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TRANSPORT MODELS FOR WAVE PROPAGATION IN SCATTERING MEDIA WITH NONLINEAR ABSORPTION*

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Abstract. This work considers the propagation of high-frequency waves in highly scattering media where physical absorption of a nonlinear nature occurs. Using the classical tools of the Wigner transform and multiscale analysis, we derive semilinear radiative transport models for the phase-space intensity and the diffusive limits of such transport models. As an application, we consider an inverse problem for the semilinear transport equation, where we reconstruct the absorption coefficients of the equation from a functional of its solution. We obtain a uniqueness result on the inverse problem.

Key words. wave propagation, nonlinear media, nonlinear absorption, semilinear diffusion equation, semilinear radiative transport equation, inverse problem

MSC codes. 35R30, 35G20, 78A60

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1. Introduction.

The derivation of kinetic models for wave propagation in highly scattering media is a classical subject [32, 15] that has received significant attention in the past two decades due to its importance in many emerging applications [11, 13, 25, 39, 41, 35]. A significant amount of progress has been made on both the mathematical justification of the derivation (such as one based on multiscale analysis of the Wigner transform) [6, 21, 24, 30, 37, 49] and the computational validation of the derived kinetic models [7, 9, 33, 47, 53]; see [1, 2, 4, 5, 12, 23, 25, 28] and references therein for additional investigations in this field. The obtained models for imaging in complex media have also been utilized in many different settings [8, 9, 13, 16].

In this work, we are interested in developing kinetic models when nonlinear absorption occurs during wave propagation [10, 38, 48, 57]. We are mainly motivated by applications where reconstructing the absorption of the underlying medium from internal or boundary observations is of practical interest. Such applications include, for instance, the case of reconstructing the two-photon absorption coefficient of biological tissues with optical or photoacoustic measurements [20, 36, 42, 44, 45, 51, 54, 55, 56].

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Our derivation will be carried out in the frequency domain, where wave propagation is described by the standard Helmholtz equation. The nonlinear absorption mechanism we are interested in is modeled as a zeroth-order perturbation to the second-order Helmholtz differential operator. This is the essential factor that makes it possible for us to perform the standard calculations in this field for our nonlinear problem. Even though this is a mathematically less attractive nonlinearity to study, the derivation does provide us a formal justification of the semilinear radiative transport model (see, for instance (2.20)), used in many applications. We refer interested readers to [22] for the derivation of transport models for a different type of nonlinearity that makes use of a mean-field approximation. Furthermore, we emphasize that the derived nonlinear transport model has a variety of important applications. In fluorescence imaging, it can be used to localize and characterize fluorescent molecules by determining their two-photon or multiphoton absorption properties.

The remainder of this paper is organized as follows. In section 2, we derive the radiative transport model for media with quadratic and higher-order absorption. We then discuss the diffusive limit of the derived transport models in section 3. As an application of the derived model, we study in section 4 an inverse medium problem for our semilinear radiative transport equation. Concluding remarks are offered in section 5.

2. Derivation of the transport equation. For simplicity, we first consider the case of quadratic nonlinear absorption. This could serve as a model of light propagation in media with two-photon absorption [51, 54]. We will then generalize the result to the case of a general polynomial nonlinearity. Let the wave field $p(z,\mathbf{x})$ be the solution to the scalar wave equation in the time-harmonic form, that is,

$$\Delta_{\mathbf{x}} p + \frac{\partial^2 p}{\partial z^2} + k^2 n^2 p = 0, \tag{2.1}$$

where $\Delta_{\mathbf{x}}$ is the transverse Laplacian in $\mathbf{x} \in \mathbb{R}^d$ $(d \geq 1)$, k is the wave number, and $n = n(z,\mathbf{x},p)$ is the refractive index. We assume that the refractive index takes the form

$$n^2 = 1 - 2\sigma V\left(\frac{z}{\ell_z}, \frac{\mathbf{x}}{\ell_{\mathbf{x}}}\right) + ik^{-1}\mu\widetilde{K}(z,\mathbf{x})|p|^2\,, \tag{2.2}$$

where V is a real bounded stationary random field with zero mean, with $\ell_{\mathbf{x}}$ and ℓ_z being the transverse and longitudinal correlation lengths of the random field, respectively. The deterministic function $\widetilde{K}$ is nonnegative and measures the strength of the second-order absorption. The parameters σ and μ are the scaling factors quantifying the amplitudes of the fluctuation and the second-order absorption, respectively. Assuming that the field p possesses a beam-like structure propagating in the z direction, we may write $p(z,\mathbf{x}) = e^{ikz}\psi(z,\mathbf{x})$ with complex amplitude $\psi(z,\mathbf{x})$ satisfying the following equation:

$$\frac{\partial^2 \psi}{\partial z^2} + 2ik\frac{\partial \psi}{\partial z} + \Delta_{\mathbf{x}}\psi + k^2(n^2-1)\psi = 0\,. \tag{2.3}$$

Let $L_{\mathbf{x}}$ and L_z be the characteristic lengths of propagation in the $\mathbf{x}$ and z directions, respectively. We rescale the variables $\mathbf{x} \mapsto L_{\mathbf{x}}\mathbf{x}$ and $z \mapsto L_z z$, and $\mathbf{x}$, z are now $\mathcal{O}(1)$. Then, with the newly defined variables, we may write (n^2-1) as

$$n^2 - 1 = -2\sigma V\left(\frac{L_z z}{\ell_z}, \frac{L_{\mathbf{x}}\mathbf{x}}{\ell_{\mathbf{x}}}\right) + ik^{-1}\mu\widetilde{K}(L_z\mathbf{z}, L_{\mathbf{x}}\mathbf{x})|p|^2\,. \tag{2.4}$$

Equation (2.3) now becomes

$$\frac{1}{L_z^2}\frac{\partial^2\psi}{\partial z^2}+2\frac{ik}{L_z}\frac{\partial\psi}{\partial z}+\frac{1}{L_{\mathbf{x}}^2}\Delta_{\mathbf{x}}\psi-2k^2\sigma V\left(\frac{L_z z}{\ell_z},\frac{L_{\mathbf{x}}\mathbf{x}}{\ell_{\mathbf{x}}}\right)\psi+ik\mu\widetilde{K}(L_z z,L_{\mathbf{x}}\mathbf{x})|\psi|^2\psi=0\,. \tag{2.5}$$

With the small aperture assumption that $L_{\mathbf{x}}\ll L_z$, we can formally approximate the above equation by the paraxial wave equation

$$i\frac{\partial\psi}{\partial z}+\frac{L_z}{2kL_{\mathbf{x}}^2}\Delta_{\mathbf{x}}\psi-kL_z\sigma V\left(\frac{L_z z}{\ell_z},\frac{L_{\mathbf{x}}\mathbf{x}}{\ell_{\mathbf{x}}}\right)\psi+i\frac{L_z}{2}\mu\widetilde{K}(L_z z,L_{\mathbf{x}}\mathbf{x})|\psi|^2\psi=0\,. \tag{2.6}$$

Our derivation works in the regime where the longitudinal propagation distance L_z is much larger than the correlation length ℓ_z and the correlation length is much larger than the wavelength, that is, $L_z\gg\ell_z$ and $\ell_z\gg\lambda:=\frac{2\pi}{k}$. We, therefore, introduce the small parameter ε and assume the scaling relations in the weak-coupling regime:

$$\frac{\ell_{\mathbf{x}}}{L_{\mathbf{x}}}=\frac{\ell_z}{L_z}=\varepsilon\ll1,\quad k\ell_{\mathbf{x}}^2=\ell_z,\quad \sigma=\frac{1}{k\ell_z}\sqrt{\varepsilon},\quad \mu\propto\frac{1}{L_z}. \tag{2.7}$$

Let us denote the rescaled wave field by $\psi_\varepsilon(z,\mathbf{x})$ and take $K(z,\mathbf{x})=L_z\mu\widetilde{K}(L_z z,L_{\mathbf{x}}\mathbf{x})$. Then the paraxial wave equation turns into

$$i\frac{\partial\psi_\varepsilon}{\partial z}+\frac{\varepsilon}{2}\Delta_{\mathbf{x}}\psi_\varepsilon-\frac{1}{\sqrt{\varepsilon}}V\left(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon}\right)\psi_\varepsilon+\frac{i}{2}K(z,\mathbf{x})|\psi_\varepsilon|^2\psi_\varepsilon=0. \tag{2.8}$$

We then take the Wigner transform of ψ_ε:

$$W_\varepsilon(z,\mathbf{x},\mathbf{k})=\int_{\mathbb{R}^d}e^{i\mathbf{k}\cdot\mathbf{y}}\psi_\varepsilon\left(\mathbf{x}-\frac{\varepsilon\mathbf{y}}{2},z\right)\overline{\psi_\varepsilon(\mathbf{x}+\frac{\varepsilon\mathbf{y}}{2},z)}\frac{d\mathbf{y}}{(2\pi)^d}. \tag{2.9}$$

