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Born and inverse Born series for scattering problems with Kerr nonlinearities

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Abstract

We consider the Born and inverse Born series for scalar waves with a cubic nonlinearity of Kerr type. We find a recursive formula for the operators in the Born series and prove their boundedness. This result gives conditions which guarantee convergence of the Born series, and subsequently yields conditions which guarantee convergence of the inverse Born series. We also use fixed point theory to give alternate explicit conditions for convergence of the Born series. We illustrate our results with numerical experiments.

Keywords: Born series, inverse Born series, Kerr nonlinearity

(Some figures may appear in colour only in the online journal)

1. Introduction

There has been considerable recent interest in inverse scattering problems for nonlinear partial differential equations (PDEs) [1–11]. There are numerous applications in various applied fields ranging from optical imaging to seismology. In general terms, the problem to be considered is to reconstruct the coefficients of a nonlinear PDE from boundary measurements. As

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in the case for linear PDEs, the fundamental questions relate to the uniqueness, stability and reconstruction of the unknown coefficients. In contrast to the linear case, which is well studied, inverse problems for nonlinear PDEs are still relatively unexplored. We note that although uniqueness and stability results for a variety of inverse problems for semilinear and quasilinear equations are known, reconstruction methods are just beginning to be developed [3, 12–15].

In this paper, we consider the inverse problem of recovering the coefficients of a nonlinear elliptic PDE with a cubic nonlinearity. This problem appears in optical physics, where the cubic term arises in the study of the Kerr effect—a nonlinear process where self-focusing of light is observed [16]. We show that it is possible to reconstruct the coefficients of the linear and nonlinear terms in a Kerr medium from boundary measurements. This result holds under a smallness condition on the measurements, which also guarantees the stability of the recovery. The reconstruction is based on inversion of the Born series, which expresses the solution to the inverse problem as an explicitly computable functional of the measured data. We note that this method has been extensively studied in the context of inverse problems for linear PDEs [17]. The extension to the nonlinear setting involves a substantial reworking of the theory, especially the combinatorial structure of the Born series itself. Our results are illustrated with numerical simulations.

The remainder of this paper is organized as follows. In section 2, we introduce the forward problem and state sufficient conditions for its solvability. The Born series is studied in section 3, the combinatorial structure of the series is characterized, and sufficient conditions for convergence are established. We also derive various estimates that are used in section 4 to obtain our main result on the convergence of the inverse Born series (IBS). In section 5, we present the results of numerical reconstructions of a two-dimensional medium. Our conclusions are presented in section 6. The appendix contains the proof of proposition 1.

2. Forward problem

We consider the Kerr effect in a bounded domain Ω in $\mathbb{R}^d$ with a smooth boundary, for $d \geqslant 2$. The scalar field u obeys the nonlinear PDE

$$\Delta u + k^2 (1 + \alpha(x)) u + k^2 \beta(x) |u|^2 u = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega, \tag{2}$$

where the wavenumber k is real and ν is the unit outward normal to $\partial\Omega$. The coefficients α and β are the linear and nonlinear susceptibilities, respectively [16] and are taken to be real valued, as is the boundary source g. It follows that u is real valued, so that $|u|^2 u = u^3$. More generally, u is complex valued, in which case our results carry over with small modifications.

We now consider the solution u_0 to the linear problem

$$\Delta u_0 + k^2 u_0 = 0 \quad \text{in} \quad \Omega, \tag{3}$$

$$\frac{\partial u_0}{\partial \nu} = g \quad \text{on} \quad \partial\Omega. \tag{4}$$

Following standard procedures, we find that the field u obeys the integral equation

$$u(x) = u_0(x) - k^2 \int_\Omega G(x,y) \left(\alpha(y) u(y) + \beta(y) u^3(y)\right) \mathrm{d}y. \tag{5}$$

Here the Green's function G obeys

$$\Delta_x G(x,y) + k^2 G(x,y) = \delta(x-y) \quad \text{in} \quad \Omega, \tag{6}$$

$$\frac{\partial G}{\partial \nu_y} = 0 \quad \text{on} \quad \partial\Omega. \tag{7}$$

We define the nonlinear operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$T(u) = u_0 - k^2 \int_\Omega G(x,y)\left(\alpha(y)u(y) + \beta(y)u^3(y)\right)\mathrm{d}y. \tag{8}$$

Note that if $u \in C(\overline{\Omega})$ is a fixed point of T, then u satisfies equation (5). The following result provides conditions for existence of a unique solution to (5).

Proposition 1. *Let $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by (8) and define μ by*

$$\mu = k^2 \sup_{x\in\Omega} \int_\Omega |G(x,y)|\mathrm{d}y. \tag{9}$$

If there exists $\gamma > 1/2$ such that

$$\|\alpha\|_\infty < \frac{2\gamma - 1}{2\mu(1+\gamma)}$$

and

$$\|\beta\|_\infty < \frac{1}{2\mu\|u_0\|^2_{C(\overline{\Omega})}(1+\gamma)^3},$$

then T has a unique fixed point on the ball of radius $\gamma\|u_0\|_{C(\overline{\Omega})}$ about u_0 in $C(\overline{\Omega})$, and fixed point iteration starting with u_0 will converge to the unique fixed point u.

The proof is given in appendix.

3. Born series

The forward problem is to compute the field u on $\partial\Omega$ given a prescribed source g on $\partial\Omega$. The solution to the forward problem is derived by iteration of the integral equation (5), beginning with the background field u_0. We thus obtain

$$\phi = K_1(\zeta) + K_2(\zeta,\zeta) + K_3(\zeta,\zeta,\zeta) + \cdots, \tag{10}$$

where $\phi = u - u_0$ and $\zeta := (\alpha,\beta)$. The forward operators

$$K_n : [L^\infty(\Omega)]^{2n} \to C(\partial\Omega \times \partial\Omega)$$

are constructed below. We will refer to (10) as the the Born series. We note that proposition 1 guarantees convergence of the Born series.

