \left[\left| {}_2F_1\left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}; x\right) \right|^2 - \frac{4\Gamma(\frac{2}{3})^6}{\Gamma(\frac{4}{3})^2 \Gamma(\frac{1}{3})^4} |x|^{2/3} \left| {}_2F_1\left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; x\right) \right|^2 \right]. \end{aligned} \quad (130)$$

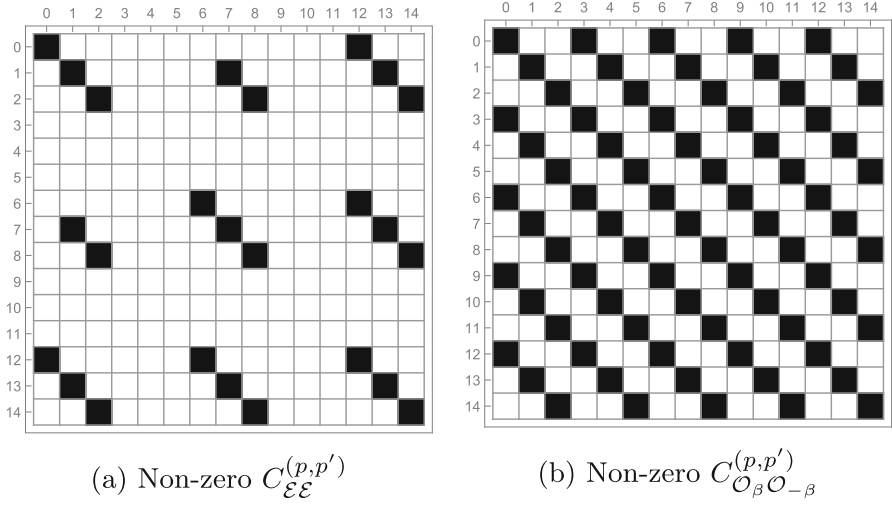


Fig. 2. The non-zero three-point function coefficients are shown. Rows and columns label (p, p') . Left: between two edge operators. Right: between two vertex operators

The expansion around $x = \bar{x} = 0$ allows us to obtain information about the primary operator spectrum and fusion rules of the operators that appear in both the $\mathcal{O}_\beta \times \mathcal{O}_{-\beta}$ and $\mathcal{E} \times \mathcal{E}$ expansions. The hypergeometric functions appearing above are regular around $x = 0$. The expansion of (130) around zero can thus be written

$$G_{34}^{21}(x) = |x|^{-4/3} \sum_{m,n=0}^{\infty} a_{m,n} x^{m/3} \bar{x}^{n/3}. \quad (131)$$

Using (129), this expansion is of the form $|x|^{-4\Delta_{\mathcal{E}}} x^{\Delta_{\mathcal{P}}+k} \bar{x}^{\bar{\Delta}_{\mathcal{P}}+\bar{k}}$, where $k, \bar{k}$ are non-negative integers. Since $\Delta_{\mathcal{E}} = 1/3$ we see that $\Delta_{\mathcal{P}}, \bar{\Delta}_{\mathcal{P}}$ can only be multiples of $1/3$. This must be equal to (127), which can now be written as

$$G_{34}^{21}(x) = |x|^{-4/3} \sum_{\substack{p,p', \\ m,n=0}}^{\infty} C_{\mathcal{E}\mathcal{E}}^{(p,p')} C_{\mathcal{O}_\beta\mathcal{O}_{-\beta}}^{(p,p')} \mathcal{F}_m^{(p)} \mathcal{F}_n^{(p')} x^{m/3} \bar{x}^{n/3}. \quad (132)$$

By comparing the last two equations, we determine the products of three-point function coefficients at any desired order. Together with the three-point coefficients determined in [7], using [33], we can uniquely determine the coefficients involving edge operators which also fuse onto vertex operators. Figure 2 shows the non-zero three-point coefficients $C_{\mathcal{E}\mathcal{E}}^{(p,p')}$ which appear in the Virasoro block expansion. The operators appearing in Fig. 2a are a subset of those in Fig. 2b, and only the operators which fuse onto both sets of operators can be discovered from (130).

The correct normalization of our operators and four-point function is ensured by

$$\begin{aligned} C_{\mathcal{E}\mathcal{E}}^{(0,0)} &\equiv C_{\mathcal{E}\mathcal{E}}^{\mathbb{1}} = 1 \\ C_{\mathcal{O}_\beta\mathcal{O}_{-\beta}}^{(0,0)} &\equiv C_{\mathcal{O}_\beta\mathcal{O}_{-\beta}}^{\mathbb{1}} = 1. \end{aligned} \quad (133)$$

Furthermore, we obtain the coefficients

$$C_{\mathcal{E}\mathcal{E}}^{(1,1)} \equiv C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} = \frac{1}{\sqrt{\lambda}} \frac{4 \cdot 2^{1/6} \cdot 3^{1/4} \cdot \sqrt{5} \pi^{3/2} \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)^3 \Gamma\left(\frac{7}{6}\right)} \quad (134)$$

$$C_{\mathcal{E}\mathcal{E}}^{(2,2)} \equiv C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}^{(2)}} = \sqrt{2}. \quad (135)$$

The complexity of these coefficients grows rapidly for larger (p, p') . The operator $\mathcal{E}^{(2)}$ can be identified with the higher order edge operator of conformal and anti-conformal dimensions $2/3$ defined in (102).

By rearranging the operators in the four-point function (130), one can easily show that the resulting four-point functions are crossing-symmetric. In particular, by exchanging operators 2 and 4, one may obtain information about the OPE of $\mathcal{O}_\beta \times \mathcal{E}$. The expansion in the cross-ratio in this channel shows logarithmic terms, which indicate the existence of degenerate operators in a logarithmic CFT. The logarithmic properties of the related $O(n)$ model have been studied, for example, in [34]. We do not investigate their relations to the BLS at this point.