It is then standard to check that $W_\varepsilon(z,\mathbf{x},\mathbf{k})$ satisfies the following Liouville equation:

$$\frac{\partial W_\varepsilon}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}W_\varepsilon+\mathcal{L}_{V,\varepsilon}W_\varepsilon+\mathcal{L}_{K,\varepsilon}W_\varepsilon=0, \tag{2.10}$$

where

$$\begin{aligned}\mathcal{L}_{V,\varepsilon}W_\varepsilon&=\frac{i}{\sqrt{\varepsilon}}\int_{\mathbb{R}^d}e^{-i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\left(W_\varepsilon\left(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\right)-W_\varepsilon\left(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right)\widehat{V}\left(\frac{z}{\varepsilon},\mathbf{p}\right)d\mathbf{p},\\ \mathcal{L}_{K,\varepsilon}W_\varepsilon&=\frac{1}{2}\int_{\mathbb{R}^d}e^{-i\mathbf{p}\cdot x}\left(W_\varepsilon\left(z,\mathbf{x},\mathbf{k}+\frac{\varepsilon\mathbf{p}}{2}\right)+W_\varepsilon\left(z,\mathbf{x},\mathbf{k}-\frac{\varepsilon\mathbf{p}}{2}\right)\right)\widehat{S}_\varepsilon(z,\mathbf{p})d\mathbf{p}\,.\end{aligned} \tag{2.11}$$

Here $S_\varepsilon(z,\mathbf{x})$ is defined as

$$S_\varepsilon(z,\mathbf{x}):=K(z,\mathbf{x})|\psi_\varepsilon(z,\mathbf{x})|^2=K(z,\mathbf{x})\int_{\mathbb{R}^d}W_\varepsilon(z,\mathbf{x},\mathbf{k})d\mathbf{k}\,, \tag{2.12}$$

while $\widehat{V}$ and $\widehat{S}_\varepsilon$ denote the Fourier transform ($\mathbf{x}\to\mathbf{p}$) of V and S_ε, respectively. We use the standard Fourier transform definition

$$\widehat{f}(\mathbf{p})=\int_{\mathbb{R}^d}e^{i\mathbf{p}\cdot\mathbf{x}}f(\mathbf{x})\frac{d\mathbf{x}}{(2\pi)^d}\,.$$

2.1. Multiscale expansion. In order to find the asymptotic limit as $\varepsilon \to 0$, we introduce $\mathbf{y} = \mathbf{x}/\varepsilon$ as the fast variable and denote $W_\varepsilon(z,\mathbf{x},\mathbf{k}) = W_\varepsilon(z,\mathbf{x},\mathbf{y},\mathbf{k})$. Formally, we write W_ε as an asymptotic expansion in ε:

$$W_\varepsilon(z,\mathbf{x},\mathbf{y},\mathbf{k}) = W_0(z,\mathbf{x},\mathbf{y},\mathbf{k}) + \sqrt{\varepsilon}W_1(z,\mathbf{x},\mathbf{y},\mathbf{k}) + \varepsilon W_2(z,\mathbf{x},\mathbf{y},\mathbf{k}) + \cdots .$$

For the linear model, the main theoretical difficulty in the rigorous derivation of the transport equation is to estimate the remainder in the above ansatz [21]. For the two-photon absorption nonlinear model, an additional difficulty is brought about by the nonlinear term $\mathcal{K}_{K,\varepsilon}W_\varepsilon$, which arises at the $\mathcal{O}(1)$ scale. Additional regularity of W_0 is required to obtain a suitable estimate. Using (2.12), we may also expand $S_\varepsilon(z,\mathbf{x}) = S_\varepsilon(z,\mathbf{x},\mathbf{y})$ accordingly as

$$S_\varepsilon(z,\mathbf{x},\mathbf{y}) = S_0(z,\mathbf{x},\mathbf{y}) + \sqrt{\varepsilon}S_1(z,\mathbf{x},\mathbf{y}) + \varepsilon S_2(z,\mathbf{x},\mathbf{y}) + \cdots ,$$

where

$$S_0(z,\mathbf{x},\mathbf{y}) = K(z,\mathbf{x})\int_{\mathbb{R}^d} W_0(z,\mathbf{x},\mathbf{y},\mathbf{k})d\mathbf{k}.$$

We can now plug the transform $\nabla_\mathbf{x} \to \nabla_\mathbf{x} + \frac{1}{\varepsilon}\nabla_\mathbf{y}$ into (2.10) to conclude that the leading- order equation at $\mathcal{O}(\varepsilon^{-1})$ implies $\mathbf{k}\cdot\nabla_\mathbf{y} W_0 = 0$. This is equivalent to

$$W_0(z,\mathbf{x},\mathbf{y},\mathbf{k}) = W_0(z,\mathbf{x},\mathbf{k}) \quad \text{and} \quad S_0(z,\mathbf{x},\mathbf{y}) = S_0(z,\mathbf{x}).$$

For the order of $\mathcal{O}(\varepsilon^{-1/2})$, we have

(2.13)
$$\mathbf{k}\cdot\nabla_\mathbf{y} W_1 + \alpha W_1 + i\int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\mathbf{y}}\left(W_0\left(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\right) - W_0\left(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right)\widehat{V}(\frac{z}{\varepsilon},\mathbf{p})d\mathbf{p} = 0,$$

where $\alpha \to 0^+$. This gives that the Fourier transform of W_1, $\widehat{W_1}$ is

$$\widehat{W_1}(z,\mathbf{x},\mathbf{p},\mathbf{k}) = \frac{(W_0(z,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}) - W_0(z,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}))\widehat{V}(\frac{z}{\varepsilon},\mathbf{p})}{\mathbf{p}\cdot\mathbf{k}+i\alpha}. \tag{2.14}$$

Finally, we derive the equation for $\mathcal{O}(1)$ terms. In order to handle the nonlinearity, we impose some regularity assumptions. Let $s > d+2$, and assume there exist positive constants C_1, C_2, and C_3 such that

$$\begin{aligned} \|W_0(z,\mathbf{x},\cdot)\|_{C^1(\mathbb{R}^d)} + \|W_0(z,\mathbf{x},\cdot)\|_{L^1(\mathbb{R}^d)} &< C_1 \quad \forall (z,\mathbf{x}) \in \mathbb{R}^{d+1}, \\ \left\|\int_{\mathbb{R}^d} W_0(z,\cdot,\mathbf{k})d\mathbf{k}\right\|_{H^s(\mathbb{R}^d)} &< C_2 \quad \forall z \in \mathbb{R}, \\ \|K(z,\cdot)\|_{H^s(\mathbb{R}^d)} &< C_3 \quad \forall z \in \mathbb{R}. \end{aligned}$$

Then we have that

$$W_0\left(z,\mathbf{x},\mathbf{k}-\frac{\varepsilon}{2}\mathbf{p}\right) + W_0\left(z,\mathbf{x},\mathbf{k}+\frac{\varepsilon}{2}\mathbf{p}\right) = 2W_0(z,\mathbf{x},\mathbf{k}) + R(z,\mathbf{x},\mathbf{k}), \tag{2.15}$$

where $|R(z,\mathbf{x},\mathbf{k})| \le C_1\varepsilon|\mathbf{p}|$. Moreover, we have that

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \frac{|\mathbf{p}||\widehat{S}_0(z,\mathbf{p})|d\mathbf{p}}{(2\pi)^d}\right|^2 &\le \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\frac{|\mathbf{p}|^2}{(1+|\mathbf{p}|^2)^{s/2}}d\mathbf{p}\int_{\mathbb{R}^d}(1+|\mathbf{p}|^2)^{\frac{s}{2}}|\widehat{S}_0(z,\mathbf{p})|^2 d\mathbf{p}\\
&= \frac{1}{(2\pi)^{2d}}\left(\int_{\mathbb{R}^d}\frac{|\mathbf{p}|^2}{(1+|\mathbf{p}|^2)^{s/2}}d\mathbf{p}\right)\|S_0(z,\cdot)\|^2_{H^s(\mathbb{R}^d)}\\
&\le C_s\|K(z,\cdot)\|^2_{H^s(\mathbb{R}^d)}\left\|\int_{\mathbb{R}^d} W_0(z,\cdot,\mathbf{k})d\mathbf{k}\right\|^2_{H^s(\mathbb{R}^d)}
\end{aligned} \tag{2.16}$$

is also uniformly bounded, where $C_s>0$ is a constant depending only on s. The last inequality holds since $s>d+2$ implies $s>d/2$. Therefore, the $\mathcal{O}(1)$ term is

$$\begin{aligned}
&\frac{\partial W_0}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}W_0 + \mathbf{k}\cdot\nabla_{\mathbf{y}}W_2\\
&\quad + i\int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\mathbf{y}}\left(W_1\left(z,\mathbf{x},\mathbf{y},\mathbf{k}+\frac{\mathbf{p}}{2}\right) - W_1\left(z,\mathbf{x},\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right)\widehat{V}\left(\frac{z}{\varepsilon},\mathbf{p}\right)d\mathbf{p}\\
&\quad + W_0(z,\mathbf{x},\mathbf{k})S_0(z,\mathbf{x}) = 0\,.
\end{aligned} \tag{2.17}$$