The forward operator K_n is an n-linear operator (multilinear of order n) on $[L^\infty(\Omega)^2]^n$. In the following, we do not denote the dependence of u_0 on the source explicitly. The first iterate in the fixed point iteration of (8) is given by

$$u_1(x) := T(u_0)(x) = u_0(x) + k^2 \int_\Omega G(x,y)\left[\alpha(y)u_0(y) + \beta(y)u_0^3(y)\right]\mathrm{d}y, \tag{11}$$

and thus K_1 is defined by

$$K_1(\zeta)(x) = k^2 \int_\Omega G(x,y)\left[\alpha(y)u_0(y) + \beta(y)u_0^3(y)\right]\mathrm{d}y. \tag{12}$$

Next we observe that the second iterate is of the form

$$u_2(x) := T(u_1)(x) = u_0(x) + k^2 \int_\Omega G(x,y)\left[\alpha(y)u_1(y) + \beta(y)u_1^3(y)\right]\mathrm{d}y. \quad (13)$$

Evidently, expansion of u_1^3 leads to terms which are multilinear in α and β. Subsequent iterates become progressively more complex. To handle this problem, we introduce the operators: $a,b : C(\overline{\Omega}) \times [L^\infty(\Omega)]^2 \to C(\overline{\Omega})$, which are defined by

$$a(v,\zeta) = k^2 \int_\Omega G(x,y)\alpha(y)v(y)\,\mathrm{d}y, \quad (14)$$

and

$$b(v,\zeta) = k^2 \int_\Omega G(x,y)\beta(y)v(y)\,\mathrm{d}y. \quad (15)$$

The above operators have tensor counterparts which are defined as follows.

Definition 1. Given $T_l = T_l(\zeta_1,\cdots,\zeta_l)$, a multi-linear operator of order l, define the $l+1$ order multilinear operators AT_l and BT_l by

$$AT_l(\zeta_1,\ldots,\zeta_l,\zeta_{l+1}) = a(T_l(\zeta_1,\ldots,\zeta_l),\zeta_{l+1})$$

and

$$BT_l(\zeta_1,\ldots,\zeta_l,\zeta_{l+1}) = b(T_l(\zeta_1,\ldots,\zeta_l),\zeta_{l+1}),$$

where a and b are given by (14) and (15) respectively.

We will also need to define the tensor product of multilinear operators.

Definition 2. Given T_j and T_l, which are multilinear operators of order j and l, respectively, define the tensor product $T_l \otimes T_j$ by

$$T_l \otimes T_j(\zeta_1,\ldots,\zeta_l,\zeta_{l+1},\ldots,\zeta_{l+j}) = T_l(\zeta_1,\ldots,\zeta_l)\,T_j(\zeta_{l+1},\ldots,\zeta_{l+j}),$$

so that $T_l \otimes T_j$ is a multilinear operator of order $l+j$.

Note that the tensor product of multilinear operators does not commute. Tensor products are extended to sums of multilinear operators by bilinearity of the tensor product. The tensor product is also associative. In this notation, we see that if v is a sum of multilinear operators, then

$$Tv = u_0 + Av + Bv \otimes v \otimes v$$

yields another sum of multilinear operators (u_0 is an operator of order zero).

Lemma 1. *Viewing the nth iterate u_n as a sum of multilinear operators, for any n we have that*

$$u_n = u_{n-1} + \textit{multilinear operators of order at least } n. \quad (16)$$

Proof. The proof is by induction. For the case $n=1$, we have that $u_1 = u_0 + Au_0 + Bu_0 \otimes u_0 \otimes u_0$, so the statement holds. Now assume that the statement holds for u_{n-1}. Then

$$u_n = u_0 + Au_{n-1} + Bu_{n-1} \otimes u_{n-1} \otimes u_{n-1}. \quad (17)$$

By the inductive hypothesis,

$$u_{n-1} = u_{n-2} + w,$$

where w is the sum of the operators of order at least $n-1$. We then have

$$\begin{aligned}u_{n-1} \otimes u_{n-1} \otimes u_{n-1} = u_{n-2} \otimes u_{n-2} \otimes u_{n-2} + u_{n-2} \otimes u_{n-2} \otimes w\\ + w \otimes u_{n-2} \otimes u_{n-2} + u_{n-2} \otimes w \otimes u_{n-2}\\ + u_{n-2} \otimes w \otimes w + w \otimes u_{n-2} \otimes w + u_{n-2} \otimes w \otimes w + w \otimes w \otimes w,\end{aligned} \tag{18}$$

so that

$$u_{n-1} \otimes u_{n-1} \otimes u_{n-1} = u_{n-2} \otimes u_{n-2} \otimes u_{n-2} + \text{multilinear operators of order at least } n-1.$$

Applying A to u_{n-1} and B to $u_{n-1} \otimes u_{n-1} \otimes u_{n-1}$, we increase the order of each by one. Hence we have that

$$u_n = u_0 + Au_{n-2} + Bu_{n-2} \otimes u_{n-2} \otimes u_{n-2} + \text{terms of order at least } n \tag{19}$$

$$= u_{n-1} + \text{terms of order at least } n. \tag{20}$$

The result follows by induction. □

Remark . Lemma 1 shows that the fixed point iteration generates a series of the form (10), which converges when the fixed point iteration converges. The operator K_n is given by the sum of all terms in the series which are homogeneous of degree n. Lemma 1 also implies that the nth iterate u_n includes all of these terms. Hence

$$u_n = u_0 + \sum_{i=1}^{n} K_i(\zeta, \dots, \zeta) + \text{terms of order at least } n+1. \tag{21}$$

3.1. General formula for the forward operators

Using tensor notation, the forward series is given by iterations of

$$Tv = u_0 + Av + Bv \otimes v \otimes v.$$

Given u_0, we have

$$\begin{aligned}u_1 &= Tu_0 = u_0 + Au_0 + Bu_0 \otimes u_0 \otimes u_0,\\ u_2 &= Tu_1 = u_0 + Au_1 + Bu_1 \otimes u_1 \otimes u_1,\\ u_{n+1} &= Tu_n = u_0 + Au_n + Bu_n \otimes u_n \otimes u_n.\end{aligned}$$

Define U_n to be the sum of the first n forward operators, that is,

$$\begin{aligned}U_n &= \sum_{i=0}^{n} K_i(\zeta_1, \dots \zeta_i)\\ &= u_0 + \sum_{i=1}^{n} K_i(\zeta_1, \dots, \zeta_i).\end{aligned}$$

