Journal of Mathematical Behavior 70 (2023) 101030

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Characterizing prospective secondary teachers' foundation and contingency knowledge for definitions of transformations

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ARTICLE INFO

Keywords:
Mathematical knowledge for teaching
Knowledge Quartet
Undergraduate
Mathematics education
Secondary Teacher Education

ABSTRACT

One promising approach for connecting undergraduate content coursework to secondary teaching is using teacher-created representations of practice. Using these representations effectively requires seeing teachers' use of mathematical knowledge in the work of teaching. We argue that the dimensions of Rowland's (2013) Knowledge Quartet, especially Foundation and Contingency, form a fruitful framework for this purpose. We contribute an analytic framework to characterize the quality of mathematical knowledge observed in the Foundation and Contingency dimensions, developed using a purposive sampling from over 300 representations. These representations all featured geometry teaching. We showcase the framework with examples of "high" and "developing" Foundation and Contingency.Then, we compare our coding along these dimensions with performance on a measure of mathematical knowledge for teaching geometry. Finally, we describe the potential for generalizing this framework to other domains, such as algebra and mathematical modeling.

Siloing subject matter risks students who silo ideas. When the students are prospective mathematics secondary teachers, and their mathematics and pedagogy courses have little overlap, the students may believe that knowledge from one course is contextually inappropriate for any other course. Indeed, despite teachers' many opportunities to learn tertiary mathematics (Hill, 2011; Tatto & Bankov, 2018) there has historically been little evidence that tertiary mathematics course-taking influences secondary teachers' pedagogy (Zazkis & Leikin, 2010). Multiple studies have documented that many secondary teachers do not find tertiary mathematics courses relevant to their careers (Goulding et al., 2003; Wasserman et al., 2018; Zazkis & Leikin, 2010), even when they have done well in the mathematics course (Wasserman & Ham, 2013). Though researchers have identified instances where tertiary mathematics can shape individual secondary mathematics teachers' decisions (e.g., Baldinger, 2018; Zazkis & Mamolo, 2011), it was not until recently that a study recorded a tertiary mathematics course's direct influence on secondary teachers' instruction (Wasserman & McGuffey, 2021). This finding was due to intentional connections between course activities and content (in real analysis) and secondary mathematics teaching.

Our work, too, is guided by the challenge and desire to integrate learning *mathematics* and learning to *teach* mathematics (cf. Baumert et al., 2010).). Mathematics encountered by prospective secondary teachers may range from geometry to algebra, to statistics, to mathematical modeling (Tatto & Bankov, 2018). We report on data from the Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE(S^2)) Project.

Abbreviations: MODULE(S^2), Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools.

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https://doi.org/10.1016/j.jmathb.2022.101030
Received 11 November 2021; Received in revised form 22 December 2022; Accepted 30 December 2022
Available online 11 January 2023

In this paper, we review a promising type of task, which we refer to as *prompts for teacher-created representations of practice* (cf. Grossman et al., 2009). These prompts can be used across content areas. Theyengage prospective secondary mathematics teachers in creating their own images of teaching practice in response to a given teaching scenario. We use the term *representation* because the teachers are representing their image of teaching in responding to these tasks. Next, we describe a framework for describing the potential range of teachers' knowledge and skill in using mathematics in teaching. This framework attends to teachers' *foundational content knowledge* and *actions contingent upon specific student thinking* (cf. Rowland, 2013; Rowland et al., 2016). The former includes teachers' personal understanding of mathematics underlying the concepts addressed in the teaching scenario. The latter incorporates how teachers frame explanations and questions in terms of student thinking. As a 'proof of concept' of analysis with this framework, we examined teachers' responses to four tasks in one content area. This area was geometry from a transformation approach. Finally, we consider how our work might generalize to other mathematical domains.

The following questions guided this study: *What foundational knowledge and contingent actions are observable in teacher-created representations of practice? How can we characterize quality of foundational knowledge and contingent actions?*

1. Background

Secondary teachers take a range of university mathematics, from first year courses in calculus, to abstract algebra, to real analysis (Tatto & Bankov, 2018; Ferrini-Mundy and Findell, 2001). By requiring these courses, programs hope that secondary teachers will teach mathematics with greater perspective and accuracy (Conference Board of the Mathematical Sciences [CBMS], 2012; Ferrini--Mundy & Findell, 2001). However, Monk (1994), who studied relationships between teacher course taking and secondary student performance, found that there was a negligible effect after the first four mathematics courses. In other words, any course taken after the first or second year, which would likely include courses specifically designed for secondary teachers, or any advanced courses, likely had little effect.

1.1. Perceived discontinuity between university mathematics and secondary teaching

As has been well-documented, many secondary teachers find their tertiary mathematics education irrelevant to their teaching. The studies of Goulding et al. (2003), Ticknor (2012), and Zazkis and Leikin (2010), among others, found that to many teachers the university curriculum appeared to focus on topics unrelated to the K-12 curriculum. Even among teachers who found their mathematical preparation useful, they are rarely able to cite specific instances of how tertiary mathematics experiences influenced their teaching (Wasserman & Ham, 2013; Zazkis & Leikin, 2010). Some prospective teachers may view the utility of university mathematics content using a "transport model", looking only at how well the exact mathematical explanations from their university mathematics courses would transfer to teaching secondary mathematics (Wasserman et al., 2018, p. 83).

1.2. Connecting university mathematics to secondary teaching via applications of mathematics to teaching

Historically, attempts to connect tertiary mathematics to secondary mathematics teaching tended to focus almost exclusively on the *mathematics* (e.g., The Panel on Teacher Training, 1971; CBMS, 2001; Kerr & Lester, 1982; Mathematical Association of America, 1983), with little attention to mathematics *teaching practice.*

We, along with multiple others, take the stance that addressing the discontinuity problem requires connecting tertiary *mathematics* and secondary mathematics *teaching practice* (Álvarez, Arnold, Burroughs, Fulton, & Kercher, 2020a; Heid, Wilson, & Blume, 2015; Lischka, Lai, Strayer, & Anhalt, 2020; Ticknor, 2012; Wasserman, Weber, Villanueva, & Mejia-Ramos, 2018; for a review, see Lai et al., in press). In our work, we take the approach of enhancing curricula for tertiary mathematics courses. To so do, we incorporate activities that *apply mathematics to teaching,* or that are *application problems* (cf. Álvarez et al., 2020a; Stylianides & Stylianides, 2010). We define such activities as tasks where prospective teachers consider a secondary teaching situation where university mathematics can be leveraged, and teachers then draw on this mathematics to respond to the situation in ways appropriate for secondary teaching.

As Álvarez et al. (2020a) observed,

> *Including applications to teaching in mathematics content courses … can advance content learning goals and meet the needs of prospective secondary mathematics teachers as they make connections between the advanced mathematics they are learning, the mathematics they will teach, and the complex human context that is central in the work of teaching (p. 17).*

Furthermore, such activities may help secondary teachers "transcend the transport model" (Wasserman et al., 2018, p. 87). The tasks allow prospective teachers to use university mathematics understandings during the work of secondary teaching.

1.3. Teacher-created representations of practice: An application of mathematics to teaching

Applying mathematics to teaching may come in different forms: analyzing students' mathematical reasoning, explaining a mathematics teacher's decision, or posing questions in response to student work. Some tasks in recent projects ask prospective secondary teachers to describe, or even write a script for, what they would say or do to explain particular secondary-level concepts or upon reviewing secondary student work (e.g., Álvarez et al., 2020a; Lai et al., (in press); Lischka, Lai, Strayer, & Anhalt, 2020; Wasserman et al., 2018). We say that the prospective secondary teachers' responses to these tasks are *teacher-created representations of* *practice* (cf. Grossman et al., 2009). We use the phrase *prompt for teacher-created representations of practice* to refer to the tasks that elicit teacher-created representations of practice.