Nevertheless, one can use $G_{32}^{41}(x) = G_{34}^{21}(1-x)$ to compute the fusion rules for $\mathcal{O}_\beta \times \mathcal{E}$, and in particular, the squares of three-point function coefficients $C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{P}}$ of all primaries $\mathcal{P}$. The expansion of $G_{34}^{21}(1-x)$ analogous to (132) allows us to obtain the following operators in the OPE

$$\mathcal{O}_\beta(z) \times \mathcal{E}(z') = C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{O}_\beta} |z - z'|^{-2/3} \mathcal{O}_\beta(z) + C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{E}_\beta} \mathcal{E}_\beta(z) + \mathcal{R}, \quad (136)$$

where $\mathcal{R}$ contains all the remaining terms in the expansion, $C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{O}_\beta} = C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}}$ and

$$\left(C_{\mathcal{O}_\beta \mathcal{E}}^{\mathcal{E}_\beta}\right)^2 = \frac{1 + \cos \beta}{2}. \quad (137)$$

The operator $\mathcal{E}_\beta$ is the $k = 1$ case of the charged edge operators defined in (115), with conformal and anti-conformal dimension $\Delta(\beta) + 1/3$.

7.2. The three-point function of the edge operator. In this section, we show how to compute the three-point function coefficient $C_{\mathcal{E}\mathcal{E}}^{(1,1)} \equiv C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}}$, which was derived in the previous section from the conformal block expansion, by applying the OPE of two edge operators. This computation is a special case of the general expansion (132), and shows the inner workings of the general method.

Using the general expression for the three-point function of a conformal primary operator and (79), we have

$$\begin{aligned} \langle \mathcal{E}(z_1) \mathcal{E}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}} &= C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} |z_{12}|^{-2/3} |z_{13}|^{-2/3} |z_{23}|^{-2/3} \\ &= C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} |z_{12}|^{-4/3} |z_{23}|^{-2/3} (1 + O(|z_{23}|)) \\ &= C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} \langle \mathcal{E}(z_1) \mathcal{E}(z_2) \rangle_{\mathbb{C}} |z_{23}|^{-2/3} (1 + O(|z_{23}|)) \\ &= \left\langle \mathcal{E}(z_1) \left[C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} |z_{23}|^{-2/3} \mathcal{E}(z_2) + O(|z_{23}|^{1/3}) \right] \right\rangle_{\mathbb{C}}. \end{aligned} \quad (138)$$

Additionally, using (85) and (96) we see that, for $\beta = \pi$,

$$\langle \mathcal{O}_\pi(z_1) \mathcal{O}_{-\pi}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}}$$

$$= |z_{12}|^{-4\lambda/5} Z_{\text{twist}} + 4\lambda |z_{12}|^{-4\lambda/5} \hat{\alpha}_{z_1|z_2;\mathbb{C}}^{z_3} \hat{\alpha}_{z_1|z_2;\mathbb{C}}^{z_4}. \quad (139)$$

The second term on the right-hand side is not divergent as $z_4 \rightarrow z_3$, while we see from (94) that $\lim_{z_4 \rightarrow z_3} |z_{34}|^{4/3} Z_{\text{twist}} = 1$, so that

$$\lim_{z_4 \rightarrow z_3} |z_{34}|^{4/3} \langle \mathcal{O}_\pi(z_1) \mathcal{O}_{-\pi}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}} = |z_{12}|^{-4\lambda/5} = \langle \mathcal{O}_\pi(z_1) \mathcal{O}_{-\pi}(z_2) \rangle_{\mathbb{C}}. \quad (140)$$

Combining these observations gives the OPE

$$\mathcal{E}(z) \times \mathcal{E}(z') = |z - z'|^{-4/3} \mathbb{1} + C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} |z - z'|^{-2/3} \mathcal{E}(z) + \mathcal{R}, \quad (141)$$

where $\mathcal{R}$ contains all the remaining terms in the expansion.

Plugging this OPE into $\langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}}$ and using (64) gives

$$\begin{aligned} & \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \mathcal{E}(z_4) \rangle_{\mathbb{C}} \\ &= \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \rangle_{\mathbb{C}} |z_{34}|^{-4/3} + C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{E}(z_3) \rangle_{\mathbb{C}} |z_{34}|^{-2/3} \\ & \quad + \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{R} \rangle_{\mathbb{C}} \\ &= |z_{12}|^{-4\Delta(\beta)} |z_{34}|^{-4/3} + C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathcal{E}} |z_{12}|^{-4\Delta(\beta)} \left| \frac{z_{12}}{z_{13}z_{23}} \right|^{2/3} |z_{34}|^{-2/3} \\ & \quad + \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{R} \rangle_{\mathbb{C}}. \end{aligned} \quad (142)$$

For $\beta = \pi$, comparing with (139) gives

$$\begin{aligned} & |z_{12}|^{-4\lambda/5} |z_{34}|^{-4/3} + C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} C_{\mathcal{O}_\pi \mathcal{O}_{-\pi}}^{\mathcal{E}} |z_{12}|^{-4\lambda/5} \left| \frac{z_{12}}{z_{13}z_{23}} \right|^{2/3} |z_{34}|^{-2/3} \\ & + \langle \mathcal{O}_\beta(z_1) \mathcal{O}_{-\beta}(z_2) \mathcal{R} \rangle_{\mathbb{C}} = |z_{12}|^{-4\lambda/5} Z_{\text{twist}} + 4\lambda |z_{12}|^{-4\lambda/5} \hat{\alpha}_{z_1|z_2;\mathbb{C}}^{z_3} \hat{\alpha}_{z_1|z_2;\mathbb{C}}^{z_4}. \end{aligned} \quad (143)$$