We know from lemma 1 that

$$u_n = U_n + w,$$

where w is a sum of multilinear operators, all of order greater than n. To find U_{n+1}, we use the iteration

$$u_{n+1} = u_0 + A(U_n + w) + B(U_n + w) \otimes (U_n + w) \otimes (U_n + w).$$

We know from lemma 1 that K_{n+1} will be the sum of all terms which are of order $n+1$. Since w contains only terms of order greater than or equal to $n+1$, after applying A or B, the result will be of higher order and hence will not be included in K_{n+1}. So any term containing w after expanding the tensor product can be dropped, and we see that all terms of K_{n+1} will be contained in the sum

$$AU_n + BU_n \otimes U_n \otimes U_n.$$

Since A and B each add one to the order, K_{n+1} will consist of AK_n and all terms of the form

$$BK_{i_1} \otimes K_{i_2} \otimes K_{i_3},$$

where the ordered triplets (i_1, i_2, i_3) are such that $i_1 + i_2 + i_3 = n$. Hence we have derived the following:

$$\begin{aligned} K_0 &= u_0, \\ K_1 &= u_0 + Au_0 + Bu_0 \otimes u_0 \otimes u_0, \\ K_{n+1} &= AK_n + B \sum_{\substack{(i_1,i_2,i_3) \\ i_1+i_2+i_3=n \\ 0\leqslant i_1,i_2,i_3 \leqslant n}} K_{i_1} \otimes K_{i_2} \otimes K_{i_3}. \end{aligned} \tag{22}$$

We note that we can count the number of such ordered triples in the above sum to be

$$C(n) = n(n+1)/2 + n + 1.$$

3.2. Bounds on the forward operators

In order to analyze the IBS, bounds on the norms of the forward operators K_i are required. We will see that to apply existing convergence results about the IBS, we need boundedness of the operators as multilinear forms. We denote by $|\cdot|_\infty$ the bound on any multilinear operator of order n, defined as follows.

Definition 3. For any multilinear operator K of order n on $[L^\infty(\Omega)]^{2n}$, we define

$$|K|_\infty = \sup_{\zeta_1,\ldots,\zeta_n \neq 0} \frac{\|K(\zeta_1, \ldots \zeta_n)\|_{C(\partial\Omega\times\partial\Omega)}}{\|\zeta_1\|_\infty \cdots \|\zeta_n\|_\infty}.$$

Note that, for two multilinear operators T_1 and T_2 of the same order, the triangle inequality,

$$|T_1 + T_2|_\infty \leqslant |T_1|_\infty + |T_2|_\infty$$

holds.

Lemma 2. *The forward operator K_n, as defined by (22), is a bounded multilinear operator from $[L^\infty(\Omega)]^{2n}$ to $C(\partial\Omega\times\partial\Omega)$ and*

$$|K_n|_\infty \leqslant \nu_n \mu^n, \tag{23}$$

where

$$\mu = k^2 \sup_{x\in\Omega}\int_\Omega |G(x,y)|\mathrm{d}y, \tag{24}$$

$$\nu_0 = \|u_0\|_{C\left(\overline{\Omega}\times\partial\Omega\right)},$$

and for all $n\geqslant 0$,

$$\nu_{n+1} = \nu_n + \sum_{\substack{(i_1,i_2,i_3)\\ i_1+i_2+i_3=n\\ 0\leqslant i_1,i_2,i_3\leqslant n}} \nu_{i_1}\nu_{i_2}\nu_{i_3}. \tag{25}$$

Proof. We first note that for the product operators in definitions 1 and 2, we have that

$$|BT_l|_\infty \leqslant \mu|T_l|_\infty,$$

$$|AT_l|_\infty \leqslant \mu|T_l|_\infty,$$

and

$$|T_l\otimes T_j|_\infty \leqslant |T_l|_\infty|T_j|_\infty.$$

The proof follows by induction. The base case clearly holds with $|K_0|_\infty = \nu_0$. Next we assume that for each $i\leqslant n$,

$$\|K_i\| \leqslant \nu_i\mu^i.$$

Using (22), we obtain

$$\begin{aligned}|K_{n+1}|_\infty &\leqslant |AK_n|_\infty + |B\sum_{\substack{(i_1,i_2,i_3)\\ i_1+i_2+i_3=n\\ 0\leqslant i_1,i_2,i_3\leqslant n}} K_{i_1}\otimes K_{i_2}\otimes K_{i_3}|_\infty\\ &\leqslant \mu|K_n|_\infty + \mu\sum_{\substack{(i_1,i_2,i_3)\\ i_1+i_2+i_3=n\\ 0\leqslant i_1,i_2,i_3\leqslant n}} |K_{i_1}|_\infty|K_{i_2}|_\infty|K_{i_3}|_\infty\end{aligned}$$

which gives, by the inductive hypothesis

$$\begin{aligned}|K_{n+1}|_\infty &\leqslant \nu_n\mu^{n+1} + \mu^{n+1}\sum_{\substack{(i_1,i_2,i_3)\\ i_1+i_2+i_3=n\\ 0\leqslant i_1,i_2,i_3\leqslant n}} \nu_{i_1}\nu_{i_2}\nu_{i_3}\\ &= \nu_{n+1}\mu^{n+1}.\end{aligned}$$

□

Lemma 3. *For the sequence $\{\nu_n\}$ defined by (25), there exist constants K and ν (both depending on ν_0 but independent of n) such that for any $n \geqslant 0$,*

$$\nu_n \leqslant \nu K^n.$$

Proof. Consider the generating function

$$P(x) = \sum_{n=0}^{\infty} \nu_n x^n.$$

We note that it suffices to prove that this power series has a positive radius of convergence. It then follows that for some positive x, the terms $\nu_n x^n \to 0$. In particular, $\nu_n x^n$ are bounded by a constant ν, which implies that

$$\nu_n \leqslant \nu (1/x)^n.$$

We now show that $P(x)$ is analytic in an interval around zero. Consider the cube of P,

$$\begin{aligned}(P(x))^3 &= \sum_{i_1,i_2,i_3} x^{i_1} x^{i_2} x^{i_3} \nu_{i_1}\nu_{i_2}\nu_{i_3} \\ &= \sum_{n=0}^{\infty} f_n x^n,\end{aligned}$$

where

$$f_n = \sum_{\substack{(i_1,i_2,i_3)\\ i_1+i_2+i_3=n \\ 0\leqslant i_1,i_2,i_3\leqslant n}} \nu_{i_1}\nu_{i_2}\nu_{i_3},$$

which appears in (25). Now, we multiply (25) by x^n and sum to obtain

$$\sum_{n=0}^{\infty} \nu_{n+1} x^n = \sum_{n=0}^{\infty} \nu_n x^n + \sum_{n=0}^{\infty} f_n x^n.$$