For instance, a prompt may present secondary students' explanations for an algebraic solution, and then ask teachers to pose questions to students to help uncover arithmetic assumptions used. Or, a prompt may present secondary students' attempts to construct a graphical display to represent the association between two quantities, and ask teachers to comment as they would to a student about the quality of the work. The teaching practices involved here—such as posing questions and interpreting student work—require mathematics to carry out well (e.g., Ball et al., 2008; Baumert et al., 2010).

Teachers' responses to such prompts can therefore showcase their recognition of mathematics relevant to the teaching scenario, as well as how they use this mathematics. Monk's (1994) results suggest that if there is a relationship between course taking and teaching, it may be nuanced. We explore the hypothesis that explicit connections to teaching are part of this nuance.

1.4. The promise of connecting university mathematics to secondary teaching practice

The potential benefits of connecting mathematics to teaching practice are both practical and methodological. First, practicing teachers may attribute their teaching moves to their experience with teacher-created representations of practice featured in university mathematics courses, as Wasserman and McGuffey (2021) found when observing and interviewing former students of their project. In another study using our project's data, teachers attributed increased confidence in their teaching practice to creating representations of practice (Lai et al., 2023).

Methodologically, teacher-created representations of practice may be a way to elicit and evaluate mathematical knowledge for teaching – the mathematical ideas, concepts, skills, and sensibilities entailed and manifested the recurrent work of teaching (Ball et al., 2008). Early in the scholarship of mathematical knowledge for teaching, Ball and Bass (2003) proposed a way to assess knowledge of mathematics relevant to teaching: by posing questions where teachers respond to a given mathematics teaching scenario. Since then, multiple projects at the secondary level seeking to assess mathematical knowledge for teaching have used this principle as well (e.g., Baumert et al., 2010; Herbst & Kosko, 2012; McCrory et al., 2012; Mohr-Schroeder et al., 2017). Teacher-created representations of practice follow this lineage, with an instantiation in curriculum rather than assessment instruments.

As data, teacher-created representations of practice may bring more nuance to describing the quality of mathematical knowledge used and how it is applied to teaching. Álvarez et al. (2020a) identified a tendency of university students to focus on computation rather than underlying concepts in sample secondary student work, a finding that led them to re-design some of their prompts for teacher-created representations of practice. Álvarez et al. (2020b) used teacher-created representations of practice to analyze teachers' propensity to validate student thinking, look for rules, and address visual representations of function from a student's perspective. Alvarez et al. 2020a; Weber et al. (2020) used teacher-created representations of practice to articulate how teachers may use known procedures when analyzing student work, and how well teachers connected different criteria for a mathematical definition.

One of the greatest needs in secondary teacher education is designing ways for teachers to learn *mathematics* and how to apply that mathematics to *teaching*. The mathematics known by teachers presents a ceiling for how well they can communicate mathematics with their students (Baumert et al., 2010). Further, as Lai et al. (in press) argued, there is a need to identify explicit ways to characterize mathematical and pedagogical qualities of applications of mathematics to teaching. In this way, mathematics teacher educators might better understand how mathematical practice can shape secondary teaching practice. This study contributes to this need. We characterize the quality of the mathematics used in teacher-created representations of practice, as well as the quality of how sample secondary students' work is taken up by teachers.

2. Conceptual perspective

2.1. Mathematical knowledge in and for teaching (MKT)

In a presidential address to the American Educational Research Association, Shulman (1986) called for attention to a "missing paradigm" problem in the study of teaching (p. 6): research that focused simultaneously on content and pedagogy. Multiple research groups responded to Shulman's address, resulting in different approaches to conceptualizing knowledge use in teaching. These include:

- Ball and colleagues' "mathematical knowledge for teaching" that elaborates Shulman's notions of content knowledge and pedagogical content knowledge (e.g., Ball & Bass, 2003; Ball et al., 2008), and which Baumert et al. (2010) re-conceptualized for the secondary level;
- Thompson and Thompson's (1996) inquiry into "mathematical knowledge for teaching" and "knowledge for conceptual teaching";
- Davis and colleagues' scholarship on "mathematics-for-teaching" (e.g., Davis & Simmt, 2006), and "mathematical knowledge in teaching" as discussed in a seminar led by Ruthven and Rowland (2007);
- Heid and collaborators' (2015) development of the notion of "mathematical understandings for secondary teaching"; and
- Rowland and colleagues' (2013, 2016) "Knowledge Quartet", which identifies and categorizes teaching routines and moments where knowledge use may be observed.

This is but a sample of scholarship on mathematical knowledge that informs teaching; it illustrates the variety of terminology in the literature.

For our project, the most salient aspect of this wide scholarship is its common thread: that teachers use content knowledge in recognizing, understanding, and responding to mathematical situations, considerations, and challenges that arise in the course of teaching mathematics. We refer to this as mathematical knowledge in and for teaching (MKT).

Synthesizing these frameworks shaped our view of how mathematical knowledge influences teaching practice. It also informed the design of our prompts for teacher-created representations of practice. First, Baumert et al. (2010) re-conceptualized Ball and colleagues' content knowledge as "a profound mathematical understanding of the curricular content to be taught" (p. 142). We have adopted a similar point of view. Second, Baumert et al. (2010) distinguished content knowledge from pedagogical content knowledge, the knowledge needed for making mathematics comprehensible to students (Shulman, 1986). These distinctions echo those made by other scholars (e.g., Ball et al., 2008; Heid et al., 2015; Rowland, 2013; Shulman, 1986). Further, Baumert et al. (2010) argued that although pedagogical content knowledge appears to have a stronger role on secondary student outcomes than content knowledge does, mathematics teacher educators must attend to content knowledge. One of Baumert et al.'s results is that a teacher's content knowledge determines how much pedagogical content knowledge the teacher will be able to learn.

One critique of Ball and colleagues' conception of knowledge for teaching is that they do not explicitly discuss its dynamic aspects. That is, teachers' knowledge is not static, but rather develops through, and perhaps even manifests itself in practice (Davis & Simmt, 2006; Ruthven & Rowland, 2007). This view supports our stance that teachers' university mathematics experiences would be enhanced by incorporating connections to teaching practice. Teacher-created representations of practice may allow mathematical knowledge to manifest.

In designing prompts for teacher-created representations of practice, we took into account the importance of supporting students through adaptive instruction (Baumert et al., 2010; Thompson & Thompson, 1996) and of cultivating mathematical practices such as defining, justifying, sense-making, or representing (Heid et al., 2015). All of our prompts for teacher-created representations of practice feature a student-level task, sample student work on that task, and a goal for teaching that involves engaging students in a mathematical practice.

Finally, in analyzing teacher-created representations of practice, we most use Rowland and colleagues' work on the Knowledge Quartet, whose details we will soon discuss. For now, we remark that the reader may wonder why this framework and not one of the many others. Our response is that we chose it for methodological reasons. We wanted a framework that articulated where exactly mathematical knowledge manifests in teaching practice, so that we would be able to identify such locations in teacher-created representations of practice. Rowland (2013) observed that this feature distinguishes the Knowledge Quartet from other frameworks for mathematical knowledge in and for teaching.

2.2. The Knowledge Quartet, its use at secondary level, and its dimensions

Soon after researchers began to conceptualize mathematical knowledge in and for teaching, multiple projects sought to link this knowledge to educational outcomes. At elementary and secondary levels, projects used assessments designed to measure teachers' mathematical knowledge and largely found associations between teachers' mathematical knowledge, learning gains, and desirable teaching qualities (e.g., Baumert et al., 2010; Hill et al., 2005; Hill et al., 2008; Rowland et al., 2000).

