

Extremizers for Adjoint Restriction to a Pair of Reflected Paraboloids

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December 21, 2023

Abstract

We consider the adjoint restriction inequality associated to the hypersurface $\{(\tau, \xi) : \tau = \pm|\xi|^2, \xi \in \mathbb{R}^d\}$ at the Stein-Tomas exponent. Extremizers exist in all dimensions and extremizing sequences are precompact modulo symmetries conditional on a certain inequality, which we verify in the case $d \in \{1, 2\}$.

1 Introduction

Fix $d \in \mathbb{N}$ and define

$$\mathcal{E}f(t, x) = \int e^{i(t, x)(|\xi|^2, \xi)} f(\xi) d\xi. \quad (1)$$

The adjoint Fourier restriction conjecture for the paraboloid is that

$$\sup_{f \in L^p} \frac{\|\mathcal{E}f\|_q}{\|f\|_p} = A_p < \infty \quad (2)$$

for $q > p$ and $q = \frac{d+2}{d}p'$. We will call p and q for which (2) holds “valid.”

Beyond boundedness, one might ask if there exists $f \in L^p$ such that $\|\mathcal{E}f\|_q/\|f\|_p = A_p$ and furthermore, if sequences $\{f_n\} \subset L^p$ such that $\lim \|\mathcal{E}f_n\|_q/\|f_n\|_p = A_p$ have any special properties.

This paper answers these questions for the adjoint restriction inequality for the union of a pair of reflected paraboloids. We prove that extremizers exist in all dimensions whenever the normal conjecture holds, and that the previously mentioned sequences are precompact modulo certain symmetries of the new operator provided $p = 2$ and the operator norm is “large enough.” Finally, we verify the hypothesis on the operator norm for $d \in \{1, 2\}$.

For the standard single paraboloid in lower dimensions ($d \in \{1, 2\}$) there are particularly satisfactory answers. Radial Gaussians are the unique extremizers for $\mathcal{E}$ at the Stein-Tomas exponent ([11], [14], [3]), and it is conjectured that this is true for all dimensions. Christ and Quilodr  n ([7]) proved that this is only possible for $p \in \{1, 2\}$.

In higher dimensions or away from the Stein-Tomas exponent, the method of profile decompositions and mass concentration have been the most successful ([4], [16], [2],

[5], [15], [20], [22]). Since the method analyzes extremizing sequences, it also yields precompactness of extremizing sequences modulo symmetries of the operator. The profile decomposition approach has yielded the existence of extremizers in all dimensions but without an explicit characterization ([20]). Stovall extended this approach in [22] to prove that extremizers exist in all dimensions for all valid exponents by truncating L^p functions in a controlled way and then applying the L^2 theory. Our proof begins with a profile decomposition mainly modelled after Killip and Viřan's formulation in [15].

There are similar results for other restriction inequalities, for example restriction to S^d . Extremizers exist and are characters multiplied by constants for $d \in \{1, 2\}$ at the Stein-Tomas exponent ([21], [9], [10]) as well as for some higher dimensions and even exponents ([6], [18]). The results establishing the precompactness of extremizing sequences of Christ and Shao in [8] ($d = 2$) and Frank, Lieb, and Sabin in [13] ($d > 2$ conditional on a conjecture) are particularly relevant to our setting. Specifically, we borrow the strategy of relating the mass of individual bubbles to the value of the operator norm as well as some computations from [13].

In this article, we address the same questions of existence of extremizers and characterization of extremizing sequences for the adjoint restriction inequality associated to the union of a pair of reflected paraboloids. Let $P = \{(\tau, \xi) : \tau = |\xi|^2\}$ and $P_- = \{(\tau, \xi) : \tau = -|\xi|^2\}$. We pull back d -dimensional Lebesgue measure through the projection map $\mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ to a measure $d\sigma$ supported on P and $d\sigma'$ supported on P_- . We will analyze the operator

$$\begin{aligned} \mathcal{E}_\pm(f, g)(t, x) &:= \mathcal{E}f(t, x) + \mathcal{E}_-g(t, x) \\ &:= \int_P e^{i(t, x)(\tau, \xi)} f(\xi) d\sigma(\tau, \xi) + \int_{P_-} e^{i(t, x)(\tau, \xi)} g(\xi) d\sigma'(\tau, \xi). \end{aligned}$$

It can also be expressed as

$$\mathcal{E}f(t, x) + \mathcal{E}_-g(t, x) = \int_{\mathbb{R}^d} e^{i(t, x) \cdot (|\xi|^2, \xi)} f(\xi) + e^{i(t, x) \cdot (-|\xi|^2, \xi)} g(\xi) d\xi.$$

When we write $f(\xi)$ or just f , we mean the function $\mathbb{R}^d \rightarrow \mathbb{C}$, whereas $f d\sigma$ will represent the measure $f d\xi$ lifted to P . Similarly, $g d\sigma'$ will represent the measure $g d\xi$ lifted to P_- .

Symmetries and Definitions

Let $\mathbf{S}_+ \subset \text{Iso}(L^p(\mathbb{R}^d))$ and $\mathbf{T}_+ \subset \text{Iso}(L^q(\mathbb{R}^{d+1}))$ be the subgroups generated by

	$Sf(\xi)$	$T\mathcal{E}f(t, x)$
Scaling	$\lambda^{d/p} f(\lambda\xi)$	$\lambda^{-(d+2)/q} \mathcal{E}f(\lambda^{-2}t, \lambda^{-1}x)$
Frequency Translation	$f(\xi - \xi')$	$e^{i(t \xi' ^2 + x \cdot \xi')} \mathcal{E}f(t, x + 2t\xi')$
Spacetime Translation	$e^{i(t_0, x_0) \cdot (\xi ^2, \xi)} f(\xi)$	$\mathcal{E}f(t + t_0, x + x_0)$

They are distinguished by the fact that $\mathcal{E} \circ \mathbf{S}_+ = \mathbf{T}_+ \circ \mathcal{E}$ and each element generates non-compact orbits in L^p or L^q .

Using these generators, we can write any symmetry $S \in \mathbf{S}_+$ as

$$Sf(\xi) = \lambda^{d/p} e^{i(t_0, x_0)(|\lambda\xi - \xi'|^2, \lambda\xi - \xi')} f(\lambda\xi - \xi'), \quad (3)$$

and the corresponding $T \in \mathbf{T}_+$ as

$$TF(t, x) = \lambda^{-(d+2)/q} e^{i(\lambda^{-2}t|\xi'|^2 + \lambda^{-1}x \cdot \xi')} F(\lambda^{-2}t + t_0, \lambda^{-1}x + x_0 + 2\lambda^{-2}t\xi'), \quad (4)$$

for some $\lambda \in \mathbb{R}^+$, $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, and $\xi' \in \mathbb{R}^d$.

Let $\mathbf{S}_- \subset \text{Iso}(L^p(\mathbb{R}^d))$ and $\mathbf{T}_- \subset \text{Iso}(L^q(\mathbb{R}^{d+1}))$ be the subgroups generated by

	$Sg(\xi)$	$T\mathcal{E}_-g(t, x)$
Scaling	$\lambda^{d/p}g(\lambda\xi)$	$\lambda^{-(d+2)/q}\mathcal{E}_-g(\lambda^{-2}t, \lambda^{-1}x)$
Frequency Translation	$g(\xi - \xi')$	$e^{i(-t \xi' ^2 + x \cdot \xi')}\mathcal{E}_-g(t, x - 2t\xi')$
Spacetime Translation	$e^{i(t_0, x_0)(- \xi ^2, \xi)}g(\xi)$	$\mathcal{E}g(t + t_0, x + x_0)$.

Note that $\mathbf{T}_- \circ \mathcal{E}_- = \mathcal{E}_- \circ \mathbf{S}_-$ here as well.

Write any $S \in \mathbf{S}_-$ as

$$Sg(\xi) = \lambda^{d/p} e^{i(-t_0, x_0)(|\lambda\xi - \xi'|^2, \lambda\xi - \xi')} g(\lambda\xi - \xi') \quad (5)$$

for some $\lambda \in \mathbb{R}^+$, $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$, and $\xi' \in \mathbb{R}^d$. Denote its matching $T \in \mathbf{T}_-$ by

$$TG(t, x) = \lambda^{-(d+2)/q} e^{i(-\lambda^{-2}t|\xi'|^2 + \lambda^{-1}x \cdot \xi')} G(\lambda^{-2}t + t_0, \lambda^{-1}x + x_0 - 2\lambda^{-2}t\xi'). \quad (6)$$

We also need notation to indicate whether or not two sequences of symmetries are asymptotically orthogonal in a weak sense.

