Syntactic Reasoning and Cognitive Load in Algebra

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In the context of proofs, researchers have distinguished between syntactic reasoning and semantic reasoning; however, this distinction has not been well-explored in areas of mathematics education below formal proof, where student reasoning and justification are also important. In this paper we draw on theories of cognitive load and syntactic versus semantic proof-production to explicate a definition for syntactic reasoning outside the context of formal proof, using illustrative examples from algebra.

Keywords: Syntactic reasoning; Semantic reasoning; Cognitive load; Algebra

In this paper we outline a framework for analyzing student reasoning in mathematics, using the distinction between syntactic and semantic reasoning. This distinction has been a helpful framework for analyzing proof construction (Weber & Alcock, 2004) but has rarely been used as a framework for analyzing reasoning or justification in other mathematical domains where formal proof is not common. Even when syntactic reasoning is referred to directly in the context of proof, it is often conflated with manipulation of symbols without understanding. In this paper, we explicate a definition for syntactic reasoning outside the context of formal proof, presenting it as a form of reasoning that is distinct from rote "symbol pushing". We then use one example from school algebra to illustrate: 1) how syntactic and semantic reasoning play critical and complementary roles in this context; 2) how leveraging syntactic reasoning can reduce cognitive load; and 3) how preferences for syntactic or semantic reasoning approaches may relate to prior knowledge and schema. Our aim is to start a conversation about the potential affordances of more explicitly attending to syntactic reasoning in task and curriculum design and instruction.

Syntactic and Semantic Reasoning

In the literature on student proof construction, a number of studies have explored syntactic versus semantic reasoning during proof production. Weber and Alcock (2004) define syntactic proof production as drawing "inferences by manipulating symbolic formulae in a logically permissible way" and semantic proof production as using "instantiations of mathematical concepts to guide the formal inferences that [the prover] draws". In some work these categories are binary and assigned to a whole proof production, but in other work (e.g., Weber and Mejia-Ramos, 2009) syntactic and semantic reasoning are conceptualized as describing different steps in a students' reasoning process, where students may switch back and forth between different approaches; the latter is the approach we aim to take.

We also point out two key distinctions between our definitions of syntactic reasoning and those that have been used in proof literature. In particular, Weber and Alcock's (2004) definition of syntactic reasoning as drawing "inferences by manipulating symbolic formulae in a logically permissible way" is for us incomplete—we are not only interested in the result of a student's calculations, but also in their reasoning[1]. A second key distinction is that while Alcock & Inglis

[1] We note that in other work, Weber (2005) distinguishes between *syntactic* and *procedural* proof production (the latter being based more on imitation of particular proof "templates" without necessarily understanding why they are valid)—this distinction is similar to, but different from our distinction between syntactic reasoning and symbolic manipulation that is not grounded in syntactic reasoning.

(2008) classify reasoning as syntactic if it takes place within the "representation system of proof," we discuss a context in which syntactic reasoning can occur beyond proof production, which we call an "abstract symbolic system."

Abstract Symbolic Systems

We situate our definition of syntactic reasoning within *abstract symbolic systems*: self-contained systems in which the symbolic objects are what Tall et al. (1999) describe as *axiomatic objects*, or objects that arise "from specifying criteria (axioms or definitions) from which properties are deduced by formal proof" (p.239). This is also related to, but more narrowly defined than, what Goldin (1998) termed a **symbolic system**—a set of conventions and implicit or explicit axioms for the use of mathematical symbols which is:

1. **Abstract**, in the sense that it does not necessarily have any "real world" or alternative representations beyond what is specified by the axioms in the system.
2. **Self-contained**, in the sense that one has all the tools one needs already within the system to be able to identify or generate equivalent objects.

Our Definitions. *Syntactic reasoning* is reasoning which justifies mathematical work by referring back to conventions and axioms *within* the abstract symbolic system. *Semantic reasoning* involves justifying mathematical work by connecting it to representations or concept images *outside* the abstract symbolic system (e.g., the concept image of multiplication as area).