Using the expression (94) for Z_{twist} , we can write

$$\begin{aligned} Z_{\text{twist}} &= \left| \frac{z_{13}z_{24}}{z_{23}z_{14}} \right|^{2/3} \left| {}_2F_1 \left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}, x \right) \right|^2 |z_{34}|^{-4/3} \\ & \quad - \left| \frac{z_{12}}{z_{23}z_{14}} \right|^{2/3} \frac{4\Gamma(\frac{2}{3})^6}{\Gamma(\frac{4}{3})^2 \Gamma(\frac{1}{3})^4} \left| {}_2F_1 \left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}, x \right) \right|^2 |z_{34}|^{-2/3}. \end{aligned} \quad (144)$$

Plugging this into (143) and observing that

$$\lim_{z_4 \rightarrow z_3} \left| \frac{z_{13}z_{24}}{z_{23}z_{14}} \right|^{2/3} \left| {}_2F_1 \left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}, x \right) \right|^2 = 1, \quad (145)$$

shows that

$$\begin{aligned} C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} C_{\mathcal{O}_\pi \mathcal{O}_{-\pi}}^{\mathcal{E}} &= -\frac{4\Gamma(\frac{2}{3})^6}{\Gamma(\frac{4}{3})^2 \Gamma(\frac{1}{3})^4} \lim_{z_3 \rightarrow z_4} \left| {}_2F_1 \left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}, x \right) \right|^2 \\ &= -\frac{4\Gamma(\frac{2}{3})^6}{\Gamma(\frac{4}{3})^2 \Gamma(\frac{1}{3})^4}. \end{aligned} \quad (146)$$

Finally, using (66), after some simplification we obtain

$$C_{\mathcal{E}\mathcal{E}}^{\mathcal{E}} = \frac{1}{\sqrt{\lambda}} \frac{4 \cdot 2^{1/6} \cdot 3^{1/4} \cdot \sqrt{5} \pi^{3/2} \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)^3 \Gamma\left(\frac{7}{6}\right)}, \quad (147)$$

which indeed coincides with (134).

8. Central Charge

Given an explicit form of a four-point function of a two dimensional CFT, together with sufficient knowledge of the operator spectrum, one can determine the central charge c of the theory. We will now use the previous result (85) for the case of the full plane to confirm that $c = 2\lambda$ in the BLS, as was derived, for instance, in [4].

In every two dimensional CFT, the two-point function of the energy-momentum tensor to leading order is fixed by conformal invariance to be

$$\langle T(z_1)T(z_2) \rangle_{\mathbb{C}} = \frac{c/2}{z_{12}^4}. \quad (148)$$

The energy-momentum tensor can be understood as the level-2 Virasoro descendant of the identity operator

$$(L_{-2}\mathbb{1})(z) = \frac{1}{2\pi i} \oint_z dw \frac{1}{w-z} T(w) = T(z), \quad (149)$$

where the integral is along any contour around the point z , and L_n are the generators of the Virasoro algebra. Its anti-holomorphic counterpart is analogously given by $\bar{T}(\bar{z}) = (\bar{L}_{-2}\bar{\mathbb{1}})(\bar{z})$.

Additionally, the OPE of two primary operators is generally given by (cf. [5], Sect. 6.6.3)

$$\begin{aligned} \mathcal{A}_1(z+\epsilon) \times \mathcal{A}_2(z) &= \sum_{\mathcal{P}} \sum_{\{k, \bar{k}\}} C_{12}^{\mathcal{P}} \beta_{12}^{\mathcal{P}\{k\}} \bar{\beta}_{12}^{\mathcal{P}\{\bar{k}\}} \epsilon^{\Delta_{\mathcal{P}} - \Delta_1 - \Delta_2 + K} \bar{\epsilon}^{\bar{\Delta}_{\mathcal{P}} - \bar{\Delta}_1 - \bar{\Delta}_2 + \bar{K}} \\ &\quad \cdot L_{-k_1} \dots L_{-k_N} \bar{L}_{-\bar{k}_1} \dots \bar{L}_{-\bar{k}_N} \mathcal{P}(z), \end{aligned} \quad (150)$$

where $C_{ij}^{\mathcal{P}}$ are three-point function coefficients, $K = \sum_{k_j \in \{k\}} k_j$ with $k_j \in \mathbb{N}$ is the descendant level, and $\beta_{ij}^{\mathcal{P}\{k\}}, \bar{\beta}_{ij}^{\mathcal{P}\{\bar{k}\}}$ are numerical coefficients that depend on the central charge and the conformal dimensions of the involved operators and are fully determined by the Virasoro algebra. The outer sum runs over all primary operators $\mathcal{P}$, and the inner sum is over all subsets $\{k\}, \{\bar{k}\}$ of the natural numbers. (This was the basis of the analysis of Sect. 7.)

Since the identity operator has non-zero OPE coefficient for both $\mathcal{O}_{\beta} \times \mathcal{O}_{-\beta}$ and $\mathcal{E} \times \mathcal{E}$, we can use (85) to obtain the central charge c by identifying the level-2 descendant of the identity.

We achieve this by applying the OPE twice to (85) and evaluating it in two equivalent ways. First, we expand the expression

$$\langle \mathcal{O}_{\beta}(z+\epsilon) \mathcal{O}_{-\beta}(z) \mathcal{E}(z' + \epsilon') \mathcal{E}(z') \rangle_{\mathbb{C}} \quad (151)$$

analytically around zero for $\epsilon, \bar{\epsilon}, \epsilon', \bar{\epsilon}'$. We then identify the term of order $(\epsilon\epsilon')^{-\Delta(\beta)-1/3+2}$ with the contribution from the algebraic expansion (150) at the same order in ϵ, ϵ' , which is

$$(\epsilon\epsilon')^{-\Delta(\beta)-1/3+2} C_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathbb{1}} C_{\mathcal{E}\mathcal{E}}^{\mathbb{1}} \beta_{\mathcal{O}_\beta \mathcal{O}_{-\beta}}^{\mathbb{1}\{2\}} \beta_{\mathcal{E}\mathcal{E}}^{\mathbb{1}\{2\}} \langle (L_{-2}\mathbb{1})(z)(L_{-2}\mathbb{1})(z') \rangle_{\mathbb{C}}. \quad (152)$$

Generically, one expects contributions like $(L_{-1}\mathcal{A}^{(3,0)})(L_{-1}\mathcal{A}^{(3,0)})$ and $\mathcal{A}^{(6,0)}\mathcal{A}^{(6,0)}$ to appear, where $\mathcal{A}^{(p,p')}$ are primary operators of conformal dimensions $(p/3, p'/3)$. However, the previous analysis has shown their relevant three-point coefficients vanish (see e.g. Fig. 2a).

