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Chromatic symmetric functions of Dyck paths and q-rook theory[☆]

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ABSTRACT

The chromatic symmetric function (CSF) of Dyck paths of Stanley and its Shareshian–Wachs q-analogue have important connections to Hessenberg varieties, diagonal harmonics and LLT polynomials. In the, so called, abelian case they are also curiously related to placements of non-attacking rooks by results of Stanley and Stembridge (1993) and Guay-Paquet (2013). For the q-analogue, these results have been generalized by Abreu and Nigro (2021) and Guay-Paquet (private communication), using q-hit numbers. Among our main results is a new proof of Guay-Paquet's elegant identity expressing the q-CSFs in a CSF basis with q-hit coefficients. We further show its equivalence to the Abreu–Nigro identity expanding the q-CSF in the elementary symmetric functions. In the course of our work we establish that the q-hit numbers in these expansions differ from the originally assumed Garsia–Remmel q-hit numbers by certain powers of q. We prove new identities for these q-hit numbers, and establish connections between the three different variants.

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1. Introduction

Ever since their introduction in 1995 in [25], the chromatic symmetric functions have been this mysterious object combining the misleading simplicity of graphs with the powerful tools of symmetric functions. Graph colorings present some of the hardest problems in combinatorics,[1] and nice formulas there qualify as miracles rather than general rules. It is thus even more appealing that the chromatic symmetric functions, and their q-generalizations, are a source of beautiful results and striking conjectures.[2] The chromatic symmetric functions have found significant connections beyond combinatorics — to *Hessenberg varieties* [24], *diagonal harmonics,CM*, and *Macdonald polynomials* [2,17].

In this paper we bring to light such an unusually nice combinatorial formula, relating the q-*rook theory* which comes from generalizations of permutations and their inversions, and chromatic symmetric functions for Dyck paths of bounce two, aka abelian case. We give an elementary proof of the strikingly elegant identity of Guay-Paquet (Theorem 1.3) which expresses the chromatic symmetric function for an arbitrary path given by partition λ in terms of the chromatic symmetric functions for rectangles with coefficients the very combinatorial q-hit numbers. Along the way we establish numerous new identities for q-hit and q-rook numbers, give an elementary proof of Theorem 1.2, and pose many conjectures stemming from our findings. Our ultimate goal is to understand the chromatic symmetric functions with more relations and connections, which could lead not only to a proof of the e-positivity Conjecture 1.1, but also to a combinatorial interpretation of these coefficients. The technique of symmetry-breaking used in our proof of Theorem 1.3 could be extended beyond the abelian case as long as there is a suitable conjectured expression for the coefficients in the e-basis.

1.1. Definitions and main results

Let G be a graph with vertices $\{v_1, v_2, \ldots, v_n\}$ that are totally ordered $v_1 < v_2 < \cdots < v_n$. In [25], Stanley defined the chromatic symmetric function (CSF) $X_G(\mathbf{x})$ of G as

$$X_G(\mathbf{x}) = \sum_{\kappa:V\to\mathbb{P},\ \text{proper}} \mathbf{x}^{\kappa} = \sum_{\kappa:V\to\mathbb{P},\ \text{proper}} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots,$$

where $\mathbb{P} = \{1, 2, 3, \ldots\}$, $\mathbf{x} = (x_1, x_2, \ldots)$, and the sum is over the proper colorings of the vertices of G.

Stanley and Stembridge [26] conjectured that the chromatic symmetric functions expand with positive coefficients in the basis $\{e_\mu\}$ of elementary symmetric functions for the graphs coming from Dyck paths in the following way. Given a Dyck path d from $(0, 0)$ to (n, n), let $G(d)$ be the graph with vertices $\{1 \ldots n\}$ and edges (i, j), $i < j$ if and only if the cell (i, j) is below the path d (see Fig. 1). These are also the incomparability graphs of *unit interval orders* or graphs obtained from *Hessenberg sequences*.

Shareshian–Wachs [24] introduced a quasisymmetric version of $X_G(\mathbf{x})$ defined by

$$X_G(\mathbf{x}, q) = \sum_{\kappa:V\to\mathbb{P},\ \text{proper}} q^{\operatorname{asc}(\kappa)} \mathbf{x}^{\kappa},$$

where $\operatorname{asc}(\kappa)$ is the number of edges $\{v_i, v_j\}$ of G with $i < j$ and $\kappa(v_i) < \kappa(v_j)$.

[1] Informally, but also formally as an NP-complete problem.

[2] Most notably the e-positivity Conjecture 1.1 of Stembridge–Stanley [26], refined further by Shareshian–Wachs [24].

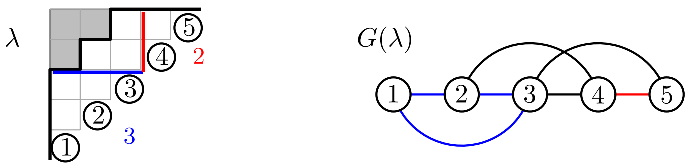


Fig. 1. The Dyck path $d = \mathsf{n}^3\mathsf{enenee}^2$ associated to the partition $\lambda = (2, 1) \subset 2 \times 3$ and the corresponding graph $G(\lambda)$. Each cell below the path d corresponds to an edge of the graph. An example of a proper coloring would be $\kappa(1) = 2, \kappa(2) = 3, \kappa(3) = 1, \kappa(4) = 2, \kappa(5) = 3$ which has $\mathrm{asc}(\kappa) = 4$.

For the graphs $G(d)$ coming from Dyck paths, the quasisymmetric function $X_{G(d)}(\mathbf{x}, q)$ is actually symmetric and Shareshian–Wachs gave a refinement of the Stanley–Stembridge conjecture for this Catalan family of graphs.

Conjecture 1.1 (*Stanley–Stembridge, Shareshian–Wachs*)**.** *Let d be a Dyck path. Then the coefficients of $X_{G(d)}(\mathbf{x}, q)$ in the elementary basis are in $\mathbb{N}[q]$.*

This conjecture has been verified independently and by different techniques by Cho–Huh [7], Harada–Precup [18], and Abreu–Nigro [1] for the case of so-called *abelian* Dyck paths (corresponding to abelian Hessenberg varieties), which are defined as Dyck paths d of from $(0, 0)$ to $(m+n, m+n)$ of the form $\mathsf{n}^m w(\lambda)\mathsf{e}^n$ where $w(\lambda)$ is the encoding in north (n) and east (e) steps of the partition $\lambda \subset n \times m$ (see Fig. 4(a)). We denote the associated graph by $G(\lambda)$ and the chromatic symmetric function by $X_\lambda(\mathbf{x}, q) := X_{G(\lambda)}(\mathbf{x}, q)$.

The symmetric functions $X_\lambda(\mathbf{x}, q)$ corresponding to abelian Dyck paths are deeply related to the *q-rook theory* of Garsia–Remmel [11] as we illustrate with the next two identities that use the following notation

$$[n]_k = [n][n-1]\cdots[n-k+1], \quad [n]! = [n]_n, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_k}{[k]!},$$

where $[x] = (1-q^x)/(1-q)$.

We define q-*hit numbers* of *rectangular boards* of size $n \times m$ that we denote as $H_j^{m,n}(\lambda)$ by a change of basis Eq. (2.1) involving the *Garsia–Remmel q-rook numbers*. These q-hit numbers are polynomials in q, satisfying $\sum_{j=0}^n H_j^{m,n}(\lambda) = [m]_n$, and at $q = 1$ give the number of placements of n non-attacking rooks in an $n \times m$ board ($n \leq m$) with j rooks in the board of λ. We show that these q-hit numbers $H_j^{m,n}(\lambda)$ are symmetric polynomials in $\mathbb{N}[q]$ and are realized by a statistic defined by Dworkin [8] (see Theorem 2.9). In the case of a square board $m = n$, these q-hit numbers $H_j^n(\lambda) := H_j^{n,n}(\lambda)$ are up to a power of q equal to the *Garsia–Remmel q-hit numbers* (Proposition B.1) which are symmetric unimodal polynomials in $\mathbb{N}[q]$ realized by different statistics by Haglund and Dworkin (see [16]).

Abreu–Nigro gave an expansion of $X_\lambda(\mathbf{x}, q)$ in the elementary basis in terms of q-hit numbers of square boards. This result is a q-analogue of a special case of a result of Stanley–Stembridge [26, Thm. 4.3].

Theorem 1.2 (*Abreu–Nigro [1]*)**.** *Let λ be partition inside an $n \times m$ board with $\ell(\lambda) = k \leq \lambda_1$. Then*

$$X_\lambda(\mathbf{x}, q) = [k]! H_k^{m+n-k}(\lambda) \cdot e_{m+n-k,k} + \sum_{j=0}^{k-1} q^j\,[j]!\,[m+n-2j]\,H_j^{m+n-j-1}(\lambda) \cdot e_{m+n-j,j}.$$

Our first main result is an elementary proof of an unpublished identity of Guay-Paquet.[3,4] In Section 4 that expands $X_\lambda(\mathbf{x}, q)$ in terms of chromatic symmetric functions for rectangular shape with coefficients given by the *q-hit numbers* of *rectangular boards* defined above.

3 Private communication [12].

4 This identity was independently found by Lee and Soh [20, Thm. 24] after this article was posted.

This result appears as a q-analogue of a special case of [13, Prop. 4.1 (iv)].

Theorem 1.3 (*Guay-Paquet [12]*). *Let λ be partition inside an $n \times m$ board ($n \leq m$). Then*

$$X_\lambda(\mathbf{x}, q) = \frac{1}{[m]_n} \sum_{j=0}^{n} H_j^{m,n}(\lambda) \cdot X_{m^j}(\mathbf{x}, q).$$

Our second result is a direct elementary proof of Theorem 1.2 following from Theorem 1.3 and using **our third result** which is the definition and properties of the q-hit numbers $H_*^{*,*}(\lambda)$, including a deletion–contraction relation, that we derive in Sections Section 3 and in Appendix A.

1.2. Old and new methods

The original proofs of the two statements above use a linear relation satisfied by $X_{G(d)}(\mathbf{x}, q)$ called the *modular relation* [1,3,13]. Our proof of Theorem 1.3 uses a simple inductive approach on both the size $m+n$ of the graph and the number M of variables (see Lemma 4.3). Our approach ignores/breaks the symmetry of the chromatic symmetric function by splitting the function as a polynomial in x_k, whose coefficients are polynomials in $x_1, \ldots, x_{k-1}$. The ultimate identities are derived from identities of the coefficients; the q-hit numbers. Such an approach could work in a more general setting if the coefficients in the expansion have some recursive combinatorial structure. Moreover, following the recursion it could be extended to a bijection, similar to RSK. The bottlenecks in this approach are the necessary new q-hit identities, which we derive after extensive use of generating functions.[5] The derivation of Theorem 1.2 follows from other q-hit identities, which can be proven also using deletion–contraction on q-hit and q-rook numbers. Note that deletion–contraction on the classical CSFs itself is not directly applicable due to the inhomogeneity of the relation.

Along the way we prove new q-hit identities (Section 3 and the Appendix) and unravel a mystery on different combinatorial statistics leading to different kinds of q-hit numbers (see Section 7.1 and the Appendix) that have been mixed up in the literature. In particular, we establish new relations of *q-rook numbers* and *q-hit numbers* (Lemmas 3.3, 3.4, 3.5, and 5.5) that develop further the q-rook theory of rectangular boards [21].

As a Corollary to the fact that Theorems 1.2 and 1.3 are in essence linear relation between chromatic symmetric functions, we establish that the same linear relation holds of the **unicellular LLT polynomials**, see Section 6.1.

1.3. Organization

In Sections 2 and 3 we give the definitions of q-hit numbers and prove the necessary identities used later on. Our elementary proof of Theorem 1.3 is in Section 4, and the proof of Theorem 1.2 is in Section 5. In Section 6 we discuss variations on these problems, expansions in other bases like CSFs for staircase shapes, applications to LLT polynomials, and some conjectures.

In Appendix A, B, and E we present the Garsia–Remmel q-hit numbers and their relation to the q-hit numbers appearing in Theorems 1.2,1.3, and the deletion–contraction relations for each variant. Appendix C, D have the proofs of Theorem 2.9 and the symmetry of the q-hit numbers, respectively.

2. Background on q-rook theory

For the rest of the paper, we assume m and n are non-negative integers with $m \geq n$.

[5] A fully combinatorial/bijective proof would be highly desirable and could completely unravel the combinatorics for CSFs in the abelian case. See Section 7.

2.1. q-rook numbers

Rook placements are a generalizations of permutation diagrams, and their q-analogues keep track of the number of inversions. We now summarize important definitions and properties used later in relation to the chromatic symmetric functions. In the Appendix we include the proofs and further properties.

Definition 2.1 (*q-rook Numbers [11]*)**.** Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ the *Garsia–Remmel q-rook numbers* are defined as

$$R_k(\lambda) = \sum_p q^{\mathrm{inv}(p)},$$

where the sum is over all placements p of k non-attacking rooks on λ and $\mathrm{inv}(p)$ is the number of cells of λ that are not occupied by a rook or directly west or north of a rook (see Fig. 3(a)).

Proposition 2.2 (*Garsia–Remmel [11]*)**.** *Given a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ we have that*

$$F(x;\lambda) := \sum_{k=0}^{\ell} R_k(\lambda)[x]_{\ell-k} = \prod_{i=1}^{\ell}[x+\lambda_{\ell-i+1}-i+1],$$

in particular $R_\ell(\lambda) = \prod_{i=1}^{\ell}[\lambda_{\ell-i+1}-i+1]$.

2.2. q-Hit numbers

The q-hit numbers are defined in terms of the q-rook numbers by a change of basis. Let $(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i)$ denote the *q-Pochhammer symbol.*

Definition 2.3 (*[21, Def. 3.1, Prop. 3.5]*)**.**

For λ inside an $n \times m$ board, we define the *q-hit polynomial* of λ by

$$P(x;\lambda) = \sum_{i=0}^{n} H_i^{m,n}(\lambda)x^i := \frac{q^{-|\lambda|}}{[m-n]!}\sum_{i=0}^{n} R_i(\lambda)\,[m-i]!(-1)^i q^{mi-\binom{i}{2}}(x;q)_i, \tag{2.1}$$

where the coefficients $H_i^{m,n}(\lambda)$ are the *q-hit numbers* associated to λ. Equivalently, we have that for every k

$$H_k^{m,n}(\lambda) = \frac{q^{\binom{k}{2}-|\lambda|}}{[m-n]!}\sum_{i=k}^{n} R_i(\lambda)\,[m-i]!\begin{bmatrix} i \\ k \end{bmatrix}(-1)^{i+k}q^{mi-\binom{i}{2}}, \tag{2.2}$$

and

$$R_k(\lambda) = q^{|\lambda|-mk}\frac{[m-n]!}{[m-k]!}\sum_{i=k}^{n} H_i^{m,n}(\lambda)\begin{bmatrix} i \\ k \end{bmatrix}_{q^{-1}}. \tag{2.3}$$

Notation 2.4. *For square boards with $n = m$, we denote the q-hit number by $H_j^m(\lambda)$.*

Remark 2.5. For the case $n = m$, Garsia–Remmel defined q-hit numbers $\widetilde{H}_k^n(\lambda)$ by the relation

$$\sum_{i=0}^{n} \widetilde{H}_i^n(\lambda)x^i = \sum_{i=0}^{n} R_i(\lambda)[n-i]! \prod_{k=n-i+1}^{n}(x-q^k). \tag{2.4}$$

One can show that the Garsia–Remmel q-hit numbers and our q-hit numbers differ by a power of q (see Proposition B.1).

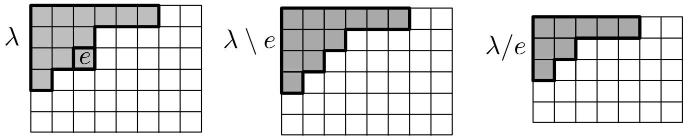


Fig. 2. Example of the deletion and contraction of the board of a partition λ.

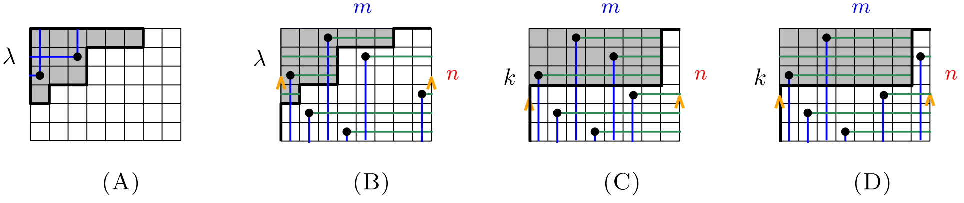


Fig. 3. Example of the statistics of (A) a q-rook number and (B) a q-hit number. (C)(D) Examples of the cases of q-hit numbers of a rectangle $(m-1)^k \subset n \times m$ for Proposition 2.17.

The q-hit numbers satisfy the following deletion–contraction relation that is proved in Appendix E. Given a shape λ and a corner cell e in λ, $\lambda \backslash e$ denotes the shape obtained after deleting the cell e in λ, and λ/e denotes the shape obtained after deleting in λ the row and column containing e. See Fig. 2 for an example.

