How do college students conceptualize algebraic properties?

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Here we explore how college students across different courses may conceptualize symbolic algebraic properties. This work draws on the theory of Grundvorstellungen (GVs) as a tool to analyse how learner conceptions do or do not align with some desired goals of instruction. In analysing interviews, several categories of conceptions, or descriptive GVs, emerged, which may be a helpful first step in understanding learners' thinking and improving instruction on algebraic properties.

Keywords: Algebraic properties, Syntactic reasoning, Equivalence, Algebraic transformation.

Mathematical properties justify transformation across mathematical domains, particularly those that rely on symbolic representation such as algebra. However, learners often do not use them in mathematically valid ways (e.g., Hoch & Dreyfus, 2004; Mok, 2010) and instruction may not address properties explicitly enough (e.g., Barnett & Ding, 2019), despite the central role that algebraic properties play in mathematical transformation. In this paper we focus on learners' ability to identify parallel syntactic structure between symbolic properties and mathematical objects such as expressions, and we explore how this may relate to their conception of symbolic properties in algebra.

Properties and forms

Here we define a *property* as any mathematical statement that may be used to transform a symbolic object into an equivalent one with a different form. Thus, two examples are: 1) the definition of negative exponents, written, e.g., as: $x^{-n} = \frac{1}{x^n}$ for $x \neq 0$; and 2) the statement about two equivalent equations: $A - B = C \leftrightarrow A = B + C$. The key point is that 1) could be used to replace an expression of the form x^{-n} with one of the form $\frac{1}{x^n}$ (or vice versa), and 2) could be used to replace an equation of the form $A - B = C$ with one of the form $A = B + C$ (or vice versa). The term *symbolic properties* denotes properties as represented by symbolic representations. Properties are made up of smaller sub-structures (e.g., each side of a formal property statement can be viewed as a separate object), which are often referred to colloquially during instruction as "*forms*" (e.g., the "form" x^{-n}, $\frac{1}{x^n}$, $A - B = C$, and $A = B + C$, in properties (1) and (2) above).

Our work addresses a gap in the literature about learners' use of mathematical properties. Existing research has focused on classifying what types of errors learners make when using properties to simplify or solve (e.g., Hoch & Dreyfus, 2004; Mok, 2010); on learners' justifications for why properties are true; or on learners' ability to derive properties from arithmetic patterns (e.g., Hunter et al., 2022). Some limited work has focused on learners' structure sense for specific properties, such as the distributive property (e.g., Schüler-Meyer, 2017), but here we aim to describe conceptions of properties and forms more generally. This is important because using and understanding symbolic properties and forms is a critical skill for transforming symbolic representations (Kieran, 2011).

Theoretical framework

Grundvorstellungen

We approach symbolic properties from the lens of prescriptive and descriptive *Grundvorstellungen* (GVs) (“fundamental conceptions”). *Prescriptive* GVs describe conceptions that we *hope* for learners to acquire from instruction (vom Hofe, 1995); whereas *descriptive* GVs describe conceptions that learners *actually* hold. Comparing these two types of GVs is intended to guide curriculum/instruction (Greefrath et al., 2016), by considering how prescriptive and descriptive GVs align after instruction. Neither descriptive nor prescriptive GVs are intended to be static, nor to present a monolithic view of what it means to learn a concept; rather, GVs are intended to evolve with research over time. We begin by describing two related *prescriptive* GVs for symbolic properties (Figure 1).

Equivalence-Preserving GV	Symbolic properties describe a method for replacing one symbolic object (e.g., expression, equation) with another equivalent one, based on some context-dependent pre-existing definition of equivalence (e.g., insertion equivalence; Prediger & Zwetzschler, 2013).
Mapping GV	In order for equivalence to be preserved, the following criteria must be met: The form on one side of the symbolic property must be mapped bijectively (one-to-one, so that no symbols in the symbolic object or the form are left out or used more than once) to the symbolic object (e.g., expression, equation) so that: 1) A unified subexpression[1] is mapped to each variable in the form; 2) All other symbols are mapped to notation in the form with the same syntactic meaning (e.g., different notation for multiplication can be mapped to one another).

