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Some mathematicians' perceived and envisioned instructional relationships in secondary teaching and teaching secondary teachers

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Abstract
When it comes to content courses for prospective secondary teachers, mathematics faculty's intentions conflict with teachers' experiences: while faculty aim to influence teachers' future teaching, many teachers find these courses irrelevant to teaching. In this study, we investigate mathematics faculty's goals for content courses for prospective secondary teachers and how these goals connect to their vision of secondary teaching. This study makes two contributions. First, drawing on data from an interview study ($n=5$) of faculty who have taught secondary teachers, we articulate these mathematicians' rationales with respect to teaching secondary teachers, particularly in capstone courses. Second, we contribute a conceptual framework that supports inferences about connections between content course activities and instructors' beliefs about secondary teaching. We conclude by suggesting that the dual triad that depicts our framework may support professional development and programmatic design.

Keywords Mathematics teacher education · Mathematics teacher educators · Content courses for secondary pre-service teachers

Teacher educators come from a variety of backgrounds, including mathematics (Beswick & Goos, 2018). In conversations we have had, over the past decade, with various mathematicians who teach practicing or prospective secondary teachers, even in courses specifically designed for these teachers as part of masters-level courses or undergraduate capstone courses, one recurring theme is the challenge of connecting advanced mathematics to secondary teaching. Some mathematicians at once acknowledge the complexity of secondary teaching, and also that they have not personally taught secondary students. Yet these mathematicians must have an image of secondary learning and teaching, if only from their experience as parents now or students years ago (Lortie, 1975). It is difficult to imagine that these mathematicians could suspend this image entirely from their teaching of secondary

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teachers. How do mathematicians envision secondary teaching? How might their notions shape decisions they make when teaching prospective or practicing secondary teachers, particularly in courses designed for teachers? In this study, we sought to learn from mathematics faculty who were willing to speak how their courses for prospective or practicing secondary teachers could connect to secondary teaching.

Teaching teachers is complex work involving specialized knowledge that is not well understood (Beswick & Goos, 2018; Chauvot, 2009; Masingila et al., 2018; Zaslavsky & Leikin, 2004; Zopf, 2010). At the secondary level, many mathematics teachers perceive an incongruity between university mathematics courses and secondary mathematics teaching (e.g., Goulding et al., 2003; Zazkis & Leikin, 2010). Yet a number of mathematics faculty who teach secondary teachers do want their courses to impact secondary teaching (Lai et al., in press). There is a disconnect between mathematicians' intentions and secondary teachers' actual experience.

In the introduction to a special issue of this journal, Beswick and Goos (2018) observed:

> Although some aspects of affect, principally beliefs, have begun to feature in work on [mathematics teacher educators'] knowledge … its role has been largely incidental. Understanding the beliefs that underpin the practice of [mathematics teacher educators] must be at least as important as understanding those that influence the work of mathematics teachers (p. 425).

Beswick and Goos (2018) advocated for research into the beliefs of those who teach teachers, including the instructors' beliefs about mathematics teaching and learning. Furthermore, Li and Superfine (2018) called for research on how mathematicians design courses for prospective teachers. We take up their call. To address the problem of disconnect in secondary mathematics teacher preparation, we investigate mathematicians' beliefs about secondary teaching and their goals for mathematical courses designed for prospective and practicing secondary teachers. Because tasks convey curriculum (Doyle, 1983), and mathematicians have a tradition of conveying mathematics via problems (e.g., Arnold, 2015), we focus on how mathematicians design tasks for such courses.

Our overarching purpose is to examine how mathematicians connect their beliefs about secondary teaching to the design of tasks for prospective or practicing secondary teachers. We use "design" to include goals, activities that serve these goals, and evidence of learning relative to these goals. Using interviews with five mathematicians, we address:

1. What short-term and long-term goals do mathematics faculty attend to?
2. Which instructional relationships do mathematicians

    a. *Perceive, in their tertiary instruction, when discussing goals of their courses?* Perceived instructional relationships refer to relationships among the mathematician as instructor, the enrolled secondary prospective or practicing teachers in their course, and tertiary course content.
    b. *Envision, in secondary instruction, when discussing goals of their tertiary courses?* Envisioned instructional relationships refer to relationships among the enrolled teachers as secondary teachers, the secondary students to be enrolled in their course, and secondary mathematics content.

In reporting our work on these questions, we propose and test a theoretical framework of an extended teaching triad (cf. Cohen et al., 2003; Jaworski, 2003; Lampert, 2001; Leikin et al., 2017). We seek to model instructors' rationales of how instructional relationships drive instruction. We illustrate the model using interviews of selected mathematicians who taught university mathematics courses that aimed to connect to secondary mathematics and its teaching.

Our primary conceptual innovation is illustrating a specific-to-mathematics-teacher education connection between the dynamics and goals of instruction. We argue that our framework provides a potentially useful way to capture and compare mathematics faculty's beliefs as to their instructional intentions and enactment.

Research context

In the USA, there are more than 1,300 organizations that offer certification programs, including a number of higher education institutions. Each organization determines its own requirements on how to comply with state policy (Tatto et al., 2018). Some prospective secondary teachers obtain certification through undergraduate programs. In some states, practicing secondary teachers may not teach calculus at their school unless they hold a master's degree. As a result, some practicing secondary teachers matriculate in master's programs.

Both undergraduate and master's programs often require *methods* and *content*, where the former focus on pedagogy and the latter are mathematics courses. Content courses are primarily taught by mathematics faculty (e.g., Goos & Bennison, 2018). Mathematics faculty may also teach combined content and methods courses (e.g., Buchbinder & McCrone, 2020).

Addressing secondary school topics in university content courses and variation within

There are no standardized, national curricula for mathematics teacher educators to use (Cohen, 2010; Zaslavsky, 2007). Various organizations have proposed guidelines for teacher education (e.g., Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences (2012); National Council of Teachers of Mathematics, 2020). Yet translating guidelines into instructional materials is an open-ended task in the extreme, and higher education faculty enjoy relative autonomy in designing their courses. Hence, unsurprisingly, variation prevails in the US system.

To illustrate the variety, we use the results of two surveys of US programs. Cox et al.'s (2013) findings suggested that a number of US institutions offer capstone courses, and that 56% of these courses among institutions surveyed had the goal that "[teachers] take an in-depth look at some mathematical topics which are particularly important in secondary mathematics" (p. 4). Tatto and Bankov (2018) found that among US secondary programs, about half address axiomatic geometry; between one-fifth and one-third address geometry topics in the secondary school curriculum; and between one-fifth and one-fourth address pattern, relation, and function topics in the secondary school curriculum. No more than a third of programs addressed any particular school topic. Thus, a majority of capstone courses may address secondary school mathematics in some way, but that no more than a third of programs address any particular topic.

Goals: Mathematical content, mathematical practices, connecting to teaching practice

What do faculty strive to impart in content courses? Broadly, mathematics faculty in the USA have repeatedly expressed the need for secondary teachers to have strong *mathematical content* and deep appreciation of *mathematical practices* (e.g., Bass, 2005; Cox et al., 2013; Cuoco, 2001; Tucker et al., 2015). In the details are variations, even among mathematics faculty (Lai, 2019). Some teach secondary mathematics from an advanced standpoint (Murray & Star, 2013). Others focus more on practices than particular content (e.g., Bass, 2017; Libeskind, 2008).

We highlight *mathematical content* and *mathematical practices* because they are ubiquitous across educational policy documents across the world. We distinguish them because this distinction is made in the USA and internationally (e.g., Department of Education, 2003; Department for Education, 2014; Ministry of Education, 2007; National Council for Curriculum and Assessment, 2020; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). By *content*, we indicate particular concepts, skills, or applications. By *mathematical practices*, we mean disciplinary habits of mind such as constructing argumentation, valuing precision, or problem-solving.

In recent years, there has been a movement to incorporate connections from content courses not just to secondary *mathematics* but also to secondary mathematics *teaching practice* (Lai et al., in press). Wasserman (2018) metaphorically termed the content addressed in university as "nonlocal," whereas secondary mathematics is "local" to secondary teachers. He argued, "[T]he knowledge gained from nonlocal mathematics must serve as mathematically powerful understandings not (only) for their knowledge of nonlocal mathematics, but for the teachers' understanding of the local mathematics they teach" (p. 122). Baumert et al. (2010) found that teachers' pedagogical content knowledge predicted secondary student outcomes more than content knowledge. Baumert et al. nonetheless found that a teacher's content knowledge shaped the extent of pedagogical content knowledge they could learn. In light of this research, as well as the documented disconnect perceived by secondary teachers, it is encouraging that multiple projects have now released materials for content courses with explicit connections to secondary teaching practice (e.g., Álvarez et al., 2020; Bremigan et al, 2011; Hauk et al., 2018; Heid et al., 2015; Lischka et al., 2020; Sultan & Artzt, 2010; Wasserman et al., 2018; for a review, see Lai et al., in press).