Lemma 2.6. *We have the following deletion–contraction relation:*

$$H_j^{m,n}(\lambda) = H_j^{m,n}(\lambda \backslash e) + q^{|\lambda/e|-|\lambda|+j+m-1}\left(H_{j-1}^{m-1,n-1}(\lambda/e) - qH_j^{m-1,n-1}(\lambda/e)\right).$$

Guay-Paquet [12] defined the rectangular q-hit numbers using a statistic similar to Dworkin's statistic [8] for the Garsia–Remmel q-hit numbers and we present this definition next, illustrated in Fig. 3(b).

Definition 2.7 (*Statistic for the q-hit Numbers*)**.** Let λ be a partition inside an $n \times m$ board. Given a placement p of n non-attacking rooks on an $n \times m$ board, with exactly j rooks inside λ, let $\mathrm{stat}(p)$ be the number of cells c in the board such that:

(i) there is no rook in c,
(ii) there is no rook above c on the same column, and either,
(iii) if c is in λ then the rook on the same row of c is in λ and to the right of c or
(iv) if c is not in λ then the rook on same row of c is either in λ or to the right of c.

Remark 2.8. Intuitively, this statistic $\mathrm{stat}(p)$ counts the number of remaining cells in the $n \times m$ board after: wrapping this board on a vertical cylinder and each rook of p cancels the cells south in its column and the cells east in its row until the border of λ.

Theorem 2.9. *Let λ be a partition inside an $n \times m$ board and $j = 0, \ldots, n$ then*

$$H_j^{m,n}(\lambda) = \sum_p q^{\mathrm{stat}(p)}, \tag{2.5}$$

where the sum is over all placements p of n non-attacking rooks on an $n \times m$ board, with exactly j rooks inside λ.

The proof for Theorem 2.9 is given in Appendix C.

Remark 2.10. Note that the Garsia–Remmel q-hit numbers have a very similar description in [16] (attributed to Dworkin) using a different attacking rule for the rooks. Our proof of Theorem 2.9 in Appendix A follows by reducing to the case of the Garsia–Remmel q-hit numbers. See also Section 7.1 for more details.

Moreover, for each partition λ, the statistic $\mathrm{stat}(\cdot)$ is *Mahonian*.

Corollary 2.11. *Let λ be a partition inside an $n \times m$ board, then*

$$\sum_{j=0}^{n} H_j^{m,n}(\lambda) = [m]_n.$$

Proof. Set $k = 0$ in (2.3) and since $R_0(\lambda) = q^{|\lambda|}$, we obtain

$$\sum_{j=0}^{n} H_j^{m,n}(\lambda) = q^{-|\lambda|} R_0(\lambda)[m]_n = [m]_n. \quad \square$$

Example 2.12. Consider the partition $\lambda = (6, 3, 3, 1)$ inside a 6×8 board. In Figs. 3(a) and 3(b), we present an example of a placement p of two rooks on λ with $\mathrm{inv}(p) = 7$ and an example of a placement p' of six rooks on the 6×8 board with two hits on λ and $\mathrm{stat}(p') = 13$, respectively.

We finish this section with some results for q-hit numbers. The next two results show the relation between the q-hit numbers when we change the dimensions of the board.

Lemma 2.13. *Let λ be a partition inside an $n \times m$ board. Then*

$$H_j^{m,n}(\lambda) = \frac{1}{[m-n]!} H_j^{m,m}(\lambda).$$

Proof. Since $R_i(\lambda) = 0$ for $i = n + 1, \ldots, m$ then (2.2) becomes,

$$H_j^{m,n}(\lambda) = \frac{q^{\binom{j}{2}-|\lambda|}}{[m-n]!} \sum_{i=j}^{m} R_i(\lambda)\,[m-i]! \begin{bmatrix} i \\ j \end{bmatrix} (-1)^{i+j} q^{mi-\binom{i}{2}} = \frac{1}{[m-n]!} H_j^{m,m}(\lambda). \quad \square$$

Lemma 2.14. *Let λ be a partition inside an $(n-1) \times m$ board. Then*

$$H_j^{m,n}(\lambda) = [m+1-n]\, H_j^{m,n-1}(\lambda).$$

Proof. We apply (2.2) to $H_j^{m,n}(\lambda)$
and use the fact $R_n(\lambda) = 0$ since $\lambda \subset (n-1) \times m$ to obtain

$$\begin{aligned} H_j^{m,n}(\lambda) &= [m+1-n]\, \frac{q^{\binom{j}{2}-|\lambda|}}{[m-n+1]!} \sum_{i=j}^{n-1} R_i(\lambda)\,[m-i]! \begin{bmatrix} i \\ j \end{bmatrix} (-1)^{i+j} q^{mi-\binom{i}{2}} \\ &= [m+1-n]\, H_j^{m,n-1}(\lambda), \end{aligned}$$

where we used (2.1) again for $H_j^{m,n-1}(\lambda)$ to obtain the desired formula. $\square$

Finally, we give formulas for q-rook numbers and q-hit numbers of rectangular shapes.

Proposition 2.15.

$$R_k(a^b) = q^{(a-k)(b-k)} \frac{[a]_k\,[b]_k}{[k]!}. \tag{2.6}$$

Proof. The result follows from the recurrence $R_k(a^b) = [b]\,R_{k-1}((a-1)^{b-1}) + q^b R_k((a-1)^b)$ or from the fact that up to a power of q and $(q-1)^k$, this is the number of rank k matrices of size $a \times b$ over a finite field with q elements [15, Thm. 1], formulas for which can be found in [22, Sec. 1.7]. □

Proposition 2.16.

$$H_k^N(m^j) = q^{(N-j-m+k)k}\,[m]_k\,[N-j]!\,\frac{[N-m]_{j-k}\,[j]_{j-k}}{[j-k]!}. \tag{2.7}$$

Proof. We compute the q-hit number directly using the statistic in (2.5). We claim that

$$H_k^N(m^j) = R_k(m^k) \cdot R_{j-k}((N-m)^j) \cdot R_{N-j}((N-j)^{N-j}).$$

To show this, let us denote by B the $N \times N$ board. Then there are $j-k$ rows occupied by rooks right of the shape m^j. These rooks cancel the respective rows from the m^j shape. The overall contribution to the q-hit number from the k rooks on the remaining shape m^k is $R_k(m^k) = [m]_k$. The overall contribution to the q-hit number from the $j-k$ rooks placed to the right of the shape m^j is $R_{j-k}((N-m)^j)$. The j rooks placed on the first j rows of the board B cancel as many columns in the shape N^{N-j} consisting of the last $N-j$ rows of the board B.

By Proposition 2.2, the overall contribution to the q-hit number from placing the remaining $N-j$ rooks in the remaining shape $(N-j)^{N-j}$ is $R_{N-j}((N-j)^{N-j}) = [N-j]!$. This proves the claim and the result follows by using the formula in (2.6) for $R_k(a^b)$. □

Proposition 2.17.

$$H_r^{m,n}((m-1)^k) = \begin{cases} q^k[m-k][m-1]_{n-1} & r = k, \\ [k][m-1]_{n-1} & r = k-1, \\ 0 & otherwise. \end{cases}$$

Proof. Since every row of the $n \times m$ board has a rook and the last column has at most one rook then the rooks can only "hit" the shape $(m-1)^k$ $r = k$ or $r = k-1$ times.

When $r = k$, the first k cells of the last column are not cancelled in any rook placement so they contribute to stat$(\cdot)$. The contribution to the q-hit number from the k rooks placed on the shape $(m-1)^k$ is $R_k((m-1)^k)$. These rooks cancel k columns in the shape of the last $n-k$ rows. Then the contribution to the q-hit number of placing $n-k$ rooks on the remaining shape $(m-k)^{n-k}$ is $R_{n-k}((m-k)^{n-k})$. See Fig. 3(c). Thus,

$$H_k^{m,n}((m-1)^k) = q^k R_k((m-1)^k) \cdot R_{n-k}((m-k)^{n-k}) = q^k[m-1]_k[m-k]_{n-k} = q^k[m-k][m-1]_{n-1}.$$

When $r = k-1$, there is a rook on one of the first k cells of the mth column which cancels all the cells of its corresponding row $i = 1, \ldots, k$ and the cells below the rook in its column. This rook contributes $i-1$ to the statistic (the cells above the rook in the mth column). There contribution to the q-hit number from the $k-1$ rooks placed on the remaining shape $(m-1)^{k-1}$ (without row i) is $R_{k-1}((m-1)^{k-1})$. The k rooks on the first k rows cancel the cells of their columns in the shape of the last $n-k$ rows. Then the contribution to the q-hit number of placing $n-k$ rooks on the remaining shape $(m-k)^{n-k}$ is $R_{n-k}((m-k)^{n-k})$. See Fig. 3(d). Summing over all $i = 1, \ldots, k$ we obtain that

$$H_{k-1}^{m,n}((m-1)^k) = [k]R_{k-1}((m-1)^{k-1}) \cdot R_{n-k}((m-k)^{n-k}) = [k][m-1]_{k-1}[m-k]_{n-k} = [k][m-1]_{n-1}. \quad \square$$

3. New q-rook and q-hit identities

The proofs of Theorems 1.2 and 1.3 rely on many new identities between q-hit numbers, and their equivalent q-rook versions. While the q-hit and q-rook identities are independent of the chromatic symmetric function theory, we first prove them using elementary combinatorial methods and univariate generating functions.

For brevity, we will denote by $\lambda/^c j$ the partition obtained from λ by removing its jth column, by $\lambda/^r i$ the partition obtained by removing its ith row, and by $\lambda/(i,j)$ the partition obtained from λ by removing its jth column and its ith row. Moreover, we denote $\ell = \ell(\lambda)$ and the conjugate partition of λ by λ'.

3.1. Identities on q-rook numbers

The following q-rook identities can be proven directly using the generating function identity in Proposition 2.2 of Garsia–Remmel. We start with a simple algebraic identity.

Lemma 3.1. *Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,*

$$q^{\lambda_1}[x]\frac{F(x-1;\lambda)}{F(x;\lambda)} = [x-\ell+\lambda_1] - \sum_{j=1}^{\lambda_1} q^{\lambda_1-j}\prod_{t=1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]}.$$

Proof. We use induction on $\ell(\lambda)$ and apply Proposition 2.2. For $\ell(\lambda)=1$, we have

$$[x]\frac{F(x-1;\lambda)}{F(x;\lambda)} = [x-1] + \sum_{j=1}^{\lambda_1} q^{-j} - q^{-j}\frac{[x+\lambda_1-1]}{[x+\lambda_1]} = q^{-\lambda_1}\frac{[x+\lambda_1-1]}{[x+\lambda_1]}([x+\lambda_1]-[\lambda_1]).$$

Next, expanding the RHS of the above identity and doing standard manipulations gives

$$[x-\ell+\lambda_1] - \sum_{j=1}^{\lambda_1} q^{\lambda_1-j}\prod_{t=1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]} =$$

$$= q^{\lambda_1-\lambda_2}\frac{[x+\lambda_1-\ell]}{[x+\lambda_1-\ell+1]}\left([x+\lambda_2-(\ell-1)] - \sum_{j=1}^{\lambda_2} q^{\lambda_2-j}\prod_{t=2}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]}\right).$$

By induction hypothesis the parenthetical on the RHS above is $q^{\lambda_2}[x]F(x-1;\tilde{\lambda})/F(x;\tilde{\lambda})$ where $\tilde{\lambda} = (\lambda_2, \ldots, \lambda_\ell)$. Using $\tilde{\lambda}_t = \lambda_{t+1}$ for the reindexing, we obtain the result. □

Next, we find the following q-analogue of the derivative of $F(x;\lambda)$.

Lemma 3.2. *We have the following formula for the q-analogue of the derivative of $F(x;\lambda)$:*

$$DF(x;\lambda) := \sum_{k=0}^{\ell}[k]R_k(\lambda)[x]_{\ell-k} = q^{\ell-x}([x]F(x-1;\lambda) - [x-\ell]F(x;\lambda)).$$

Proof. Following the notation of Proposition 2.2,

$$F(x;\lambda) := \sum_{k=0}^{\ell} R_k(\lambda)[x]_{\ell-k} = \prod_{i=1}^{\ell}[x+\lambda_{\ell-i+1}-i+1].$$

Note that

$$[x]_{\ell-k} - [x-1]_{\ell-k} = [x-1]_{\ell-k-1}\left(\frac{1-q^x-1+q^{x-\ell+k}}{1-q}\right) = \frac{[x]_{\ell-k}}{[x]}\frac{q^{x-\ell}(q^k-q^\ell)}{1-q}$$
$$= \frac{[x]_{\ell-k}}{[x]}q^{x-\ell}([\ell]-[k]).$$

Apply this identity to each term in the following difference

$$F(x;\lambda) - F(x-1;\lambda) = \sum_{k=0}^{\ell} R_k(\lambda)\frac{[x]_{\ell-k}}{[x]}q^{x-\ell}([\ell]-[k]),$$

so that

$$[x](F(x;\lambda)-F(x-1;\lambda)) = q^{x-\ell}[\ell]F(x;\lambda) - q^{x-\ell}DF(x;\lambda),$$

where $DF(x;\lambda) := \sum_{k=0}^{\ell}[k]R_k(\lambda)[x]_{l-k}$. This gives an equation for $DF(x;\lambda)$ which we solve as

$$DF(x;\lambda) = -q^{\ell-x}([x]-q^{x-\ell}[\ell])F(x;\lambda)+q^{\ell-x}[x]F(x-1;\lambda) = q^{\ell-x}([x]F(x-1;\lambda)-[x-\ell]F(x;\lambda)),$$

giving us the desired formula. □

The following result shows the relationship between q-rook numbers for partitions obtained from deleting a column of λ.

Lemma 3.3. *For all i fixed,*

$$\sum_j q^{m-j}R_i(\lambda/^c j) = R_i(\lambda)[m-i] - R_{i+1}(\lambda)(q^m - q^{m-i-1}).$$

Proof. Multiplying on both sides by $[x]_{\ell-i}$, the above claim is equivalent to the generating function identity:

$$\begin{aligned}\sum_j q^{m-j}F(x;\lambda/^c j) &= \sum_i (R_i(\lambda)[m-i] - R_{i+1}(\lambda)(q^m-q^{m-i-1}))[x]_{\ell-i}\\ &= \sum_i R_i(\lambda)\left([m-i][x]_{\ell-i} - (q^m-q^{m-i})[x]_{\ell-i+1}\right)\\ &= \sum_i R_i(\lambda)[x]_{\ell-i}[m+x-\ell] - q^m[x][x-1]_{\ell-i}\\ &= [m+x-\ell]F(x;\lambda) - q^m[x]F(x-1;\lambda),\end{aligned}$$

where we use the observation that

$$\begin{aligned}[m-i][x]_{\ell-i} - (q^m-q^{m-i})[x]_{\ell-i+1} &= [x]_{\ell-i}\frac{(1-q^{m-i})-(q^m-q^{m-i})(1-q^{x-\ell+i})}{1-q}\\ &= [x]_{\ell-i}\frac{1-q^{m-i}-q^m+q^{m-i}+q^{m+x-\ell+i}-q^{m+x-\ell}}{1-q} = [x]_{\ell-i}[x+m-\ell]-q^m[x]_{\ell-i+1}.\end{aligned}$$

We have that

$$F(x;\lambda/^c j) = \prod_{i=1}^{\lambda'_j}[x+\lambda_i-1-\ell+i]\prod_{i=\lambda'_j+1}^{\ell}[x+\lambda_i-\ell+i] = F(x;\lambda)\prod_{i=1}^{\lambda'_j}\frac{[x-1+\lambda_i-\ell+i]}{[x+\lambda_i-\ell+i]}.$$

Using Lemma 3.1 and that $\lambda/^c j = \lambda$ for $j > \lambda_1$,

$$\begin{aligned}\sum_{j=1}^m q^{m-j}F(x;\lambda/^c j) &= F(x;\lambda)\left([m-\lambda_1] + q^{m-\lambda_1}\sum_j q^{\lambda_1-j}\prod_{i=1}^{\lambda'_j}\frac{[x-1+\lambda_i-\ell+i]}{[x+\lambda_i-\ell+i]}\right)\\ &= F(x;\lambda)\left([m-\lambda_1]+q^{m-\lambda_1}[x-\ell+\lambda_1] - q^{m-\lambda_1}q^{\lambda_1}[x]\frac{F(x-1;\lambda)}{F(x;\lambda)}\right)\\ &= F(x;\lambda)[x-\ell+\lambda_1+m-\lambda_1] - q^m[x]F(x-1;\lambda),\end{aligned}$$

which is what we wanted to show and completes the proof. □

We also have the following relationship between q-rook numbers obtained from removing a single row from λ.