Figure 1: Two Related Prescriptive GVs for Symbolic Properties

Operational vs. Structural Conceptions and Extracted vs. Stipulated Definitions

In this paper we describe how we have analysed student interview data qualitatively in order to generate *descriptive* GVs for symbolic properties. In describing learners’ conceptions, this work was also influenced by an initial theoretical stance that focused on attending to operational vs. structural conceptions (Sfard, 1992) and extracted vs. stipulated definitions (Edwards & Ward, 2004). A learner with an *operational* conception views properties as a process of computation, while a learner with a *structural* conception views them as abstract objects (e.g., canonical representations of particular algebraic structures). A learner with a structural conception sees objects as reified processes (e.g., the form $a(b+c)$ is seen as an object, and not just as the process of adding b and c and then multiplying a by the result), however when learners view something as an object that is not the reification of any process, this is called a *pseudostructural* conception (Sfard, 1992, p. 75). The operational/structural distinction is related to the prescriptive Mapping GV of Symbolic Properties, which focuses on a learner’s ability to conceptualize forms within a property structurally as an object.

Extracted definitions are created to describe observed usage of a term (e.g., a learner may extract a meaning for a property from how it is used during computation in instruction). In contrast, *stipulated* definitions are stated explicitly—to determine if something fits the definition one must consult the definition directly (Edwards & Ward, 2004). This distinction is related to the Equivalence-Preserving GV of Symbolic Properties, as one key stipulated feature of properties is that they preserve equivalence (and the type of equivalence that is preserved is also based on a stipulated definition).

Methods

Cognitive interviews were conducted on items from the Algebra Concept Inventory (Wladis et al., 2018). 102 interviews were conducted with college students in the US in 18 different courses ranging from elementary algebra (which focuses on mathematical objects from Algebra I in secondary school) to linear algebra; many different courses were included with the aim of capturing as broad a range of GVs as possible. In addition, subjects were diverse in terms of gender, race/ethnicity, national origin, and English language learner status. Interviews were analysed using thematic analysis (Braun & Clarke, 2006). This analysis was influenced by an initial theoretical stance focused on noticing how students' responses may reflect extracted vs. stipulated definitions (Edwards & Ward, 2004) and operational vs. structural (Sfard, 1992) conceptions, as well as the extent to which students appeared to show evidence of Equivalence-Preserving or Mapping GVs. Analysis of interviews led to a more nuanced emergent framework of learners' conceptions, or descriptive GVs, of symbolic properties.

Results and discussion

Analysis of student interviews led to the generation of a framework that describes *descriptive* GVs of symbolic properties (Figure 2). In this two-by-two framework, operational vs. structural conception categories relate to how closely learners' GVs align with a Mapping GV and extracted vs. stipulated definition categories relate to how closely their GVs align with an Equivalence-Preserving GV (Figure 1). We now illustrate the framework by presenting a few examples.

	Extracted Definition	**Stipulated Definition**
Operational Conception of Properties	**Pseudo-process GV:** Learners see properties as a cue to a computational process, and their approaches are extracted from prior experience rather than based on stipulated definitions. They often draw on surface structure rather than syntactic meaning. For example, learners may conceptualize the distributive property as an instruction to "take what is on the outside of the parentheses and put it next to each thing on the inside", regardless of the specific operations involved.	**Process GV:** Learners see properties as a cue to a computational process, but attend to syntactical meanings and/or equivalence as a justification (e.g., checking for appropriate operations in the expression; checking that original and resulting expressions are insertionally equivalent). However, they may struggle to conceptualize properties as objects to which structures in the expression or equation can be mapped one-to-one, and as result may have difficulty generalizing the use of properties to more syntactically complex symbolic representations.
Structural Conception of Properties	**Pseudo-object GV:** Learners conceptualize a property as something that requires mapping to the specific forms in the property, but the mapping is still somewhat ill-defined and/or based on extracted notions, such as what "looks right".	**Object GV:** Learners conceptualize the property as an object, such as a canonical form, to which the specific mathematical object (i.e., expression, equation, etc.) must be mapped one-to-one, in such a way that preserves syntactic meaning. They recognize that it is these criteria that preserve equivalence.

Figure 2: Framework to Categorize Descriptive GVs for Symbolic Properties

Operational conceptions

In this section, we present some brief examples of interviews in which the learner appeared to be drawing on operational conceptions of symbolic properties. This first student, Iota, was enrolled in an introductory statistics course that had school algebra as a pre-requisite, and was given a series of seven questions with the form given in Figure 3, each presenting a different type of expression. Expressions used in other versions of this item included: Q1: $(2x+1)2$; Q2: $x-(2x+1)$; Q3: $2(2x \div 1)$; Q4: $2(x \cdot y)$; Q5: $(2x+1)^2$; and Q7: $2(xy)$.