The emerging movement to connect university content courses to secondary teaching practice largely post-dates the major studies establishing teachers' perceived disconnect (Goulding et al., 2003; Leikin et al., 2017). There is also some evidence that although some US mathematicians may be interested in making explicit connections to teaching practice, they also find this challenging (Lai, 2019). This study examines mathematicians' rationales about content course connections to secondary teaching practice, through enactment of specific tasks they see as exemplifying course goals. We leverage the autonomy that US faculty enjoy in designing their own courses, in an environment with soft policy guidance but no requirements.

Background

Mathematics teacher educators and mathematics faculty

Beswick and Goos (2018) considered mathematics teacher educators to be "anyone engaged in the education or development of teachers of mathematics" (p. 418). Under this definition, mathematics teacher educators include instructors of content courses. This inclusion is followed by others (e.g., Leikin et al., 2017; Li & Superfine, 2018; Tatto et al., 2018). Another reason to consider mathematics faculty to be mathematics teacher educators, even research mathematicians, is the need for inclusive collaboration across disciplinary boundaries in teacher education (Goos & Bennison, 2018).

In this study, we use *mathematicians* and *mathematics faculty* interchangeably; all participants received doctoral degrees in mathematics. For clarity, we only use *teacher* to refer to secondary level teachers, and we only use *instructor* to refer to tertiary level instructors.

Goals of mathematicians who teach teachers: a sparse area of research

Hoffman and Even (2018) noted, "The existing literature concerning mathematicians' positions regarding academic studies of teachers is rather limited. It mainly comprises forewords appearing in mathematics textbooks intended for teachers … and position papers" (p. 3–100). To address this gap, they interviewed five research mathematicians who taught advanced mathematics courses to practicing secondary school teachers in Israel. These mathematicians wished to convey the essence of mathematics as a discipline, and for teachers to understand how mathematics is done, as well as the practical and theoretical worth of mathematics. These themes resemble those of Leikin et al. (2017), who interviewed four mathematicians teaching at Israeli institutions. They also capture most but not all themes found by Hodge et al. (2010), who asked seven mathematicians to "describe his or her vision of a 'good' secondary mathematics teacher" (p. 649). The remaining themes of Hodge et al. (2010) are that teachers should have mathematical confidence, have good presentation skills, and treat secondary students as humans. However, when these mathematicians were then asked to describe how various advanced mathematics courses contributed to teachers developing the traits of a "good" secondary mathematics teacher, all mentioned mathematical content and practices, but none cited working with secondary students or confidence.

There is a scarcity of studies of mathematicians' views of the role of mathematics courses for teachers. Neither Hoffman and Even (2018) nor Leikin et al. (2017) reference other studies of mathematicians' views of the goals of teacher education. Hodge et al. (2010) does not have a section addressing prior literature. More generally, there is also a lack of empirical studies on teacher educators' decisions (Karsenty et al., 2021).

The studies of Leikin, Hoffman, and colleagues contribute systematic elaboration of mathematicians' views of how mathematicians wish to influence secondary teachers' notions of mathematics. Yet teachers may have obligations outside the discipline; they must also attend to students, organizational norms, and perhaps society (Ball, 1993; Herbst & Chazan, 2020). We are interested in mathematician's views on how university mathematics courses can influence (not only) teachers' views of *mathematics*, but also teachers' *mathematics teaching*.

Studying mathematics teacher educators' knowledge and beliefs

Designing and teaching content courses requires specialized mathematical knowledge (Beswick & Chapman, 2012; Tzur, 2001; Zopf, 2010). Studying mathematics teacher educators' knowledge can benefit from using tools for studying mathematics teaching (Chauvot, 2009). Our work, like Li and Superfine's (2018) work, draws on Brown's (2009) notion of teaching as design. Like Zopf (2010), we use Cohen et al.'s (2003) teaching triad, which involves relationships among the teacher, students, content, and context.

Harnessing knowledge draws on beliefs and goals. For instance, the mathematics teacher educators in Li and Superfine's (2018) study believed that prospective primary teachers may not appreciate conceptual understanding. These educators then drew upon their knowledge of teaching teachers to foster conceptual knowledge. Zazkis and Mamolo's (2018) amalgamation of a teacher educator aimed to develop teachers' mathematical awareness. This educator then used Knowledge of the Mathematical Horizon (Zazkis & Mamolo, 2011) when selecting tasks and responding to teachers' thinking. Appova and Taylor's (2019) expert teacher educators aimed for prospective primary teachers to develop particular orientations toward teaching; to do so, they drew on extensive knowledge of instructional resources and student cognition.

Studies of mathematics teacher educators' beliefs as they play out in practice are comparatively rare (Karsenty et al., 2021). Moreover, most studies of mathematics teacher educators take place at the elementary level. When they take place at the secondary level, they do not focus on conveying specific mathematical knowledge (e.g., Karsenty et al., 2021). We view our work as similar to Li and Superfine's (2018) and Appova and Taylor's (2019) studies, in that we investigate beliefs and goals, but different in that we focus on secondary education. Like Karsenty et al. (2021), we are interested in how mathematics teacher educators' beliefs inform their instruction, but we focus on conveying specific mathematics rather than on mathematics only as a lens.

Conceptual perspective

Instructors' knowledge, beliefs, and goals shape the enactment of tasks (Schoenfeld, 2010; Stein et al., 2007). Instructors' beliefs and goals mediate their use of knowledge in teaching (Brown, 2009). Our purpose is to examine beliefs about secondary teaching, and goals for content courses for secondary teachers. Our sense of "belief" aligns with Schoenfeld's (1992) characterization that beliefs are "understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (p. 358), where mathematical behavior is extended to include mathematics instruction.

Karsenty (2020), Konuk (2018), and Prediger et al. (2019) reviewed the role and development of frameworks for examining the practice of mathematics teacher education. They suggest the following strategies: extending existing frameworks for teaching; considering analogies between teaching and teacher education; nesting aspects of teaching into the content that teacher educators intend to convey; and unpacking how teacher educators, as they engage in educating teachers, evolve their personal understandings. Here, we extend an existing framework for teaching—a teaching triad—via analogies and nesting.

Instructional relationships in a teaching triad

Multiple scholars have proposed triadic conceptions for teaching (e.g., Herbst & Chazan, 2012; Jaworski, 1992; Leikin et al., 2017). Here, we model instructional relationships in a triad following Cohen et al. (2003) and Lampert (2001). These relationships (see Fig. 1) include those among the teacher, students, content, and context. Teachers' practice constitutes noticing and responding to these relationships, which shape students' content practice. We use this triad because it has been used to analyze the work of mathematics teacher education (e.g., Zopf, 2010), it focuses on the persons involved, and the edges are relatively well defined.

The edge between students and content reflects students' beliefs, attitudes, and knowledge about the content; here, *learning mathematical practice* occurs. The edge between teacher and students captures teachers' beliefs, attitudes and knowledge about students, as well as teachers' interactions with students. (It also captures students' beliefs, attitudes, and knowledge about the teacher and other students, but we focus on teachers and instructors here.) The edge between teacher and content shows the teacher's beliefs, attitudes, and knowledge about the content, including personal mathematical experiences and choices of what to address when teaching. The edge between a teacher and mathematical practice defines how a teacher may influence and be influenced by the student's learning. All of the relationships a teacher operates in—with students, content, and mathematical practice—together form their *teaching practice*. Finally, relationships do not occur in a vacuum, but rather in a context; one cannot separate interactions from the context in which they occur (Hawkins, 1967/2002). And so, a box circumscribes these relationships.

Projecting and extending this teaching triad

When teaching teachers, there are two triads in play. The instructor directly *perceives* the first triad in their tertiary teaching, and in their mind *envisions* a second triad in a secondary school setting. *Perceived relationships* describe those relationships among the mathematician, the prospective teachers in the mathematician's course, and the course content. *Envisioned relationships* are those projected among future teachers, secondary students, and secondary content, as envisioned by the mathematician. Figure 2 depicts these relationships schematically.

Secondary teachers and content are common to both sets of relationships. We conceptualize the mathematics faculty projecting the teachers they currently teach into envisioned classrooms. We visualize the mathematics faculty projecting the content they teach onto the content that secondary teachers may draw on in the future. The intended content of a university content course overlaps with, but does not subsume, secondary mathematics content. This overlap accounts for the possibility of treating a secondary mathematics topic at greater depth than may be possible or expected at the secondary level. The content node privileges the teachers' and teacher educators' viewpoint, rather than the secondary students'.

The context at play may differ across perceived and envisioned relationships. But since these contexts are related for each mathematician, and also for simplicity, we only drew one box in Fig. 2. The dotted line indicates separation, if partial, across the contexts.