Lemma 3.4. *For all fixed k,*

$$\sum_{i=1}^{n} q^{i-1+\lambda_i} R_k(\lambda/^r\, i) = [n]R_k(\lambda) - [k]R_k(\lambda).$$

Proof. First of all, notice that we can replace n with ℓ since for $i > \ell$, $\lambda/^r\, i = \lambda$, $\lambda_i = 0$ and $[n] - \sum_{i=\ell+1}^{n} q^{i-1} = [\ell]$. Multiplying by $[y]_{\ell-k} = [y][y-1]_{(\ell-1)-k}$ on both sides and summing over all k, the identity is equivalent to

$$[y]\sum_{i=1}^{\ell} q^{i-1+\lambda_i} \sum_k R_k(\lambda/^r\, i)[y-1]_{(\ell-1)-k} = [n]\sum_k R_k(\lambda)[y]_{\ell-k} - \sum_k [k]R_k(\lambda)[y]_{\ell-k}.$$

Thus, the identity is then equivalent to the generating function identity

$$\begin{aligned}
&[y]\sum_{i=1}^{\ell} q^{i-1+\lambda_i} F(y-1, \lambda/^r\, i)\\
&= [\ell]F(y;\lambda) - DF(y;\lambda) = [\ell]F(y,\lambda) + q^{-y+\ell}[y-\ell]F(y,\lambda) - q^{\ell-y}[y]F(y-1,\lambda).
\end{aligned}$$

We have that for $i \le \ell$

$$\begin{aligned}
&F(y-1, \lambda/^r\, i)\\
&= \prod_{j=1}^{\ell-1} [y-1+(\lambda/^r\, i)_j - (\ell-1)+i]\\
&= \prod_{j=1}^{i-1} [y-1+\lambda_j-\ell+1+j] \prod_{j=i+1}^{\ell} [y-1+\lambda_j-(\ell-1)+j-1]\\
&= \frac{1}{[y-1+\lambda_i-\ell+i]} \prod_{j=1}^{i-1} \frac{[y+\lambda_j-\ell]}{[y-1+\lambda_j-\ell]} F(y-1,\lambda),
\end{aligned}$$

and so we reach the following equivalent identity

$$\begin{aligned}
&[y]\sum_{i=1}^{\ell} \frac{q^{y-\ell+i-1+\lambda_i}}{[y-1+\lambda_i-\ell+i]} \prod_{j=1}^{i-1} \frac{[y+\lambda_j-\ell]}{[y-1+\lambda_j-\ell]} F(y-1,\lambda)\\
&= q^{y-\ell}[\ell]F(y,\lambda) + [y-\ell]F(y,\lambda) - [y]F(y-1,\lambda).
\end{aligned}$$

We can rewrite it as

$$\begin{aligned}
&[y]\sum_{i=1}^{\ell} \frac{[y-\ell+i+\lambda_i] - [y-1-\ell+i+\lambda_i]}{[y-1+\lambda_i-\ell+i]} \prod_{j=1}^{i-1} \frac{[y+\lambda_j-\ell+j]}{[y-1+\lambda_j-\ell+j]} F(y-1,\lambda)\\
&= q^{y-\ell}[\ell]F(y,\lambda) + [y-\ell]F(y,\lambda) - [y]F(y-1,\lambda),
\end{aligned}$$

which reduces again to

$$\begin{aligned}
&[y]F(y-1,\lambda)\sum_{i=1}^{\ell} \left(\prod_{j=1}^{i} \frac{[y+\lambda_j-\ell+j]}{[y-1+\lambda_j-\ell+j]} - \prod_{j=1}^{i-1} \frac{[y+\lambda_j-\ell+j]}{[y-1+\lambda_j-\ell+j]} \right)\\
&= [y]F(y,\lambda) - [y]F(y-1,\lambda).
\end{aligned}$$

After canceling the terms in the telescoping sum on the LHS, the identity reduces to

$$[y]F(y-1,\lambda)\left(\prod_{j=1}^{\ell} \frac{[y+\lambda_j-\ell+j]}{[y-1+\lambda_j-\ell+j]} - 1 \right) - [y]F(y,\lambda) + [y]F(y-1,\lambda)$$

$$= [y](F(y,\lambda) - F(y-1,\lambda)) - [y]F(y,\lambda) + [y]F(y-1,\lambda) = 0,$$

which completes the proof. □

Lemma 3.5. *We have the following identity for q-rook numbers:*

$$\sum_{(i,j)\in\lambda} q^{i-j+\lambda_i} R_k(\lambda/(i,j)) = q[k+1]R_{k+1}(\lambda).$$

Proof. Translating this identity into generating functions, we have that it is equivalent to

$$\sum_{k=0}^{\ell-1}\sum_{(i,j)\in\lambda} q^{i-j+\lambda_i} R_k(\lambda/(i,j))[x]_{\ell-1-k} = \sum_{k=0}^{\ell-1} q[k+1]R_{k+1}(\lambda)[x]_{\ell-k-1} = \sum_{t=0}^{\ell} q[t]R_t(\lambda)[x]_{\ell-t},$$

which can be rewritten, with the help of Lemma 3.2, in terms of the F function as

$$\sum_{(i,j)\in\lambda} q^{i-j+\lambda_i} F(x;\lambda/(i,j)) = qDF(x;\lambda) = -q^{\ell-x+1}[x-\ell]F(x;\lambda) + q^{1+\ell-x}[x]F(x-1;\lambda). \qquad (3.1)$$

Now, we notice that if $\mu = \lambda/(i,j)$

$$\begin{aligned} F(x;\lambda/(i,j)) &= \prod_{t=1}^{\ell-1}[x+\mu_t-\ell+1+t] \\ &= \prod_{t=1}^{i-1}[x+\lambda_t-1-\ell+1+t]\prod_{t=i}^{\lambda'_j-1}[x+\lambda_{t+1}-1-\ell+1+t] \\ &\quad\times\prod_{t=\lambda'_j}^{\ell-1}[x+\lambda_{t+1}-\ell+1+t] \\ &= F(x;\lambda)\frac{\prod_{t=i+1}^{\lambda'_j}[x+\lambda_t-1-\ell+t]}{\prod_{t=i}^{\lambda'_j}[x+\lambda_t-\ell+t]}. \end{aligned}$$

Note that $q^{x-1-\ell+i+\lambda_i} = [x+\lambda_i-\ell+i] - [x+\lambda_i-\ell+i-1]$, so we can rewrite this as

$$\begin{aligned} &q^{i-j+\lambda_i}F(x;\lambda/(i,j)) \\ &= q^{-x+1+\ell-j}F(x;\lambda)([x+\lambda_i-\ell+i]-[x+\lambda_i-\ell+i-1])\frac{\prod_{t=i+1}^{\lambda'_j}[x+\lambda_t-1-\ell+t]}{\prod_{t=i}^{\lambda'_j}[x+\lambda_t-\ell+t]} \\ &= q^{-x+1+\ell-j}F(x;\lambda)\left(\prod_{t=i+1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]} - \prod_{t=i}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]}\right). \end{aligned}$$

Fixing j and summing over all possible $i = 1\ldots\lambda'_j$ we get telescoping cancellations and so

$$\sum_{(i,j)\in\lambda} q^{i-j+\lambda_i}F(x;\lambda/(i,j)) = \sum_j q^{-x+1+\ell-j}F(x;\lambda)\left(1-\prod_{t=1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]}\right).$$

Substituting this into the LHS of (3.1), the needed identity transforms to the equivalent

$$\sum_j q^{1-j+\ell-x}F(x;\lambda)\left(1-\prod_{t=1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]}\right) = qDF(x;\lambda)$$

$$= q^{\ell-x+1}\left([x]F(x-1;\lambda) - [x-\ell]F(x;\lambda)\right),$$

which is equivalent to

$$[x]\frac{F(x-1;\lambda)}{F(x;\lambda)} = [x-\ell] + \sum_{j=1}^{\lambda_1} q^{-j} - q^{-j}\prod_{t=1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]}.$$

This last identity simplifies as

$$q^{\lambda_1}[x]\frac{F(x-1;\lambda)}{F(x;\lambda)} = [x-\ell+\lambda_1] - \sum_{j=1}^{\lambda_1} q^{\lambda_1-j}\prod_{t=1}^{\lambda'_j}\frac{[x+\lambda_t-1-\ell+t]}{[x+\lambda_t-\ell+t]},$$

and is equivalent to the formula in Lemma 3.1, which completes the proof. □

3.2. q-Hit identities

We now translate the above rook identities into q-hit identities using the relationship from Definition 2.3. Let

$$G^{m,n}(x;\lambda) = \sum_{k=0}^{n} H_k^{m,n}(\lambda)(q^x)^k.$$

Then we have the following equivalences in terms of the generating functions

$$\begin{aligned} G^{m,n}(x;\lambda) &:= \sum_{i=0}^{n}\sum_{k=0}^{i}\frac{q^{\binom{k}{2}-|\lambda|}}{[m-n]!}R_i(\lambda)\,[m-i]!\begin{bmatrix} i\\ k\end{bmatrix}(-1)^{i+k}q^{mi-\binom{i}{2}}q^{xk} \qquad (3.2)\\ &= \frac{q^{-|\lambda|}}{[m-n]!}\sum_{i=0}^{n}R_i(\lambda)\,[m-i]!q^{mi-\binom{i}{2}}(-1)^i\sum_{k=0}^{i}q^{\binom{k}{2}}\begin{bmatrix} i\\ k\end{bmatrix}(-q^x)^k\\ &= \frac{q^{-|\lambda|}}{[m-n]!}\sum_{i=0}^{n}R_i(\lambda)\,[m-i]!q^{mi-\binom{i}{2}}(-1)^i\prod_{k=0}^{i-1}(1-q^{x+k}). \end{aligned}$$

Lemma 3.6. *For every λ inside an $n\times m$ board and for every k*

$$\sum_{j=1}^{m} q^{m+n-j-\lambda'_j}H_k^{m-1,n}(\lambda/^c j) = [m-n]H_k^{m,n}(\lambda)q^{n-k}.$$

Proof. The collection of identities for $k=0,\dots,n$ is equivalent to the following generating function identities

$$\begin{aligned} &\sum_{k=0}^{n}\sum_{j=1}^{m} q^{m+n-j-\lambda'_j}H_k^{m-1,n}(\lambda/^c j)q^{xk}\\ &\quad = [m-n]\sum_{k=0}^{n}H_k^{m,n}(\lambda)q^{n-k}q^{xk}\sum_{j=1}^{m}q^{m+n-j-\lambda'_j}G^{m-1,n}(x;\lambda/^c j) = q^n[m-n]G^{m,n}(x-1;\lambda). \end{aligned}$$

Using (3.2) and expanding in the q^x-polynomial basis $(q^x;q)_k = \prod_{k=0}^{i-1}(1-q^{x+k})$ for $i=0,\dots,n$, the G-identity is equivalent to the following rook identity for every i:

$$\sum_{j=1}^{m} q^{m+n-j-\lambda'_j}\frac{q^{-|\lambda|+\lambda'_j}}{[m-1-n]!}\sum_{i=0}^{n}R_i(\lambda/^c j)\,[m-1-i]!q^{mi-i-\binom{i}{2}}(-1)^i\prod_{k=0}^{i-1}(1-q^{x+k})$$

$$= q^n[m-n]\frac{q^{-|\lambda|}}{[m-n]!}\sum_{i=0}^{n} R_i(\lambda)\,[m-i]!q^{mi-\binom{i}{2}}(-1)^i\prod_{k=0}^{i-1}(1-q^{x-1+k}),$$

which simplifies as

$$\sum_{j=1}^{m} q^{m-j}\sum_{i=0}^{n} R_i(\lambda/^c j)\,[m-1-i]!q^{mi-i-\binom{i}{2}}(-1)^i\prod_{k=0}^{i-1}(1-q^{x+k})$$
$$= \sum_{i=0}^{n} R_i(\lambda)\,[m-i]!q^{mi-\binom{i}{2}-i}(-1)^i q^i\prod_{k=0}^{i-1}(1-q^{x-1+k}).$$

Notice that

$$q^i(q^{x-1};q)_i = (q^i-q^{x+i-1})(q^x;q)_{i-1} = (q^i-1)(q^x;q)_{i-1}+(q^x;q)_i,$$

and so the RHS above expands in the $(q^x;q)_i$ basis as:

$$\sum_{i=0}^{n} R_i(\lambda)\,[m-i]!q^{mi-\binom{i+1}{2}}(-1)^i((q^i-1)(q^x;q)_{i-1}+(q^x;q)_i)$$
$$= \sum_{i=0}^{n}(-1)^i(q^x;q)_i\left(R_i(\lambda)\,[m-i]!q^{mi-\binom{i+1}{2}} - R_{i+1}(\lambda)\,[m-i-1]!q^{mi+m-\binom{i+2}{2}}(q^{i+1}-1)\right).$$

Therefore, this is equivalent to the following q-rook identity for every i:

$$\sum_{j=1}^{m} q^{m-j}R_i(\lambda/^c j)\,[m-1-i]!q^{mi-\binom{i+1}{2}}$$
$$= \left(R_i(\lambda)\,[m-i]!q^{mi-\binom{i+1}{2}} - R_{i+1}(\lambda)\,[m-i-1]!q^{mi+m-\binom{i+2}{2}}(q^{i+1}-1)\right).$$

Simplifying last expression, we obtain

$$\sum_{j=1}^{m} q^{m-j}R_i(\lambda/^c j) = R_i(\lambda)[m-i] - R_{i+1}(\lambda)(q^m-q^{m-i-1}),$$

which is exactly Lemma 3.3. □

Lemma 3.7. *Let $k \le n \le m$ be fixed, and $\lambda \subset n\times m$. We have the following q-hit identity:*

$$[m-n+1]\sum_{i=1}^{n} q^{i-1}H_k^{m,n-1}(\lambda/^r i) = H_k^{m,n}(\lambda)q^k[n-k] + H_{k+1}^{m,n}(\lambda)[k+1].$$

Proof. Multiplying both sides by $(q^x)^k$ and summing over all k, the identity becomes equivalent to the following generating function identity:

$$[m-n+1]\sum_{i=1}^{n} q^{i-1}G^{m,n-1}(x;\lambda/^r i) = \sum_{k=0}^{n-1} H_k^{m,n}(\lambda)(q^{x+1})^k[n-k] + \sum_{k=0}^{n-1} H_{k+1}^{m,n}(\lambda)[k+1]q^{xk}.$$

To prove the above identity, we will use the following difference operator and its properties:

$$\Delta F(x) := \frac{F(x+1)-F(x)}{q^x(q-1)}, \quad \Delta(q^x)^p = [p](q^x)^{p-1},$$
$$\Delta(q^x,q)_p = -[p](q^{x+1},q)_{p-1} = -\frac{[p](q^x;q)_p}{1-x}.$$

Notice also that $q^k[n-k] = [n]-[k]$ and $[k+1]q^{xk} = \Delta q^{x(k+1)}$. Therefore, we have

$$\begin{aligned}\sum_{k=0}^{n-1} H_k^{m,n}(\lambda)\left(q^{x+1}\right)^k [n-k] &= \sum_{k=0}^{n-1} H_k^{m,n}(\lambda) q^{xk}[n] - \sum_{k=0}^{n-1} H_k^{m,n}(\lambda)[k]q^{xk} \\ &= [n]G^{m,n}(x;\lambda) - q^x \Delta G^{m,n}(x;\lambda)\end{aligned}$$

and

$$\sum_{k=0}^{n-1} H_{k+1}^{m,n}(\lambda)[k+1]q^{xk} = \Delta G^{m,n}(x;\lambda).$$

Thus, the generating function identity is equivalent to

$$[m-n+1]\sum_{i=1}^{n} q^{i-1} G^{m,n-1}(x;\lambda/^r i) = [n]G^{m,n}(x;\lambda) - q^x \Delta G^{m,n}(x;\lambda) + \Delta G^{m,n}(x;\lambda).$$

Using the formula

$$(1-q^x)\Delta G^{m,n}(x;\lambda) = \frac{q^{-|\lambda|}}{[m-n]!}\sum_{k=0}^{n}[k]R_k(\lambda)[m-k]!q^{mk-\binom{k}{2}}(-1)^{k-1}\underbrace{(1-q^x)(q^{x+1},q)_{k-1}}_{(q^x;q)_k},$$

we can express the generating function identity in terms of q-rook numbers generating function as:

$$\begin{aligned}&[m-n+1]\sum_{i=1}^{n} q^{i-1}\frac{q^{-|\lambda|+\lambda_i}}{[m-n+1]!}\sum_{k=0}^{n} R_k(\lambda/^r i)[m-k]!q^{mk-\binom{k}{2}}(-1)^k (q^x,q)_k \\ &= [n]\frac{q^{-|\lambda|}}{[m-n]!}\sum_{k} R_k(\lambda)[m-k]!q^{mk-\binom{k}{2}}(-1)^k(q^x,q)_k \\ &\quad + \frac{q^{-|\lambda|}}{[m-n]!}\sum_{k=0}^{n}[k]R_k(\lambda)[m-k]!q^{mk-\binom{k}{2}}(-1)^{k-1}(q^x;q)_k.\end{aligned}$$

For each k, the coefficients at $(q^x;q)_k$ coincide, after canceling common factors, and reducing then to the q-rook identities from Lemma 3.4

$$\sum_{i=1}^{n} q^{i-1+\lambda_i} R_k(\lambda/^r i) = [n]R_k(\lambda) - [k]R_k(\lambda). \quad \square$$

Lemma 3.8. *We have the following identity*

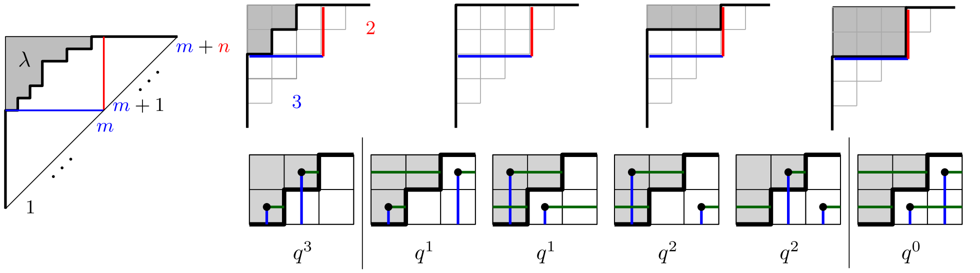
$$q^k \sum_{(i,j)\in\lambda} q^{i+(m-j-\lambda'_j)} H_k^{m-1,n-1}(\lambda/(i,j)) = [k+1]H_{k+1}^{m,n}(\lambda).$$