In order to close the equation, we still need to add the orthogonal relation between W_0 and W_2, that is, $\mathbb{E}[\mathbf{k}\cdot\nabla_{\mathbf{y}}W_2]=0$. Hence, we have

$$\begin{aligned}
&\frac{\partial W_0}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}W_0\\
&\quad + \mathbb{E}\left[i\int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\mathbf{y}}\left(W_1\left(z,\mathbf{x},\mathbf{y},\mathbf{k}+\frac{\mathbf{p}}{2}\right) - W_1\left(z,\mathbf{x},\mathbf{y},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right)\widehat{V}\left(\frac{z}{\varepsilon},\mathbf{p}\right)d\mathbf{p}\right]\\
&\quad + W_0(z,\mathbf{x},\mathbf{k})S_0(z,\mathbf{x}) = 0\,.
\end{aligned} \tag{2.18}$$

Let R be the correlation function of V, and assume that the power spectrum satisfies

$$\mathbb{E}[\widehat{V}(z,\mathbf{p})\widehat{V}(z,\mathbf{q})] = \widehat{R}(\mathbf{p})\delta(\mathbf{p}+\mathbf{q})\,. \tag{2.19}$$

Then, as $\varepsilon\to 0^+$, the expectation term in (2.18) converges weakly to

$$\begin{aligned}
&\mathbb{E}\left[i\int_{\mathbb{R}^d} e^{-i\mathbf{p}\cdot\mathbf{x}/\varepsilon}\left(W_1\left(z,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}+\frac{\mathbf{p}}{2}\right) - W_1\left(z,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right)\widehat{V}\left(\frac{z}{\varepsilon},\mathbf{p}\right)d\mathbf{p}\right]\\
&\to 4\pi\int_{\mathbb{R}^d}\widehat{R}(\mathbf{p}-\mathbf{k})[W_0(z,\mathbf{x},\mathbf{k}) - W_0(z,\mathbf{x},\mathbf{p})]\delta(|\mathbf{k}|^2-|\mathbf{p}|^2)d\mathbf{p}\,.
\end{aligned}$$

Therefore, the final radiative transport equation of W_0 is

$$\begin{aligned}
&\frac{\partial W_0}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}}W_0 + 4\pi\int_{\mathbb{R}^d}\widehat{R}(\mathbf{p}-\mathbf{k})[W_0(z,\mathbf{x},\mathbf{k}) - W_0(z,\mathbf{x},\mathbf{p})]\delta(|\mathbf{k}|^2-|\mathbf{p}|^2)d\mathbf{p}\\
&\quad + \left(K(z,\mathbf{x})\int_{\mathbb{R}^d} W_0(z,\mathbf{x},\mathbf{k}')d\mathbf{k}'\right)W_0(z,\mathbf{x},\mathbf{k}) = 0\,.
\end{aligned} \tag{2.20}$$

Physically, the last term on the left-hand side of (2.20) accounts for quadratic absorption, which indicates that the absorption coefficient linearly depends on the angular average of $W_0(\mathbf{x},z,\mathbf{k})$. Although $W_0(z,\mathbf{x},\mathbf{k})$ is not guaranteed to be a nonnegative quantity, the angular average $\int_{\mathbb{R}^d} W_0(z,\mathbf{x},\mathbf{k})d\mathbf{k} = \lim_{\varepsilon\to 0}|\psi_\varepsilon|^2$ is always nonnegative.

2.2. Extension to higher-order absorption. We now extend the previous result to the case of general polynomial absorption. This could be a physical model for the propagation of light in media with multiphoton absorption [14]. The absorption term in the refractive index is modeled by

$$n^2 = 1 - 2\sigma V\left(\frac{z}{\ell_z}, \frac{\mathbf{x}}{\ell_{\mathbf{x}}}\right) + ik^{-1}\mu \sum_{l=0}^{L} \widetilde{K}_l(z,\mathbf{x})|p|^{2l}, \tag{2.21}$$

where $\widetilde{K}_l$ stands for the absorption strength of $(l+1)$th order. Following the same derivation of the paraxial wave equation, we have a new Liouville equation in the form of (2.10) where the only modification is the term

$$S_\varepsilon(\mathbf{x}, z) = \sum_{l=0}^{L} K_l(z,\mathbf{x})|\psi_\varepsilon(z,\mathbf{x})|^{2l}, \quad K_l(z,\mathbf{x}) := \mu L_z \widetilde{K}_l(L_z z, L_{\mathbf{x}}\mathbf{x})\,. \tag{2.22}$$

Assuming that K_l and W_0 are sufficiently regular, we can follow the same procedure and obtain the radiative transport equation for W_0 in the setting of the following polynomial absorption:

(2.23)
$$\begin{aligned}\frac{\partial W_0}{\partial z} &+ \mathbf{k}\cdot\nabla_{\mathbf{x}} W_0 + 4\pi \int_{\mathbb{R}^d} \widehat{R}(\mathbf{p}-\mathbf{k})[W_0(z,\mathbf{x},\mathbf{k}) - W_0(z,\mathbf{x},\mathbf{p})]\delta(|\mathbf{k}|^2 - |\mathbf{p}|^2)d\mathbf{p} \\ &+ \left(\sum_{l=0}^{L} K_l(z,\mathbf{x})\left[\int_{\mathbb{R}^d} W_0(z,\mathbf{x},\mathbf{k}')d\mathbf{k}'\right]^l\right) W_0(z,\mathbf{x},\mathbf{k}) = 0\,.\end{aligned}$$

2.3. The nonlinear radiative transport equation in the standard form. For a monochromatic solution $W_0(z,\mathbf{x},\mathbf{k})$ which is supported on $\|\mathbf{k}\| = 1$, the above radiative transport equation becomes

$$\begin{aligned}\frac{\partial W_0}{\partial z} &+ \mathbf{k}\cdot\nabla_{\mathbf{x}} W_0 + 4\pi \int_{\mathbb{S}^{d-1}} \widehat{R}(\mathbf{p}-\mathbf{k})[W_0(z,\mathbf{x},\mathbf{k}) - W_0(z,\mathbf{x},\mathbf{p})]d\mathbf{p} \\ &+ \left(\sum_{l=0}^{L} K_l(z,\mathbf{x})\left[\int_{\mathbb{S}^{d-1}} W_0(z,\mathbf{x},\mathbf{k}')d\mathbf{k}'\right]^l\right) W_0(z,\mathbf{x},\mathbf{k}) = 0\,,\end{aligned} \tag{2.24}$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$. It can further be put in the standard form

$$\frac{\partial W_0}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}} W_0 + \Sigma_a(\langle W_0\rangle)W_0 + \Sigma_s(W_0 - \mathcal{K}W_0) = 0. \tag{2.25}$$

Here the total energy of the field at $\mathbf{x}$ is denoted by

$$\langle W_0\rangle := \int_{\mathbb{S}^{d-1}} W_0(z,\mathbf{x},\mathbf{k})d\mathbf{k}$$

and

$$\Sigma_a(\langle W_0\rangle) = \sum_{l=0}^{L} \Sigma_{a,l}\langle W_0\rangle^l \tag{2.26}$$

is the effective absorption coefficient with $\Sigma_{a,l}(z,\mathbf{x}) = K_l(z,\mathbf{x})$ being the absorption coefficient of $(l+1)$th order. The scattering coefficient is $\Sigma_s(\mathbf{k}) := 4\pi\int_{\mathbb{S}^{d-1}} \widehat{R}(\mathbf{p}-\mathbf{k})d\mathbf{p}$, and the scattering operator $\mathcal{K}$ is defined as

$$\mathcal{K}u(z,\mathbf{x},\mathbf{k}) := \int_{\mathbb{S}^{d-1}} p(\mathbf{k},\mathbf{k}')u(z,\mathbf{x},\mathbf{k}')d\mathbf{k}'\,, \tag{2.27}$$

where the scattering phase function

$$p(\mathbf{k},\mathbf{k}') = \frac{4\pi\widehat{R}(\mathbf{k}-\mathbf{k}')}{\Sigma_s(\mathbf{k})}.$$

For the problem to be physical, the absorption strengths K_l $(0 \le l \le L)$ should all be nonnegative.

3. The diffusion limit.

We now study the diffusion limit of the transport equation for $W_0(z,\mathbf{x},\mathbf{k})$ with general polynomial nonlinear absorption. We assume that the physical domain Ω is bounded and convex with smooth boundary $\partial\Omega$. We focus on the following nonlinear radiative transport equation:

$$\begin{aligned} &\frac{\partial W_0}{\partial z} + \mathbf{k}\cdot\nabla_{\mathbf{x}} W_0 + \Sigma_a(\langle W_0\rangle)W_0 + \Sigma_s(W_0 - \mathcal{K}W_0) = 0 && \text{in } X,\\ &W_0(z,\mathbf{x},\mathbf{k}) = 0 && \text{on } \Gamma_-,\\ &W_0(0,\mathbf{x},\mathbf{k}) = f(\mathbf{x}) && \text{on } X_0, \end{aligned} \tag{3.1}$$

where $X = (0,T)\times\Omega\times\mathbb{S}^{d-1}$, $\Gamma_- := \{(z,\mathbf{x},\mathbf{k}) \in (0,T)\times\partial\Omega\times\mathbb{S}^{d-1} \mid \mathbf{n_x}\cdot\mathbf{k} < 0\}$, and $\mathbf{n_x}$ is the unit outward normal vector at $\mathbf{x}\in\partial\Omega$, $X_0 = \Omega\times\mathbb{S}^{d-1}$.