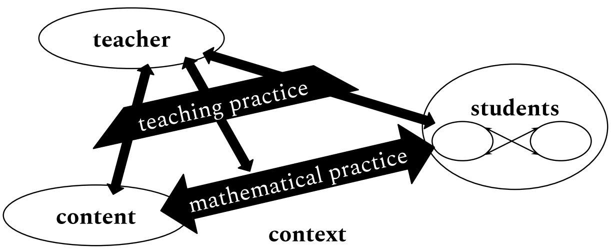


Fig. 1 Model of instructional relationships based on Lampert (2001) and Cohen et al. (2003)

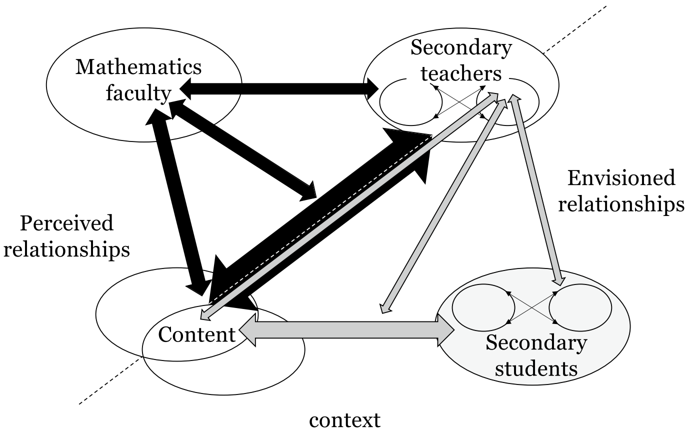


Fig. 2 Perceived and envisioned instructional relationships

Overall, teaching is design, and design is goal-directed (Brown, 2009). As Schoenfeld (2010) showed, teachers make decisions by weighing potential consequences in terms of implicit or explicit goals. When teaching teachers, long-term goals can involve ideas and practices that endure into teachers' future teaching. Shorter-term goals may involve teachers' relationship with content taught. Enacting these goals involves offering curricular activities and noticing teachers' engagement. Figure 3 shows this viewpoint, which we view as akin to an internal theory of change (cf. Reinholz & Andrews, 2020). It is internal in the sense that it is local to each faculty.

Finally, we note that envisioned relationships are similar to, but not the same as, Chauvot's (2009) category of knowledge of context for a mathematics teacher educator, which is based on Grossman's (1990) category of knowledge of context. For Chauvot and Grossman, knowledge of context is knowledge of the actual context in which teachers work. For us, envisioned relationships are the instructors' personal understandings of secondary teaching that may shape the ways that they engage with teaching secondary teachers. These understandings may or may not be informed by experiences in prospective teachers' future school districts. Analogously to how knowledge of context can mediate actions and goals taken on by a teacher educator, we posit that mathematics faculty's envisioned relationships can mediate their actions and goals.

Data and method

The primary aim of this study was to better understand the goals mathematics faculty attend to, as well as how these goals inform instructional decisions, in content courses for secondary mathematics teachers. We used a multiple case study design (Yin, 2008) to illustrate the varying rationales the mathematicians developed in such content courses.

Participants and context for recruitment

The five participants for this study were mathematics faculty members at various undergraduate institutions across the USA. All participants received doctoral degrees in

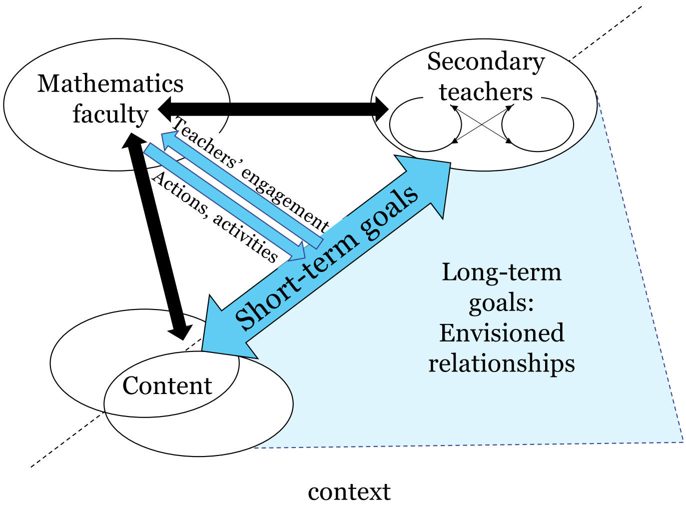


Fig. 3 Perceived and envisioned teaching triads in terms of long- and short-term goals of mathematicians for teacher education

mathematics and published in peer-reviewed research journals in their listed area of expertise. All names are pseudonyms. Table 1 summarizes participant information.

All participants had experience teaching at least one content course or combined content-methods for secondary teachers, as described in Table 1. All participants were recruited from a series of professional development workshops for mathematicians focused on mathematical knowledge for teaching (Ball et al., 2008) and the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). Some, but not all, of these workshops were co-organized by one of the authors of this paper. One to four years had lapsed between participants' workshop attendance and this study.

Because the aims of the workshops and this study overlap, and because context and knowledge influence instruction, we now describe the workshop in brief. The workshop was two days in duration. On the first day, a workshop leader reviewed Standards for Mathematical Practice in the Common Core and elaborated upon various Progressions Documents (Common Core Standards Writing Team, n.d.) that structured the grade-level content standards in the Common Core. Participants then reviewed primary, middle, and secondary level tasks for coherence with content standards at those levels. Another leader then discussed examples of mathematicians' involvement in education, such as reviewing school level textbooks. The second day made the argument that the mathematical knowledge entailed in teaching is specialized knowledge. To do so, a leader (the first author) engaged participants in tasks designed to assess mathematical knowledge for teaching, similar to those in Ball et al. (2008), and discussed foundational research in this area (e.g., Hill et al., 2007).

Data collection

Prior to interviewing, each participant was asked to submit materials (such as worksheets) for a "favorite problem" used in a content course for secondary teachers that "exemplified

Table 1 Background information for participating mathematics faculty

Name	Mathematical expertise	Content course taught	Years teaching content courses for prospective or practicing secondary teachers[a]	Years teaching at the post-secondary level[a]	Other experiences teaching teachers and/or K-12 students
Frank[b]	Partial differential equations (PhD 2014)	Modern geometry, addressing Euclidean and non-Euclidean geometry	1 year	Interviewed during first year teaching full time at post-secondary level	Four years experience teaching secondary mathematics. Assisted with content-focused professional development for secondary mathematics teachers
Anne	Dynamical systems (PhD 1992)	Secondary mathematics for teaching, focused on algebraic concepts	Interviewed during first term teaching this population	11^+ years	Taught one-semester class for elementary teachers, workshops for middle school teachers, and mathematics literacy work with high school students and teachers
Kelly	Convex and discrete geometry (PhD 2006)	Combined content and methods course, addressing Euclidean geometry, non-Euclidean geometry, and trigonometry	6–10 years	11^+ years	Participated in in-service professional development sessions. Presented at regional and national conferences
Mark[b]	Number Theory (PhD 2007)	Algebraic concepts of secondary mathematics	1 year	6–10 years	Participated in outreach activities in secondary math classes
Pam	Applied analysis (PhD 2012)	Functions and modeling	1 year	3–5 years	Four years experience teaching secondary mathematics

[a] Options given were: First time, 1 year, 2 years, 3–5 years, 6–10 years, 11^+ years

[b] Taught practicing secondary teachers. The other participants taught prospective secondary teachers

[the participant's] philosophy for teaching such courses." We asked for such a task so that we could probe into participants' enactment and design in the context of course goals.

We conducted semi-structured interviews over Skype. The protocol (see Appendix A) focused on (1) goals intended for the favorite task, (2) how the task supports these goals, and (3) how mathematicians evaluated the success of goals. We posed impromptu follow-up questions as appropriate to explore responses.

Data analysis

We recorded and transcribed each interview. Authors then conducted individual case and cross-case analyses as described below.

Analysis for Research Questions 1 and 2: Identifying goals and instructional relationships. To identify goals, we performed content analysis (Weber, 1990) to identify categories for goal statements. Initial categories were mathematical "content" and "practices," as the protocol used these terms. However, some statements did not fit these categories. We proposed additional categories based on our experience with mathematics teacher education programs, and used constant comparative analysis to refine them (Miles et al., 2018).

To identify instructional relationships that mathematics faculty perceive and envision, we performed deductive coding (Miles et al., 2018) to identify instances of perceived and envisioned relationships. We used terminology such as shown in Fig. 4.

We operationalized these relationships based on Cohen et al.'s (2003) and Lampert's (2001) descriptions. For example, the code "M → TC" applied to instances where mathematicians sought to influence teacher learning through actions and activities, and "M ← TC" applied to instances where a mathematician was observing, evaluating, or being influenced teachers' learning.

We note that at times, mathematicians mentioned ways they envisioned teachers seeking to influence student learning ("T → SC") but they did not make an explicit, separate statement about their envisioned students' interactions with content. An example of such a statement is "we expect teachers to now lead their students to do real mathematics, by which I mean struggle with problems for which they don't have an obvious solution path." (Pam). In this case, we inferred that she envisioned secondary students engaging in this kind of struggle, and hence, we assigned an inferred "SC" code. Not all statements coded "T → SC" were associated with "SC" codes, for instance, generic statements about teachers asking mathematical questions to students.