Distinguishing syntactic reasoning from "symbol pushing"

We have observed a tendency in the mathematics education community to frame syntactic reasoning as normatively undesirable, and semantic reasoning as normatively superior (see e.g., Easdown, 2009; Weber & Alcock, 2004). This is likely a reaction to approaches to teaching in which students are taught to manipulate symbols without connecting them to relevant underlying mathematical reasoning (e.g., Stacey, 2010). However, it is important to distinguish between syntactic reasoning versus symbolic manipulation that is disconnected from relevant logical reasoning.[2] We contend that the former is an essential component of mathematical reasoning, complementary to semantic reasoning, and necessary to manage the cognitive load of syntactically complex tasks. In our framework, syntactic and semantic reasoning are viewed as two essential and complementary components of mathematical reasoning.

For our definition of syntactic reasoning, it is not enough that a student manipulates symbols, even correctly—they must show evidence of reasoning within an abstract symbolic system. For example, a student may correctly transform with the distributive property in the following way: $2(3x + 5) = 2 \cdot 3x + 2 \cdot 5$. However, what reasoning this student has used is not clear from this work alone. Here are some examples of explanations that we would and would not consider to be syntactic reasoning (these examples are fictional, but have been written to mimic common forms of explanation that college students have provided in other empirical work):

1. ***Student 1 (neither syntactic reasoning nor semantic reasoning):*** *You take whatever is on the outside of the parentheses, and put it next to each thing inside the parentheses.*
2. ***Student 2 (syntactic reasoning):*** $2(3x + 5)$ *has the form* $a(b + c)$ *if we let* $a = 2$, $b = 3x$ *and* $c = 5$. *We know from the distributive property that* $a(b + c) = ab + ac$,

[2] We use the modifier "relevant" because students who are "symbol pushing" often use complex or abstract forms of reasoning that are not connected to the relevant mathematical-logical justification.

so we know that $2(3x + 5) = 2 \cdot 3x + 2 \cdot 5$ *by substituting* 2*,* $3x$ *and 5 into the correct variables in the distributive property.*

3. ***Student 3 (semantic reasoning):*** *When you have two terms that are doubled, it doesn't matter if you add first and then double the result or double each term separately. Because either way you are still doubling each individual term.*

Student 1 describes a procedure but does not draw on any conventions or axioms of the abstract symbolic system in their justification, so this is not considered syntactic reasoning by itself. Student 2, in contrast, is drawing only on conventions and axioms given in the abstract symbolic system to justify their work. Unlike the others, Student 3 has drawn on meanings for multiplication and addition that go beyond what is given in the axiomatic system; this student is using other definitions of addition (e.g., as combining parts or terms) and multiplication (e.g., as doubling) to justify their work. Some readers may prefer the semantic explanation to the syntactic one or vice versa. We note only that both draw on distinct forms of reasoning, and that each approach directs cognitive resources towards something different. The syntactic approach directs resources in working memory towards parsing complex syntax so that a learner can view complex expressions as "having the form" $a(b + c)$. On the other hand, the semantic approach directs cognitive resources towards understanding *why* the distributive property can be derived from specific conceptions of addition and multiplication. However, the working memory resources necessary to process this semantic explanation may increase significantly as the subexpressions which represent a, b and c become progressively more syntactically complex. We likely do not expect students to justify the distributive property every single time that they use it, just as we do not expect students to semantically justify every sum or product every time that they calculate. Ideally, students would be able to employ *both* forms of reasoning when solving problems, switching *back and forth* between them strategically, in ways that both maximize their understanding of the underlying mathematics, and keep the cognitive load of a particular problem to manageable levels.

Cognitive Load and Strategic Selective Attention

There are several key features of cognitive load theory. Firstly, it characterizes learning as the process of encountering novel information, processing it in working memory, and then (to the extent that it is perceived as useful) encoding it in some way into long term memory (see e.g., Kalyuga, 2010). During this process of learning novel material, elements which are encountered during learning are encoded into mental schema[3], which vary in size and complexity based on the learner's expertise in a specific knowledge domain (de Groot, 2014; Ericsson & Kintsch, 1995; Sweller et al., 2019). The number of elements that can be held in working memory does not vary for "novices" vs. "experts" in a given domain; rather, it is the complexity of the mental schema that make up the individual elements which varies based on a learner's level of expertise in a particular area (de Groot, 2014; Ericsson & Kintsch, 1995; Sweller et al., 2019).