Rowland et al.'s (2000) study, of 150 London-based primary teachers, motivated the following puzzle: "If superior content knowledge really does make a difference when [teaching mathematics], it ought somehow to be observable in the *practice* of the knowledgeable teacher" (p. 17; emphasis in the original). They wanted to know where to observe knowledge so as to "frame a coherent, content-focused discussion" (Rowland, 2013, p. 21) between a teacher and someone who was observing the teacher for the purpose of giving feedback to the teacher about their teaching. Rowland and his colleagues then undertook an observation and video study using a purposive sample of 12 participants drawn from a pool of 149 primary teachers. The sample reflected a range of outcomes based on performance on an assessment of their mathematical knowledge. Using a grounded theory approach (Glaser & Strauss, 1967; as cited in Rowland, 2013), they generated four broad categories of codes for teachers' knowledge and knowledge use in teaching.

Since the conclusion of Rowland's initial studies, researchers have validated these categories as describing knowledge use beyond primary level and beyond the UK. Weston, Kleve, & Rowland, 2012 reported on a cross-national study to determine whether the Knowledge Quartet would be a feasible framework for analyzing secondary level data. Although they identified more codes for moments of teaching where mathematical knowledge is observable, the four broad categories from the primary level study remained stable across a team of 15 researchers from 7 countries working with 55 episodes of secondary teaching. These categories make the four dimensions of the Knowledge Quartet.

Two dimensions are *Foundation* (knowledge and understanding of mathematical ideas, the nature of mathematics, as well as principles of mathematical pedagogy) and *Contingency* (the ability to respond to unanticipated events ranging from network outages to learners' alternative strategies). The moments of teaching where Foundation is observable include "awareness of purpose", "identifying errors", "overt display of subject knowledge", and "use of mathematical terminology" (Rowland et al., 2016, p. 1). The moments of teaching where Contingency is observable include "deviation from agenda", "responding to students' ideas", "use of opportunities" (p. 1). The remaining dimensions are *Transformation* (presenting ideas to learners) and *Connection* (cohering ideas over time); these are observable in moments such as "choice of examples" or "decisions about sequencing", respectively (p. 1).

We highlight Foundation and Contingency because we focus most on them. We now discuss our view of teacher-created representations of practice, how we operationalized Foundation and Contingency, and why our analysis used only these dimensions rather than all four.

2.3. Teacher-created representations of practice

As Amador et al. (2017) noted, there is power in having prospective teachers "take on an active role as *designers* (rather than *viewers*) of classroom scenes" (p. 160; emphasis in the original). We view teacher-created representations of practice as the result of teachers' design work with samples of secondary student work and a statement of an intended goal (e.g., "to advance students' understanding of how a definition connects to a procedure").

Teacher-created representations of practice can give a window into teachers' conceptions of mathematics teaching as well as their use of mathematics in teaching. We posit that when teachers share an image of envisioned teaching using sample student work, they – like teachers with classroom artifacts – draw on their knowledge and commitments (Brown, 2009).

2.4. Characterizing Foundation and Contingency in teacher-created representations of practice

Our study and Rowland and colleagues' studies have key similarities and also critical differences. Rowland and colleagues used videos of teaching across multiple topics in multiple schools. We examined teacher-created representations of practice responding to a limited set of prompts. Hence, we found it useful to delimit and elaborate the dimensions we used: Foundation and Contingency. We now discuss our operationalizations and then say why our analysis did not consider the remaining dimensions.

2.4.1. Foundation

We delimited the Foundation dimension to knowledge of mathematics, because of our interest in content coursework. Second, the dependence of Foundation on *mathematical understanding* suggested that we be theoretically clear about a conception of mathematical understanding. We used Simon's (2006) characterization: mathematical understanding is the "learned anticipation of the logical necessity of a particular pattern or relationship" (p. 364). For instance, we consider understanding mathematical procedures to include relating that procedure to underlying definitions or concepts, as well as anticipating to do so when explaining procedures or troubleshooting a use of a procedure.

2.4.2. Contingency

We delimited Contingency to the ability to use *given* student thinking in teacher-created representations of practice. By *given*, we mean the student contributions explicitly provided in the prompt for teacher-created representations of practice rather than, say, imagined by a prospective teacher. We constrained Contingency in this way because all our prompts provided sample secondary student thinking, and we posited that integrating *given* student thinking approximated for a prospective teacher an encounter with potentially unexpected student contributions.

2.4.3. Documenting the potential range of Foundation and Contingency knowledge use

Rowland and colleagues' work results in a framework for identifying instances where mathematical knowledge may be used in teaching, but it does not result in a framework for characterizing variation of such use. Weston (2013) used the Knowledge Quartet dimensions to quantify mathematical knowledge in teaching for the purpose of informing programs of teacher education. As she noted, she sought to use "consistent observation-based data across multiple trainees and multiple lessons in order to inform [teacher preparation] programs, *rather than to support individual teacher development*" (p. 289; italics ours). Our purpose is to characterize the potential range of knowledge use in dimensions of the Knowledge Quartet apparent in individual teachers' work, in ways that may support teachers' development.

2.5. The choice to use only Foundation and Contingency rather than all four dimensions

We focused on Foundation and Contingency in our work because they are the most visible in the teacher-created representations of practice we analyzed. In other words, the evidence from the teacher-created representations of practice was available and appropriate. Our results reported in this article also suggest that the evidence is sufficient for the purpose of supporting teachers' development. Teacher-created representations of practice provide a snapshot of teacher's use of knowledge and sample student work. The scope of the prompts is rich enough for at least some proportion of teachers to demonstrate mathematical understanding in the sense of Simon (2006). The prompts are also expansive enough for at least some proportion of teachers to explicitly address the student work. We note that while the argument of available and appropriate may be supported by the design of tasks, an argument of sufficiency can only be made *post hoc*, rather than *a priori*. For this reason, we point to our present work as part of the argument.

We originally set out to use all four dimensions. We found that whereas we had robust evidence for Foundation and Contingency, our evidence for Transformation and Connection seemed thin. In particular, Foundation is identified through teachers' use of mathematical terminology, their awareness of their purpose, and their command of the mathematics. Contingency can be examined through teachers' responses to student ideas. We were able to identify these moments consistently across teacher-created representations of practice. However, Transformation relies on teachers' choice of examples of instructional materials, but these were highly constrained by our prompts for teacher-created representations of practice. Connection involves ideas over time, and teacher-created representations of practice often focused on a relatively small chunk of teaching.

Nonetheless, Foundation and Contingency as we use them do echo the distinction between content knowledge and pedagogical content knowledge, while also benefiting from Rowland's work to articulate where to identify these areas. Ultimately, we do believe it would be possible to detect Transformation and Connection, and potentially ranges of these dimensions, in teacher-created representations of practice. However, the prompts for these representations would have to be designed differently than the ones that we used.

3. Context

This study was conducted as part of the MODULE(S^2) Project, which seeks to connect university mathematics coursework in secondary teacher preparation programs to secondary teaching practice (n.d.). MODULE(S^2) materials address the areas of algebra, geometry, mathematical modeling, and statistics. Throughout all MODULE(S^2) materials, there are opportunities for teachers to apply mathematics to teaching. Example activities include considering student thinking, discussing common student conceptions, and connecting to learning standards. These opportunities arise at least once per every two intended weeks of curriculum material use.

Each set of materials was written by an authorship team composed of mathematicians, mathematics educators, and practicing secondary mathematics teachers. Teams ranged from 3 to 5 persons each. The second author of this article co-wrote geometry materials, and the first author co-wrote algebra materials. The authorship teams collectively developed common writing standards regarding prompts for representations of practice and approach to content. Materials were initially trialed by members of the authorship teams, and later used by instructors external the authorship teams. Authorship teams revised materials through iterative cycles throughout the project grant period in part informed by instructor feedback and project goals (Bryk et al., 2015).