Proof. We show that this result follows from Lemma 3.5. By (2.2),

$$\begin{aligned}[k+1]H_{k+1}^{m,n}(\lambda) &= [k+1]\frac{q^{\binom{k+1}{2}-|\lambda|}}{[m-n]!}\sum_{t=k+1}^{n} R_t(\lambda)[m-t]!\begin{bmatrix} t \\ k+1\end{bmatrix}(-1)^{t+k+1}q^{mt-\binom{t}{2}} \\ &= \frac{q^{\binom{k+1}{2}-|\lambda|}}{[m-n]!}\sum_{t'=k}^{n-1}[t'+1]R_{t'+1}(\lambda)[m-t'-1]!\begin{bmatrix} t' \\ k\end{bmatrix}(-1)^{t'+k}q^{mt'+m-\binom{t'+1}{2}},\end{aligned}$$

where we reindexed the sum with $t' = t-1$. Next, we apply Lemma 3.5 and exchange sums to obtain

$$[k+1]H_{k+1}^{m,n}(\lambda) = \frac{q^{\binom{k+1}{2}-|\lambda|}}{[m-n]!}\sum_{(i,j)\in\lambda} q^{m-j+\lambda_i}\sum_{t'=k}^{n-1} R_{t'}(\lambda/(i,j))[m-t'-1]!\begin{bmatrix} t' \\ k\end{bmatrix}(-1)^{t'+k}q^{mt'+m-\binom{t'+1}{2}}$$



(A) (B)

Fig. 4. (A) an abelian Dyck path λ inside an $n \times m$ board. (B) Top: the paths for $\lambda = (2, 1)$, and for the rectangles $3^0, 3^1, 3^2$ inside a 2×3 board. (B) Bottom: the six placements of 2 rooks in 2×3 divided by how many rooks "hit" (2, 1) (in gray) and the associated statistic to each rook placement.

$$= q^k \sum_{(i,j)\in\lambda} q^{i+m-j-\lambda'_j} \frac{q^{\binom{k}{2}} - |\lambda/(i,j)|}{[m-n]!} \sum_{t'=k}^{n-1} R_{t'}(\lambda/(i,j))[m-t'-1]! \begin{bmatrix} t' \\ k \end{bmatrix} (-1)^{t'+k} q^{(m-1)t'-\binom{t'}{2}}$$
$$= q^k \sum_{(i,j)\in\lambda} q^{i+m-j-\lambda'_j} H_k^{m-1,n-1}(\lambda/(i,j)),$$

where we also used that $|\lambda/(i,j)| = |\lambda| - \lambda_i - \lambda'_j + 1$ and (2.2) for $H_k^{m-1,n-1}(\lambda/(i,j))$. $\square$

4. The Guay-Paquet q-hit identity

In this section we give our main result, a proof of Theorem 1.3 using the q-rook theory identities from Section 3. We start by giving an example of this elegant identity.

Example 4.1. For $\lambda = (2, 1)$ inside a 2×3 board, looking at Fig. 4(b), we see that $H_0^{3,2}(\lambda) = q^0 = 1$, $H_1^{3,2}(\lambda) = 2q + 2q^2$, $H_2^{3,2}(\lambda) = q^3$. One can verify that

$$X_{21}(\mathbf{x}, q) = \frac{1}{[3][2]} \left(X_{3^0}(\mathbf{x}, q) + (2q^2 + 2q) X_{3^1}(\mathbf{x}, q) + q^3 X_{3^2}(\mathbf{x}, q) \right).$$

Remark 4.2. By Lemma 2.13, the identity in Theorem 1.3 can be rewritten as

$$X_\lambda(\mathbf{x}, q) = \frac{1}{[m]!} \sum_{j=0}^{n} H_j^m(\lambda) \cdot X_{m^j}(\mathbf{x}, q). \tag{4.1}$$

Our main self-contained proof is an inductive argument, where the induction is applied both on the size $m+n$ of the graph, and also on the number of variables in the symmetric polynomials. Namely, we consider the chromatic symmetric functions in variables $x_1, \dots, x_M$ and each monomial appearing as a particular assignment of the variables (i.e. colors) to the vertices. That is, the vertices $1, \dots, N = m+n$ are colored $\{1, \dots, M\}$. For simplicity, we denote by $X_\lambda^N(M)$ the chromatic symmetric polynomial $X_{G(\lambda)}(x_1, \dots, x_M; q)$ where the graph $G(\lambda)$ has N vertices. We will use induction on both M and n, m when necessary, driven by the following recursion.

Recall that $\lambda/(i,j)$ is the partition obtained by removing the ith row and the jth column from λ. Moreover, we denote by λ/i the partitions obtained by removing from λ the ith column, for $i = 1, \dots, m$, or the $(m+n-i+1)$th row, for $i = m+1, \dots, m+n$.

Lemma 4.3. *For $\lambda \subset n \times m$ we have the following recursion*

$$X_\lambda^{m+n}(M) = X_\lambda^{m+n}(M-1) + x_M \sum_{i=1}^{m+n} q^{m+n-i-\lambda'_i} X_{\lambda/i}^{m+n-1}(M-1)$$
$$+ x_M^2 \sum_{(i,j)\in\lambda} q^{i-1+(m+n-j-\lambda'_j)} X_{\lambda/(i,j)}^{m+n-2}(M-1).$$

Proof. In the abelian case, i.e. when $\lambda \subset n \times m$, the graph $G(\lambda)$ consists of a clique with vertices $\{1, \dots, m\}$, a clique with vertices $\{m+1, \dots, m+n\}$ and a bipartite graph in between with edges $(i, m+j)$ for each (i,j) in $\overline{\lambda}$ (the complement of λ in $n \times m$). Therefore, a coloring of this graph has at most two vertices of the same color. If the colors used are in $\{1, \dots, M\}$, there are three cases for the appearances of color M:

1. No vertex is colored M, this term contributes $X_\lambda^{m+n-1}(M-1)$ to $X_\lambda^{m+n}(M)$.
2. Only one vertex is colored M. Suppose this vertex is in column j (from left) and row $i = N-j$ (from top to bottom). It creates ascents with all vertices above it but not in λ, giving $N-j-\lambda'_j$ ascents. Deleting this vertex corresponds to deleting its row and column (only one would be a row/column of λ) and we get a graph on $N-1$ vertices with shape λ/j (deleting either row $N-j$ from λ, or column j from λ). The remaining vertices and their ascents are not affected by this, so all their possible colorings contribute
$$x_M \sum_{j=1}^{N} q^{N-j-\lambda'_j} X_{\lambda/j}^{m+n-1}(M-1).$$
3. Two vertices are colored M. Suppose that the lower one is in column j and the higher one is in row i (counting from the top), necessarily with $(i,j) \in \lambda$. The lower vertex contributes $N-j-\lambda'_j$ ascents with the "visible" vertices above it. The higher vertex contributes $i-1$ ascents, the number of vertices above it, giving a total of $N-j-\lambda'_j+i-1$ ascents. We can remove these two vertices, by removing row i and column j from λ and decreasing N by 2. Again, the remaining vertices and their ascents are not affected, so these terms contribute
$$x_M^2 \sum_{(i,j)} q^{N-j-\lambda'_j+i-1} X_{\lambda/(i,j)}^{m+n-2}(M-1). \quad \square$$

For rectangular shapes $\lambda = (m^k)$, Lemma 4.3 simplifies to give the following recursive expansion.

Lemma 4.4.

$$X_{m^k}^{m+n}(M) = X_{m^k}^{m+n}(M-1) + x_M \Big(q^{n-k}[m] X_{(m-1)^k}^{m+n-1}(M-1) + [k] X_{m^{k-1}}^{m+n-1}(M-1)$$
$$+ q^k [n-k] X_{m^k}^{m+n-1}(M-1) \Big) + x_M^2 q^{n-k}[k][m] X_{(m-1)^{k-1}}^{m+n-2}(M-1).$$

Proof. This follows by carefully applying Lemma 4.3 to the shape $\lambda = m^k$. If $i \in [1, m]$ then $\lambda/i = (m-1)^k$ and $N-i-\lambda'_i = m+n-i-k$. If $i \in [m+1, m+n-k]$, then it is not a row or column of λ and we have $\lambda/i = m^k$ with $m+n-i-0$ ascents. If $i \in [m+n-k+1, m+n]$ then $\lambda/i = m^{k-1}$ and there are $m+n-i$ ascents. For $(i,j) \in \lambda$ we always have $\lambda/(i,j) = (m-1)^{k-1}$ and the ascents are $m+n-j-k+i-1$. Summing over the row/column indices in the given intervals we get the desired ascent statistics as the given q-integers. $\square$

Proof of Theorem 1.3. Translating Theorem 1.3 into chromatic symmetric polynomials, we want to prove that for every M we have

$$X_\lambda^{m+n}(M) = \frac{1}{[m]_n} \sum_{j=0}^{n} H_j^{m,n}(\lambda) \cdot X_{m^j}^{m+n}(M). \tag{4.2}$$

If $M < m$ then both $X_\lambda^{m+n}(M) = 0$ and $X_{m^j}^{m+n}(M) = 0$ since there is no proper coloring of the lower complete graph on m vertices. Otherwise, if $M = 1$ then $m = 1$ and so $n = 1$ and if $\lambda = \emptyset$ then $X_\lambda^2(M) \neq 0$ with $X_{m^0}(1) = 0$ and $H_1^{1,1}(\emptyset) = 0$, and the identity is satisfied. If $\lambda = (1)$ then $X_{(1)}^2(1) = x_1^2$, $[m]_n = [1]_1 = 1$, $H_0^{1,1}((1)) = 0$ and $H_1^{1,1}((1)) = 1$ with $m^1 = (1)$, and the identity is again trivially satisfied. This completes the initial conditions for the recursion in Lemma 4.3.

We will prove identity (4.2) by induction on M. As the argument above shows, the identity is trivially satisfied for $M = 1$. Suppose that (4.2) is true for $M - 1$, every $m \geq n$, and every shape $\lambda \subset m^n$. Naturally, if $M < m$ then both sides become trivially 0.

The rest of the proof is as follows. We apply Lemma 4.4 to each term $X_{m^j}(M)$ appearing in the RHS of (4.2). We also apply Lemma 4.3 to the LHS of (4.2), and the inductive hypothesis to $X_\mu^\star(M-1)$ for all appearing terms, where $\star$ means any value $\leq m+n$.

Applying Lemma 4.4 to each term $X_{m^j}(M)$ appearing in the RHS in (4.2), we obtain

$$\frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\cdot X_{m^j}^{m+n}(M) = \frac{1}{[m]_n}\sum_k H_k^{m,n}(\lambda)\left(X_{m^k}^{m+n}(M-1) + x_M\left(q^{n-k}[m]X_{(m-1)^k}^{m+n-1}(M-1)\right.\right.$$
$$\left.\left. +[k]X_{m^{k-1}}^{m+n-1}(M-1) + q^k[n-k]X_{m^k}^{m+n-1}(M-1)\right) + x_M^2 q^{n-k}[k][m]X_{(m-1)^{k-1}}^{m+n-2}(M-1)\right).$$

We now apply Lemma 4.3 to the LHS in (4.2). We split the sum in the linear x_M term into $j \in [1, m]$, when $\lambda/j = \lambda/^c j \subset (m-1)\times n$ (removing column j), and then $i = m+n+1-j \in [1, n]$, where $\lambda_j' = 0$ and $m+n-j-\lambda_j' = i-1$ and $\lambda/i \subset (n-1)\times m$. We then apply the inductive hypothesis to each $X_\mu^\star(M-1)$ appearing with the corresponding rectangular frame, obtaining

$$\begin{aligned}
X_\lambda^{m+n}(M) &= X_\lambda^{m+n}(M-1) \\
&\quad + x_M\sum_{j=1}^{m+n} q^{m+n-j-\lambda_j'}X_{\lambda/j}^{m+n-1}(M-1) + x_M^2\sum_{(i,j)\in\lambda} q^{i-1+(m+n-j-\lambda_j')}X_{\lambda/(i,j)}^{m+n-2}(M-1) \\
&= \frac{1}{[m]_n}\sum_k H_k^{m,n}(\lambda)X_{m^k}^{m+n}(M-1) + x_M\sum_k \frac{\sum_{j=1}^m q^{m+n-j-\lambda_j'}H_k^{m-1,n}(\lambda/^c j)}{[m-1]_n}X_{(m-1)^k}^{m+n-1}(M-1) \\
&\quad + x_M\sum_k \frac{\sum_{i=1}^n q^{i-1}H_k^{m,n-1}(\lambda/i)}{[m]_{n-1}}X_{m^k}^{m+n-1}(M-1) \\
&\quad + x_M^2\sum_k \frac{\sum_{(i,j)\in\lambda} q^{i-1+(m+n-j-\lambda_j')}H_k^{m-1,n-1}(\lambda/(i,j))}{[m-1]_{n-1}}X_{(m-1)^k}^{m+n-2}(M-1).
\end{aligned}$$

Applying Lemmas 3.6, 3.7 and 3.8 to the sums of q-hit numbers above, we get that

$$\begin{aligned}
X_\lambda^{m+n}(M) &= \frac{1}{[m]_n}\sum_k H_k^{m,n}(\lambda)X_{m^k}^{m+n}(M-1) + x_M\sum_k \frac{q^{n-k}[m-n]H_k^{m,n}(\lambda)}{[m-1]_n}X_{(m-1)^k}^{m+n-1}(M-1) \\
&\quad + x_M\sum_k \frac{q^k[n-k]H_k^{m,n}(\lambda) + [k+1]H_{k+1}^{m,n}(\lambda)}{[m-n+1][m]_{n-1}}X_{m^k}^{m+n-1}(M-1) \\
&\quad + x_M^2\sum_k \frac{q^{n-1-k}[k+1]H_{k+1}^{m,n}(\lambda)}{[m-1]_{n-1}}X_{(m-1)^k}^{m+n-2}(M-1).
\end{aligned}$$

Simplifying the factors $\frac{[m-1]_n}{[m-n]} = \frac{[m]_n}{[m]}$, $[m-n+1][m]_{n-1} = [m]_n$ and $[m-1]_{n-1} = \frac{[m]_n}{[m]}$, and grouping the terms with $H_k^{m,n}(\lambda)$, we can rewrite the above identity as

$$X_\lambda^{m+n}(M) = \frac{1}{[m]_n}\sum_k H_k^{m,n}(\lambda)\times\left(X_{m^k}^{m+n}(M-1) + x_M q^{n-k}[m]X_{(m-1)^k}^{m+n-1}(M-1)\right.$$
$$\left. + x_M(q^k[n-k]X_{m^k}^{m+n-1}(M-1) + [k]X_{m^{k-1}}^{m+n-1}(M-1)) + x_M^2 q^{n-k}[k][m]X_{(m-1)^{k-1}}^{m+n-2}(M-1)\right)$$

$$= \frac{1}{[m]_n} \sum_k H_k^{m,n}(\lambda) X_{m^k}^{m+n}(M),$$

where we recognized the sum in the parentheses as the RHS of the recursion for rectangular CSF, namely Lemma 4.4. This completes the induction. □

5. The Abreu–Nigro expansion in the elementary basis

In this section we show that Guay-Paquet's identity (Theorem 1.3) is equivalent to Abreu–Nigro's identity (Theorem 1.2). We start by giving a proof of Abreu–Nigro's identity for rectangular shapes.

Lemma 5.1 (*Abreu–Nigro's formula for rectangles*).

$$X_{m^k}(\mathbf{x}, q) = [k]! H_k^{m+n-k}(m^k) \cdot e_{m+n-k,k} + \sum_{r=0}^{k-1} q^r\, [r]!\, [m+n-2r]\, H_r^{m+n-r-1}(m^k) \cdot e_{m+n-r,r}.$$

In order to prove this case of the Abreu–Nigro identity we need the following result.