If the conformal dimensions of a pair of operators are equal, it can be shown that $\beta_{\mathcal{A}_1 \mathcal{A}_2}^{\mathbb{1}\{2\}} = 2\Delta_{\mathcal{A}_1}/c$, where $\Delta_{\mathcal{A}_1} = \Delta_{\mathcal{A}_2}$ is the conformal dimension of the operators [5]. We also note that $C_{\mathcal{A}_1 \mathcal{A}_2}^{\mathbb{1}}$ denotes the normalization of non-zero two-point functions, which is canonically chosen to be 1. Every quantity in (152) has thus been determined.

The analytic expansion of (151) yields (at the desired order)

$$(\epsilon\epsilon')^{-\Delta(\beta)-1/3+2} \frac{1}{30} \frac{1 - \cos \beta}{(z - z')^4}. \quad (153)$$

Using (148) and (149), (152) becomes (dropping the powers of ϵ and ϵ')

$$\frac{2\Delta(\beta)}{c} \frac{2\Delta_{\mathcal{E}}}{c} \langle T(z)T(z') \rangle = \frac{2}{3c} \frac{\lambda}{10} \frac{1 - \cos \beta}{(z - z')^4}, \quad (154)$$

where we used $\Delta(\beta) = \frac{\lambda}{10}(1 - \cos \beta)$, $\Delta_{\mathcal{E}} = 1/3$. Comparing (153) to (154) confirms the result that the BLS with intensity λ has central charge $c = 2\lambda$.

9. Conclusions and Future Work

In this work we identified all scalar operators that couple to the layering vertex operators $\mathcal{O}_\beta$. This leaves open the question of the nature of the operators with non-zero spin. Perhaps the most interesting is the operator with $k = 3, k' = 0$ and zero charge, which has dimensions $(1, 0)$. This is a (component of a) spin-1 current that should satisfy a conservation law and generate a conserved charge. Understanding the nature and role of this current may greatly clarify the structure of the spectrum of the CFT associated to the BLS.

Another question open to investigation is the torus partition function. By further exploiting the connection to the $O(n)$ model it seems possible that this can be computed. If so it would reveal the complete spectrum and degeneracies of the theory (modulo complications resulting from the lack of unitarity of the theory).

The theory as we have presented it has a continuous spectrum because the operator dimensions depend on the continuous parameters β . This is reminiscent of the vertex operators of the free boson. There, one can compactify the boson and obtain a discrete spectrum. An analogous procedure seems available here too, where we identify the layering number with itself modulo an integer. If this is indeed self-consistent it would render the spectrum discrete, which has a number of interesting implications that we intend to explore in future work.

The largest question is what place this Brownian loop soup conformal field theory has in the spectrum of previously known conformally invariant models. It appears to be

a novel, self-consistent, and rich theory in its own right, but its connections with the free field and the $O(n)$ model suggest that it may have ties to other theories that could be exploited to greatly advance our understanding of it.

Acknowledgments We are grateful to Sylvain Ribault for insightful comments on a draft of the manuscript. We thank an anonymous referee for a careful reading of the manuscript and for several useful comments and suggestions that improved its readability.

Declarations

Conflict of Interest Statement The work of M.K. is partially supported by the NSF through the grant PHY-1820814. No new data were created during the study.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix A: Additional Lemmas and Proofs

In this section we collect all the proofs that do not appear in the main body of the paper. We first show that the correlations functions $\langle E_\varepsilon(z_1) \dots E_\varepsilon(z_n) \rangle_D$ are well defined, a necessary step to state Lemma 2.1, proved next in this appendix, and Theorem 2.3. We refer to Sect. 2 for the notation used here, the statements of Lemma 2.1, as well as the statement and proof of Theorem 2.3. Additionally, we remind the reader of the following definitions from Sect. 3.

For any $\delta > 0$, let $\mathcal{L}^\delta$ denote a Brownian loop soup in D with intensity λ and cutoff $\delta > 0$, obtained by taking the usual Brownian loop soup and removing all loops with diameter smaller than δ . We define $N_\varepsilon^\delta(z) \equiv n_z^\varepsilon(\mathcal{L}^\delta)$ and $E_\varepsilon^\delta(z) \equiv N_\varepsilon^\delta(z) - \langle N_\varepsilon^\delta(z) \rangle_D$. Note that the random variables $N_\varepsilon^\delta(z)$ and $E_\varepsilon^\delta(z)$ are well defined because of the cutoffs $\varepsilon > 0$ and $\delta > 0$. The next lemma shows that, if we consider n -point functions of E_ε^δ for $n \geq 2$, the δ cutoff can be removed without the need to renormalize the n -point functions.