Definition 1.1. Let $\{S_n\}, \{S'_n\} \subset \mathbf{S}_+$ and $\{R_n\}, \{R'_n\} \subset \mathbf{S}_-$ be the symmetries associated with the parameters $(\lambda_n, t_n, x_n, \xi_n)$, $(\lambda'_n, t'_n, x'_n, \xi'_n)$, $(\kappa_n, s_n, y_n, \eta_n)$, and $(\kappa'_n, s'_n, y'_n, \eta'_n)$ respectively. Then we say $\{S_n\} \perp \{R_n\}$ ($\{S_n\}$ is asymptotically orthogonal to $\{R_n\}$) if

1. $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\kappa_n} \in \{0, \infty\}$; or
2. $\lim_{n \rightarrow \infty} |\frac{\lambda_n}{\kappa_n} \eta_n + \xi_n| = \infty$; or
3. $\lim_{n \rightarrow \infty} |t_n - (\frac{\lambda_n}{\kappa_n})^2 s_n| + |x_n - \frac{\lambda_n}{\kappa_n} y_n - 2\frac{\lambda_n}{\kappa_n} s_n(\eta_n + \frac{\lambda_n}{\kappa_n} \xi_n)| = \infty$.

We say $\{S_n\} \perp \{S'_n\}$ if

1. $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda'_n} \in \{0, \infty\}$; or
2. $\lim_{n \rightarrow \infty} |\frac{\lambda_n}{\lambda'_n} \xi'_n - \xi_n| = \infty$; or
3. $\lim_{n \rightarrow \infty} |t_n - (\frac{\lambda_n}{\lambda'_n})^2 t'_n| + |x_n - \frac{\lambda_n}{\lambda'_n} x'_n + 2\frac{\lambda_n}{\lambda'_n} t'_n(\xi'_n - \frac{\lambda_n}{\lambda'_n} \xi_n)| = \infty$.

We define $\{R_n\} \perp \{R'_n\}$ in the same way.

Results

Let

$$A_p^\pm := \sup_{f,g \in L^p} \frac{\|\mathcal{E}f + \mathcal{E}_-g\|_q}{(\|f\|_p^p + \|g\|_p^p)^{1/p}}.$$

Since

$$\begin{aligned} \|\mathcal{E}f_1 + \mathcal{E}_-f_2\|_q &\leq \|\mathcal{E}f_1\|_q + \|\mathcal{E}_-f_2\|_q \\ &\leq A_p (\|f_1\|_p + \|f_2\|_p) \leq 2^{1/p'} A_p (\|f_1\|_p^p + \|f_2\|_p^p)^{1/p}, \quad (7) \end{aligned}$$

$A_p^\pm < \infty$ provided $A_p < \infty$. The theorem assumes only that the exponents p, q are valid. The second is proven for the Stein-Tomas exponent.

Theorem 1.2. 1. For all $d \in \mathbb{N}$,

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_p \leq A_p^\pm < 2^{1/p'} A_p.$$

2. If $d \in \{1, 2\}$,

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_p < A_p^\pm.$$

Theorem 1.3. 1. Let $\{(f_n, g_n)\} \subset L^2 \times L^2$ be such that $\|f_n\|_2^2 + \|g_n\|_2^2 = 1$ for all n and

$$\lim_{n \rightarrow \infty} \|\mathcal{E}f_n + \mathcal{E}g_n\|_q = A_2^\pm.$$

If

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_2 < A_2^\pm,$$

then there exist $\{Q_n\} \subset \mathbf{S}_+ \cap \mathbf{S}_-$ and $f, g \in L^2$ such that along a subsequence,

- (a) $\|f - Q_n f_n\|_2 \rightarrow 0$;
- (b) $\|g - Q_n g_n\|_2 \rightarrow 0$; and therefore
- (c) $\|\mathcal{E}f + \mathcal{E}_-g\|_q = A_2^\pm$.

2. For all $d \in \mathbb{N}$, there exist $f, g \in L^2$ normalized so that $\|f\|_2^2 + \|g\|_2^2 = 1$ such that

$$\|\mathcal{E}f + \mathcal{E}_-g\|_q = A_2^\pm.$$

The theorems immediately imply the following corollary.

Corollary 1.4. For $d \in \{1, 2\}$, extremizing sequences for $\mathcal{E}_\pm$ are precompact modulo the action of $\mathbf{S}_+ \cap \mathbf{S}_-$.

Our primary tool is the identity $\mathcal{E}_- f = \overline{\mathcal{E} \tilde{f}}$ where $\tilde{f}(\xi) = \overline{f(-\xi)}$. For an extremizing sequence $\{(f_n, g_n)\}$, we apply a standard profile decomposition modeled on the one from [15] to each sequence independently. We then bound the interactions between bubbles whose symmetries are asymptotically orthogonal in the sense of Definition 1.1. From this we deduce that almost all of the L^2 mass of $\{f_n\}$ and $\{g_n\}$ come from single bubbles whose symmetries are not asymptotically orthogonal.

There are two cases. Either the bubbles are translated to infinity in frequency space, or they remain bounded. An inequality from [13] shows that under the hypothesis of Theorem 1.3 part 1, we must be in the second case and therefore we can extract a convergent subsequence. We verify this condition computationally for $d \in \{1, 2\}$. If the bubbles translate off to infinity, no sequence of symmetries can make them both converge. However, we know the operator norm exactly in this case and are able to construct extremizers for $\mathcal{E}_\pm$ from extremizers for $\mathcal{E}$, completing the proof of Theorem 1.3.

In this paper, we can only calculate useful lower bounds on the operator norm of $\mathcal{E}_\pm : \ell^p(L^p) \rightarrow L^q$ in dimensions where we know extremizers exist for $\mathcal{E} : L^p \rightarrow L^q$ and have a particularly simple form, such as Gaussians. This is conjectured to hold in all dimensions at the Stein-Tomas exponent, but based on the result of Christ and Quilodr  n ([7]), it seems unlikely that this will be possible for other valid exponents. In addition, without further simplification the numerical computation must be run for each dimension individually rather than once for the general case.

Acknowledgements

This project was suggested and overseen by Betsy Stovall and supported in part by NSF DMS-1653264. The author would like to thank her for many helpful conversations and invaluable guidance in the writing of this paper.

2 Bounds on $A_p^\pm$

The purpose of this section is to prove Theorem 1.2 part 1. The first lemma resembles an observation of Allaire ([1]).

Lemma 2.1. Let $g \in L^q(\mathbb{R}^{1+d})$ and let $\varepsilon > 0$ be sufficiently small. Then there exists $\lambda_0 > 0$ such that

$$\left| \int |\Im e^{i(-t|\eta|^2 + x \cdot \eta)} g|^q dt dx - \frac{1}{2\pi} \int_0^{2\pi} \int |\Im e^{i\theta} g|^q dt dx d\theta \right| < \varepsilon$$

for all $|\eta| > \lambda_0$.

Proof. Let $g' \in C_c^\infty(\mathbb{R}^{d+1})$ be such that $\|g - g'\|_q < \varepsilon/4$. By the triangle inequality,

$$\begin{aligned} & \left| \int |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g|^q dt dx - \frac{1}{2\pi} \int_0^{2\pi} \int |\Im e^{i\theta} g|^q dt dx d\theta \right| \\ & \leq \left| \int |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g'|^q dt dx - \frac{1}{2\pi} \int_0^{2\pi} \int |\Im e^{i\theta} g'|^q dt dx d\theta \right| + \frac{\varepsilon}{2}. \end{aligned}$$

Hence it suffices to prove the result for $g \in C_c^\infty$.