It is easy to check that the left hand side is simply $(P(x) - \nu_0)/x$, and so the above yields

$$(P(x) - \nu_0)/x = P(x) + (P(x))^3.$$

Thus we have

$$x(P(x))^3 + (x-1)P(x) + \nu_0 = 0. \tag{26}$$

This polynomial in P is singular, so it is not clear that it has a root at $x = 0$. However, if we differentiate with respect to x, we obtain

$$P'(x) = -\frac{(P(x))^3 + P(x)}{3x(P(x))^2 + x - 1} \tag{27}$$

with $P(0) = \nu_0$. Since the right hand side is an analytic function of x and P in a neighborhood of $(0, \nu_0)$, the ordinary differential equation (27) (together with an initial condition) has a unique analytic solution in a neighborhood of $x = 0$ (see theorem 4.1 of [18]). Integration of (27) combined with the initial condition implies that this solution satisfies (26), and hence its coefficients must satisfy (25). □

Proposition 2. *The forward operator K_n, given by (22), is a bounded multilinear operator from $[L^\infty(\Omega)]^{2n}$ to $C(\partial\Omega\times\partial\Omega)$, and*

$$|K_n|_\infty \leqslant \nu(K\mu)^n, \tag{28}$$

where

$$\mu = k^2 \sup_{x\in\Omega}\int_\Omega |G(x,y)|\mathrm{d}y. \tag{29}$$

Here ν and K, which are the constants in the bound on the sequence $\{\nu_n\}$, depend on $\nu_0 = \|u_0\|_{C(\partial\Omega\times\partial\Omega)}$.

Corollary. *The Born series*

$$u = u_0 + \sum_{n=1}^{\infty} K_n(\zeta,\ldots,\zeta),$$

where K_n are given by (22), converges in $C(\overline{\Omega})$ for

$$\|\zeta\|_\infty \leqslant \frac{1}{K\mu}.$$

4. IBS

The inverse problem is to reconstruct the coefficients α and β from measurements of the scattering data $\phi = u - u_0$ on $\partial\Omega$. We proceed by recalling that the IBS is defined as

$$\tilde{\zeta} = \mathcal{K}_1\phi + \mathcal{K}_2(\phi) + \mathcal{K}_3(\phi) + \cdots, \tag{30}$$

where the data $\phi \in C(\partial\Omega\times\partial\Omega)$. The IBS was analyzed in [19, 20]. The inverse operators $\mathcal{K}_m$ are defined by

$$\mathcal{K}_1(\phi) = K_1^+(\phi), \tag{31}$$

$$\mathcal{K}_2(\phi) = -\mathcal{K}_1(K_2(\mathcal{K}_1(\phi),\mathcal{K}_1(\phi))), \tag{32}$$

$$\mathcal{K}_m(\phi) = -\sum_{n=2}^{m}\sum_{i_1+\cdots+i_n=m}\mathcal{K}_1 K_n(\mathcal{K}_{i_1}(\phi),\ldots,\mathcal{K}_{i_n}(\phi)), \tag{33}$$

where K_1^+ is the regularized pseudoinverse of K_1. See [17] for a review of the IBS.

The bounds on the forward operators in proposition 2 in combination with theorems 2.2 and 2.4 of [19], yield the following results on the convergence and approximation error of the IBS. The constants ν and μ in [19] correspond to $\nu K\mu$ and $K\mu$ in (28). We denote by $\|\mathcal{K}_1\|$ the operator norm of $\mathcal{K}_1$ as a map from $C(\partial\Omega\times\partial\Omega)$ to $L^\infty(\Omega)$.

Theorem 1 (convergence of the IBS). *If $\|\mathcal{K}_1\phi\|_\infty < r$, where the radius of convergence r is given by*

$$r = \frac{1}{2K\mu}\left[\sqrt{16C^2+1} - 4C\right],$$

$C = \max\{2, \|\mathcal{K}_1\|\nu K\mu\}$ and ν, K are defined in lemma 3, then the IBS (30) converges.

Theorem 2 (approximation error). *Suppose that the hypotheses of theorem 1 hold and that the Born and inverse Born series converge. Let* $\tilde{\zeta}$ *denote the sum of the IBS and* $\zeta_1 = \mathcal{K}_1\phi$. *Setting* $\mathcal{M} = \max\left\{\|\zeta\|_\infty, \|\tilde{\zeta}\|_\infty\right\}$, *we further assume that*

$$\mathcal{M} < \frac{1}{K\mu}\left(1 - \sqrt{\frac{\nu K\mu\|\mathcal{K}_1\|}{1+\nu K\mu\|\mathcal{K}_1\|}}\right), \tag{34}$$

then the approximation error can be estimated as follows:

$$\begin{aligned}\left\|\zeta - \sum_{m=1}^{N}\mathcal{K}_m(\phi)\right\|_\infty &\leqslant M\left(\frac{\|\zeta_1\|_\infty}{r}\right)^{N+1}\frac{1}{1-\frac{\|\zeta_1\|_\infty}{r}} \\ &+ \left(1 - \frac{\nu K\mu\|\mathcal{K}_1\|}{(1-K\mu\mathcal{M})^2} + \nu K\mu\|\mathcal{K}_1\|\right)^{-1}\|(I - \mathcal{K}_1 K_1)\zeta\|_\infty,\end{aligned}$$

where

$$M = \frac{2\mu K}{\sqrt{16C^2+1}}.$$

5. Numerical reconstructions

In this section, we present several numerical simulations to test the reconstruction method in two dimensions. We note that the restriction to two dimensions is for simplicity and is not fundamental. We solve the nonlinear PDE

$$\Delta u + k^2(1+\alpha(x))u + k^2\beta(x)u^3 = 0 \quad \text{in} \quad \Omega, \tag{35}$$

$$\frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega, \tag{36}$$

using a Galerkin finite element method as implemented in the FEniCS library in Python. The domain Ω is taken to be the unit disk and we set the wavenumber $k = 1$. The finite element mesh was selected automatically by FEniCS. The boundary source g is taken to be $g(x) = g_0\delta(x-y)$, where $y \in \partial\Omega$ and g_0 is the strength of the source. The delta function is approximated by a Gaussian for numerical computations. The forward operators K_n are constructed according to algorithm 1.