Analysis for Research Question 3: Describing how relationships may drive instruction. After completing the above, we performed thematic analysis (Miles et al., 2018) to identify long-term goals, short-term goals, actions and activities, and perceptions of teacher engagement presented by each mathematician and across mathematicians. By actions and activities, we mean descriptions of mathematicians' enactment of their favorite problem, beyond the problem text itself. We revisited interview excerpts coded with instructional relationships and looked at interviews pairwise to identify similarities and differences. Considering cases pairwise allowed us to perceive unique features of the cases more acutely than examining cases individually.

After completing pairwise comparisons, we revisited each individual participant's case and together wrote a draft description of how their goals informed instruction. We constructed narratives and diagrams to represent our understanding of each participant's case. We used the process of writing longer descriptions (in narratives) and more telegraphic phrases (in diagrams) to clarify our articulations of each case. Throughout, we refined

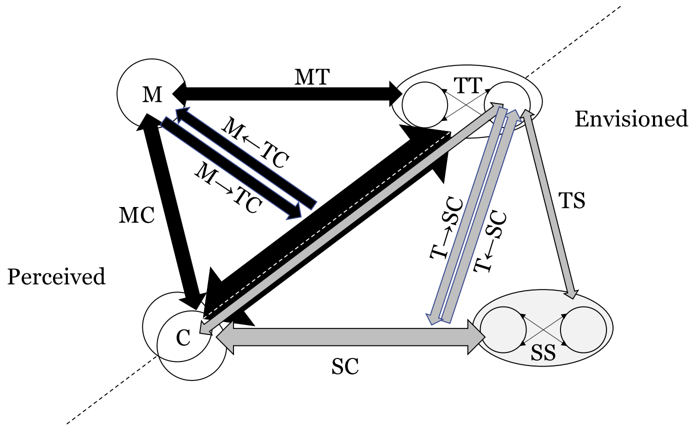


Fig. 4 Terminology for instructional relationships

long-term goals, followed by short-term goals. Once a short-term goal was proposed, we reviewed coded actions and activities for accuracy with our proposed descriptions. We refined short-term goals as needed. We edited narratives, diagrams, and occasionally coding, for precision, accuracy, and coherence.

Results

We organize the results by the research question addressed. We finish this section by describing how envisioned and perceived relationships may drive mathematicians' instruction of content courses. To do so, we present four models derived from participant interviews.

Short-Term and Long-Term Goals

Table 2 summarizes goal types. All participants discussed goals about content and practices. Participants brought up two additional types of goals, unprompted. *Pedagogical* goals, which involved setting up teachers for teaching decisions once they were in the field, were mentioned by all but Anne. Goals regarding *productive disposition*, consistent with the National Academy of Sciences' (2001) definition, were identified by two participants.

Our first two goal types are consistent with findings in the literature (Hodge et al., 2010; Leikin et al., 2017; Hoffman & Even, 2018). The third and fourth goal types do not appear in previous studies as a potential outcome of content courses taken for secondary teachers, although they appear in various policy documents.

Perceived and envisioned instructional relationships, and the potential lack of attention to envisioned student interactions

Table 3 summarizes results. For perceived relationships, all participants mentioned M→TC (in which mathematicians seek to influence teacher learning), M←TC (in which

Table 2 Goal types and number of participants identifying goals of each type

Goal Type	Description for coding	Illustrative interview excerpt	#Mathematicians
Mathematical Content	Responses involving math concepts, skills, or procedures	"… it gets at using quadratics in a novel way, it's sort of at the right level."	5
Mathematical practices	Responses involving practices in the discipline of mathematics	"The primary goal is to give [teachers] an experience with problem-solving."	5
Pedagogical	Responses involving teaching practices or decisions in future secondary teaching	"Thinking about when this would or wouldn't be appropriate with students, connecting it back to the [secondary] classroom, you know, what would the student be doing that you might pose this to them? When might you bring it up to the whole class? What would you expect them to get out of it? So making that switch from being a student to being a teacher."	4
Productive disposition	Responses involving mathematics in combination with teachers' self-efficacy	"…give [teachers] the confidence that, 'Oh, I can think about these [mathematical issues] and maybe work them out myself'"	2

mathematicians observe or are influenced by teacher learning), MC (in which mathematicians discuss their personal experience with mathematics or choices of content and practices to emphasize when teaching), TC (in which teachers engage with course content and learning occurs), and TT (in which teachers interact with each other). These results are consistent with the interview design.

Kelly, who taught the combined content-methods course, uniquely mentioned MT (interactions with teachers outside of content), when discussing rapport with teachers ("[The program size] makes for great personal relationship with the [teachers]").

For envisioned relationships, all participants mentioned TC (the envisioned relationships of teachers with content in the secondary setting). All participants but Anne mentioned T→SC (envisioning how secondary teachers attend to and influence secondary student learning); Anne also uniquely did not mention pedagogical goals. Frank and Mark explicitly mentioned SC (the envisioned mathematical practice of students, separate from how teachers may influence students' practice). We inferred SC codes from remaining participants (except Anne) from T→SC statements. Kelly uniquely alluded to T←SC (envisioned awareness of students' mathematical content and practices).

An envisioned TT relationship (in which teachers interact with one another in the secondary setting) was not in our framework, but arose in Anne's interview directly. She described wanting secondary teachers to learn from each other as teachers once in the field.

We found no descriptions of specifically envisioned SS (interactions among secondary students), nor of TS (envisioned relationship between teacher and students outside of mathematics). These absences are despite, as the next section discusses, mathematicians' attention to teachers' interactions with each other.

Mathematician participants' rationales for instruction

We now build on the above results to illustrate three rationales of how goals drive instruction, as interpreted from our participants' cases:

- Teachers transport personal mathematical practices to their teaching through reflection;
- Teachers learn to engage students in doing mathematics by practicing teaching; and

Table 3 Perceived and envisioned instructional relationships cited by participants

Perceived instructional relationship	#Mathematicians	Envisioned instructional relationship	#Mathematicians
MC	5	TC	5
M→TC (actions and activities)	5	T→SC	4
M←TC (teacher engagement)	5	T←SC	1
TC (short-term outcomes)	5	SC (Students' mathematical practice)	2+ (2)*
TT	5	SS	0
MT	1	TS	0
		TT	1

M = mathematics faculty, *T* = secondary teachers, *C* = content, *S* = secondary students

*Inferred from T→SC statements

- Teachers learn to learn content through emotionally supportive experiences with definitional issues that may defy resolution

We illustrate each rationale using participants' discussion of their favorite problem. Each illustration begins with highlights from the interviews.

Teachers transport personal mathematical practices to their teaching through reflection.

This rationale can be summarized: *Teachers who engage in mathematical practices can transport these experiences to support students in doing authentic mathematics, through modeled reflection on these experiences.* In using "transport," we reference Wasserman et al.'s (2018) "transport model" (p. 75), meaning to transfer explanations from a university course into a secondary course with minimal modification. By "modeled," we mean that the instructor articulates this reflection through verbal prompts and observations they pose to teachers.

This rationale is illustrated by the majority of our cases: Pam, Frank, and Kelly. We discuss the cases of Frank and Kelly. Frank and Pam's cases are similar, and Pam's perspective is discussed in the second case, so for brevity, we do not discuss Pam's case here. However, we provide a schematic summary of her case in Appendix B for interested readers. The key difference in Kelly's case, in comparison with Frank and Pam, is that she explicitly asked teachers to reflect on students' needs. Frank and Pam asked teachers to reflect only on their own processes, and only alluded to implications for secondary teaching; we do not know whether they made connections explicit in their course.

We now begin Frank's case. When Frank, who taught modern geometry, was asked to explain how his favorite problem (see Fig. 5) exemplified his goals, Frank reflected,

> What I like about it is, there's a lot of different ways you can do it. You can algebratize the problem […] then there's a nice geometric solution based on the locus definition of a circle as the set of points that a right angle would subtend given a fixed diameter […] You can write down a solution that doesn't require any algebra, and that does take thinking. I like that. The third thing I really like is […] if you look at the paths of the other corners of the square, they actually trace out a cardioid and limaçon. So there's the curiosity, there's the exploration, and a surprise, and I think that's really good for math as well. […] That particular problem bundles in so many good things together in an unexpected way.

He continued,

> The primary goal is to give the [teachers] an experience with problem solving. […] If you just ask typical homework problems, that doesn't always get at some of the longer-term habits of mind where you need to work on a problem over time. […] So,

Three Squares

Three unit squares are arranged as shown. There is a space of one unit between the two lower squares. If the top square is allowed to slide around freely while the bottom squares stay in place, what is the lower point that the bottom corner of the upper square can reach?