Lemma 5.2 (*Guay-Paquet formula for rectangles*). *For the shape $(m-1)^k \subset n \times m$ we have that,*

$$[m] X_{(m-1)^k} = q^k [m-k] X_{m^k} + [k] X_{m^{k-1}}.$$

Corollary 5.3.

$$\begin{bmatrix} m-1 \\ k \end{bmatrix} X_{m^k} = \sum_{j=0}^{k} \begin{bmatrix} m \\ j \end{bmatrix} (-1)^{k-j} q^{-(k+j)(k-j+1)/2} X_{(m-1)^j}.$$

Proof of Lemma 5.2. By Theorem 1.3 for the shape $\lambda = (m-1)^k \subset n \times m$ and the formula for the q-hit numbers $H_r^{m,n}((m-1)^k)$ from Proposition 2.17 we obtain

$$X_{(m-1)^k} = \frac{1}{[m]_n} q^k [m-1]_k [m-k]_{n-k} X_{m^k} + \frac{1}{[m]_n} [k][m-1]_{k-1} [m-k]_{n-k} X_{m^{k-1}},$$

which simplifies to

$$[m] X_{(m-1)^k} = q^k [m-k] X_{m^k} + [k] X_{m^{k-1}}. \quad \square$$

Proof of Lemma 5.1. We use induction on m and k. For the base case, note that $X_{m^0} = [m+n]! e_{m+n} = H_0^{m+n}(m^0) e_{m+n}$. Now, by Lemma 5.2 we have that

$$q^k [m-k] X_{m^k} = \left([m] X_{(m-1)^k} - [k] X_{m^{k-1}} \right).$$

Next, we use the induction hypothesis on $X_{(m-1)^k}$ and $X_{m^{k-1}}$ and we simplify our expression, so we want to prove that

$$\begin{aligned}
& q^k [m-k] \Big([k]! H_k^{m+n-k}(m^k) \cdot e_{m+n-k,k} + \sum_{r=0}^{k-1} q^r\, [r]!\, [m+n-2r]\, H_r^{m+n-r-1}(m^k) \cdot e_{m+n-r,r} \Big) \\
&= [m] \Big([k]! H_k^{m+n-k}((m-1)^k) \cdot e_{m+n-k,k} \\
&\quad + \sum_{r=0}^{k-1} q^r\, [r]!\, [m+n-2r]\, H_r^{m+n-r-1}((m-1)^k) \cdot e_{m+n-r,r} \Big) \\
&\quad - [k] \Big([k-1]! H_{k-1}^{m+n-k+1}(m^{k-1}) \cdot e_{m+n-k+1,k-1} \\
&\quad - \sum_{r=0}^{k-2} q^r\, [r]!\, [m+n-2r]\, H_r^{m+n-r-1}(m^{k-1}) \cdot e_{m+n-r,r} \Big).
\end{aligned} \tag{5.1}$$

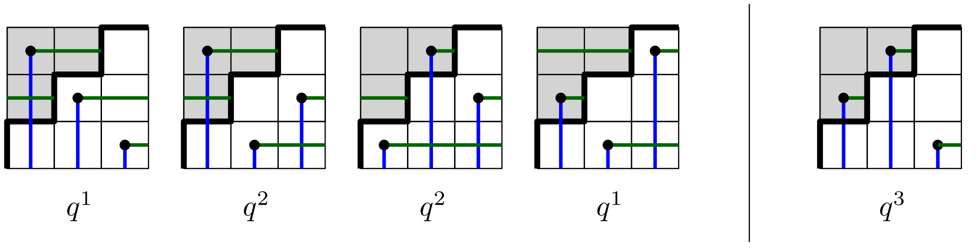


Fig. 5. Placements of 3 rooks in 3×3 that "hit" (2, 1) (in gray) once and twice and the associated statistic to each rook placement.

We do so by looking at the coefficient of $e_{m-n-r,r}$ for $0 \le r \le k$. For $r = k$, we have that

$$q^k[m-k]\,[k]!H_k^{m+n-k}(m^k) = [m]\,[k]!H_k^{m+n-k}((m-1)^k),$$

which follows from (2.7) using routine simplifications.

For $r = k-1$, we have that

$$\begin{aligned} &q^k[m-k]q^{k-1}\,[k-1]!\,[m+n-2k+2]\,H_{k-1}^{m+n-k}(m^k) \\ &\quad = [m]\,q^{k-1}\,[k-1]!\,[m+n-2k+2]\,H_{k-1}^{m+n-k}((m-1)^k) - [k]\,[k-1]!H_{k-1}^{m+n-k+1}(m^{k-1}). \end{aligned} \tag{5.2}$$

Using (2.7), we obtain that (5.2) is equivalent to

$$q\,[m-k]\,[n-k] = [m-k+1]\,[n-k+1] - [m+n-2k+1]\,. \tag{5.3}$$

For $0 \le r \le k-2$, we have that

$$\begin{aligned} &q^k\,[m-k]\,q^r\,[r]!\,[m+n-2r]\,H_r^{m+n-r-1}(m^k) \\ &\quad = [m]\,q^r\,[r]!\,[m+n-2r]\,H_r^{m+n-r-1}((m-1)^k) \\ &\qquad - [k]\,q^r\,[r]!\,[m+n-2r]\,H_r^{m+n-r-1}(m^{k-1}). \end{aligned} \tag{5.4}$$

Using (2.7), we obtain that (5.4) is equivalent to

$$q^{k-r}\,[m-k]\,[n-k] = [m-r]\,[n-r] - [m+n-r-k]\,[k-r]\,, \tag{5.5}$$

which is straightforward to verify by expanding both sides. Note that (5.3) is a particular case of (5.5) by taking $r = k-1$. $\square$

We are now ready to prove that the Guay-Paquet's identity and Abreu–Nigro's follow from each other. As a corollary, we obtain a new proof of the latter. The following example illustrates Abreu–Nigro's result.

Example 5.4. For $\lambda = (2,1)$ inside a 2×3 board, we have that for $k = 2$, $H_2^3(\lambda) = q^3$, $H_1^3(\lambda) = 2q^2 + 2q$ (see Fig. 5), and $H_0^4(\lambda) = q^3 + 3q^2 + 3q + 1$. Therefore,

$$X_{2,1}(\mathbf{x},q) = q^3(1+q)e_{3,2}(\mathbf{x}) + q(1+q+q^2)(2q^2+2q)e_{4,1}(\mathbf{x}) + (1+q+q^2+q^3)(q^3+3q^2+3q+1)e_5(\mathbf{x}).$$

Proof of Theorem 1.2.

Applying Lemma 5.1 to the RHS of the formula in Theorem 1.3, we obtain that

$$\begin{aligned} \frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\cdot X_{m^j}(\mathbf{x},q) &= \frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\Big([j]!H_j^{m+n-j}(m^j)\cdot e_{m+n-j,j}\Big) \\ &+ \frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\Big(q^r\sum_{r=0}^{j-1}[r]!\,[m+n-2r]\,H_r^{m+n-r-1}(m^j)\cdot e_{m+n-r,r}\Big). \end{aligned}$$

Now, switching the summation order, we have that

$$\frac{1}{[m]_n}\sum_{j=0}^{n} H_j^{m,n}(\lambda)\cdot X_{m^j}(\mathbf{x},q) = \sum_{r=0}^{n} e_{m+n-r,r}\frac{1}{[m]_n}[r]!H_r^{m+n-r}(m^r)H_r^{m,n}(\lambda)$$
$$+\sum_{r=0}^{n-1} e_{m+n-r,r}\frac{1}{[m]_n}\Big(q^r\sum_{j=r+1}^{n}[r]!\,[m+n-2r]\,H_r^{m+n-r-1}(m^j)H_j^{m,n}(\lambda)\Big).$$

Thus, we need to show that for $r = k = \ell(\lambda)$,

$$[m]_n\,H_k^{m+n-k}(\lambda) = H_k^{m+n-k}(m^k)H_k^{m,n}(\lambda) + q^k\sum_{j=k+1}^{n}[m+n-2k]\,H_k^{m+n-k-1}(m^j)H_j^{m,n}(\lambda)$$
$$= H_k^{m+n-k}(m^k)H_k^{m,n}(\lambda),$$

since $H_j^{m,n}(\lambda) = 0$ for $j = k+1,\ldots,n$. We also need to show that for $r < k = \ell(\lambda)$,

$$[m]_n\,q^r\,[m+n-2r]\,H_r^{m+n-r-1}(\lambda) = H_r^{m+n-r}(m^r)H_r^{m,n}(\lambda)$$
$$+\,q^r\sum_{j=r+1}^{n}[m+n-2r]\,H_r^{m+n-r-1}(m^j)H_j^{m,n}(\lambda).$$

After using (2.7), these two relations are equivalent to the following identities relating q-hit numbers of λ in square boards and rectangular boards. Thus, the Abreu–Nigro expansion for $X_\lambda(\mathbf{x},q)$ follows now from Lemma 5.5, which completes the proof. □

Lemma 5.5. *Let λ be a partition inside an $n\times m$ board and $k = \ell(\lambda)$, then*

$$\begin{bmatrix} m-k \\ n-k \end{bmatrix} H_k^{m+n-k}(\lambda) = q^{k(n-k)}\,[m+n-2k]_{m-k}\,H_k^{m,n}(\lambda), \tag{5.6}$$

and for $0 \le r < k$, we have

$$\begin{bmatrix} m-r \\ n-r \end{bmatrix} H_r^{m+n-r-1}(\lambda) = q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,H_r^{m,n}(\lambda)$$
$$+\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j \\ r \end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}H_j^{m,n}(\lambda). \tag{5.7}$$

Proof. The first relation follows from a simple combinatorial observation together with Lemma 2.13. For $k = \ell(\lambda)$, we see that the k rooks on the first k rows on the board have to be all inside λ, and all the cells outside λ in these first rows will be empty. Thus

$$H_k^N(\lambda) = H_k^{k,\lambda_1}(\lambda)q^{k(N-\lambda_1)}[N-k]!.$$

Similarly, by the proof of Lemma C.1 we have

$$H_k^{m,n}(\lambda) = H_k^{k,\lambda_1}(\lambda)q^{k(m-\lambda_1)}\begin{bmatrix} m+n-2k \\ n-k \end{bmatrix}.$$

Substituting these formulas in each side of (5.6) we get the desired identity.

For the second relation, we switch gears and use a deletion–contraction relation of q-hit numbers (Lemma 2.6). The idea is to use induction of the size of λ, the deletion–contraction relation for q-hit numbers, and deduce the identity by matching coefficients at each q-hit number.

Remark 5.6. An alternative proof appears in Appendix F, where the q-hits are expressed in terms of q-rooks and matching coefficients at each q-rook reduces to q-binomial identities.

Let us start with the RHS of (5.7). Denote by

$$\begin{aligned}B_r^{m,n}(\lambda) &:= q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,H_r^{m,n}(\lambda)\\ &\quad+\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}H_j^{m,n}(\lambda).\end{aligned}$$

If we apply the deletion–contraction relation in Lemma 2.6 to the q-hit numbers, we have that

$$\begin{aligned}B_r^{m,n}(\lambda) &= q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,H_r^{m,n}(\lambda\backslash e)\\ &\quad+q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,q^{|\lambda/e|-|\lambda|+r+m-1}H_{r-1}^{m-1,n-1}(\lambda/e)\\ &\quad-q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,q^{|\lambda/e|-|\lambda|+r+m}H_{r}^{m-1,n-1}(\lambda/e)\\ &\quad+\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}H_j^{m,n}(\lambda\backslash e)\\ &\quad+\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}q^{|\lambda/e|-|\lambda|+j+m-1}H_{j-1}^{m-1,n-1}(\lambda/e)\\ &\quad-\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}q^{|\lambda/e|-|\lambda|+j+m}H_{j}^{m-1,n-1}(\lambda/e).\end{aligned}\tag{5.8}$$

For the LHS of (5.7), we denote $C_r^{m,n}(\lambda) := \begin{bmatrix} m-r\\ n-r\end{bmatrix}H_r^{m+n-r-1}(\lambda)$. Then by Corollary E.2,

$$\begin{aligned}C_r^{m,n}(\lambda) &= \begin{bmatrix} m-r\\ n-r\end{bmatrix}H_r^{m+n-r-1}(\lambda\backslash e)\\ &\quad+\begin{bmatrix} m-r\\ n-r\end{bmatrix}q^{|\lambda/e|-|\lambda|+r+m+n-r-2}\left(H_{r-1}^{m+n-r-2}(\lambda/e)-qH_r^{m+n-r-2}(\lambda/e)\right).\end{aligned}$$

That is, we have the following deletion–contraction relation for the C's:

$$C_r^{m,n}(\lambda) = C_r^{m,n}(\lambda\backslash e)+q^{|\lambda/e|-|\lambda|+m+n-2}\left(C_{r-1}^{m-1,n-1}(\lambda/e)-q\frac{[m-r]}{[m-n]}C_r^{m-1,n}(\lambda/e)\right).\tag{5.9}$$

Now, we want to compare the two expressions in (5.8) and (5.9). By *inductive hypothesis*,

$$\begin{aligned}C_r^{m,n}(\lambda\backslash e) &= q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,H_r^{m,n}(\lambda\backslash e)\\ &\quad+\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}H_j^{m,n}(\lambda\backslash e) = B_r^{m,n}(\lambda\backslash e).\end{aligned}$$

These terms appear both in (5.8) and in (5.9), and so they cancel. We have left to show that

$$\begin{aligned}&q^{|\lambda/e|-|\lambda|+m+n-2}\left(C_{r-1}^{m-1,n-1}(\lambda/e)-q\frac{[m-r]}{[m-n]}C_r^{m-1,n}(\lambda/e)\right)\\ &= q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,q^{|\lambda/e|-|\lambda|+r+m-1}H_{r-1}^{m-1,n-1}(\lambda/e)\\ &\quad-q^{r(n-r-1)}\,[m+n-2r-1]_{m-r-1}\,q^{|\lambda/e|-|\lambda|+r+m}H_{r}^{m-1,n-1}(\lambda/e)\\ &\quad+\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}q^{|\lambda/e|-|\lambda|+j+m-1}H_{j-1}^{m-1,n-1}(\lambda/e)\\ &\quad-\sum_{j=r+1}^{n} q^{r(n-1-j)}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}q^{|\lambda/e|-|\lambda|+j+m}H_{j}^{m-1,n-1}(\lambda/e).\end{aligned}$$

This last equation simplify to the following identity:

$$
\begin{aligned}
&q^{n-2}\left(C^{m-1,n-1}_{r-1}(\lambda/e) - q\frac{[m-r]}{[m-n]}C^{m-1,n}_{r}(\lambda/e)\right)\\
&\quad = q^{r(n-r)-1}\,[m+n-2r-1]_{m-r-1}\left(H^{m-1,n-1}_{r-1}(\lambda/e) - qH^{m-1,n-1}_{r}(\lambda/e)\right)\\
&\qquad + \sum_{j=r+1}^{n} q^{r(n-1-j)+j-1}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]}\left(H^{m-1,n-1}_{j-1}(\lambda/e) - qH^{m-1,n-1}_{j}(\lambda/e)\right),
\end{aligned}
\tag{5.10}
$$

In order to show (5.10), we apply first the *inductive hypothesis* in the C's together with the relation in Lemma 2.14 to obtain their expansion in terms of q-hit numbers of the form $H^{m-1,n-1}_{j}(\lambda/e)$:

$$
\begin{aligned}
q^{n-2}C^{m-1,n-1}_{r-1}(\lambda/e) &= q^{(r-1)(n-r-1)+n-2}\,[m+n-2r-1]_{m-r-1}\,H^{m-1,n-1}_{r-1}(\lambda/e)\\
&\quad + \sum_{j=r}^{n-1} q^{(r-1)(n-2-j)+n-2}\begin{bmatrix} j\\ r-1\end{bmatrix}\frac{[m+n-r-j-2]_{m-r}}{[n-r]}H^{m-1,n-1}_{j}(\lambda/e),\\
q^{n-1}\frac{[m-r]}{[m-n]}C^{m-1,n}_{r}(\lambda/e) &= q^{r(n-r-1)+n-1}\frac{[m-r]}{[m-n]}\,[m+n-2r-2]_{m-r-2}\,H^{m-1,n}_{r}(\lambda/e)\\
&\quad + \sum_{j=r+1}^{n} q^{r(n-1-j)+n-1}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m-r]}{[m-n]}\frac{[m+n-r-j-2]_{m-r-1}}{[n-r]}H^{m-1,n}_{j}(\lambda/e).
\end{aligned}
$$

Finally, we compare the coefficients in (5.10). For $H^{m-1,n-1}_{r-1}(\lambda/e)$, we have

$$
LHS = q^{(r-1)(n-r-1)+n-2}\,[m+n-2r-1]_{m-r-1} = RHS.
$$

For $H^{m-1,n-1}_{r}(\lambda/e)$, we have

$$
\begin{aligned}
LHS &= q^{(r-1)(n-2-r)+n-2}\begin{bmatrix} r\\ r-1\end{bmatrix}\frac{[m+n-r-r-2]_{m-r}}{[n-r]}\\
&\quad - q^{r(n-r-1)+n-1}\,[m-r]\,[m+n-2r-2]_{m-r-2}\,,\\
RHS &= -q^{r(n-r)}\,[m+n-2r-1]_{m-r-1} + q^{r(n-2-r)+r}\begin{bmatrix} r+1\\ r\end{bmatrix}\frac{[m+n-2r-2]_{m-r}}{[n-r]},
\end{aligned}
$$

which simplifies to the relation $[m+n-2r-1] = q^{n-r-1}\,[m-r] + [n-r-1]$. This last relation is a straightforward verification by expanding both sides.

For $H^{m-1,n-1}_{j}(\lambda/e)$, with $r+1 \le j \le n-1$,

$$
\begin{aligned}
LHS &= q^{(r-1)(n-2-j)+n-2}\begin{bmatrix} j\\ r-1\end{bmatrix}\frac{[m+n-r-j-2]_{m-r}}{[n-r]}\\
&\quad - q^{r(n-1-j)+n-1}\begin{bmatrix} j\\ r\end{bmatrix}[m-r]\,\frac{[m+n-r-j-2]_{m-r-1}}{[n-r]},\\
RHS &= q^{r(n-2-j)+j}\begin{bmatrix} j+1\\ r\end{bmatrix}\frac{[m+n-r-j-2]_{m-r}}{[n-r]} - q^{r(n-1-j)+j}\begin{bmatrix} j\\ r\end{bmatrix}\frac{[m+n-r-j-1]_{m-r}}{[n-r]},
\end{aligned}
$$

which simplifies to the relation $[m+n-r-j-1] = q^{n-j-1}\,[m-r]+[n-j-1]$. This last relation is a straightforward verification by expanding both sides. □

6. Variations and applications

6.1. Theorems 1.2 and 1.3 In terms of unicellular LLT polynomials

Chromatic symmetric functions of Dyck paths are related to *unicellular LLT polynomials* [19] that can be defined as follows. For a Dyck path d, let $G(d)$ be the associated graph and denote

$$\mathrm{LLT}_{G(d)}(\mathbf{x}, q) := \sum_{\kappa : V(G(d)) \to \mathbb{P}} q^{\mathrm{asc}(\kappa)} \mathbf{x}^{\kappa},$$

where the sum is over all vertex colorings κ of $G(d)$ and $\mathrm{asc}(\kappa)$ is the same as in the definition of $X_G(\mathbf{x}, q)$.