Take $g \in C_c^\infty(\mathbb{R}^{d+1})$. Let $X = \lambda_0^{-1}\mathbb{Z}$ and $I = [0, 1)$. Since g is smooth and compactly supported, we can take $\lambda_0 > \varepsilon^{-1}$ large enough that

$$\int |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g|^q dt dx = \int_{\mathbb{R}^d} \sum_{\alpha \in X} \int_{\alpha + \lambda_0^{-1}I} |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g(\alpha, x)|^q dt dx + O(\varepsilon),$$

$$\int \frac{1}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g|^q d\theta dt dx = \int_{\mathbb{R}^d} \sum_{\alpha \in X} \frac{\lambda_0^{-1}}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g(\alpha, x)|^q d\theta dx + O(\varepsilon),$$

and

$$\int |g|^q dt dx = \int_{\mathbb{R}^d} \sum_{\alpha \in X} \lambda_0^{-1} |g(\alpha, x)|^q dx + O(\varepsilon).$$

Furthermore, by the change of variables $\theta = t|\eta|^2 - x \cdot \eta$,

$$\begin{aligned} & \int_{\alpha + \lambda_0^{-1}I} |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g(\alpha, x)|^q dt \\ &= |\eta|^{-2} \int_{|\eta|^2 \alpha - x \cdot \eta}^{|\eta|^2 \alpha - x \cdot \eta + \lambda_0^{-1}|\eta|^2} |\Im e^{-i\theta} g(\alpha, x)|^q d\theta \\ &= |\eta|^{-2} \left[O(1) |g(\alpha, x)|^q + \frac{\lambda_0^{-1}|\eta|^2}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g(\alpha, x)|^q d\theta \right] \\ &= O(1) |\eta|^{-2} |g(\alpha, x)|^q + \frac{\lambda_0^{-1}}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g(\alpha, x)|^q d\theta. \end{aligned}$$

Plugging this into the sum and simplifying,

$$\begin{aligned} & \int |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g|^q dt dx \\ &= \int \sum_{\alpha \in X} \left[O(1) |\eta|^{-2} |g(\alpha, x)|^q + \frac{\lambda_0^{-1}}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g(\alpha, x)|^q d\theta \right] dx + O(\varepsilon) \\ &= O(1) |\eta|^{-2} \lambda_0 \|g\|_q^q + \int \frac{1}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g|^q d\theta dt dx + O(\varepsilon). \end{aligned}$$

Hence for $|\eta| > \lambda_0 > \varepsilon^{-1}$, we have

$$\left| \int |\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} g|^q dt dx - \int \frac{1}{2\pi} \int_0^{2\pi} |\Im e^{i\theta} g|^q d\theta dt dx \right| = O(\varepsilon).$$

□

Lemma 2.2. Let $\varepsilon > 0$ be sufficiently small. Then there exists a δ such that for all $f \in L^p$ with $\|f\|_p = 1$ and $\|\mathcal{E}f\|_q > A_p - \varepsilon$, $\|\Im \mathcal{E}f\|_q > \delta$. Furthermore, $\lim_{\varepsilon \rightarrow 0} \delta =: C_p > 0$.

Proof. Suppose we had a sequence $\{f_n\}$ with $\|f_n\|_p = 1$ such that $\|\Im \mathcal{E} f_n\|_q \rightarrow 0$ and $\|\mathcal{E} f_n\|_q \rightarrow A_p$. Extremizing sequences are precompact modulo symmetries ([22]), so there exists $f \in L^p$ and symmetries $\{(S_n, T_n)\}$ such that $\|\mathcal{E} f\|_q = A_p$ and $S_n f_n \rightarrow f$ along a subsequence.

We want to prove that there exists a function g such that $\|\Im \mathcal{E} g\|_q = 0$. The quantity $\|\Im \mathcal{E} f\|_q$ is invariant under scaling and spacetime translation of f so without loss of generality assume $\mathcal{E} f_n = e^{i(t,x) \cdot (|\eta_n|^2, \eta_n)} \mathcal{E} f(t, x + 2t\eta_n)$. By hypothesis $\|\Im \mathcal{E} f_n\|_q \rightarrow 0$ so by a change of variables, $\|\Im e^{i(t,x) \cdot (-|\eta_n|^2, \eta_n)} \mathcal{E} f\|_q \rightarrow 0$.

Since $\mathcal{E} f \not\equiv 0$, $\int_0^{2\pi} \int |\Im e^{i\theta} \mathcal{E} f|^q dt dx d\theta \neq 0$. By Lemma 2.1 there exists $\varepsilon > 0$ and $\lambda_0 > 0$ such that $\|\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} \mathcal{E} f\|_q > \varepsilon$ for all $|\eta| > \lambda_0$. Therefore $\limsup |\eta_n| < \lambda_0$. The set $\{|\eta| \leq \lambda_0\}$ is compact, so by passing to a subsequence there exists an η_0 such that $\eta_n \rightarrow \eta_0$. Since $\|\Im e^{i(t,x) \cdot (-|\eta|^2, \eta)} \mathcal{E} f\|_q$ is continuous in η by the dominated convergence theorem, $\|\Im e^{i(t,x) \cdot (-|\eta_0|^2, \eta_0)} \mathcal{E} f\|_q = 0$. Changing variables and moving the frequency translation through $\mathcal{E}$, we have g a frequency translation of f such that $\|\Im \mathcal{E} g\|_q = 0$.

However, this is impossible. Examining the Schrödinger equation, we find that it is equivalent to

$$\begin{cases} \partial_t \Im u - \Delta \Re u = 0, \\ \partial_t \Re u + \Delta \Im u = 0. \end{cases}$$

Hence $\partial_t \Re \mathcal{E} g \equiv 0$ and $\Delta \Re \mathcal{E} g \equiv 0$. By Weyl's lemma $\Re \mathcal{E} g(t, \cdot)$ agrees almost everywhere with a harmonic function, but since the time derivative is also zero, $\Re \mathcal{E} g(t, \cdot)$ is the same non-zero harmonic function for all t . Such functions are not in L^q .

The above implies that there does not exist a sequence $\{(\|\mathcal{E} f_n\|_q, \|\Im \mathcal{E} f_n\|_q)\}$ in $A := \{(\|\mathcal{E} f\|_q, \|\Im \mathcal{E} f\|_q) : \|f\|_p = 1\} \subset [0, A_p] \times [0, A_p]$ that converges to $(A_p, 0)$. Then

$$C_p := \liminf_{\varepsilon \rightarrow 0} (\inf \{\|\Im \mathcal{E} f\|_q : \|f\|_p = 1, \|\mathcal{E} f\|_q > A_p - \varepsilon\}) > 0.$$

□

Proof of Theorem 1.2 part 1. First, we prove that

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_p \leq A_p^\pm.$$

Let $f \in L^p$ be an extremizer for $\mathcal{E}$ normalized so that $\|f\|_p = 1$. Let $e_1 \in \mathbb{R}^d$ be the first unit vector and define sequences

$$f_n(\xi) := f(\xi - ne_1) \quad \text{and} \quad g_n(\xi) := f(-\xi - ne_1).$$

Then by Lemma 2.1 and [13, Lemma 6.1],

$$\begin{aligned}
& \lim \int |\mathcal{E}f_n + \mathcal{E}_-g_n|^q dt dx \\
&= \lim \int \left| \mathcal{E}f_n + \overline{\mathcal{E}g_n} \right|^q \\
&= \lim \int \left| e^{i(t,x) \cdot (n^2, ne_1)} \mathcal{E}f(t, x + 2nte_1) + e^{i(t,x) \cdot (-n^2, -ne_1)} \overline{\mathcal{E}f}(t, x + 2nte_1) \right|^q \\
&= \lim \int \left| e^{i(t,x) \cdot (2n^2, 2ne_1)} \mathcal{E}f(t, x + 2nte_1) + \overline{\mathcal{E}f}(t, x + 2nte_1) \right|^q \\
&= \lim \int \left| e^{i(t,x) \cdot (-2n^2, 2ne_1)} \mathcal{E}f(t, x) + \overline{\mathcal{E}f}(t, x) \right|^q \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int |e^{i\omega} \mathcal{E}f| + |\overline{\mathcal{E}f}|^q \\
&= \frac{2^q}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} A_p^q.
\end{aligned}$$

Since $(\|f_n\|_p^p + \|g_n\|_p^p)^{1/p} = 2^{1/p}$, this proves the first inequality.

Next, following the proof of [13, Lemma 6.1] let

$$\phi(t) := \frac{1}{\pi} \int_0^\pi (1 + t \cos \theta)^{q/2} d\theta.$$

Then

$$\begin{aligned}
\phi'(t) &= \frac{q}{2\pi} \int_0^\pi (1 + t \cos \theta)^{(q-2)/2} \cos \theta d\theta \\
&= \frac{q}{2\pi} \int_0^{\pi/2} \left((1 + t \cos \theta)^{(q-2)/2} - (1 - t \cos \theta)^{(q-2)/2} \right) \cos \theta d\theta \\
&> 0.
\end{aligned}$$

Since $\phi(0) = 1$ and

$$\phi(1) = \frac{2^{q/2} \Gamma(\frac{q+1}{2})}{\sqrt{\pi} \Gamma(\frac{q+2}{2})}$$

using the integral representation of the beta function, the second inequality follows.