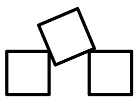


Fig. 5 Frank's favorite problem

> I'm trying to take some of the content of the geometry course and make it real in this problem that has some depth to it.

When asked what he meant by "make it real," Frank explained,

> You've got all these theorems, but how does that geometric intuition, the geometric reasoning, how does that actually connect with a particular problem, like this one that requires geometric thought in order to solve? And how does it connect with the mathematics that you'll be teaching as a high school teacher? I wanted there to be some problems that tried to bridge between abstract mathematics built on axioms and why do we have these axioms and theorems and proofs and the more typical content that they may be teaching in high school.

As Frank later said, he wanted teachers to be thinking about, "Ok, when I'm a teacher and I'm expecting my students to do problem-solving, what do I need to help my student be able to do." By "problem-solving," Frank meant that the teachers would work on a problem that contrasted with "typical homework problems" in that the problem would take more time to solve, synthesize multiple concepts, and use intuition and reasoning. Frank's statements (and Pam's) resemble Stein et al.'s (1996) descriptions of doing mathematics authentically. These statements on the surface bring to mind Woods and Weber's (2020) findings about mathematicians' goals for advanced mathematics courses. Mathematicians in this role may have a goal of teaching authentic mathematical thinking. There is a key difference in our findings: the mathematicians in Woods and Weber's study discussed their own monologue as a way to achieve this goal, whereas the mathematicians in this study focused on task design and questioning.

In elaborating how he "makes it real," Frank described asking teachers to reflect on their own processes. For instance, if he saw a teacher use coordinates, he might say, "I see you put coordinates on there. How did you to do that? Why did you think to do that?" To Frank, these questions "[get] them in the meta ideas as they're kind of doing the problem, so that they can think about, what are you going to do with your students so they can see that that would be a way of doing the problem." In other words, he asked teachers to reflect on a strategy they used so that they might later help secondary students see when this strategy might apply.

We then asked how Frank might teach his favorite problem in the future. He replied:

> If we're getting at bridging it should also be some reflection about how it could be used in their own classrooms or what, what'd they learn about problem-solving by solving this problem, reflecting on, as teachers, what are the skills they needed to do this. So they can be thinking about, 'Ok, when I'm a teacher and I'm expecting my students to do problem-solving, what do I need to help my student be able to do.' […] Or give them the [Common Core Standards of Mathematical Practice] and say, 'Which ones did you need when you using when we worked on this problem?'

As Frank concluded, when teachers articulate their own processes, they might "see what they're doing in problem-solving so that they're aware of it because you're not always aware of it if it's not made clear." By modeling reflection questions, he hoped that teachers could extrapolate moves for their future teaching.

Altogether, Frank wanted teachers to see that mathematical practices and strategies could transport to their teaching. To set up this transfer, he prompted teachers "[get] in the meta ideas" that might support future students. These meta ideas included particular strategies, and also considering why one might choose a strategy. He saw part of his role as verbalizing these "meta ideas" to render them visible to teachers. We infer that Frank's own relationship with mathematics—appreciating curiosity, exploration, and surprise—influenced his task design. His problem contrasts with "typical" problems, which he saw as deficient for showcasing mathematical practices, theorems, and intuition in secondary mathematics. Figure 6 summarizes this rationale along with a thumbnail of the instructional relationships apparent in his interview.

We now turn to the case of Kelly, who taught a combined methods/content course focused on geometry and trigonometry. Her responses showed explicit prompts to compare and contrast teachers' and students' processes. Her goals included experiencing and understanding mathematical practices, especially those in the Common Core Standards for Mathematical Practice (SMP). Figure 7 shows her favorite problem.

When asked what she would say and do to promote her intended goals in her favorite problem, Kelly described switching perspectives from learner to teacher, especially for the goal of understanding the SMP:

> I think providing some closure with actually having them talk about which [practices] they were using. […]Thinking about when this would or wouldn't be appropriate with students, connecting it back to the classroom, you know, what would the student be doing that you might pose this to them? When might you bring it up to the whole class? What would you expect them to get out of it? So making that switch from being a student to being a teacher.

In this statement, Kelly acknowledged potential similarities and differences in teachers' and students' experiences. She raised decisions irrelevant to a strictly personal exploration, such as timing for whole class discussions, or the aims of such discussions.

When asked to articulate her goals, Kelly said,

> The most important goal I have for my pre-service teachers, is to experience the mathematical processes and to really solve some problems for themselves so they can

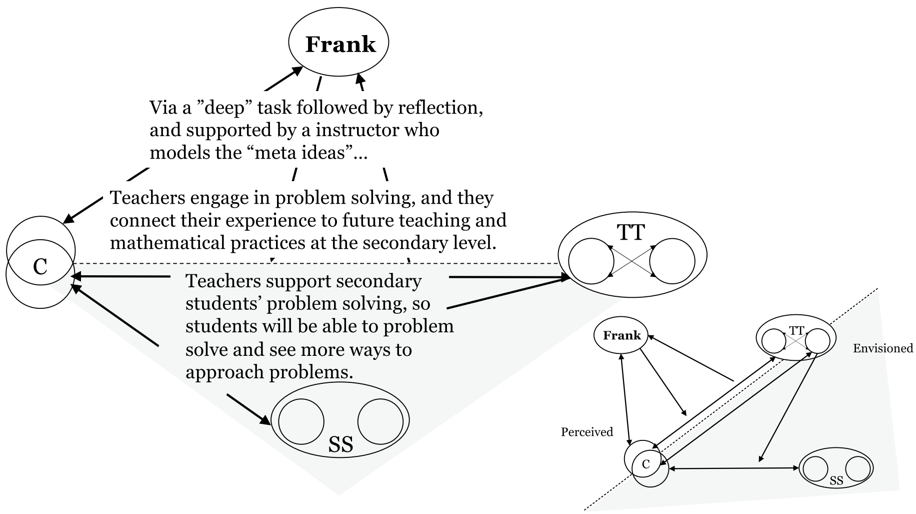


Fig. 6 Summary of Frank's rationale with thumbnail of perceived and envisioned instructional relationships. Note the different angle of large triads relative to the small; this was done for ease of reading

Area Formulas

This activity is intended to help you draw connections between ideas in geometry, including some that students first encountered in middle school.

Write the area formulas for a square, rectangle, trapezoid and triangle. Explain connections you see, and then write one general area formula that would apply to as many of these shapes as possible and explain how you know it works for all the shapes.

Fig. 7 Summary of Kelly's favorite problem

> talk about them. [...] I want my students to really have in-depth experience with that in a couple of different situations so that they can transfer it.

To understand what she meant by "transfer," we use her response about the impact of her course:

> I think a lot of where it's building to is having them as they write their unit plans, have their antennae up to look for connections, so to speak. To not be content to just take every lesson as completely separate from the ones before. To be looking for ways to connect things, to be able to create a big picture for students that they can fit this into. [...] Then [during student teaching] we can talk about some of those things there and it's like, "I love how you brought these connections together, you definitely built on what was happening the day before, you were able to do some consolidation."

Kelly emphasized the importance of teachers' seeing connections, so that teachers can create mathematical coherence for students.

When asked how she could tell whether teachers were seeing connections, she discussed teachers' content and affect. She then raised another "main goal":

> I'm looking to see that they are able to connect pieces, that they are able to abstract and make sense of some of the different pieces, because usually, even if they don't figure out how to fit the triangle into that, usually they can figure out the rectangle and the trapezoid and how those different pieces work. [...] That's where the follow up in class is important and getting to see their *gasp oh! I didn't think about that!* (emphasis in the original)
>
> And I think this goes back to one of my main goals, is really recognizing that they will continue to grow and they need to put themselves in situations in which they would continue to grow. Whether that's conferences or webinars or reading things, having some of those aha moments at the end where "I didn't realize that myself!" and it's so good that I have other people around me who can point these things out.

To Kelly, one of the tasks' key affordances is that teachers can learn from others. In so doing, teachers might see the benefit of "other people around [them]." We note it is unclear whether "others" includes teachers, though it may.

Overall, Kelly envisioned teachers who recognized their capacity for mathematical growth, and who knew to look for and convey mathematical connections to students. Kelly's intentional structuring of course activities including the favorite problem, with opportunities for mathematical experience and reflection, indicates Kelly's intention to impact teachers' own mathematical practice as well as their attention to students' mathematical practice. Figure 8 depicts her rationale and a thumbnail of perceived and envisioned interactions.

Kelly and Frank, as well as Pam, all envisioned teachers who supported students' mathematical practices, such as reasoning, problem-solving, and making connections. These mathematicians sought out opportunities to engage teachers in mathematical practices. Then, they modeled reflection processes with the hope that teachers would transport these processes to their teaching. Their cases showcase different strategies for modeling reflection: comparing personal and peer approaches (Kelly), articulating strategies and rationale for strategies (Frank), and reflecting on students' needs in contrast to teachers' needs (Kelly). Frank was further motivated by contrasts between their observations and their envisioned teaching. Namely, teachers may not have experience with complex tasks, and some teachers may not be able to explain why some procedures worked. Using reflection may not be surprising; guidance for mathematicians suggests that teachers be "constantly required to reflect on their reasoning," so that tertiary experiences can be "models for their own future classrooms" (CBMS, 2012, p. 56).