In materials written after the adoption of the writing standards for the MODULE(S^2) project, prompts for teacher-created representations of practice describe a teaching scenario that specify a student-level task, a goal for the scenario involving engaging students in a mathematical practice. Prompts also include images of secondary student work created by secondary students or a portion of a classroom discussion that came up while working on the student-level task. Classroom teachers on authorship teams collected samples of actual student work to represent in applications of mathematics to teaching.

The project advertised the materials to listservs and professional networks of university faculty in mathematics and mathematics education. Instructors from seven universities agreed to participate in this study. They received in-person professional development the summer prior to teaching as well as ongoing support via online video calls during the academic year. Instructors reported that these sessions, along with the materials themselves, supported successful implementation.

3.1. *Mathematical context for teacher-created representations of practice*

The data analyzed for this study are prospective secondary teachers' responses to prompts for teacher-created representations of practice included in the MODULE(S^2) geometry materials. These materials take a transformation perspective, which is characterized by defining congruence and similarity via transformations (Usiskin & Coxford, 1972). The transformations critical to congruence and similarity are reflections, rotations, translations, dilations, and their compositions. The data come from units on *Congruence Transformations* and *Similarity Transformations*. There were three units in total in the MODULE(S^2) geometry materials; a unit on *Axiomatic Systems* preceded the others. We did not include the representations of practice from *Axiomatic Systems* because they were written prior to the adoption of the writing standards.

MODULE(S^2) geometry materials featured activities that emphasized the "logical necessity" (Simon, 2006, p. 365) of connecting the definition of each transformation type to its construction and identification. The materials advised instructors to use these activities prior to assigning prompts for teacher-created representation of practice in each unit. As well, instructors were asked to have teachers discuss the student-level task in the representations of practice, and provided activities to do so. These activities were intended to help teachers notice features of high-quality responses to the student-level task. To our knowledge, all instructors enacted these activities.

3.2. *Prompts for teacher-created representations of practice used in this study*

There were four prompts for creating representations across the two transformation units. Two prompts asked teachers to write narratives detailing how a classroom discussion might proceed, and two asked teachers to video-record themselves as they would respond to the students whose work is provided. These prompts asked prospective teachers to envision a response to the students that would move students toward understanding connections between definitions and constructions of images of relevant transformations. Table 1 summarizes the prompts given to prospective teachers, and Fig. 1 shows images of student work from some of the prompts.

Table 1
Description of prompts for teacher-created representations of practice.

Order	Medium	Response Format	Mathematical Content
1	Written	Written plan for class discussion	Constructing reflections
2	Video	Video response spoken to learners	Constructing rotations
3	Written	Written plan for class discussion	Distinguishing dilation from similarity
4	Video	Video response spoken to learners	Constructing dilations

Sample secondary student work given to prospective teachers

<table>
<tr><td colspan="2">(a)</td><td colspan="2">(b)</td></tr>
<tr><td colspan="2">As students are working on rotations of a flag, you observe two students with the following work completed.</td><td colspan="2">As students are working on constructing reflections, you observe two students with the following work completed.</td></tr>
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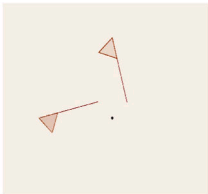
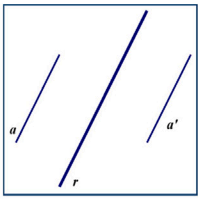
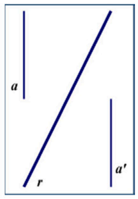
<tr><td></td><td></td><td>
</td><td>
</td></tr>
</table>

Fig. 1. Sample secondary student work given to prospective teachers.

4. Data & method

4.1. Overview

We sought to develop a framework for characterizing the variation in Foundation and Contingency use observed in teacher-created representations of practice. To do so, the authors analyzed teacher-created representations of practice in two rounds of coding. These efforts resulted in the operationalizations of Foundation and Contingency described earlier in Section 2.4 as well as the framework to be discussed later in this section.

4.2. Sampling

For this study, we selected data from eight cohorts across seven different teacher preparation programs in tertiary institutions, located in different regions of the US, that piloted the MODULE(S^2) geometry materials in one-semester content courses for prospective secondary mathematics teachers. The total dataset included more than 300 teacher-created representations of practice. The first round of coding used 75 teacher-created representations from 48 teachers (2 representations/teacher × 27 teachers + 1 representation/ teacher × 21 teachers) from the first year of data collection as well as initial data from the second year of data collection. The second round of coding used a purposive sample of 31 teacher-created representations of practice (4 representations x 7 teachers + 3 representations x 1 teacher) drawn from the representations of 62 teachers with completed pre- and post-geometry assessments from the full set of data from the second year. The rounds were sequential. There were 18 months between the conclusion of the first round and the beginning of the second round. During the second round, we continued to refine operationalizations of Foundation and Contingency generated in the first round of coding. We also continued to refer to the Knowledge Quartet website to ensure consistency in how we interpreted the constructs. There was an overlap of 2 representations from 1 teacher between data in the first round and second round. We coded these representations *de novo*. After the second round of coding concluded, we cross-checked second round and first round coding of overlapping representations, and found that they were consistent for both Foundation and Contingency.

We designed the purposive sample using a similar rationale to Rowland's (2013) selection for developing Knowledge Quartet codes: to document the range of potential knowledge use. Rowland (2013) used teachers' performance on an assessment of their mathematical knowledge as a proxy for teachers' knowledge in teaching, and selected a sample that "reflected the range of outcomes" (p. 19). Analogously, the purposive sample for this study was determined using performance on a subset of the Geometry Assessments for Secondary Teachers (GAST) instrument (Mohr-Schroeder et al., 2017). We now give details about the GAST and how we used GAST pre/post results to determine sampling. We blinded ourselves to teachers' GAST performance until after we completed coding. None of us had scored the pre/post-tests. A scorer collated samples for us without disclosing scores.

4.2.1. GAST instrument

There are 26 questions in total on the original GAST. We consulted our project's advisory board, which included Mohr-Schroeder, the lead researcher for GAST development, about using only questions from the instrument addressing content in the MODULE(S^2) geometry materials. Based on this consultation, our pre/post-forms contained 14 questions, with a total score of 22. Among the teachers whose data we collected for the second round, 61 had completed both pre- and post-semester GAST forms. The maximum pre-assessment score attained was 14. The maximum post-assessment score attained was 17.

As context for interpreting the GAST scores, we note that the original GAST forms were validated to measure mathematical knowledge for teaching geometry using a sample of predominantly practicing teachers. Our sample was strictly prospective teachers. The mean score on the full GAST form was 20 points out of a maximum of 30 (Mohr-Schroeder et al., 2017). Hence, our sample's maximum score may be comparable to their average score. Mohr-Schroeder et al. did not compare prospective and practicing teacher performance in their sample. However, using a different instrument to measure mathematical knowledge for teaching geometry, Herbst and colleagues reported that practicing secondary teachers score, on average, higher than preservice teachers do. Moreover, practicing teachers with more experience teaching geometry courses perform better than those with less experience (Herbst & Kosko, 2012; Milewski et al., 2019). Milewski et al. (2019) reported that most prospective teachers' scores were comparable to those in the lower half of practicing teachers' scores. It is not surprising that the teachers in our sample scored, overall, in a lower range than those in Mohr-Schroeder et al.'s sample.

4.2.2. Sampling using GAST

To document the range of potential knowledge use in teaching, we assigned "high performance" and "low performance" thresholds for each item on the GAST. GAST items used in this study were scored out of 1, 2, or 4 points. Thresholds were as follows. For 1 point items, "high performance" was 1 point; for 2 point items, "high performance" was 2 points; and for 4 point items, "high performance" was 3 or 4 points. Otherwise the score was categorized as "low performance".