Lemma A.1. *For any collection of points $z_1, \dots, z_n \in D$ at distance greater than 2ε from each other, with $n \geq 2$, the following limit exists:*

$$\langle E_\varepsilon(z_1) \dots E_\varepsilon(z_n) \rangle_D := \lim_{\delta \rightarrow 0} \langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D. \quad (\text{A1})$$

Moreover, for all $\varepsilon, \delta > 0$ sufficiently small,

$$\langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D = \langle E_\varepsilon(z_1) \dots E_\varepsilon(z_n) \rangle_D. \quad (\text{A2})$$

Proof. For each $j = 1, \dots, n$, we can write

$$N_\varepsilon^\delta(z_j) = M_\varepsilon^\delta(z_j) + R_\varepsilon^\delta(z_j), \quad (\text{A3})$$

where

$$M_\varepsilon^\delta(z_j) := \sum_{\ell \in \mathcal{L}^\delta} \mathbf{I}(\ell \cap B_\varepsilon(z_j) \neq \emptyset, \ell \cap B_\varepsilon(z_k) = \emptyset \forall k \neq j), \quad (\text{A4})$$

$$R_\varepsilon^\delta(z_j) := \sum_{\ell \in \mathcal{L}^\delta} \mathbf{I}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \text{ and } \ell \cap B_\varepsilon(z_k) \neq \emptyset \text{ for at least one } k \neq j), \quad (\text{A5})$$

where $\mathbf{I}(\cdot)$ denotes the indicator function.

Now consider values of $\delta < \min_{k,m}(|z_k - z_m| - 2\varepsilon)$ with $k, m = 1, \dots, n$ and $m \neq k$, then all the loops from $\mathcal{L}$ that intersect $B_\varepsilon(z_j)$ and at least one other disk $B_\varepsilon(z_k)$ must have diameter larger than δ . Therefore, for δ sufficiently small, $R_\varepsilon^\delta(z_j)$ does not depend on δ and we can drop the superscript and write $R_\varepsilon(z_j)$ instead.

Defining $m_\varepsilon^\delta(z_j) := \langle M_\varepsilon^\delta(z_j) \rangle_D$ and $r_\varepsilon(z_j) := \langle R_\varepsilon(z_j) \rangle_D$, for values of δ sufficiently small we can write

$$\begin{aligned} \langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D &= \langle [M_\varepsilon^\delta(z_1) - m_\varepsilon^\delta(z_1) + R_\varepsilon(z_1) - r_\varepsilon(z_1)] E_\varepsilon^\delta(z_2) \dots E_\varepsilon^\delta(z_n) \rangle_D \\ &= \langle [M_\varepsilon^\delta(z_1) - m_\varepsilon^\delta(z_1)] E_\varepsilon^\delta(z_2) \dots E_\varepsilon^\delta(z_n) \rangle_D \\ &\quad + \langle [R_\varepsilon(z_1) - r_\varepsilon(z_1)] E_\varepsilon^\delta(z_2) \dots E_\varepsilon^\delta(z_n) \rangle_D. \end{aligned} \quad (\text{A6})$$

$M_\varepsilon^\delta(z_1)$ is independent of $E_\varepsilon^\delta(z_j)$ for all $j \neq 1$, so

$$\langle [M_\varepsilon^\delta(z_1) - m_\varepsilon^\delta(z_1)] E_\varepsilon^\delta(z_2) \dots E_\varepsilon^\delta(z_n) \rangle_D = 0 \quad (\text{A7})$$

and

$$\langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D = \langle [R_\varepsilon(z_1) - r_\varepsilon(z_1)] E_\varepsilon^\delta(z_2) \dots E_\varepsilon^\delta(z_n) \rangle_D. \quad (\text{A8})$$

Proceeding in the same way for all values of $j = 2, \dots, n$, we obtain

$$\langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D = \langle [R_\varepsilon(z_1) - r_\varepsilon(z_1)] \dots [R_\varepsilon(z_n) - r_\varepsilon(z_n)] \rangle_D, \quad (\text{A9})$$

which is independent of δ . □

Proof of Lemma 2.1. The random variables $(N_\varepsilon^\delta(z_1), \dots, N_\varepsilon^\delta(z_n))$ are jointly Poisson. If we let $\mathbf{v} = (v_1, \dots, v_n)$ be an n -dimensional vector with components $v_j = 0$ or 1 , following [35] we see that their joint distribution is captured by

$$N_\varepsilon^\delta(\mathbf{v}) := |\{\ell : \text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j : v_j = 1, \ell \cap B_\varepsilon(z_j) = \emptyset \forall j : v_j = 0\}|, \quad (\text{A10})$$

where $N_\varepsilon^\delta(\mathbf{v})$ is itself a Poisson random variable with parameter $\lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j : v_j = 1, \ell \cap B_\varepsilon(z_j) = \emptyset \forall j : v_j = 0)$. More precisely, using Theorem 2 of [35], we can write the joint probability generating function of $(N_\varepsilon^\delta(z_1), \dots, N_\varepsilon^\delta(z_n))$ as

$$\begin{aligned} h(x_1, \dots, x_n) &:= \left\langle x_1^{N_\varepsilon^\delta(z_1)}, \dots, x_n^{N_\varepsilon^\delta(z_n)} \right\rangle \\ &= \exp \left[\lambda \sum_{\substack{I \text{ subset } \{1, \dots, n\} \\ |I| \geq 1}} \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I, \ell \cap B_\varepsilon(z_j) = \emptyset \forall j \notin I) \right. \\ &\quad \cdot \left. \left(\prod_{j \in I} x_j - 1 \right) \right]. \end{aligned} \quad (\text{A11})$$

Letting $\mathcal{D}_k := \frac{\partial}{\partial x_k} - \lambda \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_k) \neq \emptyset)$, using (99) we have

$$\langle E_\varepsilon^\delta(z_1) \dots E_\varepsilon^\delta(z_n) \rangle_D = \prod_{k=1}^n \mathcal{D}_k h(x_1, \dots, x_n) \Big|_{x_k=1}. \quad (\text{A12})$$