Across these cases, the mathematicians' attention to teachers' interactions aligns with mathematician Julie Fredericks's observation that instructors "should be really intentional about managing the ideas that come up," when using "inquiry-based" instruction (Johnson et al., 2013, p. 756). Yet we also note that the mathematicians' attention to envisioned student interactions was essentially absent. Kelly alluded to "what would your students be doing," but this generic phrase could potentially refer only to individual student work. The mathematicians gave rich, articulated images of teachers' interactions with mathematics, and their influence on these interactions. We do not have evidence of similarly vivid images in their envisioned teaching.

Teachers learn to engage students in doing mathematics by practicing teaching

We summarize this rationale as: *Teachers develop the skills for engaging future students in doing mathematics through opportunities to simulate teaching practice.* The cases of

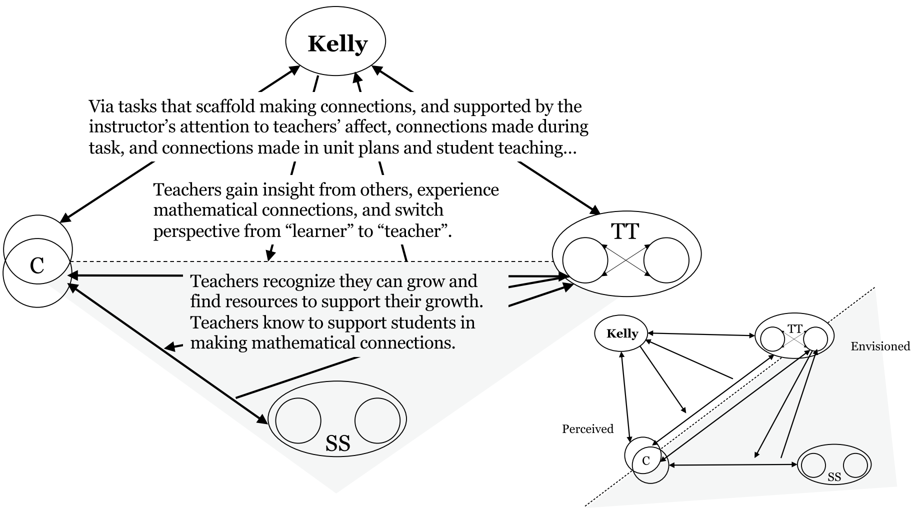


Fig. 8 Summary of Kelly's rationale with thumbnail of perceived and envisioned instructional relationships

Pam and Mark exhibited this rationale. We detail Pam's case and for brevity, do not include Mark's case here. Appendix C contains a summary of Mark's case for interested readers.

While describing her goals, Pam, who taught a course on functions and modeling, mentioned "practice teaching." Although "practice teaching" was separate from her favorite problem, we found it noteworthy and asked her to elaborate.

> PAM: I want them to have authentic experiences doing mathematics. […] I want them to pull the explanations out of their students. I want them to encourage their students to do authentic mathematics. Those are the three [goals]. And the methods are doing problems themselves, and doing practice teaching in the class.
> INTERVIEWER: When you say "practice teaching," do you mean something in your own class, or?
> PAM: I mean my classroom. Mini teaching, so the rest of us pretend to be the students, and they just teach us.

The above statement corroborates our previous claim that Pam had a long-term goal of teachers guiding students through authentic mathematical experiences. Pam then explained how "practice teaching" supported her goals:

> I think the practice teaching helps, obviously in that they're practicing doing what we want them to do. But I also think it's in particular helpful because […] it's sometimes easier to see something working [or not] when someone else is doing it. When you yourself are making a certain mistake, you're so in your own mind […] You're imagining that it's going to work, so you tell yourself it's working. When you see someone else try it, then you see, "Oh, gosh, that's what I do and it doesn't really work," or, "That was an opportunity where that teacher could have taken that question the student had and followed that path, but they brushed it off, and gosh I do that too." I think that's helpful.

Then, saying that she would "play the really inquisitive student," she explained:

> […] When a teacher is leading a group of young students, the students tend to just follow the teachers' lead. And sometimes you have that wonderful experience when you're teaching when you have a student who pushes you, and I think that's more common in college courses, but when you're teaching younger students, the students often just sort of fall back on, "I'll just follow the teacher wherever they're going." […] When a teacher gives a half-baked explanation, is there a student who says, "But wait, what about such and such?" I'd play that role in hopes that they see that there are sometimes missed opportunities in the classroom because we're sort of in a rush, or we have a program and we're just going robotically through the program.

Overall, Pam envisioned teachers seeing opportunities to delve into exploration and explanation. In her role as "inquisitive student," she simulated when teachers might pause and consequently observe opportunities to explore and explain. Figure 9 represents Pam's rationale.

Pam (and Mark) envisioned teachers' influence on secondary students' mathematical practice, and designed simulations of teaching. During simulations, Pam (and Mark) distinguished their role as enforcing their goals. To do so, Pam played the "inquisitive student." We point out that for Pam (and Mark), they were motivated by a particular vision of teaching mathematics, and that they both highlighted the strategy of question asking. We suggest Pam (and Mark)'s role is crucial to the enactment of this activity, because their role reflects their vision of secondary teaching.

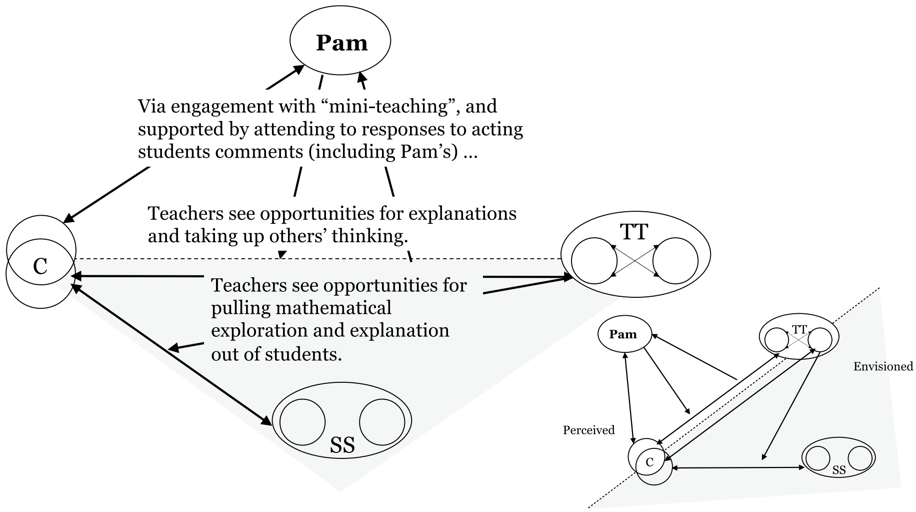


Fig. 9 Pam's rationale

Teachers learn to learn content through emotionally supportive experiences with definitional issues that may defy resolution.

According to this rationale, *Teachers must learn to learn how to approach learning in addition to learning content, and teachers can do so through emotionally supportive experiences where they grapple with an issue that may not be universally resolvable.* This rationale was illustrated by Anne's case.

In explaining her content course goals, Anne, who taught a course focused on algebraic concepts, said:

> There are very many mathematical issues that arise when in K-12 mathematics. If you're learning, once you've mastered the K-12 mathematics, you're able to use it well and do problems. These issues, you don't even think about these issues. […] So I teach for instance, calculus, which is also taught in K-12, then there are issues there that can be confusing. I mean that are real mathematical issues that we always just don't go into depth because we don't have time. Students haven't ever had the opportunity to think those issues. These are things that come up when they're teaching. So I can't hope to cover all those issues […] That's one of my most important goals: […] to make [teachers] realize that these issues are there and to give them the confidence that, "Oh, I can think about these and maybe work them out myself."