We will focus on power spectra of the form $\widehat{R}(\mathbf{p}-\mathbf{k}) = \widehat{R}(\mathbf{p}\cdot\mathbf{k})$, in which case the scattering coefficient

$$\Sigma_s(\mathbf{k}) \equiv \Sigma_s$$

is a constant, and the scattering phase function is $p(\mathbf{k},\mathbf{k}') = 4\pi\widehat{R}(\mathbf{k}\cdot\mathbf{k}')/\Sigma_s$. Assume the scattering phase function $p(\mathbf{k}\cdot\mathbf{k}')$ is bounded below and above by positive constants $\underline{\theta},\overline{\theta} \ge 0$ such that

$$\underline{\theta} \le p(\mathbf{k},\mathbf{k}') \le \overline{\theta}\,. \tag{3.2}$$

Note that $\underline{\theta}$ satisfies the condition $\nu_{d-1}\underline{\theta} \le 1$, where ν_{d-1} is the measure of the $(d-1)$ dimensional unit sphere. For simplicity, we require that the initial condition $f(\mathbf{x}) \in L^\infty(\Omega\times\mathbb{S}^{d-1})$ is $\mathbf{x}$-dependent and nonnegative. The absorption coefficients obey the condition

$$0 < \underline{\Sigma_{a,l}} \le \Sigma_{a,l}(z,\mathbf{x}) \le \overline{\Sigma_{a,l}} < \infty \tag{3.3}$$

for some constants $\underline{\Sigma_{a,l}}$ and $\overline{\Sigma_{a,l}}$.

We need the following lemma to have a well-posed problem.

Lemma 3.1 (Kellogg [34]). *Let $\mathcal{M}$ be a bounded convex open subset of a real Banach space, and $F : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ is a compact continuous map which is continuously Fréchet differentiable on $\mathcal{M}$. If* (i) *for each $m \in \mathcal{M}$, 1 is not an eigenvalue of $F'(m)$, and* (ii) *for each $m \in \partial\mathcal{M}$, $m \ne F(m)$, then F has a unique fixed point in $\mathcal{M}$.*

We define the space using $dS(\mathbf{x})$ to denote the surface measure of $\partial\Omega$ by

$$\begin{aligned} \mathcal{W}_p = \Big\{ &u \in L^p((0,T)\times\Omega\times\mathbb{S}^{d-1}); \frac{\partial u}{\partial z} + \mathbf{k}\cdot\nabla u \in L^p((0,T)\times\Omega\times\mathbb{S}^{d-1});\\ &u(0,\cdot,\cdot) \in L^p(\Omega\times\mathbb{S}^{d-1}); u|_{\Gamma_-} \in L^p((0,T)\times\Gamma_-, |\mathbf{n_x}\cdot\mathbf{k}|\,dz\,d\mathbf{k}dS(\mathbf{x})) \Big\} \end{aligned}$$

and the equipped norm by

$$\|u\|^p_{\mathcal{W}_p} = \|u\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^{d-1})} + \|\partial_z u + \mathbf{k}\cdot\nabla u\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^{d-1})}.$$

THEOREM 3.2. *Let the initial condition $f(\mathbf{x})$ satisfy*

$$\text{(i)} \qquad \|f\|_\infty \le \frac{1}{\nu_{d-1}} \inf_{(z,\mathbf{x})\in(0,T)\times\Omega} \frac{\Sigma_{a,k-1}}{k\Sigma_{a,k}} \quad \forall 1 \le k \le L$$

or

$$\text{(ii)} \qquad \begin{cases} \nu_{d-1}\Sigma_a'(\|f\|_\infty)\|f\|_\infty \le 2\Sigma_s\underline{\theta} \\ \nu_{d-1}\underline{\theta} \le 1 \end{cases} \qquad \forall (z,\mathbf{x}) \in (0,T)\times\Omega,$$

where Σ_a' is the Fréchet derivative of Σ_a, that is,

$$\Sigma_a'(m) = \sum_{l=1}^{L} \Sigma_{a,l}(z,\mathbf{x}) l m^{l-1}\,.$$

Then (3.1) *admits a unique nonnegative solution in $\mathcal{W}_\infty((0,T)\times\Omega\times\mathbb{S}^{d-1})$.*

Proof. Let $F : m \mapsto \phi$ be the map defined through the relation $\langle\phi\rangle = Fm$, where ϕ solves the transport equation

$$\begin{aligned} \frac{\partial\phi}{\partial z} + \mathbf{k}\cdot\nabla\phi + \Sigma_a(m)\phi + \Sigma_s(\phi - \mathcal{K}\phi) &= 0 && \text{in } X, \\ \phi(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-, \\ \phi(0,\mathbf{x},\mathbf{k}) &= f(\mathbf{x}) && \text{on } X_0\,. \end{aligned} \tag{3.4}$$

We denote by $\mathcal{M}$ the set

$$\mathcal{M} = \{m \in L^\infty((0,T)\times\Omega) \cap L^2((0,T)\times\Omega) \mid 0 \le m \le \|f\|_\infty + \delta\}$$

with $\delta > 0$ being arbitrary. It is straightforward to check that $\mathcal{M}$ is convex, bounded, and closed under the usual L^2 topology. For any $m \in \mathcal{M}$, we have that $\Sigma_a(m)$ is nonnegative. Therefore, by the maximum principle for the linear transport equation (3.4) (see, for instance, [19]), $\langle\phi\rangle \in \mathcal{M}$. This shows that $F : \mathcal{M} \to \mathcal{M}$. To show that F is a continuous operator, we denote by ϕ_1 and ϕ_2 the solutions to (3.4) corresponding to $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}$, respectively. We then introduce $w = \phi_1 - \phi_2$. It is then clear that w solves the following linear transport equation:

$$\begin{aligned} \frac{\partial w}{\partial z} + \mathbf{k}\cdot\nabla w + \Sigma_a(m_1)w + \Sigma_s(w - \mathcal{K}w) &= (\Sigma_a(m_2) - \Sigma_a(m_1))\phi_2 && \text{in } X, \\ w(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-, \\ w(0,\mathbf{x},\mathbf{k}) &= 0 && \text{on } X_0\,. \end{aligned} \tag{3.5}$$

By standard linear theory [19], this equation admit a unique $w \in \mathcal{W}_2$ such that

$$\begin{aligned} \|w\|_{\mathcal{W}_2((0,T)\times\Omega\times\mathbb{S}^{d-1})} &\le C\|(\Sigma_a(m_2) - \Sigma_a(m_1))\phi_2\|_{L^2((0,T)\times\Omega\times\mathbb{S}^{d-1})} \\ &\le C\Sigma_a'(\|f\|_\infty + \delta)\|f\|_\infty\|m_2 - m_1\|_{L^2((0,T)\times\Omega)} \end{aligned} \tag{3.6}$$

for some constant $C > 0$. Using the averaging lemma [27], we obtain that there exists a constant $C' > 0$ such that

$$\begin{aligned} \|Fm_1 - Fm_2\|_{H^{1/2}((0,T)\times\Omega)} = \|\langle w\rangle\|_{H^{1/2}((0,T)\times\Omega)} &\le C'\|w\|_{\mathcal{W}_2((0,T)\times\Omega\times\mathbb{S}^{d-1})} \\ &\le CC'\Sigma_a'(\|f\|_\infty + \delta)\|f\|_\infty\|m_2 - m_1\|_{L^2((0,T)\times\Omega)}. \end{aligned} \tag{3.7}$$

Combining this with the Kondrachov embedding theorem, we have shown that $F : \mathcal{M} \to \mathcal{M}$ is a continuous compact operator. By the Schauder fixed point theorem, there exists a fixed point for $\mathcal{M}$, and hence (3.1) has a nonnegative solution.

To prove the uniqueness of the solution, we use Lemma 3.1. We first observe that for any fixed point m^* of F, it must satisfy the conditions: $m^* \le \|f\|_\infty$ and $m^* > 0$. This is due to the fact that f is strictly positive. Hence, there are no fixed points on the boundary $\partial\mathcal{M}$. Next, we show that $F'(m)$ cannot have 1 as its eigenvalue. Let ϕ be the solution to (3.4) with $m \in \mathcal{M}$ and δm being a perturbation such that $m + \delta m \in \mathcal{M}$, and denote by w the solution to the following equation:

$$
\begin{aligned}
\frac{\partial w}{\partial z} + \mathbf{k}\cdot\nabla w + \Sigma_a(m)w + \Sigma_s(w - \mathcal{K}w) &= -\Sigma_a'(m)\delta m\phi && \text{in } X,\\
w(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-,\\
w(0,\mathbf{x},\mathbf{k}) &= 0 && \text{on } X_0\,.
\end{aligned}
\tag{3.8}
$$

Then the Fréchet derivative $F'(m)[\delta m] = \langle w\rangle$. Suppose $F'(m)$ indeed has 1 as its eigenvalue and $\langle w\rangle$ as the corresponding nonzero eigenfunction. Then $F'(\langle w\rangle) = \langle w\rangle$ and

$$
\begin{aligned}
\frac{\partial w}{\partial z} + \mathbf{k}\cdot\nabla w + \Sigma_a(m)w + \Sigma_s(w - \mathcal{K}w) &= -\Sigma_a'(m)\langle w\rangle\phi && \text{in } X,\\
w(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-,\\
w(0,\mathbf{x},\mathbf{k}) &= 0 && \text{on } X_0\,.
\end{aligned}
\tag{3.9}
$$