Using an induction argument, one can show that

$$\sum_{\substack{I \text{ subset } \{1, \dots, n\} \\ |I| \geq 1}} \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I, \ell \cap B_\varepsilon(z_j) = \emptyset \forall j \notin I) \quad (\text{A13})$$

$$\begin{aligned} &= \emptyset \forall j \notin I \left(\prod_{j \in I} x_j - 1 \right) \\ &= \sum_{\substack{I \text{ subset } \{1, \dots, n\} \\ |I| \geq 1}} \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I) \prod_{j \in I} (x_j - 1). \end{aligned} \quad (\text{A14})$$

Hence,

$$\begin{aligned} &h(x_1, \dots, x_n) \\ &= \exp \left[\lambda \sum_{\substack{I \text{ subset } \{1, \dots, n\} \\ |I| \geq 1}} \mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I) \prod_{j \in I} (x_j - 1) \right] \\ &= 1 + \sum_{r=1}^{\infty} \lambda^r \sum_{\substack{I_1, \dots, I_r \\ \text{subsets of } \{1, \dots, n\}}} \prod_{l=1}^r \frac{1}{m(I_l)!} \\ &\quad \cdot \left(\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I_l) \prod_{j \in I_l} (x_j - 1) \right), \end{aligned} \quad (\text{A15})$$

where the second sum is over all *unordered* collections of subsets of $\{1, \dots, n\}$ not necessarily distinct (i.e., over multiset), and we have used the fact that the number of ways in which an unordered collection of r elements can be ordered is

$$\frac{r!}{\prod_{l=1}^r m(I_l)!}, \quad (\text{A16})$$

where $m(I_l)$ is the multiplicity of I_l in the multiset.

Considering the structure of the last expression, the definition of the differential operator $\mathcal{D}_k$, and the fact that in (A12) all derivatives $\frac{\partial}{\partial x_k}$ are evaluated at $x_k = 1$, we can differentiate term by term. It is clear that in the right-hand side of (A12) the only terms that survive are those for which the derivatives saturate the variables x_k . Moreover, Lemma A.1 implies that terms of the type $\mu^{\text{loop}}(\text{diam}(\ell) > \delta, \ell \cap B_\varepsilon(z_k))$ cannot be present in the right-hand side of (A12) because otherwise the limit $\delta \rightarrow 0$ would not exist. (One can reach the same conclusion by looking at (A15) and observing that terms containing subsets that are single points, i.e. $I_l = \{z_k\}$, disappear when applying $\mathcal{D}_k$.) These considerations single out all partitions Π of $\{1, \dots, n\}$ whose elements have cardinality at least 2.

Therefore, we obtain

$$\begin{aligned} \langle E_\varepsilon(z_1) \dots E_\varepsilon(z_n) \rangle_D &= \lim_{\delta \rightarrow 0} \prod_{k=1}^n \mathcal{D}_k h(x_1, \dots, x_n) \Big|_{x_k=1} \\ &= \sum_{\{I_1, \dots, I_r\} \in \Pi} \lambda^r \prod_{l=1}^r \mu_D^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \ \forall j \in I_l), \end{aligned} \quad (\text{A17})$$

which concludes the proof. $\square$

We conclude this appendix with two lemmas, the first one used in the proof of Lemma 2.2, the second one used in the proof of Theorem 6.2.

Lemma A.2. *For any $s > 0$ we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\vartheta_\varepsilon}{\vartheta_{\varepsilon/s}} = s^{2/3}. \quad (\text{A18})$$

Proof. If (A18) holds for $s < 1$, for $s > 1$, letting $r = 1/s$, by scale invariance we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\vartheta_\varepsilon}{\vartheta_{\varepsilon/s}} = \lim_{\varepsilon \rightarrow 0} \frac{\vartheta_{\varepsilon/r}}{\vartheta_\varepsilon} = r^{-2/3} = s^{2/3}. \quad (\text{A19})$$

Hence, it is enough to prove (A18) for $s < 1$ and so in the rest of the proof we assume that $s < 1$. We use the notation introduced in the proof of Lemma 2.2 and further let $\vartheta(\varepsilon, s)$ denote the probability of a three-arm event in an annulus with inner radius ε and outer radius s . (In particular, $\vartheta(\varepsilon, 1) \equiv \vartheta_\varepsilon$.) We will show that

$$\lim_{\varepsilon \rightarrow 0} \frac{\vartheta(\varepsilon, 1)}{\vartheta(\varepsilon, s)} = s^{2/3}. \quad (\text{A20})$$

By scale invariance, this implies that

$$\lim_{\varepsilon \rightarrow 0} \frac{\vartheta(\varepsilon, 1)}{\vartheta(\varepsilon/s, 1)} = s^{2/3}, \quad (\text{A21})$$

as desired. For any $\varepsilon < s < 1$, we will let $\vartheta(\varepsilon, 1 \mid \varepsilon, s)$ denote the conditional probability of a three-arm event in $A_{\varepsilon,1}(0)$, given the existence of a three-arm event in $A_{\varepsilon,s}(0)$. The existence of the limit in (A20) follows from the scale invariance of the scaling limit of percolation. Using the notation introduced in the proof of Lemma 2.2, the scale invariance of the percolation scaling limit implies the scale invariance of ν^ε , which allows us to write

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\vartheta(\varepsilon, 1)}{\vartheta(\varepsilon, s)} &= \lim_{\varepsilon \rightarrow 0} \vartheta(\varepsilon, 1 \mid \varepsilon, s) \\ &= \lim_{\varepsilon \rightarrow 0} \int \mathbb{P}(H|\mathcal{I}) d\nu^{\varepsilon/s}(\mathcal{I}) \\ &= \int \mathbb{P}(H|\mathcal{I}) d\nu(\mathcal{I}) =: L, \end{aligned} \quad (\text{A22})$$

where H is the event that a loop responsible for a three-arm event in $A_{\varepsilon/s,1}(0)$ reaches $\partial B_{1/s}(0)$, thus producing a three-arm event in $A_{\varepsilon/s,1/s}(0)$, which has the same probability as a three-arm event in $A_{\varepsilon,1}(0)$. Now that we know that the limit exists, (A20) can be obtained as in the proof of the second limit in Equation (4.28) of Proposition 4.9 of [36]. We repeat the argument here for the reader's convenience. It is known that $\vartheta_\varepsilon = \varepsilon^{2/3+o(1)}$, where $o(1)$ goes to zero as $\varepsilon \rightarrow 0$, so that

$$\lim_{n \rightarrow \infty} \frac{\log \vartheta(s^n, 1)}{n} = \log s^{2/3}. \quad (\text{A23})$$