Second, we prove that

$$A_p^\pm < 2^{1/p'} A_p.$$

Note that for a nontrivial sequence $\{(f_n, g_n)\}$ to achieve

$$\lim_{n \rightarrow \infty} \frac{\|\mathcal{E}f_n + \mathcal{E}_-g_n\|_q}{(\|f_n\|_p^p + \|g_n\|_p^p)^{1/p}} = 2^{1/p'} A_p,$$

it must approach equality for every inequality in (7). Therefore, aiming for a contradiction, we assume that there exists a sequence $\{(f_n, g_n)\}$ normalized in $\ell^p(L^p)$ such

that $\|\mathcal{E}f_n\|_q \rightarrow A_p$, $\|\mathcal{E}g_n\|_q \rightarrow A_p$, and $\|\mathcal{E}f_n - \mathcal{E}_-g_n\|_q \rightarrow 0$. Expanding the integral using the identity $\mathcal{E}_-g = \widetilde{\mathcal{E}g}$,

$$\|\mathcal{E}f_n - \mathcal{E}_-g_n\|_q^q = \int (|\Re \mathcal{E}f_n - \Re \widetilde{\mathcal{E}g_n}|^2 + |\Im \mathcal{E}f_n + \Im \widetilde{\mathcal{E}g_n}|^2)^{q/2}.$$

Hence $\|\Im \mathcal{E}f_n + \Im \widetilde{\mathcal{E}g_n}\|_q \rightarrow 0$. If we multiply f_n and g_n by i and use the linearity of $\mathcal{E}$ and $\mathcal{E}_-$, we see that

$$\begin{aligned} \|\mathcal{E}f_n - \mathcal{E}_-g_n\|_q^q &= \|\mathcal{E}(if_n) - \mathcal{E}_-(ig_n)\|_q^q \\ &= \int (|-\Im \mathcal{E}f_n + \Im \widetilde{\mathcal{E}g_n}|^2 + |\Re \mathcal{E}f_n + \Re \widetilde{\mathcal{E}g_n}|^2)^{q/2} \end{aligned}$$

and hence $\|\Im \mathcal{E}f_n - \Im \widetilde{\mathcal{E}g_n}\|_q \rightarrow 0$ as well. However, by the triangle inequality and Lemma 2.2

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\Im \mathcal{E}f_n - \Im \widetilde{\mathcal{E}g_n}\|_q &\geq \lim 2\|\Im \widetilde{\mathcal{E}g_n}\|_q - \|\Im \mathcal{E}f_n + \Im \widetilde{\mathcal{E}g_n}\|_q \\ &= \lim 2\|\Im \widetilde{\mathcal{E}g_n}\|_q \\ &\geq 2C_p A_p, \end{aligned}$$

which is a contradiction. Therefore no such sequence $\{(f_n, g_n)\}$ exists and $A_p^\pm < 2^{1/p'} A_p$. $\square$

3 Bubble pairing and orthogonality

Let $p = 2$ and $q = \frac{2(d+2)}{d}$.

We will use the following formulation of the L^2 profile decomposition for the Schrödinger equation.

Proposition 3.1. [15, Theorem 4.7] Let $\{f_n\} \subset L^2(\mathbb{R}^d)$ be bounded. Then after passing to a subsequence, there exist

1. $J^* \in \mathbb{N} \cup \{\infty\}$;
2. functions $\phi^j \in L^2$ for all $j < J^*$;
3. remainders $w_n^J \in L^2$ for all $J < J^*$;
4. and sequences $\{\lambda_n^j\} \subset \mathbb{R}^+$, $\{(t_n^j, x_n^j)\} \subset \mathbb{R}^{d+1}$, and $\{\xi_n^j\} \subset \mathbb{R}^d$ defining $\{S_n^j\} \subset \mathbf{S}_+$ for all $j < J^*$ and all n

such that for all $J < J^*$

$$f_n = \sum_{j=1}^J (\lambda_n^j)^{d/p} e^{i(t_n^j, x_n^j) \cdot (|\lambda_n^j \xi - \xi_n^j|^2, \lambda_n^j \xi - \xi_n^j)} \phi^j(\lambda_n^j \xi - \xi_n^j) + w_n^J, \quad (8)$$

$$\lim_{J \rightarrow J^*} \limsup_{n \rightarrow \infty} \|\mathcal{E}w_n^J\|_q = 0, \quad (9)$$

$$\sup_J \lim_{n \rightarrow \infty} \left[\|f_n\|_2^2 - \sum_{j=1}^J \|\phi^j\|_2^2 - \|w_n^J\|_2^2 \right] = 0, \quad (10)$$

$$\sup_J \lim_{n \rightarrow \infty} \left[\|\mathcal{E}f_n\|_q^q - \sum_{j=1}^J \|\mathcal{E}\phi^j\|_q^q - \|\mathcal{E}r_n^J\|_q^q \right] = 0, \quad (11)$$

and for all $j \neq k$,

$$\{S_n^j\} \perp \{S_n^k\}. \quad (12)$$

The same holds for $\mathcal{E}_-$ with (8) replaced by

$$g_n = \sum_{j=1}^J (\kappa_n^j)^{d/p} e^{i(-s_n^j, y_n^j) \cdot (\kappa_n^j \xi - \eta_n^j)^2, \kappa_n^j \xi - \eta_n^j)} \psi^j(\kappa_n^j \xi - \eta_n^j) + r_n^J. \quad (13)$$

Let f_n, g_n be bounded sequences in L^2 and apply the profile decompositions of (8) and (13) respectively. If the J^* are different, take the larger of the two and pad the other sum with zero bubbles.

Lemma 3.2. Let $\{S_n^j\}$ and $\{R_n^j\}$ be the sequences of symmetries associated with the decompositions of $\{f_n\}$ and $\{g_n\}$ respectively. If $\{S_n^j\} \not\perp \{R_n^k\}$, then $\{S_n^j\} \perp \{R_n^{k'}\}$ for all $k' \neq k$.

The proof is a long, unenlightening calculation, so we omit it for readability.

Lemma 3.3. Let $\{S_n^j\}, \{S_n^{j'}\} \subset \mathbf{S}_+$. Let $\phi^j, \phi^{j'} \in L^2$. If $\{S_n^j\} \perp \{S_n^{j'}\}$, then

$$\lim_n \int |\mathcal{E}S_n^j \phi^j| |\mathcal{E}S_n^{j'} \phi^{j'}|^{q-1} + |\mathcal{E}S_n^j \phi^j|^{q-1} |\mathcal{E}S_n^{j'} \phi^{j'}| = 0.$$

Proof. By the symmetry of the statement, we need only consider the first term in the limit. Let $\varepsilon > 0$ and let $\Phi^j \in C_{cpct}^\infty(\mathbb{R}^{d+1})$ be such that $\|\Phi^j - \mathcal{E}\phi^j\|_q = O(\varepsilon)$ (resp. $\Phi^{j'}$). Expanding, changing variables, and dealing with the remainders by Hölder,

$$\begin{aligned} & \int |\mathcal{E}S_n^j \phi^j| |\mathcal{E}S_n^{j'} \phi^{j'}|^{q-1} \\ &= \int (\lambda_n^j)^{-(d+2)/q} (\lambda_n^{j'})^{-(d+2)/q'} |\Phi^j((\lambda_n^j)^{-2}t + t_n^j, (\lambda_n^j)^{-1}x + x_n^j + 2(\lambda_n^j)^{-2}t\xi_n^j)| \\ & \quad |\Phi^{j'}((\lambda_n^{j'})^{-2}t + t_n^{j'}, (\lambda_n^{j'})^{-1}x + x_n^{j'} + 2(\lambda_n^{j'})^{-2}t\xi_n^{j'})|^{q-1} + O(\varepsilon) \\ &= \int (\lambda_n^j)^{-(d+2)/q} (\lambda_n^{j'})^{(d+2)/q} |\Phi^j\left(\left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)^2 t + t_n^j, \left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)x + x_n^j + 2\left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)\xi_n^j\right)| \\ & \quad |\Phi^{j'}(t + t_n^{j'}, x + x_n^{j'} + 2t\xi_n^{j'})|^{q-1} + O(\varepsilon) \\ &= \int \left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)^{(d+2)/q} |\Phi^j\left(\left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)^2 t + t_n^j, \left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)x + x_n^j + 2\left(\frac{\lambda_n^{j'}}{\lambda_n^j}\right)\xi_n^j\right)| \\ & \quad |\Phi^{j'}(t + t_n^{j'}, x + x_n^{j'} + 2t\xi_n^{j'})|^{q-1} + O(\varepsilon). \end{aligned}$$

By Hölder's inequality, we see that

$$\int |\mathcal{E}S_n^j \phi^j| |\mathcal{E}S_n^{j'} \phi^{j'}|^{q-1} \leq \left(\frac{\lambda_n^{j'}}{\lambda_n^j} \right)^{(d+2)/q} \|\Phi^j\|_\infty \|(\Phi^{j'})^{q-1}\|_1 + O(\varepsilon)$$

and

$$\int |\mathcal{E}S_n^j \phi^j| |\mathcal{E}S_n^{j'} \phi^{j'}|^{q-1} \leq \left(\frac{\lambda_n^{j'}}{\lambda_n^j} \right)^{-(d+2)/q'} \|\Phi^j\|_1 \|(\Phi^{j'})^{q-1}\|_\infty + O(\varepsilon).$$

Therefore if $\lim \frac{\lambda_n^{j'}}{\lambda_n^j} \in \{0, \infty\}$ (i.e. $\{S_n^j\} \perp \{S_n^{j'}\}$ is satisfied by condition 1),

$$\int |\mathcal{E}S_n^j \phi^j| |\mathcal{E}S_n^{j'} \phi^{j'}|^{q-1} = O(\varepsilon).$$

Taking $\varepsilon \rightarrow 0$, this proves the claim.