The function $\mathrm{LLT}_{G(d)}(\mathbf{x}, q)$ is actually symmetric (see [2, Sec. 3.1]) and the symmetric functions $X_{G(d)}(\mathbf{x}, q)$ and $\mathrm{LLT}_d(\mathbf{x}, q)$ are related via a plethystic substitution discovered independently by Carlsson–Mellit [5, Prop. 3.4] and Guay-Paquet [14, Lemma 172]:

$$\mathrm{LLT}_{G(d)}(\mathbf{x}, q) = (q-1)^n X_{G(d)}[\mathbf{x}/(q-1), q],$$

where n is the size of the Dyck path. As a consequence of this connection, any linear relation in one family implies the same relation in the other one. Since Theorems 1.2 and 1.3 yield a linear relation among certain chromatic symmetric functions, we immediately obtain the same relations for the corresponding unicellular LLT polynomials.

Corollary 6.1. *Let λ be partition inside an $n \times m$ board with $\ell(\lambda) = k \le \lambda_1$. Then*

$$\begin{aligned}\mathrm{LLT}_{G(\lambda)}(\mathbf{x}, q) = {} & \frac{1}{[m+n-k]!} H_k^{m+n-k}(\lambda) \cdot \mathrm{LLT}_{K(m+n-k,k)}(\mathbf{x}, q) \\ & + \sum_{j=0}^{k-1} q^j \frac{[m+n-2j]}{[m+n-j]!} H_j^{m+n-j-1}(\lambda) \cdot \mathrm{LLT}_{K(m+n-j,j)}(\mathbf{x}, q),\end{aligned}$$

where $K(a, b)$ is the disjoint union of complete graphs on vertices $\{1, \ldots, a\}$ and $\{a+1, \ldots, a+b\}$.

Proof. Since $X_{K(a,b)}(\mathbf{x}, q) = [a]![b]!e_{a,b}$, then Theorem 1.2 yields a linear relation among $X_{G(\lambda)}(\mathbf{x}, q)$, $X_{K(m+n,0)}(\mathbf{x}, q), \ldots, X_{K(m+n-k,k)}(\mathbf{x}, q)$. The result then follows from the fact that every linear relation among a set of chromatic symmetric functions of Dyck paths has a corresponding relation among unicellular LLT polynomials [2, Prop. 55]. □

Corollary 6.2. *Let λ be a partition inside an $n \times m$ ($n \le m$) board. Then*

$$\mathrm{LLT}_{G(\lambda)}(\mathbf{x}, q) = \frac{1}{[m]_n} \sum_{j=0}^{n} H_j^{m,n}(\lambda) \cdot \mathrm{LLT}_{G(m^j)}(\mathbf{x}, q).$$

Proof. Theorem 1.3 yields a linear relation among $X_\lambda(\mathbf{x}, q), X_{m^0}(\mathbf{x}, q), \ldots, X_{m^n}(\mathbf{x}, q)$. The result then follows by the same fact as in the proof above. □

6.2. The staircase basis

Let $V^{m,n} := \mathrm{span}_{\mathbb{Q}(q)}\{X_\lambda \mid \lambda \text{ inside an } n \times m \text{ board}\}$. Consider the set of chromatic symmetric functions given by rectangular shapes $\mathcal{R}^{m,n} := \left\{X_{m^j}(\mathbf{x}, q)\right\}_{j=0}^{n}$. This set is actually a basis for $V^{m,n}$.

Corollary 6.3. *The set $\mathcal{R}^{m,n}$ forms a basis for the $\mathbb{Q}(q)$-vector space $V^{m,n}$.*

To prove this result we need the following result.

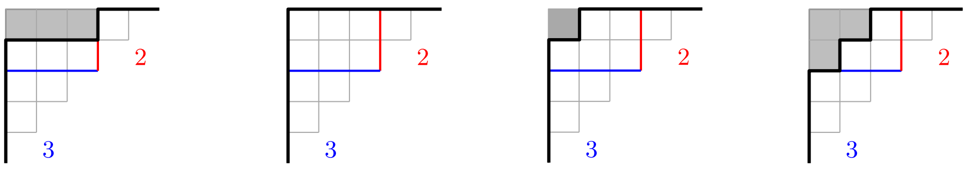


Fig. 6. Left: the path for $\lambda = 3$ inside a 2×3 board. Right: the three staircase paths $\delta_0, \delta_1, \delta_2$ for $m = 3$ and $n = 2$.

Proposition 6.4. *Let λ be a partition inside the $n \times m$ board and let $k = \ell(\lambda) \le \lambda_1$*

$$X_\lambda(\mathbf{x}, q) = [k]!H_k^{m+n-k}(\lambda)s_{2^k1^{m+n-2k}}(\mathbf{x}) + \sum_{i=0}^{k-1} s_{2^i1^{m+n-2i}}(\mathbf{x})\Big([k]!H_k^{m+n-k}(\lambda) + \sum_{j\ge i}^{k-1} q^j[j]![m+n-2j]H_j^{m+n-j-1}(\lambda)\Big). \tag{6.1}$$

Proof. This expansion follows by combining expressing the elementary symmetric functions in the Schur basis $e_{n-j,j} = \sum_{i=0}^{j} s_{2^i1^{n-2i}}$ in Theorem 1.2. □

Proof of Corollary 6.3. By Theorem 1.3 we have that $\mathcal{R}^{m,n}$ span $V^{m,n}$.

By Theorem 1.2 the leading term $X_{m^j}(\mathbf{x}, q)$ in the e-basis is $H_j^{m+n-j}(m^j)e_{m+n-j,j}$. Since $j \le m$ we can certainly place j rooks on m^j and see that $H_j^{m+n-j}(m^j) \ne 0$. Note that the full formula is given in Proposition 2.16. Thus the transition matrix between $\mathcal{R}^{m,n}$ and the $e_{m+n-j,j}$'s is upper triangular with nonzero diagonal entries, so they are linearly independent and form a basis. □

Guay-Paquet considered another basis for the space $V^{m,n}$. This basis is called the *staircase basis* since it is indexed by the staircase partitions, which are partitions inside an $n \times m$ board of the form $\delta_j = (j, j-1, \ldots, 1)$, for $j = 0, \ldots, n$ (see Fig. 6). Let $\mathcal{S}^{m,n} := \left\{X_{\delta_j}(\mathbf{x}, q)\right\}_{j=0}^{n}$.

Proposition 6.5 ([12]). *The set $\mathcal{S}^{m,n}$ forms a basis for the $\mathbb{Q}(q)$-vector space $V^{m,n}$.*

Proof. We look at the expansion in Theorem 1.2 for δ_j. In this case, $H_j^{m+n-j}(\delta_j) \ne 0$ because we can place j rooks on the main diagonal of δ_j and place $m+n-2j$ remaining rooks outside δ_j (for instance, in the main diagonal of the square board). Therefore, the leading term of $X_{\delta_j}(\mathbf{x}, q)$ is $H_j^{m+n-j}(\delta_j)e_{m+n-j,j} \ne 0$. That is, $\mathcal{S}^{m,n}$'s transfer matrix with the $V^{m,n}$ basis given by $\{e_{m+n-j,j}\}$ is upper triangular with nonzero diagonal and hence is also a basis. □

Since the coefficients of $X_\lambda(\mathbf{x}, q)$ in the $\mathcal{R}^{m,n}$ basis involve q-hit numbers, we also want to study the coefficients appearing in the decomposition of $X_\lambda(\mathbf{x}, q)$ in this new basis $\mathcal{S}^{m,n}$. That is, we want to understand the coefficients $a_j^{m,n}(\lambda, q)$ in the expression

$$X_\lambda(\mathbf{x}, q) = \sum_{j=0}^{n} a_j^{m,n}(\lambda, q) \cdot X_{\delta_j}(\mathbf{x}, q). \tag{6.2}$$

Calculations suggest that up to a sign and a power of q, these coefficients are in $\mathbb{N}[q]$.

Conjecture 6.6. *Let λ be a partition inside an $n \times m$ board ($n \le m$) and j be fixed, then $a_j^{m,n}(\lambda, q)$ is a Laurent polynomial in q whose coefficients are integers of the same sign.*

Example 6.7. For $\lambda = 3$ inside a 2×3 board, we have

$$q^2X_3(\mathbf{x}, q) = -(q+1)X_\varnothing(\mathbf{x}, q) + (q^2+q+1)X_1(\mathbf{x}, q) + 0 \cdot X_{21}(\mathbf{x}, q).$$

A possible approach to Conjecture 6.6 is looking at the change of bases between the rectangular basis $\mathcal{R}^{m,n}$ and the staircase basis $\mathcal{S}^{m,n}$. Let us start showing that the coefficients of the change of bases determine $a_j^{m,n}(\lambda, q)$.

Proposition 6.8. $a_j^{m,n}(\lambda, q) = \dfrac{1}{[m]_n} \displaystyle\sum_{i=0}^{n} H_i^{m,n}(\lambda) a_j^{m,n}(m^i, q).$

Proof. This follows from basic linear algebra. By Theorem 1.3 and (6.2), we have that

$$\begin{aligned} X_\lambda(\mathbf{x}, q) &= \frac{1}{[m]_n} \sum_{k=0}^{n} H_k^{m,n}(\lambda) \cdot X_{m^k}(\mathbf{x}, q) \\ &= \frac{1}{[m]_n} \sum_{k=0}^{n} H_k^{m,n}(\lambda) \sum_{j=0}^{n} a_j^{m,n}(m^k, q) X_{\delta_j}(\mathbf{x}, q). \end{aligned}$$

Next, we exchange the order of summation to obtain

$$X_\lambda(\mathbf{x}, q) = \sum_{j=0}^{n} X_{\delta_j}(\mathbf{x}, q) \frac{1}{[m]_n} \sum_{k=0}^{n} H_k^{m,n}(\lambda) a_j^{m,n}(m^k, q).$$

The result then follows by extracting the coefficient of $X_{\delta_j}(\mathbf{x}, q)$ on both sides above. □

This means that in order to prove Conjecture 6.6 it suffices to verify it for $a_j^{m,n}(m^k, q)$.

Remark 6.9. We have the following $(n+1) \times (n+1)$ matrix and its inverse

$$\mathcal{A} := \left(a_j^{m,n}(m^k, q)\right)_{0 \le j,k \le n}, \quad \mathcal{H} := \mathcal{A}^{-1} = \frac{1}{[m]_n} \left(H_j^{m,n}(\delta_k)\right)_{0 \le j,k \le n}.$$

Moreover, the q-hit numbers $H_j^{n,n}(\delta_n)$ are the **q-Eulerian polynomials** defined by

$$A_{n,j+1}(q) := \sum_{\substack{w \in \mathfrak{S}_n \\ exc(w)=j}} q^{\mathrm{stat}(w)},$$

where stat is inv or maj indices (see [4,23]). Also, one can check that the specialization

$$H_j^{n,n}(\delta_{n-r})\big|_{q=1} = A_{n,j}^{(r)},$$

where the $A_{n,j}^{(r)}$ are the **r-excedence numbers** that count the number of permutations in $\mathfrak{S}_n$ with j r-excendences $\{i \mid w(i) \ge i + r\}$ (see [9,10]).

Note that the coefficients $a_j^{m,n}(\lambda, q)$ add up to one.

Proposition 6.10. *Let λ be a partition inside an $n \times m$ ($n \le m$) board. Then* $\displaystyle\sum_{j=0}^{n} a_j^{m,n}(\lambda, q) = 1.$

Proof. By Proposition 6.8 and Corollary 2.11 it is enough to show the result for the $a_j^{m,n}(m^k, q)$. That is, to show that the columns of the matrix $\mathcal{A}$ from Remark 6.9 add up to one. This follows from basic linear algebra since the columns of the inverse matrix $\mathcal{A}^{-1} = \mathcal{H}$ also add up to one by Corollary 2.11. □

6.3. Theorem 1.3 In terms of q-rook numbers

Theorem 1.3 expresses $X_\lambda(\mathbf{x},q)$ as a linear combination of the functions $X_{m^j}(\mathbf{x},q)$ where the coefficients are normalized q-hit numbers. Since the latter are defined in (2.2) in terms of q-rook numbers it is natural to give an expression for $X_\lambda(\mathbf{x},q)$ involving q-rook numbers $R_i(\lambda)$.

Definition 6.11. For non-negative integers m, n, and i, with $m \geq n$ and $0 \leq i \leq n$, define

$$Y_i^{m,n}(\mathbf{x},q) := \sum_{k=0}^{i} (-1)^{i-k} \begin{bmatrix} i \\ k \end{bmatrix} q^{\binom{k}{2}} X_{m^k}(\mathbf{x},q).$$

Corollary 6.12. *Let λ be a partition inside an $n \times m$ board ($n \leq m$). Then*

$$X_\lambda(\mathbf{x},q) = \frac{1}{[m]!} \sum_{i=0}^{n} [m-i]! q^{mi-\binom{i}{2}-|\lambda|} R_i(\lambda) \cdot Y_i^{m,n}(\mathbf{x},q).$$

Proof. We use (2.2) in Theorem 1.3 and change the order to summation to obtain:

$$\begin{aligned} X_\lambda(\mathbf{x},q) &= \frac{1}{[m]_n} \sum_{k=0}^{n} \frac{q^{\binom{k}{2}-|\lambda|}}{[m-n]!} \sum_{i=k}^{n} [m-i]! \begin{bmatrix} i \\ k \end{bmatrix} (-1)^{i+k} q^{mi-\binom{i}{2}} R_i(\lambda) X_{m^k}(\mathbf{x},q) \\ &= \frac{1}{[m]!} \sum_{i=0}^{n} [m-i]! q^{mi-\binom{i}{2}-|\lambda|} R_i(\lambda) \cdot \left(\sum_{k=0}^{i} (-1)^{i-k} \begin{bmatrix} i \\ k \end{bmatrix} q^{\binom{k}{2}} X_{m^k}(\mathbf{x},q) \right). \end{aligned}$$

The result follows by noting that the sum in parenthesis on the RHS is exactly $Y_i(\mathbf{x},q)$. □

The following example shows that the $Y_i^{m,n}(\mathbf{x},q)$ are not m-positive.

Example 6.13. For $m = 3$, $n = 2$ and $i = 1$, we have that

$$Y_1^{3,2}(\mathbf{x},q) = X_{3^1}(\mathbf{x},q) - X_{3^0}(\mathbf{x},q).$$

Expanding in the m-basis, we have that

$$\begin{aligned} Y_1^{3,2}(\mathbf{x},q) = &\left(-q^{10} - 4q^9 - 9q^8 - 14q^7 - 12q^6 - 2q^5 + 11q^4 + 16q^3 + 11q^2 + 4q\right) m_{1,1,1,1,1} \\ &+ \left(q^6 + 3q^5 + 5q^4 + 5q^3 + 3q^2 + q\right) m_{2,1,1,1}. \end{aligned}$$

However, in the e-basis we have that

$$\begin{aligned} &= \left(q^6 + 3q^5 + 5q^4 + 5q^3 + 3q^2 + q\right) e_{4,1} \\ &+ \left(-q^{10} - 4q^9 - 9q^8 - 14q^7 - 17q^6 - 17q^5 - 14q^4 - 9q^3 - 4q^2 - q\right) e_5. \end{aligned}$$

Indeed, it appears that the expansion of the $Y_i^{m,n}$ in the elementary basis yields palindromic polynomials with all positive or all negative coefficients.

Conjecture 6.14. *Consider the functions $Y_i(\mathbf{x},q)$ expanded in the e-basis*

$$Y_i(\mathbf{x},q) = \sum_{k=0}^{i} c_k(q) e_{m+n-k,k}.$$

Then $(-1)^{i-k} c_k(q) \in \mathbb{N}[q]$ and are palindromic.

7. Final remarks

7.1. A tale of two variants of q-hit numbers

To our surprise, the q-hit numbers appearing in Theorems 1.2 and 1.3 are not exactly the Garsia–Remmel q-hit numbers denoted by $\widetilde{H}_j^n(\lambda)$ but instead, they are off by a power of q (see Proposition B.1). Thus, these two variants of q-hit numbers satisfy different versions of deletion–contraction (see Appendix E).