We computed the percentage of "high-pre-score"-"high-post-score" pairs and "low-pre-score"-"low-post-score" pairs. In theory, someone's GAST responses could receive 100% "high-pre-score"-"high-post-score" if, on each item, their responses had a "high performance" score in both pre- and post-tests. Someone's GAST responses could also theoretically receive 100% "low-pre-score"-"low-post-score" if, on each item, their responses had a "low performance" score in both pre- and post-tests. In reality, most teacher received a mixture of all combinations across items: high/high, low/low, low/high, and occasionally high/low.

We narrowed the pool to 20 teachers (10 +10 teachers), consisting of:

- The teachers with the 10 highest percentages of "high-pre-score"-"high-post-score" item scores, and
- The teachers with the 10 highest percentages of "low-pre-score"-"low-post-score" item scores.

In the first group, high/high percentages ranged 45–73%, and low/low percentages ranged 0–45.4%. In the second group, high/high percentages ranged 0–27%, and low/low percentages ranged 60.0–83.3%. Across the larger pool from which this purposive sample was drawn, high/high percentages ranged 0–72.7% and low/low percentages ranged 9.1–83.3%.

We then examined representations from the 8 teachers out of the 20 who had submitted all 4 representations of practice selected for this analysis (see Table 2). During analysis we realized that one assignment had been scanned incompletely. At this point in the study, we no longer had access to contact the teacher and thus eliminated this one representation from our sample. This resulted in a sample of 31 teacher-created representations of practice.

4.3. Developing a framework to characterize Foundation and Contingency

To develop a framework for characterizing foundation and contingency, the authors first considered Rowland and colleagues' descriptions of the Foundation and Contingency dimensions of the Knowledge Quartet (e.g., Rowland et al., 2016). We reflected on how these descriptions may apply to the specific teacher-created representations of practice analyzed and, at the same time, how they may apply to other domains. To do so, the authors involved researchers with expertise in a variety of mathematical domains—such as mathematical modeling, algebra, and geometry—in our discussion.

We emphasize that our codes throughout the study represent *demonstrated* use of mathematical knowledge. We do not claim that this is the totality of the teachers' knowledge; the knowledge may have been tacit but possessed. It is therefore incorrect to extrapolate about the entire body of a teacher's knowledge based on these representations of practice. However, these data may still give some insight into how mathematical knowledge use is expressed in the context of a university content course.

We determined coding in both rounds by consensus. We only finalized a code when all coders agreed on the code assigned, as well as the reasoning. The reasoning was then incorporated into the operationalizations used, and all codes made to that point were checked against the revised operationalization.

In our first round of coding, we were able to reconcile most but not all differences for Foundation codes across four levels. However, efforts to agree on distinctions among four levels of Contingency observations reached no conclusion.

Table 2
Purposive sampling of teachers ordered by "high-pre-score"-"high-post-score" item percentage.

teacher	high-high proportion*	low-low proportion**	GAST (Pre)	GAST (Post)	GAST (Difference)
GMM205	45%	27%	9	12	+ 3
GMM202	45%	27%	7	11	+ 4
GTA218	45%	36%	9	10	+ 1
GMM308	27%	64%	5	9	+ 4
GMM302	18%	64%	6	9	+ 3
GTA208	18%	73%	2	3	+ 1
GMM206	17%	83%	1	6	+ 5
GMM201	11%	67%	3	9	+ 6

*high-high proportion = proportion of "high-pre-score"-"high-post-score" GAST item percentage.
* *low-low proportion = proportion "low-pre-score"-"low-post-score" GAST item percentage

In the second round of coding, for both Foundation and Contingency, we used two categories of "High" and "Developing". We anticipated that having two categories would result in greater reliability among coders, because in the first round, we were most unable to draw clear distinctions among middle and lowest categories. We began by coding demonstrated Foundation. In contrast to the first round, where nearly half the codes needed reconciliation, only 4 codes for 31 representations needed reconciling. We repeated this process for demonstrated Contingency. Contingency coding required reconciling 12 codes across the 31 representations.

5. Results

Our primary result is a framework for characterizing demonstrated high and developing knowledge in the Foundation and Contingency dimensions in teacher-created representations of practice created in the context of a university mathematics course. Fig. 2(a) shows this framework across domains. Fig. 2(b) specializes the framework to definitions of geometric transformations. Fig. 3 shows assigned codes for all representations of practice in the purposive sample. Table 3 shows the four combinations of High and Developing codes.

We now describe the four combinations of High or Developing Foundation and Contingency with selected teacher-created representations of practice. To do so, we take advantage of the fact that there is one representation of practice that showcases all four combinations. This is the written prompt in Fig. 3, Column 2. Fig. 4 shows the full text of this prompt (the student work is similar to that in Fig. 1). Fig. 5 shows a definition for reflection suggested in the curriculum materials.

5.1. High Foundation/High Contingency

Before we begin, we remind the reader that all illustrations are based on responses to the prompt shown in Fig. 4. While we ground our descriptions in details of the chosen prompt, we also highlight aspects that generalize across the category (High/High, Developing/High, High/Developing, or Developing/Developing). After each specific description, we step back and summarize the category's signal qualities.

In our framework, the quality of demonstrated Foundation is characterized by linking constructions to the definition, and the quality of Contingency is characterized by integrating student work into the work of connecting constructions and definitions. GMM302′s representation of practice was coded as High Foundation/High Contingency. GMM302 began their representation of

Framework for Characterizing Foundation and Contingency Knowledge

(a) Across domains

	FOUNDATION	CONTINGENCY
HIGH	Recognizes logical necessities in patterns and relationships in the intended mathematics.	Frames questions or explanations about patterns and relationships in terms of given student thinking.
DEVP	Does not explain patterns or relationships in the intended mathematics.	Superficial use of student thinking in explanations of patterns or relationships.

(b) Within definitions of transformations, in geometry

	FOUNDATION	CONTINGENCY
HIGH	Recognizes the logical necessity connecting the definition of a transformation to ways of constructing a transformation image *Examples:* ● Explains a method of construction by marking points on a preimage and then "applying the definition to each of the marked points" to obtain the image ● Reasons that an attempted image is incorrect by showing that it does not satisfy the transformation definition	Frames questions or explanations about connection between construction and definition in terms of given students' thinking *Examples:* ● Directs attention to student work to understand the idea that all properties of the definition must be followed to produce a correct construction ● Engages students in selecting locations in sample student work, and reasoning whether the definition is satisfied
DEVP	Explicitly or implicitly treats the definition of a transformation as separate from constructing images, and/or demonstrates lack of understanding of definition *Examples:* ● Describes a method for constructing, and never mentions any definition ● Provides incorrect definition	Superficial use of student thinking in explanation of connection between construction and definition *Examples:* ● Evaluates student work as "right" or "wrong"; does not cite work otherwise ● Provides a correct explanation that does not reference student work ● Only uses own extrapolations of student thinking, and does not incorporate student contributions provided in prompts for creating representation of practice

Fig. 2. Framework for Characterizing Foundation and Contingency Knowledge.