Using the notations $\Sigma_t = \Sigma_a(m) + \Sigma_s$ and $R = \Sigma_s\mathcal{K}w - \Sigma_a'(m)\langle w\rangle\phi$, we can write the solution to (3.9) as

$$
\begin{aligned}
w(z,\mathbf{x},\mathbf{k}) &= \int_0^{z\wedge\tau_-(\mathbf{x},\mathbf{k})} \exp^{-\int_0^s \Sigma_t(z-l,\mathbf{x}-l\mathbf{k})dl} R(z-s,\mathbf{x}-s\mathbf{k},\mathbf{k})ds\\
&= \int_0^{z\wedge\tau_-(\mathbf{x},\mathbf{k})} \Sigma_t(z-s,\mathbf{x}-s\mathbf{k})e^{-\int_0^s \Sigma_t(z-l,\mathbf{x}-l\mathbf{k})dl}\frac{R(z-s,\mathbf{x}-s\mathbf{k},\mathbf{k})}{\Sigma_t(z-s,\mathbf{x}-s\mathbf{k})}ds\,,
\end{aligned}
$$

which then gives the bound

$$
\begin{aligned}
|w(z,\mathbf{x},\mathbf{k})| &\le \int_0^{z\wedge\tau_-(\mathbf{x},\mathbf{k})} \Sigma_t(z-s,\mathbf{x}-s\mathbf{k})e^{-\int_0^s \Sigma_t(z-l,\mathbf{x}-l\mathbf{k})dl}\left|\frac{R(z-s,\mathbf{x}-s\mathbf{k},\mathbf{k})}{\Sigma_t(z-s,\mathbf{x}-s\mathbf{k})}\right|ds\\
&\le \left(1 - e^{-\int_0^{z\wedge\tau_-(\mathbf{x},\mathbf{k})}\Sigma_t(z-l,\mathbf{x}-l\mathbf{k})dl}\right)\sup_{0\le s\le z\wedge\tau_-(\mathbf{x},\mathbf{k})}\left|\frac{R(z-s,\mathbf{x}-s\mathbf{k},\mathbf{k})}{\Sigma_t(z-s,\mathbf{x}-s\mathbf{k})}\right|\\
&\le \gamma \sup_{(0,T)\times\Omega\times\mathbb{S}^{d-1}}\left|\frac{R(z,\mathbf{x},\mathbf{k})}{\Sigma_t(z,\mathbf{x})}\right|
\end{aligned}
\tag{3.10}
$$

for some $0 < \gamma < 1$. Here $a\wedge b = \min(a,b)$ and $\tau_-(\mathbf{x},\mathbf{k})$ is the distance from $\mathbf{x}$ to Γ_- in the direction of $-\mathbf{k}$. The next step is to show that $\sup_{(0,T)\times\Omega\times\mathbb{S}^{d-1}}\left|\frac{R(z,\mathbf{x},\mathbf{k})}{\Sigma_t(z,\mathbf{x})}\right| \le \|w\|_\infty$, which, when combined with the bound in (3.10), leads to the bound $|w(z,\mathbf{x},\mathbf{k})| \le \gamma\|w\|_\infty$ (and hence $w = 0$). This contradicts the assumption that $\langle w\rangle$ is the eigenfunction corresponding to the eigenvalue 1 of F'. We derive the bound under the following two assumptions in the theorem:

(a) When condition (i) is satisfied, we deduce from it that

$$\nu_{d-1}\Sigma_a'(m)\|f\|_\infty \le \Sigma_a(m)\,.$$

Meanwhile, we also have that

$$|R(z,\mathbf{x},\mathbf{k})| \le \Sigma_s \|w\|_\infty + \nu_{d-1}\Sigma_a'(m)\|f\|_\infty \|w\|_\infty .$$

Combining the above two bounds gives us

$$|R(z,\mathbf{x},\mathbf{k})| \le (\Sigma_s + \Sigma_a(m))\|w\|_\infty = \Sigma_t \|w\|_\infty .$$

Therefore, we have $\sup_{(0,T)\times\Omega\times\mathbb{S}^{d-1}} \left|\frac{R(z,\mathbf{x},\mathbf{k})}{\Sigma_t(z,\mathbf{x})}\right| \le \|w\|_\infty$.

(b) For the case when condition (ii) is satisfied, we first observe that

$$R(z,\mathbf{x},\mathbf{k}) = \Sigma_s\left(\mathcal{K}w - \underline{\theta}\langle w\rangle\right) + \left(\Sigma_s\underline{\theta} - \Sigma_a'(m)\phi\right)\langle w\rangle$$

implies

$$\left|\frac{R(z,\mathbf{x},\mathbf{k})}{\Sigma_t(z,\mathbf{x})}\right| \le \frac{\Sigma_s(1-\nu_{d-1}\underline{\theta}) + \nu_{d-1}|\Sigma_s\underline{\theta} - \Sigma_a'(m)\phi|}{\Sigma_t(z,\mathbf{x})}\|w\|_\infty . \tag{3.11}$$

When $\|f\|_\infty$ satisfies

$$\|f\|_\infty \le \frac{1}{\nu_{d-1}}\frac{2\Sigma_s\underline{\theta}}{\Sigma_a'(\|f\|_\infty + \delta)} \quad \forall (z,\mathbf{x}) \in (0,T)\times\Omega, \tag{3.12}$$

we obtain that $\Sigma_s(1-\nu_{d-1}\underline{\theta}) + \nu_{d-1}|\Sigma_s\underline{\theta} - \Sigma_a'(m)\phi| \le \Sigma_s$, which, when combined with (3.11), implies that

$$\left|\frac{R(z,\mathbf{x},\mathbf{k})}{\Sigma_t(z,\mathbf{x})}\right| \le \|w\|_\infty .$$

Since $\delta > 0$ is arbitrary, taking $\delta \to 0$, the condition (3.12) becomes

$$\|f\|_\infty \le \frac{1}{\nu_{d-1}}\frac{2\Sigma_s\underline{\theta}}{\Sigma_a'(\|f\|_\infty)} \quad \forall (z,\mathbf{x}) \in (0,T)\times\Omega\,,$$

which is simply (ii).

The proof is now complete. ∎

To study the diffusion limit, we introduce a small parameter ϵ and denote by W_ϵ the solution to the following scaled radiative transport equation:

$$\begin{aligned} \frac{\partial W_\epsilon}{\partial z} + \frac{1}{\epsilon}\mathbf{k}\cdot\nabla W_\epsilon + \Sigma_a(\langle W_\epsilon\rangle)W_\epsilon + \frac{1}{\epsilon^2}\Sigma_s\left(W_\epsilon - \mathcal{K}W_\epsilon\right) &= 0 && \text{in } X,\\ W_\epsilon(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-,\\ W_\epsilon(0,\mathbf{x},\mathbf{k}) &= f(\mathbf{x}) && \text{on } X_0\,. \end{aligned} \tag{3.13}$$

We have the following corollary using condition (ii) in Theorem 3.2.

Corollary 3.3. *If ϵ is sufficiently small such that*

$$\Sigma_a'(\|f\|_\infty)\|f\|_\infty \le \frac{1}{\nu_{d-1}}\frac{2\underline{\theta}\Sigma_s}{\epsilon^2},$$

then (3.13) *admits a unique nonnegative solution in $\mathcal{W}_\infty((0,T)\times\Omega\times\mathbb{S}^{d-1})$. Moreover, the solution satisfies $\|W_\epsilon\|_\infty \le \|f\|_\infty$.*

3.1. Asymptotic expansion. Let ϵ be sufficiently small such that (3.13) admits a unique solution. We formally expand the solution in powers of ϵ:

$$W_\epsilon(z,\mathbf{x},\mathbf{v}) = W_0(z,\mathbf{x},\mathbf{v}) + \epsilon W_1(z,\mathbf{x},\mathbf{v}) + \epsilon^2 W_2(z,\mathbf{x},\mathbf{v}) + \phi_\epsilon(z,\mathbf{x},\mathbf{v})\,. \tag{3.14}$$

Let $\mathcal{I}$ denote the identity operator. We then substitute the above expansion into (3.13). Matching the equations at orders ϵ^{-2}, ϵ^{-1}, and ϵ^0 gives the following system:

$$\begin{aligned} (\mathcal{I}-\mathcal{K})W_0 &= 0,\\ \Sigma_s(\mathcal{I}-\mathcal{K})W_1 + \mathbf{k}\cdot\nabla W_0 &= 0,\\ \frac{\partial W_0}{\partial z} + \Sigma_s(\mathcal{I}-\mathcal{K})W_2 + \mathbf{k}\cdot\nabla W_1 + \Sigma_a(W_0)W_0 &= 0\,. \end{aligned} \tag{3.15}$$