While working on this project, we have encountered two different variants of Dworkin's statistic for the Garsia–Remmel q-hit numbers $\widetilde{H}_n^j(\lambda)$ for $\lambda \subset n \times n$. In [8, Sec. 7, Fig. 3] Dworkin gives a statistic with a rule that is the transpose of the statistic in Definition 2.7. However, this statistic actually yields our q-hit numbers $H_j^n(\lambda)$ (this can be seen from the change of basis to q-rook numbers and because the latter stay invariant under conjugation) and *not* the Garsia–Remmel q-hit numbers $\widetilde{H}_j^n(\lambda)$ as claimed in [8]. In [16, Fig. 5], Haglund and Remmel give a statistic similar to Dworkin that they attribute to him (see Definition A.2) that *does* yield the Garsia–Remmel q-hit numbers. The authors in [16] give a weight-preserving bijection between their version of Dworkin's statistic and Haglund's statistic for $\widetilde{H}_j^n(\lambda)$ from [15] thus proving the validity of their version of Dworkin's statistic.

See Appendices A and B for more details on the Garsia–Remmel q-hit numbers and their relation to our q-hit numbers. And see Appendix C for a proof of Theorem 2.9 which shows the statistic in Definition 2.7 that yields the q-hit numbers $H_{m,n}^j(\lambda)$.

For a recent q-analogue of hit numbers for general boards, not just *Ferrers boards*, see [21].

7.2. A conjectured deletion–contraction relation for q-hit numbers

We use the deletion–contraction of q-hit numbers (Lemma 2.6) in our proof of Theorem 1.2. It appears that the q-hit numbers satisfy a similar deletion–contraction relation with simpler powers of q. For more details in another deletion–contraction relation of q-hit numbers see Appendix E.

Conjecture 7.1. *Let λ be a partition inside an $n \times m$ board and e be an outer corner of λ. Then we have the following recursion: $H_j^{m,n}(\varnothing) = [m]_n\, \delta_{j,0}$ and*

$$qH_j^{m,n}(\lambda) = H_j^{m,n}(\lambda \backslash e) + q^m H_{j-1}^{m-1,n-1}(\lambda/e) - H_j^{m-1,n-1}(\lambda/e).$$

7.3. Combinatorial proof of Theorem 1.3

Guay-Paquet's proof of Theorem 1.3 sketched in [12] is based on the idea of dual basis from linear algebra. He shows that the vector space $V^{m,n}$, together with the basis $\mathcal{R}^{m,n}$, has a dual vector space $V_{m,n}^* := \operatorname{span}_{\mathbb{Q}(q)}\big(P(x;\lambda)/[m]_n \mid \lambda \subset n \times m\big)$ with dual basis $\{x^i \mid i = 0, \ldots, n\}$. Now, the dual basis coefficients are given by the normalized q-hit numbers $H_i^{m,n}(\lambda)/[m]_n$ as shown (up to normalization) in (2.1).

In contrast, our proof of Theorem 1.3 uses q-rook theory, it would be interesting to find a bijective proof of this result relating colorings with rook placements.

Specifically, we can rewrite the identity as

$$[m]_n X_\lambda(\mathbf{x}, q) = \sum_{k=0}^{n} H_k^{m,n}(\lambda) X_{m^k}(\mathbf{x}, q).$$

Matching monomials in $\mathbf{x}$, powers of q, and interpreting $[m]_n = H_0^{m,n}(\emptyset)$ we are looking for a bijection φ_ν for every $\nu = (2, \ldots, 2, 1, \ldots)$ as follows.

From:

Pairs of a rook placement $P_0^{m,n}(\emptyset)$ of n rooks on $n \times m$ board with $\operatorname{inv}(P)$ inversions and a proper coloring $\kappa(\lambda, \nu)$ of G_λ of type ν and $\operatorname{asc}(\kappa)$ total ascents.

Into:

Triples $k, P_k^{m,n}(\lambda), \kappa(m^k, \nu)$ consisting of an integer $k \in [0, n]$, a rook placement on $n \times m$ with exactly k rooks inside λ, and a proper coloring of G_{m^k} of type ν,

such that

$$\mathrm{asc}(\kappa(m^k, \nu)) + \mathrm{inv}(P_k^{m,n}) = \mathrm{asc}(\kappa) + \mathrm{inv}(P).$$

7.4. Combinatorial proof of Theorem 1.2

There are other rules for the elementary basis expansion of $X_\lambda(\mathbf{x}, q)$. In particular, Cho–Huh [7] give an expansion in terms of *P-tableaux* of shape $2^j 1^{m+n-2j}$ such that there is no $s \geq j+2$ such that $(a_{i,1}, a_{s,1}) \in \lambda$ for all $i \in \{\ell+1, \dots, s-1\}$ (see [24, Sec. 6] for details on P-tableaux). We denote such set of P-tableaux by $\mathcal{T}'((m+n-j, j))$. Let

$$c_j^{m,n}(q) := \sum_{T \in \mathcal{T}'(m+n-j,j)} q^{\mathrm{inv}_{G(\lambda)}(T)}.$$

For the definitions of P-tableau and $\mathrm{inv}_G(T)$ see [24, Sec. 6].

It would be interesting to find a weight-preserving bijection that shows that

$$c_j^{m,n}(q) = \begin{cases} [j]!H_j^{m+n-j}(\lambda) & \text{if } j = \ell(\lambda), \\ q^j[j]![m+n-2j]H_j^{m+n-j-1}(\lambda) & \text{if } j < \ell(\lambda). \end{cases}$$

For instance, for the case $j = \ell(\lambda)$, we need to establish a weight-preserving bijection between the rook placements with j rooks inside λ, together with some labeling of the ones inside λ to account for $[j]!$, and $\mathcal{T}'((m+n-j, j))$.

7.5. Extending Theorems 1.2 and 1.3 to bicolored graphs

Both Theorem 1.2 and Theorem 1.3 are q-analogues of a special cases of respective results by Stanley–Stembridge [26, Thm. 4.3] and by Guay-Paquet [13, Prop. 4.1 (iv)] for *bicolored graphs*. It would be interesting to find a q-analogue of these more general results for bicolored graphs G. However, for such graphs G the function $X_G(\mathbf{x}, q)$ may not be symmetric.

7.6. Beyond the abelian case

We have studied the chromatic symmetric function $X_{G(d)}(\mathbf{x}, q)$ for Dyck paths d of bounce two, aka the abelian case [18]. Recently Cho–Hong [6] verified Conjecture 1.1 when $q = 1$ for Dyck paths of bounce three. Their expansion is in terms of certain P-tableaux. For these Dyck paths, it would be interesting to find an e-expansion involving q-rook theory or extending Theorem 1.3.

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Appendix

In this appendix we give more details on the two variants of the statistic on the Garsia–Remmel q-hit numbers, their relation, and their deletion–contraction relations.

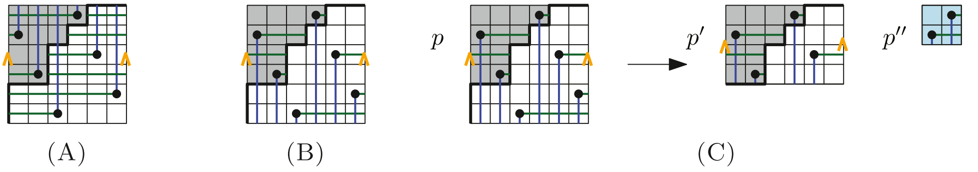

Fig. 7. (A) For the same rook placement p, examples of (A) the statistic $\mathrm{stat}_{\mathrm{D}}(p)$ for the Garsia–Remmel q-hit numbers, and (B) of the statistic $\mathrm{stat}(p)$ for our q-hit numbers. Note how the empty cells in the calculation of one statistic correspond to the double crossings in the other. Moreover in each placement $\mathrm{stat}(p)-\mathrm{cross}(p)=j\cdot m-|\lambda|$. (C) Illustration of bijection between rook placement counted in $H_j^{m,m}(\lambda)$ and rook placements counted in $H_j^{m,n}(\lambda)$ and $R_{m-n}((m-n)^{m-n})$, respectively.

Appendix A. Garsia-Remmel q-hit numbers and Dworkin's statistic

We start by defining the original version of the q-hit numbers given by Garsia–Remmel that is different than our q-hit numbers. Recall that $m \geq n$.

Definition A.1 (*[11]*)**.** For λ inside an $n \times n$ board, we define the Garsia–Remmel *q-hit polynomial* of λ by

$$\sum_{i=0}^{n} \widetilde{H}_i^{n,n}(\lambda)x^i := \sum_{i=0}^{n} R_i(\lambda)\,[n-i]! \prod_{k=n-i+1}^{n} (x-q^k). \tag{A.1}$$

Garsia and Remmel [11, Theorem 2.1] showed that $\widetilde{H}_i^{n}(\lambda) := \widetilde{H}_i^{n,n}(\lambda)$ is a polynomial with nonnegative coefficients and Dworkin [8] and Haglund [15] gave different statistics realizing these q-hit numbers. We focus on Dworkin's statistic (as presented in [16], see Section 7.1) since it is very similar to the statistic in Definition 2.7.

Definition A.2 (*Dworkin's Statistic for the q-hit Numbers [16]*)**.** Let λ be a partition inside an $n \times m$ board. Given a placement p of n non-attacking rooks on an $n \times n$ board, with exactly j inside λ, let $\mathrm{stat}_{\mathrm{D}}(p)$ be the number of cells c in the $n \times m$ board such that

(i) there is no rook in c,
(ii) there is no rook below c on the same column, and either,
(iii) if c is in λ then the rook on the same row of c is either outside λ or else to the left of c; or
(iv) if c is not in λ then the rook on same row of c is not in λ and to the left of c.

Example A.3. Consider the partition $\lambda = (4,3,2,2)$ inside a 6×6 board. In Fig. 7(a), we present a rook placement p of six rooks on the 6×6 board with three hits on λ where $\mathrm{stat}_{\mathrm{D}}(p) = 4$.

Theorem A.4 (Dworkin *[8,16]*)**.** *Let λ be a partition inside an $n \times n$ board and $j = 0, \dots, n$ then*

$$\widetilde{H}_j^{n}(\lambda) = \sum_{p} q^{\mathrm{stat}_{\mathrm{D}}(p)},$$

where the sum is over all placements p of n non-attacking rooks on an $n \times n$ board with exactly j rooks inside λ.

Appendix B. Relation between q-hit numbers $H_j^n(\lambda)$ and $\widetilde{H}_j^n(\lambda)$

The next result shows that our q-hit numbers and the Garsia–Remmel q-hit numbers are off by a power of q. We show this from the respective definitions of each q-hit numbers from the generating polynomials of q-rook numbers. We will ultimately show Theorem 2.9 by showing that for a rook placement p, the statistics $\mathrm{stat}(p)$ and $\mathrm{stat}_{\mathrm{D}}(p)$ are off by the same power (see Lemma C.2).

Proposition B.1.

$$\widetilde{H}_j^m(\lambda) := q^{|\lambda|-jm} H_j^m(\lambda). \tag{B.1}$$

Proof. We compare (2.1) and (A.1). Substituting xq^{-m} and multiplying by $q^{|\lambda|}$ in (2.1), rewriting the factor $q^{mi-\binom{i}{2}} = q^{\sum_{k=0}^{i-1} m-k}$ and rewriting the q-Pochhammer symbol we have

$$\begin{aligned}\sum_{j=0}^{m} H_j^m(\lambda) q^{|\lambda|}(xq^{-m})^j &= \sum_{i=0}^{m} R_i(\lambda)[m-i]!(-1)^i \prod_{k=0}^{i-1} q^{m-k}(1-xq^{-m}q^k)\\ &= \sum_{i=0}^{m} R_i(\lambda)[m-i]! \prod_{k=0}^{i-1}(x-q^{m-k}),\end{aligned}$$

which is the RHS of (A.1) for $m = n$. Comparing coefficients at x^j we obtain the desired identity. □

Appendix C. Proof of Theorem 2.9

Let $\widehat{H}_j^{m,n}(\lambda)$ be the sum in the RHS of (2.5). The next result is an analogue of Lemma 2.13 for $\widehat{H}_j^{m,n}(\lambda)$.

Lemma C.1. *Let λ be a partition inside an $n \times m$ board. Then*

$$\widehat{H}_j^{m,n}(\lambda) = \frac{1}{[m-n]!} \widehat{H}_j^{m,m}(\lambda).$$

Proof. We claim that

$$\widehat{H}_j^{m,n}(\lambda) = \widehat{H}_j^{m,n}(\lambda) \cdot R_{m-n}((m-n)^{m-n}). \tag{C.1}$$

The result then follows since by Proposition 2.2, $R_{m-n}((m-n)^{m-n}) = [m-n]!$. This q-factorial corresponds to the q^{stat}-weighted enumeration of rook placements in a $(m-n)\times(m-n)$ board. Let p be a rook placement on an $m \times m$ board with j rooks inside $\lambda \subset m \times n$ and p' be the placement obtained by restricting p to the top m rows. Then the bottom $m-n$ rows contain $m-n$ rooks outside λ and after removing the n columns occupied by rooks from the top n column we obtain a placement p'' of $m-n$ rooks on an $(m-n)\times(m-n)$ board. This gives a bijection $p \mapsto (p', p'')$ between the rook placements on the LHS and pairs of rook placements from the RHS of (C.1). See Fig. 7(c) Moreover, the bijection is weight-preserving. That is

$$\mathrm{stat}(p) = \mathrm{stat}(p') + \mathrm{stat}(p'') = \mathrm{stat}(p') + \mathrm{inv}(p''),$$

where $\mathrm{inv}(p'')$ is the statistic of the q-rook numbers. This weight-preserving bijection gives the desired result. □

The next lemma shows that for a fixed rook placement the statistics $\mathrm{stat}(\cdot)$ and $\mathrm{stat}_{\mathrm{D}}(\cdot)$ are related.

Lemma C.2. *Let λ be a partition inside an $m \times m$ board. Given a placement p of m non-attacking rooks on an $m \times m$ board with j rooks inside λ then*

$$\mathrm{stat}(p) - \mathrm{stat}_{\mathrm{D}}(p) = j \cdot m - |\lambda|.$$

Example C.3. Consider the partition $\lambda = (4,3,2,2)$ inside a 6×6 board. Fig. 7(b) illustrates an example of a placement p of six rooks on the 6×6 board with $j = 3$ hits on λ with $\mathrm{stat}(p) = 11$. Fig. 7(a) illustrates for the same rook placement p that $\mathrm{stat}_{\mathrm{D}}(p) = 4$. Note that

$$\mathrm{stat}(p) - \mathrm{stat}_{\mathrm{D}}(p) = 7 = 3 \cdot 6 - |\lambda|.$$

The proof of Lemma C.2 is postponed to the end of the section. We now use this lemma to complete the proof of our main result of this appendix.

Proof of Theorem 2.9. By Lemma C.2 we have that

$$\widetilde{H}_j^{m,m}(\lambda) = q^{|\lambda|-jm}\widehat{H}_j^{m,m}(\lambda).$$

Next, by comparing this identity with Proposition B.1 we conclude that $H_j^{m,m}(\lambda) = \widehat{H}_j^{m,m}(\lambda)$. Finally, combining this with both Lemmas 2.13 and C.1 we conclude that $H_j^{m,n}(\lambda) = \widehat{H}_j^{m,n}(\lambda)$ as desired. □

The rest of the section is devoted to the proof of Lemma C.2. We need the following definition.

Definition C.4 (*Crossing Statistic for the q-hit Numbers*)**.** Let λ be a partition inside an $m \times m$ board. Given a placements p of m non-attacking rooks on an $n \times m$ board, with exactly j rooks inside λ, let $\mathrm{cross}(p)$ be the number of cells c in the $m \times m$ board such that

(i) there is no rook in c,
(ii) there is a rook on the same column and below c,
(ii) if c is in λ then there is a rook on the same row in λ to the right of c,
(iii) if c is not in λ then either there is a rook on the same row in λ or a rook on the same row to the right of c.

In other words, $\mathrm{cross}(p)$ is the number of cells that have double crossings after the rook cancellations used to obtain $\mathrm{stat}(p)$. See Remark 2.8 and Fig. 7(b).

Example C.5. For the rook placement p in Example C.3 and Fig. 7(b) we have that $\mathrm{stat}(p) = 11$ and $\mathrm{cross}(p) = 4$.

First observe that for a rook placement p on the $m \times m$ board we have that $\mathrm{stat}_{\mathrm{D}}(p) = \mathrm{cross}(p)$, since the rays in stat and $\mathrm{stat}_{\mathrm{D}}$ are complementary to each other and the crossings in one directly correspond to the empty boxes in the other. Therefore Lemma C.2 follows from the next lemma.

Lemma C.6. *Let λ be a partition inside an $m \times m$ Given a placement p of m non-attacking rooks on an $m \times m$ board with j rooks inside λ then*

$$\mathrm{stat}(p) - \mathrm{cross}(p) = j \cdot m - |\lambda|. \tag{C.2}$$

Proof. We proceed by induction on $|\lambda|$ for $\lambda \subset m \times m$. When $\lambda = \emptyset$ we only have rook placements for $j = 0$, and then it is clear that,

$$\mathrm{stat}(p) = \mathrm{stat}_{\mathrm{D}}(p) = \mathrm{cross}(p) = \mathrm{inv}(w),$$

the number of inversions of the permutation w corresponding to the rook diagram. Thus the identity (C.2) is verified.