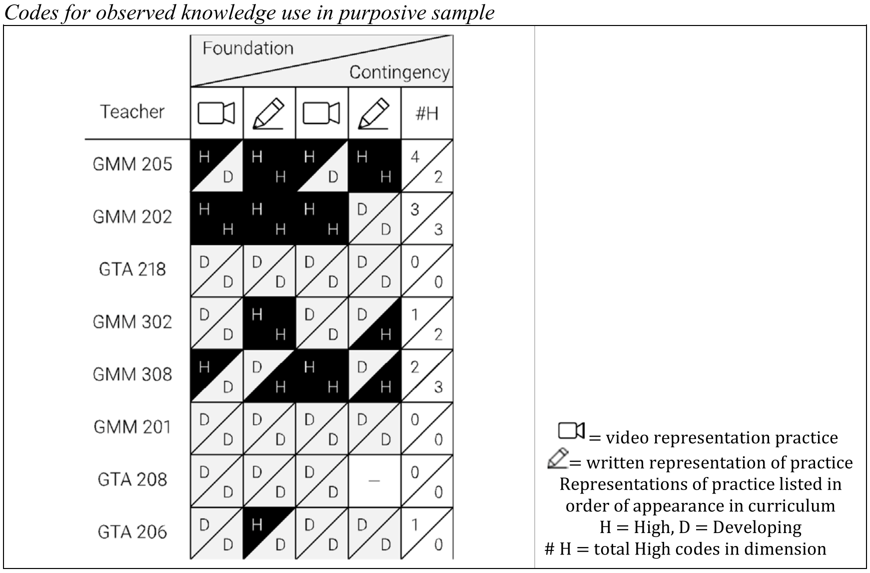


Fig. 3. Codes for observed knowledge use in purposive sample.

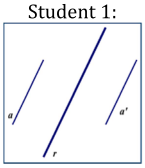
Table 3
Foundation/Contingency code combinations.

Combination Foundation/Contingency	Teachers whose representations of practice were coded with this combination (how many)
High F/High C	GMM205(2), GMM202(3), GMM302(1), GMM308(1)
Developing F/High C	GMM302(1), GMM308(2)
High F/Developing C	GMM205(2), GTA206(1)
Developing F/Developing C	GMM202(1), GTA218(4), GMM302(2), GMM201(4), GTA208(3), GTA206(3)

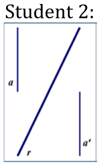
Summary of prompt given to teachers to create a written representation of practice

Students in your 9th grade math class have defined the isometric transformations and are working on performing transformations in preparation for exploration of the properties of the transformations. As students are working on reflections of a segment across a line, you observe two students with the following work completed.

Student 1:



Student 2:



Write a paper in which you clearly describe a plan for how you will conduct a whole class discussion which will allow you to elicit student thinking about these reflections, with specific use of the two example students' work, and move the class toward understanding connections between methods of reflection and the definition of reflection. Your plan for the discussion should include discussion questions, descriptions of the ways you anticipate that students might respond to questions, and any appropriate tasks that will move student thinking forward. Your response should indicate your understandings about the definition of reflection and ways in which this is applied to various methods of construction.

Fig. 4. Summary of prompt given to teachers to create a written representation of practice.

practice:

To start, I would draw the student responses and our definition of Reflection on the board. [.] Pointing to the first response, [I would ask,] if we were to draw a line between the points P and the corresponding P's, what can we tell about the line segments made by P and P'? As students respond, I draw and make the corresponding changes to the figure on the board (see Fig. 6(a)).

After describing some potential responses from students, GMM302 prompted students to link construction and definition: “What is

Version of class definition of reflection

A reflection across a line L is a transformation that, for every point P in the plane:

- $P'=P$ if P is on L
- L is the perpendicular bisector of segment PP' if P is not on L.

Note: These materials use the convention that P' refers to the image of a preimage P under the transformation discussed.

Fig. 5. Version of class definition of reflection.

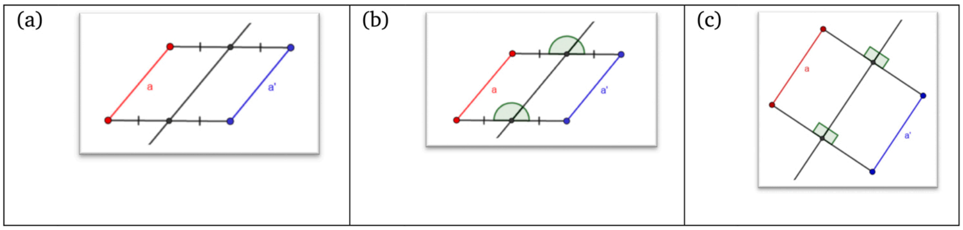


Fig. 6. Board drawings proposed by GMM302 to link construction to definition.

it we know about our line of reflection in regard to our definition of reflection?" GMM302 then marked the angles (see Fig. 6(b)) and asked students questions to review the two defining properties of perpendicular bisectors (bisecting, and with perpendicular angles). GMM302 posed, "Since our main problem here is the angles, how might we approach this in a way that results in right angles instead?" Finally, after drawing a correct reflection (see Fig. 6(c)) but without evaluating it as such to the students, GMM302 asked, "Looking at our new figure, does this hold true to the definition of a Reflection?" GMM302 concluded, after describing potential responses, "Yes, it does hold true. So, we know [segment] a' is the reflection of [segment] a across the given line."

We coded this representation as High Foundation/High Contingency because GMM302 created tight connections from incorrect and correct images to the definition, positioned students to engage with these links, and did so while centering student work. Other examples that were coded as High Foundation/High Contingency also exhibited close connections between the definition of the transformation and its construction. In each case, the definition was used as a tool to check the correctness of the provided student work (as in the case of GMM 302) or to develop explanations through questioning. For example, a teacher might provide an image and ask questions of the class relating to each portion of the definition of the related transformation, either with specific mention of the definition by the teacher or by asking questions that would lead to the definition being provided by students in the class.

5.2. Developing Foundation/High Contingency

GMM308's response was coded Developing Foundation/High Contingency. GMM308 began their representation of practice by analyzing the student work, and suggesting what may have been going on in the students' mind that led to these constructions.

> *It looks as though they have drawn lines across the line of reflection from each point to the reflected point. I believe that they have thought that since it is reflected that the distance from the line of reflection is now opposite for each point (the point on top of the reflected image is the same distance as the point on the bottom of the pre-image and vice versa).*

In this way, GMM308 exemplified the notion of interacting reflectively with student thinking (Thompson, 2000). GMM308 then described linking the work to the definition:

> *I would use [Student 2's work] to discuss with students how this attending to some points of the definition, but not quite (sic). They have used the idea of the same distance from the line of reflection, but it was utilized incorrectly. I would use this to be able to discuss with students how this doesn't fully fit the definition of a reflection and how we can fix that. We would work as a class to improve the original reflection and make sure it fits all of the necessary components of the definition needed.*

This representation of practice exemplified High Contingency—it identified specific connections *from given student work* to the definition of reflection, including the role of perpendicular bisectors. However, it did not articulate the reasoning about perpendicular bisectors precisely, and so we assigned a Developing Foundation code. This was the case for GMM308's presented analysis of both provided samples of student work.

This pattern was repeated in each representation coded Developing Foundation/High Contingency. In each, the response begins by drawing out the thinking provided by one of the students related to the appropriate transformation. Then, the responses envision questions to students where pieces of the definition for the transformation are provided but not connected to the construction of the transformation. By starting with the given student work, these responses demonstrate High Contingency. Unclear or inaccurate connection between the definition and the construction of the transformation leads the response to be coded with Developing Foundation. In our data, there were two ways for a connection to be coded as Developing Foundation with High Contingency. First, as demonstrated by GMM308, the reasoning about the definition is imprecise. Second, a teacher may reason about the definition incorrectly.

5.3. High Foundation/Developing Contingency

In our framework, demonstrating High Foundation requires clearly linking the construction to the definition whereas demonstrating High Contingency relies on the use of student work to make the connection between the definition and construction. GTA206′s representation of practice was coded High Foundation/Developing Contingency.