<u>Q6</u>: Which of the following could result from using the **<u>distributive property</u>** to rewrite the expression $(x + 2)(3x + 7)$? a. $x + 2 \cdot 3x + 7$ b. $x \cdot 3x + 2 \cdot 7$ c. $x + 2 \cdot 3x + 2 \cdot 7$ d. $(x + 2) \cdot 3x + (x + 2) \cdot 7$ e. None of the above. f. I don't know the distributive property.

Figure 3: Task discussed with Iota during the interview

For each item, Iota stated that the distributive property could be used to rewrite the expression: They[1] correctly chose d for Q6, and equivalent expressions that could be conceptualized as the result of the distributive property for Q1 $(2x \cdot 2 + 1 \cdot 2)$ and Q2 $(x - 2x - 1)$; but they also incorrectly chose "results" of the distributive property for Q3 $(2 \cdot 2x \div 2 \cdot 1)$, Q4 $(2x \cdot 2y)$, Q5 $((2x)^2 + 1^2)$ and Q7 $(2x2y)$. The specific answers Iota chose suggest that they may have an operational conception of the distributive property, perhaps something like "The distributive property is an instruction to take what is outside the brackets and apply it to each 'thing' inside the brackets". However, Iota's ability to conceptualize $(x + 2)$ as a unified subexpression within $(x + 2)(3x + 7)$ that could be "distributed" to each term in $3x + 7$ is an unusual and syntactically sophisticated skill, suggesting that Iota is also capable of thinking structurally. When asked about their thinking on Q4 $(2(x \cdot y))$, Iota explained "Because obviously two can distribute [makes motion with fingers as though moving two from left to right twice] with the one in parentheses. So two in the front can distribute to $2x$ multiply by $2y$. So it's gonna be $2x$ multiply by $2y$ [repeats distributive motion with fingers]—that's the result." In this excerpt, Iota is focused on describing computation, but is not considering a potential need to verify or justify the mathematical validity that computation, which is consistent with an operational GV. Thus, at this moment, Iota appears to be drawing on a pseudo-process GV. We see more of this when, later in the interview, the interviewer asks Iota what the distributive property is:

Interviewer:	What is the distributive property?
Iota:	Distribute property is like that you can use the main number or main groups to distribute to each of another number or another groups.
Interviewer:	So is that like here [highlighting $(x + 2)$ in Q6], is $x + 2$ the main number?
Iota:	It's a main group. Yes.
Interviewer:	And then you apply that to each of the ones [motions to $3x$ and 7 in Q6]
Iota:	Yes.
Interviewer:	Okay. So I noticed that this one [highlights + in expression $(3x + 7)$ in Q6] has a plus sign in between them. Is the distributive property only for the plus sign or could it also be subtraction? Could it be multiplication or division?
Iota:	So, yeah, it could be subtraction, multiplication... Could be any sign, but when you calculate, when you are doing it, you have to do with that own sign.

Here Iota again appears to view the distributive property as a process, in which whatever is outside the brackets is multiplied by each "group" inside the brackets, with the original operation between the multiple "groups" inside the brackets preserved. Here again Iota appears to be drawing on a pseudo-process GV of the distributive property. In contrast, when Iota was interviewed about Q7 $(2(xy))$, they start to reveal some evidence of a process view:

[1] In this paper we use the pronoun "they" as a singular non-gendered pronoun, as the gender of the subject is not the focus of this research. This use of the term "they" is an accepted use in English (see e.g., Merriam-Webster., n.d.).

Iota: Sometimes when I see these kind of questions, at first I may think its right answer is A ($2x2y$), but what I normally do is I double check the answer. So I create some equations and I double check it, it's incorrect. So for this case, I create like x is *3*. Okay, let me type it now, y is *2*.

x=3, y=2

2(3*2) = 2*6 =12

2*3*2*2 = 24

I think it's wrong. So I say no.... I don't know why, but this is very tricky question for me... So *x* and *y* multiply each other should be do before multiply the one outside. …I don't know, it's not look like a distributive property for me. It's look like the way to calculate is you do the xy first because in parentheses, and after you get the result of xy you do with the number *2*. So I don't think this one is like a distributive property... to be honest, I don't know why. I don't think it's A, but I just feel it's not.