Now we may assume that, after passing to a subsequence in n , $\lim \frac{\lambda_n^{j'}}{\lambda_n^j} = R \in (0, \infty)$. By scaling, we may also assume that $\lambda_n^j = r_n^{-1}$ such that $\lim r_n = R$ and $\lambda_n^{j'} = 1$ for all n .

Let $\varepsilon > 0$ and let $f, g \in C_{cpt}^\infty$ be such that $\|f - \phi^j\|_2 = O(\varepsilon)$ and $\|g - \phi^{j'}\|_2 = O(\varepsilon)$. By the boundedness of $\mathcal{E}$, the triangle inequality, and Hölder,

$$\begin{aligned} \int |\mathcal{E}S_n^j \phi^j| |\mathcal{E}S_n^{j'} \phi^{j'}|^{q-1} &\leq \int |\mathcal{E}S_n^j f| |\mathcal{E}S_n^{j'} g|^{q-1} + O(\varepsilon) \\ &\leq \|(\mathcal{E}S_n^j f)(\mathcal{E}S_n^{j'} g)\|_{s_\delta} \|(\mathcal{E}S_n^{j'} g)^{q-2}\|_{s'_\delta} + O(\varepsilon) \\ &= \|(\mathcal{E}S_n^j f)(\mathcal{E}S_n^{j'} g)\|_{s_\delta} \|\mathcal{E}S_n^{j'} g\|_{s'_\delta(q-2)}^{q-2} + O(\varepsilon), \end{aligned}$$

where $\delta > 0$ is sufficiently small (to be determined), $s_\delta = \frac{q+\delta}{2}$, and $t'_\delta = \frac{ds_\delta(q-2)}{d+2}$. As a function of δ , $s'_\delta(q-2) = \frac{(q+\delta)(q-2)}{q+\delta-2}$ is continuous near zero and for $\delta = 0$, $s'_0(q-2) = q$. Since the exponent pair $(2, \frac{2(d+2)}{d})$ is in the interior of the known range of restriction estimates (e.g. [23]), there exists a $\delta_0 > 0$ such that the operators $\mathcal{E}$ and $\mathcal{E}_-$ are bounded from $L^{t_{\delta_0}}$ to $L^{s'_{\delta_0}(q-2)}$. Let

$$N_n := \frac{\text{dist}(\text{supp } S_n^j f, \text{supp } S_n^{j'} g)}{\text{diam}(\text{supp } S_n^j f) + \text{diam}(\text{supp } S_n^{j'} g)}.$$

This number is well-defined since f and g have compact support. Note that since $\{r_n\}$ is bounded, $N_n \sim \text{dist}(\text{supp } S_n^j f, \text{supp } S_n^{j'} g)$ with some n -independent constants. Continuing the estimate using scaled bilinear restriction ([23, Corollary 1.3]) and the scaling assumption,

$$\begin{aligned} \|(\mathcal{E}S_n^j f)(\mathcal{E}S_n^{j'} g)\|_{s_\delta} \|\mathcal{E}S_n^{j'} g\|_{s'_\delta(q-2)}^{q-2} &\lesssim N_n^{d-\frac{d+2}{s_\delta}} \|f\|_2 \|g\|_2 \|S_n^{j'} g\|_{t_{\delta_0}}^{q-2} \\ &\lesssim_{R,f,g} N_n^{d-\frac{d+2}{s_{\delta_0}}}. \end{aligned}$$

Since $s_{\delta_0} > \frac{d+2}{d}$ and $N_n \sim |r_n \xi_n^{j'} - \xi_n^j|$ for large enough n , taking $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ proves the claim when $\{S_n^j\} \perp \{S_n^{j'}\}$ is satisfied by 2 but not 1.

We turn to the final case where neither 1 nor 2 is satisfied, but condition 3 is. We make the same assumptions on scaling as in the previous case. Changing variables,

$$\int |\mathcal{E} S_n^j \phi^j| |\mathcal{E} S_n^{j'} \phi^{j'}|^{q-1} = \int R^{\frac{d+2}{q}} |\mathcal{E} \phi^j(A_n(t, x) + b_n)| |\mathcal{E} \phi^{j'}|^{q-1}$$

where

$$A_n = \begin{pmatrix} r_n^2 & 0 \\ 2r_n(r_n \xi_n^j - \xi_n^{j'}) & r_n I_d \end{pmatrix} \text{ and } b_n = \begin{pmatrix} t_n^j - r_n^2 t_n^{j'} \\ x_n^j - r_n x_n^{j'} + 2r_n t_n^{j'} (\xi_n^{j'} - r_n \xi_n^j) \end{pmatrix}.$$

Since condition 2 is not satisfied, we may pass to a subsequence in n so that $\|A_n\|_{op} \sim 1$. Condition 3 implies $|b_n| \rightarrow \infty$, so $|A_n(t, x) + b_n| \rightarrow \infty$ uniformly on compact sets. Approximating $\mathcal{E} \phi^j$ and $\mathcal{E} \phi^{j'}$ in L^q by functions $F, G \in C_{cpt}^\infty$, we see that

$$\lim \int R^{\frac{d+2}{q}} |F(A_n(t, x) + b_n)| |G|^{q-1} = 0$$

by dominated convergence. The lemma follows by taking better and better approximations. $\square$

Lemma 3.4. Let $\{S_n^j\} \subset \mathbf{S}_+$ and $\{R_n^j\} \subset \mathbf{S}_-$. Let $\phi^j, \psi^j \in L^2$. If $\{S_n^j\} \perp \{R_n^j\}$, then

$$\lim \int |\mathcal{E} S_n^j \phi^j| |\mathcal{E}_- R_n^j \psi^j|^{q-1} + |\mathcal{E} S_n^j \phi^j|^{q-1} |\mathcal{E}_- R_n^j \psi^j| = 0.$$

Proof. As usual, we can express S_n^j and R_n^j in the form of (3) and (5):

$$\begin{aligned} S_n^j \phi^j(\xi) &= (\lambda_n^j)^{d/p} e^{i(t_n^j, x_n^j) \cdot (|\lambda_n^j \xi - \xi_n^j|^2, \lambda_n^j \xi - \xi_n^j)} \phi^j(\lambda_n^j \xi - \xi_n^j), \\ R_n^j \psi^j(\xi) &= (\kappa_n^j)^{d/p} e^{i(s_n^j, y_n^j) \cdot (|\kappa_n^j \xi - \eta_n^j|^2, \kappa_n^j \xi - \eta_n^j)} \psi^j(\kappa_n^j \xi - \eta_n^j). \end{aligned}$$

Define the sequence $\{U_n^j\} \subset \mathbf{S}_+$ by

$$Q_n^j f(\xi) := (\kappa_n^j)^{d/p} e^{i(s_n^j, y_n^j) \cdot (|\kappa_n^j \xi + \eta_n^j|^2, \kappa_n^j \xi + \eta_n^j)} f(\kappa_n^j \xi + \eta_n^j).$$

By a change of variables,

$$\begin{aligned} \overline{\mathcal{E}_- R_n^j f}(t, x) &= \int e^{-i(t, x) \cdot (-|\xi|^2, \xi)} \overline{R_n^j f}(\xi) d\xi \\ &= \int e^{i(t, x) \cdot (|\xi|^2, -\xi)} (\kappa_n^j)^{d/p} e^{i(s_n^j, y_n^j) \cdot (|\kappa_n^j \xi - \eta_n^j|^2, -\kappa_n^j \xi + \eta_n^j)} \overline{f}(\kappa_n^j \xi - \eta_n^j) d\xi \\ &= \int e^{i(t, x) \cdot (|\xi|^2, \xi)} (\kappa_n^j)^{d/p} e^{i(s_n^j, y_n^j) \cdot (|\kappa_n^j \xi + \eta_n^j|^2, \kappa_n^j \xi + \eta_n^j)} \widetilde{f}(\kappa_n^j \xi + \eta_n^j) d\xi \\ &= \mathcal{E} Q_n^j \widetilde{f}(t, x). \end{aligned}$$

The symmetries Q_n^j have the same dilation and spacetime translation as R_n^j , only with negative frequency translation. From this it is clear that $\{S_n^j\} \perp \{R_n^j\} \iff \{S_n^j\} \perp \{Q_n^j\}$ and $|\mathcal{E}_- R_n^j f| = |\mathcal{E} Q_n^j \widetilde{f}|$. Thus the claim follows from the previous lemma. $\square$

We will need the following elementary calculus lemma.