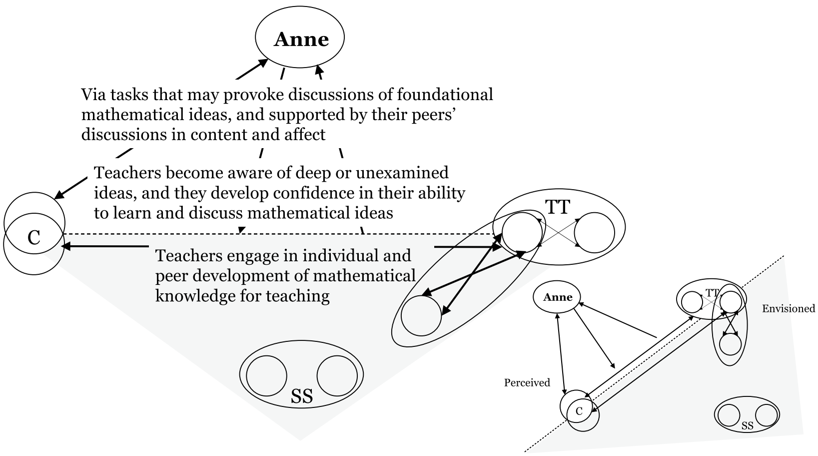
Anne believed that mathematical knowledge for teaching extended beyond what can be covered in a teacher's mathematical education. She sought strategies to increase teachers' confidence in learning new ideas. One strategy was to learn independently ("work them out myself"). She later added a second strategy: "In their teaching career […] one of the most important ways that they can deal with all these issues that come up is by talking to other teachers that are their colleagues." The set up for all these strategies was a task that showcased the ambiguity of the concept of "variable." We summarize her rationale and favorite problem now, in Fig. 10, and then discuss her enactment.

In Anne's favorite problem, teachers read a number of mathematical "vignettes" then developed a definition for "variable" with peers. When asked how she assessed teachers' progress toward the intended goals, Anne described monitoring the content of discussions, for instance, that an expression could itself be considered a variable ("I want them to see, well if I write $c + 5c + 7x - y$ and x and y are variables, then that whole expression should be a variable"). When asked about how engaging in the task might build confidence, she explained:

> I hope it builds their confidence that they get, we have all these vignettes and they're like, "Oh my goodness, I have no idea which is I would call the variable and which one I wouldn't." And then there's a really rich discussion that comes out of it.

After describing possible interpretations of variable, she continued, "Having a discussion builds confidence if the conversation is constructive, you know, if it's not people putting each other down but listening to all the ideas." She concluded:

> I'm hoping that I teach in a way that they are talking about these things with each other and learn how to do that. Right? And, constructively, and you know, some of those conversations can get emotional at times.



The Meaning of Variable

Teachers divide into groups. Each group is given a number of cards containing a mathematical "vignette" containing expressions with variables. Teachers are asked to use these examples to come up with a definition of "variable" as it is used in mathematics.

Example "vignettes" included:

- The commutative law of addition states: $a + b = b + a$ for all real numbers a and b.

  Examples of expressions include: a, b, $a + b$

- Delilah scored 73, 66, 84, and 77 on her unit tests and her average score on these tests including the final was 77. Let x be her score on the final, so $\frac{73 + 66 + 84 + 77 + x}{5} = 77$.

  Examples of expressions include: x, 77 + x, $\frac{73 + 66 + 84 + 77 + x}{5}$

Fig. 10 Summary of Anne's favorite problem and rationale, with a thumbnail of perceived and envisioned instructional relationships

Overall, Anne monitored tone and content, looking for evidence that the classroom environment where teachers "listen[ed] to all the ideas" and it was acceptable to be "emotional." To bolster confidence, she alluded to the difficulty of the problem even for mathematicians:

> You ask a mathematician, they'd have the same issue. They look at all these things, 'I don't know what is a variable and what isn't.' In that sense that builds confidence too, that it's like, 'Oh, we can build our own concept of what a variable is.'

In Anne's view, teachers had the agency to define "variable," and this agency can lead to confidence. Her emphasis echoes that of mathematician Lee Gibson, in wanting university students to develop "the intellectual courage to take on a new challenge" (Johnson et al., p. 753).

Anne believed that if prospective teachers had a positive, collaborative experience of unpacking a deep issue that may defy a universal definition, then they would be more likely to have the confidence to seize opportunities to learn new content as practicing teachers, including with other teachers. This belief is reminiscent of Zaslavsky's (2008) observation that teacher educators seek for teachers to "cope with conflicts, dilemmas, and problem situations" (p. 95). It is also a contrast to typical approaches to connecting university mathematics to secondary teaching. For instance, Murray and Baldinger (2018), citing Barwell (2005), assert, "[W]hen it comes to mathematical language, 'Any ambiguity, that is, any possibility of more than one interpretation for a mathematical expression arises from sloppy use of language *rather than any uncertainty of mathematical ideas*" (p. 118)'" (p. 406, emphasis ours). Yet a key feature of Anne's favorite problem is that "variable" appears to defy a universal definition, usable in any context.

Discussion

We investigated mathematics faculty's design of tasks used in content courses via their attention to perceived and envisioned instructional relationships. The contribution of this study is twofold. First, to our knowledge, there has been little examination of how mathematics faculty conceive connections between content and secondary mathematics *teaching practice*, as opposed to secondary *mathematics*. Our results include illustrations of mathematics faculty members' rationales for connecting content course activities to envisioned secondary teaching. Moreover, we featured a rationale that contrasts with previous findings (Anne's case). Second, we provide a methodological contribution: an extended teaching triad that focuses on the role of content. This teaching triad highlighted the relative paucity of envisioned relationships as compared to the density of perceived relationships. We suggest that this dual triad can be used as a professional development tool as well as programmatic design tool for teacher education.

This study's design is limited in that it captures only a snapshot of participants' teaching, rather than their rationale for activities across an entire course. It is also limited in the instructor sample, who all were recruited from a particular series of workshops. Nonetheless, we observe that no participants made mention of specific workshop events during interviews, nor did "favorite problems" resemble any tasks used in the workshop. The workshop did address mathematical practices, and these discussions may have bolstered their relevance to the participants. The sample is likely unusual in that two of the five, Pam and Frank, had multiple years of secondary teaching experience. However, their rationales for the importance of cultivating teachers' mathematical practice resemble those in in the CBMS's (2012) *Mathematical Education of Teachers II*, a publication written to guide mathematics departments.

We did not have measures of participants' professed goals. Some participants expressed uncertainty as to whether their intended goals were achieved. Our interpretations are only based on the participants' self-reporting.

It is possible that the faculty envisioned secondary students' interactions about mathematics, but that they left these ideas tacit in their interviews. After all, we did not probe specifically about these interactions. On the other hand, it is striking to have a complete absence across all our interviewees of statements describing interactions among students. We hypothesize that even if faculty do envision secondary students' interactions, their images may not be vivid enough to bring out in content course instruction, or the faculty may have chosen to prioritize other aspects of envisioned secondary teaching.

Seeing known rationales in new ways

Our work contributes different ways to see the rationales of connections through mathematical practice and learning through experiences of uncertainty. The rationale of connections via mathematical practice is given by mathematics teacher educator researchers and policy for mathematicians alike (e.g., Baldinger, 2018; Murray & Baldinger, 2018; CBMS, 2012; Hoffman & Even, 2018, 2019).

Yet we now problematize this rationale in light of one of our findings: instructors' potential lack of attention to envisioned student interactions. As Philipp et al. (2007) suggested, observing a mode of instruction does not guarantee enacting such instruction. They argued that teachers must focus on student thinking to develop more sophisticated beliefs about mathematics teaching and learning. Moreover, cultivating fertile environments for mathematical practices often entails setting up productive interactions among students about the content (e.g., Shaughnessy et al., 2021). Here, based on the mathematicians' descriptions, the teachers were participating in mathematical practice—but they also did not focus on any secondary student thinking, let alone interactions among secondary students, except for possibly in Kelly's case.

We suggest that the main impact of reflections on mathematical practice may be more about cultivating teachers' belief in the relevance of mathematical practice, more than it is about being able to enact instruction that promotes practice. In other words, it is more about the teacher's relationship to content than it is about the teacher's capacity to shape students' relationships with content. Beliefs are important, of course, but beliefs do not determine action; knowledge of mathematics and knowledge of teaching practices are also needed (e.g., Blömeke et al., 2015).

The rationale of learning to learn mathematics, illustrated by Anne, bears a resemblance to the literature on using uncertainty in mathematics teaching. Yet Anne's focus on "learning to learn" through a task that may be unresolvable distinguishes her approach. In Zaslavsky (2008), for instance, uncertainties about powers of complex numbers can be ultimately resolved by means of concepts from complex analysis. In Anne's task, the notion that even active research mathematicians may not immediately know how to define a variable was key to supporting teachers' mathematical agency.

The envisioned teacher–teacher relationship in the dual triad

We contribute to the literature the envisioned teacher–teacher relationship in the secondary setting. Anne was the sole participant to mention this relationship directly, and she viewed attention to this relationship in the university setting as a precursor to teachers' engaging with each other in the secondary setting. This relationship is not one we had originally conceived, nor is it explicitly in Leikin et al. (2017) extension of Zaslavsky and Leikin's (2004) teacher educators' triad. Yet practicing teachers do learn from each other in professional learning communities. This finding prompts the question of how teacher-to-teacher relationships, present and future, can be leveraged explicitly in content course design to promote professional growth.