Now note that $\vartheta(s^n, 1)$ can be written as

$$\vartheta(s^n, 1) = \frac{\vartheta(s^n, 1)}{\vartheta(s^n, s)} \frac{\vartheta(s^{n-1}, 1)}{\vartheta(s^{n-1}, s)} \cdots \frac{\vartheta(s, 1)}{1}, \quad (\text{A24})$$

which implies that

$$\frac{\log \vartheta(s^n, 1)}{n} = \frac{1}{n} \sum_{j=1}^n \log \frac{\vartheta(s^j, 1)}{\vartheta(s^j, s)}. \quad (\text{A25})$$

Since $s < 1$, using (A22), we have

$$\lim_{j \rightarrow \infty} \log \frac{\vartheta(s^j, 1)}{\vartheta(s^j, s)} = \log L. \quad (\text{A26})$$

By convergence of the Cesàro mean, the right-hand side of (A25) converges to $\log L$, so (A25) and (A23) imply $\log L = \log s^{2/3}$, which concludes the proof. $\square$

Proof of Lemma 6.1. This proof is similar to that of Lemma 2.1. With the notation introduced in the proof of Lemma 2.1, we have that

$$\left\langle E_\varepsilon^{(k_1); \delta}(z_1) \cdots E_\varepsilon^{(k_n); \delta}(z_n) \right\rangle_D = \prod_{j=1}^n \mathcal{D}_j^{k_j} h(x_1, \dots, x_n) \Big|_{x_j \equiv 1}. \quad (\text{A27})$$

Considering the structure of (A15), the definition of the differential operator $\mathcal{D}_j$, and the fact that in (A27) all derivatives $\frac{\partial}{\partial x_j}$ are evaluated at $x_j = 1$, it is clear that in the

right-hand side of (A27) the only terms that survive are those for which the derivatives saturate the variables x_j . Moreover, the structure of (A15) implies that all terms containing subsets that are single points, i.e. $I_l = \{z_j\}$, disappear when applying D_j . These considerations imply that the only non-zero terms are those corresponding to multisets $M \in \mathcal{M}$. Note also that, when $\frac{\partial}{\partial x_j}$ is applied k_j times to $h(x_1, \dots, x_n)$, as prescribed by $\mathcal{D}_j^{k_j}$ it produces a multiplicative factor $k_j!$ for each $j = 1, \dots, n$.

Therefore, if the vector $\mathbf{k} = (k_1, \dots, k_n)$ is such that $\mathcal{M} = \emptyset$, we obtain

$$\begin{aligned} \langle E_\varepsilon(z_1) \dots E_\varepsilon(z_n) \rangle_D &= \lim_{\delta \rightarrow 0} \prod_{j=1}^n \mathcal{D}_j^{k_j} h(x_1, \dots, x_n) \Big|_{x_j \equiv 1} \\ &= \left(\prod_{j=1}^n k_j! \right) \sum_{M \in \mathcal{M}} \lambda^{\sum_{S \in M} m_M(S)} \prod_{S \in M} \frac{1}{m_M(S)!} \left(\mu_D^{\text{loop}}(\ell \cap B_\varepsilon(z_j) \neq \emptyset \forall j \in I_S) \right)^{m_M(S)}, \end{aligned} \quad (\text{A28})$$

otherwise we get zero, as required. $\square$

References

1. Lawler, G.F., Werner, W.: The Brownian loop soup. *Probab. Theory Relat. Fields* **128**(4), 565–588 (2004). <https://doi.org/10.1007/s00440-003-0319-6>
2. Symonzik, K.: Euclidean quantum field theory. I. Equations for a scalar model. *J. Math. Phys.* **7**(3), 510–525 (1966). <https://doi.org/10.1063/1.1704960>
3. Freivogel, B., Kleban, M.: A conformal field theory for eternal inflation. *JHEP* **12**, 019 (2009). <https://doi.org/10.1088/1126-6708/2009/12/019>. [arXiv:0903.2048](https://arxiv.org/abs/0903.2048) [hep-th]
4. Camia, F., Gandolfi, A., Kleban, M.: Conformal correlation functions in the Brownian loop soup. *Nucl. Phys. B* **902**, 483–507 (2016). <https://doi.org/10.1016/j.nuclphysb.2015.11.022>
5. Di Francesco, P., Mathieu, P., Sénéchal, D.: *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer, New York (1997). <https://doi.org/10.1007/978-1-4612-2256-9>
6. Camia, F., Gandolfi, A., Peccati, G., Reddy, T.R.: Brownian loops, layering fields and imaginary Gaussian multiplicative Chaos. *Commun. Math. Phys.* **381**(3), 889–945 (2021). <https://doi.org/10.1007/s00220-020-03932-9>
7. Camia, F., Foit, V.F., Gandolfi, A., Kleban, M.: Exact correlation functions in the Brownian Loop Soup. *J. High Energy Phys.* **2020**(7), 1 (2020). [https://doi.org/10.1007/jhep07\(2020\)067](https://doi.org/10.1007/jhep07(2020)067)
8. Foit, V.F., Kleban, M.: New Recipes for Brownian Loop Soups (2020). [arXiv:2007.01869](https://arxiv.org/abs/2007.01869) [math-ph]
9. Borcherds, R.E.: Vertex algebras, Kac-Moody algebras, and the monster. *Proc. Nat. Acad. Sci.* **83**, 3068–3071 (1986). <https://doi.org/10.1073/pnas.83.10.3068>
10. Borcherds, R.E.: Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.* **109**(1), 405–444 (1992). <https://doi.org/10.1007/bf01232032>
11. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys. B* **241**, 333–380 (1984). [https://doi.org/10.1016/0550-3213\(84\)90052-X](https://doi.org/10.1016/0550-3213(84)90052-X)
12. Henkel, M.: *Conformal Invariance and Critical Phenomena*. Springer, Berlin (1999). <https://doi.org/10.1007/978-3-662-03937-3>
13. Cardy, J.: *Conformal Field Theory and Statistical Mechanics* (2008). [arXiv:0807.3472](https://arxiv.org/abs/0807.3472) [cond-mat.stat-mech]
14. Ribault, S.: *Conformal Field Theory on the Plane* (2014). [arXiv:1406.4290](https://arxiv.org/abs/1406.4290) [hep-th]
15. Mohaupt, T.: The Free Massless Scalar Field on the Complex Plane, pp. 49–62 (2022). <https://doi.org/10.1017/9781108611619.009>
16. Segal, G.B.: In: Bleuler, K., Werner, M. (eds.) *The Definition of Conformal Field Theory*, pp. 165–171. Springer, Dordrecht (1988). https://doi.org/10.1007/978-94-015-7809-7_9
17. Frenkel, I.B., Lepowsky, J., Meurman, A.: *Vertex Operator Algebras and the Monster*. Pure and Applied Mathematics. Academic Press, San Diego (1989)
18. Frenkel, I.B., Huang, Y.-Z., Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules. *Memoirs Am. Math. Soc.* **104**(494), 1 (1993). <https://doi.org/10.1090/memo/0494>