Lemma 3.5. There exists $C_n > 0$ such that for all $a_1, \dots, a_n \in \mathbb{C}$,

$$\left| \sum_{j=1}^n a_j^q - \sum_{j=1}^n |a_j|^q \right| \leq C_n \sup_{1 \leq j \neq j' \leq n} |a_j| |a_{j'}|^{q-1}.$$

Proof. First, let $a, b \in \mathbb{C}$ be such that $|a| \geq |b|$. Let $F(z) = z^q$. Then by the triangle inequality and the mean value theorem,

$$\begin{aligned} ||a + b|^q - |a|^q - |b|^q| &\leq ||a + b|^q - |a|^q| + |b|^q \\ &\leq |b| \sup_{|z| \leq 2|a|} |F'(z)| + |b|^q \\ &= q 2^{q-1} |b| |a|^{q-1} + |b|^q \\ &\lesssim \max\{|a| |b|^{q-1}, |a|^{q-1} |b|\}. \end{aligned}$$

Now take any $a_1, \dots, a_n \in \mathbb{C}$. We can expand telescopically and apply the base case to show that

$$\begin{aligned} \left| \sum_{j=1}^n a_j^q - \sum_{j=1}^n |a_j|^q \right| &= \left| \sum_{k=2}^n \left(\left| \sum_{j=1}^k a_j \right|^q - \left| \sum_{j=1}^{k-1} a_j \right|^q - |a_k|^q \right) \right| \\ &\leq \sum_{k=2}^n \left| \left| \sum_{j=1}^k a_j \right|^q - \left| \sum_{j=1}^{k-1} a_j \right|^q - |a_k|^q \right| \\ &\lesssim \sup_{2 \leq k \leq n} \max \left\{ \left| \sum_{j=1}^{k-1} a_j \right|^{q-1} |a_k|, \left| \sum_{j=1}^{k-1} a_j \right| |a_k|^{q-1} \right\}. \end{aligned}$$

By the triangle inequality and the pseudo-triangle inequality $|a + b|^{q-1} \lesssim |a|^{q-1} + |b|^{q-1}$, we obtain the result. $\square$

Proposition 3.6. Let $\{(f_n, g_n)\} \subset L^2 \times L^2$ be bounded. Then there exist $J^* \in \mathbb{N} \cup \{\infty\}$ and decompositions

$$f_n = \sum_{j=1}^J S_n^j \phi^j + r_n^J \quad \text{and} \quad g_n = \sum_{j=1}^J R_n^j \psi^j + w_n^J$$

for $J \leq J^*$ that satisfy all the conclusions in Proposition 3.1 and such that there exist partitions $A_J \cup B_J = \{1, \dots, J\}$ such that

$$\begin{aligned} \lim_{J \rightarrow J^*} \limsup_{n \rightarrow \infty} &\| \mathcal{E} f_n + \mathcal{E}_- g_n \|_q^q - \sum_{j \in A_J} \| \mathcal{E} S_n^j \phi^j + \mathcal{E}_- R_n^j \psi^j \|_q^q \\ &- \sum_{j \in B_J} \| \mathcal{E} \phi^j \|_q^q - \sum_{j \in B_J} \| \mathcal{E}_- \psi^j \|_q^q - \| \mathcal{E} r_n^J \|_q^q - \| \mathcal{E}_- w_n^J \|_q^q = 0. \end{aligned} \quad (14)$$

Furthermore, for $J_1 \leq J_2 \leq J^*$ we have $A_{J_1} \subset A_{J_2}$.

Proof. Let $J < J^*$ and $j \leq J$. By Lemma 3.2, there exists at most one $j' \leq J$ such that $\{S_n^j\} \not\subset \{R_n^{j'}\}$. If such a j' exists we may assume that $j = j'$ by rearranging the sequence $\{\psi^j\}_1^J$ and adding j to the set A_J . If no such j' exists, we add j to B_J . By the triangle inequality,

$$\begin{aligned}
& \left| \|\mathcal{E}f_n + \mathcal{E}_-g_n\|_q^q - \sum_{j \in A_J} \|\mathcal{E}S_n^j\phi^j + \mathcal{E}_-R_n^j\psi^j\|_q^q \right. \\
& \quad \left. - \sum_{j \in B_J} \|\mathcal{E}\phi^j\|_q^q - \sum_{j \in B_J} \|\mathcal{E}_-\psi^j\|_q^q - \|\mathcal{E}r_n^J\|_q^q - \|\mathcal{E}_-w_n^J\|_q^q \right| \\
& \leq \int \left| |\mathcal{E}w_n^J + \mathcal{E}_-r_n^J + \sum_{j=1}^J \mathcal{E}S_n^j\phi^j + \mathcal{E}_-R_n^j\psi^j|^q - \sum_{j \in A_J} |\mathcal{E}S_n^j\phi^j + \mathcal{E}_-R_n^j\psi^j|^q \right. \\
& \quad \left. - \sum_{j \notin A_J} (|\mathcal{E}S_n^j\phi^j|^q + |\mathcal{E}_-R_n^j\psi^j|^q) - |\mathcal{E}w_n^J|^q - |\mathcal{E}_-r_n^J|^q \right|. \quad (15)
\end{aligned}$$

Thanks to Lemma 3.5, Lemmas 3.3 and 3.4, the quasi-triangle inequality, Hölder, and the fact that $\mathcal{E}$ is bounded, we continue the estimate of (15) with

$$\begin{aligned}
\lim_{n \rightarrow \infty} \text{RHS} & \lesssim \lim_{n \rightarrow \infty} \sup_{j, j' \leq J, \epsilon_1 \neq \epsilon_2 \in \{1, q-1\}} \int |\mathcal{E}S_n^j\phi^j + \mathcal{E}_-R_n^{j'}\psi^{j'}|^{\epsilon_1} |\mathcal{E}w_n^J + \mathcal{E}_-r_n^J|^{\epsilon_2} \\
& \lesssim \lim_{n \rightarrow \infty} \sup_{j, j' \leq J, \epsilon_1 \neq \epsilon_2 \in \{1, q-1\}} \|\mathcal{E}S_n^j\phi^j + \mathcal{E}_-R_n^j\psi^j\|_q^{\epsilon_1} \|\mathcal{E}w_n^J + \mathcal{E}_-r_n^J\|_q^{\epsilon_2} \\
& \lesssim \lim_{n \rightarrow \infty} \|\mathcal{E}w_n^J\|_q + \|\mathcal{E}w_n^J\|_q^{q-1} + \|\mathcal{E}_-r_n^J\|_q + \|\mathcal{E}_-r_n^J\|_q^{q-1}.
\end{aligned}$$

By (9), the claim is proved. $\square$

4 Existence of Extremizers

Now we use the profile decomposition and information about the operator norm to find a pair of bubbles that accounts for the full L^2 mass in the limit.

Proof of Theorem 1.3 part 1. Assume that

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_p < A_p^\pm$$

as in the statement of the theorem.