Thus, during the process of problem solving, how much cognitive load a particular task requires will vary based on the existing mental schema of the learner. A task that is perceived to have a dozen elements by a "novice" might be perceived as all fitting into a single mental schema for an "expert" in that domain (and thus from a cognitive load perspective, would take up only one element of working memory for the "expert"). In addition, the information encoded in mental schema may be automated, so that it is no longer processed consciously. This reduces

[3] We note that the term "schema" is used differently in cognitive load literature and should not be conflated with more specific usage in mathematics education research (e.g., APOS theory [Dubinsky, 1991]).

the cognitive load for the learner, but also removes much of the work of solving problems from their conscious mental awareness (Cooper & Sweller, 1987; Sweller, 2011). This may have consequences for both learners and instructors: for instructors who have automated particular complex schema, attempting to explain these processes which they no longer execute consciously can significantly increase their cognitive load during instruction (Lee & Kalyuga, 2014), and also make it difficult for instructors to accurately gauge the cognitive load that a "novice" (who does not have these automated schema) might experience from the same problem.

Strategic selective attention

Because syntactic and semantic approaches can have different implications for cognitive load, switching back and forth between these two approaches may be one way of strategically managing cognitive load within a single task. This could be described by theories of selective attention. *Selective attention* is a measure of the extent to which someone is able to filter out irrelevant information during the problem-solving process and focus only on the aspects of the problem that are salient to the task at hand (Broadbent, 1958; Treisman, 1964). Research has established that selective attention may be a key skill in mathematical problem-solving and is related to students' working memory (see e.g., Arán Filippetti & Richaud, 2016; Campos et al., 2013). Most use of selective attention in the mathematics education literature focuses on it as a learner's ability to filter out completely irrelevant information (e.g., on a one-step problem about red apples, ignoring information about green apples), but we focus on a related but slightly different aspect of selective attention, which we term ***strategic selective attention***: a student's ability to *temporarily* ignore information that is not relevant to the *current* step in solving a problem (but which may be relevant at another step, or to interpreting the answer, etc.). For example, a student may temporarily ignore concept images that are helpful for reasoning semantically during a problem-solving step that is focused on syntactic reasoning (or vice versa), in order to lower their cognitive load.

Because strategic selective attention allows a student to temporarily ignore information that is not relevant to the current step, it narrows the amount of information that must be held and processed in working memory and can therefore be critical once mathematical problems become more complex. Thus, learning to work in valid ways with abstract symbolic systems by focusing on syntactic reasoning could be key to helping students reduce the cognitive load of many standard mathematics problems. To illustrate how strategic use of syntactic reasoning could help to reduce the cognitive load of mathematical problems, we present a few examples of how a standard algebra problem might be justified. We do not contend that any particular choice of when to reason syntactically versus semantically is right or wrong—this may vary for different people in different contexts.

Illustrative Example

As a starting point for discussion, we consider the following problem, which is a standard question in school algebra, typically introduced in 8th or 9th grade in the U.S.:

Example 1. Simplify $\frac{6x^2+2x}{2x}$ completely (assume $x \neq 0$).

We have chosen this example because it is fairly accessible, but student errors are nonetheless quite common, often involving invalid "cancelling" procedures, for example (Malle, 1993):

$$\frac{6x^2 + \cancel{2x}}{\cancel{2x}} = 6x^2$$

Researchers have explained this invalid form of 'cancelling' in a variety of ways: generalizing from a limited set of examples where this heuristic holds to contexts where it no longer holds (Matz, 1982, p.26), and attending to visual similarities and patterns on the page (Erlwanger, 1973; Kirshner & Awtry, 2004). We note that if a student is performing this cancelling approach, they are *not drawing on important syntactic meanings within this abstract symbolic system*. For example, one critical syntactic meaning is that the numerator $6x^2 + 2x$ is a single unified object, and cannot be partially canceled, as is done in the work above. Another critical syntactic meaning is that canceling represents the replacement of a single fraction of the form $\frac{a}{a}$ with the equivalent expression 1 (assuming $a \neq 0$); it does not represent a "disappearance" of the objects being "cancelled".