Using the dual triad for professional development and programmatic design

Frameworks can be used for professional development design for mathematics teacher educators, as well as in programmatic design (Karsenty, 2020). Even (2008) proposed that the education of educators may be, in its essence, about *knowtice*: "the integration of knowledge and practice" (p. 9). We envision that our dual triad could be used to facilitate the *knowtice* of mathematics teacher educators, whether they teach methods or content, or both. The dual triad can be used to help bring awareness to where mathematics teacher educators attend or not, and where it may be worth increasing attention.

Further, if those involved with separate content and method courses are making programmatic decisions, or determining how to collaborate, they could use the dual triad to coordinate their roles. A content course may not be able to address all possible relationships in the dual triad: the balance of pedagogy and mathematics is hard to coordinate because trying to do both can risk diluting one or the other (Suzuka et al., 2009), and they rely on different disciplinary bases (Goos & Bennison, 2018). However, programmatically, all possible relationships need to be addressed. A methods and content course instructor could use this diagram as a way to broker interactions (cf. Goos & Bennison, 2018).

We suggest that our dual triad has an advantage over other suggested triads, because it makes explicit the connections between all nodes, rather than nesting an entire triad into a superordinate node. This explicitness could be a potential advantage for clarifying design decisions.

We echo Li and Superfine's (2018) recommendation for further research into how mathematics teacher educators make connections between the activities they use and teachers' future teaching. We believe that the superposition of rationales and a teaching triad supports such research. Using the superposition helped us to articulate explicitly the connections that mathematics faculty saw between content courses and secondary teaching, and in a way that builds on an existing framework for studying teaching. The superposition could potentially be used by education researchers in collaboration with mathematicians to articulate and refine such rationales over time, and acknowledge differences. Such studies would follow the inventive tradition of Nardi's (2007) bringing together or mathematicians and mathematics education, or Goos and Bennison's (2018) study of brokering. Such studies could also contribute to the field in bringing together mathematicians and mathematics educators for research in teacher education.

We conclude with optimism. The participating mathematics faculty's perspectives tell a promising counter-story to the usual narrative of disconnection at the secondary level.

All five participants in this study wanted or articulated connections from their course design to their vision of secondary teaching. Although the participants were recruited from a workshop on teacher education, we nonetheless take their interest in teacher education as a promising existence proof. The participants' visions of secondary teaching involved problem-solving, reasoning, and teachers' collaborative learning. These ideals are shared by education faculty and secondary school leaders, as evidenced by guiding documents for improving and assessing elementary and secondary mathematics education in the USA and internationally (e.g., Department of Education, 2003; NCTM, 2020; NCCA, 2020). We believe that it is time to change from a deficit narrative to an asset-based narrative with respect to the mathematical education of secondary teachers. Rather than looking for discontinuity, let us focus on how mathematics faculty's beliefs and instruction can promote deep and productive continuity from mathematical preparation to secondary teaching.

Appendix A

The first set of questions was designed to probe the participants' experiences in teaching content courses for secondary mathematics teachers:

1. What are the most important goals you have had for students (pre- or in-service teachers) in math courses for teachers?
2. What kind of experiences or knowledge have you drawn upon to carry these out in your teaching, especially the times that you felt like your teaching particularly supported those goals?
3. How does your use of your favorite problem exemplify teaching toward these goals?

These questions were posed simultaneously to participants via Skype chat, and participants were asked to talk for 5 min in response to the questions.

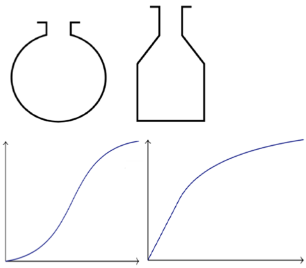
The second set of questions was designed to examine the goals, actions and activities, and attention to teacher engagement with the shared favorite task:

1. In your view, if you had to say it in one sentence, what specifically are the most important content or practice goals you have in mind when using this task? (Do you want a moment to think about this before saying this sentence?)
2. How does working on this task get at these content and practice goals?
3. What do you do when you teach to help this task get to those goals?
4. What are some ways you use to tell whether the students learned what you intended them to learn through working on this task?
5. How does set up or build upon content and practice for the week before or after that?
6. Has the way you've used this task changed over time? How did these ways support the goal?

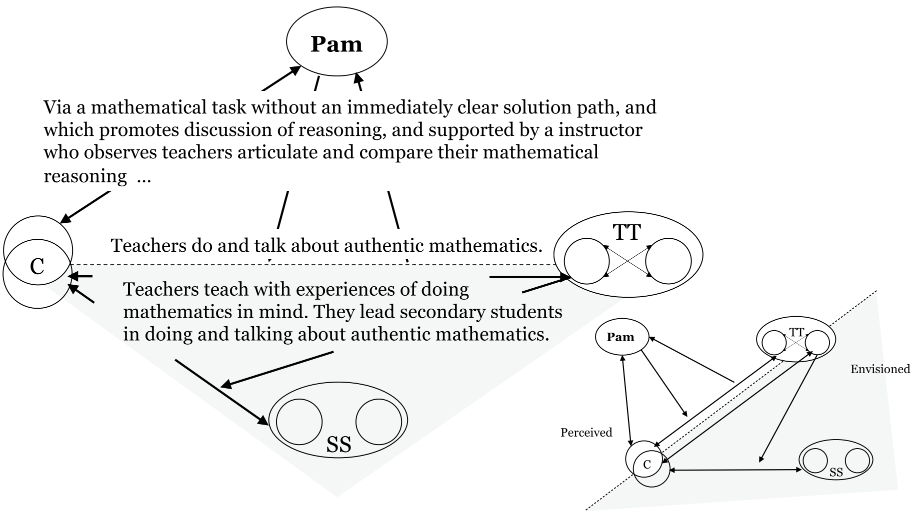
Appendix B

Pam's favorite problem was the Bottle Calibration Problem, an excerpt of which is shown below.

Bottle Calibration Problem (Excerpt)
Based on Shell Centre for Mathematical Education's (1985) task "Filling Bottles"
To calibrate a bottle, we need to know how the height of the liquid depends on the volume in the bottle. Here are [some] pictures of bottles and graphs. Choose the correct graph for each bottle. Then invent your own bottles and sketch their graphs



We summarize Pam's rationale with the schematic below. She envisioned secondary students doing and talking about authentic mathematics, meaning that teachers must practice having mathematical conversation. But, in her view, they often have not had this opportunity, nor may some teachers be able to explain the reasoning behind some procedures. She specifically sought out problems like the Bottle Calibration Problem, where mathematical exchanges could arise organically. At times, she set up these exchanges through her own pointed observations, thereby modeling actions that could be transported into her envisioned secondary teaching.



Appendix C

Mark's favorite problem was open-ended: to present an "enrichment activity" to the class that would "stimulate their students' interest and curiosity."

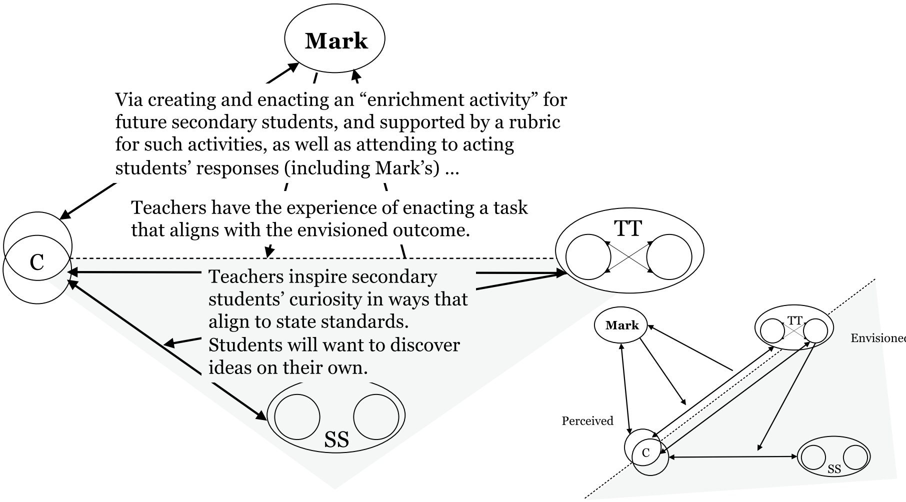
Enrichment activity for secondary students

Organize and present an enrichment activity appropriate for high school students that:

Includes clearly stated goals aligned with the [mathematics standards of US location]

Fosters an active learning environment;
Stimulates students' interest and curiosity in mathematics;
Offers students opportunities to make conjectures and test their validity.
The project will be presented to the class. Fellow teachers will play the role of secondary school students and will evaluate each project. The instructor will also evaluate each project.

We summarize Mark's rationale with the schematic below. At the beginning of the semester, Mark provided a rubric for the activity. When describing what he said and did during presentations, Mark said, "If something is left vague, or they don't address something in that rubric, then I'll ask them a question." Mark described his goals as, "I want them to try to encourage their students to be curious and to allow their students to discover things on their own as much as they can."



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Declarations

Conflict of interest Not applicable.

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