Let $\{(f_n, g_n)\} \in \ell^2(L^2)$ be such that

$$\lim_{n \rightarrow \infty} \|\mathcal{E}f_n + \mathcal{E}_-g_n\|_q = A_2^\pm \quad \text{and} \quad \|f_n\|_2^2 + \|g_n\|_2^2 = 1$$

for all n . Using Proposition 3.6, write

$$f_n = \sum_{j=1}^J S_n^j\phi^j + r_n^J \quad \text{and} \quad g_n = \sum_{j=1}^J R_n^j\psi^j + w_n^J$$

for $J < J^*$. Let $\varepsilon > 0$ and let J and N be sufficiently large dependent on ε . Then by applying (11), Hölder, and (10),

$$\begin{aligned}
(A_2^\pm)^q - \varepsilon &\leq \sum_{j \in A_J} \|\mathcal{E}S_n^j \phi^j + \mathcal{E}_- R_n^j \psi^j\|_q^q + \sum_{j \in B_J} (\|\mathcal{E} \phi^j\|_q^q + \|\mathcal{E}_- \psi^j\|_q^q) \\
&\quad + \|\mathcal{E} w_n^J\|_q^q + \|\mathcal{E}_- r_n^J\|_q^q \\
&\leq \sup_{k \in A_J} \|\mathcal{E}S_n^k \phi^k + \mathcal{E}_- R_n^k \psi^k\|_q^{4/d} \sum_{j \in A_J} \|\mathcal{E}S_n^j \phi^j + \mathcal{E}_- R_n^j \psi^j\|_q^2 \\
&\quad + \sup_{k \in B_J} \max\{\|\mathcal{E} \phi^k\|_q^{4/d}, \|\mathcal{E}_- \psi^k\|_q^{4/d}\} \sum_{j \in B_J} (\|\mathcal{E} \phi^j\|_q^2 + \|\mathcal{E}_- \psi^j\|_q^2) + O(\varepsilon) \\
&\leq (A_2^\pm)^2 \sup_{k \in A_J} \|\mathcal{E}S_n^k \phi^k + \mathcal{E}_- R_n^k \psi^k\|_q^{4/d} \sum_{k \in A_J} (\|\phi^j\|_2^2 + \|\psi^j\|_2^2) \\
&\quad + A_2^2 \sup_{k \in B_J} \max\{\|\mathcal{E} \phi^k\|_q^{4/d}, \|\mathcal{E}_- \psi^k\|_q^{4/d}\} \sum_{j \in B_J} (\|\phi^j\|_2^2 + \|\psi^j\|_2^2) + O(\varepsilon) \\
&\leq \max \left\{ (A_2^\pm)^2 \|\mathcal{E}S_n^{j_0} \phi^{j_0} + \mathcal{E}_- R_n^{j_0} \psi^{j_0}\|_q^{4/d}, A_2^2 \|\mathcal{E} \phi^{j_1}\|_q^{4/d}, A_2^2 \|\mathcal{E}_- \psi^{j_2}\|_q^2 \right\} \\
&\quad + O(\varepsilon),
\end{aligned}$$

for $j_0 \in A_J$ and $j_1, j_2 \in B_J$, chosen based on the supremum, and all $n > N$. By the second inequality in Theorem 1.2 part 1, there exists a $C > 0$ such that $A_2^q + C < (A_2^\pm)^q$. Since $\|\phi^{j_1}\|_2^2 \leq 1$ by (10),

$$A_2^2 \|\mathcal{E} \phi^{j_1}\|_q^{4/d} \leq A_2^q \|\phi^{j_1}\|_2^{4/d} < (A_2^\pm)^q - C.$$

Applying the same logic to $\mathcal{E}_- \psi^{j_2}$ and taking ε sufficiently small, this proves that the maximum is achieved by the first term and hence

$$A_2^\pm - O(\varepsilon) \leq \|\mathcal{E}S_n^{j_0} \phi^{j_0} + \mathcal{E}_- R_n^{j_0} \psi^{j_0}\|_q. \quad (16)$$

Since

$$\|\mathcal{E}S_n^{j_0} \phi^{j_0} + \mathcal{E}_- R_n^{j_0} \psi^{j_0}\|_q \leq A_2^\pm (\|\phi^{j_0}\|_2^2 + \|\psi^{j_0}\|_2^2)^{1/2},$$

we have $\|\phi^{j_0}\|_2^2 + \|\psi^{j_0}\|_2^2 = 1$ by taking $\varepsilon \rightarrow 0$. By (10),

$$\|f_n - S_n^{j_0} \phi^{j_0}\|_2 \rightarrow 0 \quad \text{and} \quad \|g_n - R_n^{j_0} \psi^{j_0}\|_2 \rightarrow 0 \quad (17)$$

Let $\{(\lambda_n, t_n, x_n, \xi_n)\}$ and $\{(\kappa_n, s_n, y_n, \eta_n)\}$ be the parameters for $\{S_n^{j_0}\} \subset \mathbf{S}_+$ and $\{R_n^{j_0}\} \subset \mathbf{S}_-$ respectively and let $\{T_n^{j_0}\} \subset \mathbf{T}_+$ and $\{U_n^{j_0}\} \subset \mathbf{T}_-$ be the associated L^q symmetries. Let $r_n = \frac{\kappa_n}{\lambda_n}$ and

$$\theta_n := -r_n^2 |\xi_n|^2 s_n - r_n y_n \cdot \xi_n - 2r_n s_n \eta_n \cdot \xi_n + s_n |\eta_n|^2 + y_n \cdot \eta_n. \quad (18)$$

By calculation,

$$\begin{aligned}
(U_n^{j_0})^{-1} T_n^{j_0} F(t, x) &= r_n^{\frac{d+2}{q}} e^{i(r_n^2 t, r_n x) \cdot (|\xi_n + r_n^{-1} \eta_n|^2, \xi_n + r_n^{-1} \eta_n) - 2i(t, x) \cdot (|\eta_n|^2, \eta_n) + i\theta_n} \\
&\quad F(r_n^2 t + t_n - r_n^2 s_n, r_n x + x_n - r_n y_n + 2r_n^2 (\xi_n + r_n^{-1} \eta_n)(t - s_n)).
\end{aligned}$$

Since $j_0 \in A_J$, Definition 1.1 implies that there exist $r_0 \in (0, \infty)$, $(t_0, x_0) \in \mathbb{R}^{d+1}$, $\xi_0 \in \mathbb{R}^d$, $\theta_0 \in [0, 2\pi)$, and a subsequence in n along which

1. $\lim r_n = r_0$,
2. $\lim \xi_n + r_n^{-1} \eta_n = \xi_0$,
3. $\lim(t_n - r_n^2 s_n, x_n - r_n y_n - 2r_n^2 s_n(\xi_n + r_n^{-1} \eta_n)) = (t_0, x_0)$, and
4. $\lim e^{i\theta_n} = e^{i\theta_0}$.

Let $V^{j_0} \in \mathbf{T}_+$ be the symmetry associated with the parameters r_n^{-1} , ξ_0 , and (t_0, x_0) , and let $W^{j_0} \in \mathbf{S}_+$ be the corresponding L^2 isometry. Assume $F, G \in C_{cct}^\infty$. Then by the dominated convergence theorem,

$$\lim \|(U_n^{j_0})^{-1} T_n^{j_0} F(t, x) - e^{-2i(t, x) \cdot (|\eta_n|^2, \eta_n)} e^{i\theta_0} V^{j_0} F(t, x)\|_q = 0.$$

By density, we can extend this to $F, G \in L^q$.

Let $\Phi := \mathcal{E}\phi^{j_0}$ and $\Psi := \mathcal{E}_-\psi^{j_0}$. By the triangle inequality,

$$\lim \|T_n^{j_0} \Phi + U_n^{j_0} \Psi\|_q - \|e^{-2i(t, x) \cdot (|\eta_n|^2, \eta_n)} e^{i\theta_0} V^{j_0} \Phi + \Psi\|_q = 0.$$

If $|\eta_n| \rightarrow \infty$,

$$\lim \|e^{-2i(t, x) \cdot (|\eta_n|^2, \eta_n)} e^{i\theta_0} V^{j_0} \Phi + \Psi\|_q^q = \frac{1}{2\pi} \int_0^{2\pi} \int |e^{i\omega} V^{j_0} \Phi + \Psi|^q dt dx d\omega$$

by Lemma 2.1. However, by [13, Lemma 6.1]

$$\frac{1}{2\pi} \int_0^{2\pi} \int |e^{i\omega} V^{j_0} \Phi + \Psi|^q dt dx d\omega \leq \frac{2^{q/2}}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} (\|\Phi\|_q^2 + \|\Psi\|_q^2)^{q/2}.$$

This contradicts the hypothesis on $A_2^\pm$ and (16) by taking ε sufficiently small. Therefore $|\eta_n| \not\rightarrow \infty$ and there exists $\eta_0 \in \mathbb{R}^d$ such that $\eta_n \rightarrow \eta_0$ along a subsequence.