This representation of practice began by posing to the class, "What do we know about reflections?" and encouraging discussion specifically focused on the definition of reflection. GTA206′s representation then hypothesized student-generated definitions that were correct but imprecise, and how they would clarify the role of the perpendicular bisector in the definition:

> *I will discuss this by showing students different examples on the board on how this comes about by drawing a perpendicular bisector between point A on Object 1 and point A′ on Object 2. I will do this with each point of the object to show that the perpendicular bisector does in fact make a right angle at the line of reflection. Doing so links the definition to the way in which one can determine whether or not two figures are in fact a reflection of each other.*

At this point, GTA206′s representation of practice turned to more examples but did not use the exact student work provided: "I will at this point use student 1 and student 2′s examples (sic), but not the exact same lines as the students drew. I will change them so that they do not feel as if I am targeting their work." GTA206′s representation of practice suggests that learners will be able to determine their errors by examining other work that is not their own. The discussion of reflections concludes with an assignment in which learners would "draw a perpendicular bisector from each original figure point to the new reflected figures."

GTA206′s representation of practice made a clear connection between the definition and the construction of the reflection, exemplifying High Foundation. However, GTA206′s representation of practice did not draw on the provided learner thinking in moving the class forward in their understanding of reflections. This representation of practice provides no evidence of knowledge for responding to potentially unexpected contributions from students, because they did not reference the given student contributions at all. Perhaps GTA206 left their knowledge of the connection tacit rather than expressed, because they prioritized the students' potential emotions. Although consideration for the learner's feelings as their work is shown is important, it is also vital to use the thinking of the students in the class in order to move their thinking forward. We coded this response as Developing Contingency, but also use this case to remind us that we are coding only proxies of knowledge.

The other examples in our sample that were coded as High Foundation/Developing Contingency were video responses. In each case, the teachers correctly described the connections between the provided student work and definitions of the appropriate transformations, demonstrating High Foundation as they evaluated the student work. In their descriptions of how they would respond to the students, they each directed students to new examples rather than the provided work and related these new examples to the definition. In this way, the teachers responded to the students without using the given student work.

5.4. Developing Foundation/Developing Contingency

A response that demonstrates Developing Foundation and Developing Contingency lacks connection between definition and construction, and furthermore does not use the given student work to address the concepts. GMM201′s representation of practice illustrates this type.

GMM201′s representation of practice opened by sharing the two student samples and asking students "if they agree or disagree with the two works." GMM201 intended to probe students for their reasoning, and give them time to share their responses with the whole class. Although potential student responses are given that include reference to the distance of a point from the line of reflection, there is no resolution or clear use of the definition of reflection.

Rather than resolving this discussion, GMM201′s representation of practice then posed a new task to students, where the students were asked to reflect provided shapes over a given line. Then, after monitoring students' work on this task, the representation of practice stated that the "teacher will provide the definition of reflection to the students, and work on another example on the board to show how the definition is related to the work." This is the only mention of a definition of reflection in GMM201′s response. Note that there was no elaboration of what the definition is or how it is used to construct a reflection. GMM201′s representation concluded by indicating that students will then be asked to revise their answers to previous problems and to determine in other ways, such as paper folding, whether or not their images are reflections.

At first read, it may seem that GMM201 used student thinking by displaying student responses and providing time for learners to make sense of the work they observe. However, this representation of practice never linked the given student work to the definition of reflection, and also makes no substantive reference to the given student work beyond displaying it. For these reasons, we coded this representation of practice as Developing Foundation/Developing Contingency.

Other examples coded in the Developing Foundation/Developing Contingency category similarly lacked substantive use of student work and explicit connection between definition and construction. The student work may be presented at some point during the representation of practice but it is not taken up beyond asking whether or not the class agrees with the work, if at all. Responses in this category did not build discussion from the initial prompts asking for agreement. In this category, the definition was either not given, or stated by the teacher with no connection to the student work samples or to construction. Students were at times directed to a new task, without connection between these pieces.

5.5. Foundation and Contingency levels in relation to GAST selection process

Recall that we selected the purposive sample in part by the proportion of “high-pre-score”- “high-post-score” GAST item percentages (shown in Table 2). We coded Foundation and Contingency for these teachers’ representations of practice. To consider the relation between our sampling procedure and the coding, we plotted their item percentage in relation to the number of High Foundation and High Contingency codes for each teacher. See Fig. 7.

Overall, as observed in Fig. 7, the proportion of “high-pre-score”-“high-post-score” GAST item percentages appear to be associated more strongly with Foundation codes than Contingency codes. This may be explained by the content of the GAST. When we analyzed GAST items administered to teachers for whether they fit the description of knowledge in the Foundation or Contingency dimensions, we only found 1 item to fit the description of Contingency: the question asked teachers to incorporate given student work. All other items assessed knowledge of particular mathematical theorems in the form of a purely mathematical question, or involved Foundation knowledge to analyze a proposed mathematical task.

Among the teachers with the highest “high-pre-score”-“high-post-score” percentages (GMM205, GMM202, GTA218; see Table 2), two exhibited high levels in the foundation and contingency dimensions on at least half of their teacher-created representations of practice. Overall, their representations of practice demonstrated greater knowledge use in Foundation and Contingency than the representations of practice from teachers selected for the highest “low-pre-score”-“low-post-score” percentages. The exceptions are GTA218 and GMM308 (selected for "low-pre-score"-"low-post-score" percentage). GTA218 only improved by one point on their post-test GAST performance; perhaps this indicates an overall difficulty with developing deeper mathematical knowledge overall in the course. This would explain a difficulty applying mathematical knowledge to the work of teaching in the representations of practice that they created, particularly if this semester was their first opportunity to apply mathematical knowledge directly to the work of teaching. Another explanation is that GTA218 did not engage with this material due to lack of interest, pressures outside of class, or other reasons.

Across all four of GMM308′s representations, two High Foundation and three High Contingency codes were assigned. As noted in the previous discussion of GMM308′s representations of practice, there is attention to potential connections between definition and construction. Developing Foundation codes were given when the connection was not precisely articulated. In other words, GMM308′s representations of practice demonstrated an attention to underlying mathematical structures, but this attention was not always expressed as precisely as needed to clearly connect the construction to the definition and thus to be coded as High Foundation.

Representations from the three teachers with the highest “low-pre-score”-“low-post-score” percentages were coded with only Developing Contingency and Foundation. Two of these teachers’ GAST scores saw the largest increase from pre- to post-test across the sample. These teachers were developing mathematical knowledge for teaching from the most basic levels, may have had difficulty applying their knowledge to teaching, despite their knowledge gain.

6. Discussion

We set out to examine observable use of knowledge in Foundation and Contingency dimensions in teacher-created representations of practice, and to characterize their variation. To do so, we analyzed a purposive sample of teacher-created representations of practice in geometry. We now address limitations of the analysis and the potential generality of the framework.

First, one limitation to coding any proxy of knowledge is that “Developing” codes may reflect the absence of evidence rather than absence of knowledge. Developing Foundation or Contingency may be related more to prioritizing different aims, such as student affect or a desire for brevity. It may also come from disengagement with the course. Yet we also observe that we collected data from courses where teachers generated definitions for transformations based on constructing images, routinely analyzed and responded to sample secondary student thinking, and discussed featured secondary student-level tasks in view of the connection to definition. These

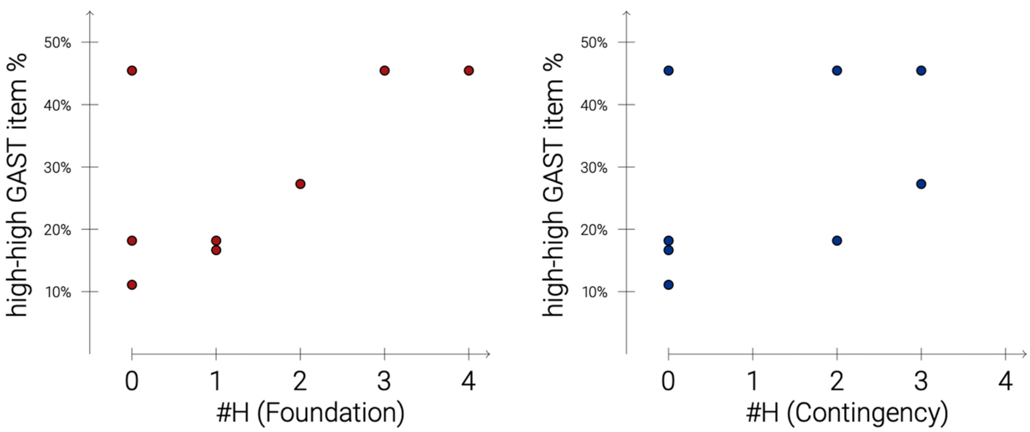


Fig. 7. Foundation and Contingency vs. proportion of high-high pre- and post-GAST item scores.

activities were offered prior to assigning the prompt for the representations of practice. Considering these curricular supports, we conclude that at least some Developing Foundation and Contingency codes indicate where a teacher's knowledge can grow, although we cannot absolutely conclude that for all responses.