Following standard procedures [15, 19], we obtain from the first two equations that

$$W_0(z,\mathbf{x},\mathbf{k}) = W_0(z,\mathbf{x}), \quad W_1(z,\mathbf{x},\mathbf{k}) = -\sum_{i=1}^{d}\frac{D_i(\mathbf{k})}{\Sigma_s}\partial_{x_i}W_0(z,\mathbf{x})\,, \tag{3.16}$$

where $D_i(v)\in L^\infty(\mathbb{S}^{d-1})$ are the unique solutions to

$$(\mathcal{I}-\mathcal{K})D_i(\mathbf{k}) = \mathbf{k}\cdot\mathbf{e}_i, \quad \int_{\mathbb{S}^{d-1}} D_i(\mathbf{k})d\mathbf{k} = 0\,, \quad i=1,2,\ldots,d. \tag{3.17}$$

Next, we integrate the third equation in (3.15) over $\mathbb{S}^{d-1}$ and utilize the fact that

$$\langle(\mathcal{I}-\mathcal{K})W_2\rangle = 0$$

to get the equation for W_0. This leads to

$$\int_{\mathbb{S}^{d-1}}\left(\frac{\partial W_0}{\partial z} + (\mathbf{k}\cdot\nabla W_1 + \Sigma_a(W_0)W_0)\right)d\mathbf{k} = 0\,.$$

Since W_0 is independent of $\mathbf{k}$, we have that W_0 solves the following diffusion equation:

$$\begin{aligned} \frac{\partial W_0}{\partial z} - \nabla\cdot\left(\frac{A}{\Sigma_s}\nabla W_0\right) + \Sigma_a(W_0)W_0 &= 0, && \text{in } (0,T)\times\Omega,\\ W_0(z,\mathbf{x}) &= 0, && \text{on } (0,T)\times\partial\Omega,\\ W_0(0,\mathbf{x}) &= f(\mathbf{x}) && \text{on } \Omega, \end{aligned} \tag{3.18}$$

where the matrix A is positive definite and is defined as

$$A_{ij} = \int_{\mathbb{S}^{d-1}}\mathbf{k}\cdot\mathbf{e}_i D_j(\mathbf{k})d\mathbf{k}\,.$$

Under our assumption that the scattering phase function $p(\mathbf{k}\cdot\mathbf{k}')$ is rotation invariant, we have that $A_{ij} = \frac{1}{d(1-g)}\delta_{ij}$, where $g\in(-1,1)$ is the anisotropy parameter.

Let ϕ_ϵ be the remainder in the expansion. Then we can check that ϕ_ϵ satisfies the following equation:

$$\begin{aligned} \partial_z\phi_\epsilon + \frac{1}{\epsilon}\mathbf{k}\cdot\nabla\phi_\epsilon + \frac{\Sigma_s}{\epsilon^2}(\mathcal{I}-\mathcal{K})\,\phi_\epsilon + \Sigma_a(\langle W_\epsilon\rangle)\phi_\epsilon &= \epsilon h_1 && \text{in } X,\\ \phi_\epsilon(z,\mathbf{x},\mathbf{k}) &= \epsilon h_2 && \text{on } \Gamma_-,\\ \phi_\epsilon(0,\mathbf{x},\mathbf{k}) &= \epsilon h_3 && \text{on } X_0\,, \end{aligned} \tag{3.19}$$

where the functions h_1, h_2, and h_3, respectively, are of the following forms:

$$\begin{aligned}
h_1(z,\mathbf{x},\mathbf{k}) &= -\partial_z W_1 - \mathbf{k}\cdot\nabla W_2 - \Sigma_a(\langle W_\epsilon\rangle)(W_1+\epsilon W_2)\\
&\quad - \epsilon\partial_z W_2 + \frac{1}{\epsilon}\left[\Sigma_a(W_0) - \Sigma_a(\langle W_\epsilon\rangle)\right] W_0,\\
h_2(z,\mathbf{x},\mathbf{k}) &= -\sum_{i=1}^{d}\frac{D_i(\mathbf{k})}{\Sigma_s}\frac{\partial W_0}{\partial x_i} - \epsilon W_2,\\
h_3(0,\mathbf{x},\mathbf{k}) &= -W_1 - \epsilon W_2\,.
\end{aligned}$$

Let $\delta := \frac{1}{2}\inf_{[0,T]\times\Omega}\Sigma_{a,0}(z,\mathbf{x})$. Then, by the assumption in (3.3), $\delta > 0$. We then have that for any $z\in[0,T)$,

$$\begin{aligned}
\|\phi_\epsilon(z,\cdot,\cdot)\|_\infty &\le \epsilon\|h_3\|_\infty e^{-\delta z} + \epsilon\int_0^z e^{-\delta(z-s)}(\|h_1\|_\infty + \delta\|h_2\|_\infty)ds\\
&\le \epsilon T(\|h_1\|_\infty + \delta\|h_2\|_\infty) + \epsilon\|h_3\|_\infty.
\end{aligned} \tag{3.20}$$

It remains to show that h_1, h_2, and h_3 are bounded. We first observe from the equations in (3.15) that

$$\|W_1\|_\infty \le C_1\|W_0\|_{C([0,T),C^1(\Omega))}, \quad \|W_2\|_\infty \le C_2\|W_0\|_{C([0,T),C^2(\Omega))}.$$

We then take the derivative with respect to z of the equations in (3.15) to deduce that

$$\|\partial_z W_1\|_\infty \le C_3\|W_0\|_{C([0,T),C^3(\Omega))}, \quad \|\partial_z W_2\|_\infty \le C_4\|W_0\|_{C([0,T),C^4(\Omega))}$$

together with

$$\|\mathbf{k}\cdot\nabla W_2\|_\infty \le C_5\|W_0\|_{C([0,T),C^3(\Omega))}\,.$$

Therefore, given that $W_0\in C([0,T),C^4(\Omega))$, we have the following estimate for the diffusion approximation:

$$\|W_\epsilon - W_0\|_\infty \le \|\phi_\epsilon\|_\infty + \epsilon\|W_1\|_\infty + \epsilon^2\|W_2\|_\infty = C(T,\|W_0\|_{C([0,T),C^4(\Omega))})\epsilon.$$

To ensure the regularity of the solution W_0, at least for a short time, we simply need the initial condition f to be smooth enough since W_0 would be smoother than the initial condition due to the diffusive nature.

We can prove the following result.

THEOREM 3.4. *Assume that $f\in C_0^4(\Omega)$ is nonnegative, and the absorption and scattering coefficients satisfy the condition*

$$\Sigma_{a,l}\in C^2(\Omega), \quad \Sigma_{a,l}\ge 0, \quad \Sigma_s > 0.$$

Then the diffusion equation (3.18) *admits a unique strong solution $W_0\in C([0,T),C^4(\Omega))$ when $1\le d\le 3$.*

Proof. Let $L = -\nabla\cdot\left(\frac{A}{\Sigma_s}\nabla\right)$. Then $-L$ is the infinitesimal generator of an analytic semigroup $G(t)$ on $L^2(\Omega)$ and $\|G(t)\|\le 1$ for all $t\ge 0$. We denote

$$\mathcal{D}(L) = H^2(\Omega)\cap H_0^1(\Omega).$$

By [43, Theorem 8.4.4] and [43, Theorem 6.3.1], we have that when $f \in \mathcal{D}(L)$, there exists a unique local strong solution W_0 to (3.18) on $(0,T)\times\Omega$, that is,

$$W_0 \in C([0,T), L^2(\Omega)) \cap C((0,T), H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1((0,T), L^2(\Omega))\,. \tag{3.21}$$

Moreover, $0 \le W_0 \le \|f\|_\infty$ by the comparison principle. Then the result of [43, Corollary 6.3.2] ensures that $\partial_z W_0$ is locally Hölder continuous for $z \in (0,T)$. Hence, $W_0(z,\mathbf{x})$ and $\partial_z W_0(z,\mathbf{x})$ are both continuous on $(0,T)\times\overline{\Omega}$. This means that $\Sigma_a(W_0)$ is also continuous. Therefore, we must have $W_0(z,\cdot)\in C^2(\Omega)$, which means W_0 is a classical solution.

Let $g(\mathbf{x}) := -Lf - \Sigma_a(f)$, $f \in C^2(\Omega)\in\mathcal{D}(L)$. By differentiating (3.18), we find that $\psi := \partial_z W_0$ satisfies the following equation:

$$\begin{aligned} \partial_z\psi + L\psi + (\Sigma_a'(W_0)W_0 + \Sigma_a(W_0))\,\psi &= 0 && \text{in } (0,T)\times\Omega,\\ \psi(z,\mathbf{x}) &= 0 && \text{on } (0,T)\times\partial\Omega,\\ \psi(0,\mathbf{x}) &= g(\mathbf{x}) && \text{on } \{0\}\times\Omega. \end{aligned} \tag{3.22}$$

Following a similar process, we can deduce that $\psi(z,\cdot)\in C^2(\Omega)$. Since $W_0(z,\cdot)\in C^2(\Omega)$ and $\partial_z W_0(z,\cdot)\in C^2(\Omega)$ for $z\in(0,T)$, we have $\partial_z W_0(z,\cdot)+\Sigma_a(W_0(z,\cdot))W_0(z,\cdot)\in C^2(\Omega)$. By classical regularity theory for elliptic equations [26], $W_0(z,\cdot)\in C^4(\Omega)$. ◻

Remark 3.5. We have assumed so far that the initial condition f is independent of the variable $\mathbf{k}$. In fact, the case of f depending on $\mathbf{k}$, that is, $f = f(\mathbf{x},\mathbf{k})$, can be treated in a similar manner by introducing another fast variable $\theta = \frac{z}{\epsilon^2}$, as in [19, section XXI.5.3]. We will not repeat the calculations here.