A "semantic" reasoning approach

We now present a semantic reasoning approach, shown in Figure 1. However, we first note that no approach to this problem in its current form can be 100% semantic, because at a minimum, syntactic reasoning is necessary to read the symbolic expression. (For example, we need to know that when a number is written next to a letter, it means multiplication.) Therefore, because no approach justifying mathematics written using symbolic representations is 100% semantic, we write the word semantic in quotes for this overall example.

Step 1	$\frac{6x^2+2x}{2x}$ $=\frac{6x^2}{2x}+\frac{2x}{2x}$	**Semantic reasoning:** Fractions can be thought of as parts of a whole, where the top number represents the number of pieces and the bottom number represents the size of the pieces. So, if we want to split these two subexpressions into two separate fractions, it will have the same meaning as long as we use the same denominator for both fractions, using the denominator that was in the original fraction.
Step 2	$=\frac{(2x)(3x)}{2x}+\frac{2x}{2x}$	**Syntactic reasoning:** Because of the generalized associative/commutative property of multiplication, we can perform multiplication using any order or grouping (as long as only multiplication is involved). So $6x^2 = 6 \cdot (x \cdot x) = (2x) \cdot (3x)$.
Step 3	$=\frac{2x}{2x} \cdot (3x)+\frac{2x}{2x}$	**Semantic reasoning:** Dividing by a number is the same as multiplying by the reciprocal of that number because both dividing by c and multiplying by $\frac{1}{c}$ can be thought of as breaking the original number up into c-many equally-sized groups, and then taking the size of just one of those groups. **Syntactic reasoning:** Combining this semantic reasoning with the generalized commutative/associative property of multiplication, we can replace $\frac{(2x)(3x)}{2x}$ with $\frac{2x}{2x}(3x)$.
Step 4	$= 1 \cdot (3x) + 1$	**Semantic reasoning:** Dividing anything by itself will always be 1, because everything goes into itself only once, so $\frac{2x}{2x} = 1$ as long as x is not zero.
Step 5	$= 3x + 1$	**Semantic reasoning:** Multiplying anything by 1 is just like taking it one time, so it does not change it.

Figure 1: Using "Semantic" Reasoning

In Figure 1, the reader will notice the label "syntactic reasoning" for the two areas where the generalized commutative/associative property of multiplication is used. Students may be asked to justify this property semantically in specific cases, but are not expected to justify the general case semantically (i.e., that when multiplying any number of factors, neither the order nor the grouping matters). The generalized case is typically justified formally using proof by induction, which is by definition a syntactic justification that is not appropriate for most K-12 (or even college) students. Thus, most students who use the generalized commutative/associative property of multiplication are using it syntactically, as a stated axiom within the abstract symbolic system within which they are working.

Hidden forms of syntactic reasoning in our "semantic" example

Embedded within each step labeled 'semantic reasoning' is hidden syntactic reasoning which is necessary in order for the semantic reasoning to be connected to the symbolic representations. In Figure 2, we now fill in some of that reasoning, making the implicit more explicit for Step 1.

Step 1	$\frac{6x^2+2x}{2x}$ $= \frac{6x^2}{2x} + \frac{2x}{2x}$	Syntactic reasoning: In this expression, we can treat $6x^2$ and $2x$ at the top of the fraction and $2x$ at the bottom of the fraction each as a unified subexpression, because of the convention that the top and bottoms of fractions should be treated as unified subexpressions, and because according to the order of operations, exponents and multiplication come before addition. So, we can think of this as $\frac{a+b}{c}$ where a represents $6x^2$, b represents $2x$, and c represents $2x$. Semantic reasoning: Fractions can be thought of as parts of a whole, where the top number represents the number of pieces and the bottom number represents the size of the pieces. So, addition involving two subexpressions at the top of a fraction $(a + b)$ represents addition of one number of pieces of the same size from another. So, if we want to split these two subexpressions into two separate fractions, it will have the same meaning as long as we use the same denominator for both, using the denominator in the original fraction.