Algorithm 1. Construction of the operators $K\ n$; A and B are given by definition 1.

```
Function compute-K(n, α, β)
    if n = 0 then
        return u_0;
    v_α := compute-K(n − 1, α(n), β(n));
    v_β := 0;
    for i_1 = 0 to n − 1 do
        for i_2 = 0 to n − i_1 − 1 do
            i_3 := n − i_2 − i_1 − 1;
            K_{i_1} := compute-K(i_1, α(1 : i_1), β(1 : i_1));
            K_{i_2} := compute-K(i_2, α(i_1 + 1 : i_1 + i_2), β(i_1 + 1 : i_1 + i_2));
            K_{i_3} := compute-K(i_3, α(i_1 + i_2 + 1 : n − 1), β(i_1 + i_2 + 1 : n − 1));
            v_β := v_β + K_{i_1} · K_{i_2} · K_{i_3};
    return  A(v_α, α(n)) + B(v_β, β(n));
```

The operators A and B are defined by the corresponding integral operators a and b. a and b are evaluated by solving a PDE. Note that only the right-hand side of the PDE changes for each evaluation of a and b. The IBS is implemented according to (31)–(33), as described in [20]. The solution to the linearized inverse problem is given in terms of the operator $\mathcal{K}_1$, which is constructed from the Tikhonov regularized pseudoinverse of the forward operator K_1. The coefficients α and β are piecewise constant. They take the values α_0 and β_0 inside a circle of radius $\sqrt{.2}$ centered at $(.3,0)$ and vanish otherwise.

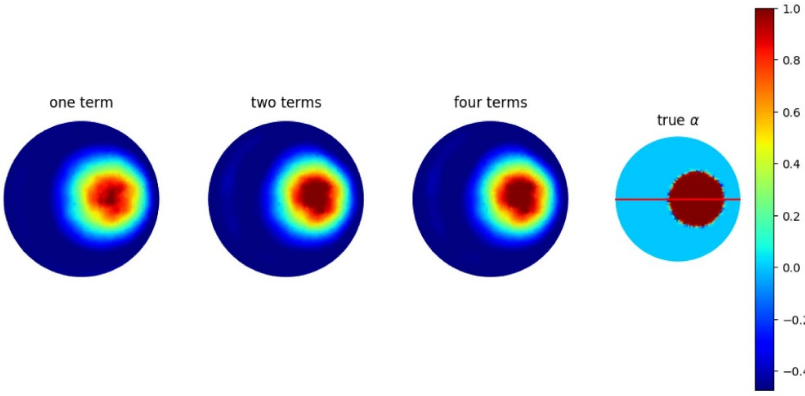
We now present a series of numerical simulations in which reconstructions are carried out to fourth order in the IBS. All reconstructions reported below make use of 16 sources and 16 detectors, which are arranged uniformly on the boundary of the domain. We first consider the problem of recovering only one coefficient, either α or β. In figure 1(A), we show the case of low contrast with $\alpha_0 = 1$ and $\beta_0 = 0$. In figure 1(B), we show the case of low contrast, with $\alpha_0 = 0$ and $\beta_0 = 1$. Figure 2 displays the corresponding cross sections along a horizontal line passing through the center of the inclusion. It can be seen that the series converges rapidly. Note that FEniCS produces a finite element approximation to the true coefficient, which explains the oscillations at the jump. Next, in figures 3 and 4, we present reconstructions at intermediate contrast with $\alpha_0 = 4$ or $\beta_0 = 4$. In this case, the series converges more slowly, and at fourth order the errors present in the linear reconstruction are largely removed. Finally, figures 5 and 6 illustrate reconstructions at high contrast with $\alpha_0 = 16$ or $\beta_0 = 16$. Evidently, in this example the IBS fails to converge. As expected, the series diverges for higher contrast inclusions.

We now consider the simultaneous reconstruction of the coefficients α and β. Here the contrast $\alpha_0 = \beta_0 = 1$ and $g_0 = 1$. The results are presented in figure 7. The quality of the reconstructions is qualitatively similar to those in figure 1.

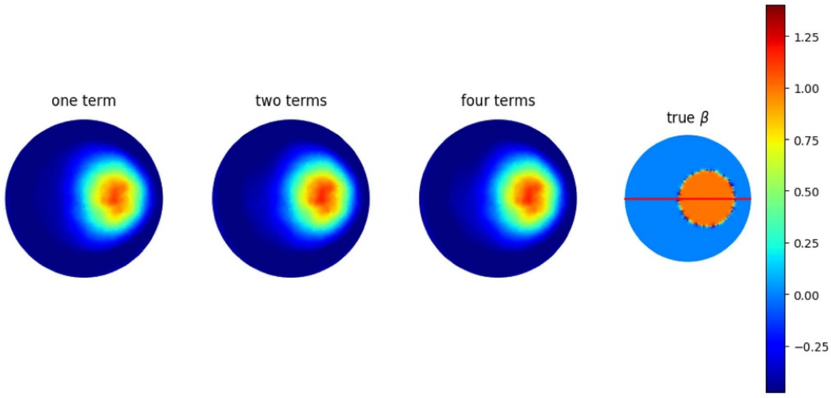
Finally, we note that due to the nonlinearity of (1), the field u depends nonlinearly on the boundary source g. The implications of this for the inverse problem are illustrated in figure 8, where $g_0 = 2$. We see that there is an improvement in the quality of the reconstruction compared to figure 7, where $g_0 = 1$.

6. Discussion

In conclusion, we have investigated the IBS for scalar waves with a cubic nonlinearity of Kerr type. We have analyzed the convergence of the series and have conducted numerical simulations to illustrate the use of the method. We note that in contrast to the Born series for

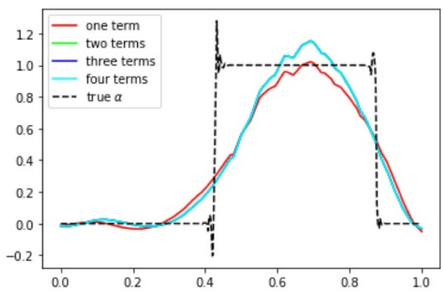


(A) Reconstruction of α with $\beta = 0$.