3.2. The case of degenerate coefficients. Let us now briefly consider the case when the problem is degenerate, that is, when the absorption coefficient can vanish in part of the domain of interest. More precisely, we relax the requirement that all $\Sigma_{a,l} > 0$ to the following:

$$\Sigma_{a,l} \ge 0, \quad \Sigma_a(\|f\|_\infty) > 0 \quad \forall (z,\mathbf{x}) \in [0,T]\times\Omega. \tag{3.23}$$

In this case, $\Sigma_a'(\|f\|_\infty) > 0$. When ϵ is sufficiently small, the scaled transport equation (3.13) admits a unique solution in $L^\infty(X)$. Let w_ϵ be the solution to the following linear transport equation:

$$\begin{aligned} \partial_z w_\epsilon + \frac{1}{\epsilon}\mathbf{k}\cdot\nabla w_\epsilon + \Sigma_a(\|f\|_\infty)w_\epsilon + \frac{\Sigma_s}{\epsilon^2}(\mathcal{I}-\mathcal{K})w_\epsilon &= 0 && \text{in } X,\\ w_\epsilon(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-,\\ w_\epsilon(0,\mathbf{x},\mathbf{k}) &= f(\mathbf{x}) && \text{on } X_0. \end{aligned} \tag{3.24}$$

Since the absorption coefficient $\Sigma_a(\|f\|_\infty) \ge \Sigma_a(\langle W_\epsilon\rangle)$, we conclude that $w_\epsilon \le W_\epsilon$ for any $\epsilon > 0$. On the other hand, as $\epsilon \to 0$, we have $w_\epsilon \to w_0$, where w_0 is the solution to

$$\begin{aligned} \partial_z w_0 - \nabla\cdot\left(\frac{A}{\Sigma_s}\nabla w_0\right) + \Sigma_a(\|f\|_\infty)w_0 &= 0 && \text{in } (0,T)\times\Omega,\\ w_0(z,\mathbf{x}) &= 0 && \text{on } (0,T)\times\partial\Omega,\\ w_0(0,\mathbf{x}) &= f(\mathbf{x}) && \text{on } \Omega. \end{aligned} \tag{3.25}$$

Hence, $\|W_\epsilon\| \ge \|w_\epsilon\| \ge \inf_{[0,T]\times\Omega} w_0 - C\epsilon$ for some $C > 0$. Because $\inf_{(0,T)\times\Omega} w_0 > 0$ strictly for ϵ sufficiently small, we conclude that W_ϵ is bounded from below by a positive number, which implies $\Sigma_a(\langle W_\epsilon\rangle)$ is also strictly positive. Then we repeat the process in (3.20) by setting $\delta = \frac{1}{2}\inf_{(0,T)\times\Omega}\Sigma_a(\langle W_\epsilon\rangle)$; instead, we obtain the same conclusion that $\|W_\epsilon - W_0\|_\infty \le C'\epsilon$ for a constant C' independent of ϵ.

4. An application in inverse problems. We now consider the following inverse medium problem as a direct application of the transport model:

$$
\begin{aligned}
\partial_z u + \mathbf{k}\cdot\nabla u + \Sigma_a(\langle u\rangle)u + \Sigma_s(\mathcal{I}-\mathcal{K})u &= 0 && \text{in } X,\\
u(z,\mathbf{x},\mathbf{k}) &= 0 && \text{on } \Gamma_-,\\
u(0,\mathbf{x},\mathbf{k}) &= f(\mathbf{x}) && \text{on } X_0,
\end{aligned}
\tag{4.1}
$$

where $f(\mathbf{x})\in L^\infty(X_0)$ is a strictly positive source function. We assume that (4.1) has a unique positive solution $u\in\mathcal{W}_\infty$.

We assume that the absorption coefficient Σ_a is not known, but we have additional data that is the density of the solution, that is,

$$
g(z,\mathbf{x}) = \langle u\rangle := \int_{\mathbb{S}^{d-1}} u(z,\mathbf{x},\mathbf{k})d\mathbf{k}\,. \tag{4.2}
$$

The inverse problem amounts to finding the unknown absorption coefficients $\Sigma_{a,l}$ from the observed data g from a given f.

We can prove the following result.

THEOREM 4.1. *Let g and $\widetilde{g}$ be data of the form* (4.2) *generated from* (4.1) *with coefficients Σ_a and $\widetilde{\Sigma}_a$, respectively. Then $g=\widetilde{g}$ implies $\Sigma_a(\langle u\rangle)=\widetilde{\Sigma}_a(\langle\widetilde{u}\rangle)$.*

Proof. Let $\delta u = u-\widetilde{u}$. We verify that for any δu, we have the identity

$$
\int_{\Omega\times\mathbb{S}^{d-1}} [\mathbf{k}\cdot\nabla\delta u]\,\frac{\delta u}{\tilde u}d\mathbf{x}d\mathbf{k} = \int_{\Omega\times\mathbb{S}^{d-1}} \mathbf{k}\cdot\nabla\frac{|\delta u|^2}{2\tilde u}d\mathbf{x}d\mathbf{k} - \int_{\Omega\times\mathbb{S}^{d-1}}\left[\mathbf{k}\cdot\nabla\frac{1}{\tilde u}\right]\frac{|\delta u|^2}{2}d\mathbf{x}d\mathbf{k} \tag{4.3}
$$

and the identity

$$
\mathbf{k}\cdot\nabla\frac{1}{\tilde u} = \frac{1}{\tilde u^2}\partial_z\tilde u + \frac{\widetilde{\Sigma_a}(\langle\tilde u\rangle)}{\tilde u} + \frac{\Sigma_s(\mathcal{I}-\mathcal{K})\tilde u}{|\tilde u|^2}\,. \tag{4.4}
$$

Using the fact that $g=\widetilde{g}$, we can also conclude that

$$
\langle\delta u\rangle = 0\,. \tag{4.5}
$$

It is also straightforward to check that δu solves the following transport equation:

$$
\partial_z\delta u + \mathbf{k}\cdot\nabla\delta u + \Sigma_a(\langle u\rangle)\delta u + \Sigma_s(\mathcal{I}-\mathcal{K})\delta u = (\widetilde{\Sigma}_a(\langle\tilde u\rangle) - \Sigma_a(\langle u\rangle))\tilde u \tag{4.6}
$$

with zero initial and incoming boundary conditions. We multiply this equation by $\frac{\delta u}{\tilde u}$ and integrate over $\Omega\times\mathbb{S}^{d-1}$ to obtain

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{S}^{d-1}}(\partial_z\delta u)\frac{\delta u}{\tilde u}d\mathbf{x}d\mathbf{k} + \int_{\Omega\times\mathbb{S}^{d-1}}\mathbf{k}\cdot\nabla\frac{|\delta u|^2}{2\tilde u}d\mathbf{x}d\mathbf{k} - \int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\delta u|^2}{2|\tilde u|^2}\partial_z\tilde u d\mathbf{x}d\mathbf{k}\\
&\quad -\int_{\Omega\times\mathbb{S}^{d-1}}\frac{\widetilde{\Sigma_a}(\langle\tilde u\rangle)+\Sigma_s}{2\tilde u}|\delta u|^2 d\mathbf{x}d\mathbf{k} + \int_{\Omega\times\mathbb{S}^{d-1}}\frac{(\Sigma_s\mathcal{K}\tilde u)|\delta u|^2}{2|\tilde u|^2}d\mathbf{x}d\mathbf{k}\\
&\quad +\int_{\Omega\times\mathbb{S}^{d-1}}\frac{\Sigma_a(\langle u\rangle)}{\tilde u}|\delta u|^2 d\mathbf{x}d\mathbf{k} + \Sigma_s\int_{\Omega\times\mathbb{S}^{d-1}}[(\mathcal{I}-\mathcal{K})\delta u]\,\frac{\delta u}{\tilde u}d\mathbf{x}d\mathbf{k}\\
&\quad = \int_{\Omega\times\mathbb{S}^{d-1}}(\widetilde{\Sigma_a}(\langle\tilde u\rangle) - \Sigma_a(\langle u\rangle))\delta u d\mathbf{x}d\mathbf{k},
\end{aligned}
\tag{4.7}
$$

where we have used the identities (4.3) and (4.4).

We first observe that since $\Sigma_a(\langle u\rangle)$ and $\widetilde{\Sigma_a}(\langle\tilde u\rangle)$ do not depend on $\mathbf{k}$, the right-hand side of (4.7) vanishes due to (4.5).