19. Kac, V.: Vertex Algebras for Beginners. University lecture series, American Mathematical Society, Providence, Rhode Island (1998)
20. Wassermann, A.J.: Operator algebras and conformal field theory. In: Chatterji, S.D. (ed.) Proceedings of the International Congress of Mathematicians, pp. 966–979. Birkhäuser Basel, Basel (1995)
21. Gabbiani, F., Fröhlich, J.: Operator algebras and conformal field theory. *Commun. Math. Phys.* **155**(3), 569–640 (1993). <https://doi.org/10.1007/bf02096729>
22. van de Brug, T., Camia, F., Lis, M.: Spin systems from loop soups. *Electron. J. Probab.* **23**, 1–17 (2018). <https://doi.org/10.1214/18-EJP200>
23. Camia, F., Garban, C., Newman, C.M.: Planar Ising magnetization field I. Uniqueness of the critical scaling limit. *Ann. Probab.* **1**, 528–571 (2015). <https://doi.org/10.1214/13-AOP881>
24. Camia, F., Newman, C.M.: Two-dimensional critical percolation: the full scaling limit. *Commun. Math. Phys.* **268**(1), 1–38 (2006). <https://doi.org/10.1007/s00220-006-0086-1>
25. Werner, W.: The conformally invariant measure on self-avoiding loops. *J. Am. Math. Soc.* **21**(1), 137–169 (2008). <https://doi.org/10.1090/S0894-0347-07-00557-7>
26. Lawler, G.F., Schramm, O., Werner, W.: The dimension of the planar Brownian frontier is 4/3. *Math. Res. Lett.* **8**(4), 401–411 (2001). <https://doi.org/10.4310/mrl.2001.v8.n4.a1>
27. Simmons, J.J.H., Cardy, J.: Twist operator correlation functions in $O(n)$ loop models. *J. Phys. A: Math. Theor.* **42**(23), 235001 (2009). <https://doi.org/10.1088/1751-8113/42/23/235001>
28. Nacu, S., Werner, W.: Random soups, carpets and fractal dimensions. *J. Lond. Math. Soc.* **83**(3), 789–809 (2011). <https://doi.org/10.1112/jlms/jdq094>
29. Camia, F., Newman, C.: SLE_6 and CLE_6 from critical percolation. In: Pinsky, M., Birnir, B. (eds.) Probability, geometry and integrable systems. MSRI Publications, vol. 55, pp. 103–130 (2008). <https://doi.org/10.1007/s00440-006-0049-7>
30. Smirnov, S., Werner, W.: Critical exponents for two-dimensional percolation. *Math. Res. Lett.* **8**(6), 729–744 (2001). <https://doi.org/10.4310/MRL.2001.v8.n6.a4>
31. van de Brug, T., Camia, F., Lis, M.: Random walk loop soups and conformal loop ensembles. *Probab. Theory Relat. Fields* **166**(1–2), 553–584 (2015). <https://doi.org/10.1007/s00440-015-0666-0>
32. Camia, F., Jan, Y.L., Reddy, T.R.: In: Vares, M.E., Fernández, R., Fontes, L.R., Newman, C.M. (eds.) Limit Theorems for Loop Soup Random Variables, pp. 219–237. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-60754-8_11
33. Headrick, M.: Mathematica packages. <http://people.brandeis.edu/~headrick/Mathematica/index.html>. Accessed 2019 August 05
34. Gorbenko, V., Zan, B.: Two-dimensional $O(n)$ models and logarithmic CFTs. *J. High Energy Phys.* **2020**(10), 1 (2020). [https://doi.org/10.1007/jhep10\(2020\)099](https://doi.org/10.1007/jhep10(2020)099)
35. Kawamura, K.: The structure of multivariate Poisson distribution. *Kodai Math. J.* **2**(3), 337–345 (1979). <https://doi.org/10.2996/kmj/1138036064>
36. Garban, C., Pete, G., Schramm, O.: Pivotal, cluster, and interface measures for critical planar percolation. *J. Am. Math. Soc.* **26**(4), 939–1024 (2013). <https://doi.org/10.1090/S0894-0347-2013-00772-9>

Communicated by J. Ding