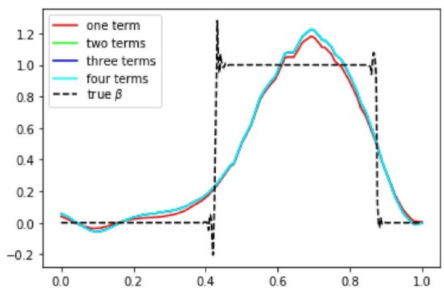


(B) Reconstruction of β with $\alpha = 0$.

Figure 1. Reconstructions of α or β at low contrast.

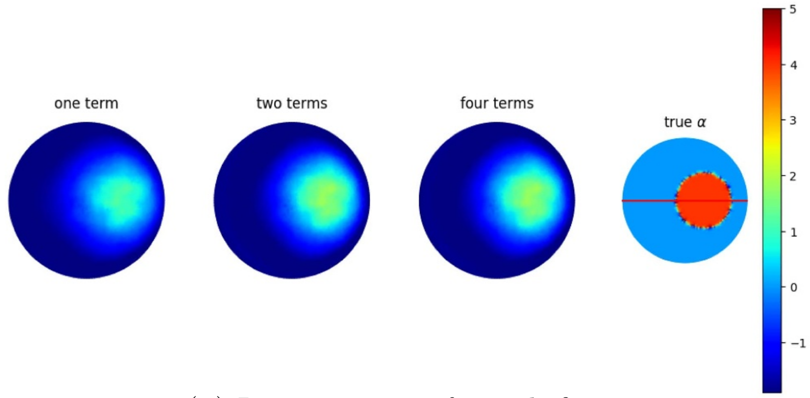


(A) Reconstruction of α with $\beta = 0$.

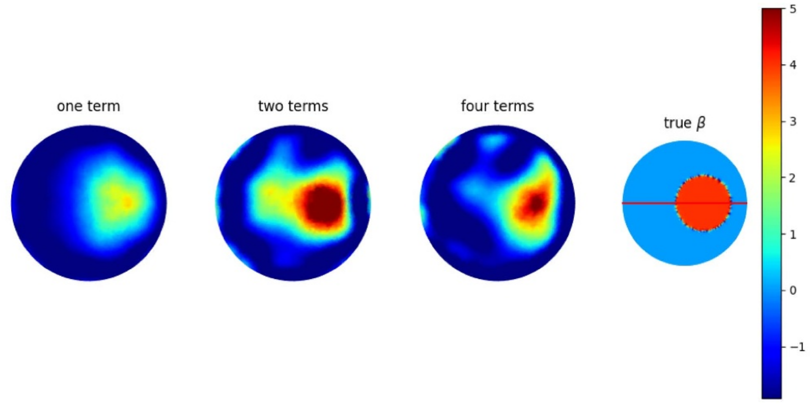


(B) Reconstruction of β with $\alpha = 0$.

Figure 2. Cross sections of α and β at low contrast.

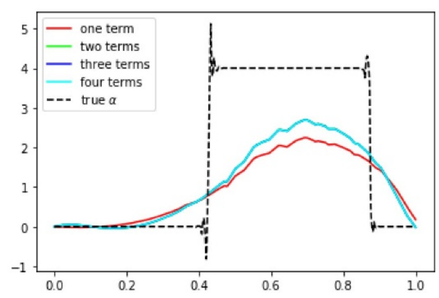


(A) Reconstruction of α with $\beta = 0$

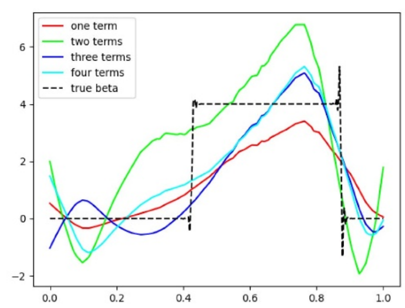


(B) Reconstruction of β with $\alpha = 0$

Figure 3. Reconstructions of α and β at intermediate contrast.

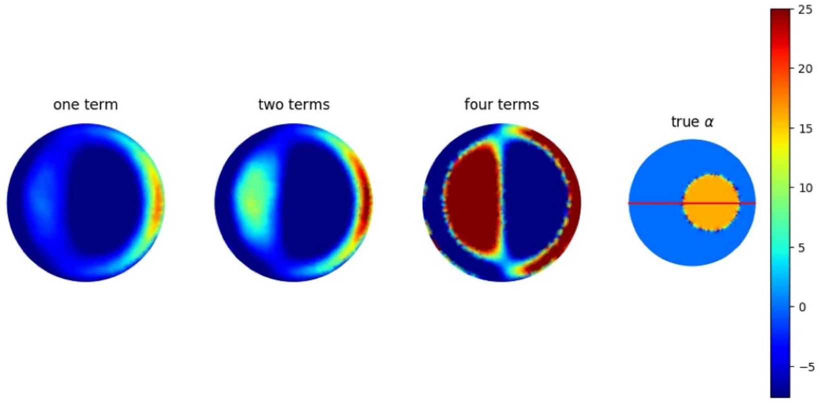


(A) Reconstruction of α with $\beta = 0$.

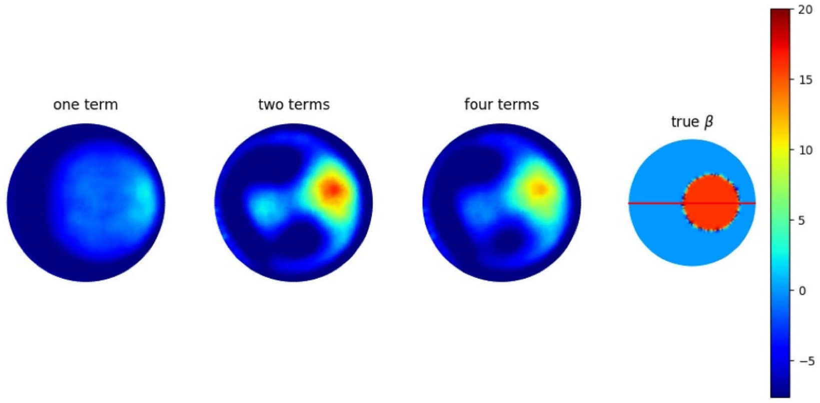


(B) Reconstruction of β with $\alpha = 0$.

Figure 4. Cross sections of α and β at intermediate contrast.