Second, we would be remiss not to issue caveats about the binary scheme of "Developing" and "High". These characterizations, like other hierarchical characterizations in the literature (e.g., Ader & Carlson, 2021; Munter, 2014; Serbin et al., 2020), are not intended to be characterizations of teachers or their ultimate potential. Rather, we present descriptions of observable features of representations of practice that point to mathematical and pedagogical areas teachers may need more support to develop, and also to showcase how exemplary knowledge may manifest.

In evaluating the robustness of our framework, we consider the limitation of our data to four prompts for representations of practice in geometry, and the potential for our framework to generalize across domains. Our framework, as reported, is tailored to the use of definition to a particular concept of geometry, and derived from the analysis of a limited number of prompts. However, we see our framework as generalizable. Its underpinnings in the Knowledge Quartet (Rowland, 2013) and mathematical understanding (Simon, 2006) are intended to apply broadly to mathematics teaching and learning. The centrality of definition to mathematics, as well as reasoning with definition and assumptions (Kitcher, 1984), suggests the potential for adapting this framework to domains with strongly structured logical systems, such as algebra.

For instance, in place of linking definition with construction methods, the framework could emphasize connecting definitions with common procedures. As an example, consider inverse functions. Multiple studies have shown that students compartmentalize the various meanings and representations of inverse functions, not understanding how one is related to the other, or to definitions. Students may not see connections between "switching-and-solving" (Vidakovic, 1996), reflecting a graph about the line $y = x$, and the definition of inverse function (Brown & Reynolds, 2007; see Paoletti et al., 2018 for a review). Yet we would hope that teachers would understand these connections, and moreover, be able to apply knowledge of this connection to their teaching (e.g., Weber et al., 2020). In Fig. 8, we interpret this situation in terms of our framework.

For domains such as mathematical modeling, which apply mathematics in phases of distinctive practices (e.g., Blum & Leiß, 2007), the framework could emphasize the rationale for each phase as well as anticipation of movement across phases, for instance, knowing that the proposal of a mathematical model can be followed by considering the real world or the results of the model, that these phases can work together to refine one's model (e.g., Czocher, 2018).

Finally, we note that our case for Foundation is stronger than our case for Contingency: we have applied the framework to more data for Foundation, due to the first round coding process. Nonetheless we believe that our case for Contingency holds promise, and that the specific attention to given student work is a contribution to the field.

7. Conclusion

We now consider our results with respect to the literature on teachers' knowledge. We see Foundation knowledge as content knowledge, pedagogical content knowledge as elicited by applications of mathematics to teaching, and Contingency as an outcome of pedagogical content knowledge. Using the Knowledge Quartet allowed us to attend precisely to how mathematics can be applied to teaching when designing prompts for teacher-created representations of practice.

As Baumert et al. (2010) noted, "one of the next great challenges for teacher research will be to determine how [content knowledge and pedagogical content knowledge] can be best conveyed" (p. 168). Teacher education must address both content knowledge and pedagogical content knowledge—and in ways that connect to teaching. Rowland (2013) posited that Foundation knowledge determines the use of mathematics in the other dimensions of the Knowledge Quartet, including Contingency.

Our work provides a proof of concept of a device that can be used in multiple university mathematics courses to integrate mathematics and its application to teaching. The representations of practice we analyzed were created by teachers as part of a suite of university mathematics courses. In the representations studied, they elicited Foundation knowledge that could be integrated with Contingency actions. We believe that these results extend across content areas. Our results speak to Monk's (1994) argument for a more nuanced understanding of how mathematical knowledge may contribute to secondary teaching. We find that mathematical knowledge, such as can be taught in a university mathematics course, can be observed while explicit connections to teaching are made. Moreover, we demonstrated ways that quality in two dimensions, Foundation and Contingency, may be articulated.

Our data corroborates Rowland's posited relationship between Foundation and Contingency: High Foundation tended to correspond with High Contingency, and Developing Foundation tended to correspond with Developing Contingency. However, a High Foundation code was not a guarantee of High Contingency. We do not have the data to know whether our results corroborate Baumert et al.'s (2010) findings for content knowledge as a precursor of pedagogical content knowledge.

Our results also indicate how far we must go to meet Baumert et al.'s (2010) challenge. There were not many High Foundation/High Contingency codes, and producing a representation of practice coded as High earlier in the term did not correspond to being coded High later in the term. Moreover, there were many Developing/Developing code combinations. One interpretation is that, like Herbst and colleagues' findings, there is on average a wide gap between prospective teachers' and practicing teachers' knowledge for teaching geometry. Another interpretation is that it is difficult to transfer mathematical teaching practices learned in one mathematical context (e.g., congruence) to another (e.g., similarity); a similar result was found at the elementary level by Morris et al. (2009). Whatever the explanation, the challenge remains for connecting mathematics to its applications in teaching, and in ways that support teachers' development across content areas.

Potential generalization of framework to inverse functions

	FOUNDATION	CONTINGENCY
HIGH	Recognizes the logical necessity of connecting the definition of inverse function to ways of constructing inverse functions, representing functions and their inverses, and determining whether a proposed function is the inverse of a given function. *Examples:* ● Relates a method of construction to the definition of inverse function. ● Explains why different conceptions of inverse function (e.g., "reverse of the function process", "switching x and y and solving for y"; see Vidakovic, 1996) are equivalent, in terms of the definition of inverse function.	Frames questions or explanations about connections among representations, constructions, and reasoning about inverse functions in terms of given students' thinking. *Examples:* ● Directs attention to various given student conceptions and connects them via the definition of inverse function. ● Engages students in comparing behavior evident in students' representations of a function and its inverse, and why these illustrate the defined relationship between an invertible function and its inverse.
DEVP	Explicitly or implicitly treats the definition of inverse function as separate from representing or constructing an inverse function, and/or demonstrates lack of understanding of the definition. *Examples:* ● Describes how to construct an inverse strictly procedurally, without mention of a definition (cf. Brown & Reynolds, 2007). ● Provides imprecise definition of inverse function.	Superficial use of student thinking in explanations of connection between definition and representations, constructions, or other reasoning. *Examples:* ● Evaluates student work as "right" or "wrong"; does not cite work otherwise ● Provides a correct explanation that does not reference student work

Fig. 8. Potential generalization of framework to inverse functions.

Funding

This research was partially funded by National Science Foundation DUE-#1726723 and #1726744.

Acknowledgments

We thank Lindsay Czap for scoring the pre- and post-tests, Cynthia Anhalt for generative discussion regarding the framework reported, and Cathy Callow-Heusser and John Sutton for advising us on sampling considerations. This research was partially funded by the US National Science Foundation DUE-1726723 and DUE 1726744. Any opinions expressed here are the views of the authors, and not of the National Science Foundation.

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