Figure 2: First Step of Expanded "Semantic" Reasoning Example, with "Hidden" Semantic Reasoning Added

There are several interesting things to note about the expanded example in Figure 2. Firstly, in each step, the semantic reasoning is dependent upon some underlying syntactic reasoning. Although this reasoning is hidden, it is still needed to complete the problem. Thus, many semantic explanations provided to students during instruction and in curriculum may actually be more syntactic than they seem and may hide key parts of justifications from students.

Secondly, this explanation reads as quite long, because it combines the necessary syntactic reasoning with added semantic explanations which are, in a sense, justifications of the properties that need to be used when justifying syntactically (in a purely syntactic example these properties might be treated as axioms that do not require justification). Thus, there is increased cognitive load needed to provide both semantic and syntactic justification simultaneously. This increased load may be desirable in some cases and undesirable in others, depending on factors which will vary from one problem-solver to another, and from one instructional context to another.

The role of cognitive load in choices to implement syntactic vs. semantic reasoning

Many readers who are "experts" in algebra (e.g., algebra instructors, mathematicians) may prefer the short "semantic" justification in Figure 2 and feel that the expanded "semantic" justification in Figure 3 carries a higher cognitive load, because it contains significant amounts of "extraneous" information. For these experts, the syntactic reasoning that has been explicitly given and broken down in detail in the expanded explanation represents prior knowledge that they have already reified into unified schema and automated. Thus, unpacking this prior knowledge by breaking it down and making it conscious requires more cognitive effort.

On the other hand, for a problem-solver who lacks some of these schema, or who has not automated them (or has automated non-normative syntactic meanings), the additional information given in the expanded explanation may be helpful, or even essential, to understanding the “semantic” example. Including these additional details may make the “semantic” justification accessible in a way that it was not before. This contrast between how an “expert” and a “novice” might experience worked examples with additional explanatory information is similar to patterns that have been found in the research literature, in which additional explanatory information improved the performance of “novices” but slowed down “experts” who did not need it (the “expertise reversal effect”; see e.g., Kalyuga, 2007; Kalyuga et al., 1998). We note, however, that the cognitive load in the expanded “semantic” justification is still quite high. We see this expanded “semantic” justification as a useful tool for unpacking many of the different kinds of knowledge that are necessary for understanding the justification of the short “semantic” example; it is *not* intended to be presented as a useful example of how this might be taught to students.

We are left with the challenge of how to teach justification without overloading students’ working memory, even in cases where they have not yet acquired the necessary schema and automated syntactic knowledge to parse a short “semantic” justification. One option is to break down the complex network of interacting information into isolated elements which can be learned separately, each with a lower cognitive load individually. Studies that have employed this kind of approach in other contexts have shown that it can help students to learn complex interdependent types of knowledge which would have a too high cognitive load if learned all at once (see e.g., Pollock et al., 2002). Further research is necessary to determine the best methods.

Conclusion

In this paper we have described how syntactic reasoning could be defined for mathematical contexts that do not use formal proof. Our definition of syntactic reasoning is distinct from mere “symbol pushing.” It requires not just manipulation of symbols, but reasoning behind symbolic manipulation that draws on specific syntactic meanings of the abstract symbolic system in which the manipulations are being conducted. Thus, in our framework, semantic reasoning does not have to be present for productive and authentic mathematical reasoning to occur.

We have illustrated how syntactic and semantic approaches to problems may impact the cognitive load of particular tasks differently, particularly with respect to the existing schema of the problem-solver. We deconstructed one particular algebra task in order to illustrate many of the hidden ways in which semantic reasoning depends on syntactic reasoning as soon as algebra is written symbolically. This “hidden” syntactic reasoning may relate to the difference in cognitive load that “experts” and “novices” experience. “Experts” may have already automated much of the syntactic reasoning that they use, whereas “novices” may have no awareness of the role that syntactic reasoning plays. “Experts” may also be unaware of their own unconscious automated processes, which may perpetuate implicit, rather than explicit, handling of syntactic reasoning in curriculum and instruction. This suggests that to maximize rich understanding and minimize cognitive load for all learners, more research is needed to understand how syntactic reasoning can best be taught to learners with varied levels of prior knowledge, and to tease out how students and instructors can leverage strategic selective attention to switch between syntactic and semantic modes of reasoning.

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