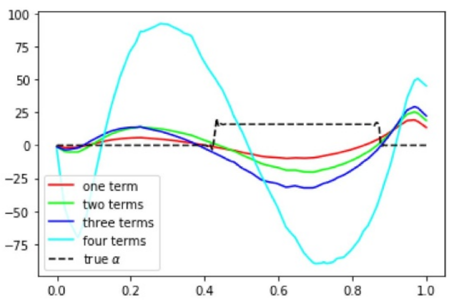


(A) Reconstruction of α with $\beta = 0$.

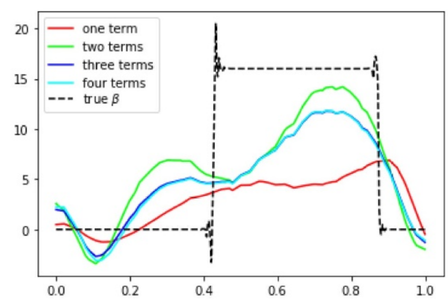


(B) Reconstruction of β with $\alpha = 0$.

Figure 5. Reconstructions of α or β at high contrast.



(A) Reconstruction of α with $\beta = 0$.



(B) Reconstruction of β with $\alpha = 0$.

Figure 6. Cross sections of α and β at high contrast.

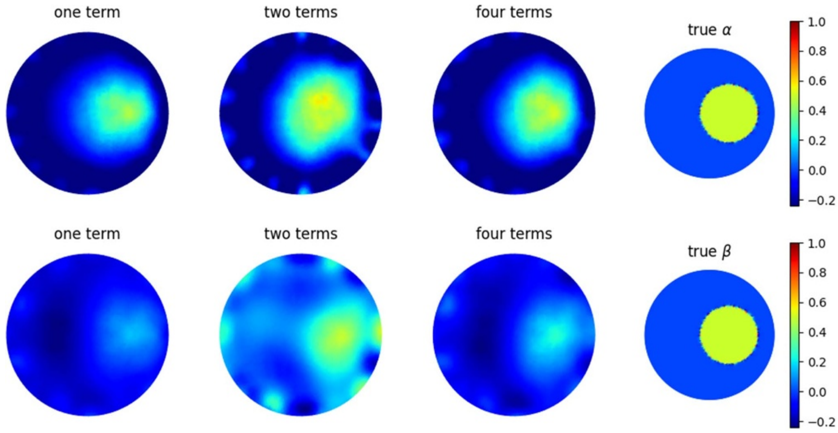


Figure 7. Simultaneous reconstruction of α and β with $g_0 = 1$.

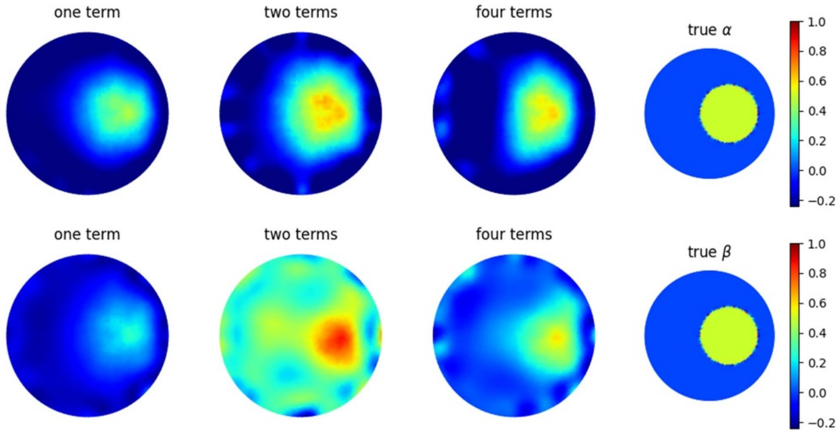


Figure 8. Simultaneous reconstruction of α and β with $g_0 = 2$.

linear PDEs, the Born series for nonlinear PDEs has a complex combinatorial structure. By analyzing this structure, we have found conditions which guarantee the convergence of the IBS.

The ideas developed in this paper provide a framework for studying inverse problems for a wide class of nonlinear PDEs with polynomial nonlinearities. The formulas and algorithm for generating the forward operators, the use of the generating functions, and the resulting reconstruction algorithm are readily generalizable to this setting and will be explored in future work.

Data availability statement

No new data were created or analyzed in this study.

Acknowledgments

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Appendix. Proof of proposition 1

In this appendix we obtain conditions for existence of a unique solution to (5) and give alternative conditions on α and β that guarantee convergence of the Born series. Define the linear operator

$$G : C^0\left(\overline{\Omega}\right) \to C^0\left(\overline{\Omega}\right)$$

by

$$G(v) = -k^2 \int_\Omega G(x,y)\, v(y)\, \mathrm{d}y.$$

Then, for u_0 in $C^0(\overline{\Omega})$, we have that T can be written as

$$T(v) = u_0 + G\left(\alpha v + \beta v^3\right). \tag{37}$$

Note that G is compact and bounded. Define μ by

$$\mu = k^2 \sup_{x\in\Omega} \int_\Omega |G(x,y)|\mathrm{d}y, \tag{38}$$

Then we have that

$$\|G(v)\| \leqslant \mu\|v\|$$

for all $v \in C^0(\overline{\Omega})$, where throughout this section the norm $\|\cdot\|$ denotes $\|\cdot\|_{C(\overline{\Omega})}$. We will make use of the following two lemmas. The first gives conditions to have a contraction.

Lemma 4. *For any $f, g \in C^0(\overline{\Omega})$ such that $\|f\|, \|g\| \leqslant R$, we have*

$$\|T(f) - T(g)\| \leqslant q\|f - g\|$$

where

$$q = \mu\left(\|\alpha\| + 3R^2\,\|\beta\|\right)$$

for μ defined by (38).

Proof. Let $f, g \in B$. Then,

$$T(f) - T(g) = k^2 \int_\Omega G(x,y)\left[\alpha(y)\left(f(y) - g(y)\right) + \beta(y)\left(f(y)^3 - g(y)^3\right)\right] \mathrm{d}y \tag{39}$$

so that

$$
\begin{aligned}
\|T(f)-T(g)\| &\leqslant \mu\left\|\alpha(f-g)+\beta\left(f^3-g^3\right)\right\| \\
&= \mu\left\|\left[\alpha+\beta\left(f^2+fg+g^2\right)\right](f-g)\right\| \\
&\leqslant \mu\left(\|\alpha\|+\left\|\beta\left(f^2+fg+g^2\right)\right\|\right)\cdot\|(f-g)\| \\
&\leqslant \mu\left(\|\alpha\|+3R^2\|\beta\|\right)\cdot\|f-g\|. \qquad (40)
\end{aligned}
$$

□

The second lemma gives us a ball which T maps into itself.