Suppose the identity (C.2) holds for all $|\lambda| \leq N$ and then for any $j = 0, \ldots, m$. Let ν be a partition of $N+1$ and $\nu = \lambda + e$, where e is a corner cell. Let p be a rook configuration with j rooks in ν, and let p' be the same rook configuration on λ (so there are j or $j-1$ rooks in λ). Note that cell e cannot be empty since there is a rook in its row, which is either in ν, and hence the rook's "arm" crosses e or is outside in which case the wrap-around also crosses e. We now consider several cases:

Case 1. Cell e has a rook and hence the horizontal arm stops at e as the border of ν. Thus there are no crossings in the row of e. Suppose there are k empty boxes in the row of e, then there are also $m-k-1$ vertical crossings in this row. Now consider p' as a configuration with $j-1$ rooks in λ. The rook in e is outside λ and this time the entire row is crossed, so all empty cells have now a horizontal line and all vertically crossed cells have now a double crossing. Thus

$$\mathrm{stat}(p') - \mathrm{cross}(p') = \mathrm{stat}(p) - k - (\mathrm{cross}(p) + m - k - 1) = \mathrm{stat}(p) - \mathrm{cross}(p) - m + 1.$$

By induction we have

$$\mathrm{stat}(p') - \mathrm{cross}(p') = m(j-1) - |\lambda| = j\dot{m} - |\nu| - m + 1,$$

and matching sides we obtain the desired identity in this case.

Case 2. The rook in the row of e is to the left of e. Then either e is a double crossing or is only a horizontal crossing. Then in the rook placement p' there is no horizontal line crossing e. If e was a double crossing in p, then it is neither a double crossing nor empty cell in p', and if e was not a double crossing in p then it became an empty cell in p'. In both cases we have

$$\mathrm{stat}(p') - \mathrm{cross}(p') = \mathrm{stat}(p) - \mathrm{cross}(p) + 1.$$

Since the number of rooks inside λ is still j then we have

$$\mathrm{stat}(p) - \mathrm{cross}(p) = \mathrm{stat}(p') - \mathrm{cross}(p') - 1 = j \cdot m - |\lambda| - 1 = j \cdot m - |\nu|.$$

This gives the desired identity in this case.

Case 3. The rook in the row of e is to right of e, so outside ν. Then again there is a horizontal line crossing e, so e is either a double crossing in p or neither a double crossing nor an empty cell in p. In both cases when we remove e from ν we either turn the double crossing on e in p to a not a double crossing in p' or from not a double crossing in e to an empty cell in p'. Thus, again

$$\mathrm{stat}(p') - \mathrm{cross}(p') = \mathrm{stat}(p) - \mathrm{cross}(p) + 1.$$

Since the number of rooks inside λ is still j then we have

$$\mathrm{stat}(p) - \mathrm{cross}(p) = \mathrm{stat}(p') - \mathrm{cross}(p') - 1 = j \cdot m - |\lambda| - 1 = j \cdot m - |\nu|.$$

This gives the desired identity in this case.

This completes the proof. □

Appendix D. Symmetry of q-hit numbers of rectangular boards

Since the Garsia–Remmel q-hit numbers are symmetric polynomials in $\mathbb{N}[q]$ [8,11,15], then so are $H_j^{m,n}(\lambda)$.

Corollary D.1. *The q-hit numbers $H_j^{m,n}(\lambda)$ are symmetric polynomials in $\mathbb{N}[q]$.*

Proof. By Theorem 2.9 we have that $H_j^{m,n}(\lambda)$ are in $\mathbb{N}[q]$. By Lemma 2.13 and Proposition B.1 we have that

$$H_j^{m,n}(\lambda) = \frac{1}{[m-n]!} q^{|jm-\lambda|} \widetilde{H}_j^{m}(\lambda).$$

Now $[m-n]!$ is a symmetric polynomial in $\mathbb{N}[q]$ and so are the Garsia–Remmel q-hit numbers $\widetilde{H}_j^{m}(\lambda)$ [15, Sec. 5]. Therefore, the result follows. □

Appendix E. Deletion–contraction for q-hit numbers

In this section we give a proof of the deletion–contraction relations for the q-hit numbers $H_j^{m,n}(\lambda)$ and $\widetilde{H}_j^{m,n}(\lambda)$.

Lemma E.1 ([8, Thm. 6.11]). *Let λ be a partition inside an $n \times m$ board and e be an outer corner of λ. Then we have the following recursion:*

$$\widetilde{H}_j^{m,n}(\lambda) = q\widetilde{H}_j^{m,n}(\lambda \backslash e) + \widetilde{H}_{j-1}^{m-1,n-1}(\lambda/e) - q^m \widetilde{H}_j^{m-1,n-1}(\lambda/e), \qquad \widetilde{H}_j^{m,n}(\varnothing) = [m]_n\, \delta_{j,0}.$$

Proof. This follows from the deletion–contraction relation for q-rook numbers [8, Thm. 6.10]

$$R_i(\lambda) = q \cdot R_i(\lambda\backslash e) + R_{i-1}(\lambda/e), \qquad R_i(\varnothing) = \delta_{i,0},$$

which follows directly from considering if a placement p of i rooks in λ has or not a rook in cell e. Substituting this rook recursion in (2.2), we obtain

$$\begin{aligned} H_k^{m,n}(\lambda) &= \frac{q^{\binom{k}{2}-|\lambda|}}{[m-n]!}\sum_{i=k}^{n}(q\cdot R_i(\lambda\backslash e)+R_{i-1}(\lambda/e))\,[m-i]!\begin{bmatrix} i\\ k\end{bmatrix}(-1)^{i+k}q^{mi-\binom{i}{2}} \\ &= \frac{q^{\binom{k}{2}-|\lambda\backslash e|}}{[m-n]!}\sum_{i=k}^{n}R_i(\lambda\backslash e)\,[m-i]!\begin{bmatrix} i\\ k\end{bmatrix}(-1)^{i+k}q^{mi-\binom{i}{2}} \\ &\quad + \frac{q^{\binom{k}{2}-|\lambda|}}{[m-n]!}\sum_{i=k}^{n}R_{i-1}(\lambda/e)\,[m-i]!\begin{bmatrix} i\\ k\end{bmatrix}(-1)^{i+k}q^{mi-\binom{i}{2}}. \end{aligned}$$

Manipulating the last expression from q-rook numbers into q-hit numbers, we obtain the following recurrence

$$H_k^{m,n}(\lambda) = H_k^{m,n}(\lambda\backslash e) + q^{m+k-1-|\lambda|+|\lambda/e|}H_{k-1}^{m-1,n-1}(\lambda/e) - q^{k+m-|\lambda|+|\lambda/e|}H_k^{m-1,n-1}(\lambda/e)$$

Now, we use (B.1) to translate this recursion into the recursion for the $\widetilde{H}$'s:

$$\begin{aligned} &q^{km-|\lambda|}\widetilde{H}_k^{m,n}(\lambda) \\ &\quad = q^{km-|\lambda\backslash e|}\widetilde{H}_k^{m,n}(\lambda\backslash e) + q^{(k-1)(m-1)-|\lambda/e|}q^{m+k-1-|\lambda|+|\lambda/e|}\widetilde{H}_{k-1}^{m-1,n-1}(\lambda/e) \\ &\quad - q^{k(m-1)-|\lambda/e|}q^{k+m-|\lambda|+|\lambda/e|}H_k^{m-1,n-1}(\lambda/e), \end{aligned}$$

which simplifies to the desired recursion. □

Proof of Lemma 2.6. Combining together (B.1) and Lemma E.1, we obtain

$$\begin{aligned} q^{|\lambda|-jm}H_j^{m,n}(\nu) &= q^{|\lambda\backslash e|-jm+1}H_j^{m,n}(\lambda\backslash e) \\ &\quad + q^{|\lambda/e|-(j-1)(m-1)}H_{j-1}^{m-1,n-1}(\lambda/e) - q^{|\lambda/e|-j(m-1)+m}\widetilde{H}_j^{m-1,n-1}(\lambda/e). \end{aligned}$$

Noticing that $|\lambda\backslash e|+1=|\lambda|$ and simplifying the expression we obtain that

$$\begin{aligned} H_j^{m,n}(\nu) &= H_j^{m,n}(\lambda\backslash e) + q^{|\lambda/e|-|\lambda|+j+m-1}H_{j-1}^{m-1,n-1}(\lambda/e) - q^{|\lambda/e|-|\lambda|+j+m}H_j^{m-1,n-1}(\lambda/e) \\ &= H_j^{m,n}(\lambda\backslash e) + q^{|\lambda/e|-|\lambda|+j+m-1}\left(H_{j-1}^{m-1,n-1}(\lambda/e) - qH_j^{m-1,n-1}(\lambda/e)\right). \quad \square \quad \square \end{aligned}$$

The previous deletion–contraction relation specializes to square boards as follows.

Corollary E.2. $H_j^{N}(\lambda) = H_j^{N}(\lambda\backslash e) + q^{|\lambda/e|-|\lambda|+j+N-1}\left[H_{j-1}^{N-1}(\lambda/e) - qH_j^{N-1}(\lambda/e)\right].$

Conjecture E.3. *Let λ be a partition inside an $n\times m$ board and e be an outer corner of λ, then we have: $P(x;\varnothing)=[m]_n$, and*

$$P(x;\lambda) = qP(x;\lambda\backslash e) + (xq^m-1)P(x;\lambda/e).$$

Appendix F. Another proof of Lemma 5.5

In this section we include the proof of (5.7) using q-binomials.

Proof (*Proof of* (5.7) *in Lemma* 5.5)**.** We use Eq. (2.2) to rewrite both the LHS and RHS in terms of q-rook numbers $R_i(\lambda)$. By q-manipulations, we have that (5.7) is equivalent to

$$\begin{bmatrix} m+n-r-i-1\\ n-r-1\end{bmatrix} = q^{(n-r-1)(r-i)}\cdot\left(\sum_{j=r}^{i}\begin{bmatrix} m+n-r-j-1\\ n-j-1\end{bmatrix}\begin{bmatrix} i-r\\ j-r\end{bmatrix}(-1)^{j-r}q^{\binom{j-r}{2}}\right). \quad (\text{F.1})$$

Now, to prove this q-binomial identity, we consider the following q-binomial identities:

$$\prod_{k=0}^{i-r-1}(1-q^k t)=\sum_{k=0}^{i-r} q^{\binom{k}{2}}\begin{bmatrix} i-r \\ k\end{bmatrix}(-1)^k t^k,$$
$$\prod_{k=0}^{m-r-1}\frac{1}{(1-q^k t)}=\sum_{k=0}^{\infty}\begin{bmatrix} m-r+k-1 \\ k\end{bmatrix} t^k.$$

We have that

$$\prod_{k=0}^{i-r-1}(1-q^k(q^a t))\prod_{k=0}^{m-r-1}\frac{1}{(1-q^k(q^b t))}=$$
$$\left(\sum_{j-r=0}^{i-r} q^{\binom{j-r}{2}}\begin{bmatrix} i-r \\ j-r\end{bmatrix}(-1)^{j-r}q^{a(j-r)}t^{j-r}\right)\cdot\left(\sum_{k=0}^{\infty}\begin{bmatrix} m-r+k-1 \\ k\end{bmatrix}q^{bk}t^k\right).$$

Taking the coefficient at t^{n-r-1} at the RHS we get

$$\sum_{j-r=0}^{i-r} q^{\binom{j-r}{2}}\begin{bmatrix} i-r \\ j-r\end{bmatrix}\begin{bmatrix} m-r+n-j-1 \\ n-j-1\end{bmatrix}q^{b(n-j-1)}t^{n-j-1}(-1)^{j-r}q^{a(j-r)}t^{j-r},$$
$$=\sum_{j=r}^{i}(-1)^{j-r}\begin{bmatrix} i-r \\ j-r\end{bmatrix}\begin{bmatrix} m-r+n-j-1 \\ n-j\end{bmatrix}q^{\binom{j-r}{2}+b(n-j-1)+a(j-r)}.$$

Setting $a=b=r-i$ and denoting by $[t^\ell]@P$ the coefficient of t^ℓ at P, we have that

$$\begin{bmatrix} m-i+n-r-1 \\ n-r-1\end{bmatrix}=[t^{n-r-1}]@\prod_{k=0}^{m-i-1}\frac{1}{(1-q^k t)}$$
$$=[t^{n-r-1}]@\prod_{k=0}^{i-r-1}(1-q^k(q^{r-i}t))\prod_{k=0}^{m-r-1}\frac{1}{(1-q^k(q^{r-i}t))}$$
$$=[t^{n-r-1}]\sum_{j-r=0}^{i-r} q^{\binom{j}{2}}\begin{bmatrix} i-r \\ j-r\end{bmatrix}\begin{bmatrix} m-r+n-j-1 \\ n-j-1\end{bmatrix}q^{(r-i)(n-j-1)}t^{n-j-1}(-1)^{j-r}q^{(r-i)(j-r)}t^{j-r}$$
$$=\sum_{j=r}^{i}(-1)^{j-r}\begin{bmatrix} i-r \\ j-r\end{bmatrix}\begin{bmatrix} m-r+n-j-1 \\ n-j-1\end{bmatrix}q^{\binom{j-r}{2}+(r-i)(j-r)+(r-i)(n-j-1)},$$

as desired. □

References

[1] A. Abreu, A. Nigro, Chromatic symmetric functions from the modular law, J. Combin. Theory Ser. A 180 (2021) 105407.

[2] P. Alexandersson, G. Panova, LLT polynomials, chromatic quasisymmetric functions and graphs with cycles, Discrete Math. 341 (12) (2018) 3453–3482.

[3] P. Alexandersson, R. Sulzgruber, A combinatorial expansion of vertical-strip LLT polynomials in the basis of elementary symmetric functions, Adv. Math. 400 (2022) 108256.

[4] F. Butler, M. Can, J. Haglund, J.B. Remmel, Rook theory notes, 2014, book project https://www2.math.upenn.edu/~jhaglund/books/rookchap1.pdf.link.

[5] E. Carlsson, A. Mellit, A proof of the shuffle conjecture, J. Amer. Math. Soc. 31 (3) (2018) 661–697.

[6] S. Cho, J. Hong, Positivity of chromatic symmetric functions associated with hessenberg functions of bounce number 3, 2019, https://arxiv.org/abs/1910.07308.

[7] S. Cho, J. Huh, On e-positivity and e-unimodality of chromatic quasisymmetric functions, SIAM J. Discrete Math. 33 (4) (2019) 2286–2315, http://dx.doi.org/10.1137/18M1216201.

[8] M. Dworkin, An interpretation for Garsia and Remmel's q-hit numbers, J. Combin. Theory Ser. A 81 (2) (1998) 149–175.
[9] S. Elizalde, Descents on quasi-stirling permutations, J. Combin. Theory Ser. A 180 (2021) http://dx.doi.org/10.1016/j.jcta.2021.105429, Paper No. 105429, 35.
[10] D. Foata, M.-P. Schützenberger, Théorie Géométrique Des Polynômes Eulériens, Vol. 138, Springer, 2006.
[11] A.M. Garsia, J.B. Remmel, q-counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41 (1986) 246–275.
[12] M. Guay-Paquet, Linear Relations Between q-Chromatic symmetric functions, Non-published, shared by private communication.
[13] M. Guay-Paquet, A modular relation for the chromatic symmetric functions of (3+1)-free posets, 2013, http://arxiv.org/abs/1306.2400.
[14] M. Guay-Paquet, A second proof of the shareshian–wachs conjecture, by way of a new hopf algebra, 2016, https://arxiv.org/abs/1601.05498.
[15] J. Haglund, q-Rook polynomials and matrices over finite fields, Adv. Appl. Math. 20 (4) (1998) 450–487.
[16] J. Haglund, J.B. Remmel, Rook theory for perfect matchings, Adv. Appl. Math. 27 (2–3) (2001) 438–481, Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).
[17] J. Haglund, A.T. Wilson, Macdonald polynomials and chromatic quasisymmetric functions, Electron. J. Combin. 27 (3) (2020) P3.37.
[18] M. Harada, M.E. Precup, The cohomology of abelian Hessenberg varieties and the stanley-stembridge conjecture, Algebr. Comb. 2 (6) (2019) 1059–1108.
[19] A. Lascoux, B. Leclerc, J.-Y. Thibon, Ribbon tableaux, Hall-littlewood functions, quantum affine algebras, and unipotent varieties, J. Math. Phys. 38 (2) (1997) 1041–1068.
[20] S.J. Lee, S.K.Y. Soh, Explicit formulas for e-positivity of chromatic quasisymmetric functions, 2022, https://arxiv.org/abs/2201.13080.
[21] J.B. Lewis, A.H. Morales, Rook theory of the finite general linear group, Exp. Math. (2018) 1–19.
[22] K.E. Morrison, Integer sequences and matrices over finite fields, J. Integer. Seq. 9 (2) (2006) Article 06.2.1, 28.
[23] J. Shareshian, M. Wachs, q-Eulerian polynomials: excedance number and major index, Electron. Res. Announc. Am. Math. Soc. 13 (4) (2007) 33–45.
[24] J. Shareshian, M.L. Wachs, Chromatic quasisymmetric functions, Adv. Math. 295 (2016) 497–551.
[25] R.P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, Adv. Math. 111 (1995) 166–194.
[26] R.P. Stanley, J.R. Stembridge, On immanants of Jacobi-Trudi matrices and permutations with restricted position, J. Combin. Theory Ser. A 62 (2) (1993) 261–279.
[27] The Sage-Combinat community, Sage-combinat: enhancing sage as a toolbox for computer exploration in algebraic combinatorics, 2008, http://combinat.sagemath.org.
[28] The Sage Developers, SageMath, the sage mathematics software system (Version 8.9), 2019, https://www.sagemath.org.