Let $\{K_n\} \subset \mathbf{S}_+ \cap \mathbf{S}_-$ be the symmetries associated with the parameters $\{(\kappa_n, s_n, y_n + 2s_n \eta_n, 0)\}$. Let $\{L_n\} \subset \mathbf{T}_+ \cap \mathbf{T}_-$ be the associated L^q (note that the definitions of the L^q symmetries coincide when the frequency translation is zero so the L_n are well-defined). Let $r_n = \frac{\kappa_n}{\lambda_n}$,

$$\theta_n := -r_n^2 s_n |\xi_n|^2 - r_n y_n \cdot \xi_n - 2r_n s_n \eta_n \cdot \xi_n,$$

and

$$\omega_n := -s_n |\eta_n|^2 - y_n \cdot \eta_n.$$

We calculate

$$\begin{aligned} & L_n^{-1} T_n^{j_0} \Phi(t, x) + L_n^{-1} U_n^{j_0} \Psi(t, x) \\ &= r_n^{\frac{d+2}{q}} e^{i\theta_n} e^{i(r_n^2 t, r_n x) \cdot (|\xi_n|^2, \xi_n)} \\ & \quad \Phi(r_n^2 t + t_n - r_n^2 s_n, r_n x + 2r_n^2 t \xi_n + (2r_n^2 s_n(r_n^{-1} \eta_n - \xi_n) + x_n - r_n y_n)) \\ & \quad + e^{i\omega_n} e^{i(t, x) \cdot (-|\eta_n|^2, \eta_n)} \Psi(t, x - 2t \eta_n). \end{aligned}$$

By Definition 1.1 and the fact that $\lim \eta_n = \eta_0$, there exist $r_0 \in (0, \infty)$, $(t_0, x_0) \in \mathbb{R}^{d+1}$, $\xi_0 \in \mathbb{R}^d$, $\theta_0 \in [0, 2\pi)$, $\omega_0 \in [0, 2\pi)$, and a subsequence in n along which

1. $\lim r_n = r_0$,
2. $\lim \xi_n = \xi_0$,
3. $\lim(t_n - r_n^2 s_n, x_n - r_n y_n + 2r_n^2 s_n(r_n^{-1} \eta_n - \xi_n)) = (t_0, x_0)$,
4. $\lim e^{i\theta_n} = e^{i\theta_0}$, and
5. $\lim e^{i\omega_n} = e^{i\omega_0}$.

Let $(W^{j_0}, Y^{j_0}) \in \mathbf{S}_+ \times \mathbf{T}_+$ be the symmetry pair associated with the parameters $(r_0^{-1}, t_0, x_0, \xi_0)$ and let $(X^{j_0}, Z^{j_0}) \in \mathbf{S}_- \times \mathbf{T}_-$ be associated with the parameters $(1, 0, 0, \eta_0)$. By approximating ϕ^{j_0} and ψ^{j_0} in C_{cpct}^∞ and applying dominated convergence, the convergence in parameters implies that

$$K_n^{-1} S_n^{j_0} \phi^{j_0} \rightarrow e^{i\theta_0} W^{j_0} \phi^{j_0} \quad \text{and} \quad K_n^{-1} R_n^{j_0} \phi^{j_0} \rightarrow e^{i\omega_0} X^{j_0} \psi^{j_0}.$$

Let $f = e^{i\theta_0} W^{j_0} \phi^{j_0}$ and $g = e^{i\omega_0} X^{j_0} \psi^{j_0}$. By (17),

$$K_n^{-1} f_n \rightarrow f \quad \text{and} \quad K_n^{-1} g_n \rightarrow g,$$

in L^2 . Finally, by the continuity of $\mathcal{E}_\pm$ from $\ell^2(L^2)$ to L^q ,

$$\|\mathcal{E}f + \mathcal{E}_-g\|_q = A_2^\pm.$$

□

Proof of Theorem 1.3 part 2. If

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_2 < A_2^\pm,$$

then the previous part proves that extremizers exist. Therefore, assume

$$\left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} 2^{1/p'} A_2 = A_2^\pm.$$

Let $f \in L^2$ be such that $\|f\|_2 = 1$ and $\|\mathcal{E}f\|_q = A_2$ ([22]). Let $g_\theta(\xi) = e^{i\theta} \overline{f}(-\xi)$. Then by the identity $\mathcal{E}_-h = \widetilde{\mathcal{E}h}$ and [13, Lemma 6.1],

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \int |\mathcal{E}f + \mathcal{E}_-g_\theta|^q dt dx d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \int |\mathcal{E}f + e^{-i\theta} \overline{\mathcal{E}f}|^q dt dx d\theta \\ &= \frac{2^q}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \|\mathcal{E}f\|_q^q \\ &= \frac{2^q}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} A_2^q. \end{aligned}$$

Since $\int |\mathcal{E}f + \mathcal{E}_-g_\theta|^q$ is continuous in θ by dominated convergence, there exists a $\theta_0 \in [0, 2\pi)$ such that

$$\|\mathcal{E}f + \mathcal{E}_-g_{\theta_0}\|_q = 2 \left(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} \right)^{1/q} A_2.$$

Therefore, by the assumption on $A_2^\pm$,

$$\frac{\|\mathcal{E}f + \mathcal{E}_-g_{\theta_0}\|_q}{(\|f\|_2^2 + \|g_{\theta_0}\|_2^2)^{1/2}} = A_2^\pm.$$

□

5 Validated numerics

Proof of Theorem 1.2 part 2. Let $f(\xi) = e^{-|\xi|^2}$. By [11, Theorem 1.1], $\|\mathcal{E}f\|_q = A_2\|f\|_2$. By the identity $\mathcal{E}_-f = \widetilde{\mathcal{E}f}$ and [13, Lemma 6.1],

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \int |\Re e^{i\theta/2} \mathcal{E}f|^q dt dx d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \int |e^{-i\theta/2} \mathcal{E}f + e^{i\theta/2} \mathcal{E}_-f|^q dt dx d\theta \\ &= \frac{2^q}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} A_2^q. \end{aligned}$$

Let $\mathcal{J}(\theta) = \int |\Re e^{i\theta} \mathcal{E}f|^q dt dx$. If we can show that $\mathcal{J}(\theta_1) \neq \mathcal{J}(\theta_2)$ for some $\theta_1 \neq \theta_2$, then by the continuity of $\mathcal{J}$ there exists a θ_0 such that

$$\mathcal{J}(\theta_0) > \frac{2^q}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})} A_2^q,$$

which will complete the proof.

Expanding $\mathcal{E}f$ and integrating in polar coordinates, we see that

$$\begin{aligned} \mathcal{J}(\theta) &= \pi^{dq/2} \int (1+t^2)^{-dq/2} e^{\frac{-q|x|^2}{4(1+t^2)}} \left| \cos \left(\theta + \frac{t|x|^2}{4(1+t^2)} \right) \right|^q dt dx \\ &= c_d \int_{-\infty}^{\infty} \int_0^{\infty} (1+t^2)^{\frac{d(1-q)}{2}} r^{d-1} e^{-qr^2} \left| \cos \left(\theta + \frac{tr^2}{4} \right) \right|^q dr dt \end{aligned}$$

where $c_d = \frac{2\pi^{d(1+q)/2}}{\Gamma(d/2)}$.

We will consider $d = 1$ as the case $d = 2$ follows *m.m.* Bounding $\cos$ by one and using Sage ([19]), we can bound the tail error

$$c_d \int_{|t|>49} \int_4^{\infty} (1+t^2)^{\frac{d(1-q)}{2}} r^{d-1} e^{-qr^2} \left| \cos \left(\theta + \frac{tr^2}{4} \right) \right|^q dr dt < 10^{-19}.$$

The techniques of interval arithmetic (e.g. [17, Chapter 3], `RIF` Sage datatype) allow us to track potential computation errors in the upper and lower Riemann sums with steps of 0.1. Sage provides

$$c_d \int_{|t| < 50} \int_0^5 (1+t^2)^{\frac{d(1-q)}{2}} r^{d-1} e^{-qr^2} \left| \cos\left(\frac{tr^2}{4}\right) \right|^q dr dt \in (23, 37)$$

and

$$c_d \int_{|t| < 50} \int_0^5 (1+t^2)^{\frac{d(1-q)}{2}} r^{d-1} e^{-qr^2} \left| \cos\left(\frac{\pi}{2} + \frac{tr^2}{4}\right) \right|^q dr dt \in (0, 0.1).$$

These intervals are further apart than twice the tail error, so $\mathcal{J}(0) \neq \mathcal{J}(\pi/2)$. $\square$

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