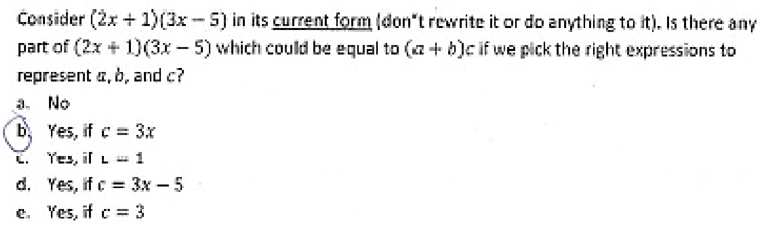
Interviewer: This strategy that you were doing, replacing *x* and *y* with numbers and seeing if they were the same: if you did that for number 6, for example, would get the same answers?

Iota: Oh, that's a good question. I don't... Yeah. Right. I don't know... I didn't... I didn't try. But... I mean, I'm just, I'm looking at it right now. Yeah, it should be the same. Because it should be only one value. Mm-hmm.

We see evidence of both process and pseudo-process GVs. Here, for the first time Iota shows evidence of the prescriptive Equivalence-Preserving GV, plugging in numbers to check whether the expression resulting from their distributive property computation in Q7 produces the same output as the original expression, at least for one value. When they see that it does not, they question their use of the distributive property to replace $2(xy)$ with $2x2y$. Thus, this appears to be evidence of a process GV. However, their approach still draws on extracted meanings and some pseudo-process GVs: they speak several times about "feeling" that the distributive property is not right here or about whether the expression "looks like" the distributive property should be used. They do not call on their process GV on the other six distributive property questions, but once the interviewer asks them whether this would be true for those questions as well, Iota is then able to draw on their knowledge of the distributive property as an equivalence-preserving transformation to see that this would be relevant for the other questions, too. Interestingly though, those items, because of the way that they "look", appear to have cued for Iota a pseudo-process rather than process GV, which would have drawn on their knowledge of the fact that the distributive property should be equivalence-preserving. It may be that Iota, and learners like them, would benefit from instruction, tasks, and assessments that focus more on checking and justifying calculation, and that aim to link the equivalence-preserving GV about properties to actual calculation procedures; future research is needed to explore this possibility.

Pseudo-object GV

We now present an interview with an elementary algebra student (whom we call Eta), where they were asked to interpret whether $(2x + 1)(3x - 5)$ could be viewed as equal to the form $(a + b)c$.


Consider $(2x + 1)(3x - 5)$ in its current form (don't rewrite it or do anything to it). Is there any part of $(2x + 1)(3x - 5)$ which could be equal to $(a + b)c$ if we pick the right expressions to represent a, b, and c?

a. No
b. Yes, if $c = 3x$
c. Yes, if $c = 1$
d. Yes, if $c = 3x - 5$
e. Yes, if $c = 3$

Figure 4: Eta considering whether $(2x + 1)(3x - 5)$ can be seen as having the form $(a + b)c$

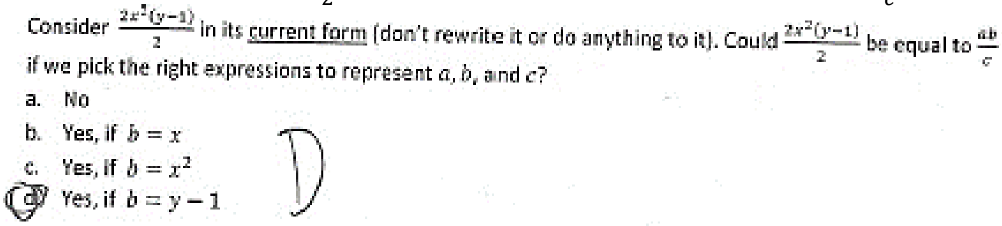
Eta: $2x$ could be a then the one would be b, then the c would be $3x$… if c is equal to $3x$

	then it would make sense…. I'm just doing it by order by the first number, second number, third number. Maybe that's not the best way, but that's what I was doing.
Interviewer:	What's being multiplied in each case [pointing to the expression]?
Eta:	Two is being multiplied by three. Two is also being multiplied by the negative five. The same thing for the one, the one is being multiplied by three and then the one is also being multiplied by the negative five.