To handle the left-hand side of (4.7), we observe that

$$\int_{\Omega\times\mathbb{S}^{d-1}} \mathbf{k}\cdot\nabla\frac{|\delta u|^2}{2\tilde u}d\mathbf{x}d\mathbf{k} = \int_{\Gamma_+}\mathbf{k}\cdot\mathbf{n}\frac{|\delta u|^2}{2\tilde u}\geq 0\,, \tag{4.8}$$

$$(\partial_z\delta u)\frac{\delta u}{\tilde u} = \frac{1}{2}\partial_z\left[\frac{|\delta u|^2}{\tilde u}\right]+\frac{1}{2}\frac{|\delta u|^2}{|\tilde u|^2}\partial_z\tilde u\,, \tag{4.9}$$

and

(4.10)
$$\Sigma_s\int_{\Omega\times\mathbb{S}^{d-1}}[(\mathcal{I}-\mathcal{K})\delta u]\frac{\delta u}{\tilde u}d\mathbf{x}d\mathbf{k} = \Sigma_s\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\delta u|^2}{\tilde u}d\mathbf{x}d\mathbf{k} - \Sigma_s\int_{\Omega\times\mathbb{S}^{d-1}}(\mathcal{K}\delta u)\frac{\delta u}{\tilde u}d\mathbf{x}d\mathbf{k}\,.$$

We can also prove the following inequality (see Appendix A):

(4.11)
$$\int_{\Omega\times\mathbb{S}^{d-1}}(\mathcal{K}\delta u)\frac{\delta u}{\tilde u}d\mathbf{x}d\mathbf{k} \leq \int_{\Omega\times\mathbb{S}^{d-1}}(\mathcal{K}\tilde u)\frac{|\delta u|^2}{2|\tilde u|^2}d\mathbf{x}d\mathbf{k} + \nu_{d-1}\frac{\kappa^2}{2}\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\delta u|^2}{\tilde u}d\mathbf{x}d\mathbf{k},$$

where $\kappa = \frac{(\overline{\theta}-\underline{\theta})}{2\sqrt{\underline{\theta}}}$, the constants $\overline{\theta}$ and $\underline{\theta}$ being defined in (3.2).

Let $M(z,\mathbf{x}) := \Sigma_a(\langle u\rangle) - \frac{\widetilde{\Sigma_a}(\langle\tilde u\rangle)}{2} + \Sigma_s\left(\frac{1-\nu_{d-1}\kappa^2}{2}\right)$. We can then deduce from (4.7), using (4.8), (4.9), (4.10), and (4.11), that

$$\frac{1}{2}\partial_z\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\delta u|^2}{\tilde u}d\mathbf{x}d\mathbf{k} + \int_{\Omega\times\mathbb{S}^{d-1}}M(z,\mathbf{x})\frac{|\delta u|^2}{\tilde u}d\mathbf{x}d\mathbf{k}\leq 0. \tag{4.12}$$

Since the coefficients $\Sigma_{a,l}$ and Σ_s are finite and both $\langle u\rangle, \langle\tilde u\rangle$ are bounded by $\|f\|_{L^\infty(X_0)}$, there exists a constant $\underline{M} = \inf_{(0,T)\times\Omega}M(z,\mathbf{x})$ such that

$$\frac{1}{2}\partial_z\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\delta u|^2}{\tilde u}d\mathbf{x}d\mathbf{k} + \underline{M}\int_{\Omega\times\mathbb{S}^{d-1}}\frac{|\delta u|^2}{\tilde u}d\mathbf{x}d\mathbf{k}\leq 0.$$

Then, by the Grönwall inequality and the initial condition $\delta u = 0$ at $z = 0$, we must have that $\delta u\equiv 0$. Therefore, $u = \tilde u$, which implies that

(4.13)
$$\Sigma_a(\langle u\rangle) = -\frac{\partial_z u + \mathbf{k}\cdot\nabla u + \Sigma_s(\mathcal{I}-\mathcal{K})u}{u} = -\frac{\partial_z\tilde u + \mathbf{k}\cdot\nabla\tilde u + \Sigma_s(\mathcal{I}-\mathcal{K})\tilde u}{\tilde u} = \widetilde{\Sigma_a}(\langle\tilde u\rangle)\,.$$

The proof is complete. ∎

The following corollary is a direct result of the comparison principle and Theorem 4.1.

COROLLARY 4.2. *Under the assumption of Theorem* 4.1*, the coefficients* $\Sigma_{a,l}$ *can be uniquely determined with finitely many data sets* $\langle u_j\rangle$, $j = 1,2,\ldots,L+1$, *if the initial conditions satisfy* $0 < f_1 < f_2 < \ldots < f_{L+1}$ *on* Ω.

5. Concluding remarks. This work describes the derivation of semilinear radiative transport models for wave propagation in highly scattering media with nonlinear absorption. While the technical aspects of the derivation are relatively standard, we believe that our work provides a theoretical justification for the semilinear radiative transport models, as well as their diffusion approximations, used in applications such as multiphoton imaging [10, 36, 44, 45, 51, 55, 56, 57].

As we have remarked before, one concrete example of the quadratic absorption we considered here is two-photon absorption in nonlinear optics [38, 48]. The radiative transport equation we derive for this case, (2.20), is different from the two-photon radiative transport equation of [40], where the phase space intensity corresponds to a two-photon entangled state of light, not two-photon absorption.

The calculation we have presented here does not generalize to media with nonlinearities such as those that arise in the Kerr effect and second harmonic generation [3, 29, 46, 52]. The derivation of transport equations for wave propagation in such media is a topic of great interest but is much more challenging due to the richness of the behavior of the corresponding wave equations [17, 18, 50]. We point to the derivation in [22] in the context of the nonlinear Schrödinger equation within the mean-field approximation and leave further investigations in this direction to future work. We note that the acousto-optic effect, in which light undergoes a frequency shift due to interaction with an acoustic wave, has also been studied in random media. Although this effect is not nonlinear in the sense considered in this paper, it is possible to develop a suitable kinetic model and associated radiative transport equations [31].

Appendix A. Proof of inequality (4.11).

Proof. Using the AM-GM inequality, we deduce that

$$\int_{\Omega\times\mathbb{S}^{d-1}} (\mathcal{K}\delta u)\frac{\delta u}{\tilde u} d\mathbf{x}d\mathbf{k} \le \int_{\Omega\times\mathbb{S}^{d-1}} (\mathcal{K}\tilde u)\frac{|\delta u|^2}{2|\tilde u|^2} d\mathbf{x}d\mathbf{k} + \frac{1}{2}\int_{\Omega\times\mathbb{S}^{d-1}} \left|\frac{\mathcal{K}\delta u}{\sqrt{\mathcal{K}\tilde u}}\right|^2 d\mathbf{x}d\mathbf{k}.$$

Since $\underline{\theta} \le p(\mathbf{k},\mathbf{k}') \le \overline{\theta}$ and $\langle \delta u\rangle = 0$, we have the preliminary estimate

$$\left|\frac{\mathcal{K}\delta u}{\sqrt{\mathcal{K}\tilde u}}\right| = \left|\frac{\mathcal{K}\delta u - \frac{\overline{\theta}+\underline{\theta}}{2}\langle \delta u\rangle}{\sqrt{\mathcal{K}\tilde u}}\right| \le \frac{(\overline{\theta}-\underline{\theta})\langle|\delta u|\rangle}{2\sqrt{\underline{\theta}\langle\tilde u\rangle}}.$$

Denote the constant $\kappa := \frac{(\overline{\theta}-\underline{\theta})}{2\sqrt{\underline{\theta}}}$. We then have

$$\int_{\Omega\times\mathbb{S}^{d-1}} \left|\frac{\mathcal{K}\delta u}{\sqrt{\mathcal{K}\tilde u}}\right|^2 d\mathbf{x}d\mathbf{k} \le \kappa^2 \int_{\Omega\times\mathbb{S}^{d-1}} \frac{\langle|\delta u|\rangle^2}{\langle\tilde u\rangle} d\mathbf{x}d\mathbf{k} = \nu_{d-1}\kappa^2\int_\Omega \frac{\langle|\delta u|\rangle^2}{\langle\tilde u\rangle}d\mathbf{x}.$$

Using the Cauchy–Schwartz inequality, we arrive at

$$\int_{\mathbb{S}^{d-1}} \frac{|\delta u(\mathbf{x},\mathbf{k})|^2}{\tilde u(\mathbf{x},\mathbf{k})} d\mathbf{k} \int_{\mathbb{S}^{d-1}} \tilde u(\mathbf{x},\mathbf{k}) d\mathbf{k} \ge \left(\int_{\mathbb{S}^{d-1}} |\delta u(\mathbf{x},\mathbf{k})| d\mathbf{k}\right)^2 = \langle|\delta u|\rangle^2.$$

Therefore, we have

$$\frac{\langle|\delta u|\rangle^2}{\langle\tilde u\rangle} \le \int_{\mathbb{S}^{d-1}} \frac{|\delta u(\mathbf{x},\mathbf{k})|^2}{\tilde u(\mathbf{x},\mathbf{k})} d\mathbf{k}.$$

This implies that

$$\int_{\Omega\times\mathbb{S}^{d-1}} (\mathcal{K}\delta u)\frac{\delta u}{\tilde u} d\mathbf{x}d\mathbf{k} \le \int_{\Omega\times\mathbb{S}^{d-1}} (\mathcal{K}\tilde u)\frac{|\delta u|^2}{2|\tilde u|^2} d\mathbf{x}d\mathbf{k} + \nu_{d-1}\frac{\kappa^2}{2}\int_{\Omega\times\mathbb{S}^{d-1}} \frac{|\delta u|^2}{\tilde u} d\mathbf{x}d\mathbf{k}.$$

This completes the proof. ∎

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