Lemma 5. *Let $r>0$ be given, and let $B=B(u_0,r)$ be the ball of radius r about u_0 in $C^0(\Omega)$. Define $R=\|u_0\|+r$. Then if*

$$\mu R\left(\|\alpha\|+R^2\|\beta\|\right)<r,$$

we have that $T(v)\in B$ for any $v\in B$, and hence T has a fixed point in B.

Proof. Assume that $v\in B$. Then, by the triangle inequality, we have that $\|v\|\leqslant R$. Then,

$$
\begin{aligned}
\|T(v)-u_0\| &= \left\|G\left(\alpha v+\beta v^3\right)\right\| && (41)\\
&\leqslant \mu\left\|\alpha v+\beta v^3\right\| && (42)\\
&\leqslant \mu\left(R\|\alpha\|+R^3\|\beta\|\right). && (43)
\end{aligned}
$$

By hypothesis, this is less than r, and hence $T(v)\in B$. □

Lemma 6. *If the hypotheses of lemma 5 hold and, additionally,*

$$\mu\left(\|\alpha\|+3R^2\|\beta\|\right)<1,$$

then T is a contraction mapping on B.

Proof. Since $\mu\|\alpha\|+3R^2\|\beta\|<1$ by assumption, by lemma 4 we have that $\mathrm{d}(T(f),T(g))\leqslant q\cdot\mathrm{d}(f,g)$ where $q<1$. Since T also maps B into itself, T is a contraction mapping on B. □

Clearly if lemma 6 holds, by the Banach fixed point theorem T will have a unique fixed point on B. Furthermore, if we start with initial function u_0 in B, then fixed-point iteration will converge to the unique fixed point, which in this case is the solution to the integral of the PDE defined by (5). The iterates of the fixed point iteration will generate the (forward) Born series.

Remark. We note that If β and α satisfy the (more restrictive) condition

$$\mu\left(\|\alpha\|+3R^2\|\beta\|\right)<\frac{r}{R},$$

then both lemmas 5 and 6 are also satisfied.

The following is well known for the linear case, see for example [21].

Proposition 3. *If $\beta=0$ and $\|\alpha\|<\frac{1}{\mu}$, then T has a unique fixed point on all of $C^0(\Omega)$.*

Proof. In this case, lemmas 5 and 6 are both satisfied if

$$\mu\|\alpha\|<\frac{r}{r+\|u_0\|}.$$

Since by assumption $\mu\|\alpha\| < 1$, there exists r_0 such that for any $r > r_0$

$$\mu\|\alpha\| < \frac{r}{r + \|u_0\|} < 1.$$

Therefore T has a unique fixed point on $B(u_0, r)$ for any $r > r_0$, so the fixed point must be unique on all of $C^0(\Omega)$. □

Proposition 4. *If* $\alpha = 0$ *and* $\beta < \frac{4}{27\mu\|u_0\|^2}$, *then* T *has a unique fixed point in the ball* $B(u_0, \|u_0\|/2)$.

Proof. Let $r = \|u_0\|/2$. This means that $R = 3\|u_0\|/2$. Then,

$$\begin{aligned}
\mu R\left(\|\alpha\| + R^2\|\beta\|\right) &= \mu R^3\|\beta\| \\
&= \mu\left(3\|u_0\|/2\right)^3\|\beta\| \\
&< \frac{27\mu\|u_0\|^3}{8} \cdot \frac{4}{27\mu\|u_0\|^2} \\
&= \frac{\|u_0\|}{2} = r.
\end{aligned}$$

Thus, the hypothesis needed for lemma 5 is satisfied. Additionally, we have that

$$\begin{aligned}
\mu\left(\|\alpha\| + 3R^2\|\beta\|\right) &= 3R^2\mu\|\beta\| \\
&= 3\left(3\|u_0\|/2\right)^2\mu\|\beta\| \\
&< \frac{27\mu\|u_0\|^2}{4} \cdot \frac{4}{27\mu\|u_0\|^2} \\
&= 1,
\end{aligned}$$

so that the condition for lemma 6 to hold is also satisfied, and hence T has a unique fixed point on $B(u_0, r)$. □

Proposition 5. *If there exists some* $\gamma > 1/2$ *such that*

$$\|\alpha\| < \frac{2\gamma - 1}{2\mu(1+\gamma)}$$

and

$$\|\beta\| < \frac{1}{2\mu\|u_0\|^2(1+\gamma)^3},$$

then T *has a unique fixed point in the ball* $B(u_0, \gamma\|u_0\|)$.

Proof. Let $r = \gamma\|u_0\|$, which means that $R = (1+\gamma)\|u_0\|$. The hypotheses then imply that lemma 5 holds, because

$$
\begin{aligned}
\mu R\left(\|\alpha\|+R^{2}\|\beta\|\right) &= \mu(1+\gamma)\|u_{0}\|\left(\|\alpha\|+(1+\gamma)^{2}\|u_{0}\|^{2}\|\beta\|\right) \\
&< \mu(1+\gamma)\|u_{0}\|\left(\frac{2\gamma-1}{2\mu(1+\gamma)}+\frac{1}{2\mu(1+\gamma)}\right) \\
&= \|u_{0}\|\left(\frac{2\gamma-1}{2}+\frac{1}{2}\right) \\
&= \gamma\|u_{0}\| = r.
\end{aligned}
$$

For lemma 6, we have that

$$
\begin{aligned}
\mu\left(\|\alpha\|+3R^{2}\|\beta\|\right) &= \mu\left(\|\alpha\|+3(1+\gamma)^{2}\|u_{0}\|^{2}\|\beta\|\right) \\
&< \mu\left(\frac{2\gamma-1}{2\mu(1+\gamma)}+(1+\gamma)^{2}\|u_{0}\|^{2}\frac{1}{2\mu\|u_{0}\|^{2}(1+\gamma)^{3}}\right) \\
&= \frac{2\gamma-1}{2(1+\gamma)}+\frac{1}{2(1+\gamma)} \\
&= \frac{\gamma}{1+\gamma} < 1.
\end{aligned}
$$

Thus T has a unique fixed point on the ball $B(u_0, \gamma\|u_0\|)$. □

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