We see evidence of Eta drawing on a pseudo-object GV by mapping subexpressions to variables in the form "in order": mapping the "first subexpression" to the first variable, etc., without attending to grammatical meaning. In $(2x + 5)(3x - 5)$, Eta initially appears not to see the second set of brackets (or not recognize them as indicating grouping) as they map subexpressions to the form $(a + b)c$; however, further questioning reveals that Eta knows that each term in $(3x - 5)$ will eventually be multiplied by each term in $(2x + 1)$. This suggests that Eta's pseudo-object GV of properties does not stem directly from a failure to recognize the syntactic role of the second set of brackets. Rather, there appears to be a disconnect between Eta's syntactic meanings of expressions, and the information they focus on when mapping to a form. Eta does not identify the current syntactic meaning of $(2x + 1)(3x - 5)$ as $2x + 1$ and $3x - 5$ being multiplied, but rather as something like $2 \cdot 3 \cdot x^2 + 2 \cdot -5 + 1 \cdot 3 \cdot x + 1 \cdot 5$. This computational view of the syntactic meaning of $(2x + 1)(3x - 5)$ (perceiving it as the result of expansion rather than its literal meaning) may obscure the structure needed to map this expression to the form $(a + b)c$. Thus, a computational view of syntactic structure may be impacting Eta's GV for symbolic properties. It may be that Eta, and learners like Eta, would benefit from instruction that explicitly highlights differences in syntactic structure of different expressions, and links this explicitly to form mapping—this may enable Eta to draw on existing knowledge of syntax, symbolic structures, and forms as objects; future research is needed to explore this.

Object GV

We now present an interview with an elementary algebra student (whom we call Theta), where they were asked to interpret whether $\frac{2x^2(y-1)}{2}$ could be viewed as equal to the form $\frac{(ab)}{c}$ (where $c \neq 0$).


Consider $\frac{2x^2(y-1)}{2}$ in its current form (don't rewrite it or do anything to it). Could $\frac{2x^2(y-1)}{2}$ be equal to $\frac{ab}{c}$ if we pick the right expressions to represent a, b, and c?

a. No

b. Yes, if $b = x$

c. Yes, if $b = x^2$

d. Yes, if $b = y - 1$

D

Figure 5: Theta mapping a multi-term expression to a variable in a form

Theta:	I felt like D was the best option because looking at a and b over c the first equation fit that like a could be $2x^2$ and b could be $y - 1$ and c could be 2.
Interviewer:	Did the parentheses impact your decision?
Theta:	Yes.
Interviewer:	How?
Theta:	Because I saw that the $y - 1$, I saw it as separate from $2x^2$. And I know that looking at the second one that a and b in order for them to be multiplied they would most likely have to have parentheses around them. And I saw $y - 1$ in parentheses so I just... looking at them all as substitutes, as soon as I saw a and b over c like I was just putting in my head okay, $2x^2$ is a, y minus 1 is b, and the two is equal to c.

Here Theta shows evidence of drawing on an object GV. They identify mathematically valid subexpressions in $\frac{2x^2(y-1)}{2}$, and identify which of these should map to which variable in the form to preserve the structure. After this excerpt, the interviewer asked Theta to identify various syntactic structures in the expression, and Theta was able to do so accurately without further prompting. Theta also specifically talks about brackets from an object view (as a grouping mechanism rather than a cue to a procedure [see Wladis et al, 2022a]) because they "separate" the $2x^2$ from the $y-1$. This suggests that Theta has an object view of syntactic structure that they are able to successfully leverage into object view of symbolic properties, because it allows them to identify the subexpression structures that will produce a one-to-one mapping from the expression $\frac{(2x^2)(y-1)}{2}$ to the form $\frac{ab}{c}$ in a way that correctly preserves the syntactic structure. Theta specifically mentions substitution in discussing how the subexpressions relate to the properties form: this suggests that Theta's notions of substitution and substitution equivalence may be related to their conceptions of properties (see Wladis et al., 2022b). Theta's discussion of this question is unusually structural compared to other students in the sample (including those in algebra as well as higher-level courses such as calculus). While we can draw no causal conclusions based on this evidence, Theta's responses indicate that, with explicit instruction, some algebra students are capable of reasoning structurally about symbolic properties. Theta was part of an intervention that was focused on explicitly teaching students the prescriptive GVs presented here (as well as others related to syntactic structure and equivalence), so this may have influenced their GV formation; ongoing research is underway to better explore this possibility.

Conclusion

In the three vignettes presented here, all three students have prior knowledge that may be productive when using symbolic properties to transform algebraic expressions or equations. However, in some cases the learners were able to draw on that prior knowledge in robust ways, and in other cases, that prior knowledge was not cued or viewed as relevant in the moment by the learners as they answered questions about how they make sense of forms and symbolic properties. It is thus important for future research to better understand the relationship between prior knowledge and different conceptions or GVs of symbolic properties as well as syntactic structure, substitution, and equivalence, since these different types of knowledge are all connected during algebraic transformation (Wladis et al., 2023).

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